Theoretical Framework for Global Dynamics and Metastability Analysis of Heavy-Tailed Systems

Chang-Han Rhee

Northwestern University

2023 INFORMS Annual Meeting, Phoenix

October 16, 2023

Based on the joint works with

Mihail Bazhba, Jose Blanchet, Bohan Chen, Sewoong Oh, Zhe Su, Xingyu Wang, and Bert Zwart

How Rare Events Characterize Global Dynamics of Heavy-Tailed Systems

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Team

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Sewoong Oh
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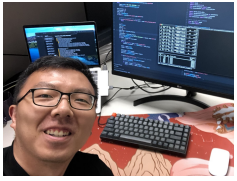
Bert Zwart
CWI



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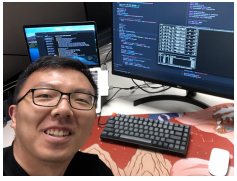

Xingyu Wang
Northwestern

(2023 Nicholson Student Paper Prize, 2nd Place)

Generalization Mystery of Deep Learning

Empirical Success of Deep Neural Networks (DNNs)

"Deep Learning is eating the world."

- Jorge Nocedal

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- Even with massive over-parametrization, DNNs generalize well.

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Training Set

Test Set image source

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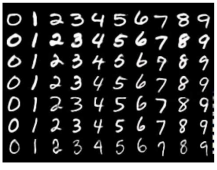

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A Central Mystery of Deep Learning

Heavy Tailed Large Deviations and Metastability

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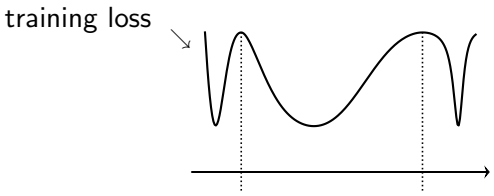

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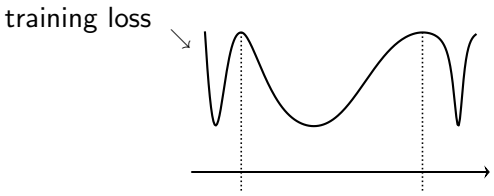

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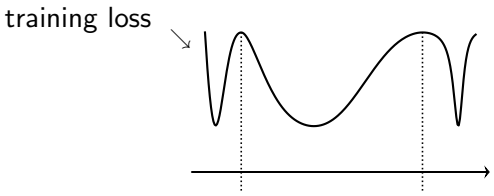

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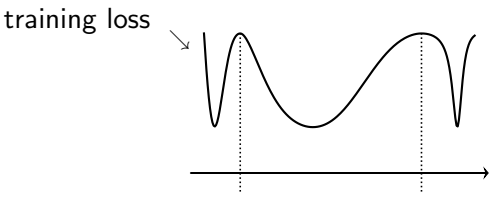

It takes more than 10^{400} time steps for SGD to escape from any of these.

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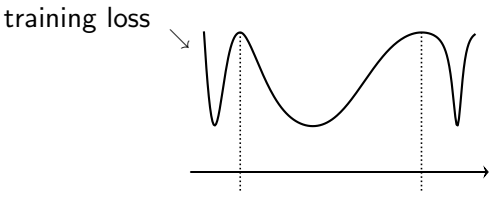

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Heavy-tailed SGD escapes local minima and prefers flat local minima.

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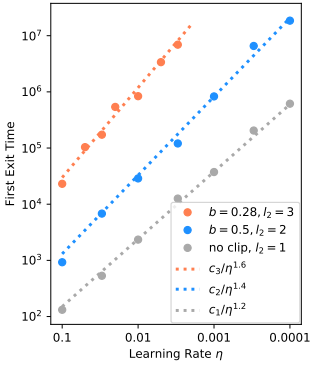
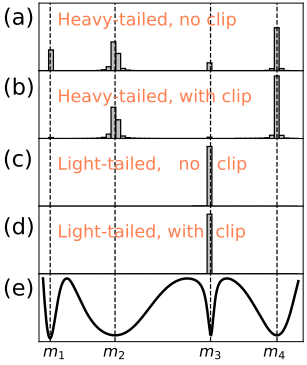
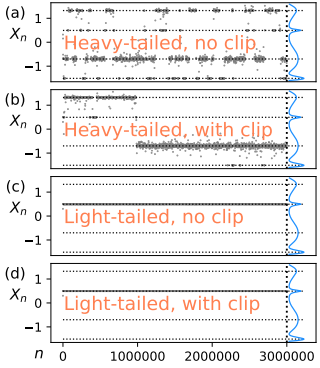
Large deviations and metastability of SGD has fascinating connection this!

Entirely Different Global Dynamics Depending on Tail Behaviors







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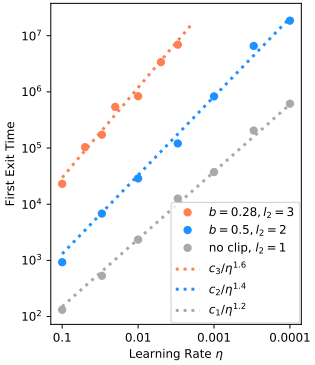

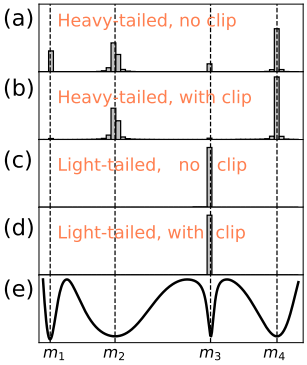

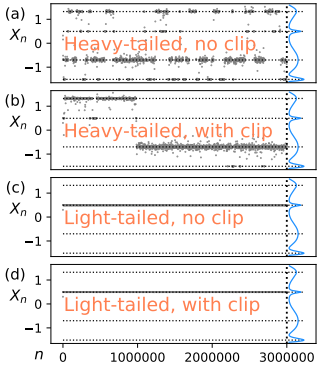

Explanation?

Heavy Tails and Catastrophe Principle

Rare Events depend on "Tail Behaviors"

Light-Tailed Distributions

- Extreme Values are Very Rare
- Normal, Exponential, etc

Heavy-Tailed Distributions

- Extreme Values are Frequent
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Instagram

Structural difference in the way systemwide rare events arise.

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Systemwide rare events

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EVERYTHING goes wrong.

(Conspiracy Principle)

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Systemwide rare events

arise because of

A FEW Catastrophes.

(Catastrophe Principle)

Structural difference in the way systemwide rare events arise.

Insurance Example: Capital Reserve

$$Y(t) = c + pt - \sum_{i=1}^{N(t)} X_i$$

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i.i.d. Claim Size

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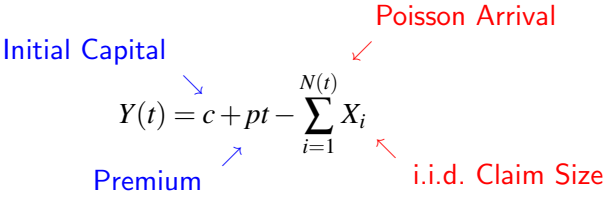

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Large n: analysis of large loss over a long time period

Typical Scenario

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Sample paths of $\bar{Y}_n$:

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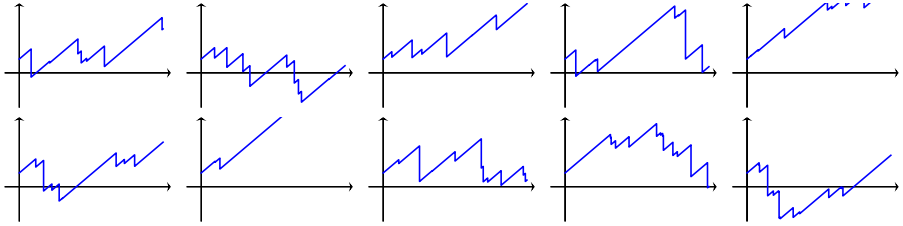

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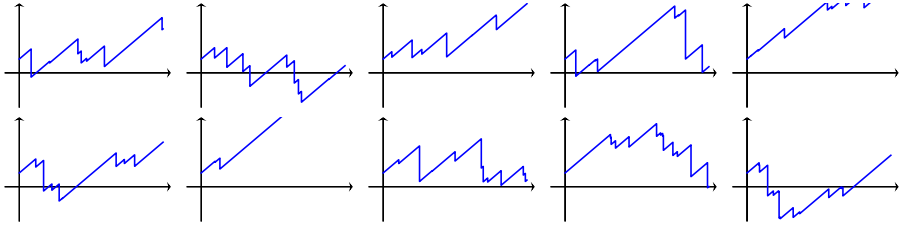

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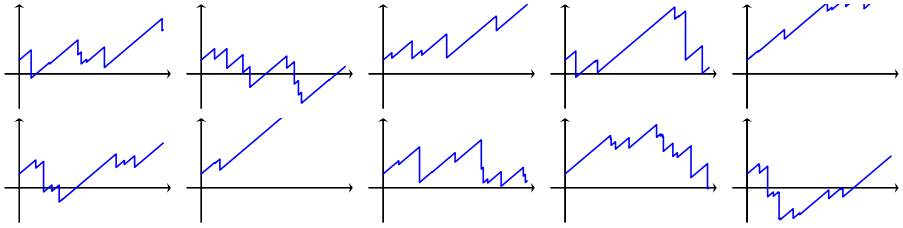

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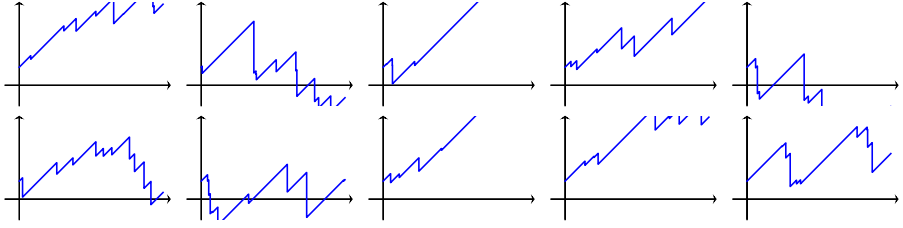

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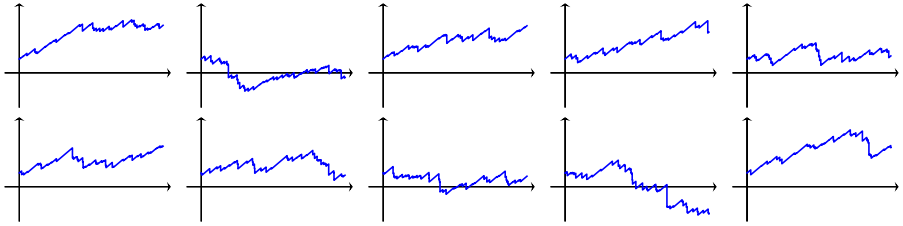
Sample paths of $\bar{Y}_n$: n=50 & claim sizes are **light-tailed**



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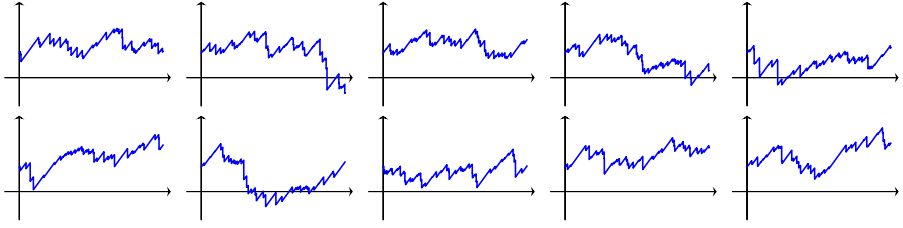

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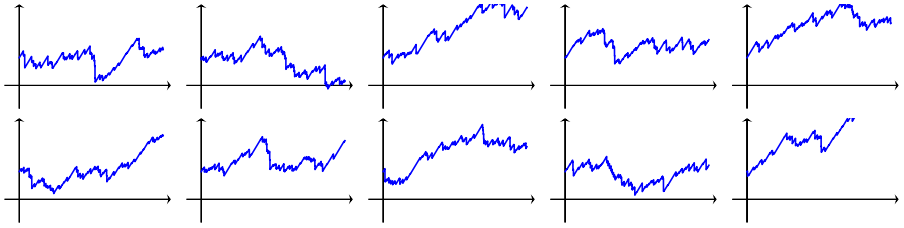

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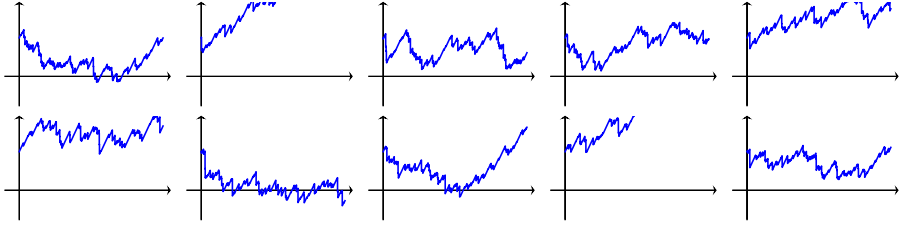

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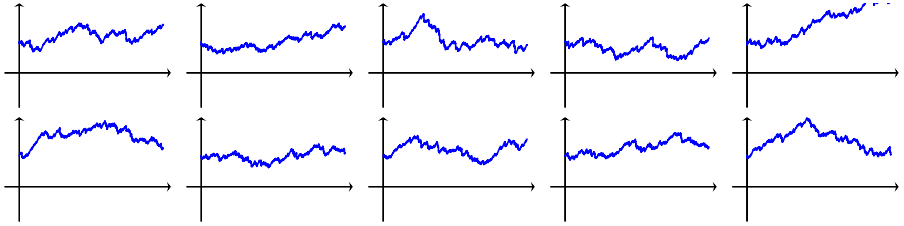
Sample paths of $\bar{Y}_n$: n=500 & claim sizes are **light-tailed**



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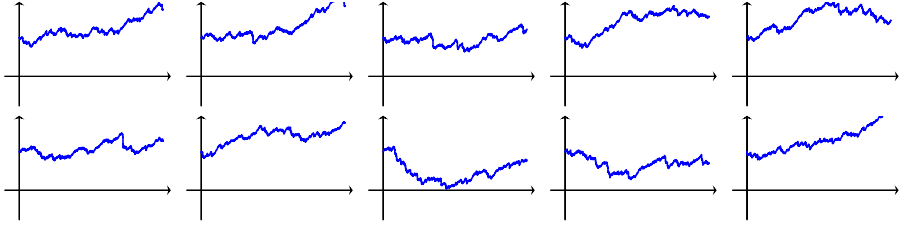

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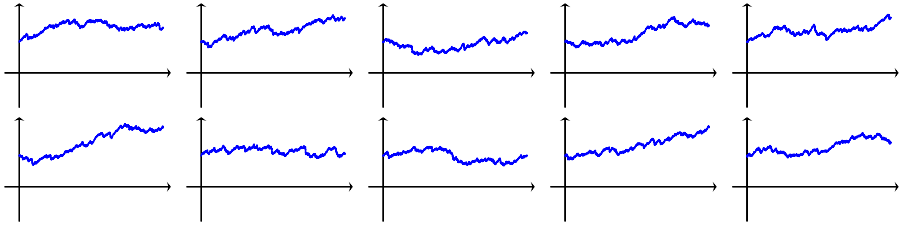
Sample paths of $\bar{Y}_n$: n=1000 & claim sizes are **light-tailed**



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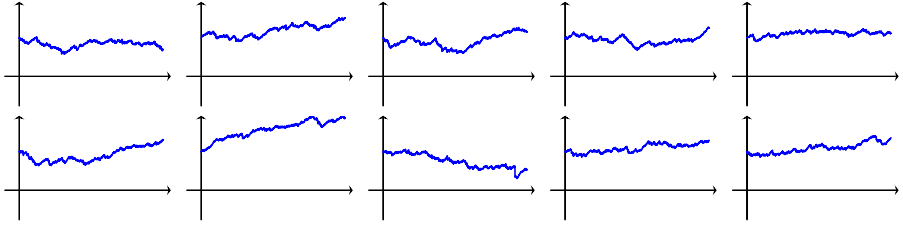

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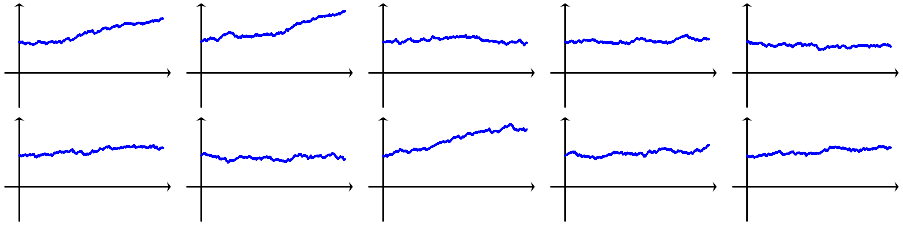
Sample paths of $\bar{Y}_n$: n=2500 & claim sizes are **light-tailed**



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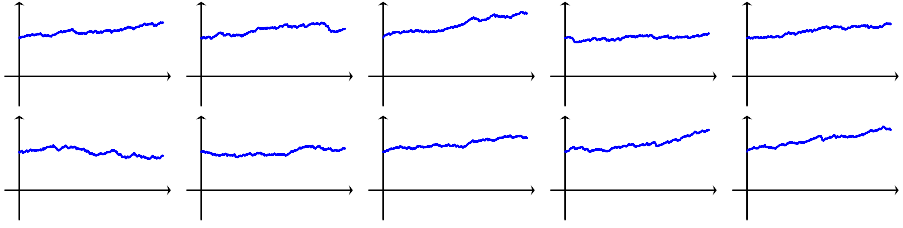

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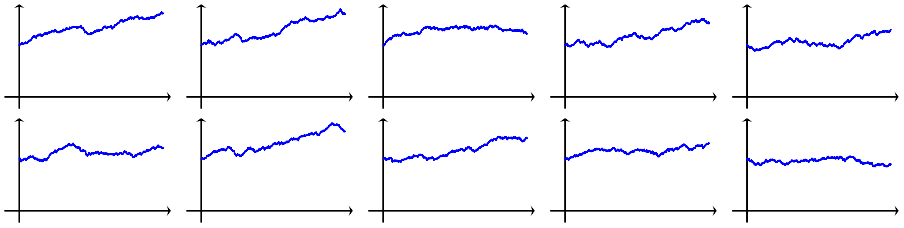
Sample paths of $\bar{Y}_n$: n=5000 & claim sizes are **light-tailed**



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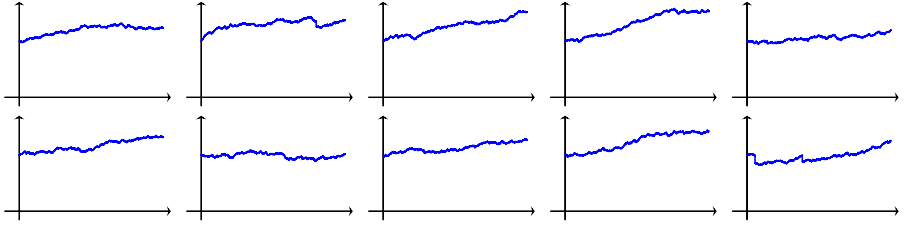

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That is, $\bar{Y}_n(t) \approx c + (p - \mathbf{E}X)t$ for large n.

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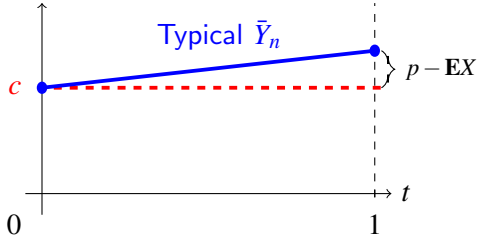

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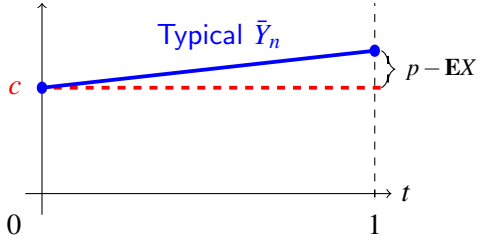

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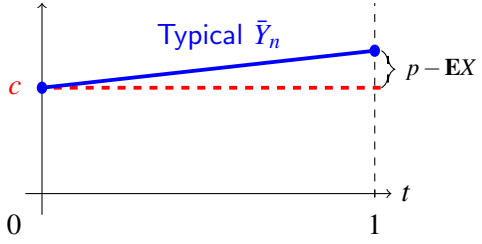

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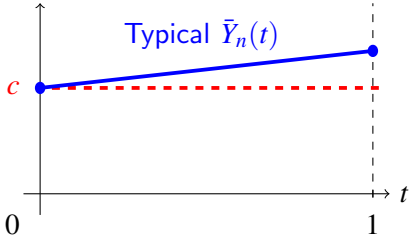
regardless of the tail distributions.

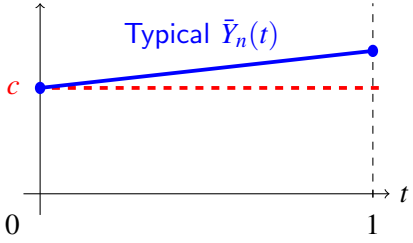
What about atypical cases?

A Rare Event: Bankruptcy



A Rare Event: Bankruptcy

Consider $B \triangleq \{\ \bar{Y}_n$ falls below 0 on $[0,1]\ \}$. (i.e., Bankruptcy)



A Rare Event: Bankruptcy

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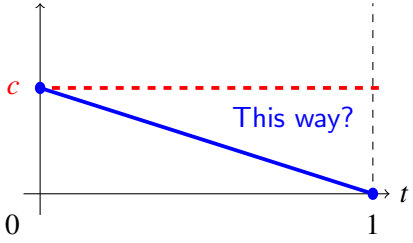
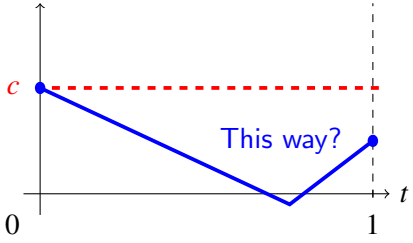


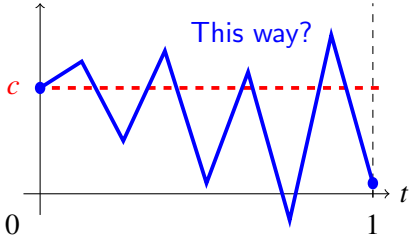
A Rare Event: Bankruptcy

Consider $B \triangleq \{\ \bar{Y}_n$ falls below 0 on $[0,1]\ \}$. (i.e., Bankruptcy)



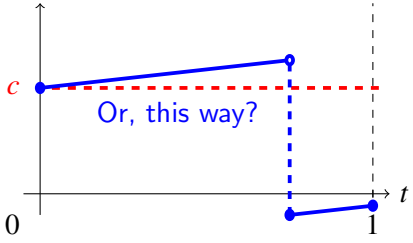
A Rare Event: Bankruptcy

Consider $B \triangleq \{\ \bar{Y}_n$ falls below 0 on $[0,1]\ \}$. (i.e., Bankruptcy)



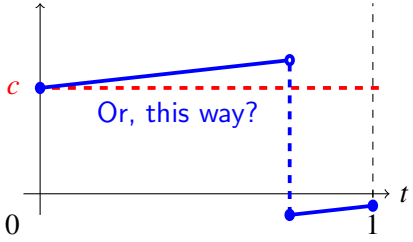
A Rare Event: Bankruptcy

Consider $B \triangleq \{\ \bar{Y}_n$ falls below 0 on $[0,1]\ \}$. (i.e., Bankruptcy)



A Rare Event: Bankruptcy

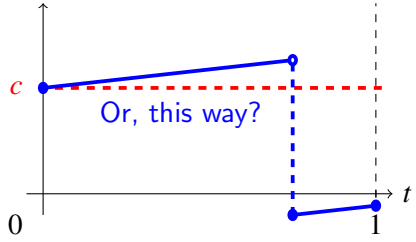
Consider $B \triangleq \{\ \bar{Y}_n$ falls below 0 on $[0,1]\ \}$. (i.e., Bankruptcy)



Are we going to see clear patterns?

