Heavy-Tailed Large Deviations and How to Eliminate Sharp Minima from SGD

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Northwestern University

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Based on the joint works with

Mihail Bazhba, Jose Blanchet, Bohan Chen, Sewoong Oh, Zhe Su, Xingyu Wang, and Bert Zwart

Team

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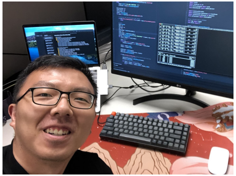

Xingyu Wang
Northwestern

Generalization Mystery of Deep Learning

Modern “Artificial Intelligence”

“AI is in a ‘golden age’ and solving problems that were once in the realm of sci-fi.”

- Jeff Bezos

Empirical Success of Deep Neural Networks (DNNs)

"Deep Learning is eating the world."

- Jorge Nocedal

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- Despite massive over-parametrization, DNNs generalize well.

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Training Set

Test Set image source

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A Central Mystery of Deep Learning

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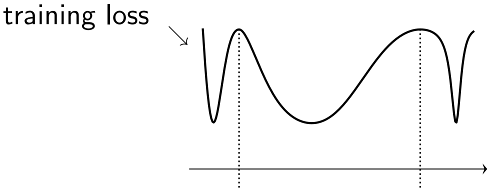

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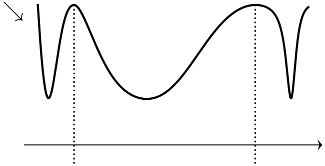

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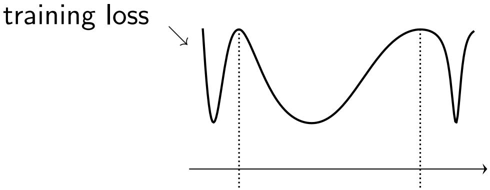

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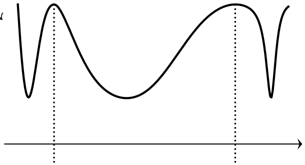

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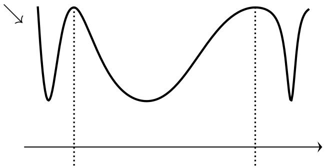

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Heavy-tailed SGD escapes local minima and prefers flat local minima.

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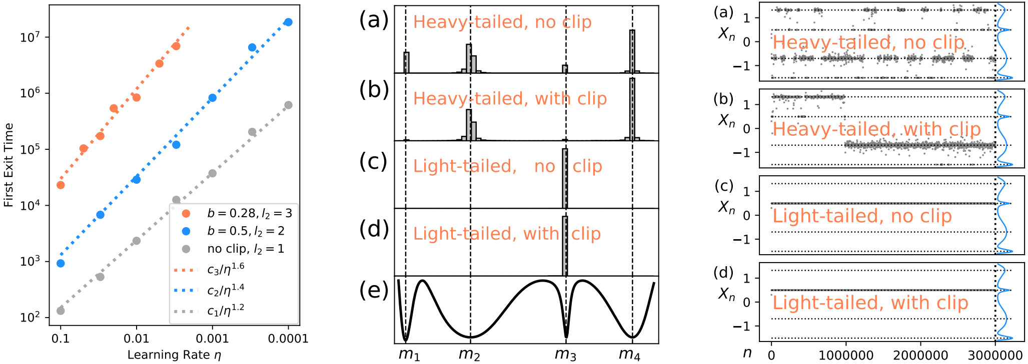
However, when SGD is heavy-tailed, one often truncates gradients.

Entirely Different Global Dynamics Depending on Tail Behaviors



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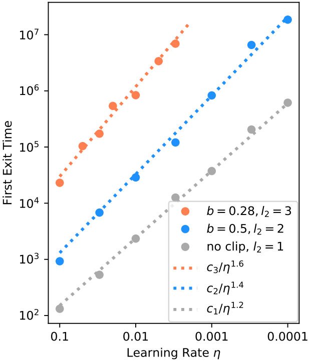
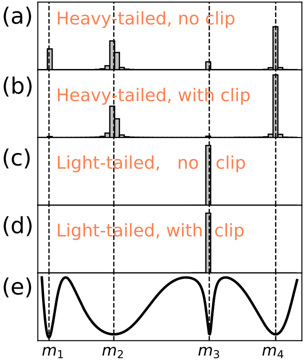
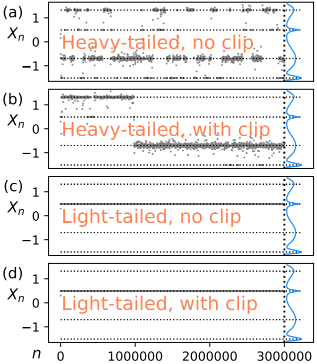
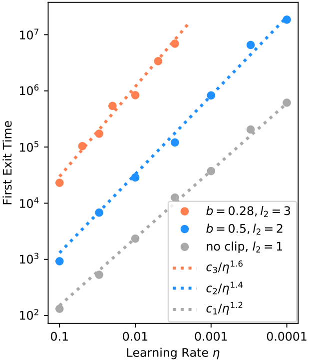


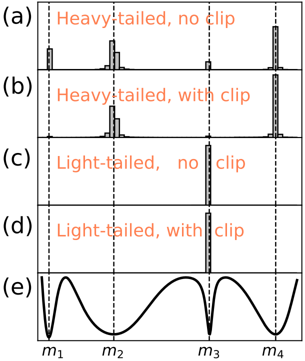


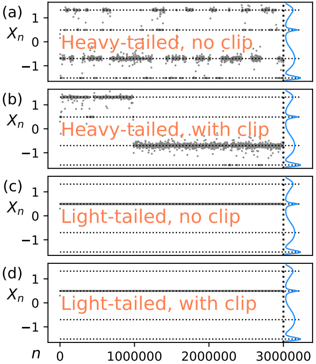


Explanation?

Entirely Different Global Dynamics Depending on Tail Behaviors







Explanation? Catastrophe Principle.

Heavy Tails and Catastrophe Principle

Rare Events depend on “Tail Behaviors”

Light-Tailed Distributions

- Extreme Values are Very Rare
- Normal, Exponential, etc

Heavy-Tailed Distributions

- Extreme Values are Frequent
- Power Law, Weibull, etc

Instagram

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Instagram

Structural difference in the way systemwide rare events arise.

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arise because

EVERYTHING goes wrong.

(Conspiracy Principle)

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Systemwide rare events

arise because of

A FEW Catastrophes.

(Catastrophe Principle)

Structural difference in the way systemwide rare events arise.

Insurance Example: Capital Reserve

$$Y(t) = c + pt - \sum_{i=1}^{N(t)} X_i$$

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Premium

i.i.d. Claim Size

Insurance Example: Capital Reserve

Poisson Arrival

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Large n: analysis of large loss over a long time period

Typical Scenario

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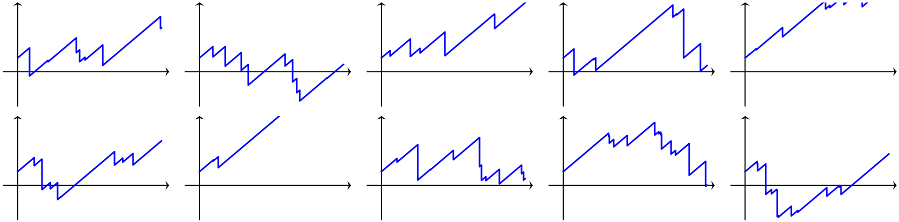

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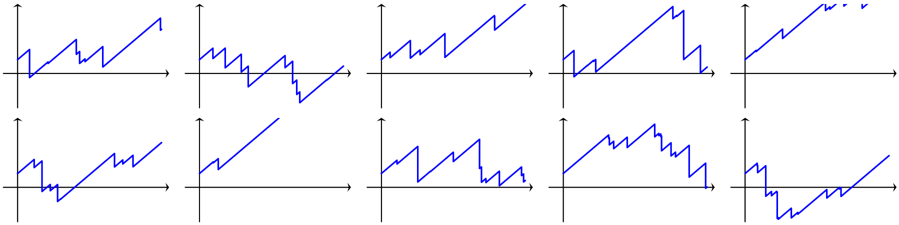

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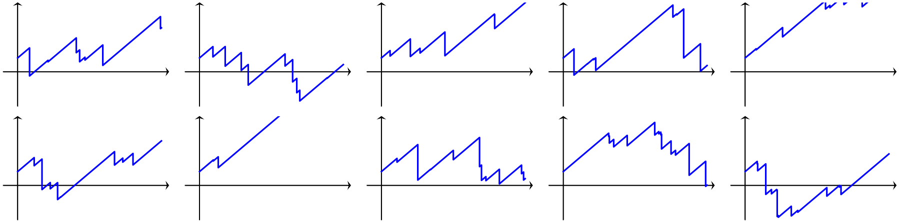

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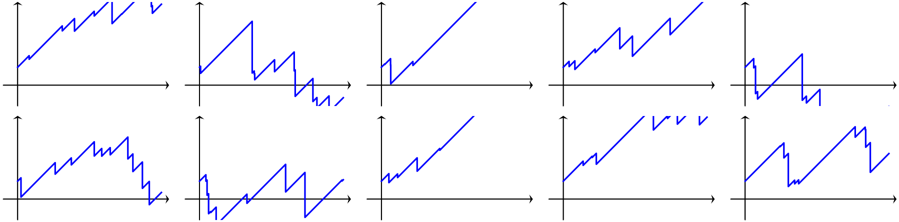

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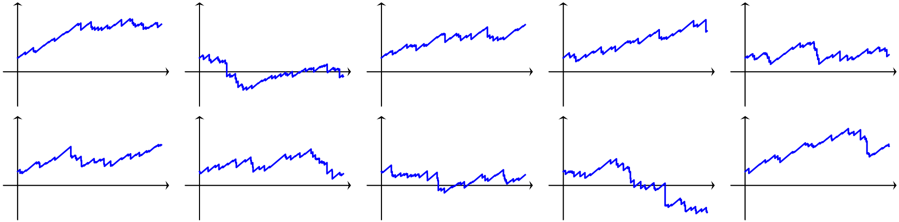
Sample paths of $\bar{Y}_n$: n=50 & claim sizes are **light-tailed**



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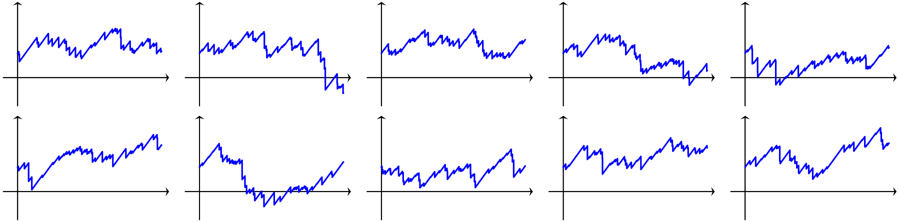

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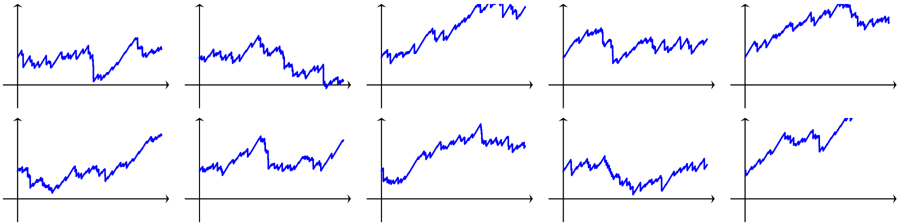
Sample paths of $\bar{Y}_n$: n=100 & claim sizes are **light-tailed**



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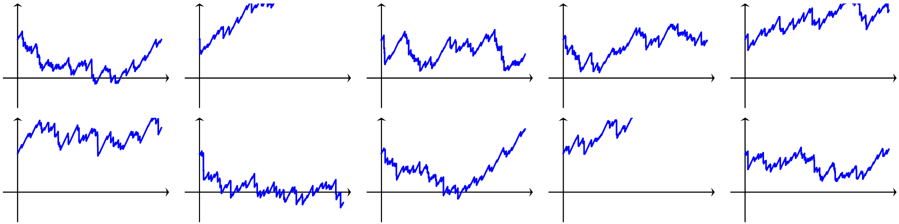

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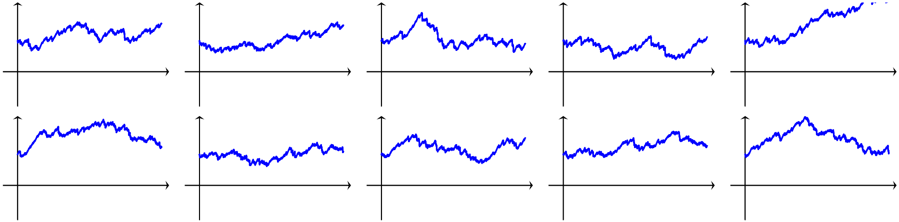
Sample paths of $\bar{Y}_n$: n=500 & claim sizes are **light-tailed**



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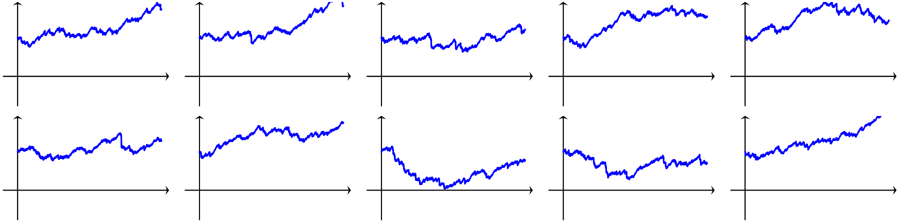

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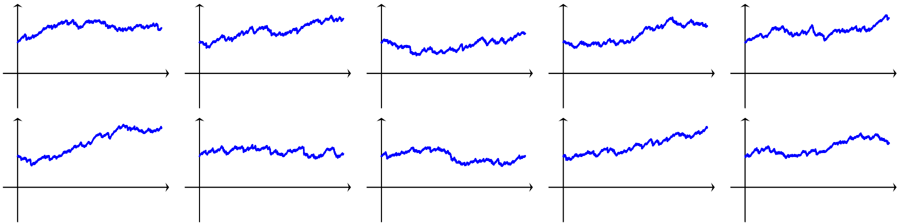
Sample paths of $\bar{Y}_n$: n=1000 & claim sizes are **light-tailed**



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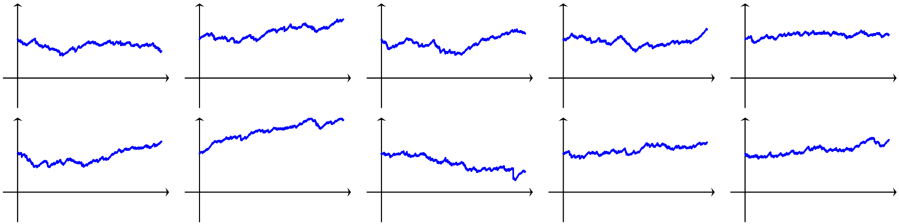

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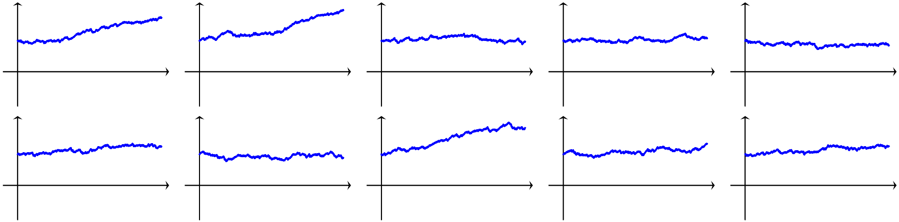
Sample paths of $\bar{Y}_n$: n=2500 & claim sizes are **light-tailed**



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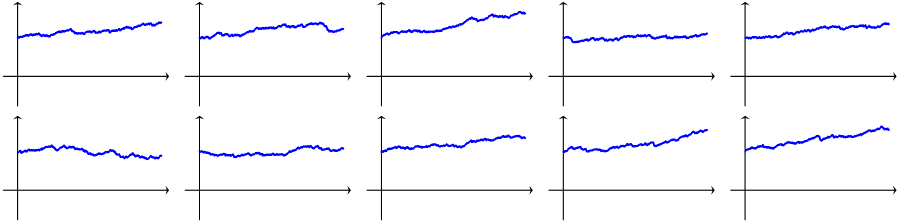

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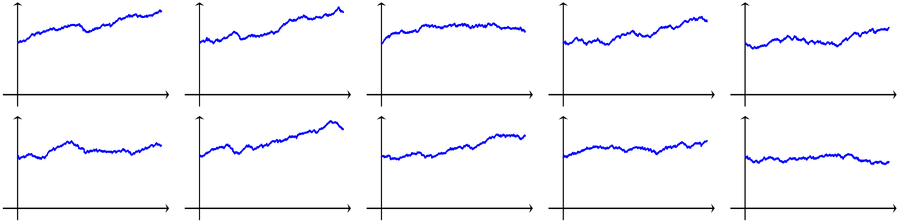
Sample paths of $\bar{Y}_n$: n=5000 & claim sizes are **light-tailed**



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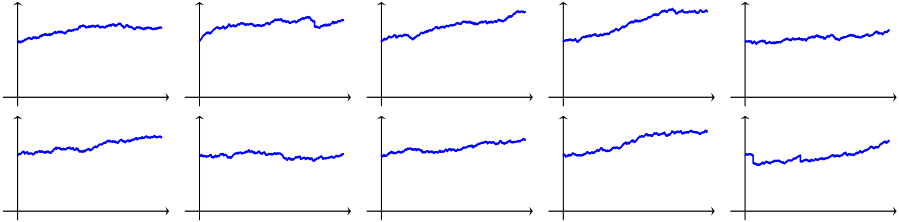

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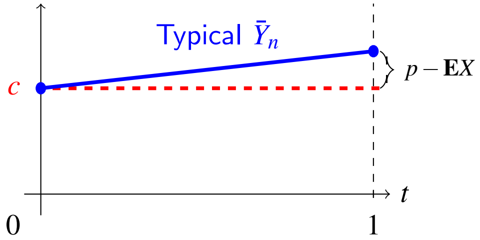

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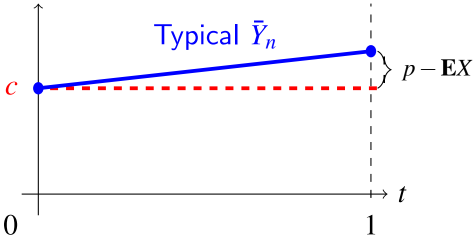

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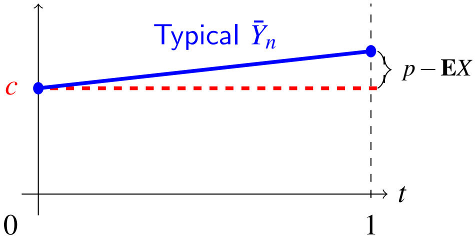
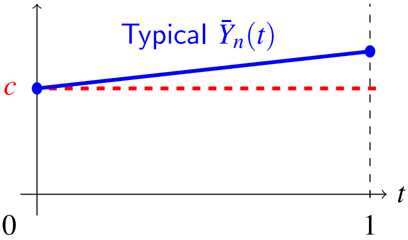


Typically, your business will flourish

regardless of the tail distributions.

What about atypical cases?

A Rare Event: Bankruptcy



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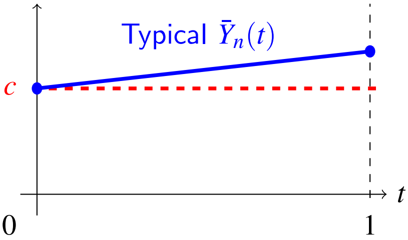
Consider $B \triangleq \{\ \bar{Y}_n$ falls below 0 on $[0,1]\ \}$. (i.e., Bankruptcy)



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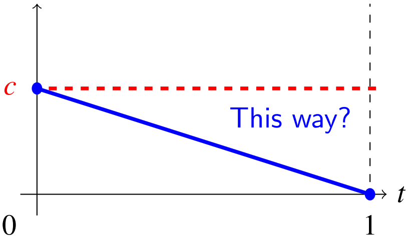

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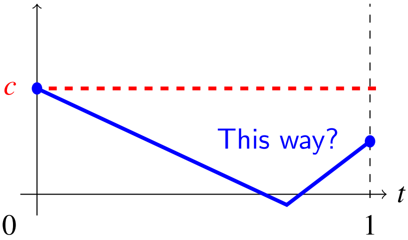

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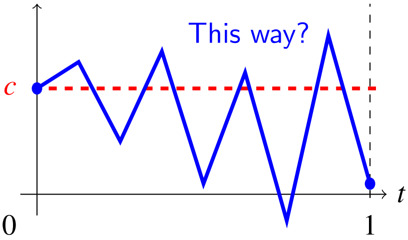

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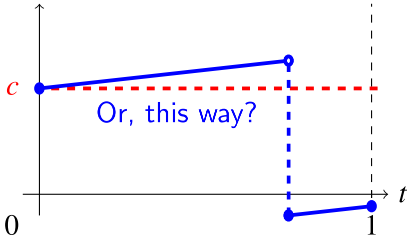

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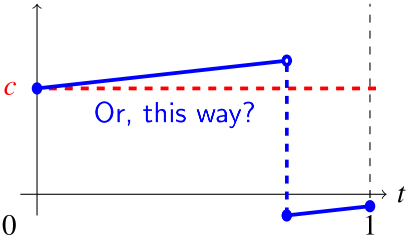

Are we going to see clear patterns?

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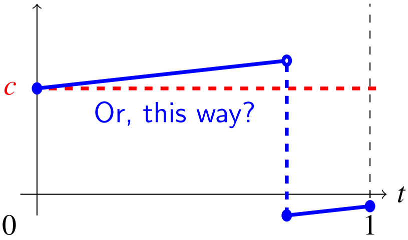

Are we going to see clear patterns?

Do they depend on the tail distributions?

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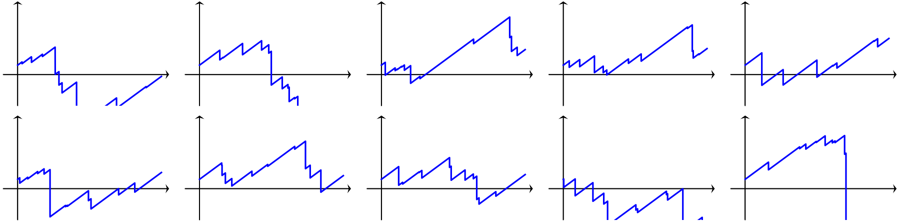
Sample paths of $\bar{Y}_{10}$ conditional on B for **light-tailed** claims:



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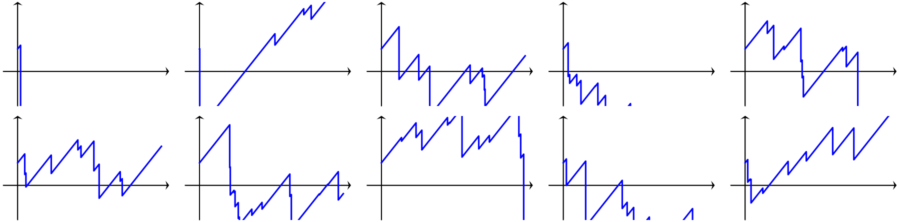

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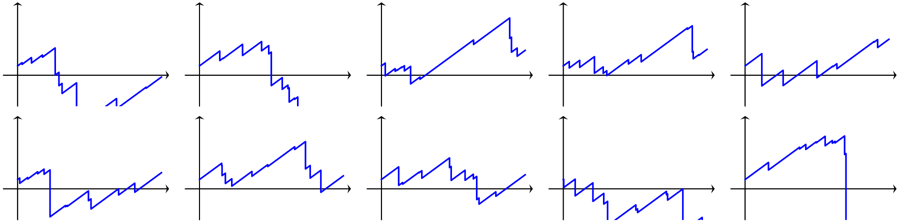

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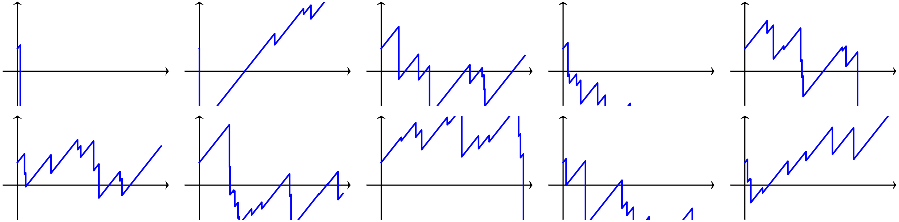

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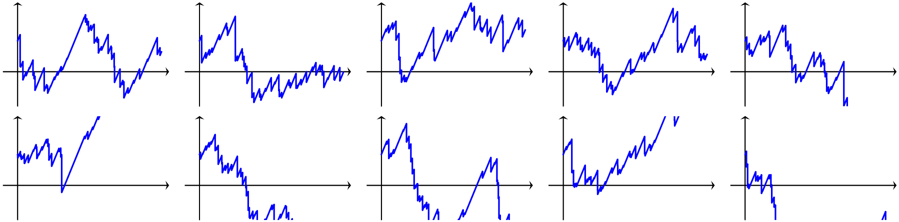
Sample paths of $\bar{Y}_{50}$ conditional on B for **light-tailed** claims:



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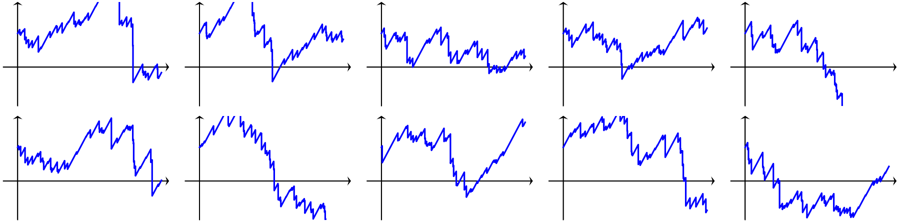
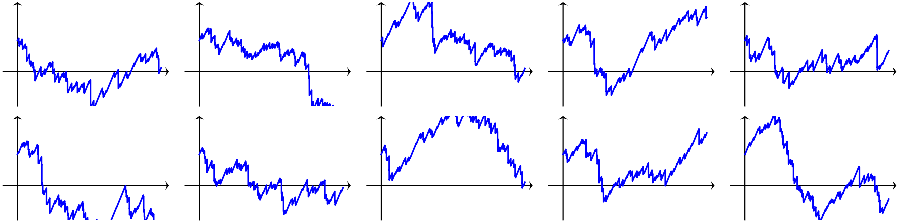


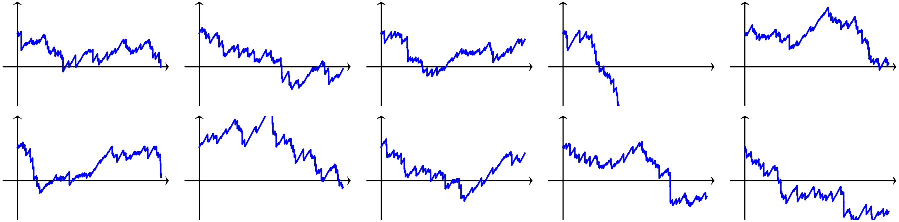
A Rare Event: Bankruptcy

Bankruptcy

Sample paths of $\bar{Y}_{100}$ conditional on B for **light-tailed** claims:



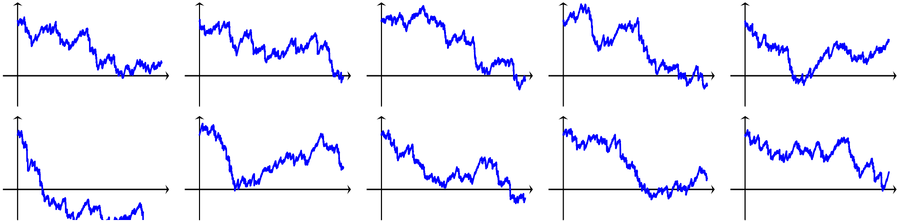
Sample paths of $\bar{Y}_{100}$ conditional on B for **heavy-tailed** claims:



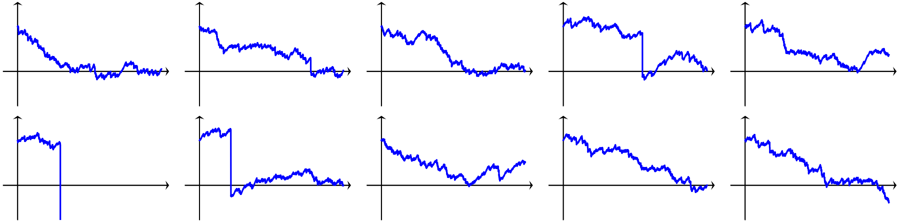
A Rare Event: Bankruptcy

Bankruptcy

Sample paths of $\bar{Y}_{500}$ conditional on B for **light-tailed** claims:



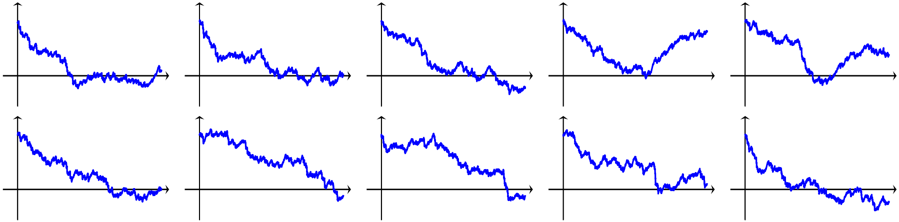
Sample paths of $\bar{Y}_{500}$ conditional on B for **heavy-tailed** claims:



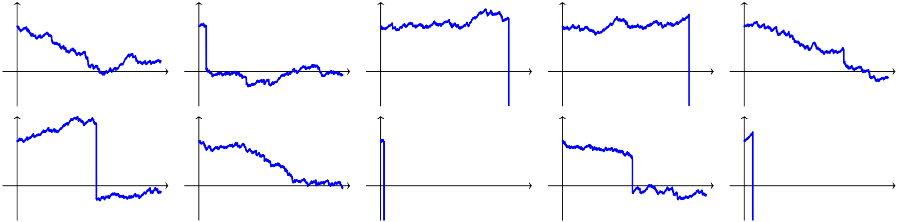
A Rare Event: Bankruptcy

Bankruptcy

Sample paths of $\bar{Y}_{1000}$ conditional on B for **light-tailed** claims:



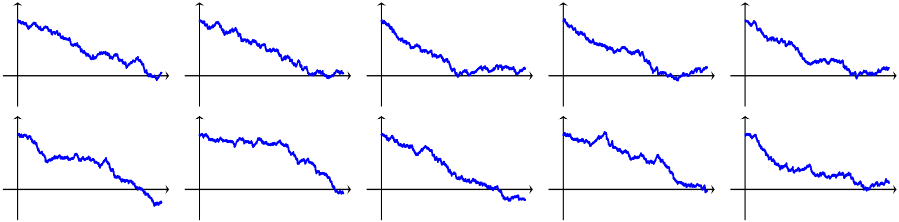
Sample paths of $\bar{Y}_{1000}$ conditional on B for **heavy-tailed** claims:



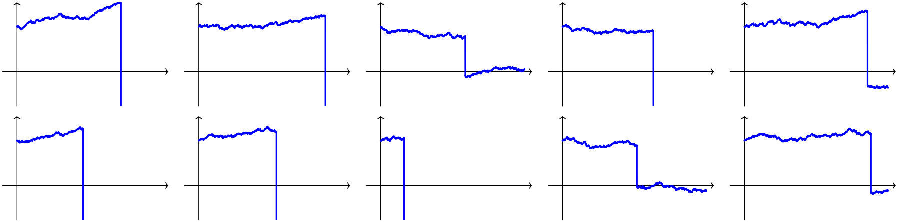
A Rare Event: Bankruptcy

Bankruptcy

Sample paths of $\bar{Y}_{2500}$ conditional on B for **light-tailed** claims:



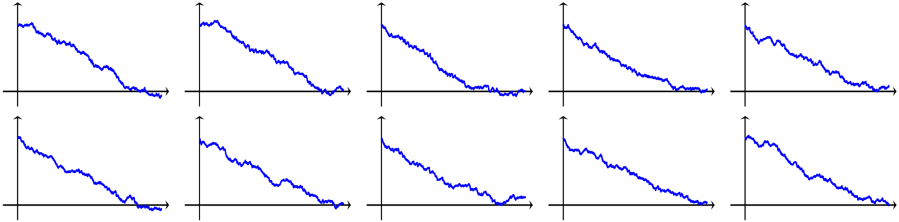
Sample paths of $\bar{Y}_{2500}$ conditional on B for **heavy-tailed** claims:



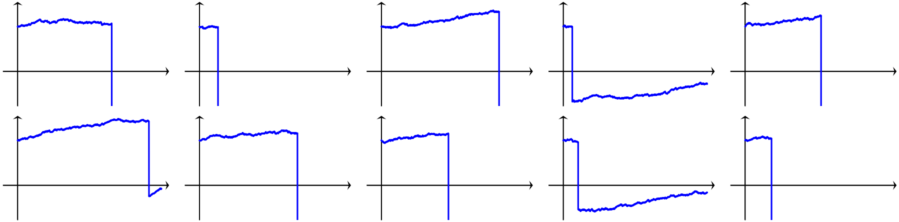
A Rare Event: Bankruptcy

Bankruptcy

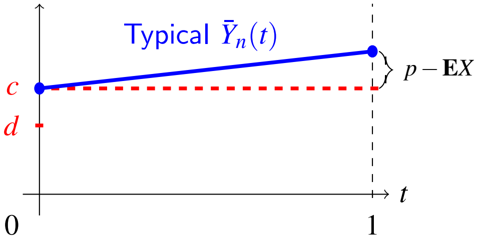
Sample paths of $\bar{Y}_{5000}$ conditional on B for **light-tailed** claims:



Sample paths of $\bar{Y}_{5000}$ conditional on B for **heavy-tailed** claims:

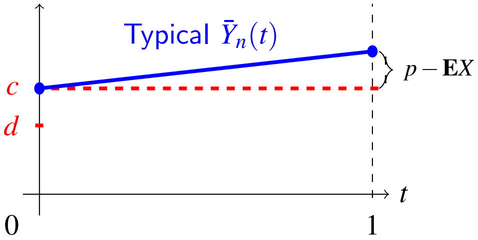


Bankruptcy Despite of Reinsurance



Bankruptcy Despite of Reinsurance

Consider $R \triangleq \{\ \bar{Y}_n$ falls below 0 on [0,1], jump sizes $\leq d\ \}$



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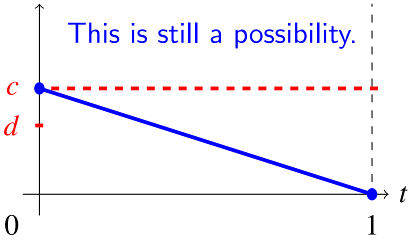

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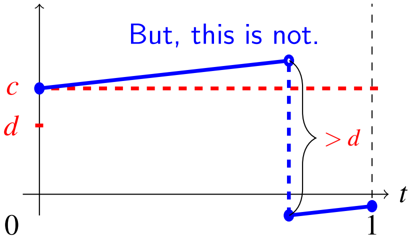

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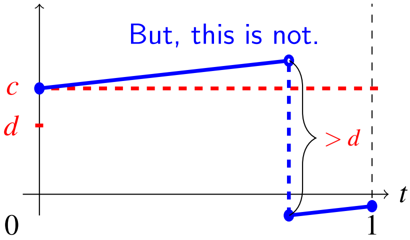
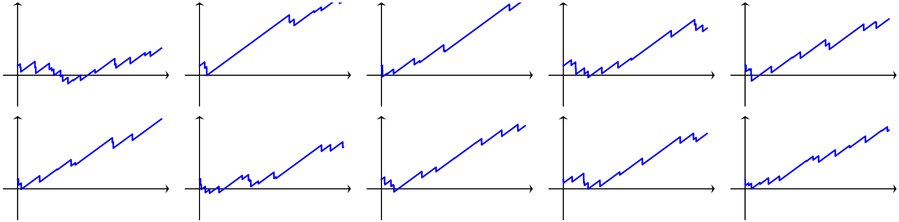


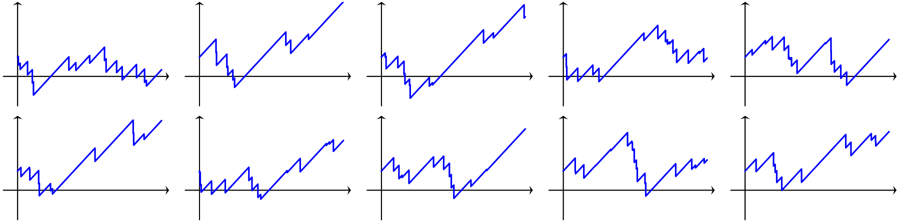
How does the pattern change in this case?

Bankruptcy Despite of Reinsurance

Sample paths of $\bar{Y}_{10}$ conditional on R for **light-tailed** claims:

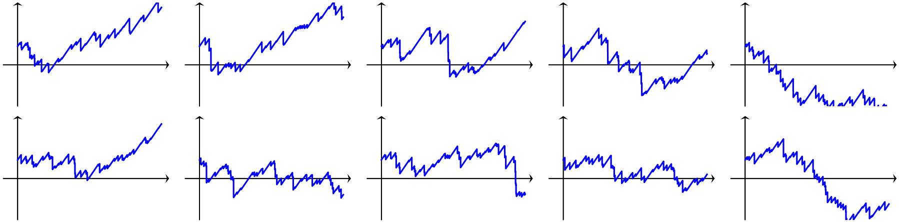


Sample paths of $\bar{Y}_{10}$ conditional on R for **heavy-tailed** claims:

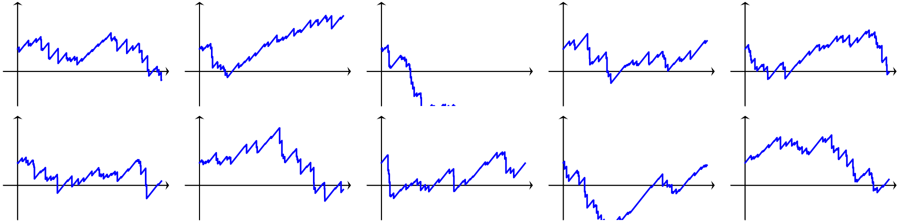


Bankruptcy Despite of Reinsurance

Sample paths of $\bar{Y}_{50}$ conditional on R for **light-tailed** claims:

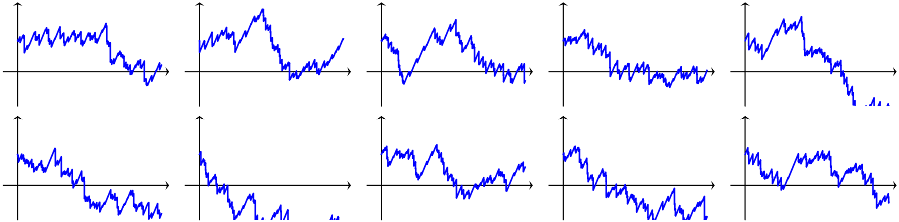


Sample paths of $\bar{Y}_{50}$ conditional on R for **heavy-tailed** claims:

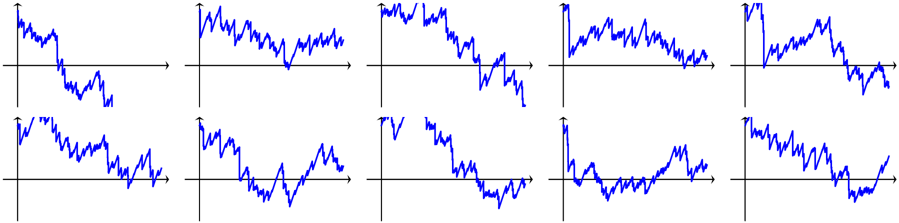


Bankruptcy Despite of Reinsurance

Sample paths of $\bar{Y}_{100}$ conditional on R for **light-tailed** claims:

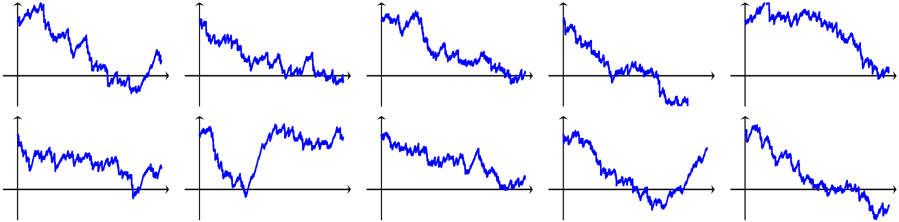


Sample paths of $\bar{Y}_{100}$ conditional on R for **heavy-tailed** claims:

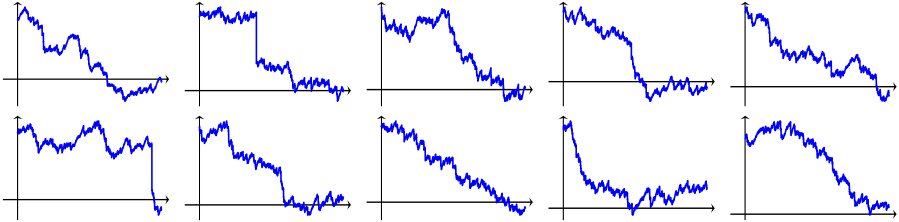


Bankruptcy Despite of Reinsurance

Sample paths of $\bar{Y}_{500}$ conditional on R for **light-tailed** claims:

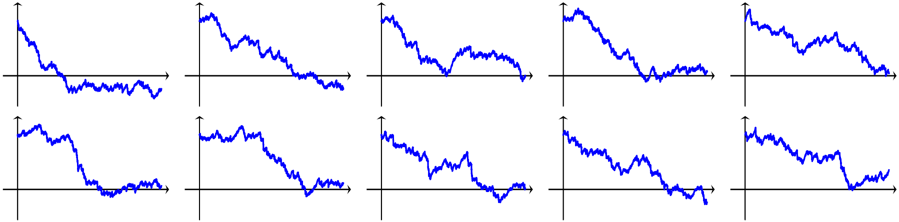


Sample paths of $\bar{Y}_{500}$ conditional on R for **heavy-tailed** claims:

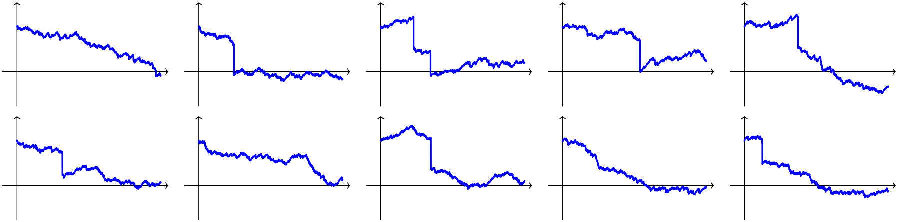


Bankruptcy Despite of Reinsurance

Sample paths of $\bar{Y}_{1000}$ conditional on R for **light-tailed** claims:

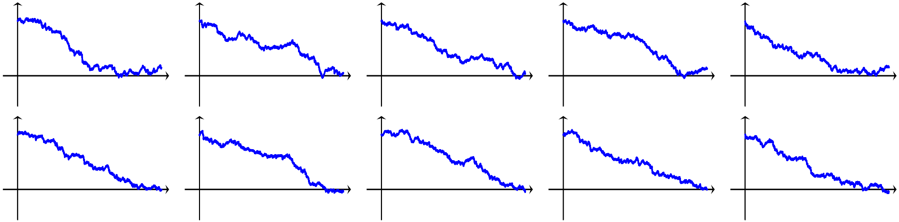


Sample paths of $\bar{Y}_{1000}$ conditional on R for **heavy-tailed** claims:

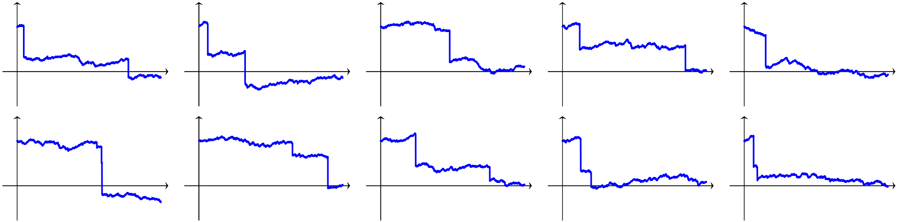


Bankruptcy Despite of Reinsurance

Sample paths of $\bar{Y}_{2500}$ conditional on R for **light-tailed** claims:

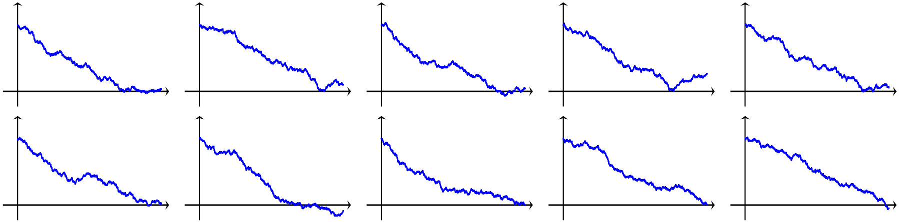


Sample paths of $\bar{Y}_{2500}$ conditional on R for **heavy-tailed** claims:

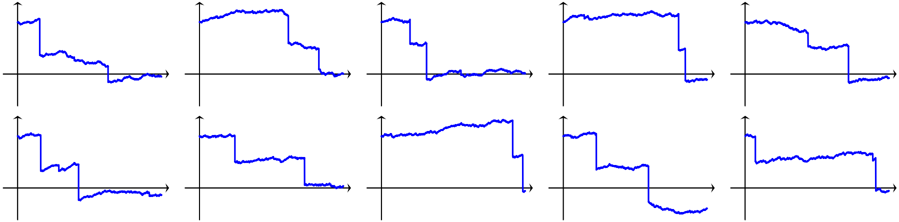


Bankruptcy Despite of Reinsurance

Sample paths of $\bar{Y}_{5000}$ conditional on R for **light-tailed** claims:

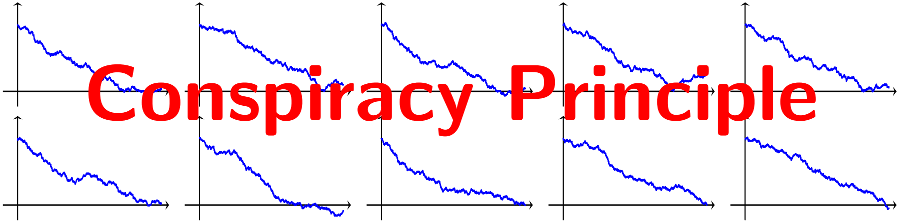


Sample paths of $\bar{Y}_{5000}$ conditional on R for **heavy-tailed** claims:

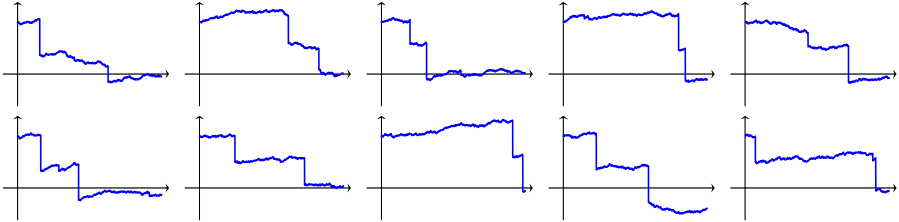


Bankruptcy Despite of Reinsurance

Sample paths of $\bar{Y}_{5000}$ conditional on R for **light-tailed** claims:



Sample paths of $\bar{Y}_{5000}$ conditional on R for **heavy-tailed** claims:



Bankruptcy Despite of Reinsurance

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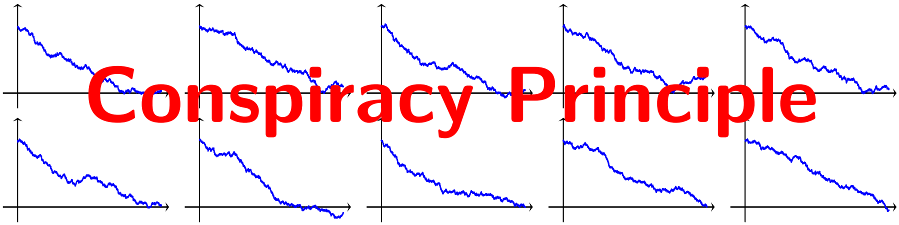
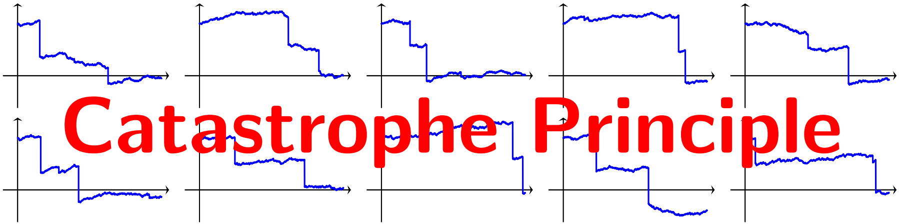


Sample paths of $\bar{Y}_{5000}$ conditional on R for **heavy-tailed** claims:



Heavy-Tailed Large Deviations:

Rigorous Characterization of Catastrophe Principle

Heavy-Tailed Large Deviations

$$\bar{S}_n(t) = \frac{1}{n}\sum_{i=1}^{\lfloor nt \rfloor} X_i,$$

X_i: centered iid r.v. with $\mathbf{P}(X_i \geq x) = x^{-(\alpha+1)}, \quad \alpha > 0$

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Theorem (R., Blanchet, Zwart, 2019)

For "general" $A \subseteq \mathbb{D}$

$$C(A^\circ) \leq \liminf_{n\to\infty} \frac{\mathbf{P}(\bar{S}_n \in A)}{n^{-\alpha \mathcal{J}(A)}} \leq \limsup_{n\to\infty} \frac{\mathbf{P}(\bar{S}_n \in A)}{n^{-\alpha \mathcal{J}(A)}} \leq C(A^-).$$

- $\mathcal{J}(A)$: *min #jumps for step functions to be inside* A
- $C(\cdot)$: *a measure*

Heavy-Tailed Large Deviations

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LD power index

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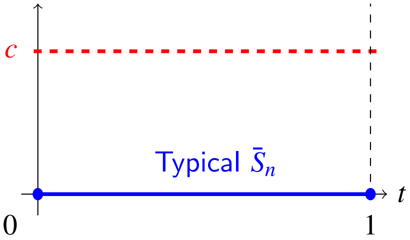
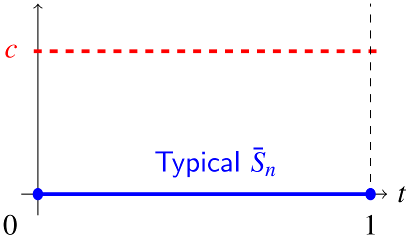
Conspiracy vs Catastrophe

$$\bar{S}_n \in \{\, f \in \mathbb{D} : \ f \text{ crosses level } c \text{ on } [0,1] \,\} = A$$



Light-Tailed Claim Size



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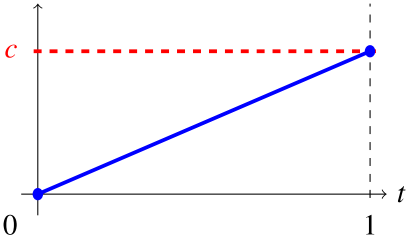

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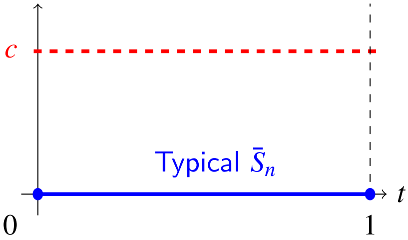

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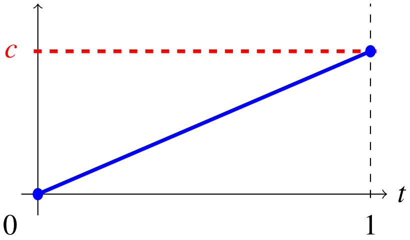

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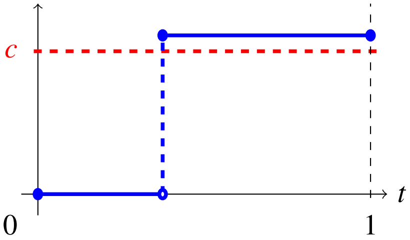

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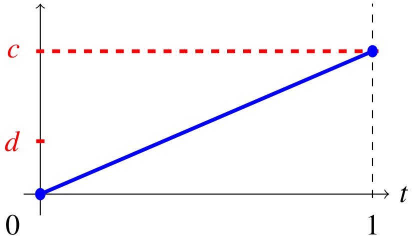

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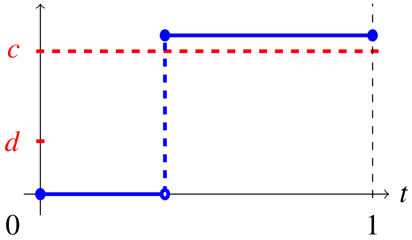

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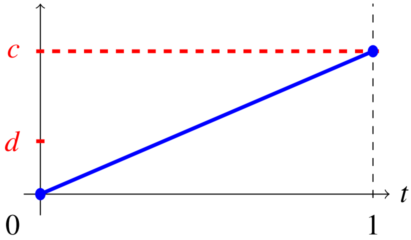

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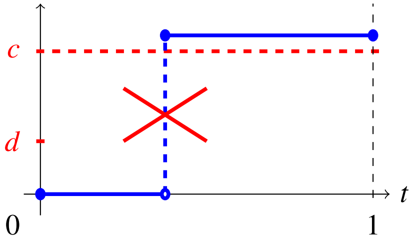

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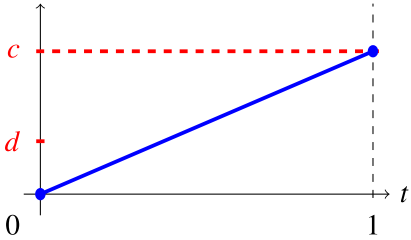
$$\bar{S}_n \in \{ f \in \mathbb{D} : f \text{ crosses level } c \text{ on } [0,1] \text{ \& jump sizes} \leq d \} = A$$



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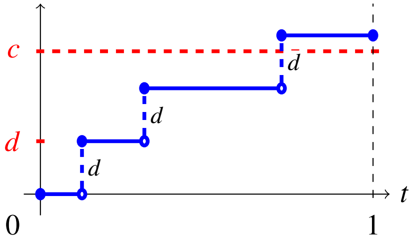
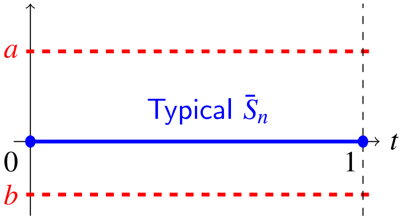


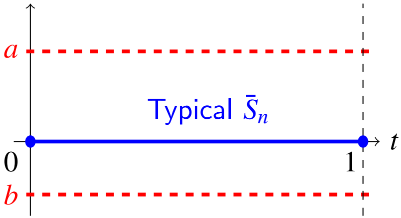
Heavy-Tailed Claim Size

Conspiracy vs Catastrophe

$$\bar{S}_n \in \{\, f \in \mathbb{D} :\ f \text{ hits below } b \text{ on } [0,1] \text{ and ends up above } a \,\} = A$$



Light-Tailed Increments



Heavy-Tailed Increments

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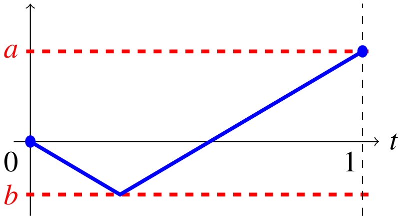

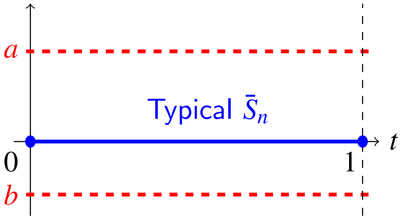

Light-Tailed Increments

Heavy-Tailed Increments

Conspiracy vs Catastrophe

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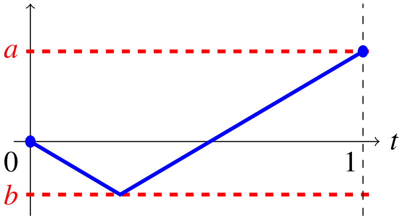

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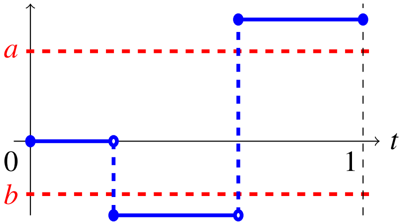
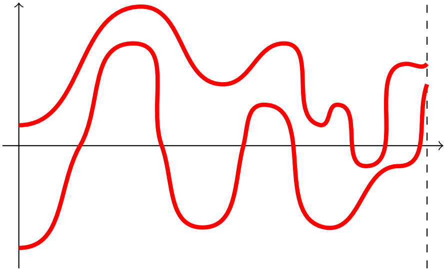


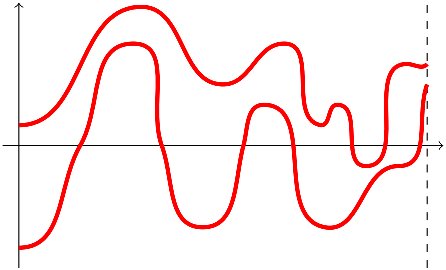
Heavy-Tailed Increments

Conspiracy vs Catastrophe

$$\bar{S}_n \in \{\, f \in \mathbb{D} : \, f \text{ lies between the two red curves} \,\}$$



Light-Tailed Increments



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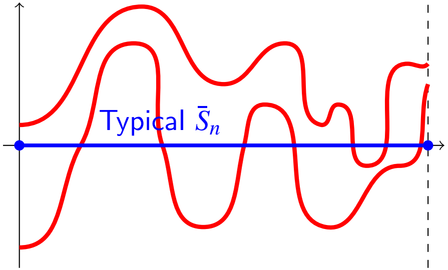

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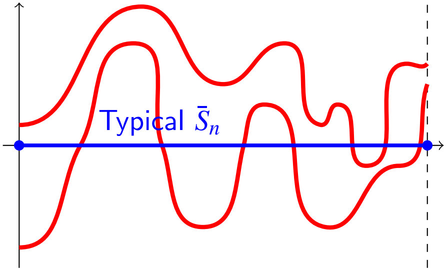
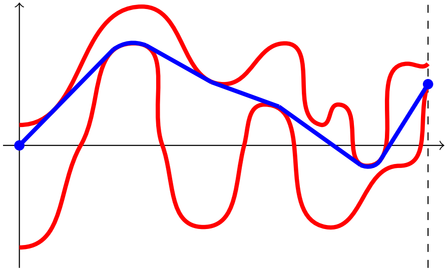


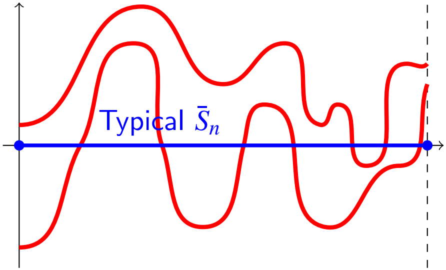
Heavy-Tailed Increments

Conspiracy vs Catastrophe

$\bar{S}_n \in \{\, f \in \mathbb{D} : \ f \text{ lies between the two red curves} \,\}$



Light-Tailed Increments



Heavy-Tailed Increments

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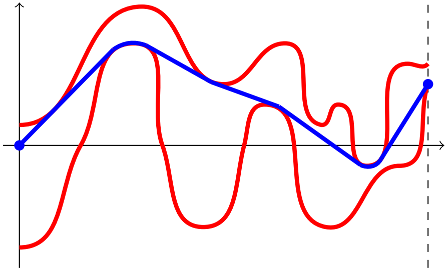

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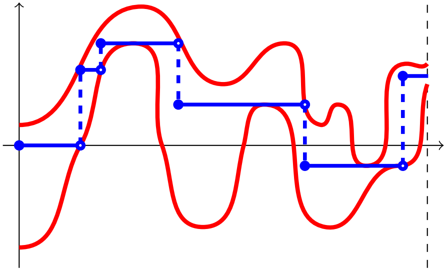
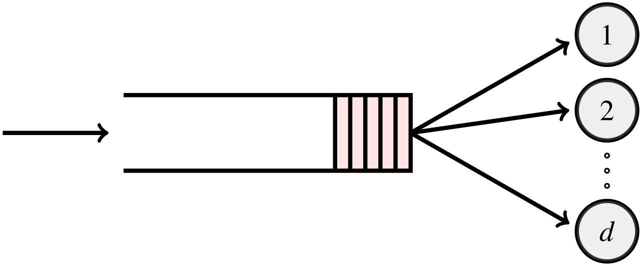