A Rare Event: Bankruptcy

Consider $B \triangleq \{\ \bar{Y}_n$ falls below 0 on $[0,1]\ \}$. (i.e., Bankruptcy)

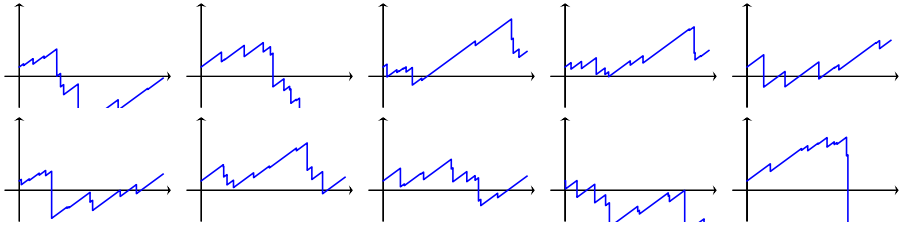


Are we going to see clear patterns?

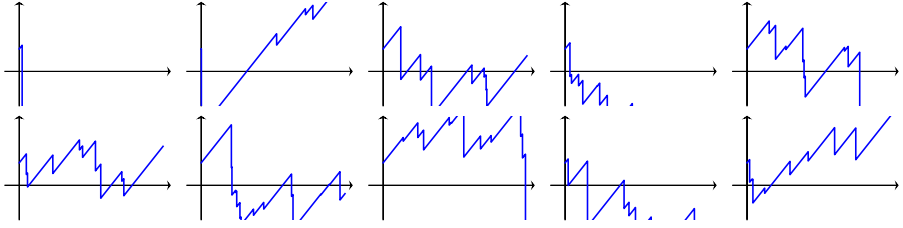
Do they depend on the tail distributions?

A Rare Event: Bankruptcy

Sample paths of $\bar{Y}_{10}$ conditional on B for **light-tailed** claims:



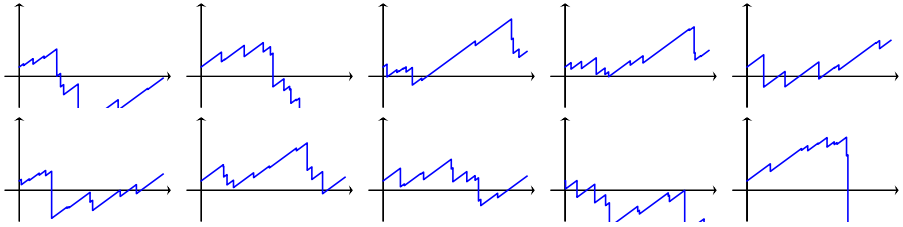
Sample paths of $\bar{Y}_{10}$ conditional on B for **heavy-tailed** claims:



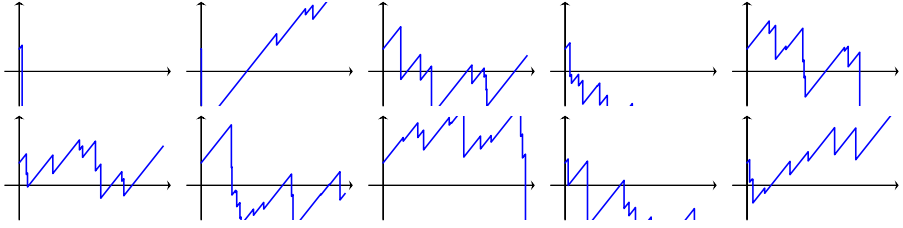
A Rare Event: Bankruptcy

Bankruptcy

Sample paths of $\bar{Y}_{10}$ conditional on B for **light-tailed** claims:



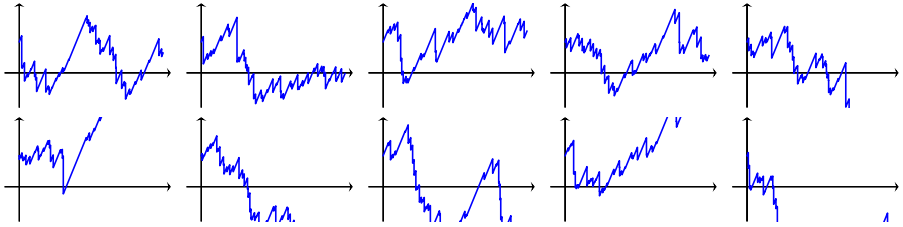
Sample paths of $\bar{Y}_{10}$ conditional on B for **heavy-tailed** claims:



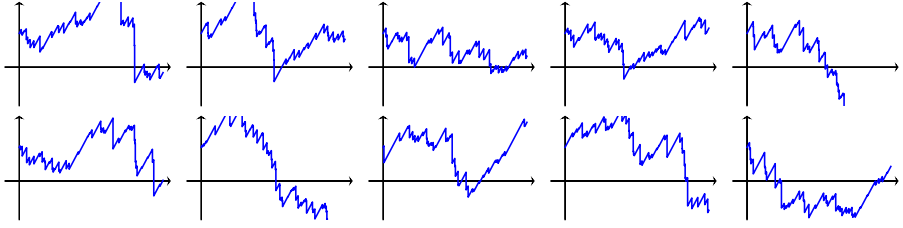
A Rare Event: Bankruptcy

Bankruptcy

Sample paths of $\bar{Y}_{50}$ conditional on B for **light-tailed** claims:



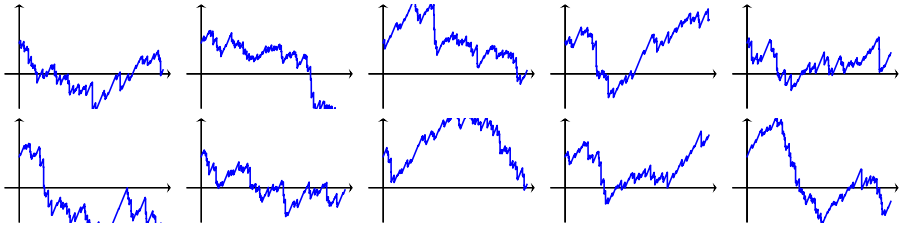
Sample paths of $\bar{Y}_{50}$ conditional on B for **heavy-tailed** claims:



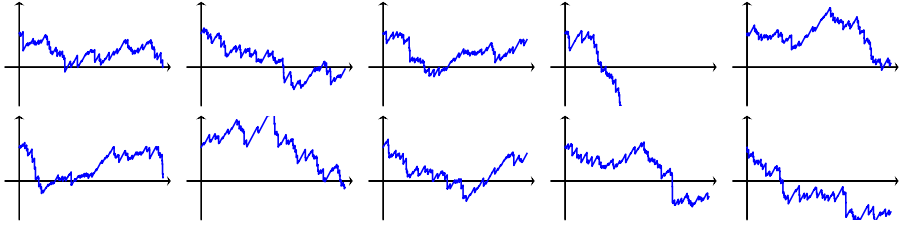
A Rare Event: Bankruptcy

Bankruptcy

Sample paths of $\bar{Y}_{100}$ conditional on B for **light-tailed** claims:



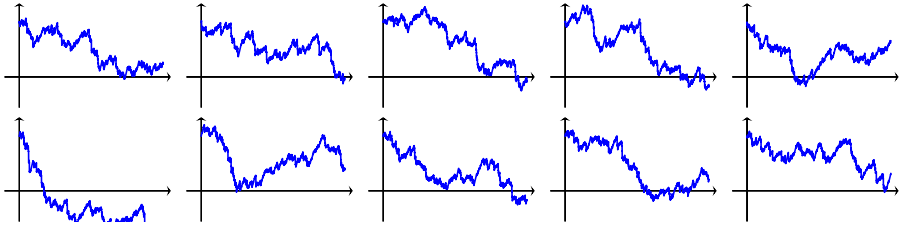
Sample paths of $\bar{Y}_{100}$ conditional on B for **heavy-tailed** claims:



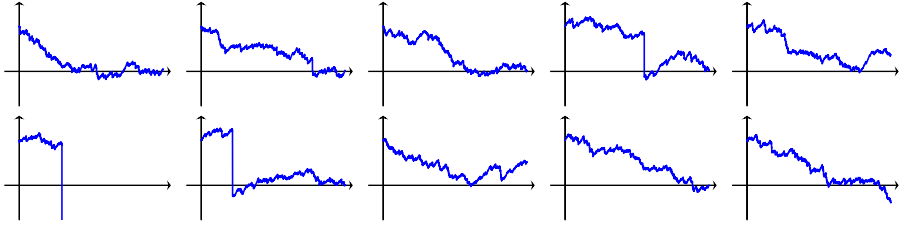
A Rare Event: Bankruptcy

Bankruptcy

Sample paths of $\bar{Y}_{500}$ conditional on B for **light-tailed** claims:



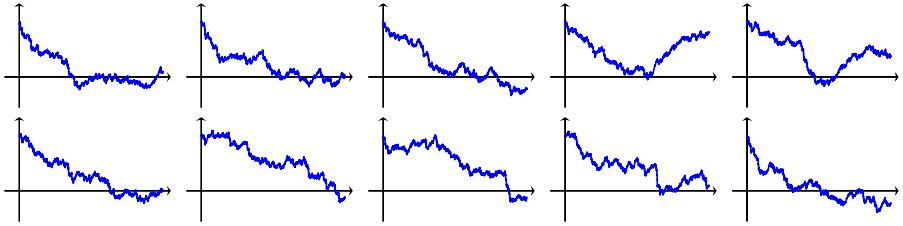
Sample paths of $\bar{Y}_{500}$ conditional on B for **heavy-tailed** claims:



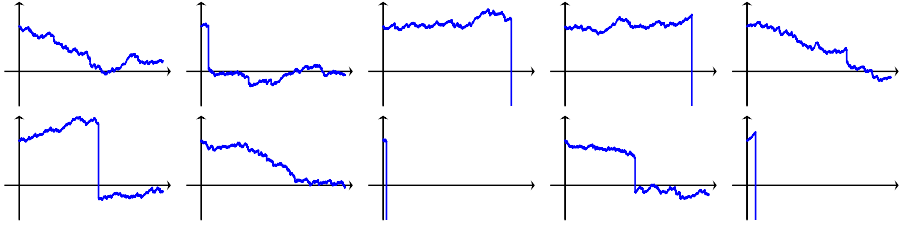
A Rare Event: Bankruptcy

Bankruptcy

Sample paths of $\bar{Y}_{1000}$ conditional on B for **light-tailed** claims:



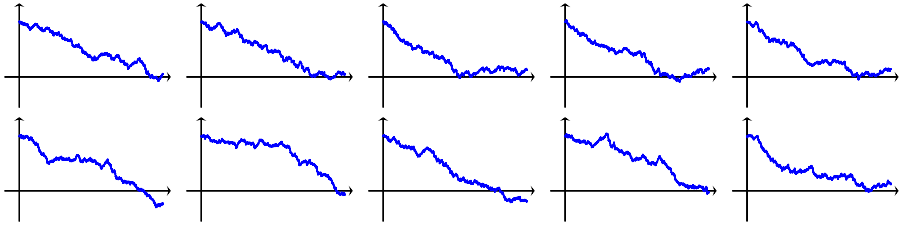
Sample paths of $\bar{Y}_{1000}$ conditional on B for **heavy-tailed** claims:



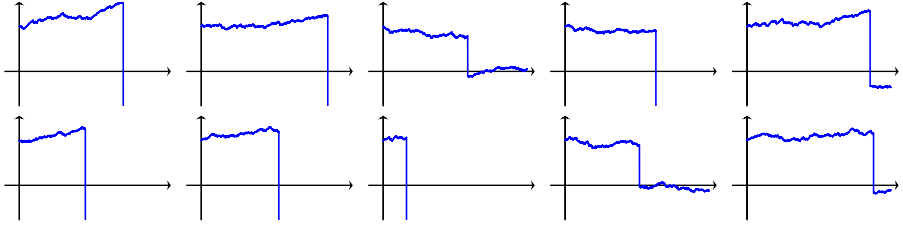
A Rare Event: Bankruptcy

Bankruptcy

Sample paths of $\bar{Y}_{2500}$ conditional on B for **light-tailed** claims:



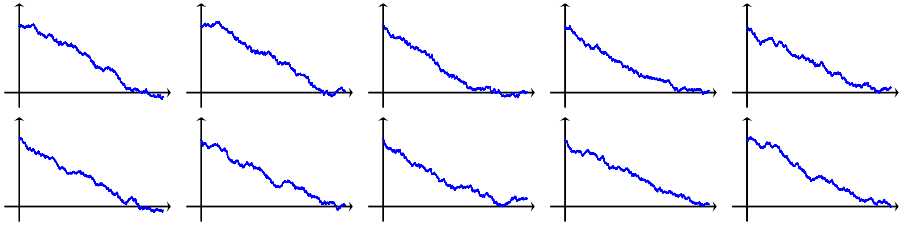
Sample paths of $\bar{Y}_{2500}$ conditional on B for **heavy-tailed** claims:



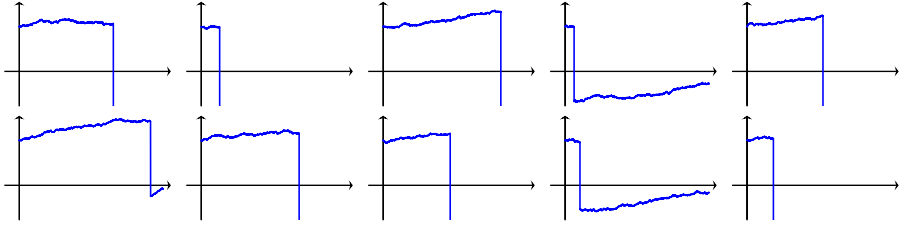
A Rare Event: Bankruptcy

Bankruptcy

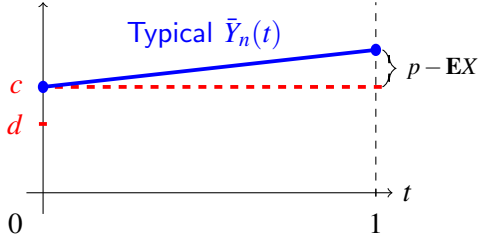
Sample paths of $\bar{Y}_{5000}$ conditional on B for **light-tailed** claims:



Sample paths of $\bar{Y}_{5000}$ conditional on B for **heavy-tailed** claims:

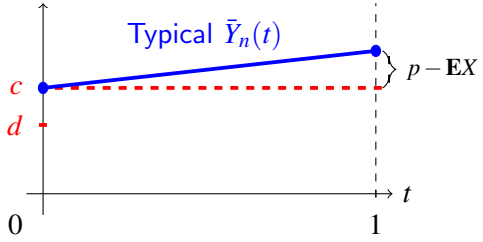


Bankruptcy Despite of Reinsurance



Bankruptcy Despite of Reinsurance

Consider $R \triangleq \{\ \bar{Y}_n$ falls below 0 on [0,1], jump sizes $\leq d\ \}$



Bankruptcy Despite of Reinsurance

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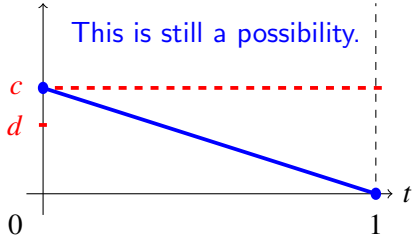

Bankruptcy Despite of Reinsurance

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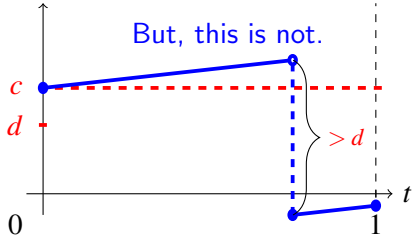

Bankruptcy Despite of Reinsurance

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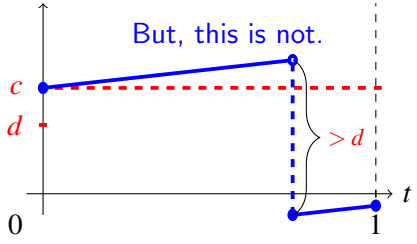
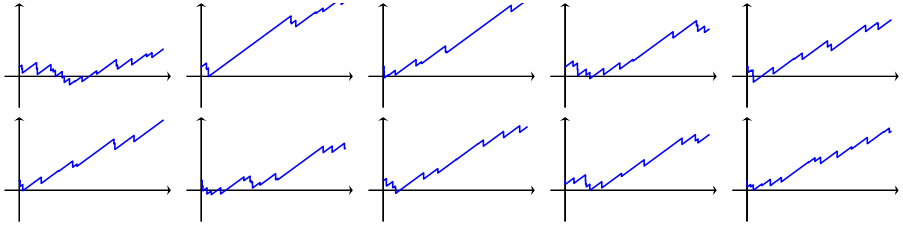


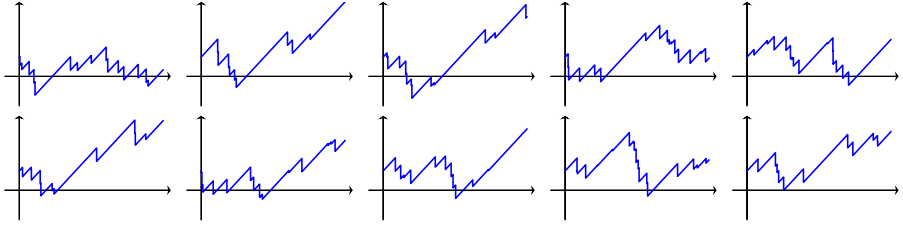
How does the pattern change in this case?

Bankruptcy Despite of Reinsurance

Sample paths of $\bar{Y}_{10}$ conditional on R for **light-tailed** claims:

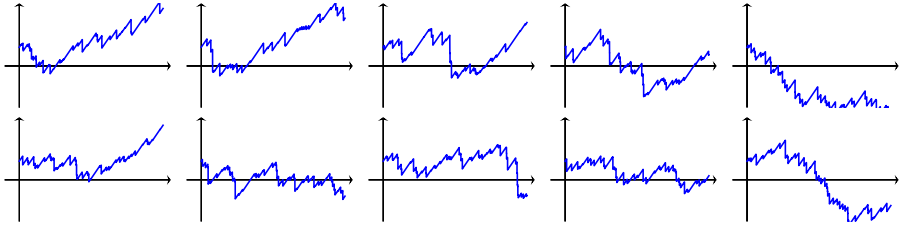


Sample paths of $\bar{Y}_{10}$ conditional on R for **heavy-tailed** claims:

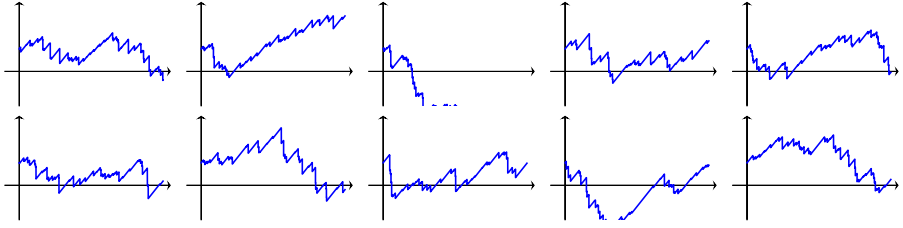


Bankruptcy Despite of Reinsurance

Sample paths of $\bar{Y}_{50}$ conditional on R for **light-tailed** claims:

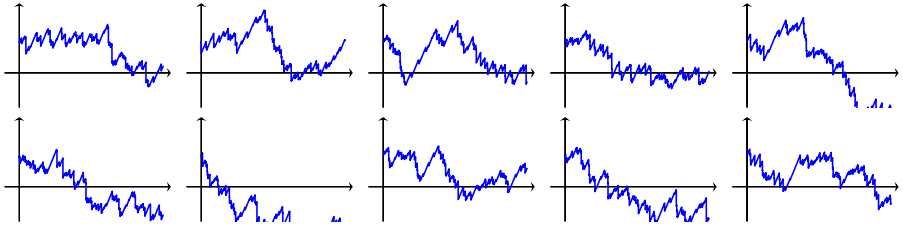


Sample paths of $\bar{Y}_{50}$ conditional on R for **heavy-tailed** claims:

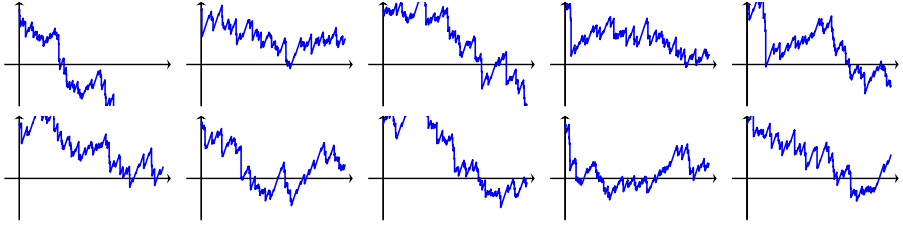


Bankruptcy Despite of Reinsurance

Sample paths of $\bar{Y}_{100}$ conditional on R for **light-tailed** claims:

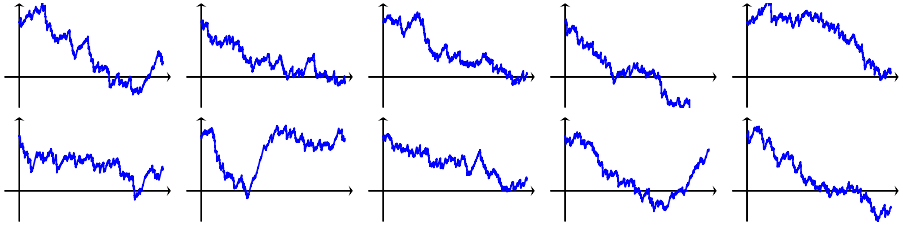


Sample paths of $\bar{Y}_{100}$ conditional on R for **heavy-tailed** claims:

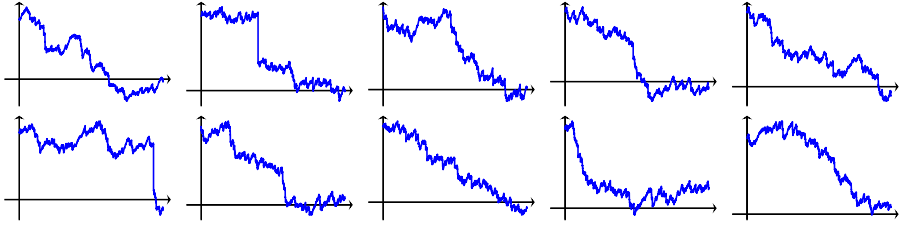


Bankruptcy Despite of Reinsurance

Sample paths of $\bar{Y}_{500}$ conditional on R for **light-tailed** claims:

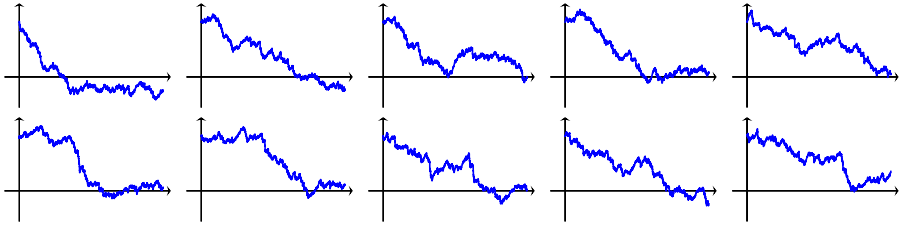


Sample paths of $\bar{Y}_{500}$ conditional on R for **heavy-tailed** claims:

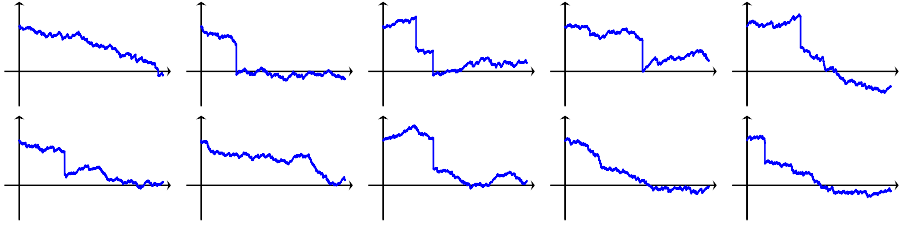


Bankruptcy Despite of Reinsurance

Sample paths of $\bar{Y}_{1000}$ conditional on R for **light-tailed** claims:

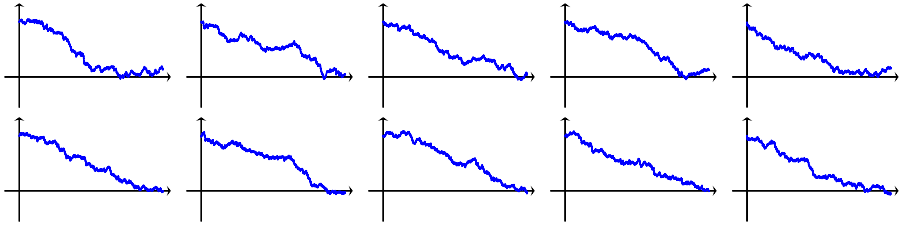


Sample paths of $\bar{Y}_{1000}$ conditional on R for **heavy-tailed** claims:

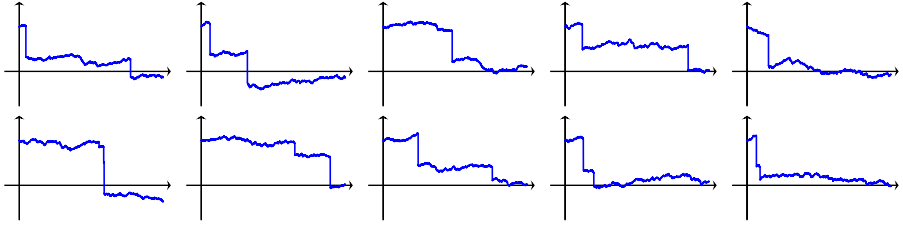


Bankruptcy Despite of Reinsurance

Sample paths of $\bar{Y}_{2500}$ conditional on R for **light-tailed** claims:

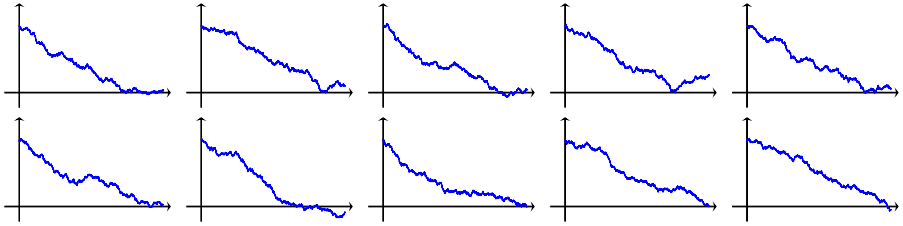


Sample paths of $\bar{Y}_{2500}$ conditional on R for **heavy-tailed** claims:

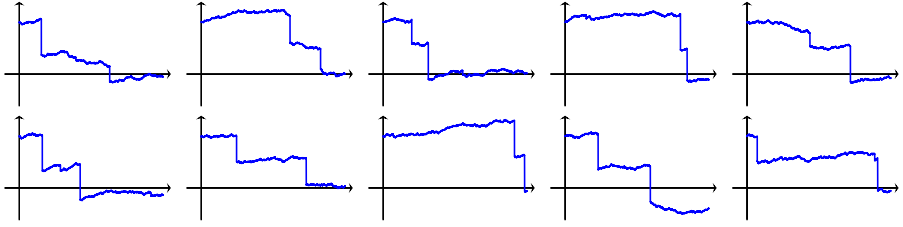


Bankruptcy Despite of Reinsurance

Sample paths of $\bar{Y}_{5000}$ conditional on R for **light-tailed** claims:

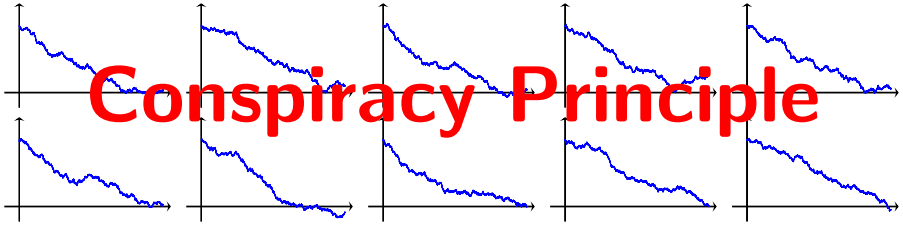


Sample paths of $\bar{Y}_{5000}$ conditional on R for **heavy-tailed** claims:

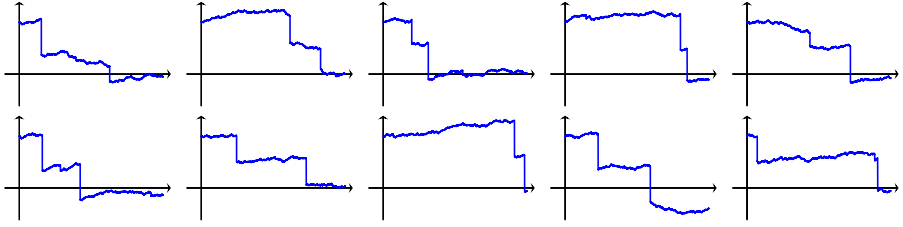


Bankruptcy Despite of Reinsurance

Sample paths of $\bar{Y}_{5000}$ conditional on R for **light-tailed** claims:

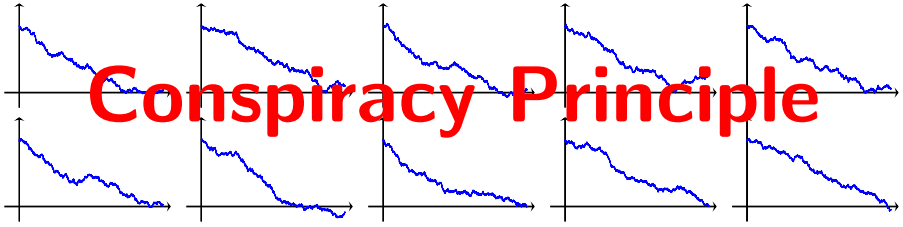


Sample paths of $\bar{Y}_{5000}$ conditional on R for **heavy-tailed** claims:



Bankruptcy Despite of Reinsurance

Sample paths of $\bar{Y}_{5000}$ conditional on R for **light-tailed** claims:



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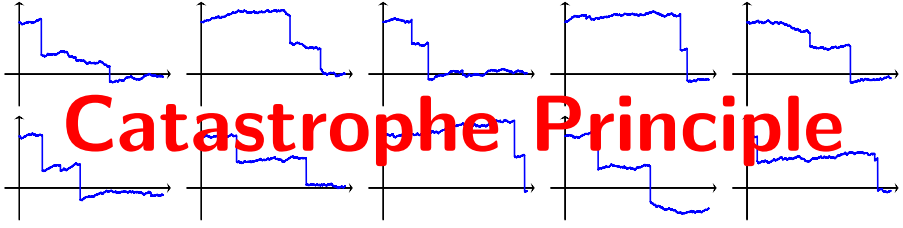

Heavy-Tailed Large Deviations:

Rigorous Characterization of Catastrophe Principle

Heavy-Tailed Large Deviations

$$\bar{S}_n(t) = \frac{1}{n}\sum_{i=1}^{\lfloor nt \rfloor} X_i, \qquad X_i\text{: centered iid r.v. with } \mathbf{P}(X_i \geq x) = x^{-(\alpha+1)}, \quad \alpha > 0$$

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Theorem (R., Blanchet, Zwart, 2019)

For "general" $A \subseteq \mathbb{D}$

$$C(A^\circ) \leq \liminf_{n\to\infty} \frac{\mathbf{P}(\bar{S}_n \in A)}{n^{-\alpha \mathcal{J}(A)}} \leq \limsup_{n\to\infty} \frac{\mathbf{P}(\bar{S}_n \in A)}{n^{-\alpha \mathcal{J}(A)}} \leq C(A^-).$$

- $\mathcal{J}(A)$*: min #jumps for step functions to be inside* A
- $C(\cdot)$*: a measure*

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LD power index

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Implication: The Catastrophe Principle

Under certain regularity conditions on A,

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Rigorous Characterization of Catastrophe Principle

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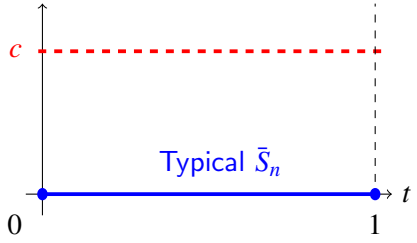
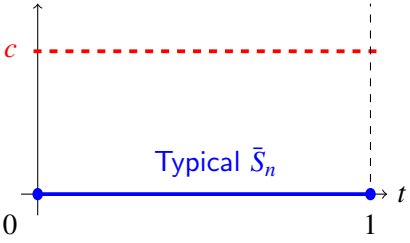


Light-Tailed Claim Size



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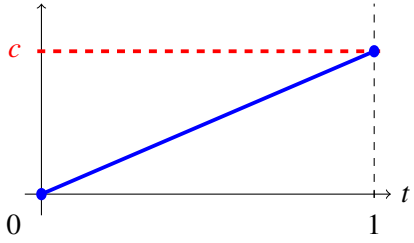

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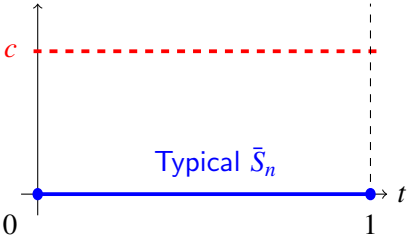

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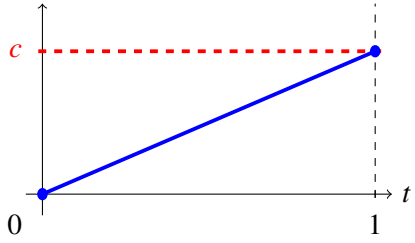

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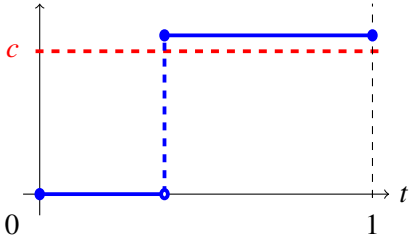

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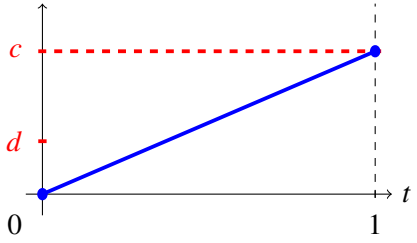

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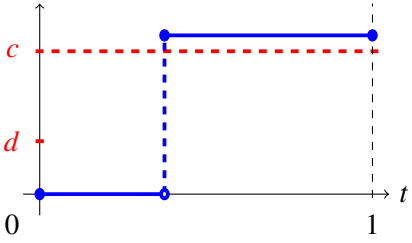

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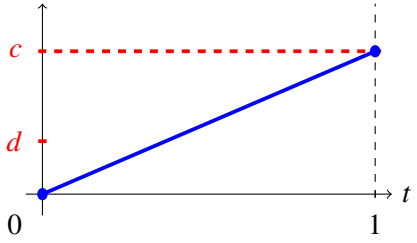

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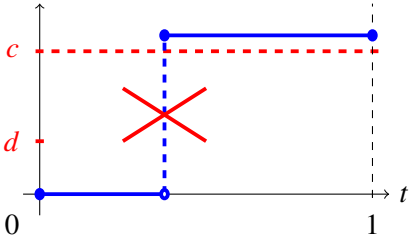

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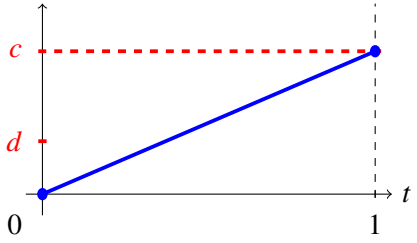

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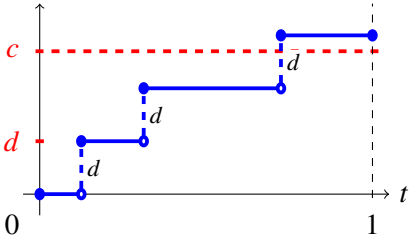

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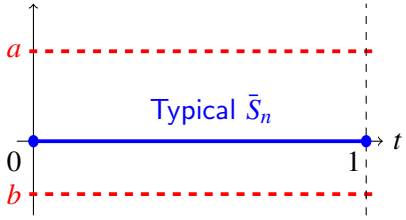

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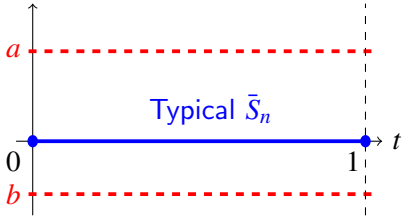

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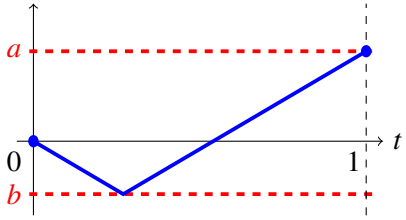

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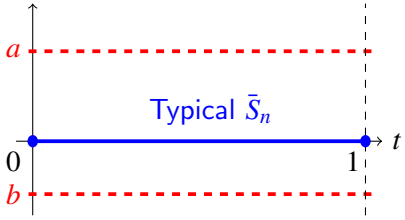

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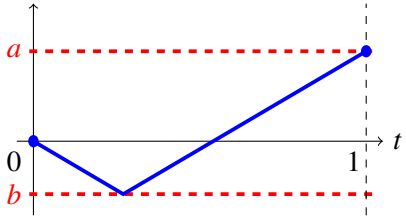

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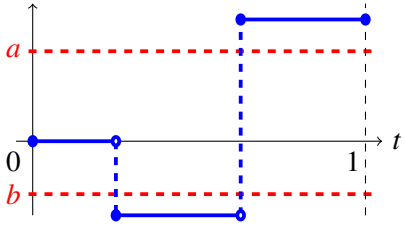

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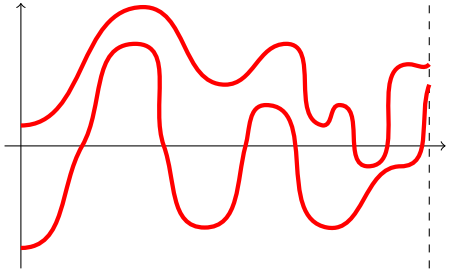

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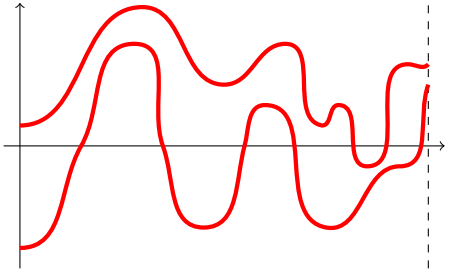

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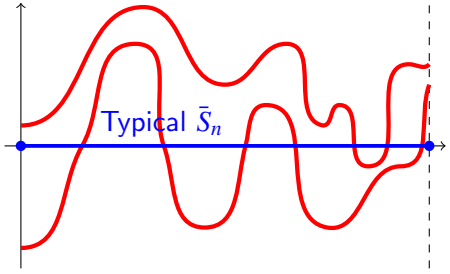

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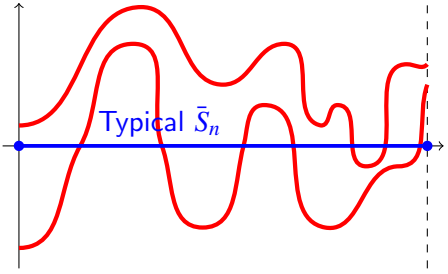

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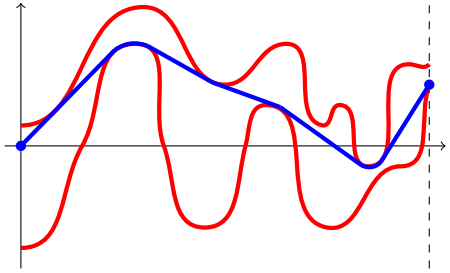

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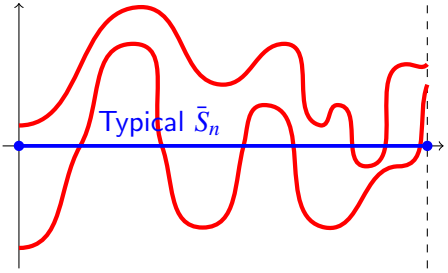

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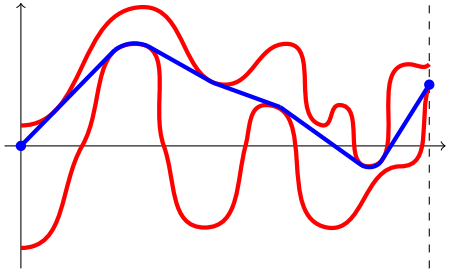

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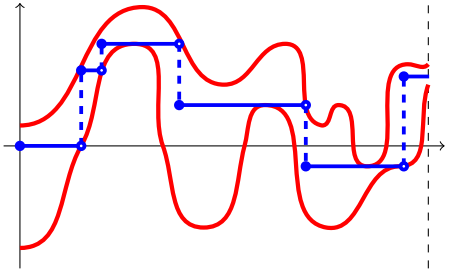
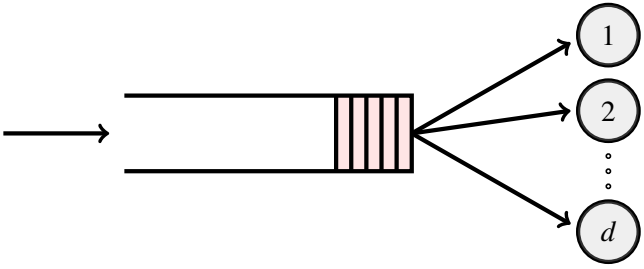


Heavy-Tailed Increments

Congestion of Multiple Server Queue:



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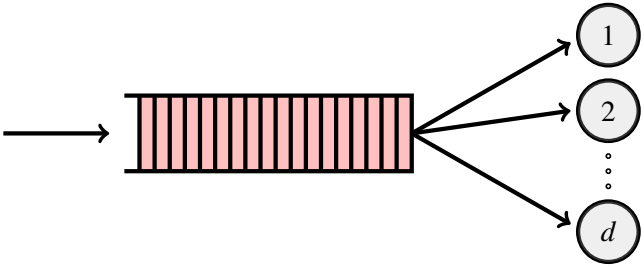
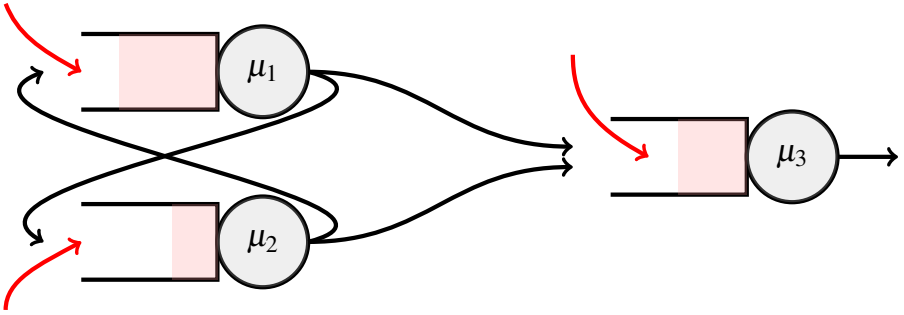


Conspiracy vs Catastrophe

Congestion of Stochastic Fluid Networks:



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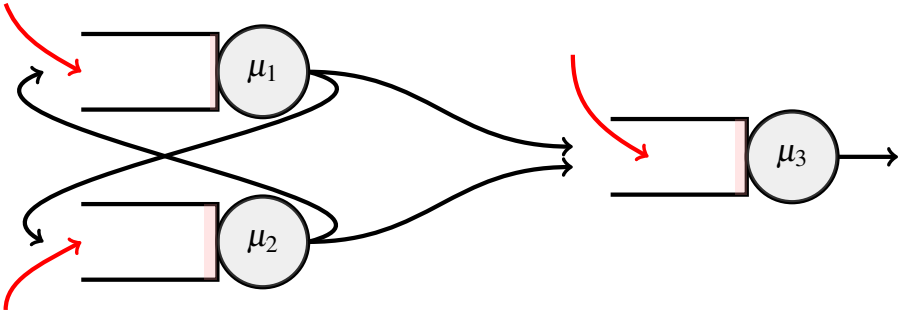

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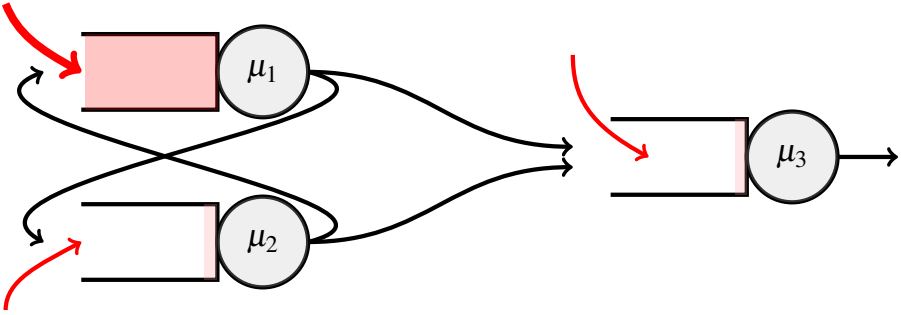

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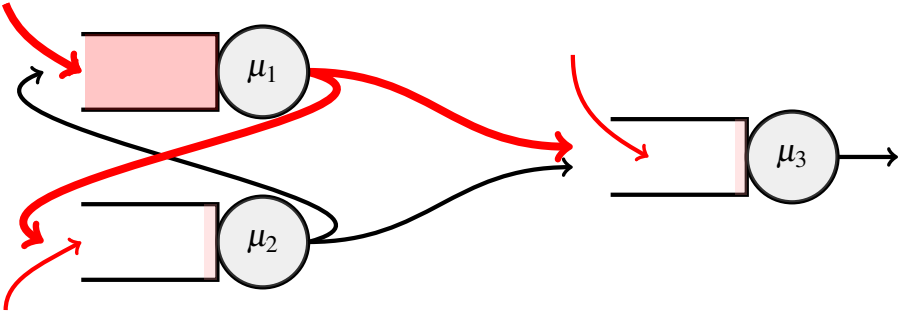

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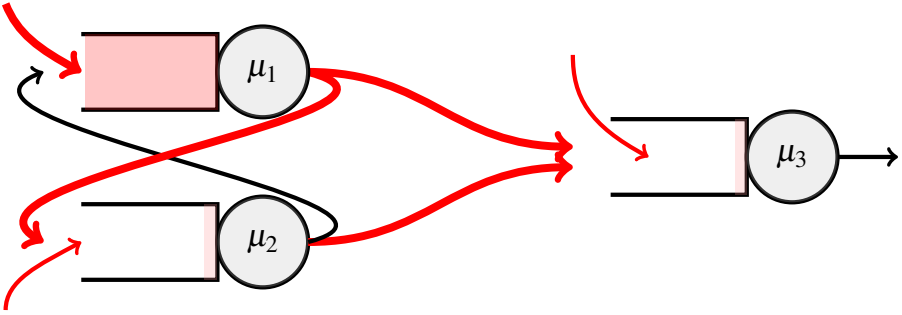

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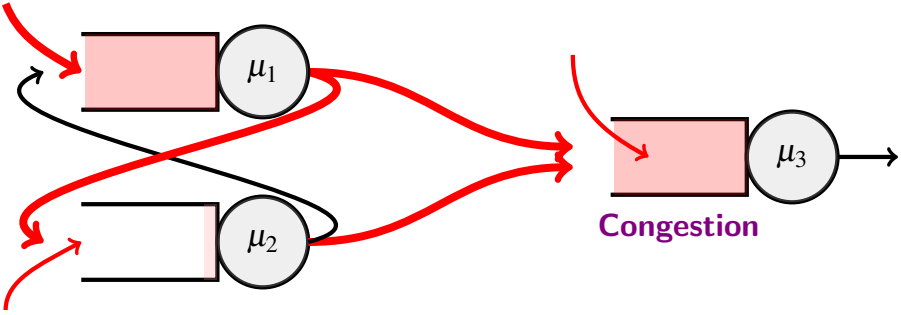

Catastrophe Principle Extends to General Heavy-Tailed Systems

Heavy-Tailed Large Deviations for

- Continuous-Time Processes
  R., Blanchet, Zwart (2019), Bazhba, Blanchet, R., Zwart (2020), Su, Wang, R. (2023+)

- Processes with Spatial and Temporal Correlations
  Chen, R., Zwart (2023+), Bazhba, Blanchet, R., Zwart (2023+), Su, R. (2023+), Wang, R. (2023+)

Minimal # Jumps added to Typical Paths Characterize the Catastrophe Principle

Open Problems Solved by Heavy-Tailed Large Deviations

- Design of strongly efficient rare-event simulation algorithm
- Multiple server queue length asymptotics
- Ruin probability under the presence of reinsurance contracts

From Large Deviations to Metastability

Stochastic Gradient Descent (SGD)

Minimizing loss function f:

$$W_{k+1} = W_k - \eta \left(f'(W_k) \right) \qquad k = 0, 1, 2, \ldots$$

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Stochastic Gradient Descent (SGD)

Minimizing loss function f:

$$W_{k+1} = W_k - \eta \left(f'(W_k) + Z_k \right) \qquad k = 0, 1, 2, \ldots$$

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$$W^{\eta}{}_{k+1} = W^{\eta}{}_{k} - \eta\left(f'(W^{\eta}{}_{k}) + Z_k\right) \qquad k = 0, 1, 2, \ldots$$

Stochastic Gradient Descent (SGD)

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$$W^{\eta}(\eta(k+1)) = W^{\eta}(\eta k) - \eta\left(f'(W^{\eta}(\eta k)) + Z_k\right) \qquad k = 0, 1, 2, \ldots$$

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Then

$$W^{\eta}(\cdot) \to w(\cdot) \qquad \text{as} \qquad \eta \to 0$$

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Then

$$W^{\eta}(\cdot) \to w(\cdot) \qquad \text{as} \qquad \eta \to 0$$

where

$$\mathrm{d}w(t) = -f'(w(t))\mathrm{d}t$$

Stochastic Gradient Descent (SGD)

Minimizing loss function f:

$$W^{\eta}(\eta(k+1)) = W^{\eta}(\eta k) - \eta\left(f'(W^{\eta}(\eta k)) + Z_k\right) \qquad k = 0, 1, 2, \ldots$$

Then

$$W^{\eta}(\cdot) \to w(\cdot) \qquad \text{as} \qquad \eta \to 0$$

Gradient Flow

where

$$\mathrm{d}w(t) = -f'(w(t))\mathrm{d}t$$

Heavy-Tailed Large Deviations for SGD

Theorem (Wang, R., 2023+)

For "general" $B \subseteq \mathbb{D}[0,T]$

$$C(B^\circ) \leq \liminf_{\eta \to 0} \frac{\mathbf{P}(W^\eta \in B)}{\eta^{\alpha \mathcal{J}(B)}} \leq \limsup_{\eta \to 0} \frac{\mathbf{P}(W^\eta \in B)}{\eta^{\alpha \mathcal{J}(B)}} \leq C(B^-).$$

- $\mathcal{J}(B)$: *min #jumps added to $w(\cdot)$ for it to be inside B*
- $C(\cdot)$: *a measure*

Heavy-Tailed Large Deviations for SGD

$$\mathbf{P}(|Z_i| \geq x) = x^{-(\alpha+1)}, \quad \alpha > 0$$

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Locally Uniform Large Deviations over Asymptotic Atom $\{A(\varepsilon) : \varepsilon > 0\}$

$\mathbb{M}$-Convergence

$$\mathbb{M}(\mathbb{S}\setminus\mathbb{C}) = \{\nu(\cdot) : \nu(\mathbb{S}\setminus\mathbb{C}^{\varepsilon}) < \infty, \forall \varepsilon > 0; \quad \nu(\cdot) \text{ Borel measure on } \mathbb{S}\setminus\mathbb{C}\}$$

Definition ($\mathbb{M}$-convergence; Lindskog, Resnick, Roy., 2014)

Let $\mu^{\eta}, \mu \in \mathbb{M}(\mathbb{S}\setminus\mathbb{C})$ for each $\eta > 0$. We say that μ^{η} converges to μ in $\mathbb{M}(\mathbb{S}\setminus\mathbb{C})$ as $\eta \to 0$ if

$$\lim_{\eta\downarrow 0} |\mu^{\eta}(f) - \mu(f)| = 0 \qquad \forall f \in \mathcal{C}(\mathbb{S}\setminus\mathbb{C}).$$

$\mathbb{M}$-Convergence

ε-fattening of $\mathbb{C}$

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"bounded continuous functions supported on $\mathbb{S}\setminus\mathbb{C}$"

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"bounded continuous functions supported on $\mathbb{S}\setminus\mathbb{C}$"

Definition (Uniform $\mathbb{M}$-convergence; Wang, R., 2023+)

Let Θ be a set of indices. Let $\mu_{\theta}^{\eta}, \mu_{\theta} \in \mathbb{M}(\mathbb{S}\setminus\mathbb{C})$ for each $\eta > 0$ and $\theta \in \Theta$. We say that μ_{θ}^{η} converges to μ_{θ} in $\mathbb{M}(\mathbb{S}\setminus\mathbb{C})$ uniformly in θ on Θ as $\eta \to 0$ if

$$\lim_{\eta\downarrow 0} \sup_{\theta\in\Theta} |\mu_{\theta}^{\eta}(f) - \mu_{\theta}(f)| = 0 \qquad \forall f \in \mathcal{C}(\mathbb{S}\setminus\mathbb{C}).$$

Portmanteau Theorem for Uniform $\mathbb{M}(\mathbb{S}\setminus\mathbb{C})$-Convergence

Theorem (Wang, R., 2023+)

Let $\mu_\theta^\eta, \mu_\theta \in \mathbb{M}(\mathbb{S}\setminus\mathbb{C})$ for each $\eta > 0$ and $\theta \in \Theta$. Suppose that for any sequence $(\theta_n)_{n\geq 1}$, there exist a sub-sequence $(\theta_{n_k})_{k\geq 1}$ and $\theta^* \in \Theta$ s.t.

$$\lim_{k\to\infty} \mu_{\theta_{n_k}}(f) = \mu_{\theta^*}(f) \qquad \forall f \in \mathcal{C}(\mathbb{S}\setminus\mathbb{C}).$$

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Then the following statements are equivalent:

- $\mu_\theta^\eta \to \mu_\theta$ *in* $\mathbb{M}(\mathbb{S}\setminus\mathbb{C})$ *uniformly in* θ *on* Θ *as* $\eta \downarrow 0$;
- *For all* $\varepsilon > 0$, F, G *bounded away from* $\mathbb{C}$,

$$\liminf_{\eta\downarrow 0} \inf_{\theta\in\Theta} \left(\mu_\theta^\eta(G) - \mu_\theta(G_\varepsilon)\right) \geq 0$$

$$\limsup_{\eta\downarrow 0} \sup_{\theta\in\Theta} \left(\mu_\theta^\eta(F) - \mu_\theta(F^\varepsilon)\right) \leq 0$$

Portmanteau Theorem for Uniform $\mathbb{M}(\mathbb{S}\setminus\mathbb{C})$-Convergence

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Then the following statements are equivalent:

- $\mu_\theta^\eta \to \mu_\theta$ *in* $\mathbb{M}(\mathbb{S}\setminus\mathbb{C})$ *uniformly in* θ *on* Θ *as* $\eta \downarrow 0$;
- *For all* $\varepsilon > 0$, F, G *bounded away from* $\mathbb{C}$, $\liminf_{\eta\downarrow 0} \inf_{\theta\in\Theta} \left(\mu_\theta^\eta(G) - \mu_\theta(G_\varepsilon)\right) \geq 0$

  $\limsup_{\eta\downarrow 0} \sup_{\theta\in\Theta} \left(\mu_\theta^\eta(F) - \mu_\theta(F^\varepsilon)\right) \leq 0$

Furthermore, they both imply

- *For all open* G *and closed* F *that are bounded away from* $\mathbb{C}$,

$$\inf_{\theta\in\Theta} \mu_\theta(G) \leq \liminf_{\eta\downarrow 0} \inf_{\theta\in\Theta} \mu_\theta^\eta(G)$$

$$\limsup_{\eta\downarrow 0} \sup_{\theta\in\Theta} \mu_\theta^\eta(F) \leq \sup_{\theta\in\Theta} \mu_\theta(F).$$

Asymptotic Atom $\{A(\varepsilon) \subseteq \mathbb{S} : \varepsilon > 0\}$ of Markov Chain $\{V_j^\eta(x) : j \geq 0\}$

For measurable $B \subseteq \mathbb{S}$, there exist $\delta_B : (0,\infty) \times (0,\infty) \to (0,\infty)$, $\varepsilon_B > 0$, and $T_B > 0$ s.t.

$$C(B^\circ) - \delta_B(\varepsilon, T) \leq \liminf_{\eta \downarrow 0} \frac{\inf_{x \in A(\varepsilon)} \mathbf{P}\left(\tau^\eta_{I(\varepsilon)^\complement}(x) \leq T/\eta;\ V^\eta_{\tau_\varepsilon}(x) \in B\right)}{\gamma(\eta)T/\eta}$$
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for any $\varepsilon \leq \varepsilon_B$ and $T \geq T_B$, where $\gamma(\eta)/\eta \to 0$ as $\eta \downarrow 0$ and δ_B's are such that

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$\{I(\varepsilon) \subseteq I : \varepsilon > 0\}$: covering of I

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$\tau_{I(\varepsilon)^\complement}$: exit time from $I(\varepsilon)$

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Also,

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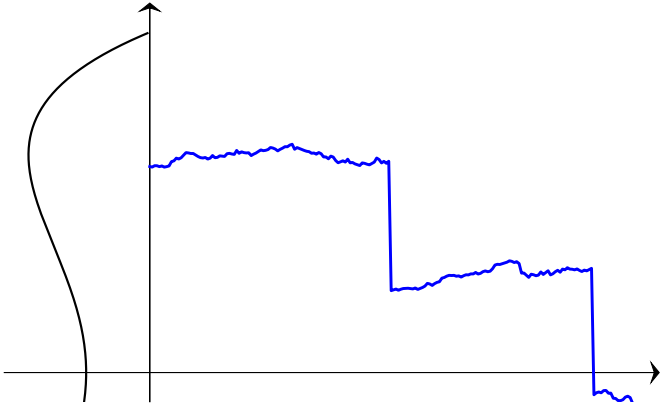
Asymptotic Atoms



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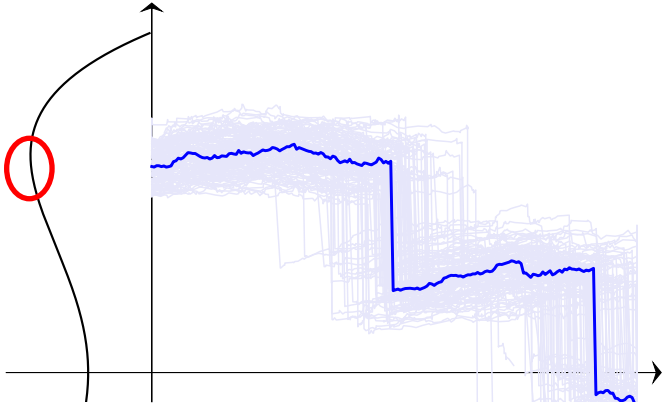

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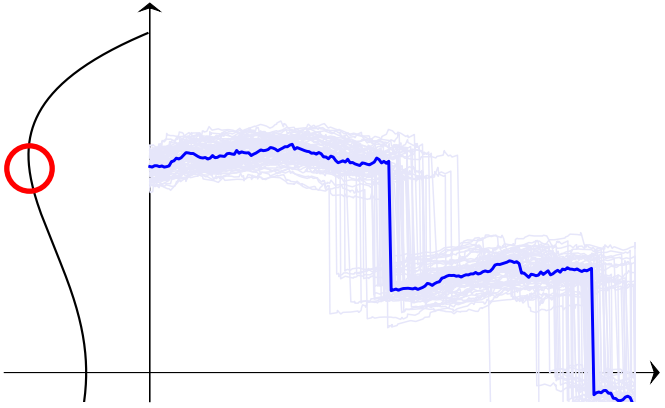

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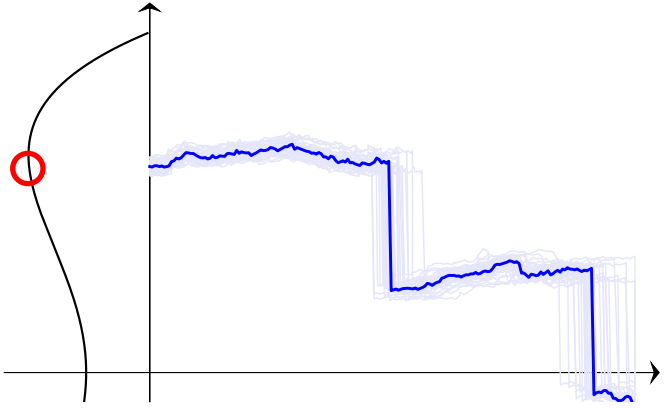

Exit Time and Location under the Presence of Asymptotic Atom

Theorem (Wang, R., 2023+)

If Markov chain $\{V_j^\eta(x) : j \geq 0\}$ *possesses an asymptotic atom* $\{A(\varepsilon) \subseteq \mathbb{S} : \varepsilon > 0\}$, *then*

$$\begin{aligned} C(B^\circ) \cdot e^{-t} &\leq \liminf_{\eta \downarrow 0} \inf_{x \in I(\varepsilon)} \mathbf{P}\big(\gamma(\eta)\tau_{I^\complement}^\eta(x) > t, V_\tau^\eta(x) \in B\big) \\ &\leq \limsup_{\eta \downarrow 0} \sup_{x \in I(\varepsilon)} \mathbf{P}\big(\gamma(\eta)\tau_{I^\complement}^\eta(x) > t, V_\tau^\eta(x) \in B\big) \leq C(B^-) \cdot e^{-t}. \end{aligned}$$

Truncated Version of Stochastic Gradient Descent

SGD

$$W_{k+1}^{\eta} = W_k^{\eta} - \qquad \eta\left(f'(W_k^{\eta}) + Z_k\right) \qquad k = 0, 1, 2, \ldots$$

Truncated Version of Stochastic Gradient Descent

SGD with Gradient Clipping

$$W_{k+1}^{\eta} = W_k^{\eta} - \varphi_c\left(\eta\left(f'(W_k^{\eta}) + Z_k\right)\right) \qquad k = 0, 1, 2, \ldots$$

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$$\varphi_c(x) = \frac{x}{|x|}\min\{c, |x|\}.$$

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Then, again,

$$W^{\eta}(\cdot) \to w(\cdot) \qquad \text{as} \qquad \eta \to 0$$

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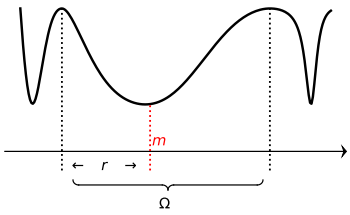
First Exit Time Analysis



gradient noise

$l = \lceil r/c \rceil$: "width" of the attraction fields,

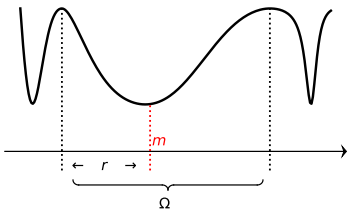
$$\mathbf{P}(|Z_i| \geq x) = x^{-(\alpha+1)}, \quad \alpha > 0$$

Theorem (Wang, R., 2023+)

Let $\tau(\eta) = \min\{j \geq 0 : W_j^\eta \notin \Omega\}$ *and* $\gamma(\eta) \sim \eta^{1+\alpha \cdot l}$

$$\tau(\eta)\gamma(\eta) \Rightarrow \mathrm{Exp}(1)$$

First Exit Time Analysis



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First Exit Time

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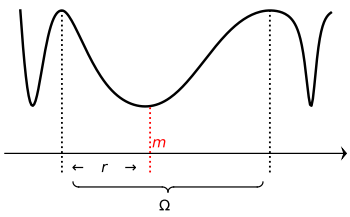

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First Exit Time

$$\tau(\eta)\gamma(\eta) \Rightarrow \mathrm{Exp}(1)$$

$\sim (1/\eta)^{1+\alpha \cdot l}$

Eliminating Sharp Local Minima with Truncated Heavy-Tails

gradient noise

l^*: "width" of the widest attraction fields, $\mathbf{P}(|Z_i| \geq x) = x^{-(\alpha+1)}, \quad \alpha > 0$

Theorem (Wang, R., 2023+)

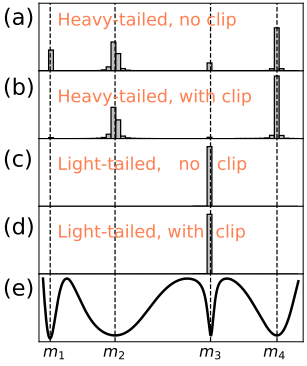
Under certain structural conditions, for any $t > 0$ and $\beta > 1 + \alpha \cdot l^*$,

$$\frac{1}{t/\eta^\beta} \int_0^{\lfloor t/\eta^\beta \rfloor} \mathbb{I}\{W^\eta_{\lfloor u \rfloor} \in \textit{sharp minima}\} du \xrightarrow{p} 0$$

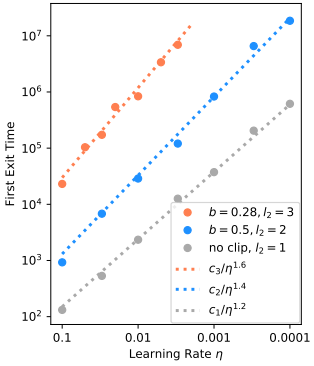
In fact, $W^\eta_{\lfloor t/\eta^{1+\alpha \cdot l^*} \rfloor}$ converges to a Markov jump processes

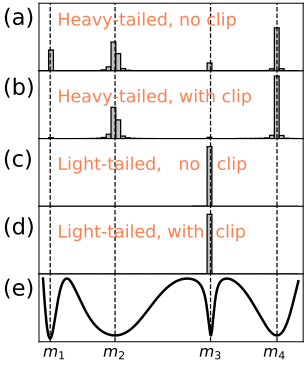
whose state space consists of wide local minima only.