Heavy-Tailed Increments

Congestion of Multiple Server Queue:



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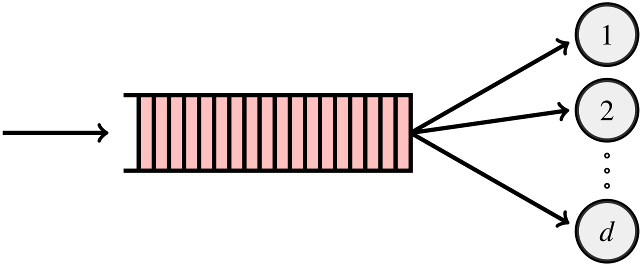
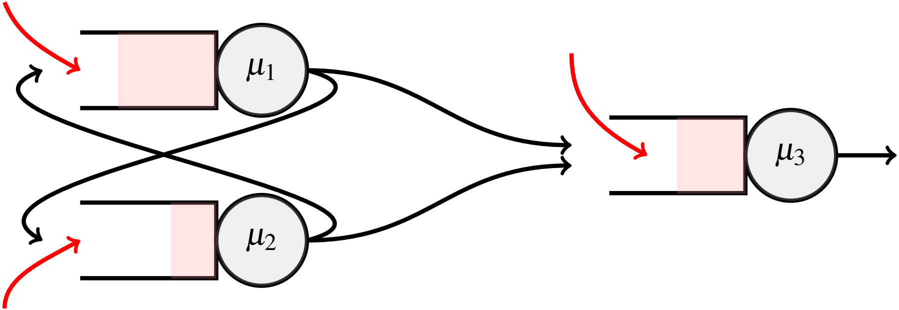


Conspiracy vs Catastrophe

Congestion of Stochastic Fluid Networks:



Conspiracy vs Catastrophe

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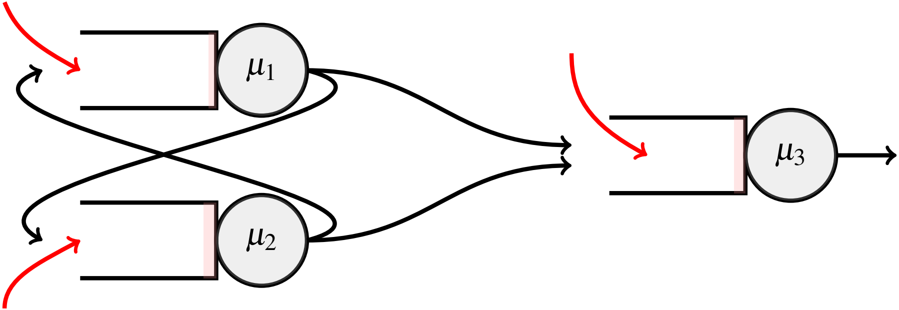

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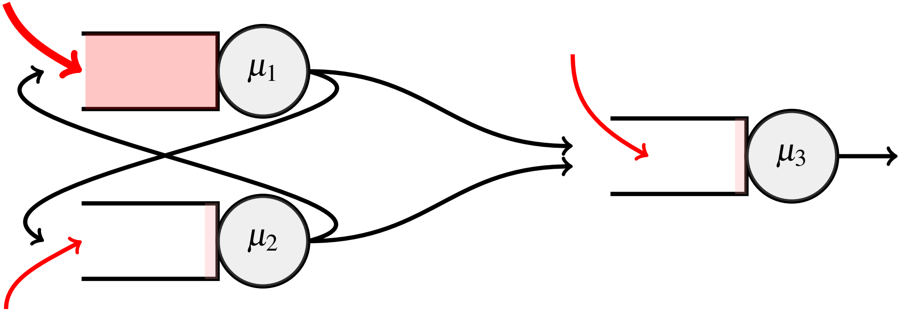

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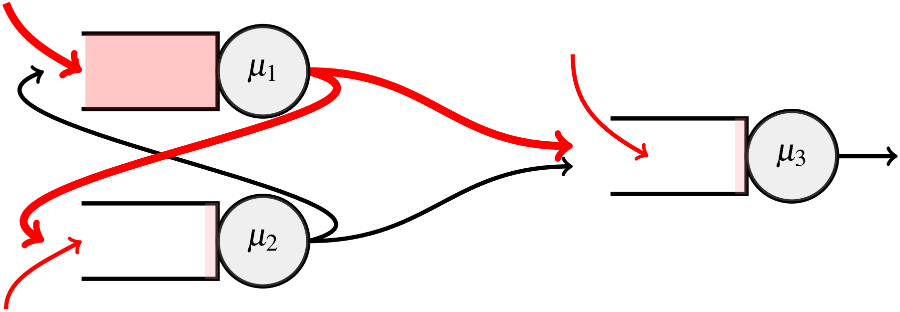

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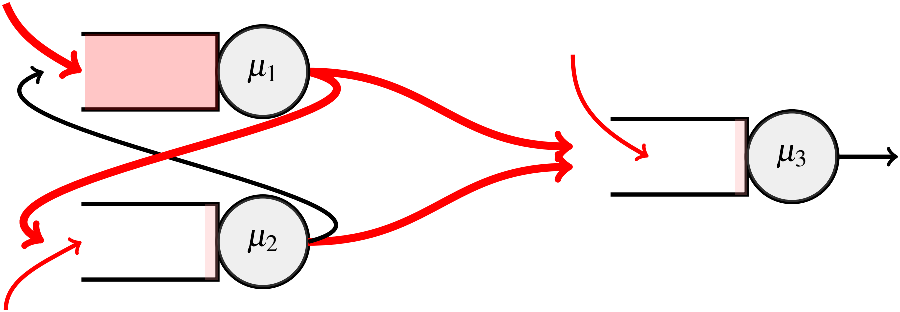

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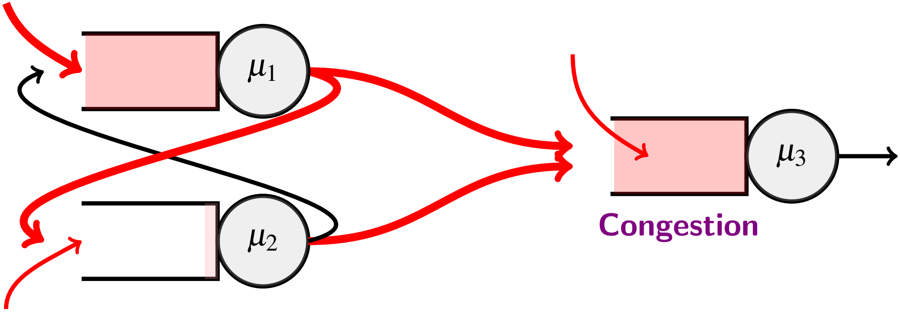

The point is

Many heavy-tailed rare events can be written as

$$\{\bar{S}_n \in A\}$$

and the decay rate is determined by $\mathcal{J}(A)$, i.e.,

$$\mathbf{P}(\bar{S}_n \in A) \sim n^{-\alpha \mathcal{J}(A)}$$

Catastrophe Principle Extends to General Heavy-Tailed Systems

Heavy-Tailed Large Deviations for

- Continuous-Time Processes
  R., Blanchet, Zwart (2019), Bazhba, Blanchet, R., Zwart (2020), Su, Wang, R. (2023+)

- Processes with Spatial and Temporal Correlations
  Chen, R., Zwart (2023+), Bazhba, Blanchet, R., Zwart (2023+), Su, R. (2023+), Wang, R. (2023+)

Minimal # Jumps added to Typical Paths Characterize the Catastrophe Principle

Challenge of Rare Event Simulation

Monte Carlo simulations as repetitive random experiments:

e.g. Coin flip: want to estimate $\mathbf{P}(\text{Head})$

- Flip the coin 100 times
- Count the number of head
- Divide by 100 and report the number

Should be reasonably close to $1/2$

Challenge of Rare Event Simulation

Monte Carlo simulations as repetitive random experiments:

e.g. Coin flip: want to estimate $\mathbf{P}(\text{Edge})$

- Flip the coin 100 times
- Count the number of Edge
- Divide by 100 and report the number

Challenge of Rare Event Simulation

Monte Carlo simulations as repetitive random experiments:

e.g. Coin flip: want to estimate $\mathbf{P}(\text{Edge})$

- Flip the coin 100 times
- Count the number of Edge : most likely to be 0
- Divide by 100 and report the number

Challenge of Rare Event Simulation

Monte Carlo simulations as repetitive random experiments:

e.g. Coin flip: want to estimate $\mathbf{P}(\text{Edge})$

- Flip the coin 100 times
- Count the number of Edge : most likely to be 0
- Divide by 100 and report the number : and hence, likely to be 0

Challenge of Rare Event Simulation

Monte Carlo simulations as repetitive random experiments:

e.g. Coin flip: want to estimate $\mathbf{P}(\text{Edge})$

- Flip the coin 100 times
- Count the number of Edge : most likely to be 0
- Divide by 100 and report the number : and hence, likely to be 0

Is 0 a useful answer?

Challenge of Rare Event Simulation

Monte Carlo simulations as repetitive random experiments:

e.g. Coin flip: want to estimate $\mathbf{P}(\text{Edge})$

- Flip the coin 100 times
- Count the number of Edge : most likely to be 0
- Divide by 100 and report the number : and hence, likely to be 0

Is 0 a useful answer? No.

e.g., Nuclear Meltdown, Large-Scale Blackout, Large Financial Loss

Challenge of Rare Event Simulation

Monte Carlo simulations as repetitive random experiments:

e.g. Coin flip: want to estimate $\mathbf{P}(\text{Edge}) \overset{Suppose}{\approx} 10^{-6}$

- Flip the coin 100 times
- Count the number of Edge
- Divide by 100 and report the number

Challenge of Rare Event Simulation

Monte Carlo simulations as repetitive random experiments:

e.g. Coin flip: want to estimate $\mathbf{P}(\text{Edge}) \overset{Suppose}{\approx} 10^{-6}$

- Flip the coin ~~100~~ a few million times
- Count the number of Edge
- Divide by the total number of flips and report the number

Challenge of Rare Event Simulation

Monte Carlo simulations as repetitive random experiments:

e.g. Coin flip: want to estimate $\mathbf{P}(\text{Edge}) \overset{Suppose}{\approx} 10^{-6}$

- Flip the coin ~~100~~ a few million times
- Count the number of Edge
- Divide by the total number of flips and report the number

Much harder than $\mathbf{P}$(Head)

Importance Sampling

- Construct an alternative universe

Importance Sampling

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- Perform experiments there

Importance Sampling

- Construct an alternative universe
- Perform experiments there
- Recover the true probability using the relationship between the two parallel universes

Importance Sampling

- Construct an alternative universe
  - i.e., Consider an "importance distribution" $\mathbf{Q}$

- Perform experiments there

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- Construct an alternative universe
  - i.e., Consider an "importance distribution" $\mathbf{Q}$

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  - i.e., Sample $\mathbb{I}^{(1)}_{\text{Edge}}, \ldots, \mathbb{I}^{(m)}_{\text{Edge}}$ from $\mathbf{Q}$

- Recover the true probability using the relationship between the two parallel universes

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- Construct an alternative universe
  - i.e., Consider an "importance distribution" $\mathbf{Q}$
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  - i.e., Sample $\mathbb{I}_{\text{Edge}}^{(1)}, \ldots, \mathbb{I}_{\text{Edge}}^{(m)}$ from $\mathbf{Q}$

    as well as the likelihood ratio $\left(\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}\mathbf{Q}}\right)^{(1)}, \ldots, \left(\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}\mathbf{Q}}\right)^{(m)}$
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- Recover the true probability using the relationship between the two parallel universes
  - i.e., Report $\frac{1}{m}\sum_{i=1}^{m} \mathbb{I}_{\text{Edge}}^{(i)} \left(\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}\mathbf{Q}}\right)^{(i)}$ as an estimate of $\mathbf{P}(\text{Edge})$

Importance Sampling for $\mathbf{P}(\bar{S}_n \in A)$

- Construct an alternative universe
  - i.e., Consider an "importance distribution" $\mathbf{Q}$

- Perform experiments there
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Importance Sampling for $\mathbf{P}(\bar{S}_n \in A)$

- Construct an alternative universe
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- Perform experiments there
  - i.e., Sample $\mathbb{I}^{(1)}_{\{\bar{S}_n \in A\}}, \ldots, \mathbb{I}^{(m)}_{\{\bar{S}_n \in A\}}$ from $\mathbf{Q}_n$

    as well as the likelihood ratio $\left(\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}\mathbf{Q}_n}\right)^{(1)}, \ldots, \left(\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}\mathbf{Q}_n}\right)^{(m)}$

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Finding a good alternative universe $\mathbf{Q}_n$ is crucial.

What is a good alternate universe $\mathbf{Q}_n$ for $\mathbf{P}(\bar{S}_n \in A)$?

General principle for making $\overbrace{\mathbb{I}_{\{\bar{S}_n \in A\}} \frac{\mathrm{d}\mathbf{P}}{\mathrm{d}\mathbf{Q}_n}}^{\text{IS estimator}}$ an efficient estimator:

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- **Choose $\mathbf{Q}_n(\cdot)$ as close to $\mathbf{P}(\cdot | \bar{S}_n \in A)$ as possible.**

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- **Choose $\mathbf{Q}_n(\cdot)$ as close to $\mathbf{P}(\cdot|\bar{S}_n \in A)$ as possible.**

- **Make sure that $\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}\mathbf{Q}_n}$ does not blow up.**

Goal: Strongly Efficient IS Estimator

$\overbrace{\mathbb{I}_{\{\bar{S}_n \in A\}} \frac{\mathrm{d}\mathbf{P}}{\mathrm{d}\mathbf{Q}_n}}^{\text{IS estimator}}$ is a **strongly efficient** estimator for $\mathbf{P}(\bar{S}_n \in A)$, if

$$\mathbf{E}^{\mathbf{Q}_n}\left(\mathbb{I}_{\{\bar{S}_n \in A\}} \frac{\mathrm{d}\mathbf{P}}{\mathrm{d}\mathbf{Q}_n}\right)^2 \quad \sim \quad \mathbf{P}(\bar{S}_n \in A)^2$$

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IS estimator

Error ^{2}

$$\mathbf{E}^{\mathbf{Q}_n}\left(\mathbb{I}_{\{\bar{S}_n \in A\}} \frac{\mathrm{d}\mathbf{P}}{\mathrm{d}\mathbf{Q}_n}\right)^2 \quad \sim \quad \mathbf{P}(\bar{S}_n \in A)^2$$

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$$\underbrace{\mathbb{I}_{\{\bar{S}_n \in A\}} \frac{\mathrm{d}\mathbf{P}}{\mathrm{d}\mathbf{Q}_n}}_{\text{IS estimator}}$$ is a **strongly efficient** estimator for $\mathbf{P}(\bar{S}_n \in A)$, if

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$\Rightarrow$ Number of simulation runs required remains bounded.

Goal: Strongly Efficient IS Estimator

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$\Rightarrow$ Number of simulation runs required remains bounded.

Considered Notoriously Hard for Heavy-Tailed Processes.

First Universal Rare-Event Simulation Scheme for Heavy-Tails

- Fix $w \in (0, 1)$ and define

$$\mathbf{Q}_n(\cdot) \triangleq w\mathbf{P}(\cdot) + (1-w)\mathbf{P}(\cdot\,|\,\bar{X}_n \in B^\gamma)$$

First Universal Rare-Event Simulation Scheme for Heavy-Tails

- Fix $w \in (0,1)$ and define

$\frac{d\mathbf{P}}{d\mathbf{Q}_n}$ not too big

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- $B^\gamma \triangleq \{$paths w/ at least $\mathcal{J}(A)$ jumps of size $> \gamma\}$

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$$\mathbf{E}^{\mathbf{Q}_n}\left(\mathbb{1}_{\{\bar{X}_n \in A\}} \frac{d\mathbf{P}}{d\mathbf{Q}_n}\right)^2 \leq \frac{1}{w}\mathbf{P}(\bar{X}_n \in A \setminus B^\gamma) + \mathbf{P}(\bar{X}_n \in A) \cdot \mathbf{P}(\bar{X}_n \in B^\gamma)$$

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First Universal Rare-Event Simulation Scheme for Heavy-Tails

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First Universal Rare-Event Simulation Scheme for Heavy-Tails

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First Universal Rare-Event Simulation Scheme for Heavy-Tails

- $B^\gamma \triangleq \{\text{paths w/ at least } \mathcal{J}(A) \text{ jumps of size } > \gamma\} \;\Rightarrow\; \mathcal{J}(A) = \mathcal{J}(B^\gamma)$

- Fix $w \in (0,1)$ and define

$$\mathbf{Q}_n(\cdot) \triangleq w\mathbf{P}(\cdot) + (1-w)\mathbf{P}(\cdot \,|\, \bar{X}_n \in B^\gamma)$$

- If we choose γ so that $\mathcal{J}(A \setminus B^\gamma) \geq \mathcal{J}(A) + \mathcal{J}(B^\gamma)$

$$\mathbf{E}^{\mathbf{Q}_n}\left(\mathbb{1}_{\{\bar{X}_n \in A\}} \frac{d\mathbf{P}}{d\mathbf{Q}_n}\right)^2 \leq \frac{1}{w}\overbrace{\mathbf{P}(\bar{X}_n \in A \setminus B^\gamma)}^{n^{-\alpha\mathcal{J}(A \setminus B^\gamma)}} + \underbrace{\overbrace{\mathbf{P}(\bar{X}_n \in A)}^{n^{-\alpha\mathcal{J}(A)}} \cdot \overbrace{\mathbf{P}(\bar{X}_n \in B^\gamma)}^{n^{-\alpha\mathcal{J}(B^\gamma)}}}_{\sim\, \mathbf{P}(\bar{X}_n \in A)^2}$$

First Universal Rare-Event Simulation Scheme for Heavy-Tails

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$$\sim \big(\mathbf{P}(\bar{X}_n \in A)\big)^2$$

First Universal Rare-Event Simulation Scheme for Heavy-Tails

- $B^\gamma \triangleq \{$paths w/ at least $\mathcal{J}(A)$ jumps of size $> \gamma\} \;\Rightarrow\; \mathcal{J}(A) = \mathcal{J}(B^\gamma)$

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Error^2 $\rightarrow$

$$\mathbf{E}^{\mathbf{Q}_n}\left(\mathbb{1}_{\{\bar{X}_n \in A\}}\frac{d\mathbf{P}}{d\mathbf{Q}_n}\right)^2 \leq \frac{1}{w}\mathbf{P}(\bar{X}_n \in A \setminus B^\gamma) + \mathbf{P}(\bar{X}_n \in A)\cdot\mathbf{P}(\bar{X}_n \in B^\gamma)$$

$$\sim \big(\mathbf{P}(\bar{X}_n \in A)\big)^2$$

$\leftarrow$ Target Quantity^2

First Universal Rare-Event Simulation Scheme for Heavy-Tails

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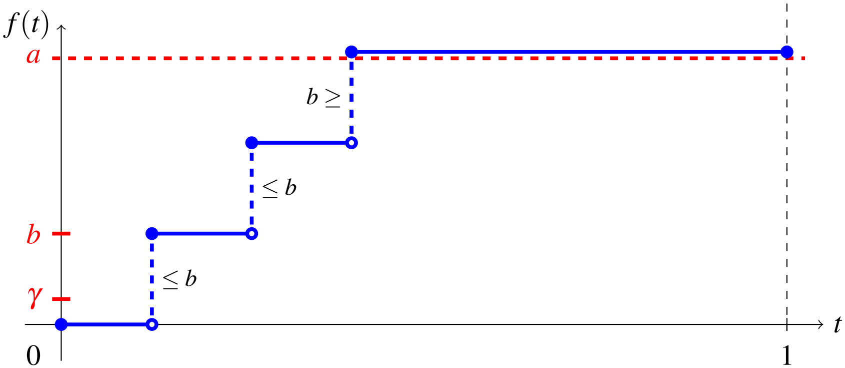
$$\sim \big(\mathbf{P}(\bar{X}_n \in A)\big)^2 \qquad \leftarrow \text{Target Quantity}^2$$

$Z_n \triangleq \mathbb{1}_{\{\bar{X}_n \in A\}}\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}\mathbf{Q}_n}$ **is strongly efficient for** $\mathbf{P}(\bar{X}_n \in A)$**!**

Chen, Blanchet, R., and Zwart (2019)

How to ensure $\mathcal{J}(A \setminus B^{\gamma}) \geq 2\mathcal{J}(A)$: Reinsurance Example



- $A \quad = \{$paths that cross level a on $[0,1]$ & jump sizes $\leq b\}$
- $B^{\gamma} \quad = \{$paths with at least 3 jumps of size $> \gamma\}$
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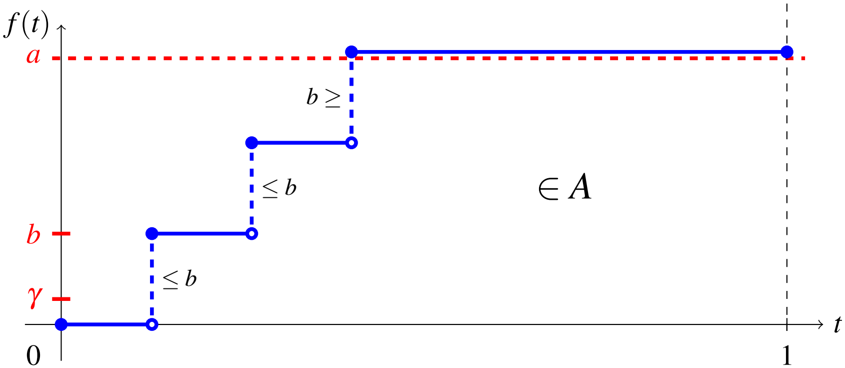

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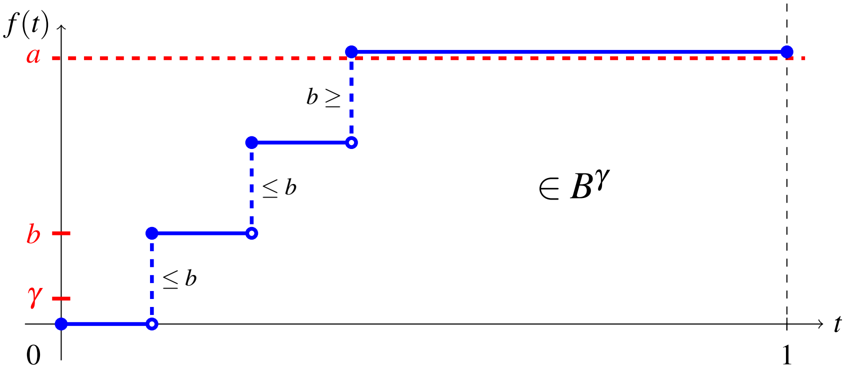

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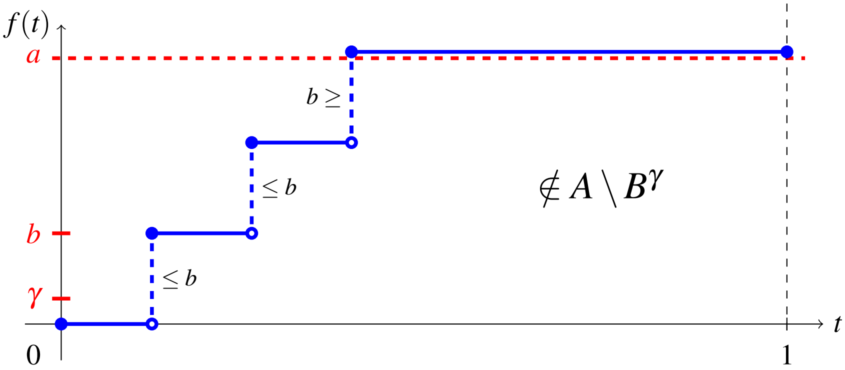

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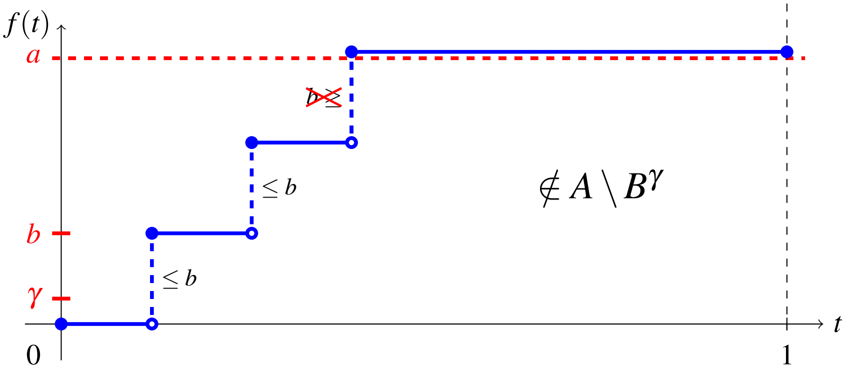

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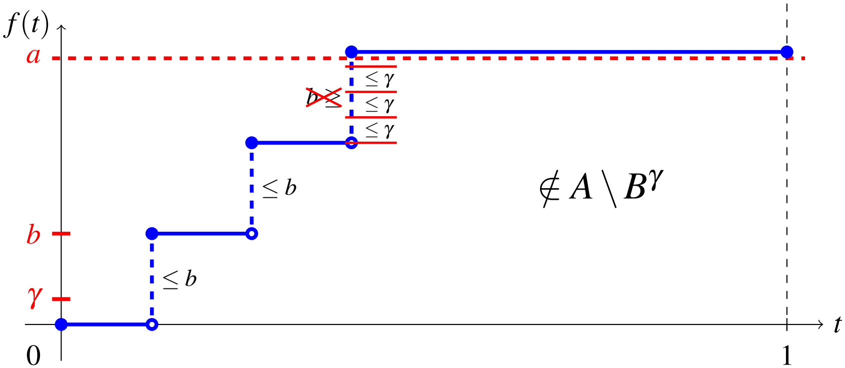

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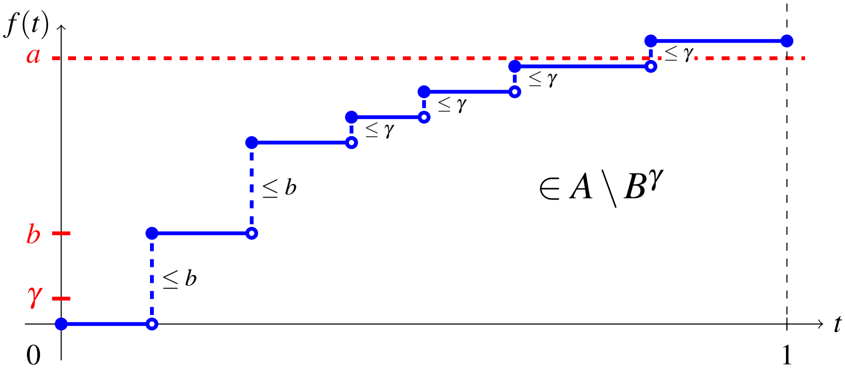

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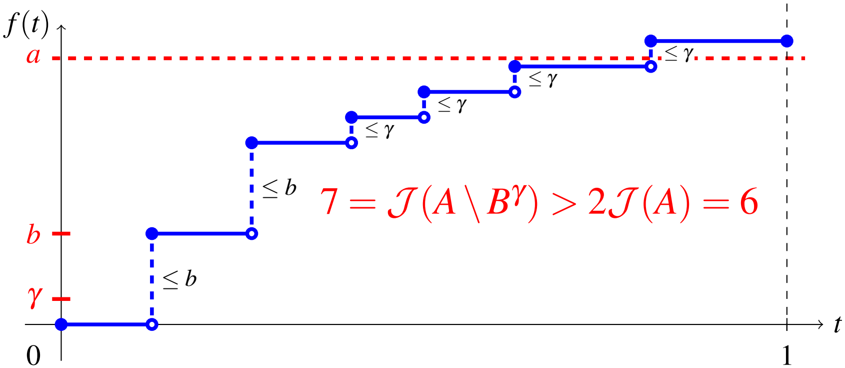

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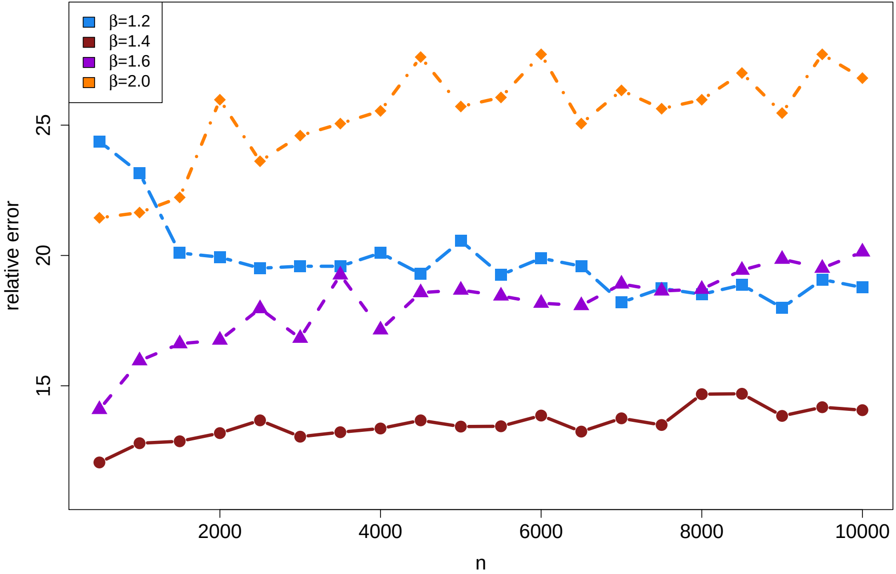
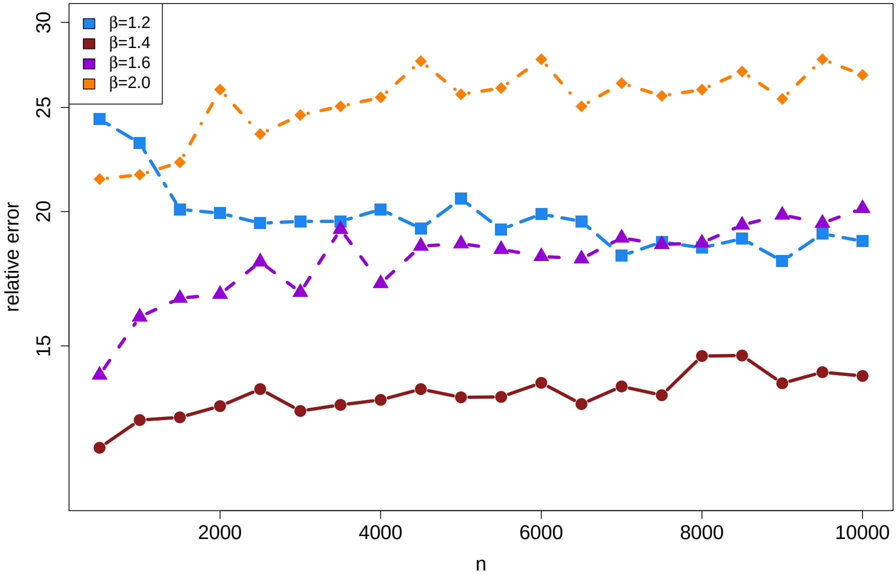
Numerical Experiments for Reinsurance Example



Numerical Results for Reinsurance Example



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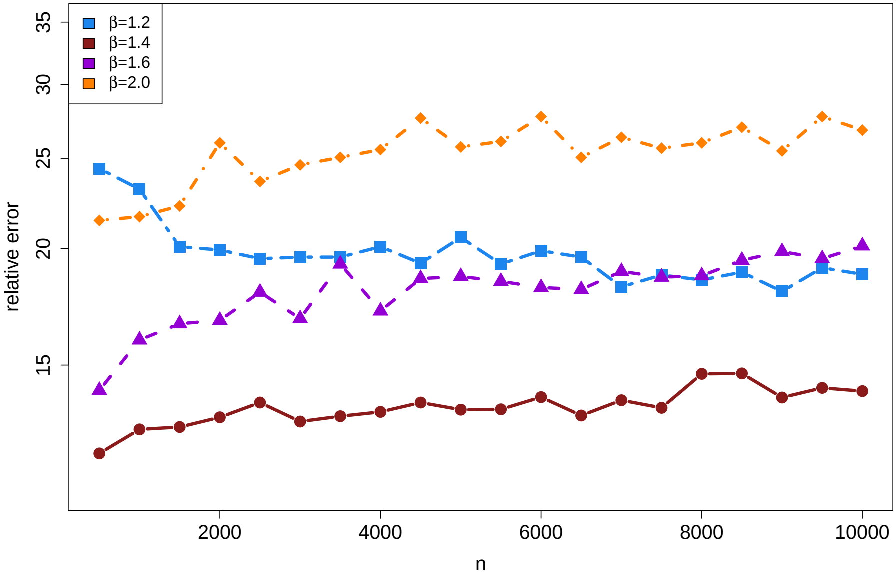

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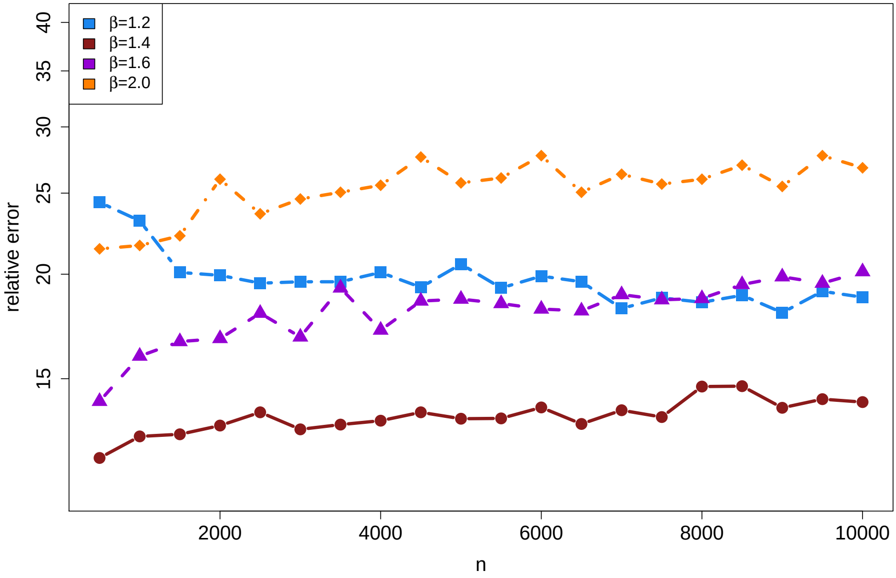

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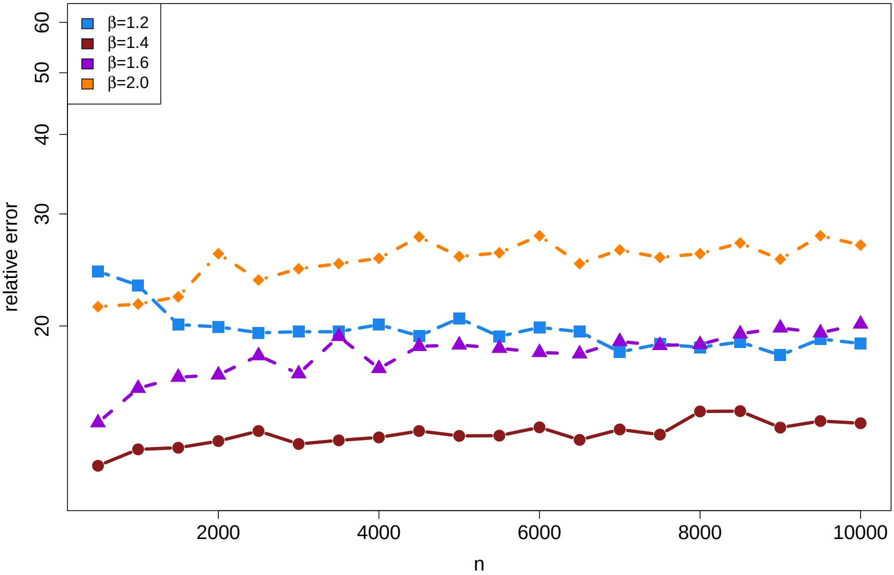

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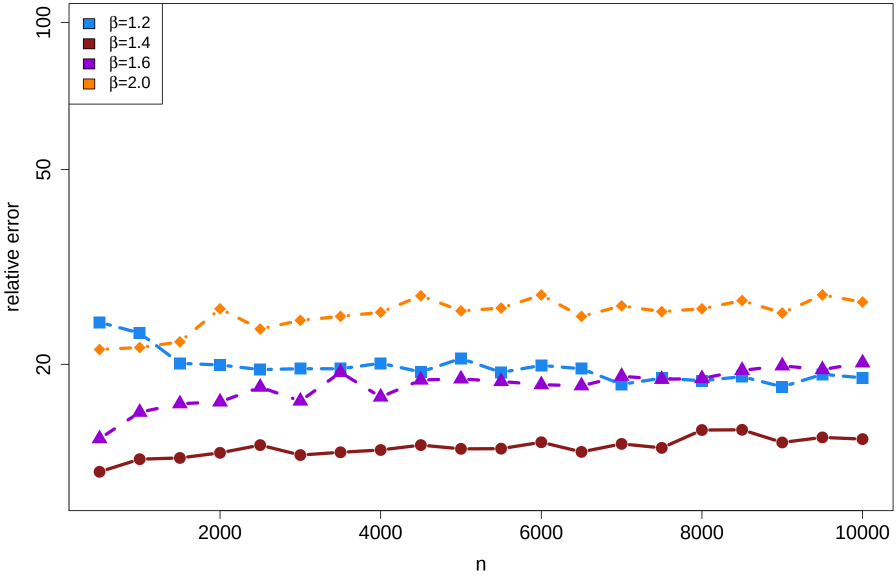

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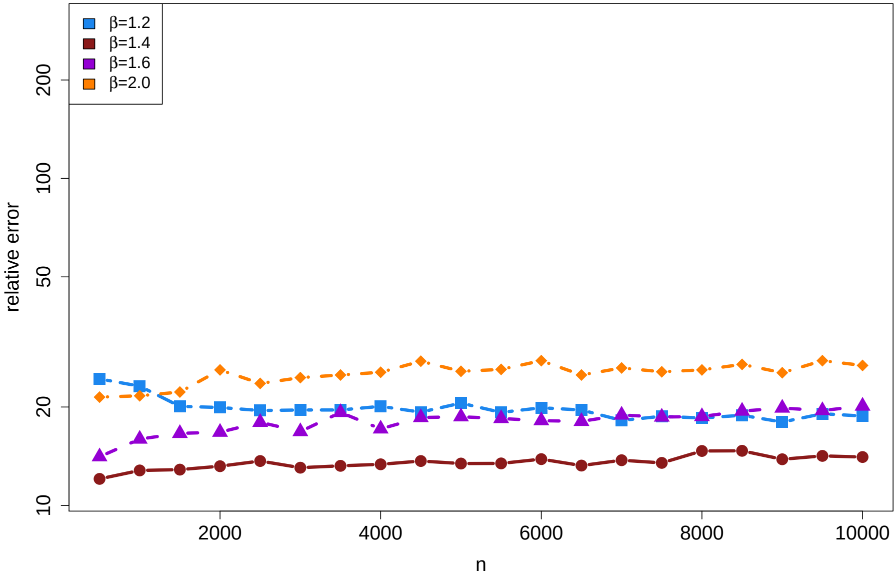

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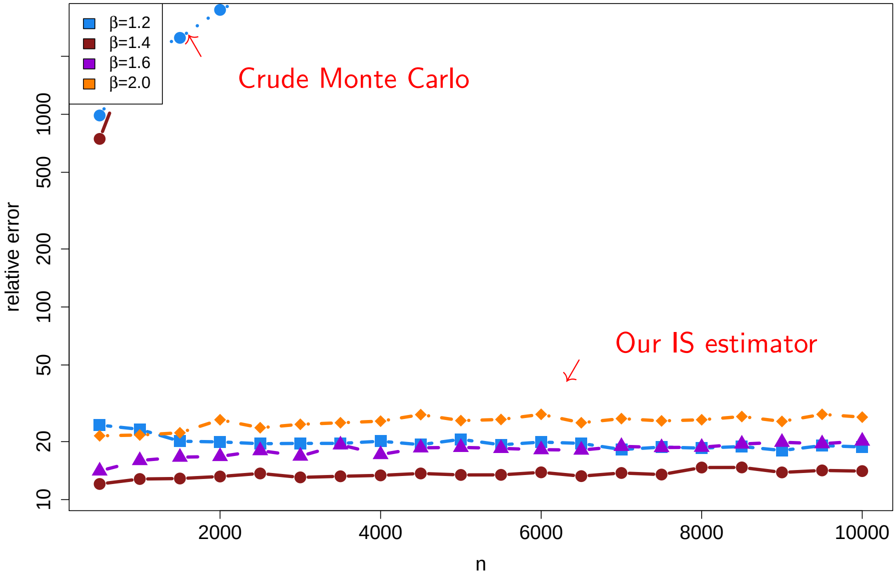

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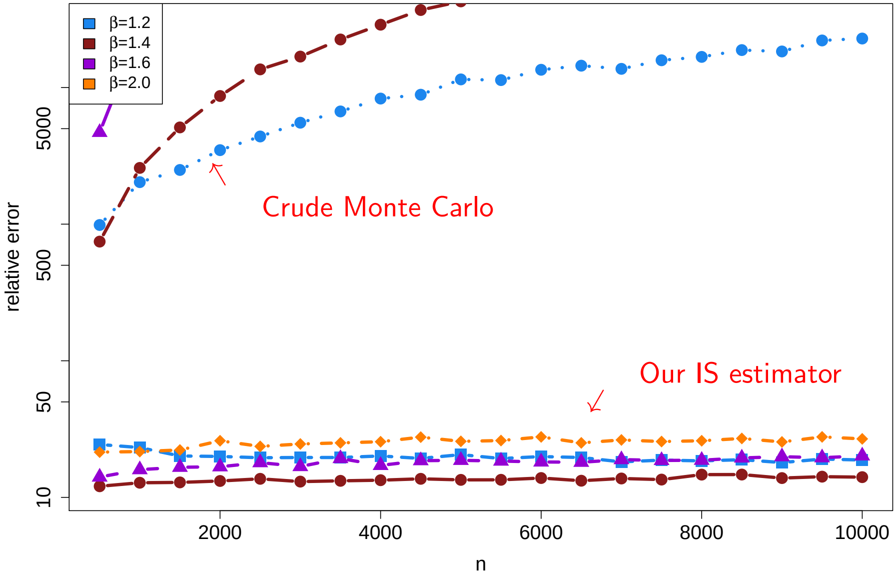

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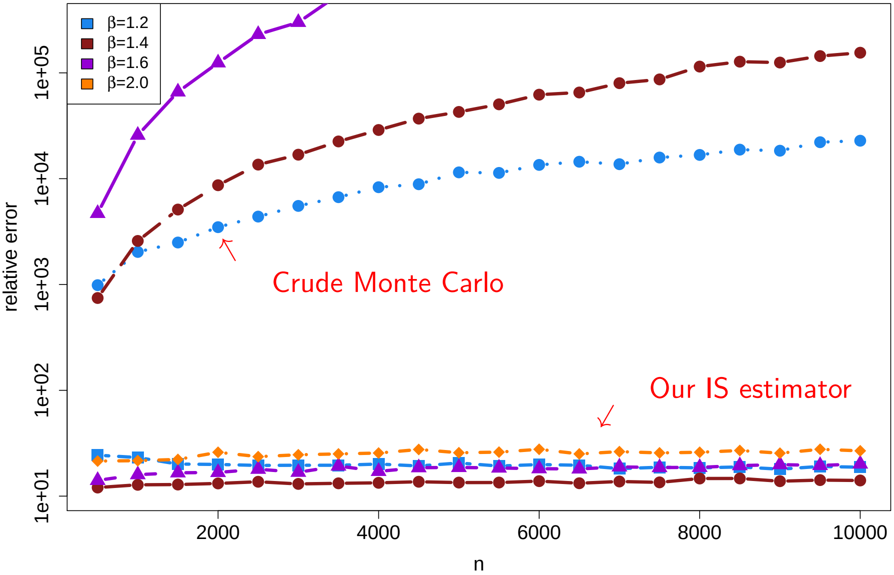

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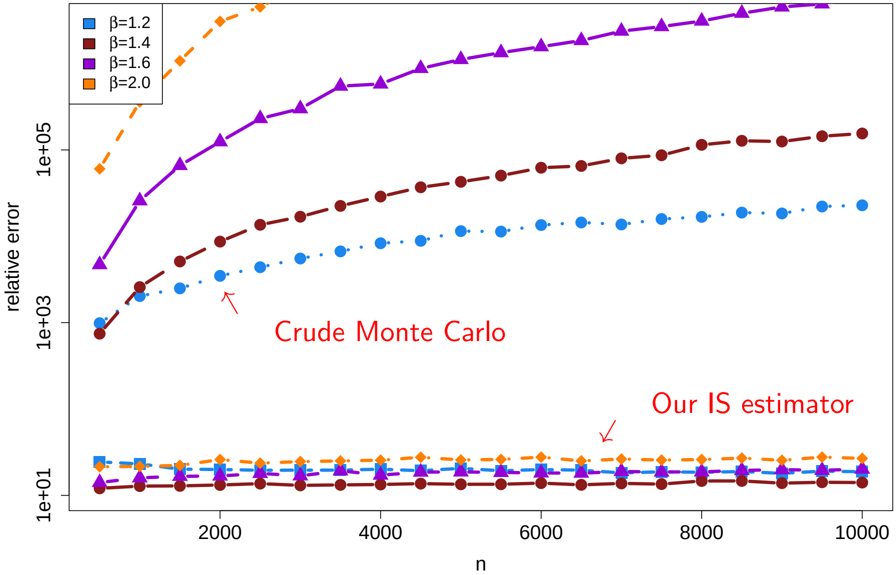

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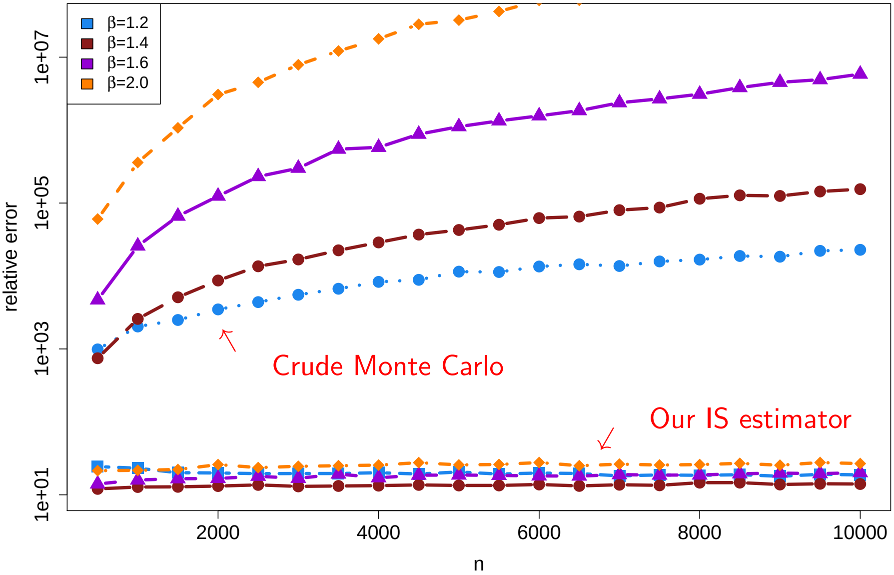

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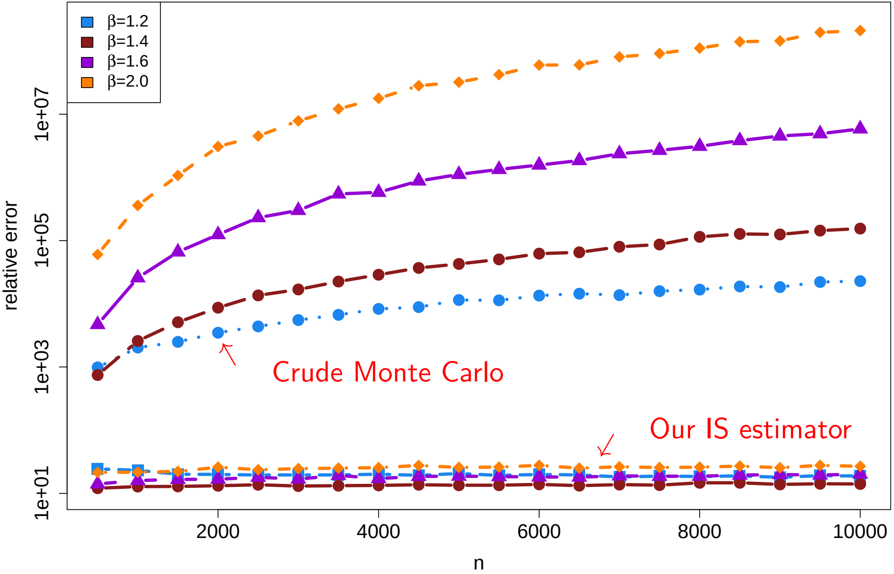

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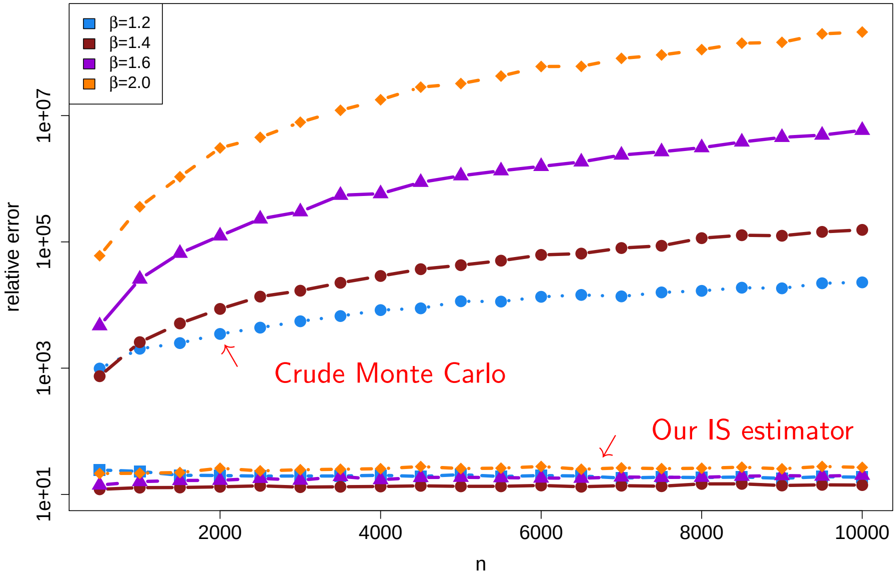
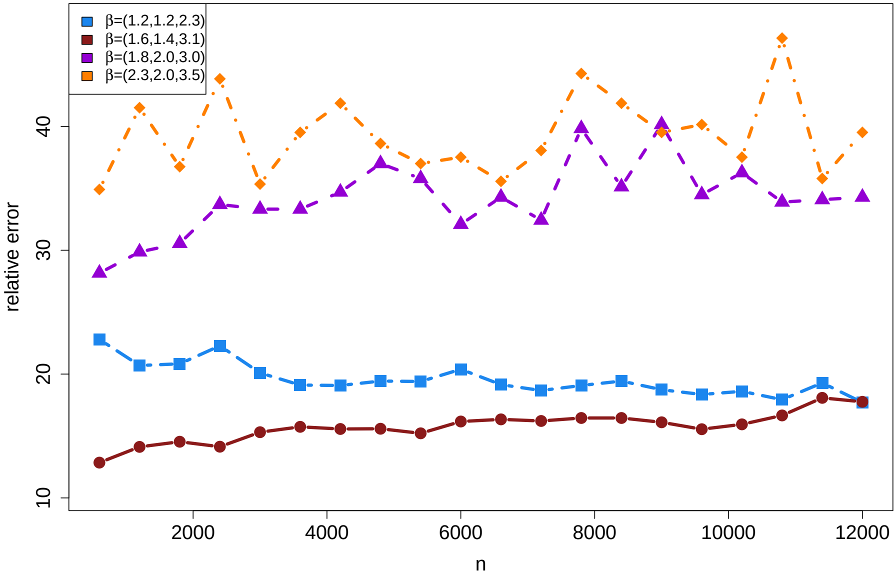


Numerical Results for Fluid Network Example



Large Deviations for Weibull Tails

$$\mathbf{P}(X_i \geq x) = \exp(-x^{\alpha}), \ \ \alpha \in (0,1)$$

What's already known: LDP w.r.t. L_1 topology

Nina Gantert (1998)

- $\limsup_{n\to\infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P}\left(\bar{S}_n \in A\right) \le -\inf_{\xi\in A^-} I(\xi)$, A^-: closure of A
- $\liminf_{n\to\infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P}\left(\bar{S}_n \in A\right) \ge -\inf_{\xi\in A^\circ} I(\xi)$, A° : interior of A

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- $\liminf_{n\to\infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P}\left(\bar{S}_n \in A\right) \ge -\inf_{\xi\in A^\circ} I(\xi)$, A°: interior of A
- $$I(\xi) = \begin{cases} \sum_{\{t:\xi(t)\neq\xi(t-)\}} \left(\xi(t)-\xi(t-)\right)^\alpha & \text{if } \xi \text{ is a nondecreasing step function} \\ \infty & \text{otherwise} \end{cases}$$

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- $\mathbf{P}\left(\bar{S}_n \in A\right) \lesssim \exp\left(-L(n)n^{\alpha}I(A^{-})\right)$, $\qquad I(A^{-}) = \inf_{\xi \in A^{-}} I(\xi)$
- $\mathbf{P}\left(\bar{S}_n \in A\right) \gtrsim \exp\left(-L(n)n^{\alpha}I(A^{\circ})\right)$, $\qquad I(A^{\circ}) = \inf_{\xi \in A^{\circ}} I(\xi)$
- $$I(\xi) = \begin{cases} \sum_{\{t:\xi(t)\neq\xi(t-)\}} \left(\xi(t)-\xi(t-)\right)^{\alpha} & \text{if } \xi \text{ is a nondecreasing step function} \\ \infty & \text{otherwise} \end{cases}$$

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- $\mathbf{P}\left(\bar{S}_n \in A\right) \gtrsim \exp\left(-L(n)n^\alpha I(A^\circ)\right)$, $\qquad I(A^\circ) = \inf_{\xi\in A^\circ} I(\xi)$
- $I(\xi) = \begin{cases} \sum_{\{t:\xi(t)\neq\xi(t-)\}} \big(\xi(t)-\xi(t-)\big)^\alpha & \text{if } \xi \text{ is a nondecreasing step function} \\ \infty & \text{otherwise} \end{cases}$
- I is a good rate function w.r.t. L_1 topology

  i.e., $d(\xi,\zeta) = \int_0^1 |\xi(s)-\zeta(s)|ds$

What's already known: LDP w.r.t. L_1 topology

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- $\mathbf{P}\left(\bar{S}_n \in A\right) \gtrsim \exp\left(-L(n)n^{\alpha}I(A^{\circ})\right)$, $\quad I(A^{\circ}) = \inf_{\xi \in A^{\circ}} I(\xi)$
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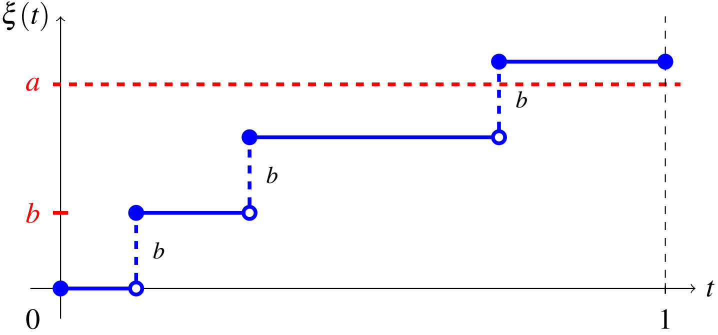
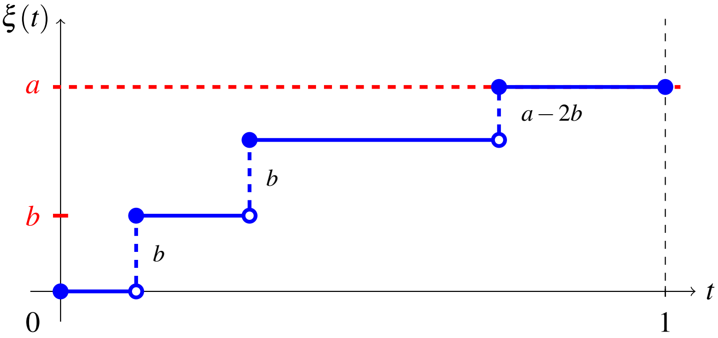
BUT

L_1 topology is weak



- $A = \{\xi \in \mathbb{D} : \sup_{t\in[0,1]} \xi(t) \geq a, \sup_{t\in[0,1]} |\xi(t) - \xi(t-)| \leq b\}$
- $\mathbf{P}\left(\bar{S}_n \in A\right) \lesssim \exp\left(-L(n)n^{\alpha} I(A^-)\right)$, $\quad I(A^-) = \inf_{\xi\in A^-} I(\xi)$
- $I(A^-) \stackrel{?}{=} b^{\alpha} + b^{\alpha} + b^{\alpha}$

L_1 topology is weak



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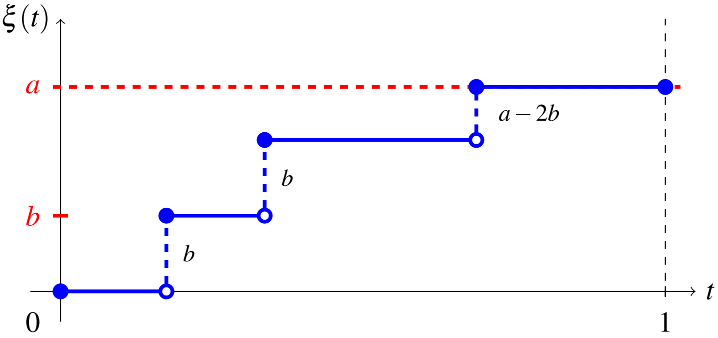

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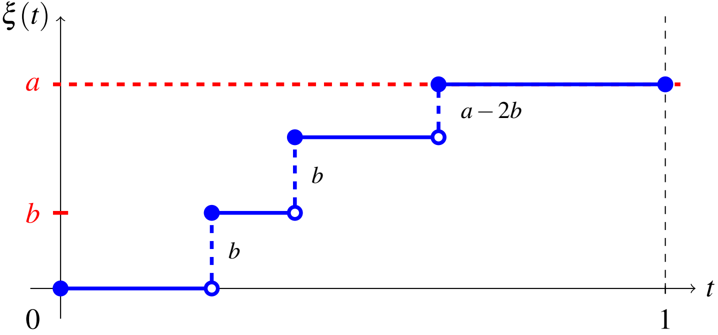

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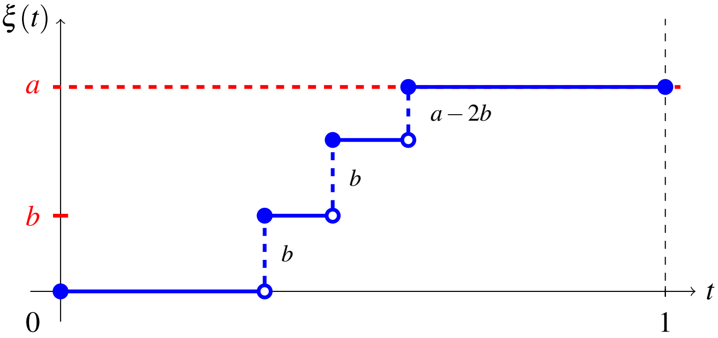

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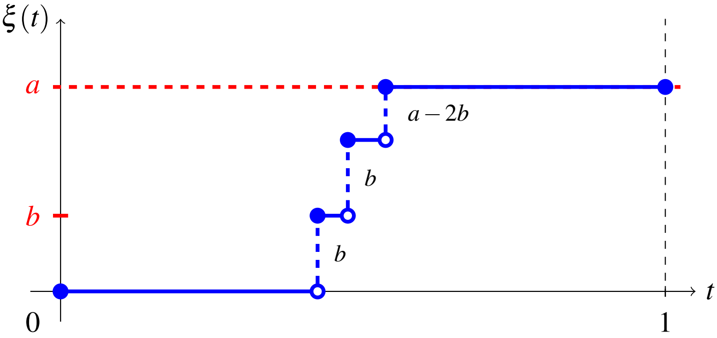