Eliminating Sharp Local Minima with Truncated Heavy-Tails



Eliminating Sharp Local Minima with Truncated Heavy-Tails





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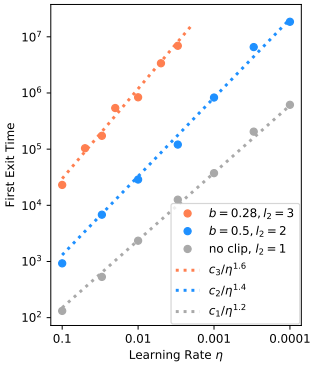

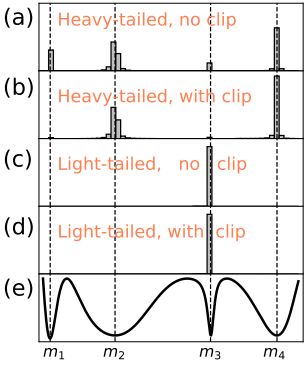

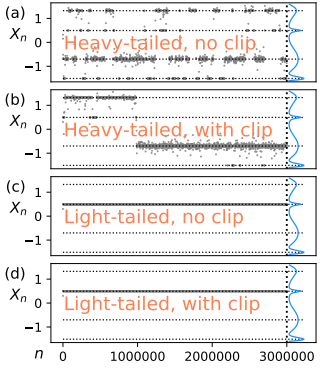

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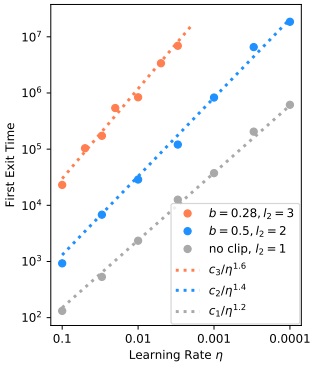
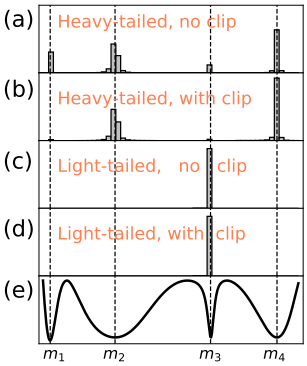
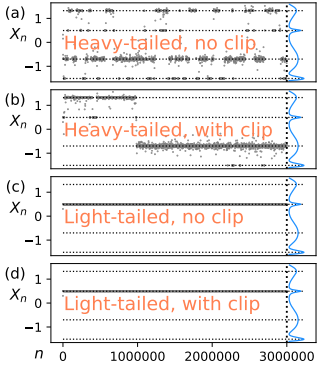
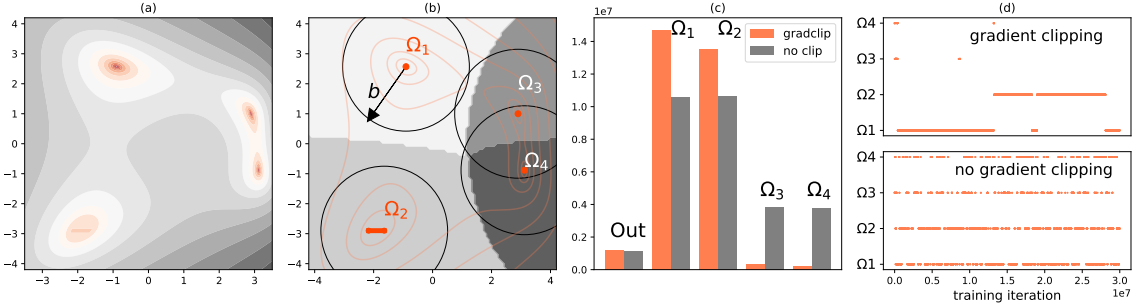






Consistent with what theory predicts!

Same Phenomena in $\mathbb{R}^2$ with More General Geometry



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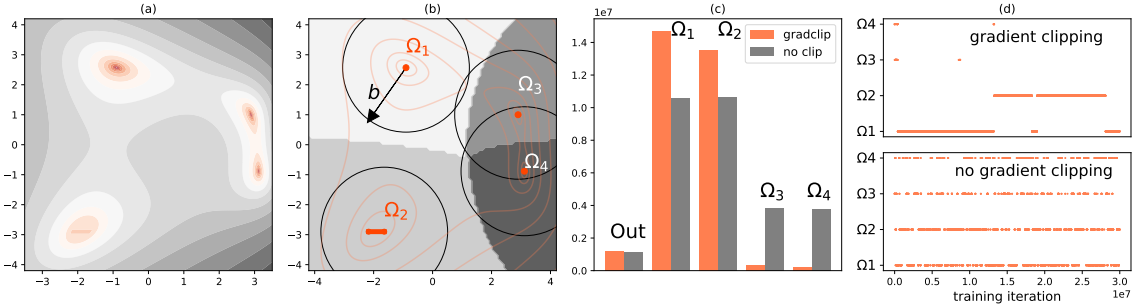

Again, consistent with what theory predicts!

In Deep Neural Network

$$\nabla \tilde{f} \quad = \nabla f_{\text{small batch}}$$

In Deep Neural Network

Stochastic Gradient

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Stochastic Gradient

$$\nabla \tilde{f}_{\text{our}} = \nabla f_{\text{small batch}} + R \cdot (\nabla f_{\text{large batch}} - \nabla f_{\text{small batch}})$$

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Stochastic Gradient

Pareto RV

$$\nabla \tilde{f}_{\text{our}} = \nabla f_{\text{small batch}} + R \cdot (\nabla f_{\text{large batch}} - \nabla f_{\text{small batch}})$$

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Heavy-Tailed Stochastic Gradient

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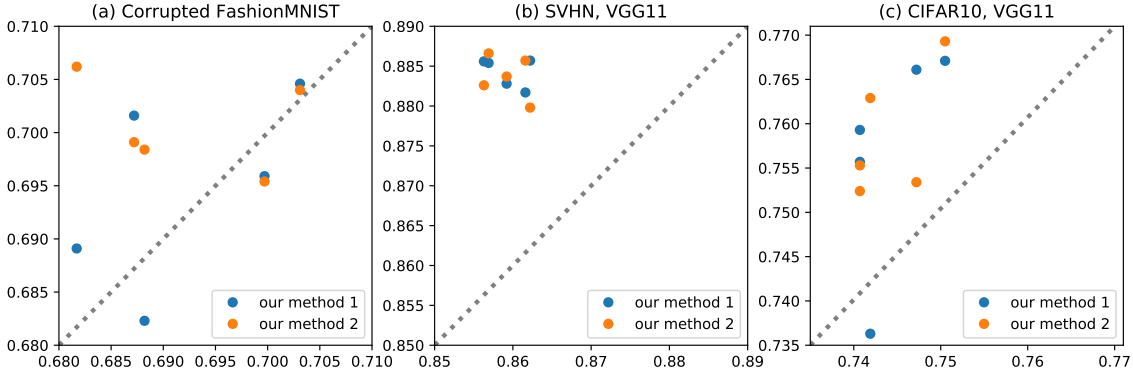
In Deep Neural Network

Heavy-Tailed Stochastic Gradient Pareto RV

$$\widetilde{\nabla f}_{\text{our}} = \nabla f_{\text{small batch}} + R \cdot (\nabla f_{\text{large batch}} - \nabla f_{\text{small batch}})$$

Test accuracy	LB	SB	SB+Clip	SB+Noise	Our 1	Our 2
FashionMNIST, LeNet	68.66%	69.20%	68.77%	64.43%	69.47%	**70.06%**
SVHN, VGG11	82.87%	85.92%	85.95%	38.85%	**88.42%**	88.37%
CIFAR10, VGG11	69.39%	74.42%	74.38%	40.50%	75.69%	**75.87%**
Expected Sharpness	LB	SB	SB+Clip	SB+Noise	Our 1	Our 2
FashionMNIST, LeNet	0.032	0.008	0.009	0.047	0.003	**0.002**
SVHN, VGG11	0.694	0.037	0.041	0.012	**0.002**	0.005
CIFAR10, VGG11	2.043	0.050	0.039	2.046	**0.024**	0.037

Improvement is Consistent



Summary

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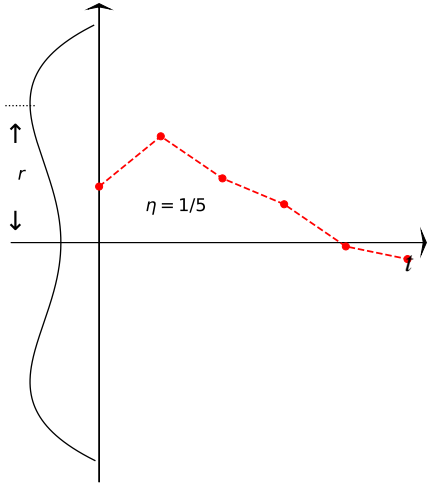
- Tail-Inflation-Truncation strategy improves SGDs' generalization performance in DNN

Gradient Flow: Law of Large Numbers



(90° rotation of) f

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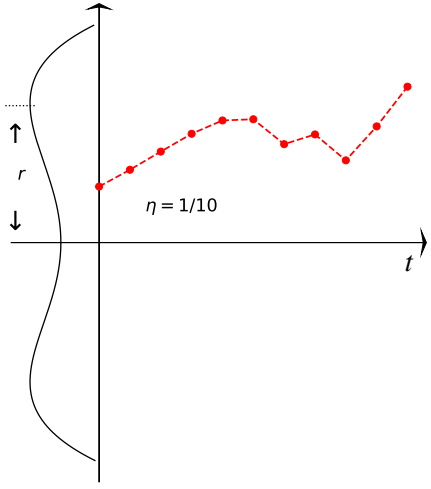

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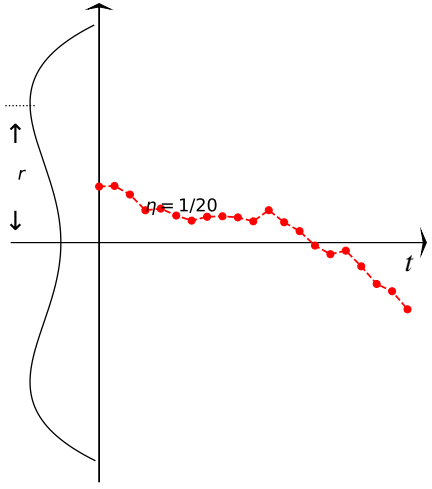

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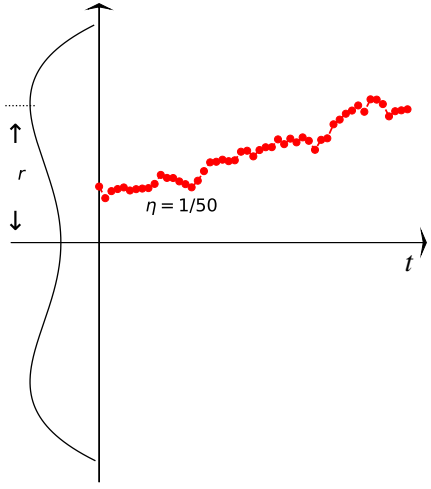

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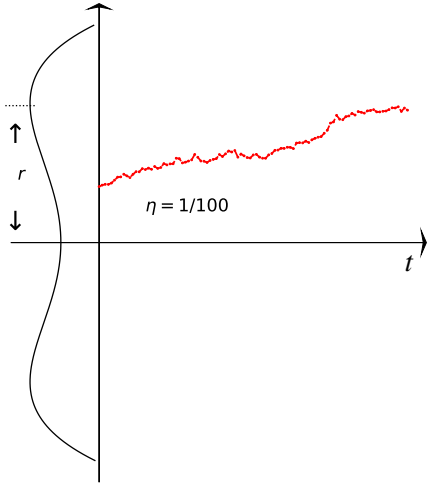

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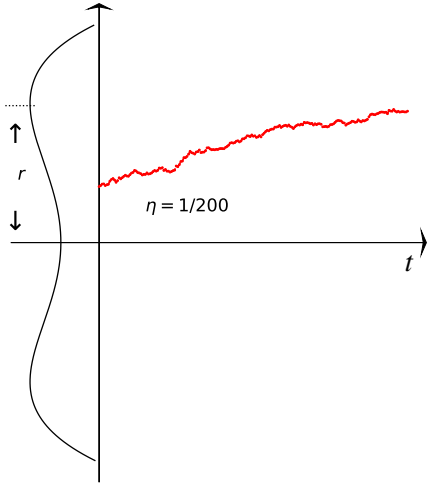

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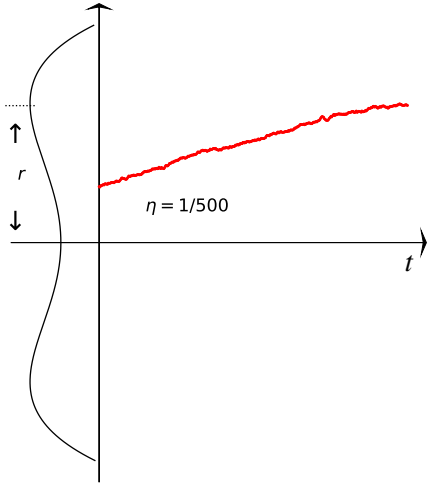

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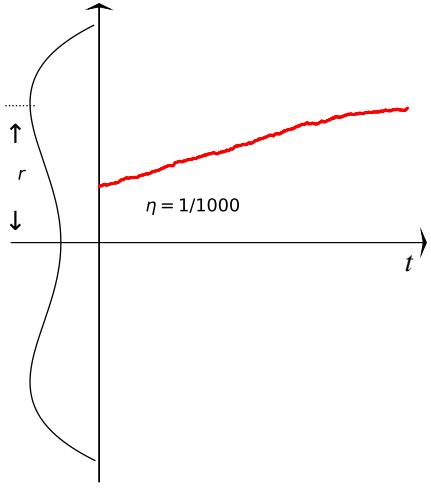

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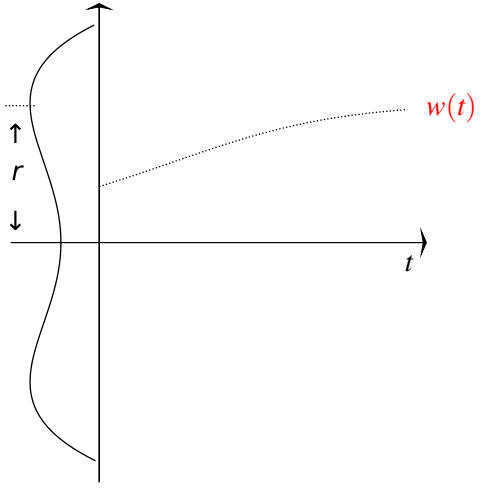

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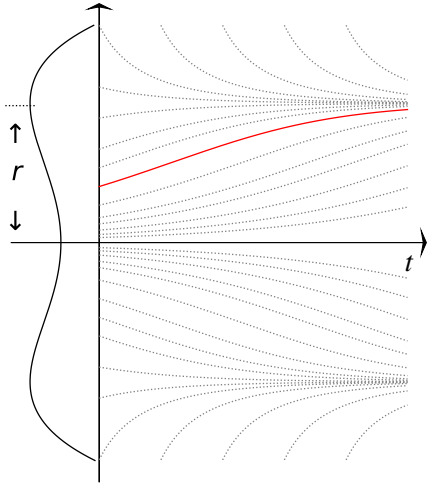

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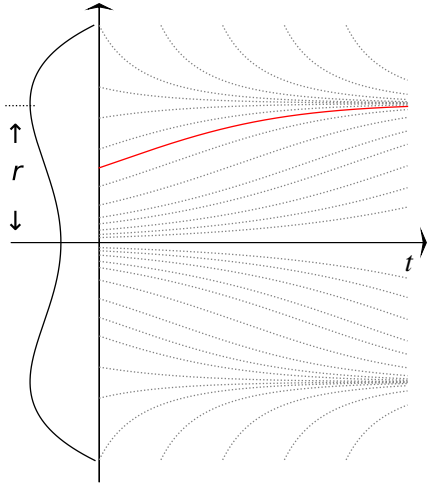

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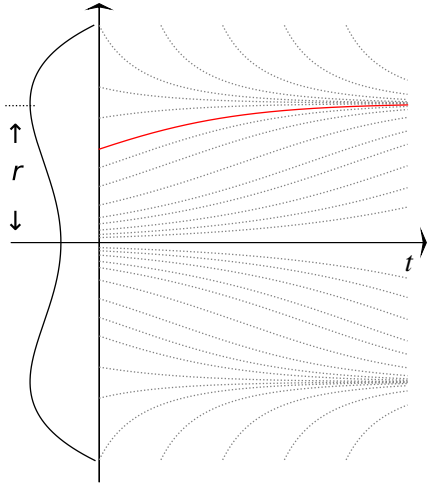

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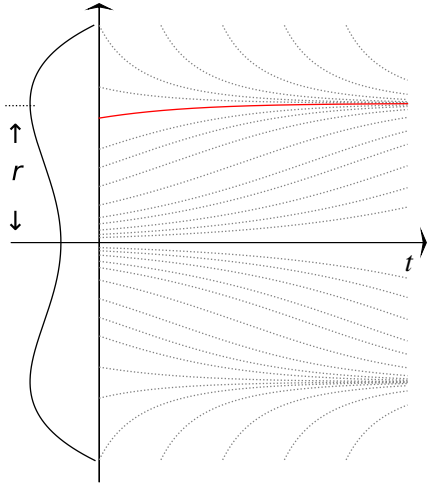

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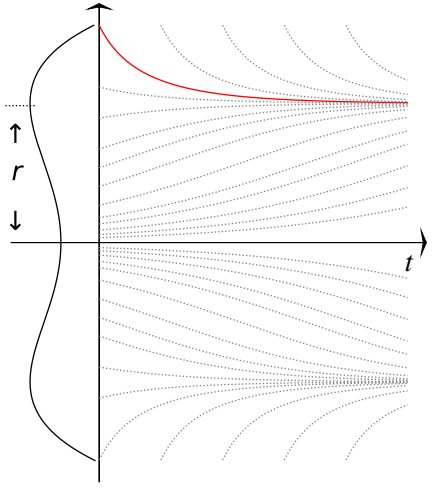

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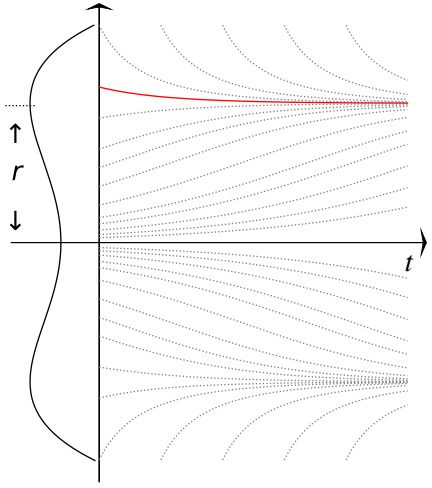

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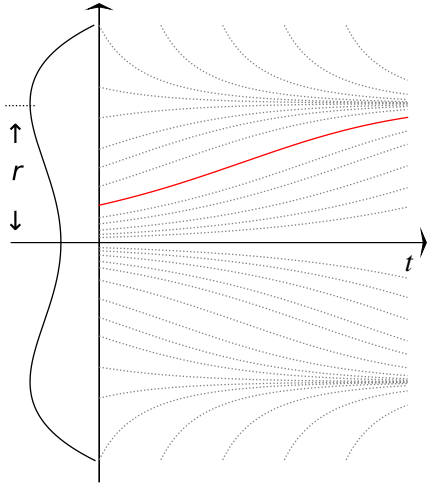

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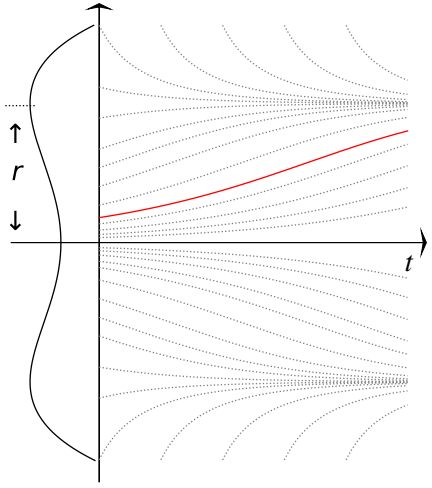

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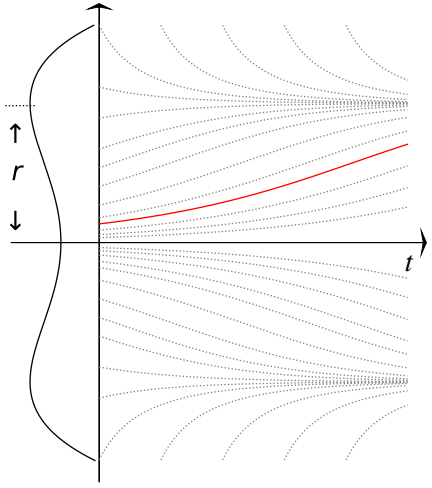

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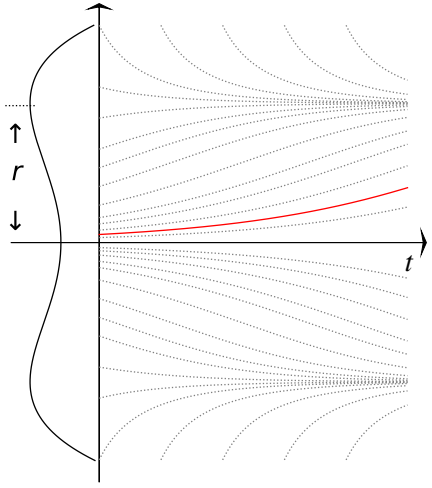

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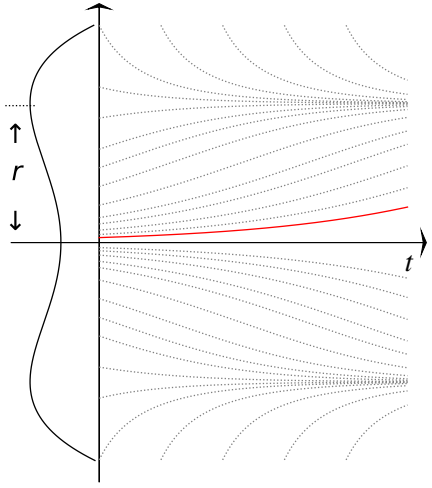