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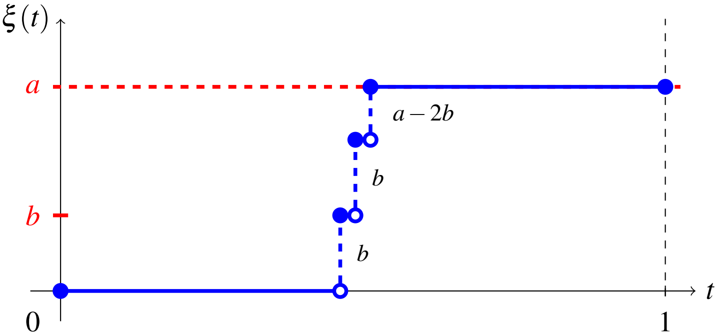

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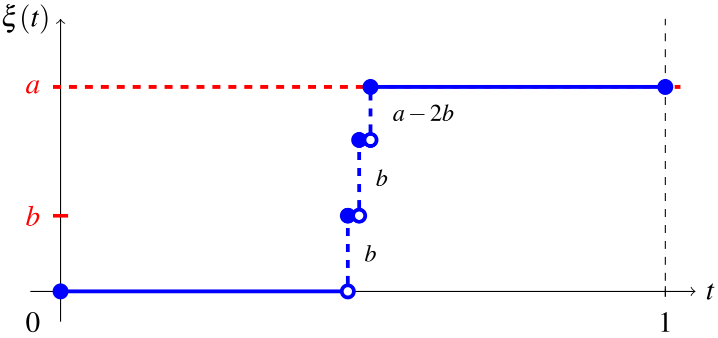

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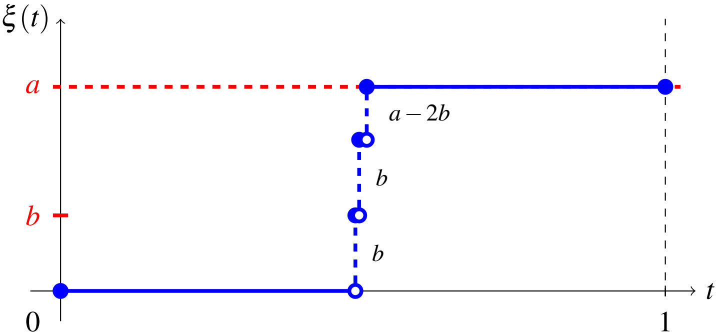

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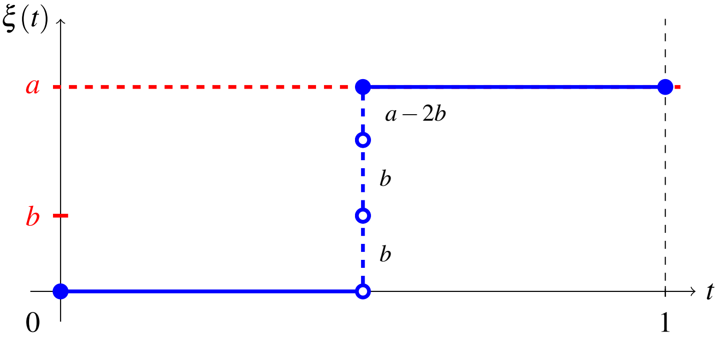

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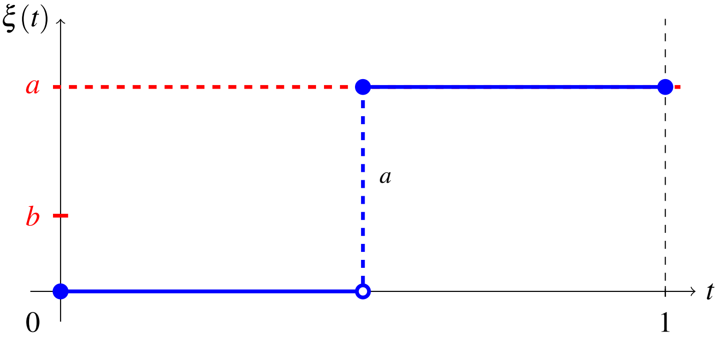

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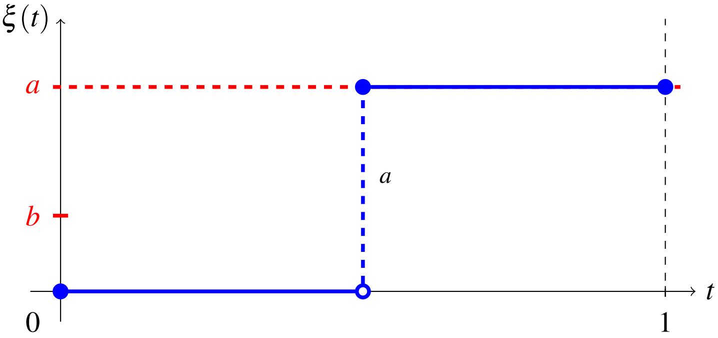

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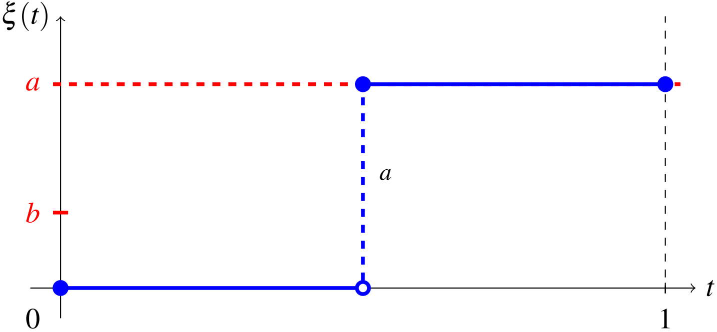

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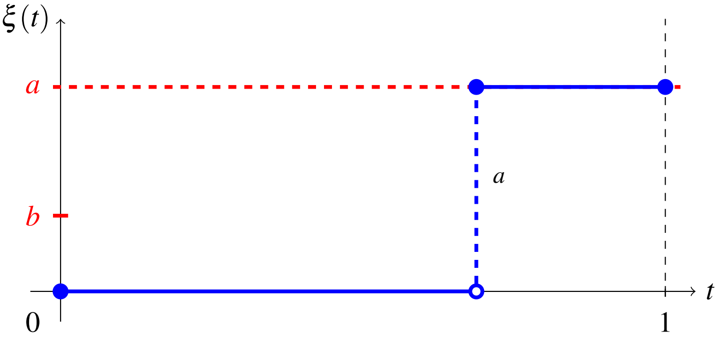

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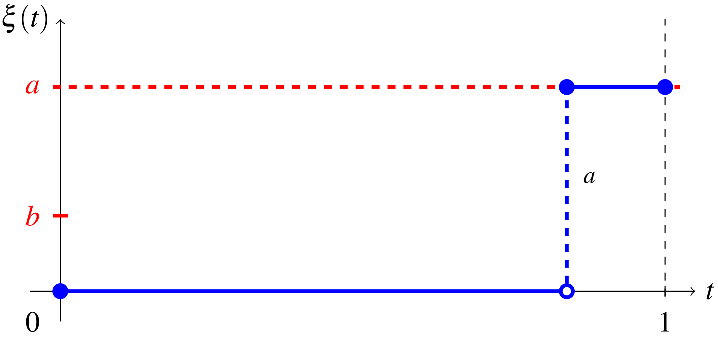

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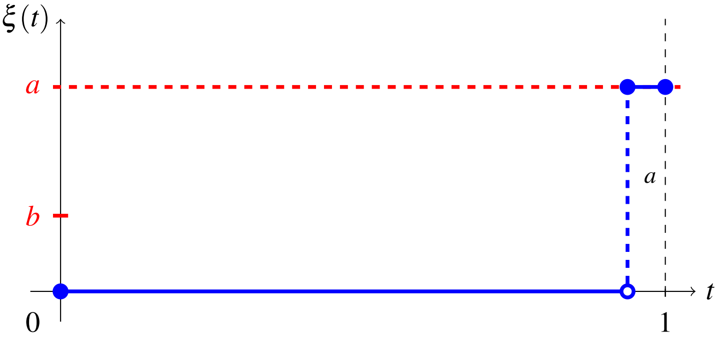

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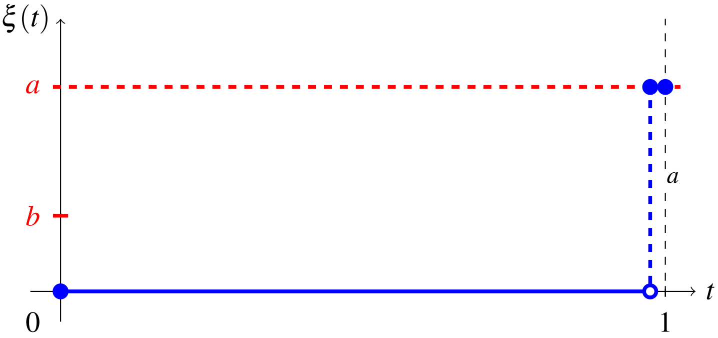

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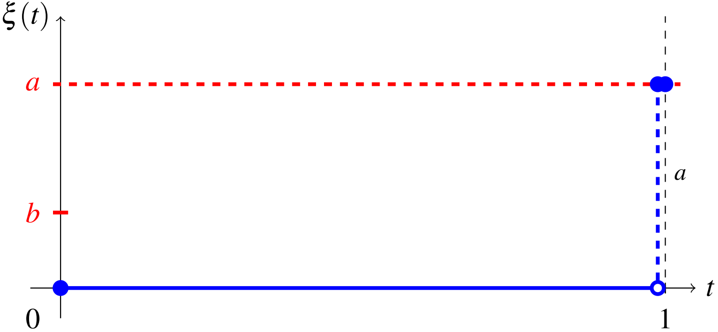

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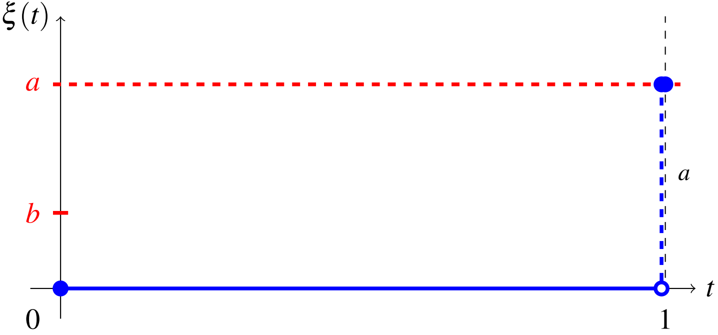

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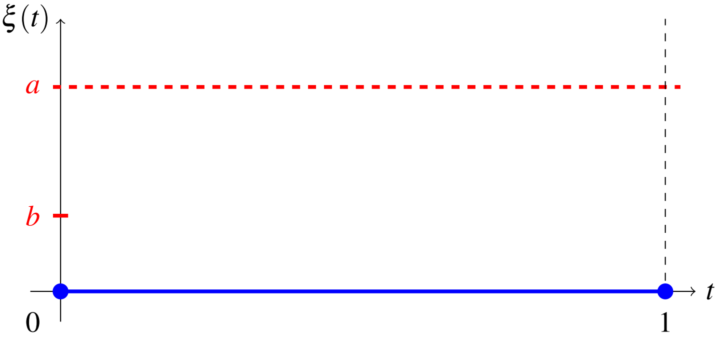

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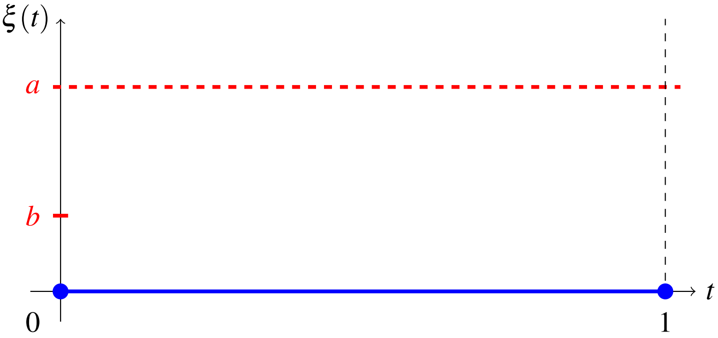

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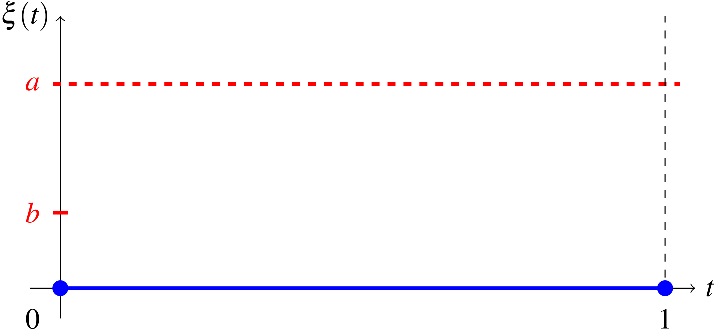

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No information!

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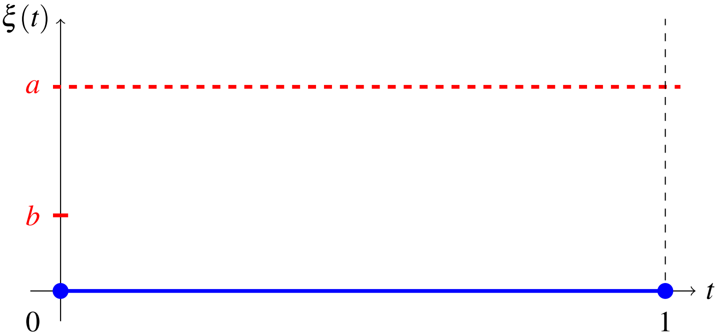

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Want a stronger topology, ideally J_1 topology!

LDP w.r.t. J_1 Topology is Impossible

Counterexample (Bazhba, Blanchet, R., Zwart 2020)

There exists a closed set $A \subseteq \mathbb{D}$ such that

$$\limsup_{n\to\infty} \frac{1}{\sqrt{n}} \log \mathbf{P}\left(\bar{S}_n \in A\right) \not\leq -\inf_{\xi\in A} I(\xi)$$

- $L(n) = 1$ and $\alpha = \frac{1}{2}$ so that $\mathbf{P}(X_1 > x) = e^{-\sqrt{x}}$
- Paths in A have m increases of size $O\left(\frac{1}{m^2}\right)$, for some m

“Extended” Large Deviation Principle

Theorem (Bazhba, Blanchet, R., Zwart 2020)

$\bar{X}_n$ satisfies an “extended LDP” w.r.t. J_1 topology, i.e.,

$$\limsup_{n\to\infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P}\left(\bar{X}_n(t) \in A\right) \leq -\lim_{\varepsilon\to 0} \inf_{\xi \in A^\varepsilon} I(\xi)$$

$$\liminf_{n\to\infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P}\left(\bar{X}_n(t) \in A\right) \geq -\inf_{\xi \in A^\circ} I(\xi)$$

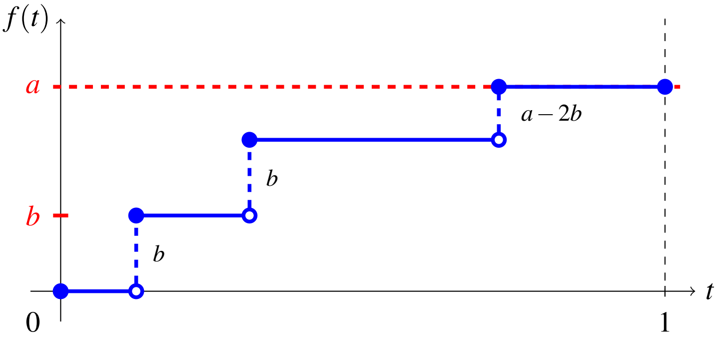
where

$$I(\xi) = \begin{cases} \sum_{\{t:\xi(t)\neq\xi(t^-)\}} \left(\xi(t) - \xi(t^-)\right)^\alpha, & \textit{if } \xi \textit{ is a nondecreasing pure jump function} \\ \infty, & o.w. \end{cases}$$

Corollary

If ϕ is Lipschitz, $\phi\left(\bar{X}_n\right)$ satisfies a LDP, if the resulting rate function is good.

Back to our old example



- $A = \{f \in \mathbb{D} : f$ crosses level a on $[0,1]$ & jump sizes $\le b\}$
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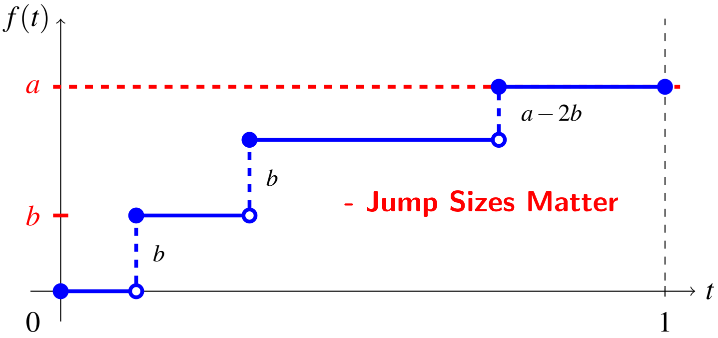

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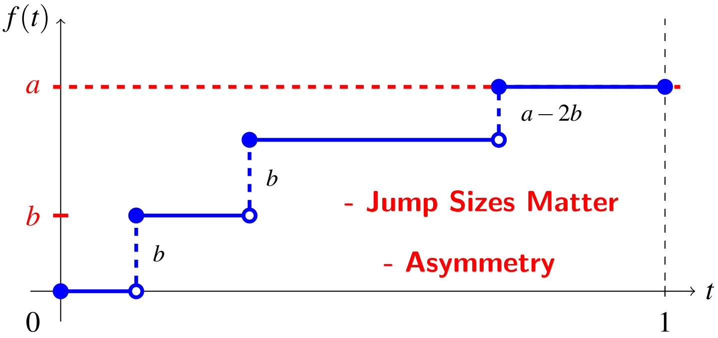

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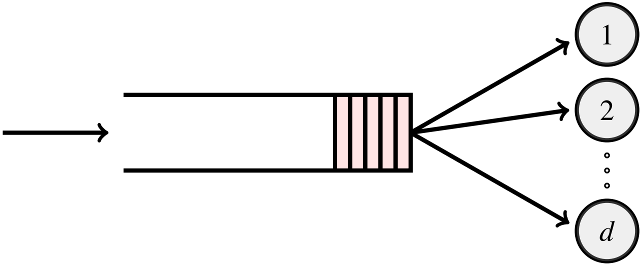
Queue Length Asymptotics for the Weibull GI/GI/d Queue

In case service time distribution is Weibull, i.e., $\mathbf{P}(S > x) \sim \exp(-x^{\alpha})$,



Tail Asymptotics? Most likely scenario?

So far, NOT EVEN a reasonable conjecture!

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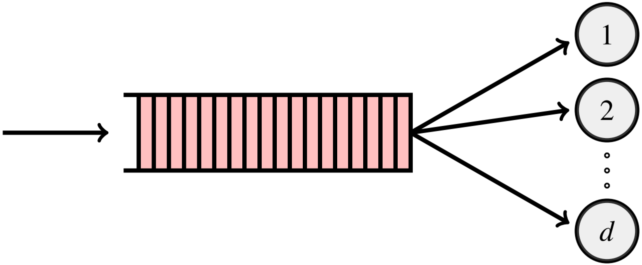

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Our Sample Path LDP Answers the Question!

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iid inter-arrival times

iid service times for server k

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Queue Length Asymptotics for the Weibull GI/GI/d Queue

If service time distribution is Weibull, i.e., $\mathbf{P}(S_1 > x) \sim \exp(-x^\alpha)$, then

$$\mathbf{P}(Q(\gamma n) > n) \sim \exp\left(-c^* n^\alpha\right)$$

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- Nontrivial Tradeoff between Size and Number of Jumps!

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Solution to Open Question Posed by Whitt (2000) and Foss (2009)

More Specifically,

If $\gamma > \frac{1}{\lambda - \lfloor \lambda \rfloor}$: job/jump sizes (of the most likely scenario) are **symmetric**

- If $\lfloor \lambda \rfloor \leq \frac{\lambda - \alpha d}{1-\alpha}$: smallest possible number of big jobs to block enough servers
  $\Rightarrow$ **same as the power law case**
- If $\lfloor \lambda \rfloor > \frac{\lambda - \alpha d}{1-\alpha}$: larger number of moderately big jobs might be more likely
  $\Rightarrow$ **qualitatively different from the power law case**

If $\gamma < \frac{1}{\lambda - \lfloor \lambda \rfloor}$: job/jump sizes may be **asymmetric** (upto 3 different sizes)

Back to Stochastic Gradient Descent

Stochastic Gradient Descent

Minimizing loss function f:

$$W_{k+1} = W_k - \eta\left(f'(W_k)\right) \qquad k = 0, 1, 2, \ldots$$

Stochastic Gradient Descent

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$$W_{k+1} = W_k - \eta \left(f'(W_k) + Z_k \right) \qquad k = 0, 1, 2, \ldots$$

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Gradient Flow

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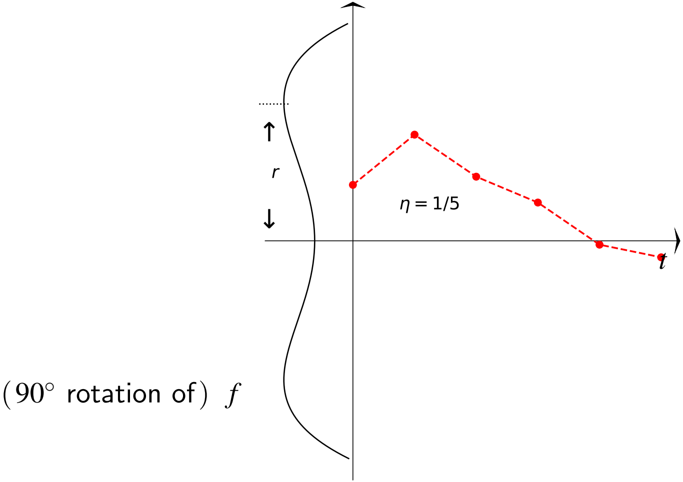
Gradient Flow: Law of Large Numbers



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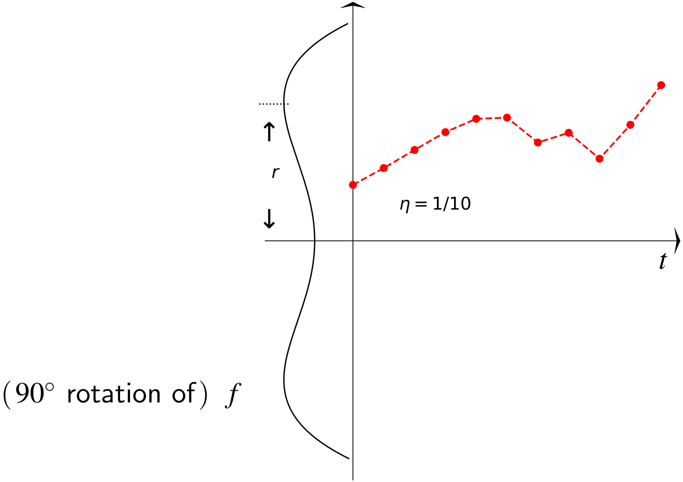

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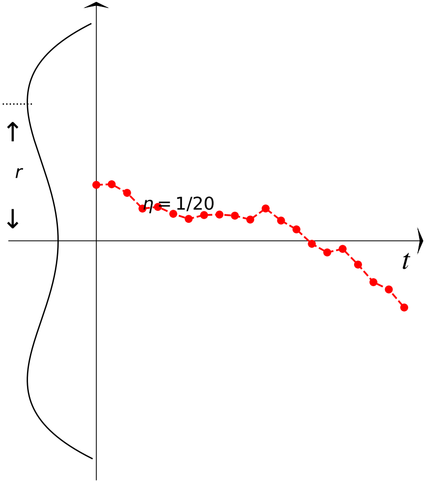

$(90^\circ$ rotation of$)$ f

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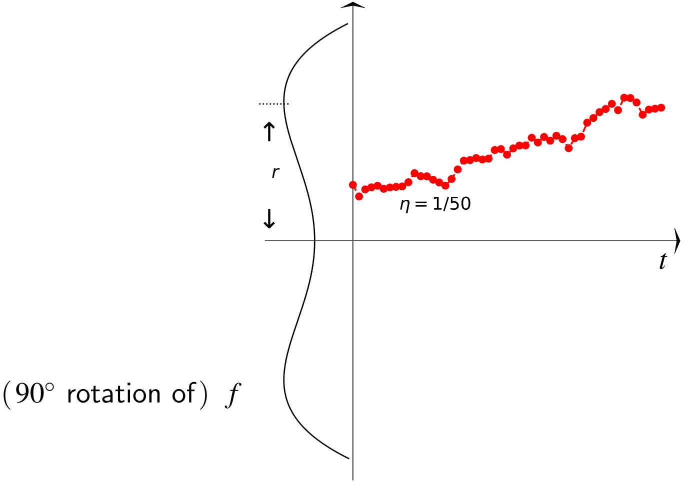

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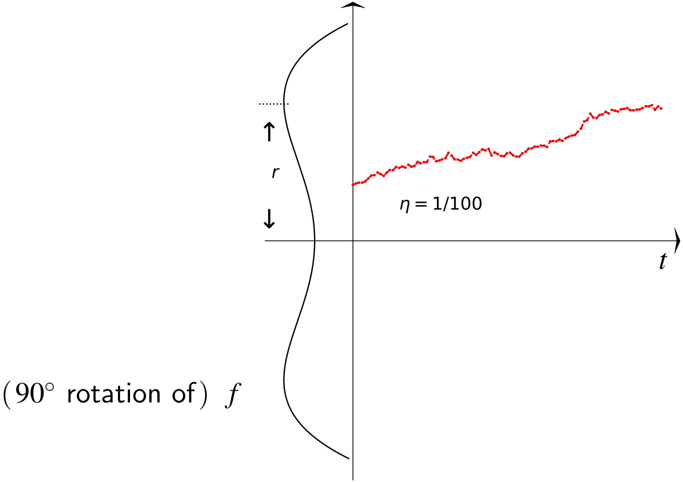

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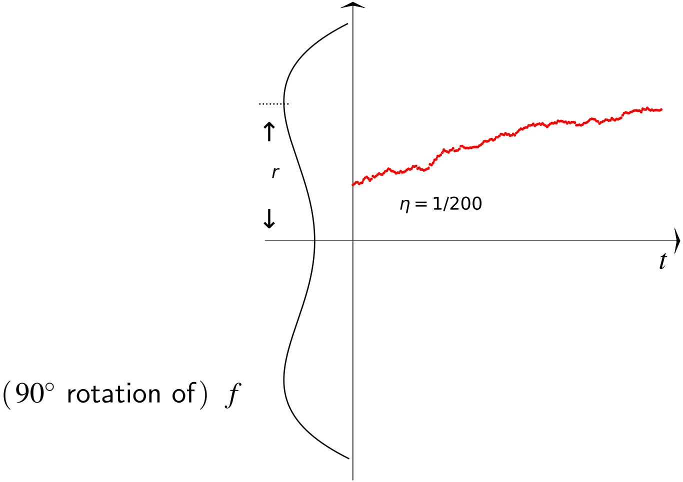

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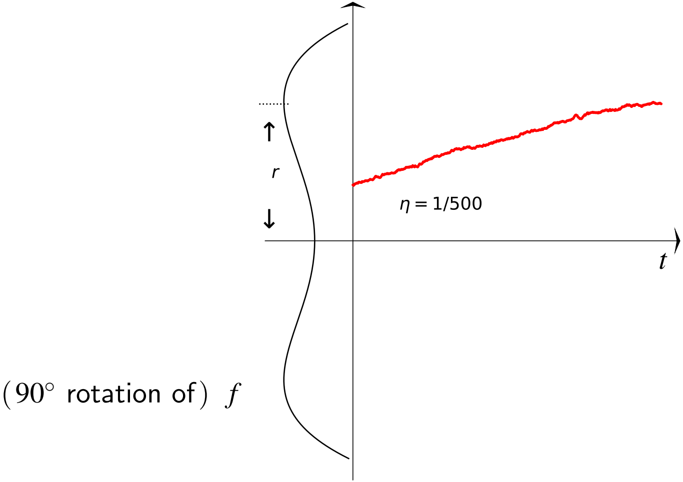

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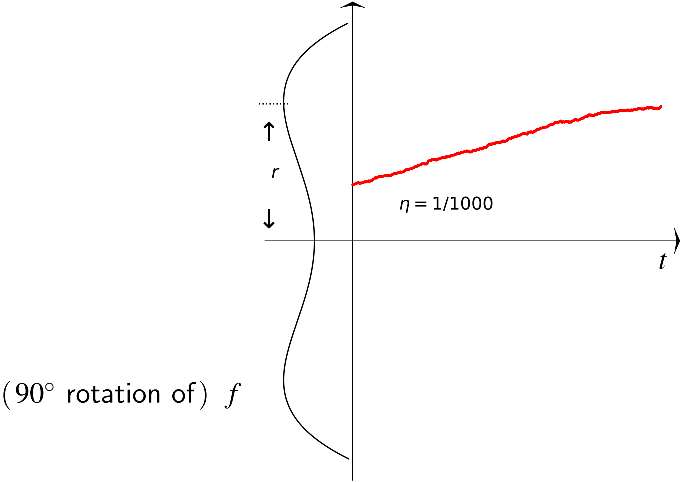

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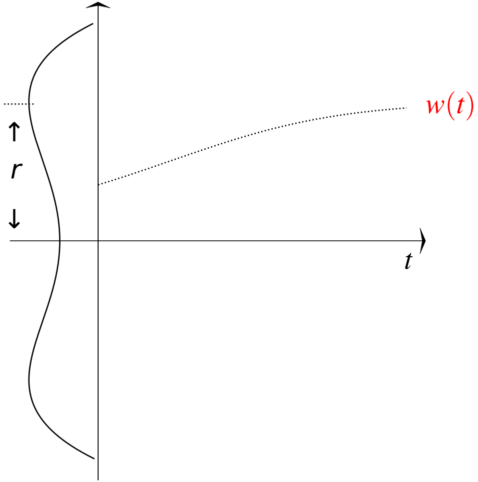

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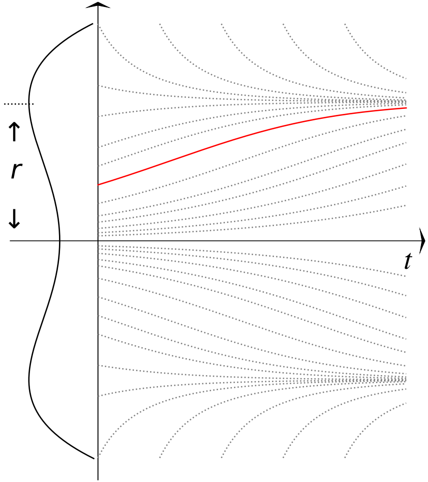

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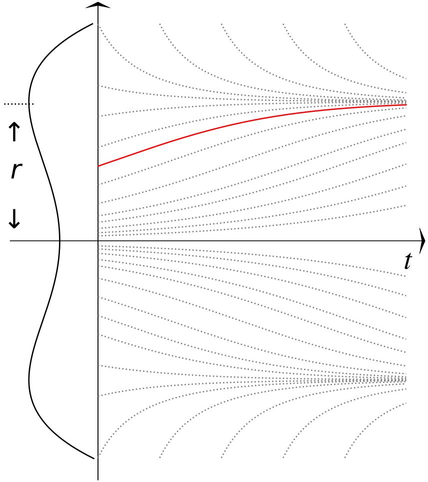

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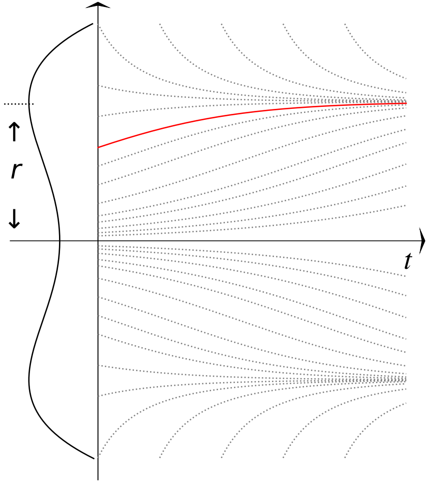

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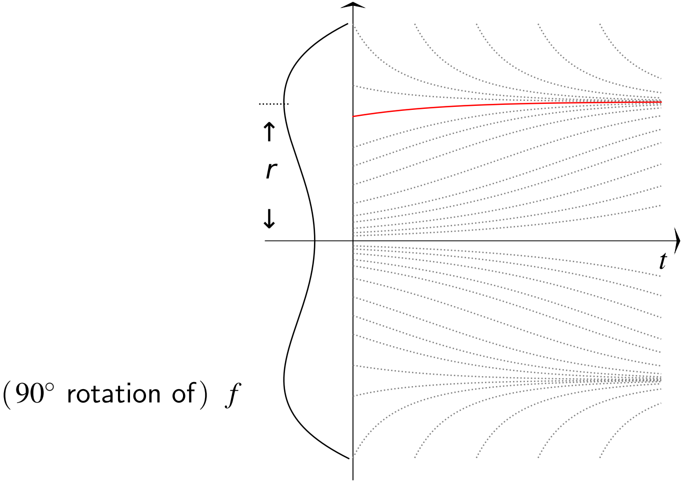

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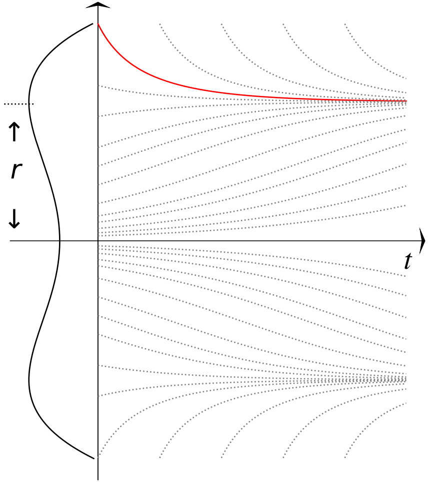

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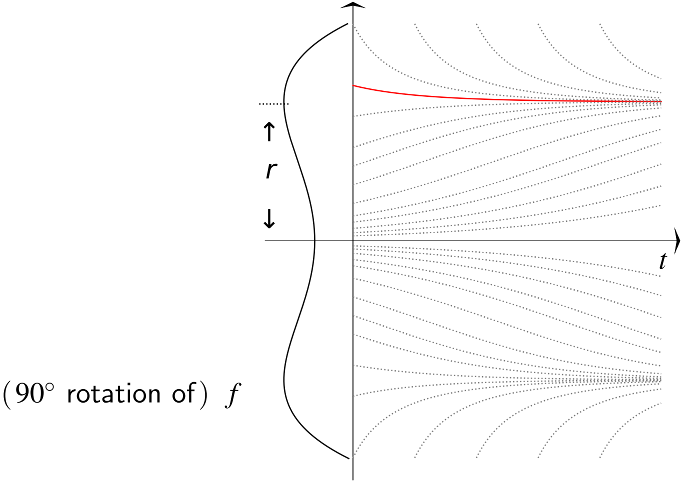

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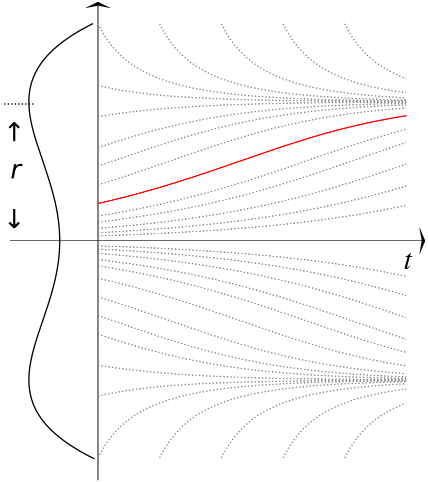

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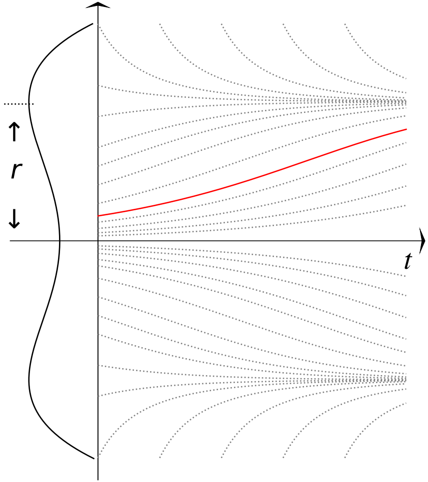

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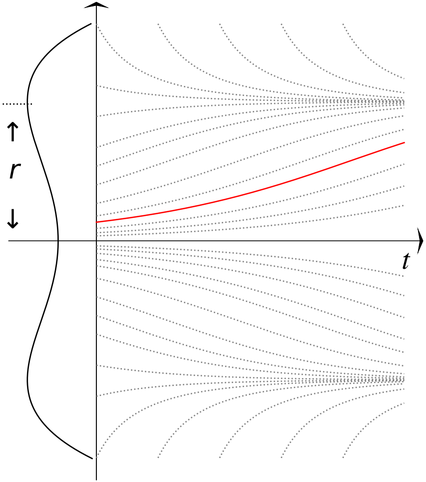

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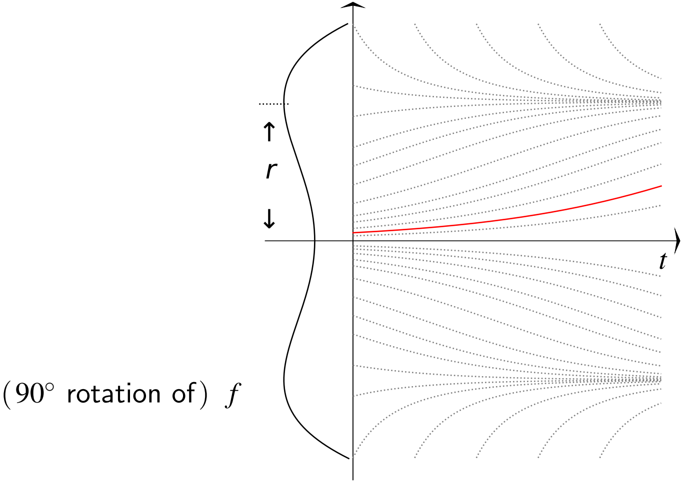

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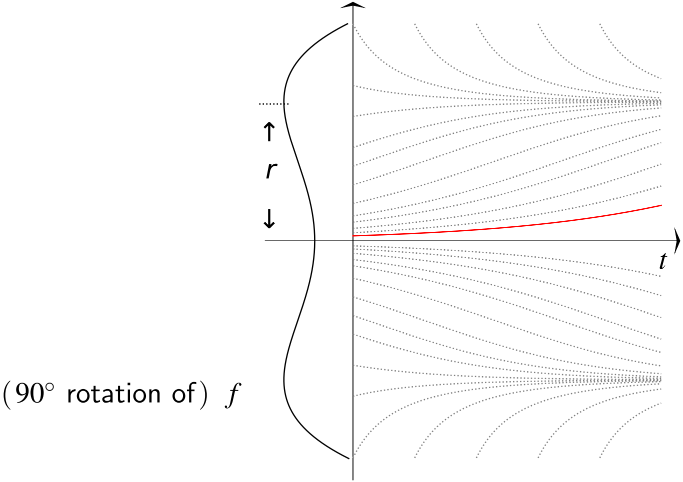

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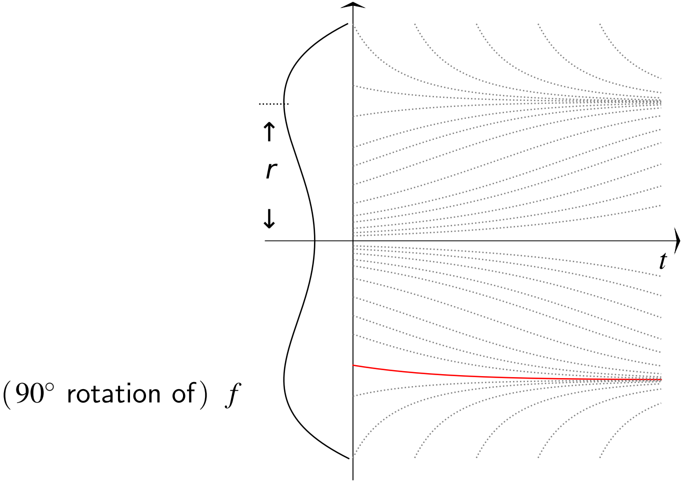

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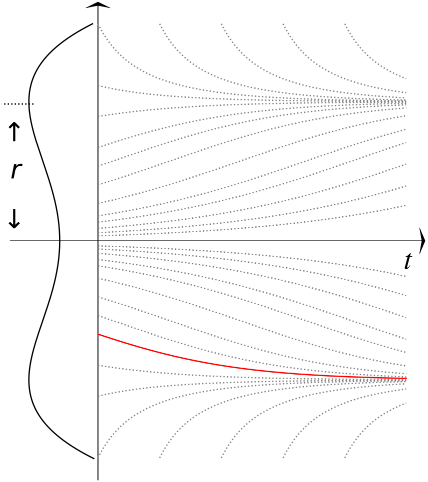

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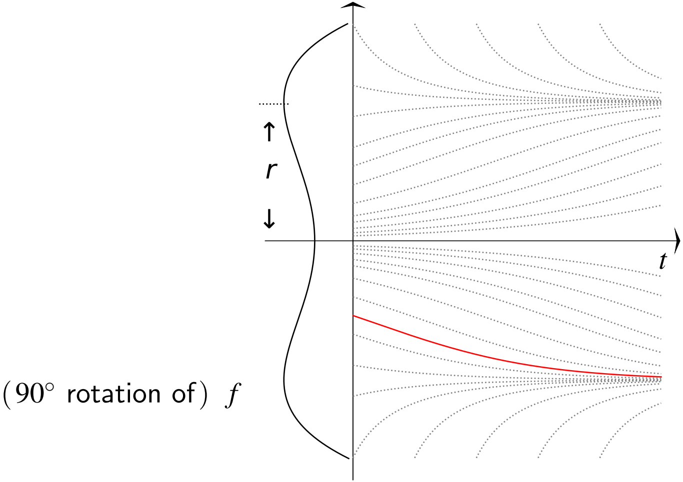

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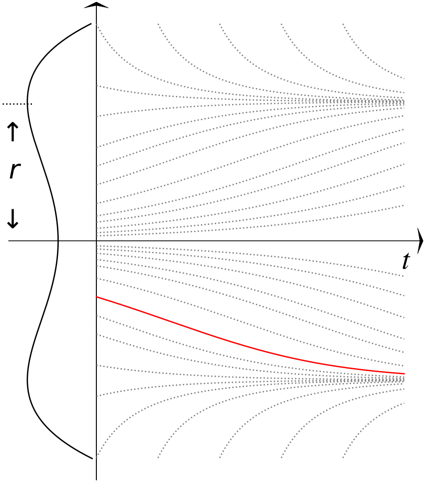

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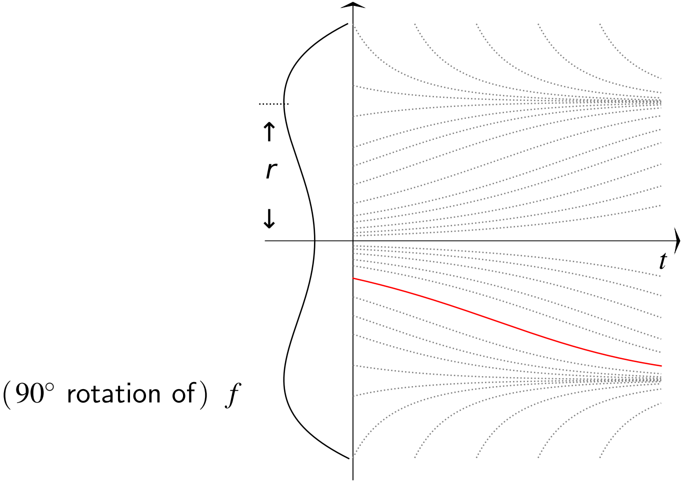
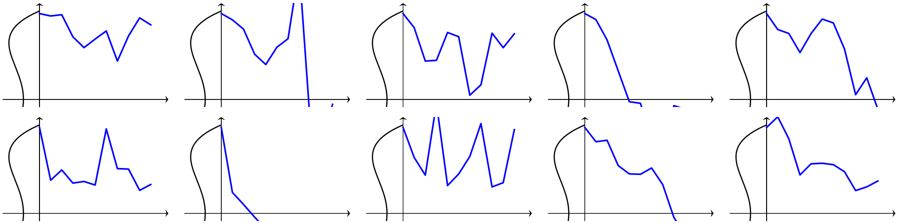


Typical Scenario

Trajectory of SGD W^η: $\eta = 1/10$ & noises are **light-tailed**



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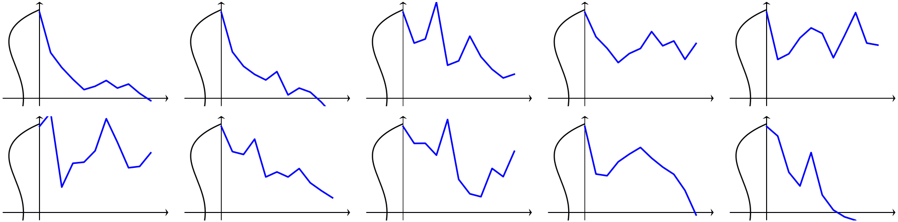

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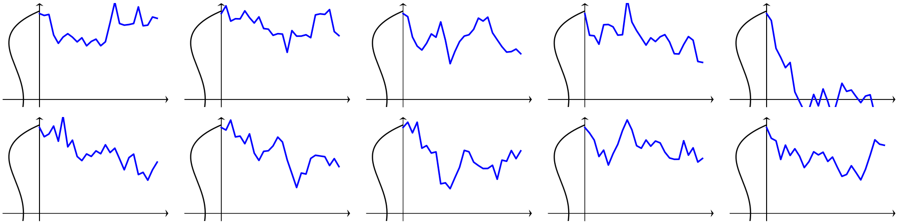

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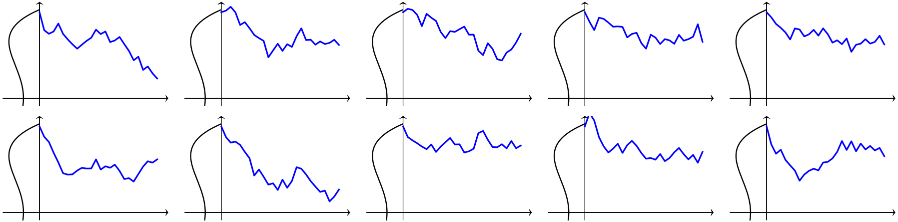
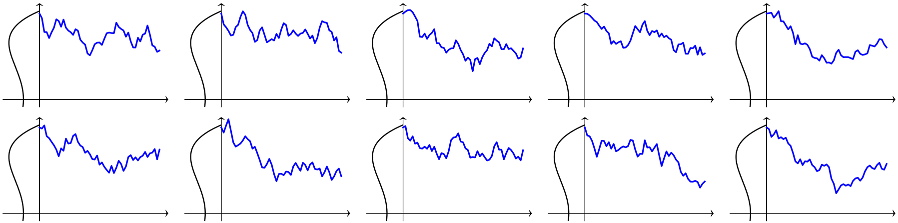
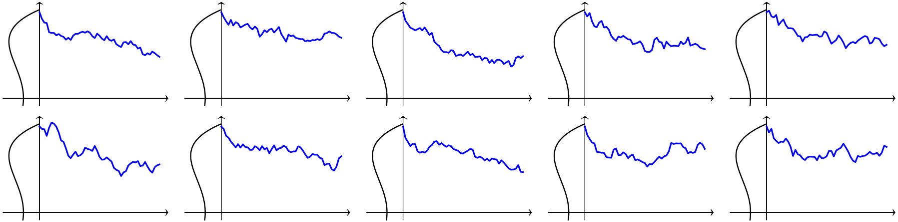


Typical Scenario

Trajectory of SGD W^η: $\eta = 1/50$ & noises are **light-tailed**

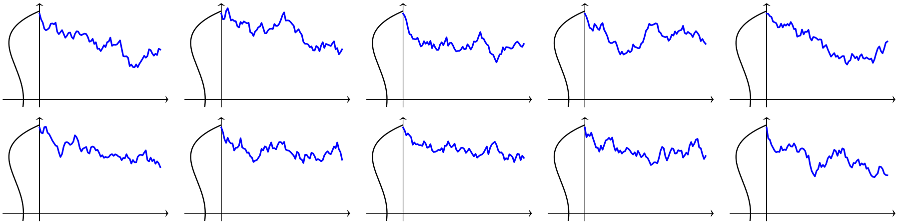


Trajectory of SGD W^η: $\eta = 1/50$ & noises are **heavy-tailed**



Typical Scenario

Trajectory of SGD W^{η}: $\eta = 1/75$ & noises are **light-tailed**



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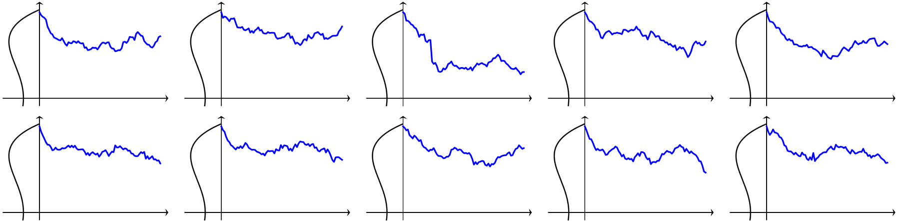
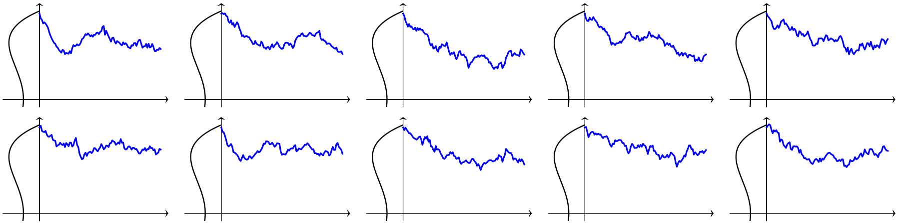


Typical Scenario

Trajectory of SGD W^{η}: $\eta = 1/100$ & noises are **light-tailed**



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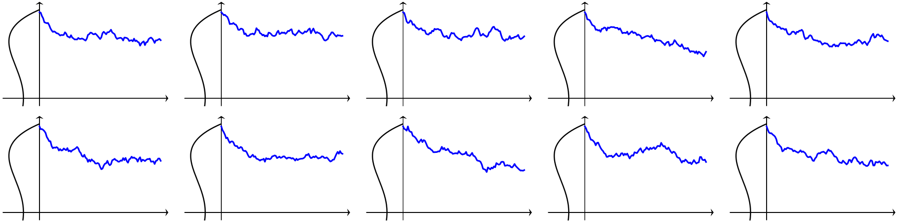
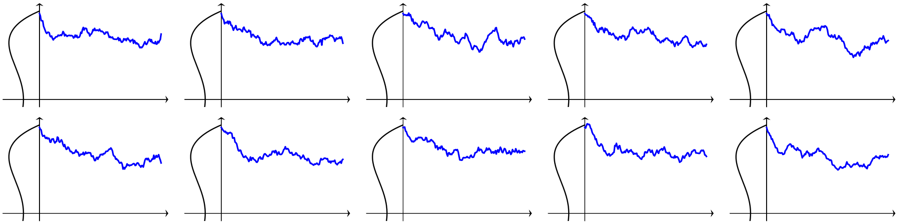


Typical Scenario

Trajectory of SGD W^{η}: $\eta = 1/150$ & noises are **light-tailed**



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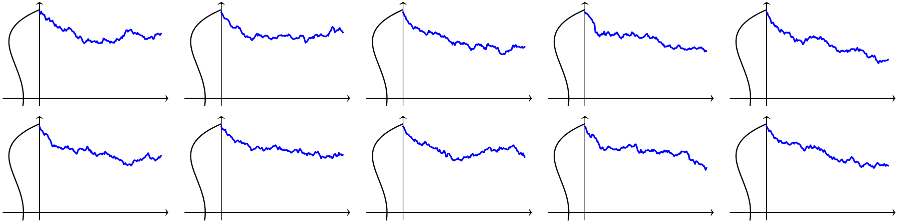
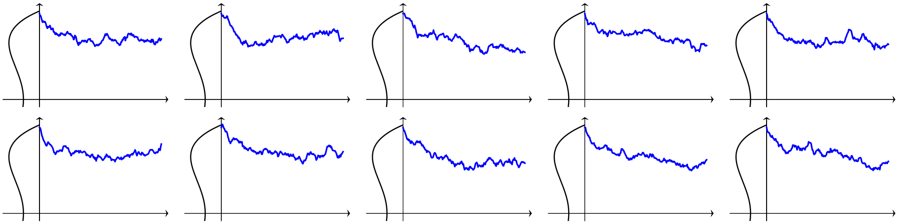


Typical Scenario

Trajectory of SGD W^{η}: $\eta = 1/200$ & noises are **light-tailed**



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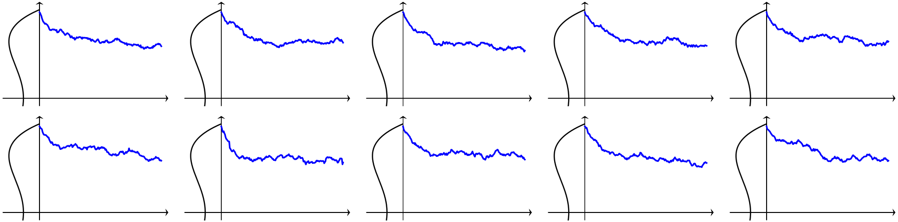

Heavy-Tailed Large Deviations for SGD

Theorem (Wang, Su, R., 2022+)

For "general" $A \subseteq \mathbb{D}$

$$C(A^\circ) \leq \liminf_{\eta \to 0} \frac{\mathbf{P}(W^\eta \in A)}{\eta^{\alpha \mathcal{J}(A)}} \leq \limsup_{\eta \to 0} \frac{\mathbf{P}(W^\eta \in A)}{\eta^{\alpha \mathcal{J}(A)}} \leq C(A^-).$$

- $\mathcal{J}(A)$: *min #jumps added to $w(\cdot)$ for it to be inside A*
- $C(\cdot)$: *a measure*

Heavy-Tailed Large Deviations for SGD

$$\mathbf{P}(|Z_i| \geq x) = x^{-(\alpha+1)}, \quad \alpha > 0$$

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Heavy-Tailed Large Deviations for SGD

gradient noise $\swarrow$

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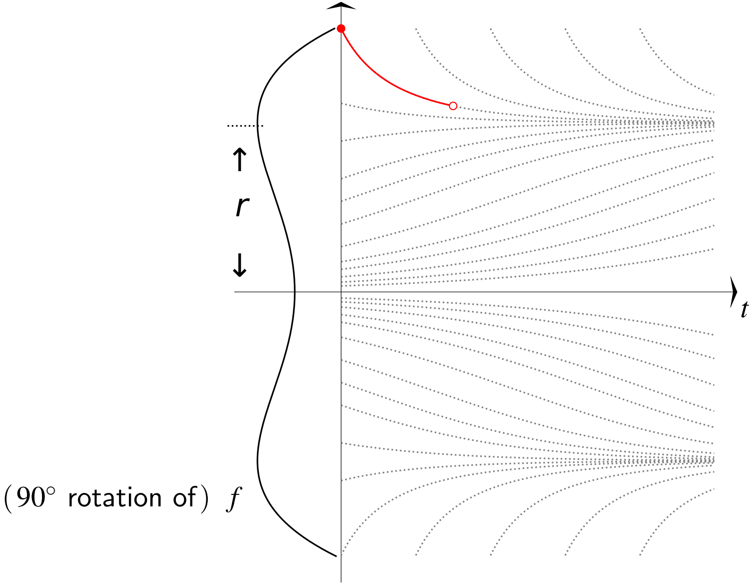
Adding Jumps to $w(\cdot)$: Piecewise Gradient Flow



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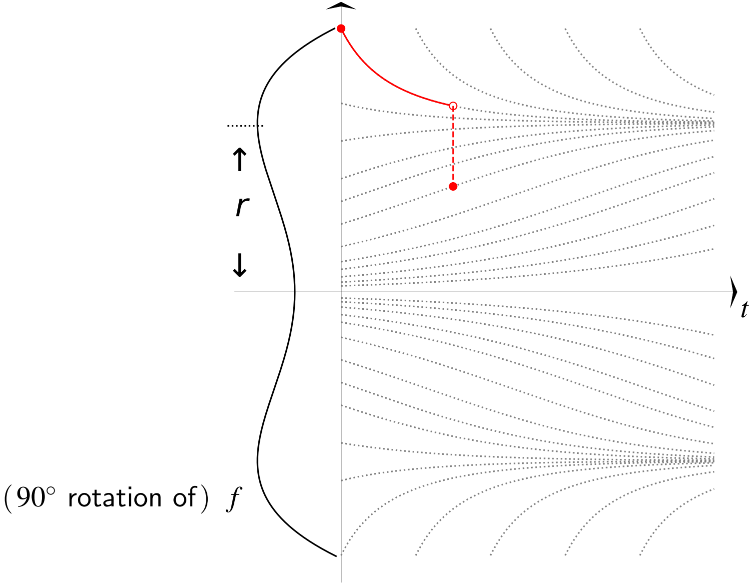

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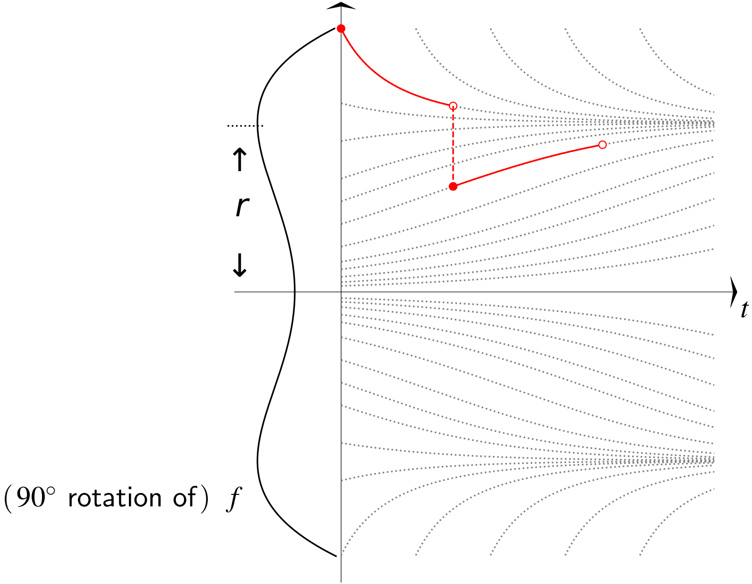

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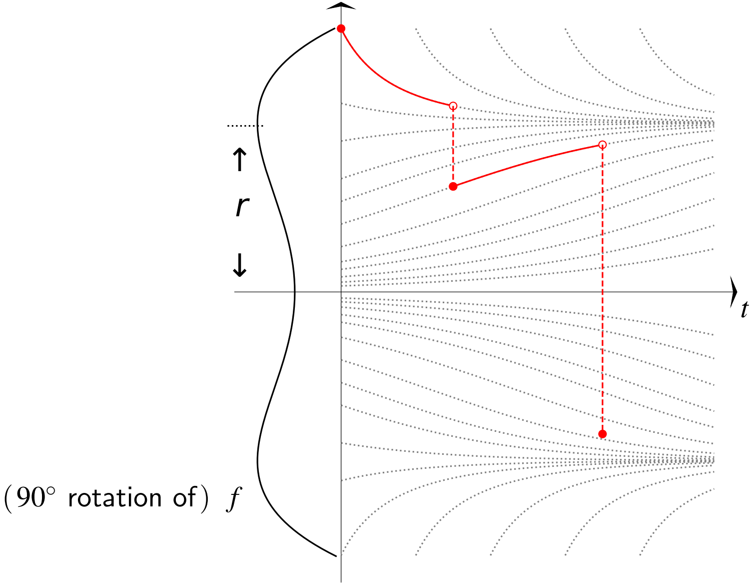