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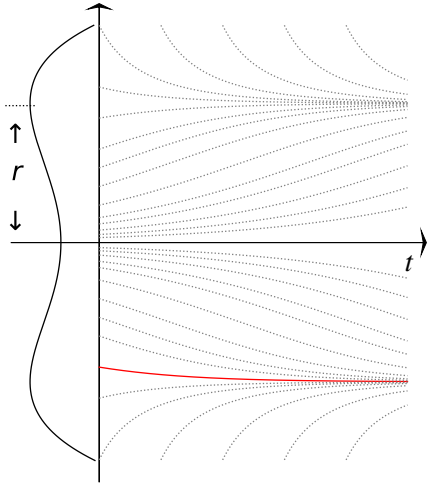

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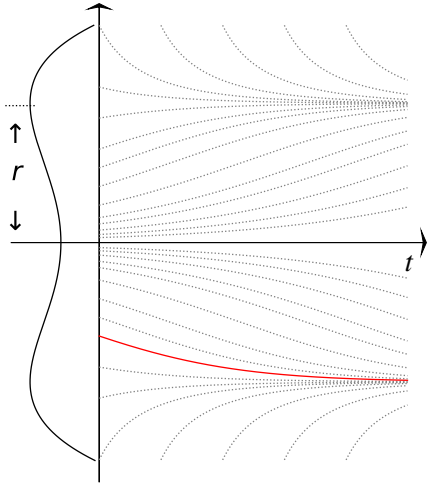

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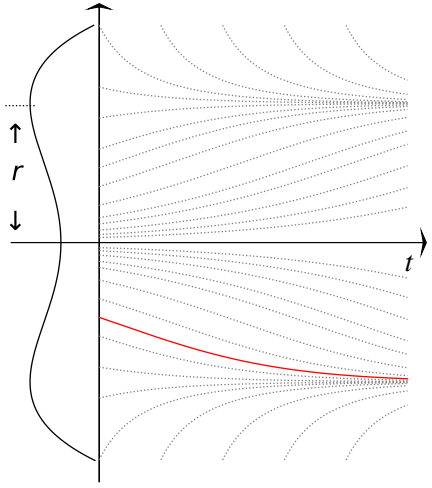

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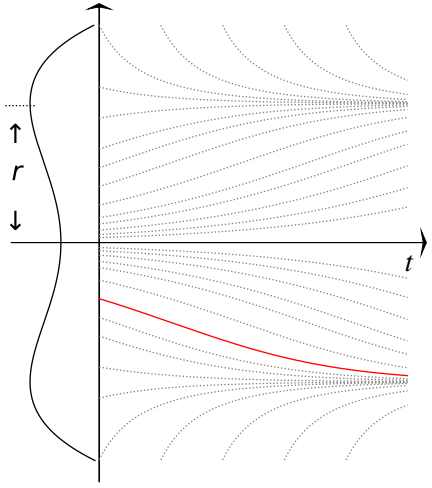

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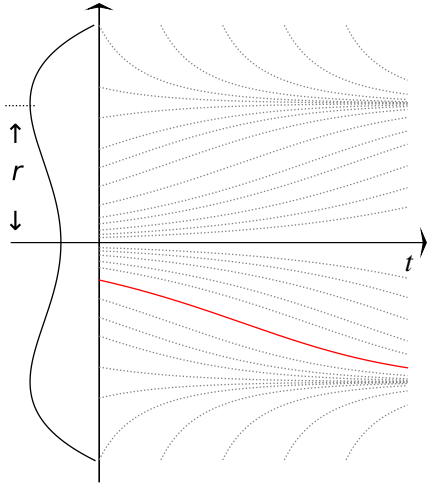

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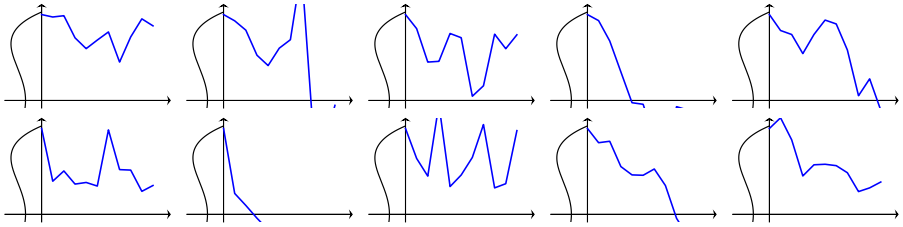
Typical Scenario

Trajectory of SGD W^η: $\eta = 1/10$ & noises are **light-tailed**



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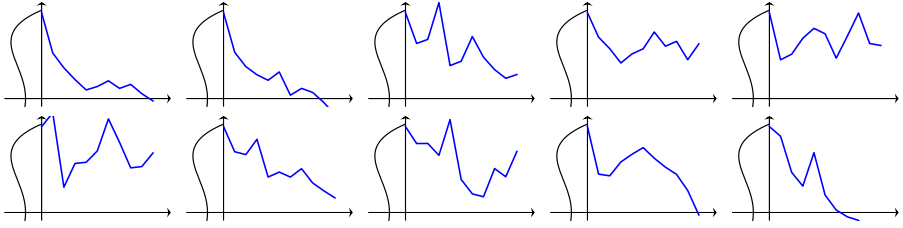
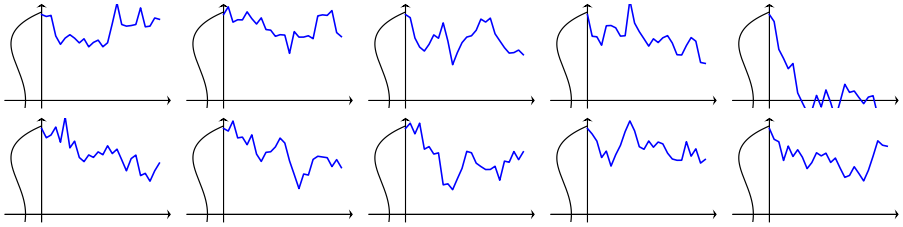


Typical Scenario

Trajectory of SGD W^η: $\eta = 1/25$ & noises are **light-tailed**



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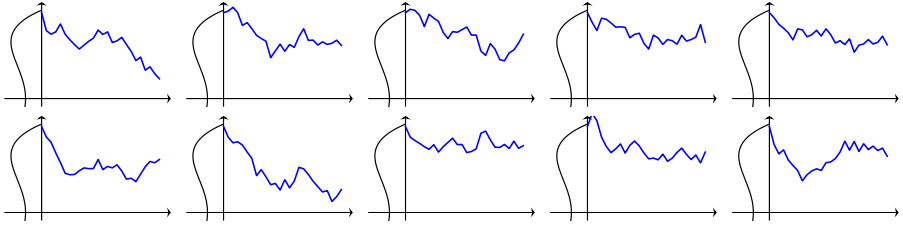
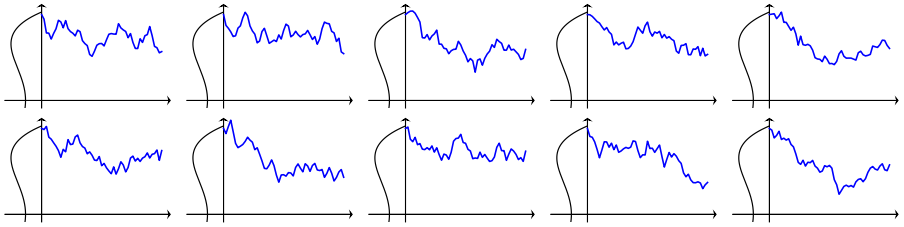


Typical Scenario

Trajectory of SGD W^η: $\eta = 1/50$ & noises are **light-tailed**



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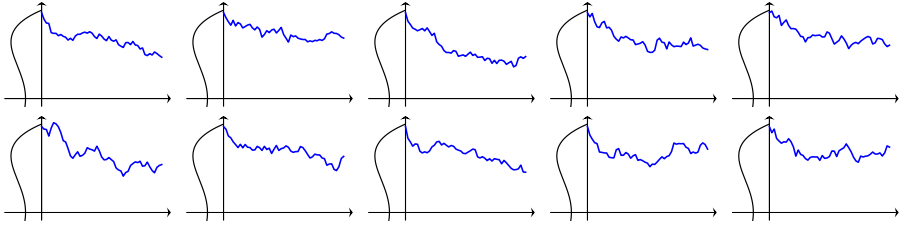
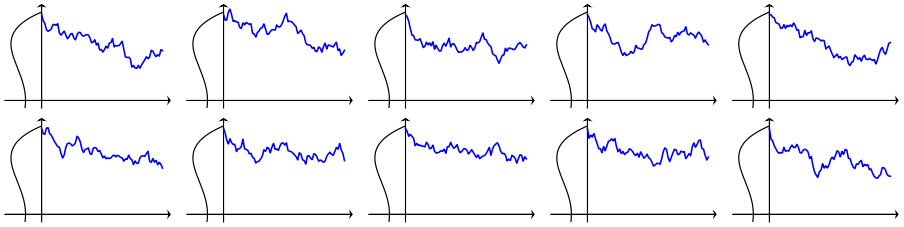
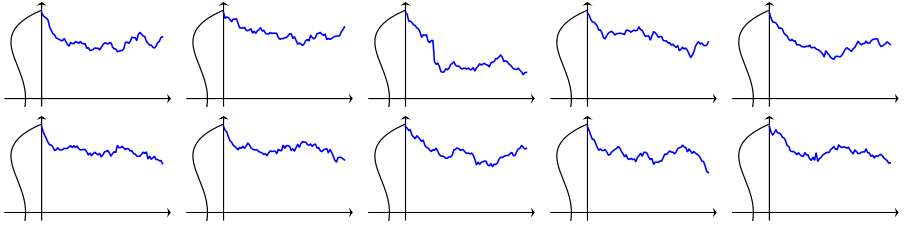


Typical Scenario

Trajectory of SGD W^η: $\eta = 1/75$ & noises are **light-tailed**

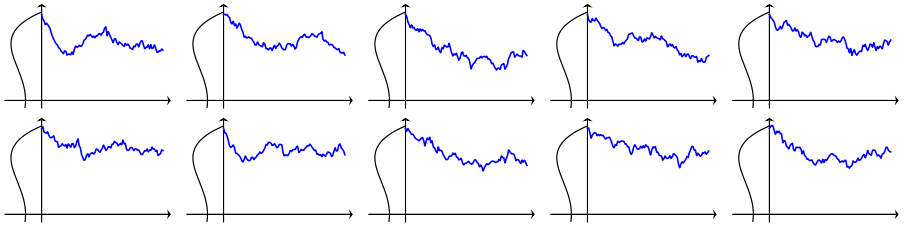


Trajectory of SGD W^η: $\eta = 1/75$ & noises are **heavy-tailed**

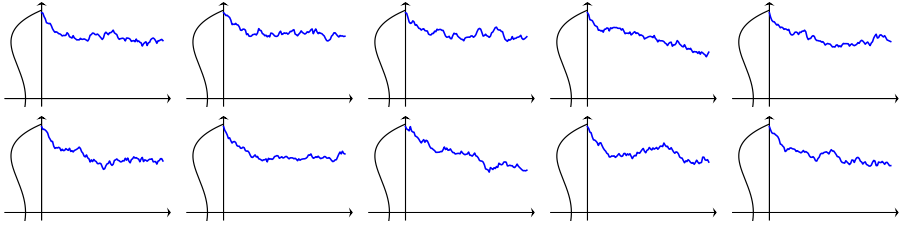


Typical Scenario

Trajectory of SGD W^η: $\eta = 1/100$ & noises are **light-tailed**

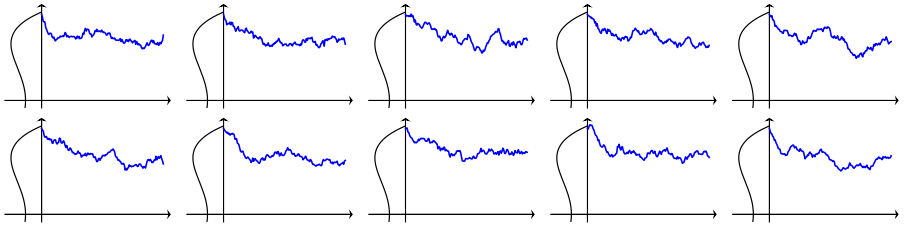


Trajectory of SGD W^η: $\eta = 1/100$ & noises are **heavy-tailed**

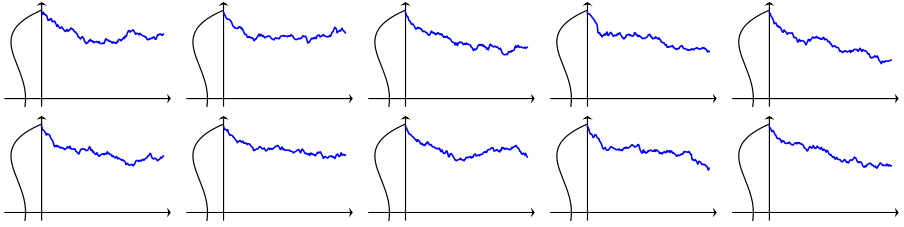


Typical Scenario

Trajectory of SGD W^η: $\eta = 1/150$ & noises are **light-tailed**

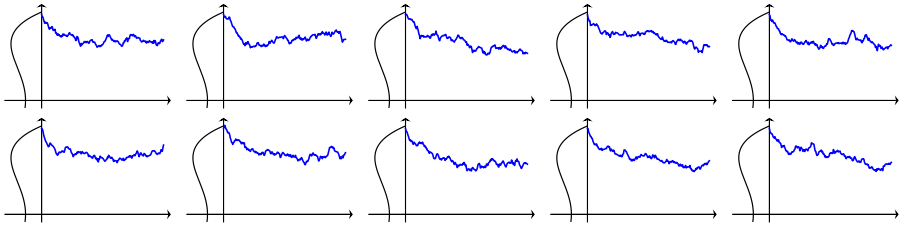


Trajectory of SGD W^η: $\eta = 1/150$ & noises are **heavy-tailed**

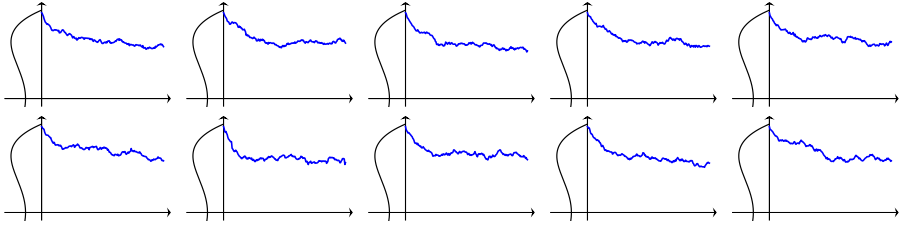


Typical Scenario

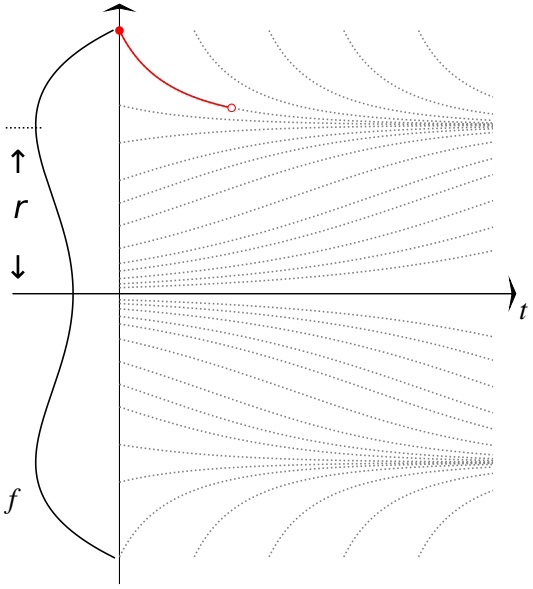
Trajectory of SGD W^η: $\eta = 1/200$ & noises are **light-tailed**



Trajectory of SGD W^η: $\eta = 1/200$ & noises are **heavy-tailed**



Adding Jumps to $w(\cdot)$: Piecewise Gradient Flow



(90° rotation of) f

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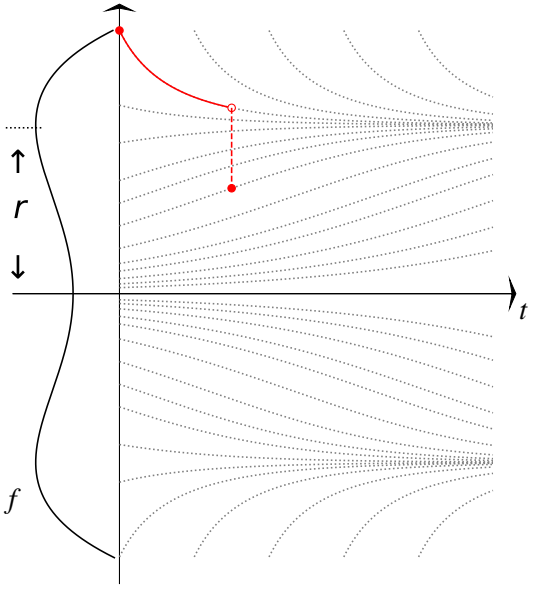

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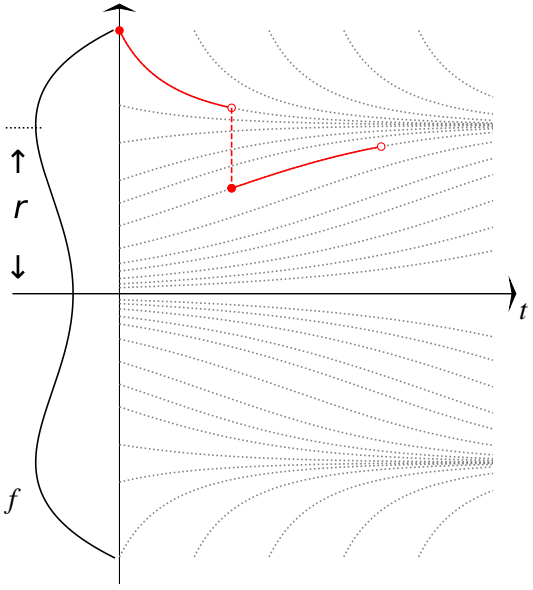

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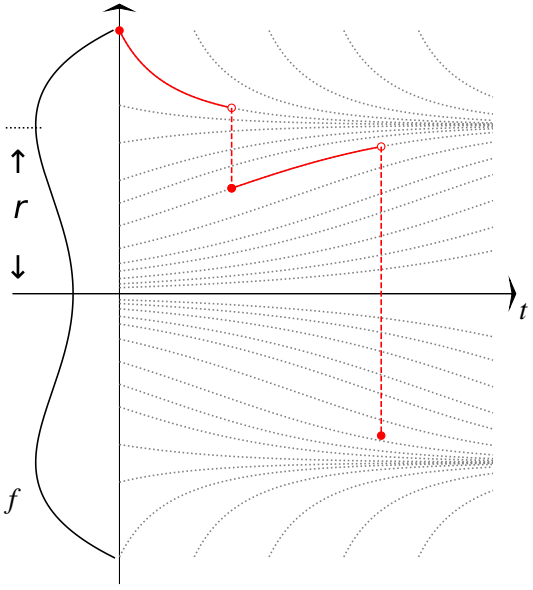

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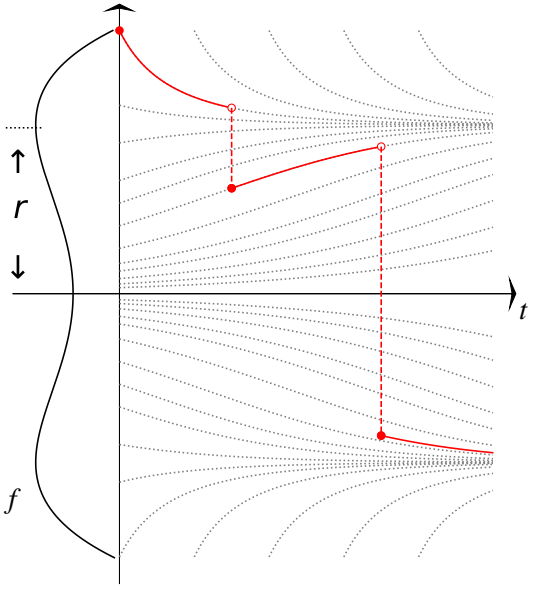

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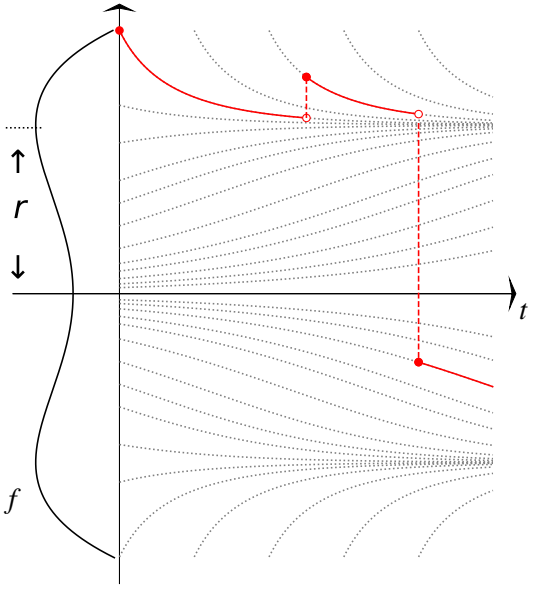

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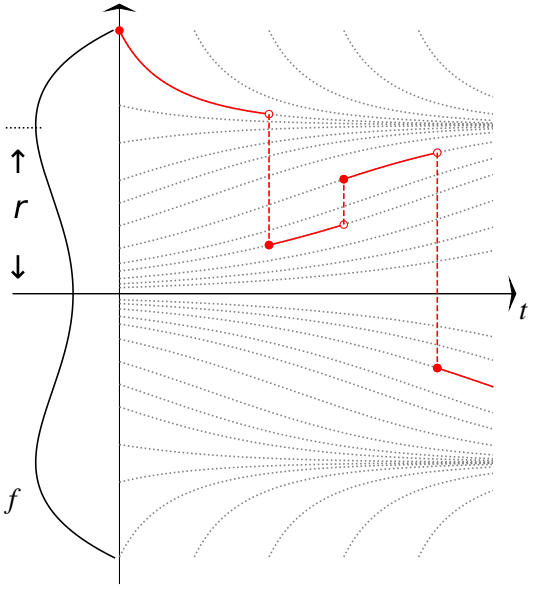
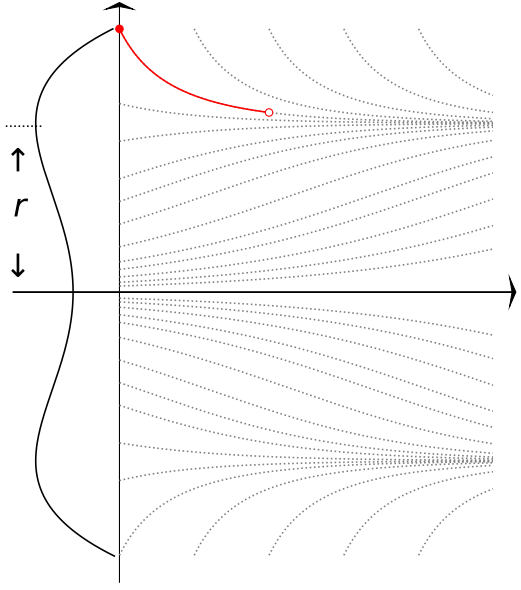


(90° rotation of) f

Catastrophe Principle: Most Likely Escape Route

Most Likely Paths for $\{W^\eta$ escapes the local minimum$\}$



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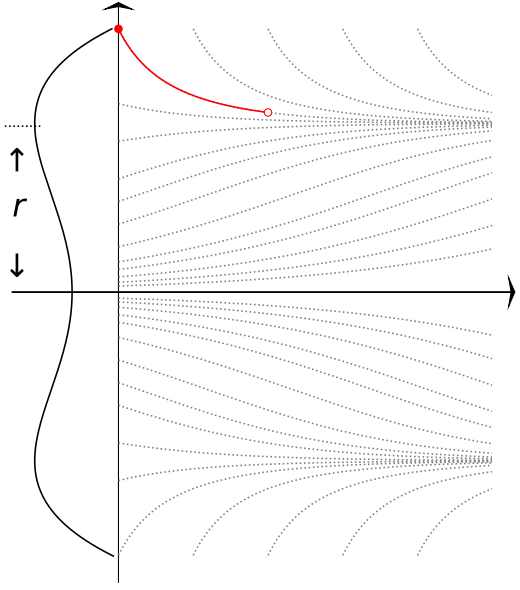

$\mathcal{J}(A) = 1$

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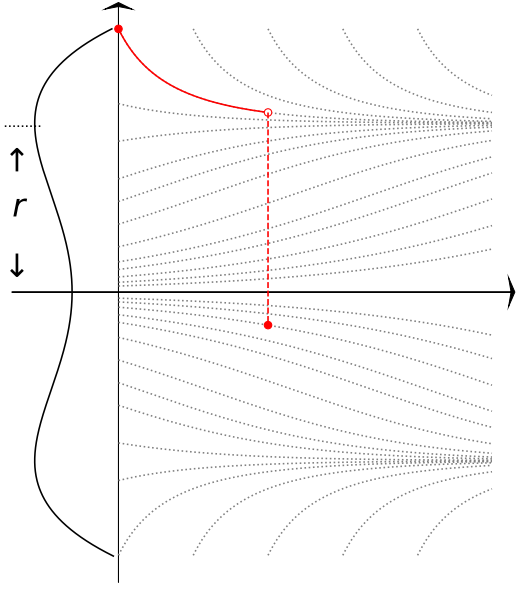

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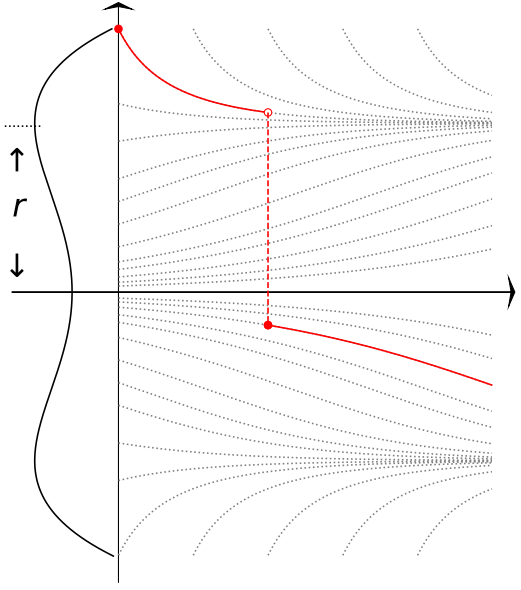

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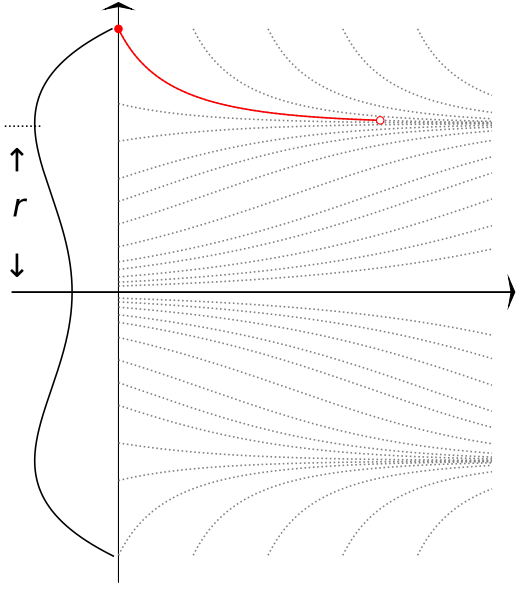

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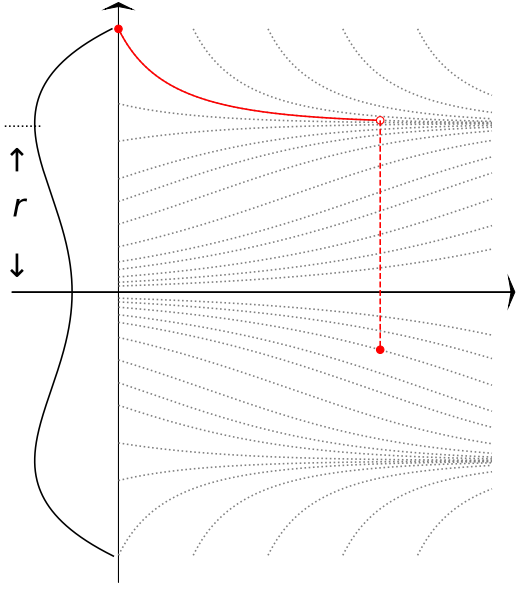

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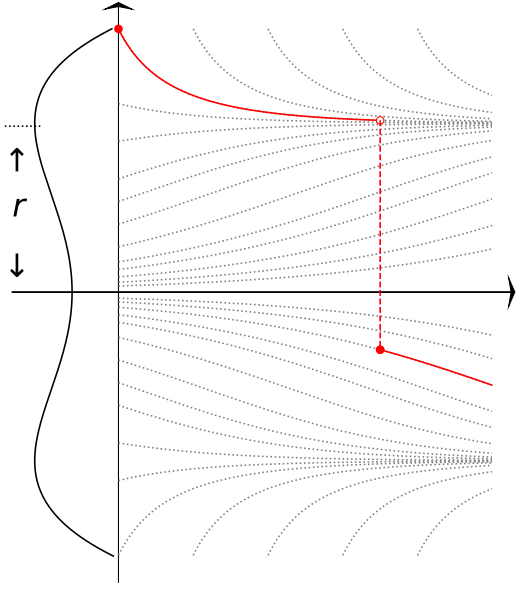

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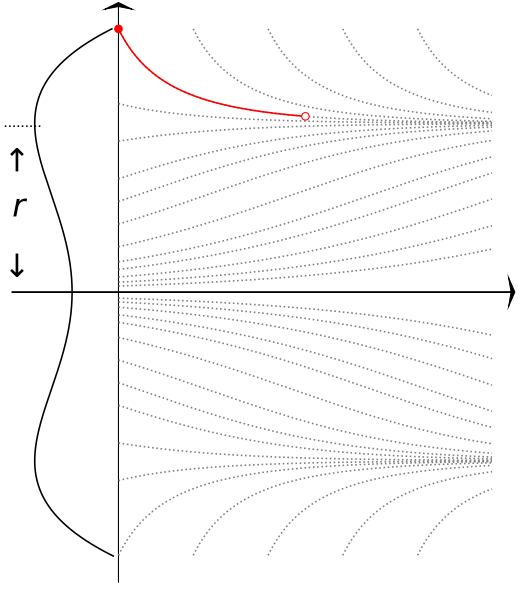

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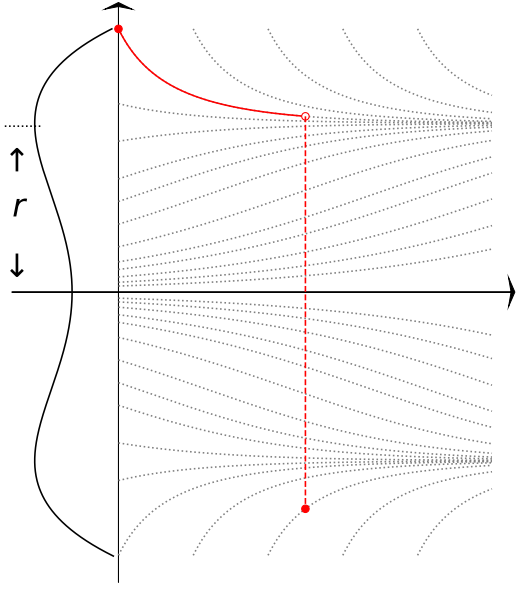

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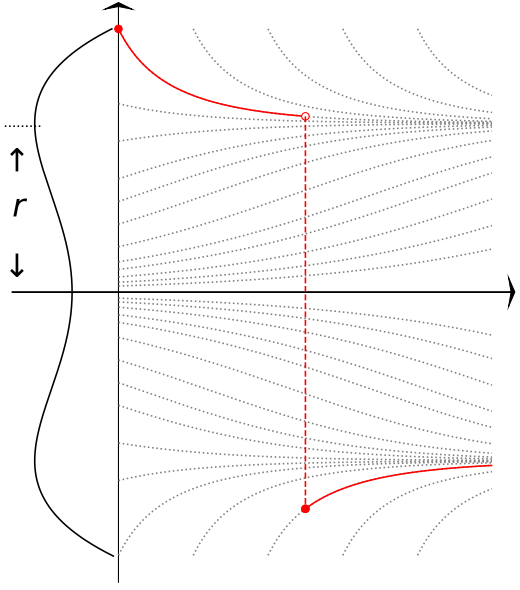

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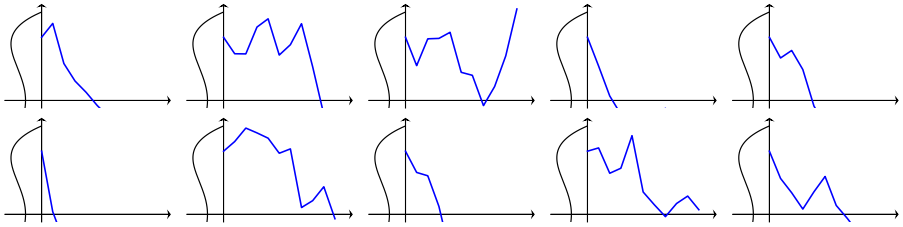
Catastrophe Principle Dictates SGD's Escape Route

Trajectory of SGD X^η conditional on exit: **light-tailed** noises with $\eta = 1/10$



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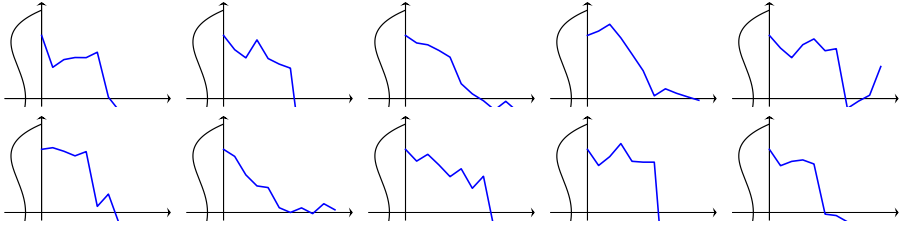
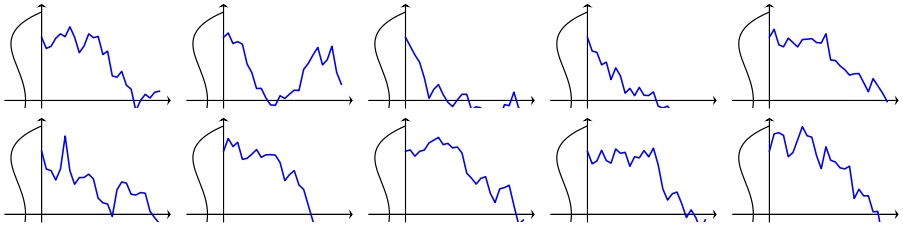


Catastrophe Principle Dictates SGD's Escape Route

Trajectory of SGD X^η conditional on exit: **light-tailed** noises with $\eta = 1/25$



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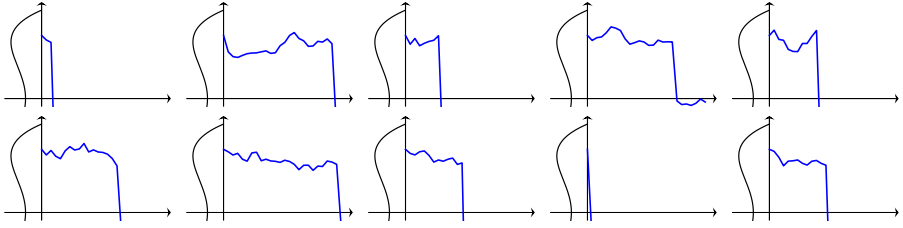
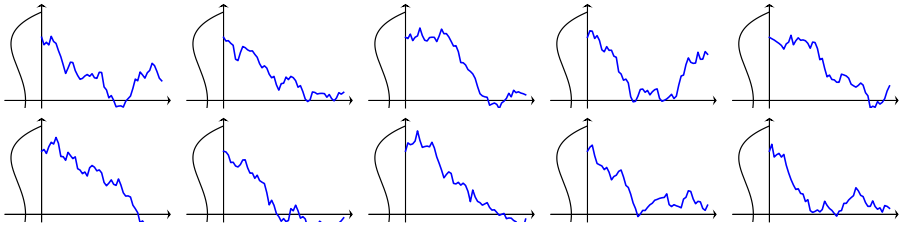


Catastrophe Principle Dictates SGD's Escape Route

Trajectory of SGD X^η conditional on exit: **light-tailed** noises with $\eta = 1/50$



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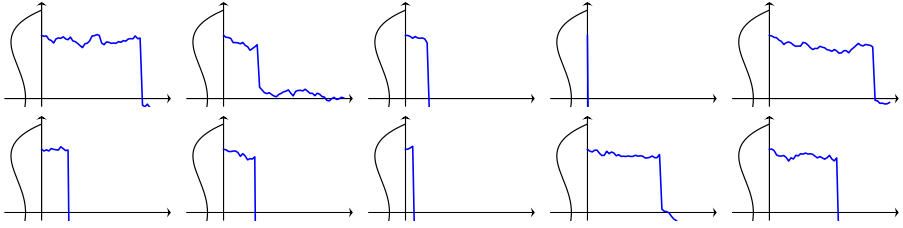

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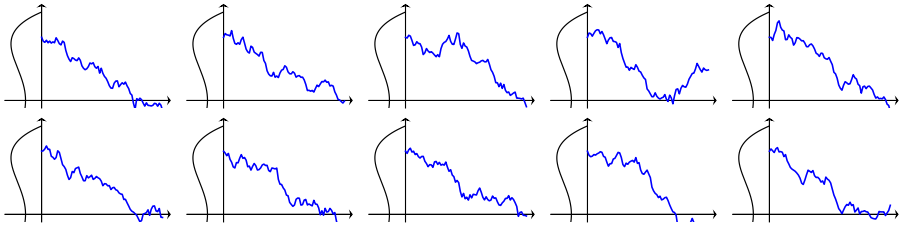

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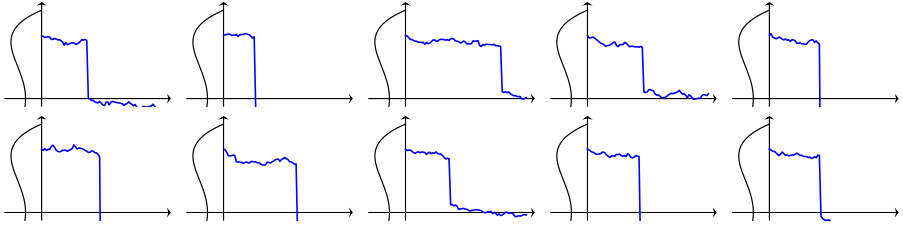

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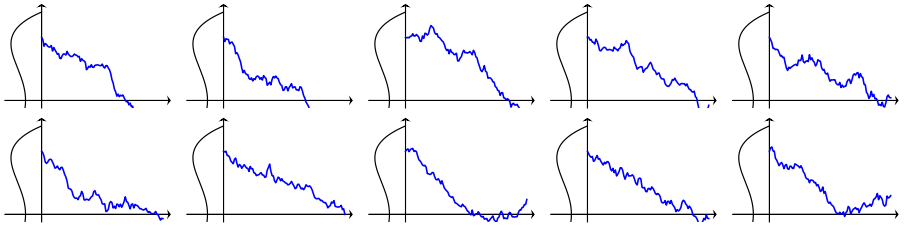

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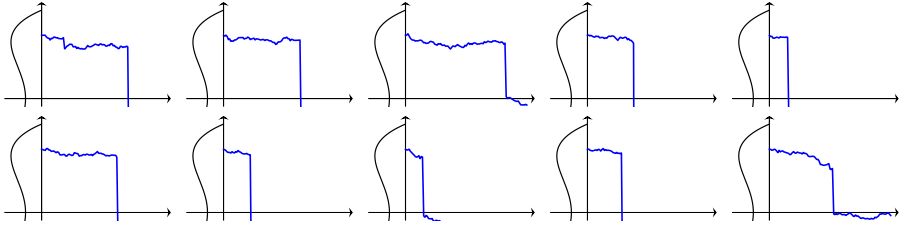

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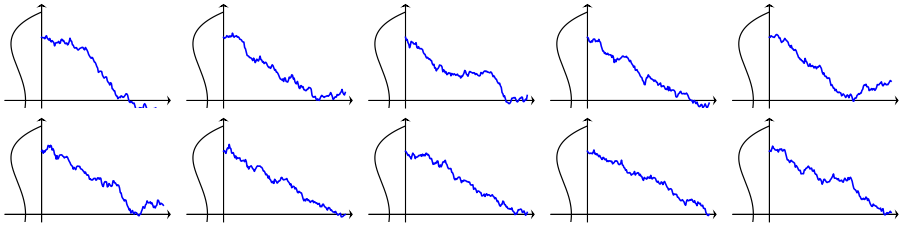

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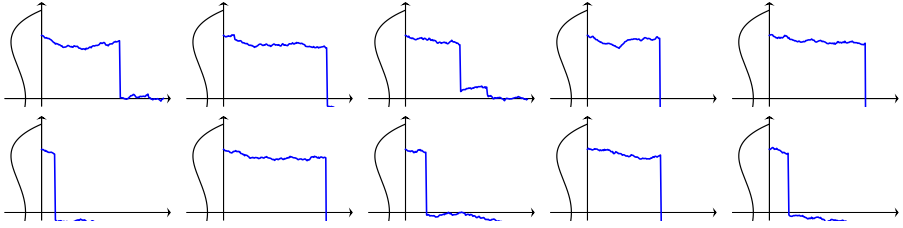

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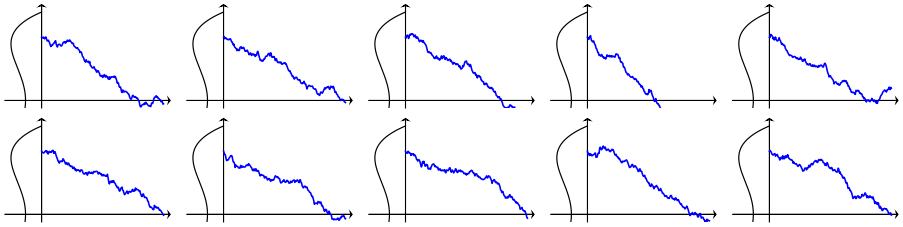

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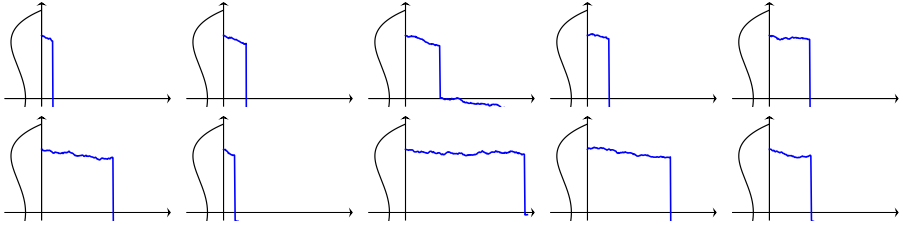

Truncated Version of Stochastic Gradient Descent

SGD

$$W_{k+1}^{\eta} = W_k^{\eta} - \qquad \eta\left(f'(W_k^{\eta}) + Z_k\right) \qquad k = 0, 1, 2, \ldots$$

Truncated Version of Stochastic Gradient Descent

SGD with Gradient Clipping

$$W_{k+1}^{\eta} = W_k^{\eta} - \varphi_c\left(\eta\left(f'(W_k^{\eta}) + Z_k\right)\right) \qquad k = 0, 1, 2, \ldots$$

Truncated Version of Stochastic Gradient Descent

SGD with Gradient Clipping

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where

$$\varphi_c(x) = \frac{x}{|x|}\min\{c, |x|\}.$$

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where

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Then, again,

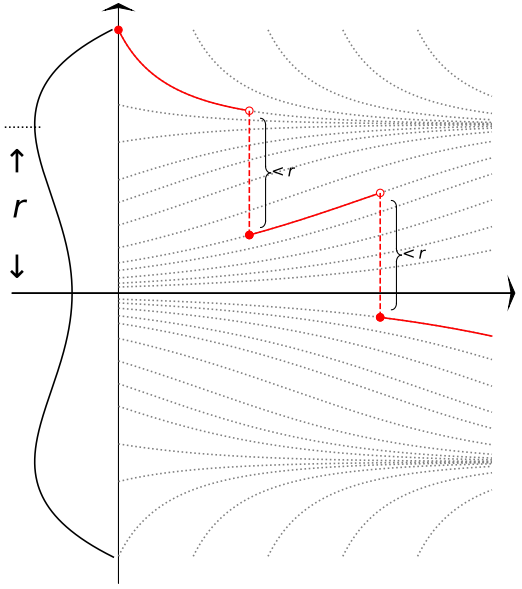
$$W^{\eta}(\cdot) \to w(\cdot) \qquad \text{as} \qquad \eta \to 0$$

where

$$\mathrm{d}w(t) = -f'(w(t))\mathrm{d}t.$$

How does $\mathcal{J}(A)$ change?