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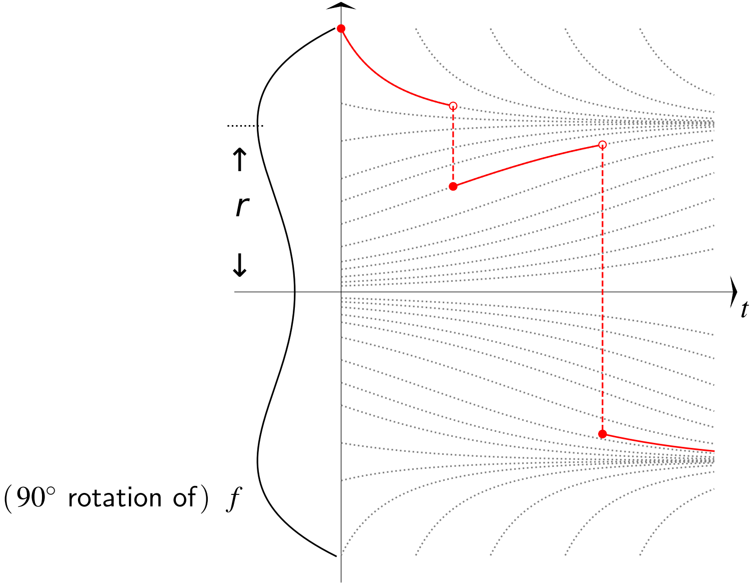

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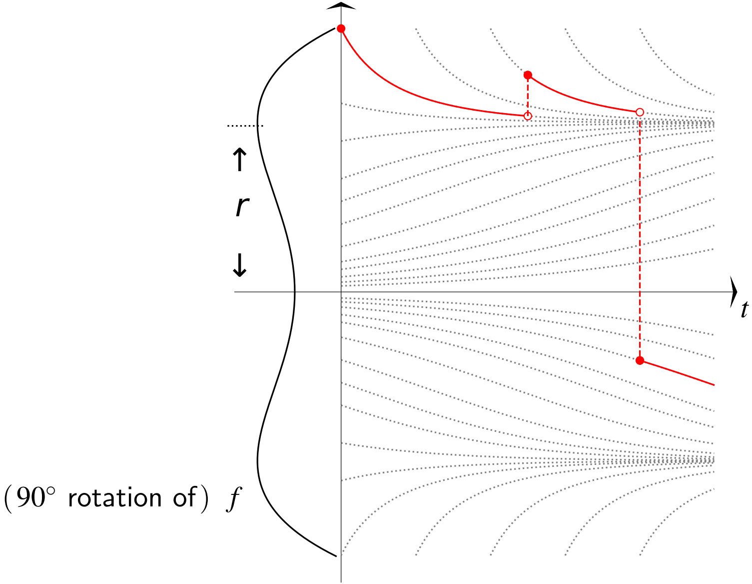

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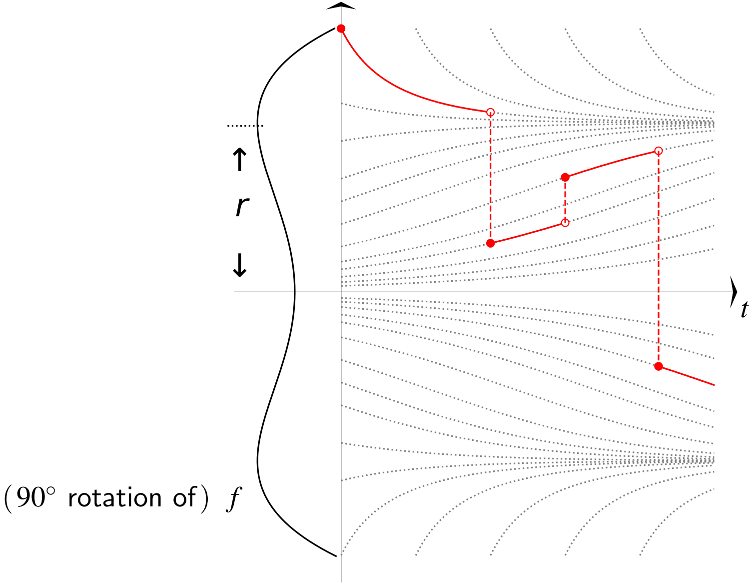

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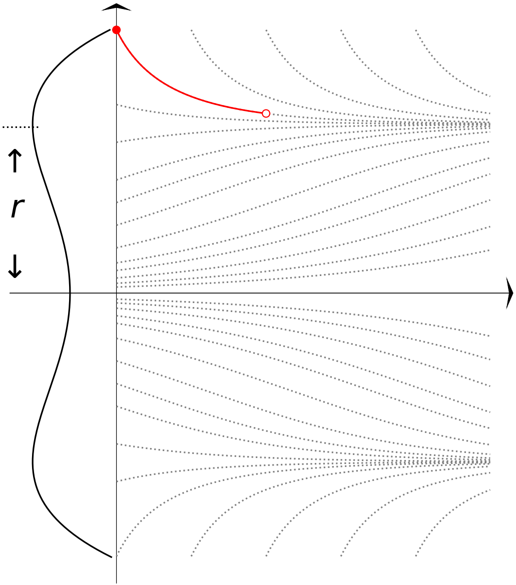
Catastrophe Principle: Most Likely Escape Route

Most Likely Paths for $\{\, W^\eta$ escapes the local minimum $\}$



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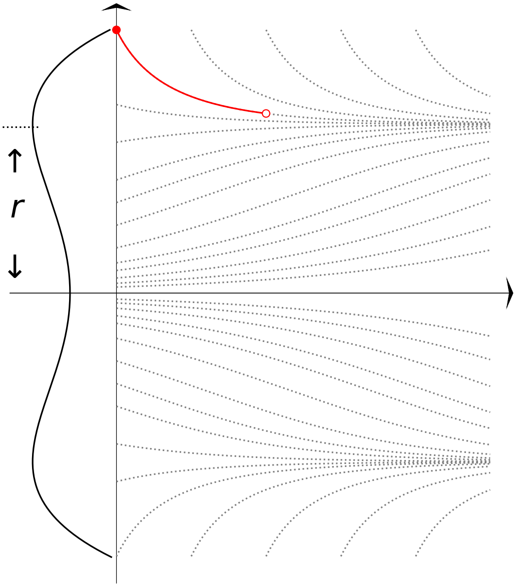

$\mathcal{J}(A) = 1$

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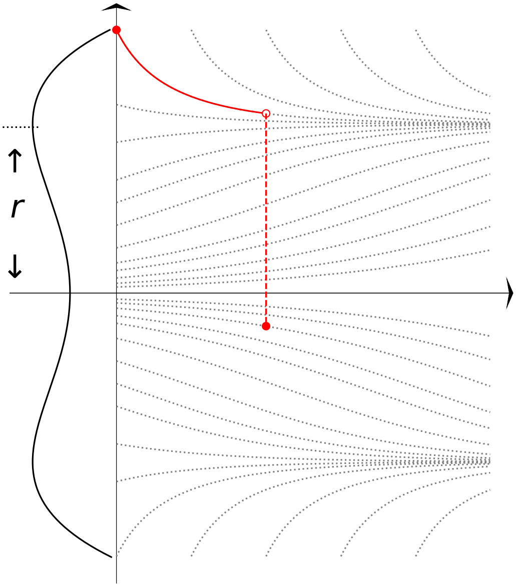

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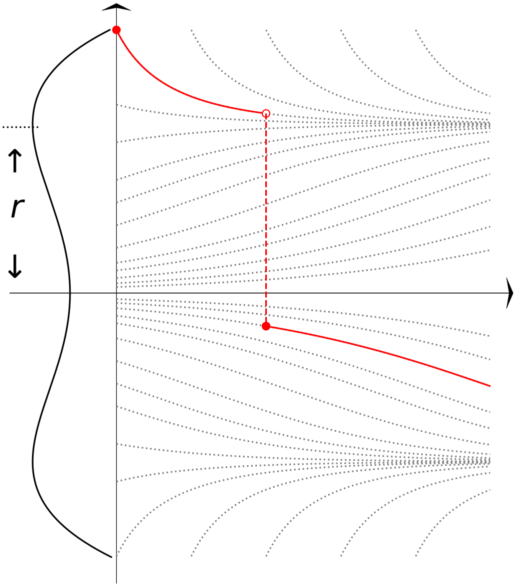

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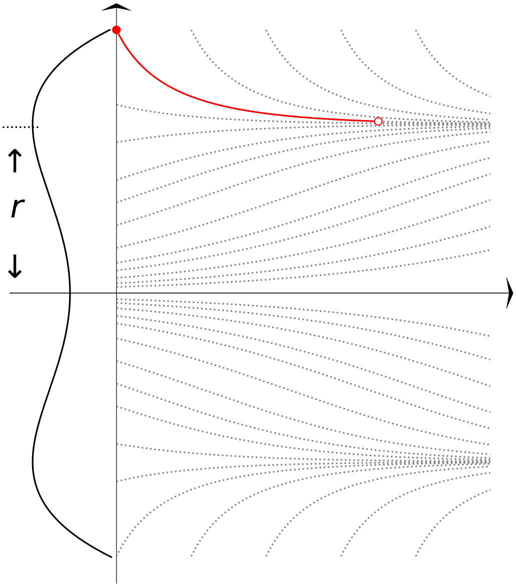

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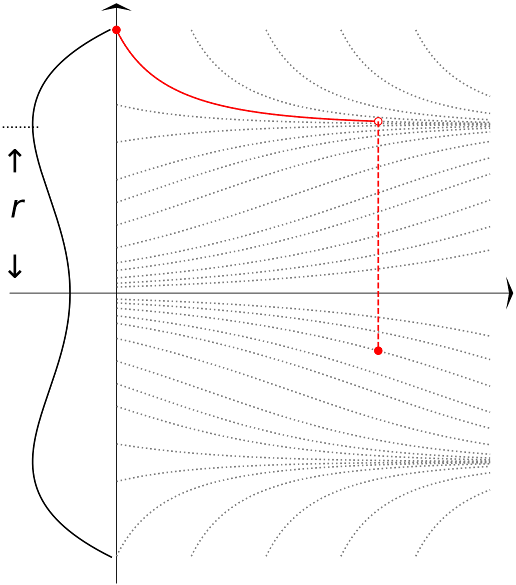

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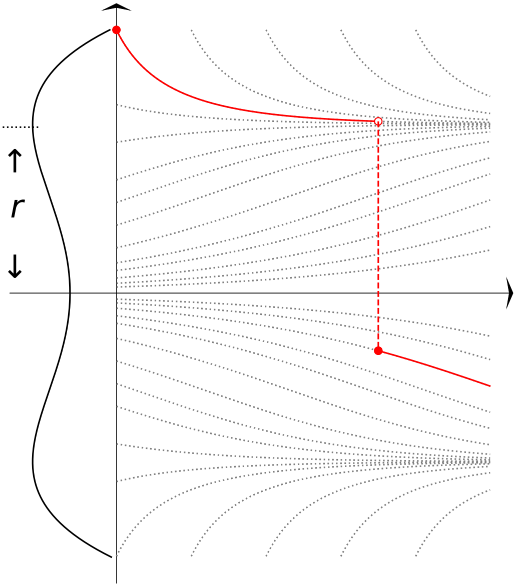

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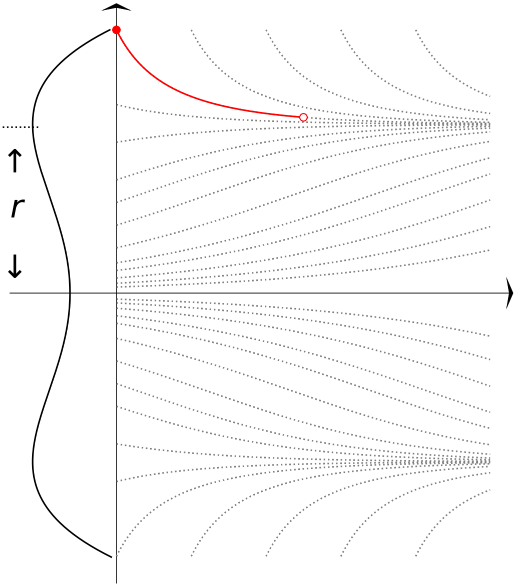

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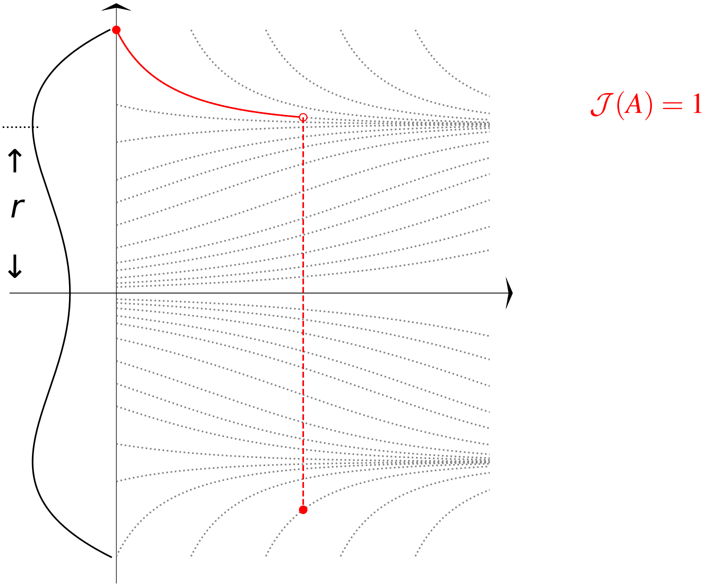

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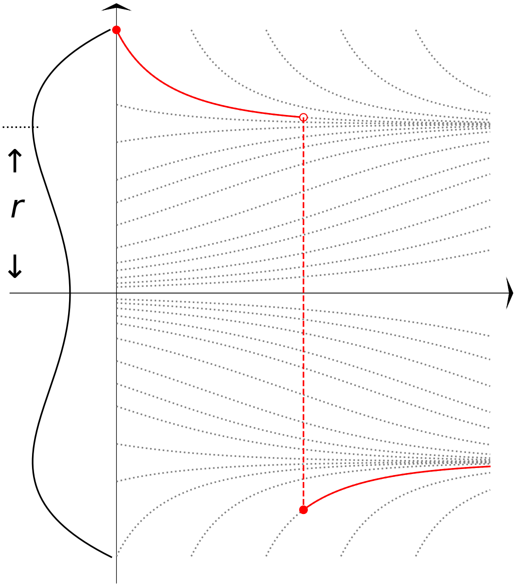
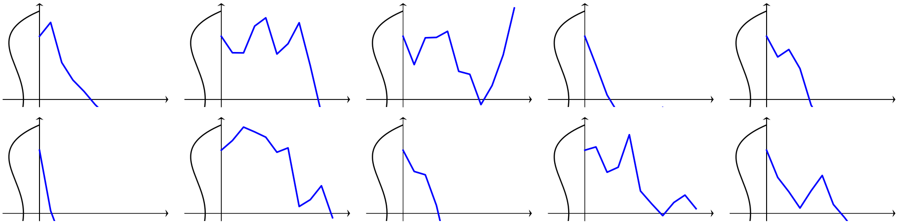


$\mathcal{J}(A) = 1$

Catastrophe Principle Dictates SGD's Escape Route

Trajectory of SGD X^η conditional on exit: **light-tailed** noises with $\eta = 1/10$



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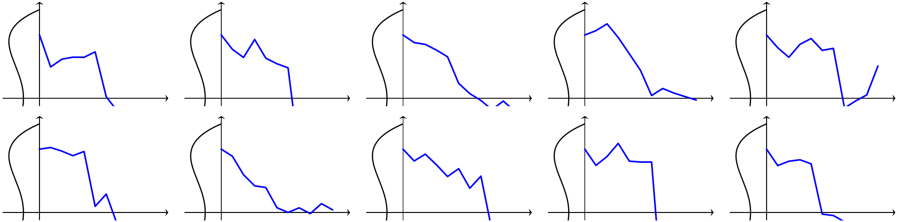
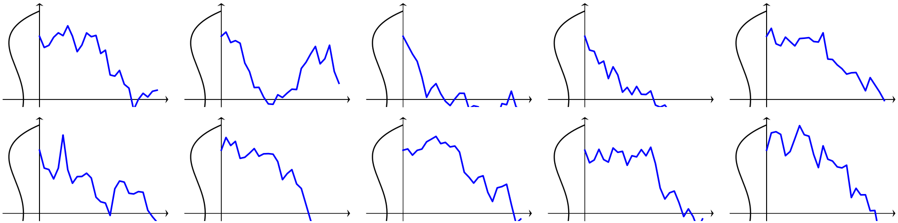


Catastrophe Principle Dictates SGD's Escape Route

Trajectory of SGD X^{η} conditional on exit: **light-tailed** noises with $\eta = 1/25$



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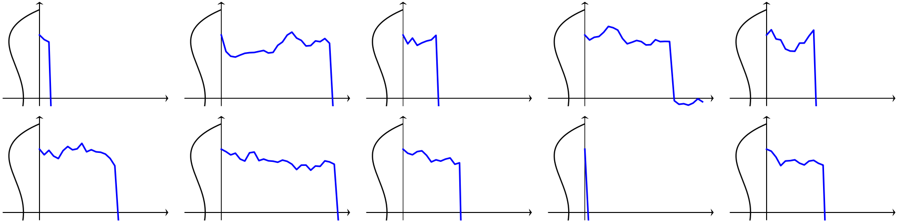

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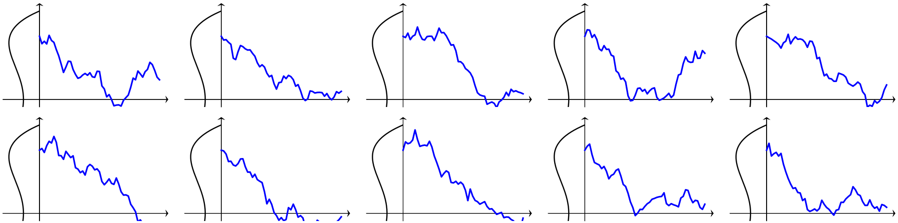

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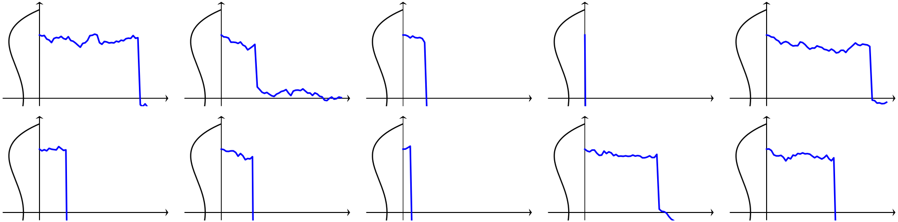

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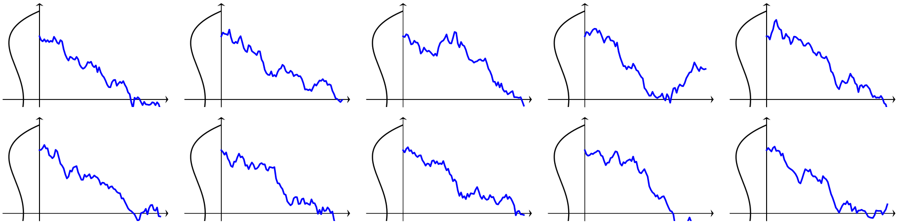

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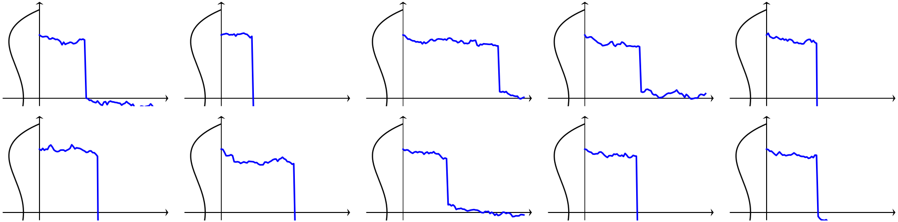

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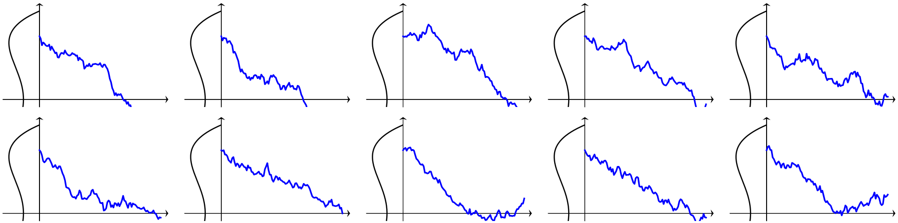

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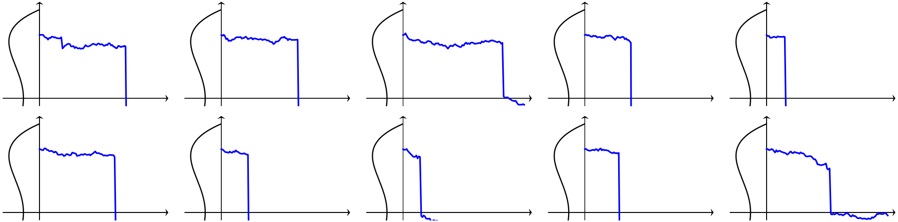

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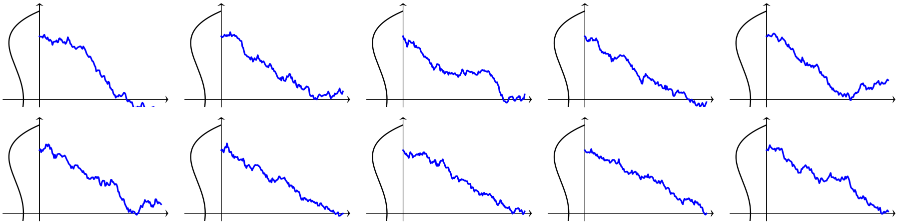

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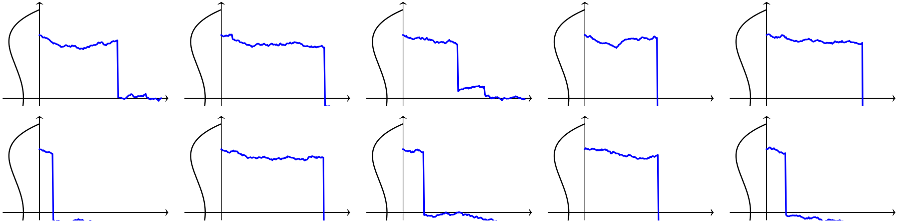

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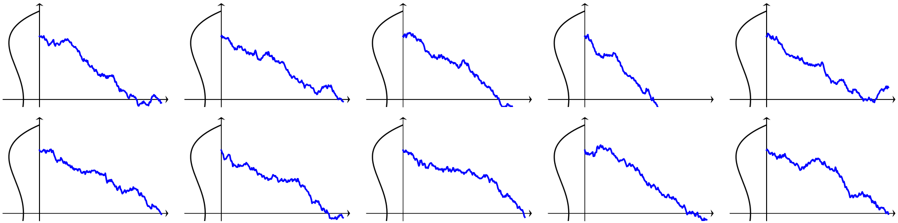

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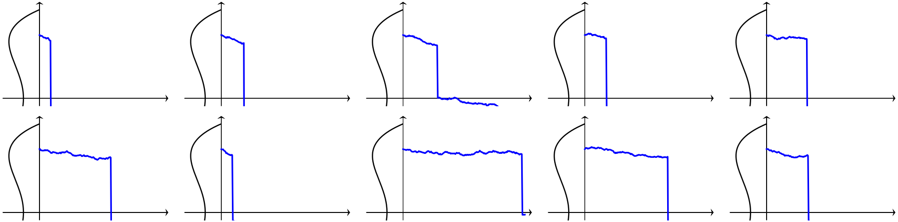


Truncated Version of Stochastic Gradient Descent

SGD

$$W_{k+1}^{\eta} = W_k^{\eta} - \quad \eta\left(f'(W_k^{\eta}) + Z_k\right) \qquad k = 0, 1, 2, \ldots$$

Truncated Version of Stochastic Gradient Descent

SGD with Gradient Clipping

$$W_{k+1}^{\eta} = W_k^{\eta} - \varphi_c\Big(\eta\big(f'(W_k^{\eta}) + Z_k\big)\Big) \qquad k = 0, 1, 2, \ldots$$

Truncated Version of Stochastic Gradient Descent

SGD with Gradient Clipping

$$W_{k+1}^{\eta} = W_k^{\eta} - \varphi_c\left(\eta\left(f'(W_k^{\eta}) + Z_k\right)\right) \qquad k = 0, 1, 2, \ldots$$

where

$$\varphi_c(x) = \frac{x}{|x|}\min\{c, |x|\}.$$

Truncated Version of Stochastic Gradient Descent

SGD with Gradient Clipping

$$W_{k+1}^{\eta} = W_k^{\eta} - \varphi_c\big(\eta\big(f'(W_k^{\eta}) + Z_k\big)\big) \qquad k = 0, 1, 2, \ldots$$

where

$$\varphi_c(x) = \frac{x}{|x|}\min\{c, |x|\}.$$

Then, again,

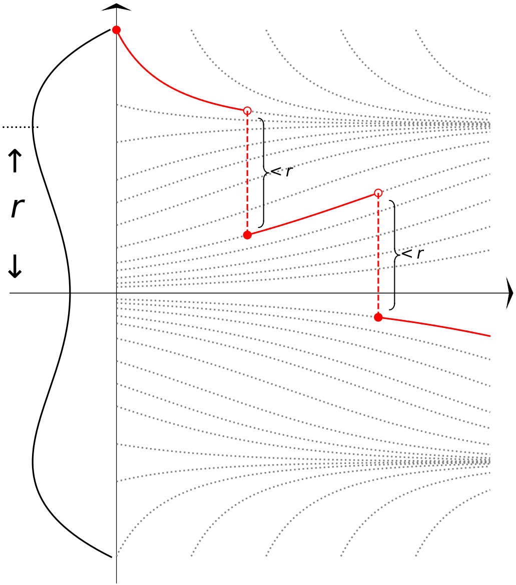
$$W^{\eta}(\cdot) \to w(\cdot) \qquad \text{as} \qquad \eta \to 0$$

where

$$\mathrm{d}w(t) = -f'(w(t))\mathrm{d}t.$$

How does $\mathcal{J}(A)$ change?

If $r \in (c, 2c)$



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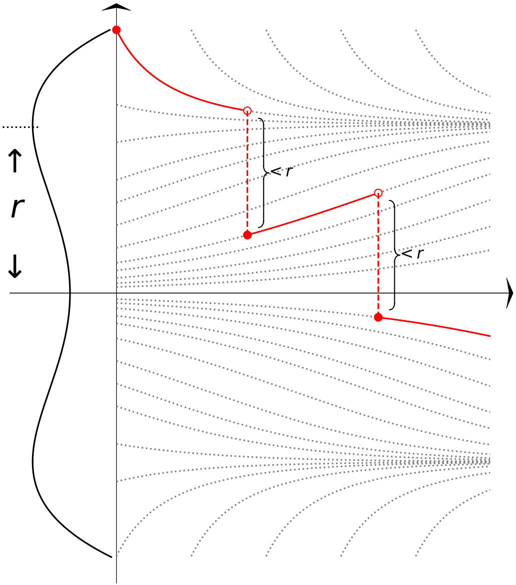

$\mathcal{J}(A) = 2$

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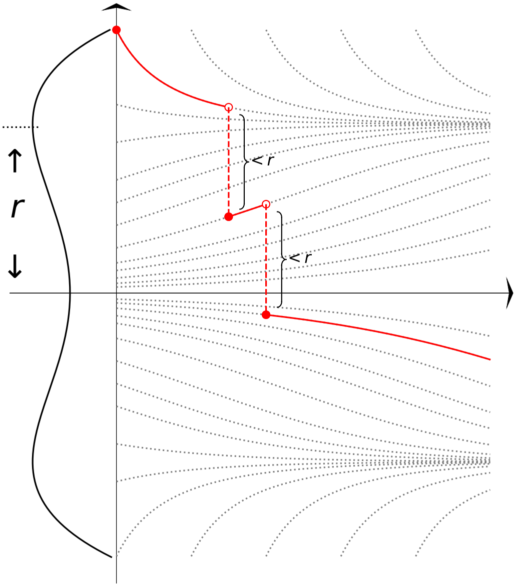

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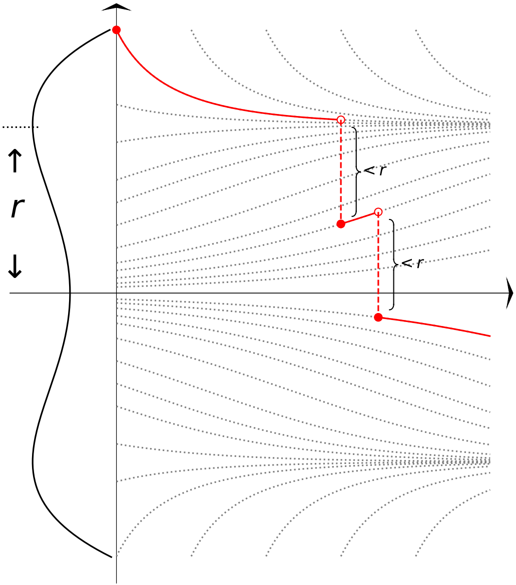
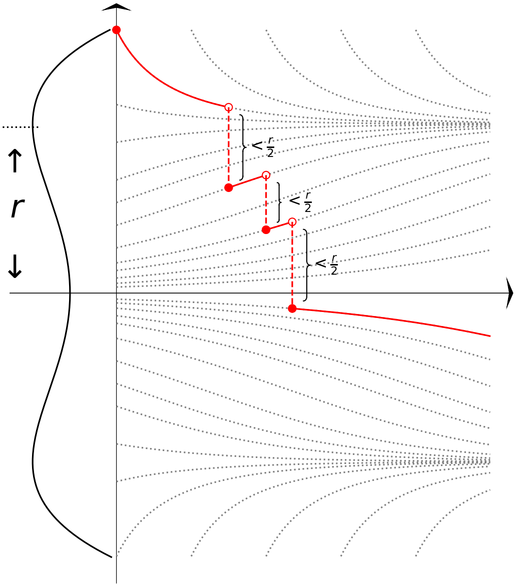


$\mathcal{J}(A) = 2$

How does $\mathcal{J}(A)$ change?