If $r \in (c, 2c)$



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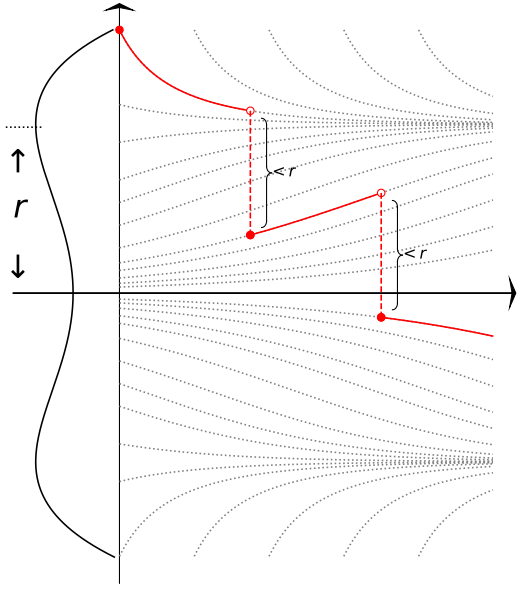

$\mathcal{J}(A) = 2$

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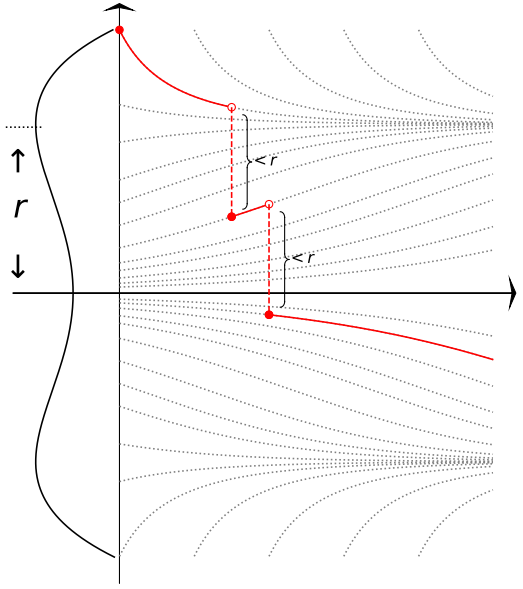
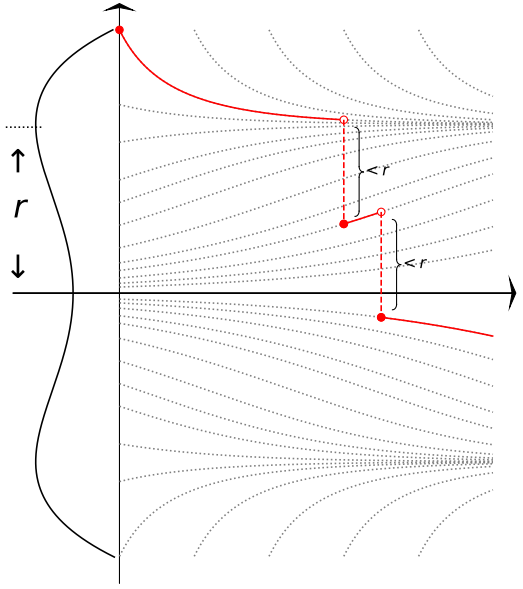


$\mathcal{J}(A) = 2$

How does $\mathcal{J}(A)$ change?

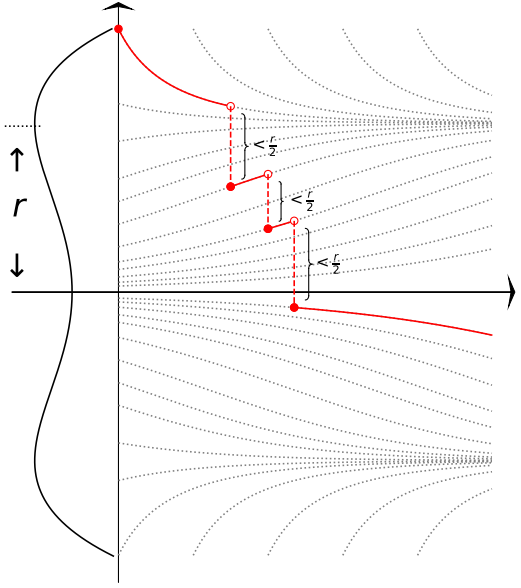
If $r \in (c, 2c)$



$\mathcal{J}(A) = 2$

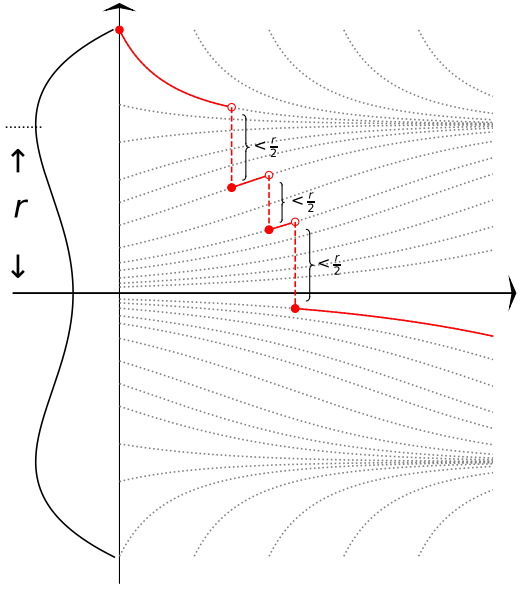
How does $\mathcal{J}(A)$ change?

If $r \in (2c, 3c)$



How does $\mathcal{J}(A)$ change?

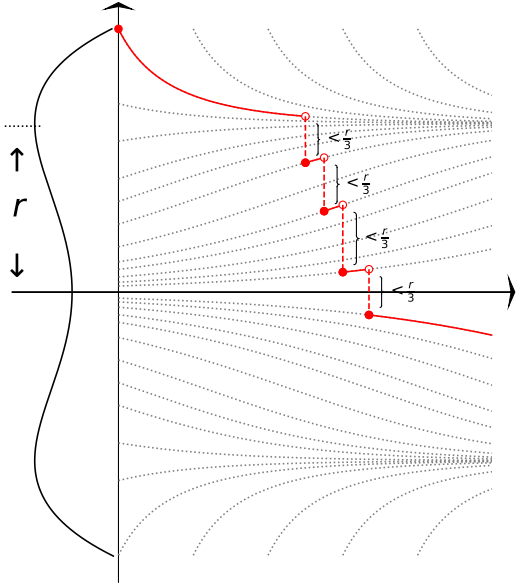
If $r \in (2c, 3c)$



$\mathcal{J}(A) = 3$

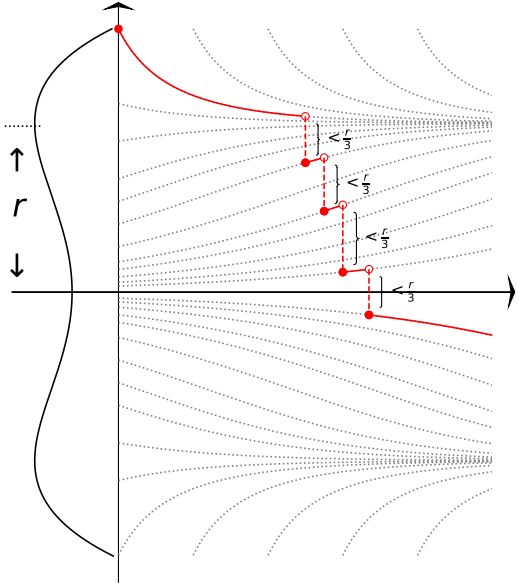
How does $\mathcal{J}(A)$ change?

If $r \in (3c, 4c)$



How does $\mathcal{J}(A)$ change?

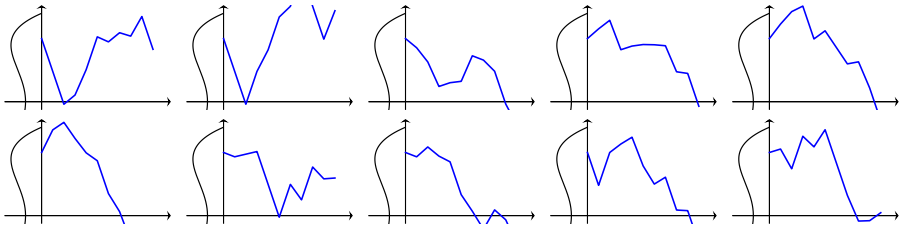
If $r \in (3c, 4c)$



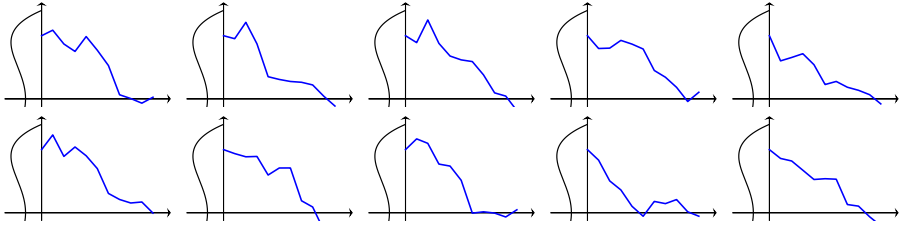
$\mathcal{J}(A) = 4$

SGD's Escaping Route under Gradient Clipping with $\mathcal{J}(A) = 2$

Trajectory of SGD X^η conditional on exit: **light-tailed** noises with $\eta = 1/10$

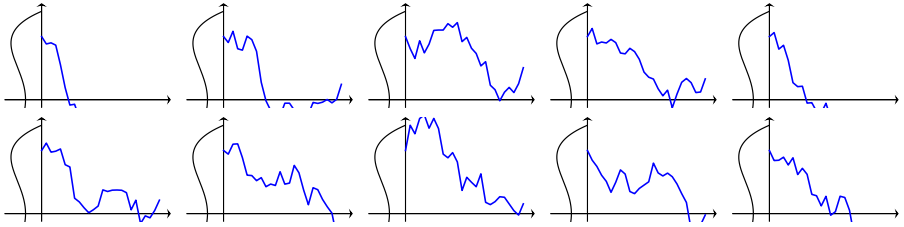


Trajectory of SGD X^η conditional on exit: **heavy-tailed** noises with $\eta = 1/10$

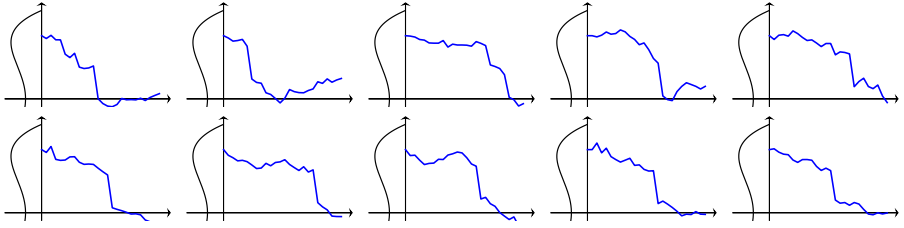


SGD's Escaping Route under Gradient Clipping with $\mathcal{J}(A) = 2$

Trajectory of SGD X^η conditional on exit: **light-tailed** noises with $\eta = 1/25$

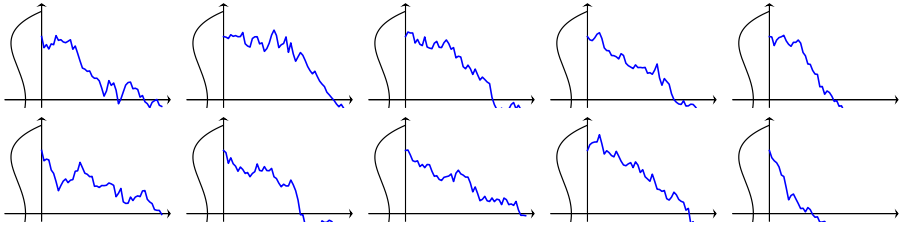


Trajectory of SGD X^η conditional on exit: **heavy-tailed** noises with $\eta = 1/25$

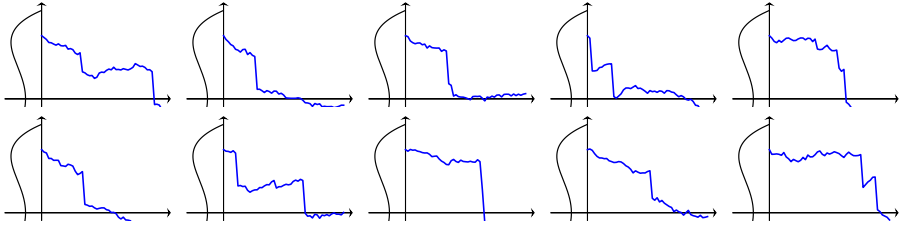


SGD's Escaping Route under Gradient Clipping with $\mathcal{J}(A) = 2$

Trajectory of SGD X^η conditional on exit: **light-tailed** noises with $\eta = 1/50$

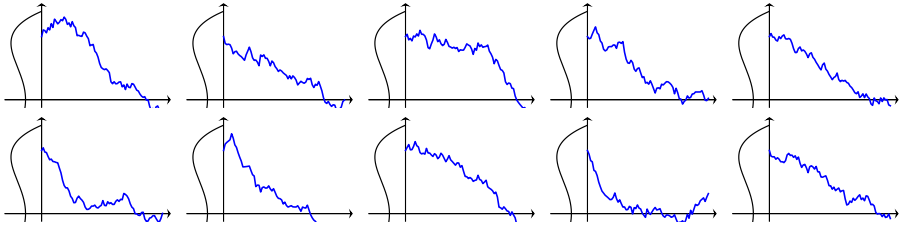


Trajectory of SGD X^η conditional on exit: **heavy-tailed** noises with $\eta = 1/10$

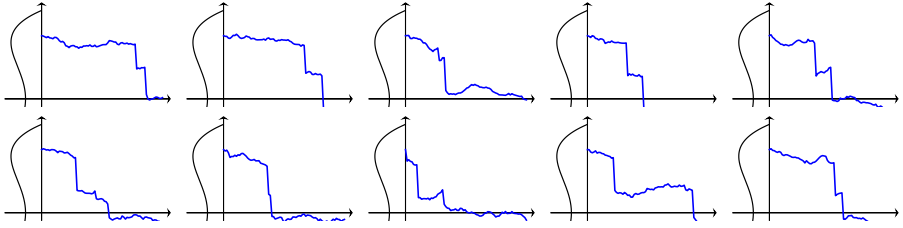


SGD's Escaping Route under Gradient Clipping with $\mathcal{J}(A) = 2$

Trajectory of SGD X^η conditional on exit: **light-tailed** noises with $\eta = 1/75$

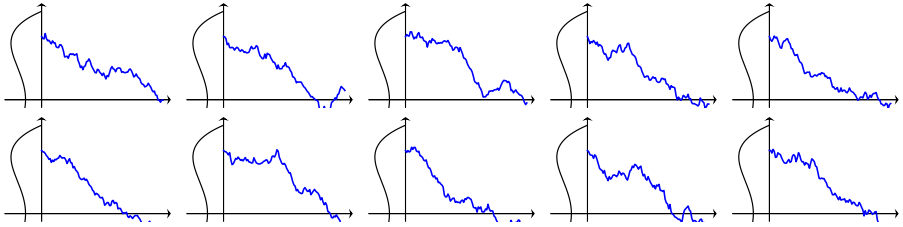


Trajectory of SGD X^η conditional on exit: **heavy-tailed** noises with $\eta = 1/75$

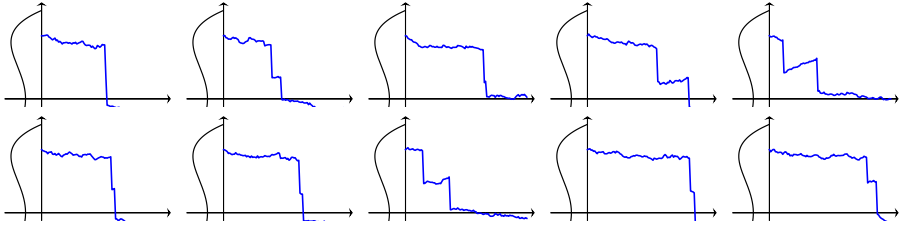


SGD's Escaping Route under Gradient Clipping with $\mathcal{J}(A) = 2$

Trajectory of SGD X^η conditional on exit: **light-tailed** noises with $\eta = 1/100$

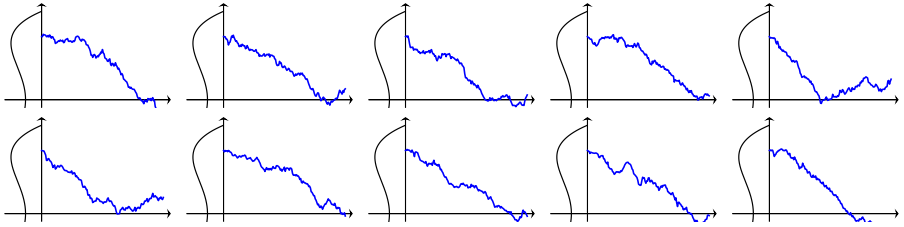


Trajectory of SGD X^η conditional on exit: **heavy-tailed** noises with $\eta = 1/100$

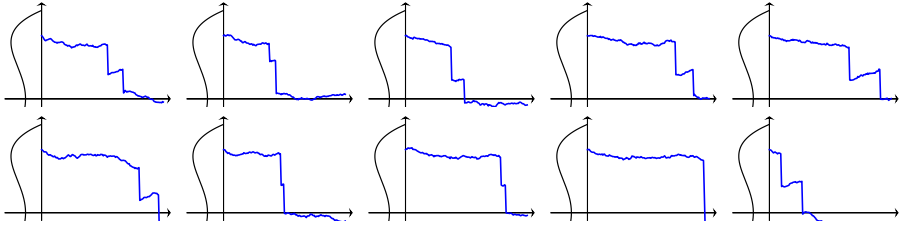


SGD's Escaping Route under Gradient Clipping with $\mathcal{J}(A) = 2$

Trajectory of SGD X^η conditional on exit: **light-tailed** noises with $\eta = 1/150$

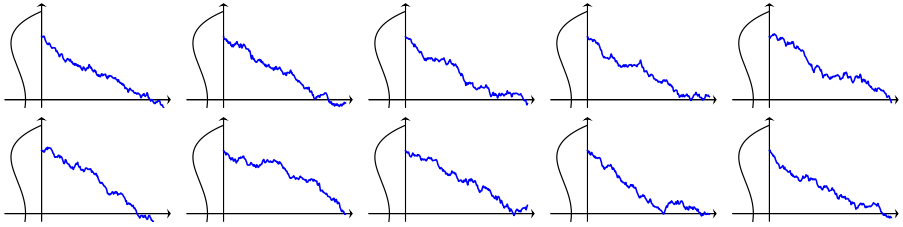


Trajectory of SGD X^η conditional on exit: **heavy-tailed** noises with $\eta = 1/150$

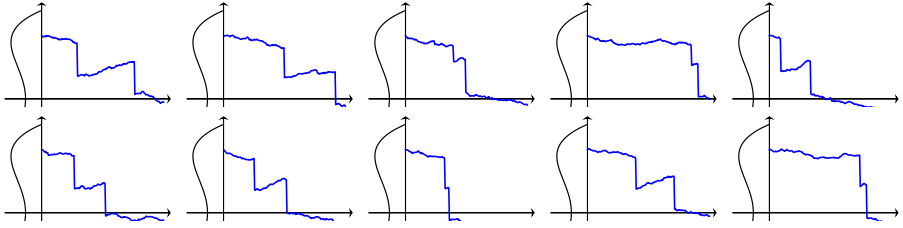


SGD's Escaping Route under Gradient Clipping with $\mathcal{J}(A) = 2$

Trajectory of SGD X^η conditional on exit: **light-tailed** noises with $\eta = 1/200$

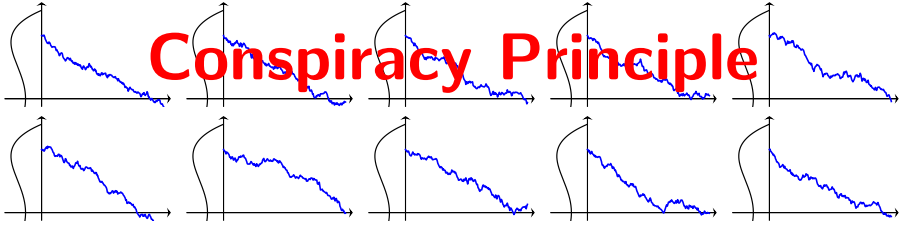


Trajectory of SGD X^η conditional on exit: **heavy-tailed** noises with $\eta = 1/200$

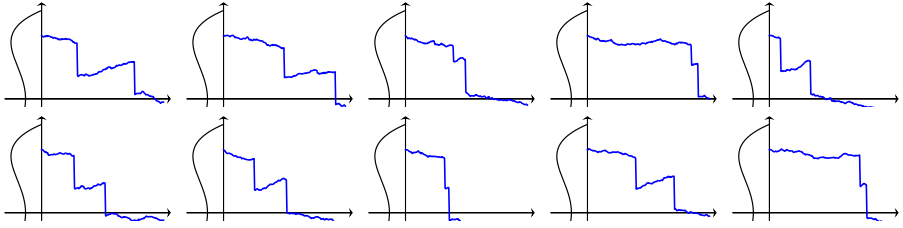


SGD's Escaping Route under Gradient Clipping with $\mathcal{J}(A) = 2$

Trajectory of SGD X^η conditional on exit: **light-tailed** noises with $\eta = 1/200$

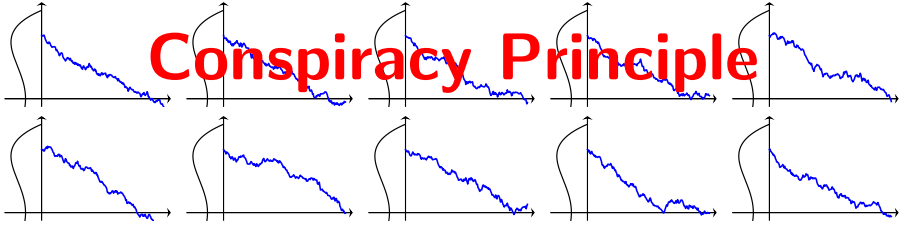


Trajectory of SGD X^η conditional on exit: **heavy-tailed** noises with $\eta = 1/200$



SGD's Escaping Route under Gradient Clipping with $\mathcal{J}(A) = 2$

Trajectory of SGD X^η conditional on exit: **light-tailed** noises with $\eta = 1/200$



Trajectory of SGD X^η conditional on exit: **heavy-tailed** noises with $\eta = 1/200$