If $r \in (2c, 3c)$



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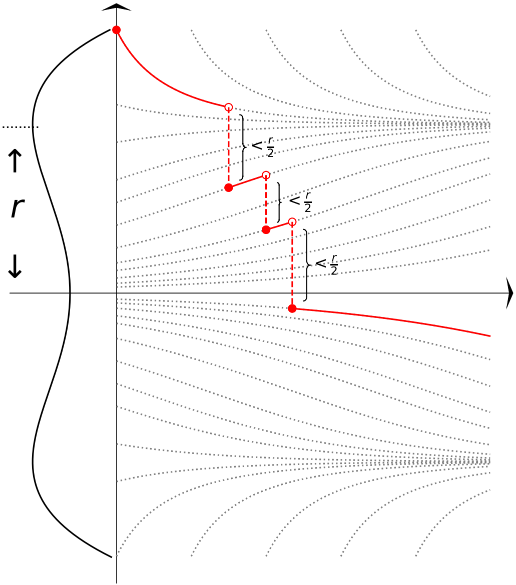
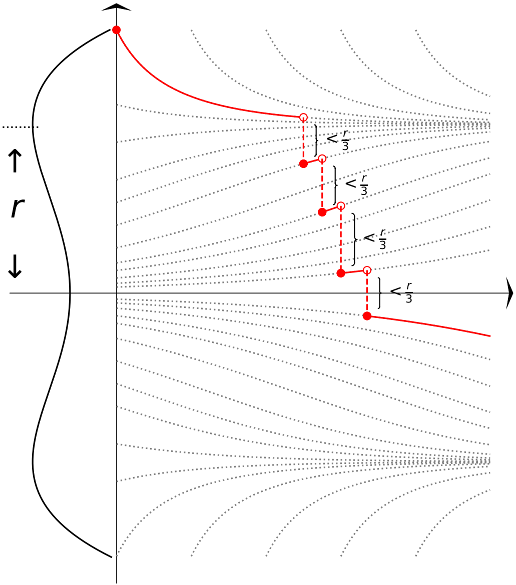
If $r \in (2c, 3c)$



$\mathcal{J}(A) = 3$

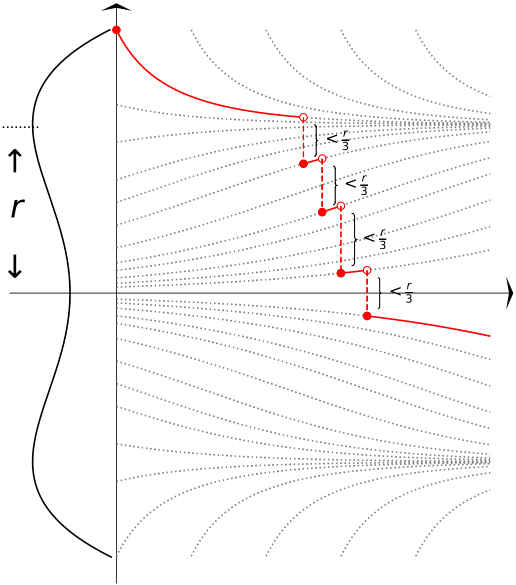
How does $\mathcal{J}(A)$ change?

If $r \in (3c, 4c)$



How does $\mathcal{J}(A)$ change?

If $r \in (3c, 4c)$



$\mathcal{J}(A) = 4$

SGD's Escaping Route under Gradient Clipping with $\mathcal{J}(A) = 2$

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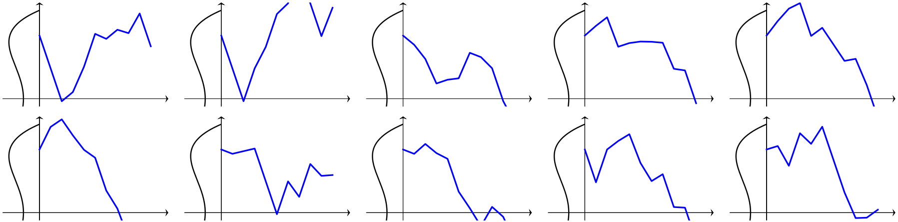
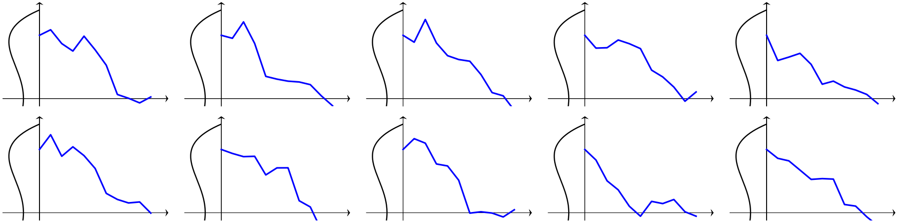
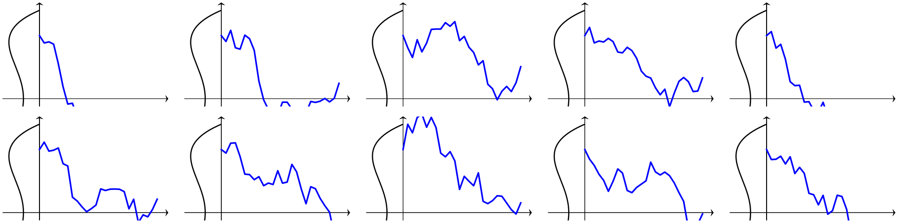


Trajectory of SGD X^η conditional on exit: **heavy-tailed** noises with $\eta = 1/10$

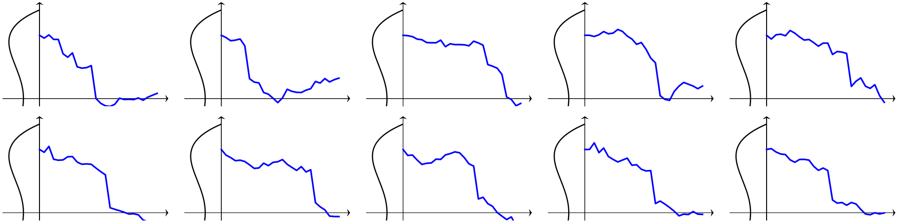


SGD's Escaping Route under Gradient Clipping with $\mathcal{J}(A) = 2$

Trajectory of SGD X^η conditional on exit: **light-tailed** noises with $\eta = 1/25$

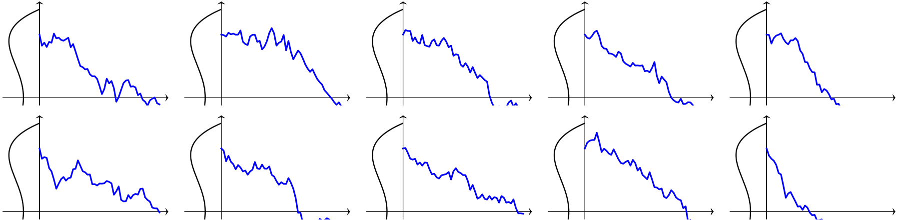


Trajectory of SGD X^η conditional on exit: **heavy-tailed** noises with $\eta = 1/25$

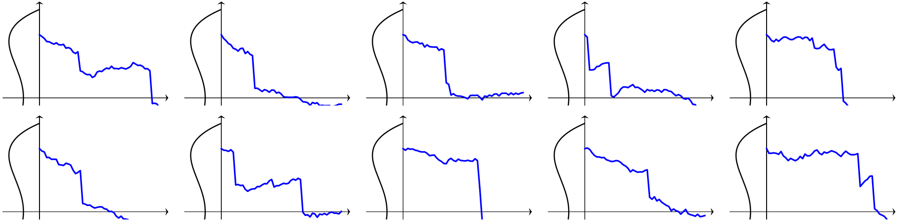


SGD's Escaping Route under Gradient Clipping with $\mathcal{J}(A) = 2$

Trajectory of SGD X^η conditional on exit: **light-tailed** noises with $\eta = 1/50$

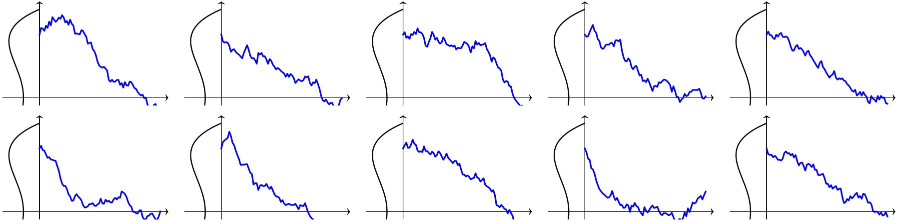


Trajectory of SGD X^η conditional on exit: **heavy-tailed** noises with $\eta = 1/10$

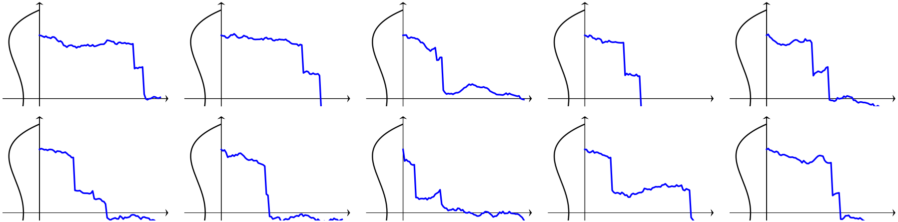


SGD's Escaping Route under Gradient Clipping with $\mathcal{J}(A) = 2$

Trajectory of SGD X^η conditional on exit: **light-tailed** noises with $\eta = 1/75$

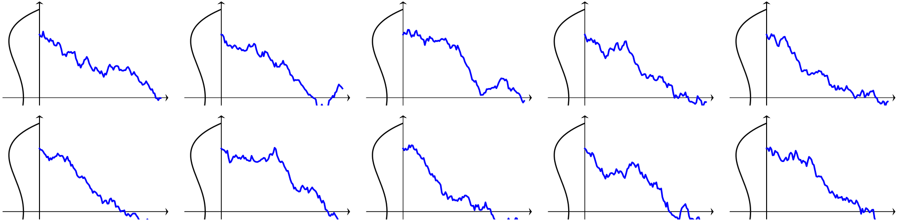


Trajectory of SGD X^η conditional on exit: **heavy-tailed** noises with $\eta = 1/75$

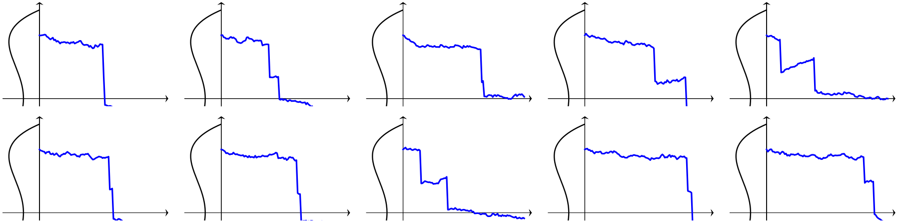


SGD's Escaping Route under Gradient Clipping with $\mathcal{J}(A) = 2$

Trajectory of SGD X^η conditional on exit: **light-tailed** noises with $\eta = 1/100$

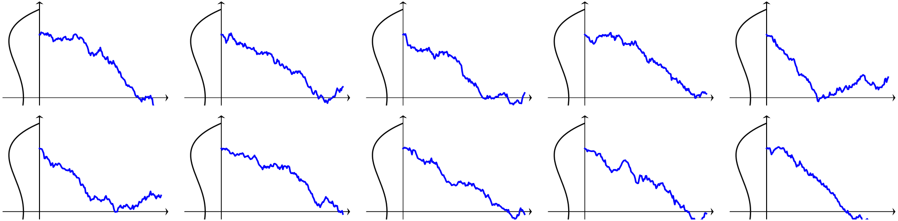


Trajectory of SGD X^η conditional on exit: **heavy-tailed** noises with $\eta = 1/100$

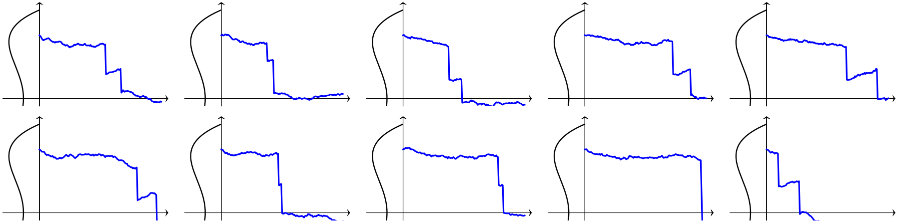


SGD's Escaping Route under Gradient Clipping with $\mathcal{J}(A) = 2$

Trajectory of SGD X^η conditional on exit: **light-tailed** noises with $\eta = 1/150$

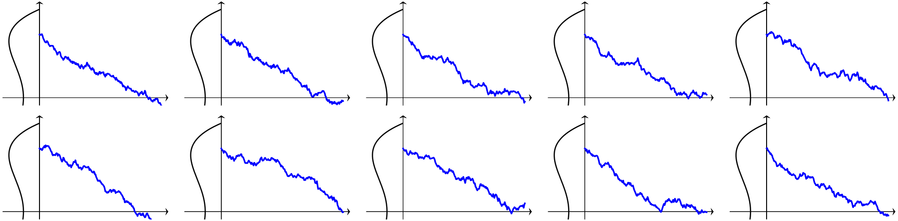


Trajectory of SGD X^η conditional on exit: **heavy-tailed** noises with $\eta = 1/150$

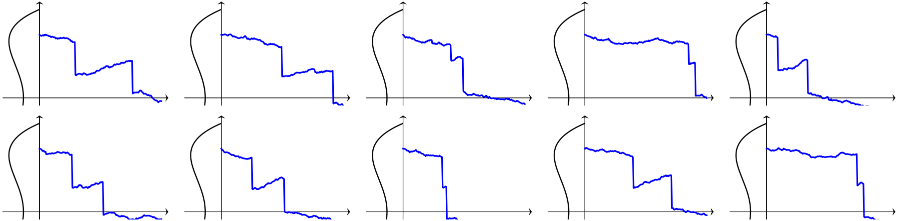


SGD's Escaping Route under Gradient Clipping with $\mathcal{J}(A) = 2$

Trajectory of SGD X^η conditional on exit: **light-tailed** noises with $\eta = 1/200$

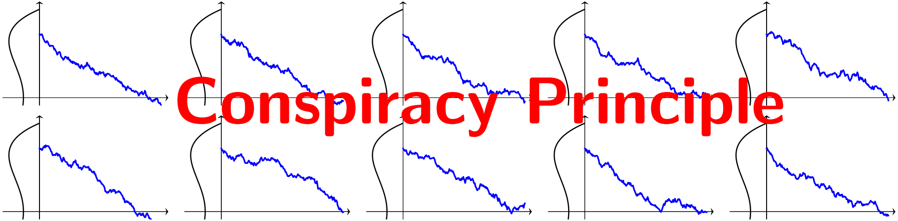


Trajectory of SGD X^η conditional on exit: **heavy-tailed** noises with $\eta = 1/200$



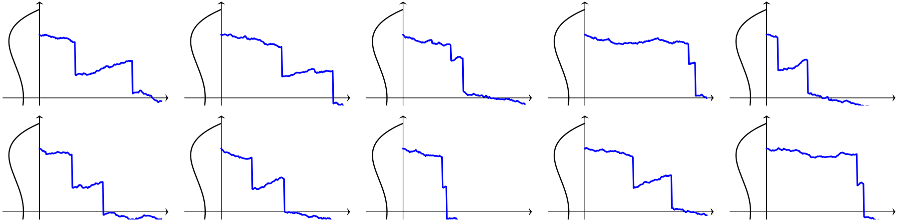
SGD's Escaping Route under Gradient Clipping with $\mathcal{J}(A) = 2$

Trajectory of SGD X^η conditional on exit: **light-tailed** noises with $\eta = 1/200$



Conspiracy Principle

Trajectory of SGD X^η conditional on exit: **heavy-tailed** noises with $\eta = 1/200$



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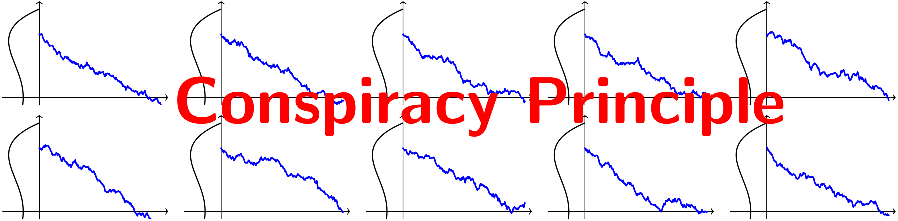

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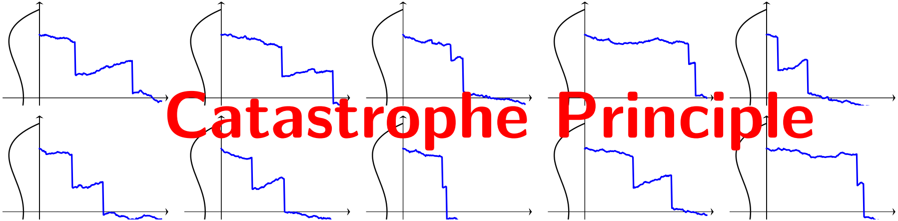
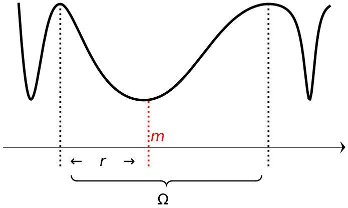


Catastrophe Principle

Metastability of SGD

First Exit Time Analysis



$l = \lceil r/c \rceil$: "width" of the attraction fields,

gradient noise

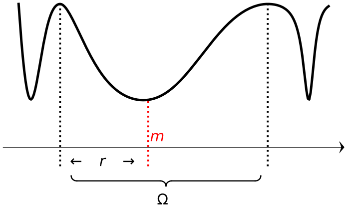
$\mathbf{P}(|Z_i| \geq x) = x^{-(\alpha+1)}, \quad \alpha > 0$

Theorem (Wang, Oh, R., 2022)

Let $\sigma(\eta) = \min\{j \geq 0 : W_j^\eta \notin \Omega\}$ *and* $\lambda(\eta) \sim \eta^{1+\alpha \cdot l}$

$$\sigma(n)\lambda(n) \Rightarrow \mathrm{Exp}(1)$$

First Exit Time Analysis



gradient noise

$l = \lceil r/c \rceil$: "width" of the attraction fields, $\mathbf{P}(|Z_i| \geq x) = x^{-(\alpha+1)}$, $\alpha > 0$

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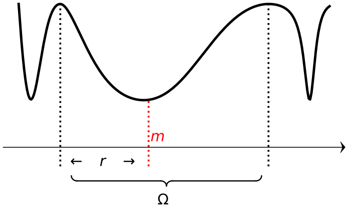
Let $\sigma(\eta) = \min\{j \geq 0 : W_j^\eta \notin \Omega\}$ *and* $\lambda(\eta) \sim \eta^{1+\alpha \cdot l}$

First Exit Time

$$\sigma(n)\lambda(n) \Rightarrow \text{Exp}(1)$$

First Exit Time Analysis



gradient noise

$l = \lceil r/c \rceil$: "width" of the attraction fields, $\mathbf{P}(|Z_i| \geq x) = x^{-(\alpha+1)}, \quad \alpha > 0$

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Let $\sigma(\eta) = \min\{j \geq 0 : W_j^\eta \notin \Omega\}$ *and* $\lambda(\eta) \sim \eta^{1+\alpha \cdot l}$

First Exit Time

$$\sigma(n)\lambda(n) \Rightarrow \mathrm{Exp}(1)$$

$\sim (1/\eta)^{1+\alpha \cdot l}$

Eliminating Sharp Local Minima with Truncated Heavy-Tails

gradient noise

l^*: "width" of the widest attraction fields, $\mathbf{P}(|Z_i| \geq x) = x^{-(\alpha+1)}, \quad \alpha > 0$

Theorem (Wang, Oh, R., 2022)

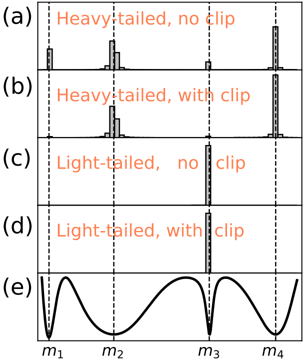
Under certain structural conditions, for any $t > 0$ and $\beta > 1 + \alpha \cdot l^*$,

$$\frac{1}{t/\eta^\beta} \int_0^{\lfloor t/\eta^\beta \rfloor} \mathbb{I}\{W^\eta_{\lfloor u \rfloor} \in \textit{sharp minima}\} du \xrightarrow{p} 0$$

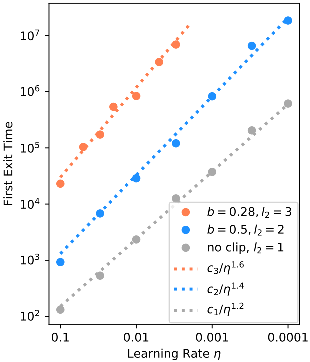
In fact, $W^\eta_{\lfloor t/\eta^{1+\alpha \cdot l^*} \rfloor}$ converges to a Markov jump processes

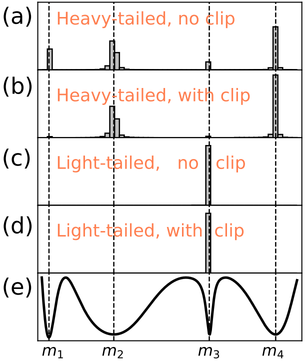
whose state space consists of wide local minima only.

Eliminating Sharp Local Minima with Truncated Heavy-Tails

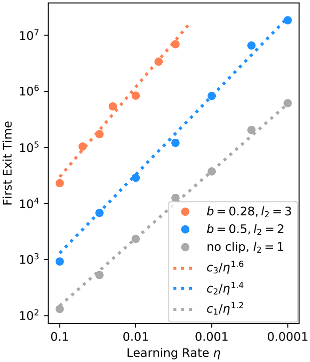


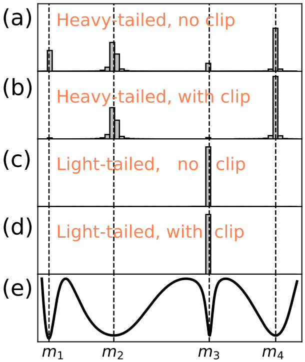
Eliminating Sharp Local Minima with Truncated Heavy-Tails

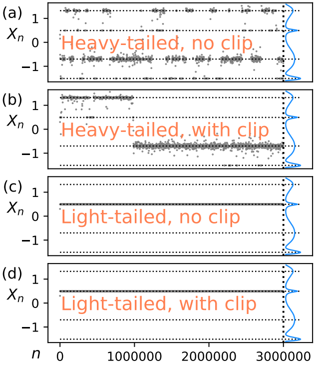




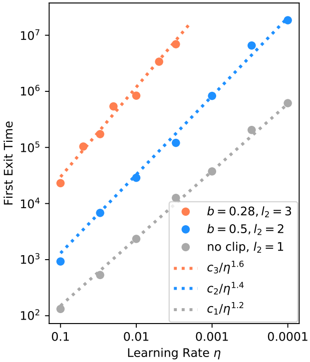
Eliminating Sharp Local Minima with Truncated Heavy-Tails

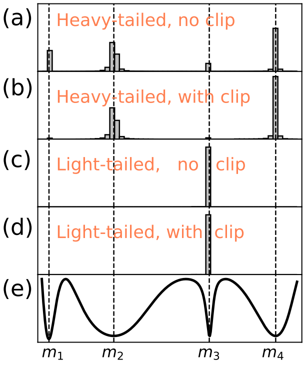


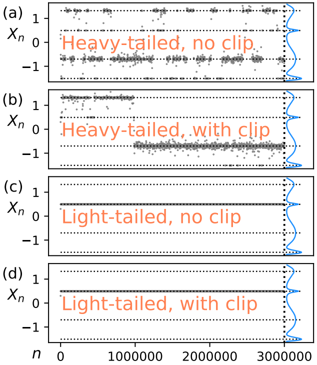




Eliminating Sharp Local Minima with Truncated Heavy-Tails

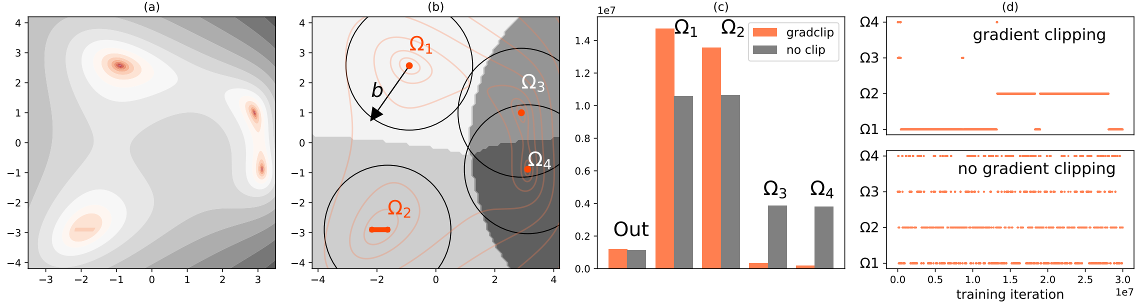




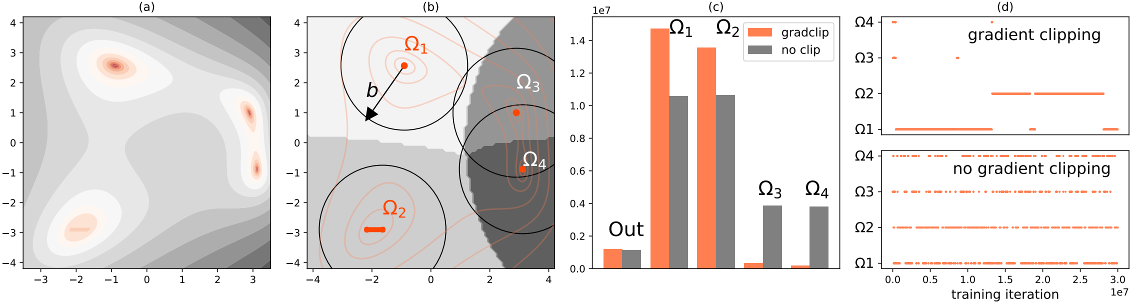


Consistent with what theory predicts!

Same Phenomena in $\mathbb{R}^2$ with More General Geometry



Same Phenomena in $\mathbb{R}^2$ with More General Geometry



Again, consistent with what theory predicts!

New Training Strategy: Tail-INflation-Truncation

Inflating Tail of Gradient Noise

$$Y = Y^{\text{Small Batch}}$$

Inflating Tail of Gradient Noise

Stochastic Gradient

$$\searrow$$

$$Y = Y^{\text{Small Batch}}$$

Inflating Tail of Gradient Noise

Stochastic Gradient

$$\searrow$$

$$Y = Y^{\text{Small Batch}} + R \cdot (Y^{\text{Large Batch}} - Y^{\text{Small Batch}})$$

Inflating Tail of Gradient Noise

Stochastic Gradient $\searrow$ Pareto RV $\swarrow$

$$Y = Y^{\text{Small Batch}} + R \cdot (Y^{\text{Large Batch}} - Y^{\text{Small Batch}})$$

Inflating Tail of Gradient Noise

Heavy-Tailed Stochastic Gradient

Pareto RV

$$Y = Y^{\text{Small Batch}} + R \cdot (Y^{\text{Large Batch}} - Y^{\text{Small Batch}})$$

Inflating Tail of Gradient Noise

Heavy-Tailed Stochastic Gradient

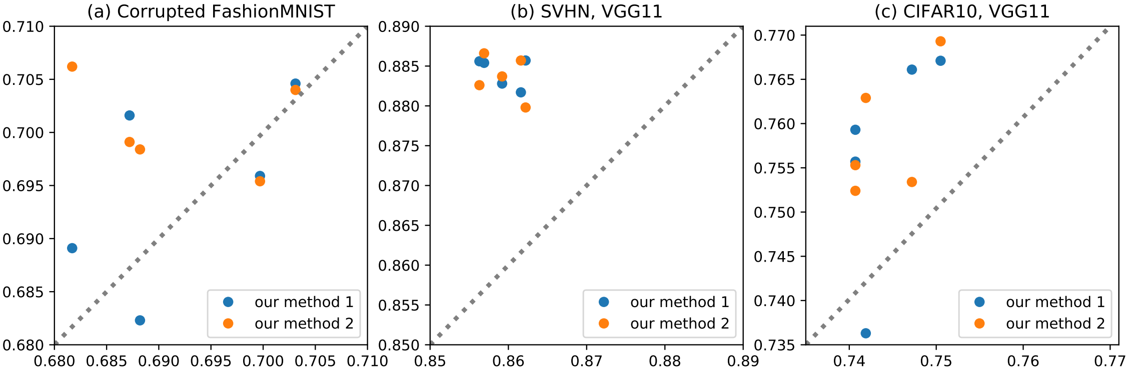
Pareto RV

$$Y = Y^{\text{Small Batch}} + R \cdot (Y^{\text{Large Batch}} - Y^{\text{Small Batch}})$$

Test accuracy	LB	SB	SB+Clip	SB+Noise	Our 1	Our 2
FashionMNIST, LeNet	68.66%	69.20%	68.77%	64.43%	69.47%	**70.06%**
SVHN, VGG11	82.87%	85.92%	85.95%	38.85%	**88.42%**	88.37%
CIFAR10, VGG11	69.39%	74.42%	74.38%	40.50%	75.69%	**75.87%**

Expected Sharpness	LB	SB	SB+Clip	SB+Noise	Our 1	Our 2
FashionMNIST, LeNet	0.032	0.008	0.009	0.047	0.003	**0.002**
SVHN, VGG11	0.694	0.037	0.041	0.012	**0.002**	0.005
CIFAR10, VGG11	2.043	0.050	0.039	2.046	**0.024**	0.037

Improvement is Consistent



Does This Actually Work with High-Volume Real-Life Data?

We will know in a few months:

- Moloco
  - top performing mobile ad tech company
  - 35B+ user impressions per month

- Launching a project to improve Moloco's conversion-rate prediction accuracy

Summary

- Rich mathematical structure in the global dynamics of heavy-tailed SGD
- Elimination of sharp local minima from SGD with truncated heavy-tailed gradient noise
- Tail-INflation-Truncation strategy improves SGD's generalization performance