# Capitalizing Catastrophe Principle in Training DNN and Other Related Topics

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Northwestern University

INI Satellite Programme on Heavy Tails in Machine Learning, The Alan Turing Institute

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Based on the joint works with

Mihail Bazhba, Jose Blanchet, Bohan Chen, Sewoong Oh, Jeeho Ryu, Insuk Seo, Zhe Su, Xingyu Wang, and Bert Zwart

# Outline

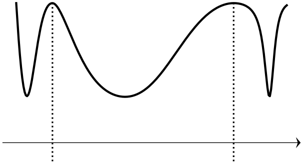
- Lecture 1: Heavy-Tailed Large Deviations Approach to Global Dynamics of SGD
  - Motivation, Phenomena, and Insights
- Lecture 2: Conspiracy vs Catastrophe Principle via Sample Path Large Deviations
  - Sample-Path Large Deviations: Light-Tailed vs Heavy-Tailed Frameworks
  - Fundamental Limitations of the Classical LDP Framework in Heavy-Tailed Systems
  - M-convergence and Related Machinery
  - Topological Considerations
- Lecture 3: Analysis of Local Stability and Global Dynamics of Heavy-Tailed SGD
  - Uniform M-convergence
  - Exit Time Analysis via Asymptotic Atom
  - Scaling Limit of Sample Paths
- Lecture 4: Capitalizing Catastrophe Principle in Training DNN & Other Related Topics
  - Tail Inflation Truncation Strategy
  - Large Deviations in M1' topology
  - Local Instability and its Connection to Edge of Stability

# Previously in this Lecture Series

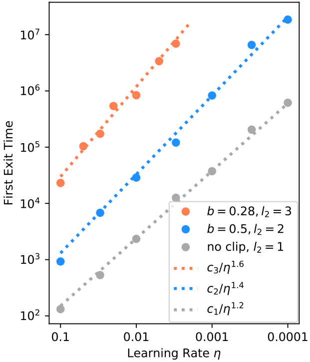
# Generalization and Sharp/Flat Local Minima

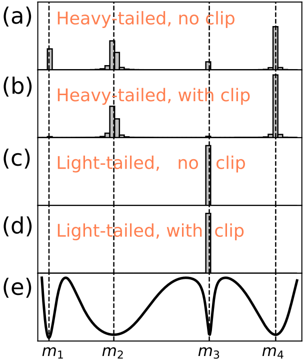
Training Set

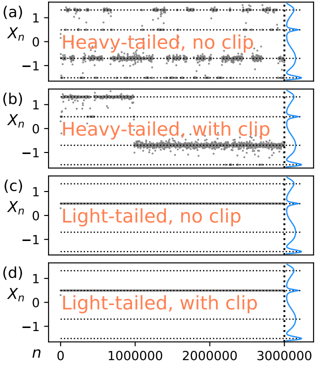
Test Set image source



# Catastrophe vs Conspiracy in SGD



# Rare Events depend on "Tail Behaviors"

**Light-Tailed Distributions**

- Extreme Values are Very Rare
- Normal, Exponential, etc

**Heavy-Tailed Distributions**

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**Systemwide rare events**

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**A FEW Catastrophes.**

**(Catastrophe Principle)**

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# Insurance Example: Capital Reserve

$$Y(t) = c + pt - \sum_{i=1}^{N(t)} X_i$$

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Large $n$: analysis of large loss over a long time period

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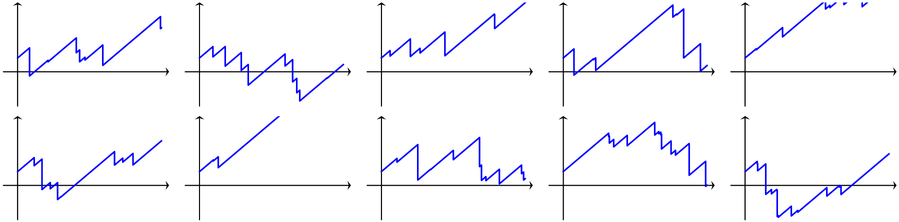

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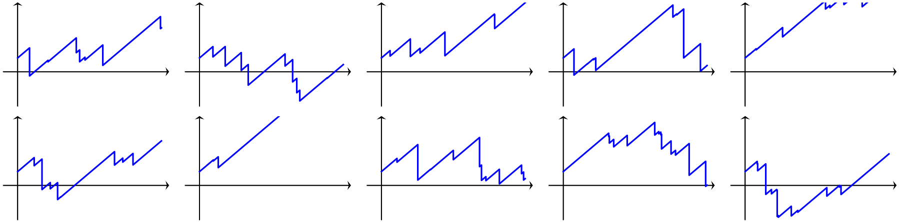

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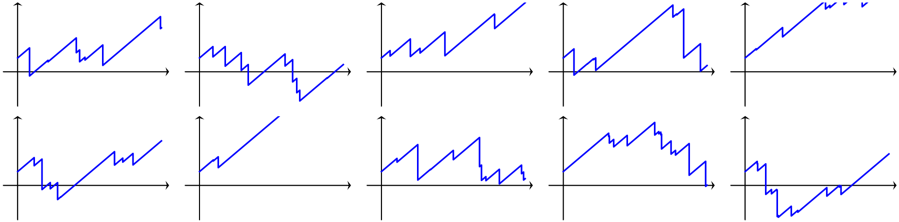

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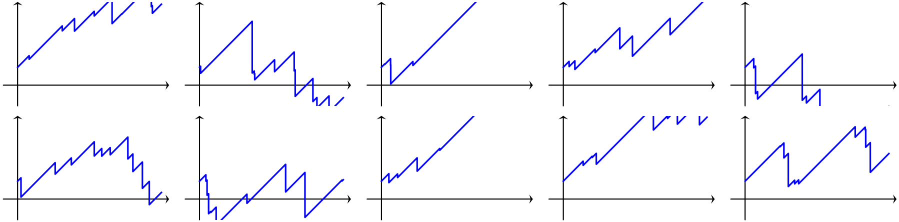

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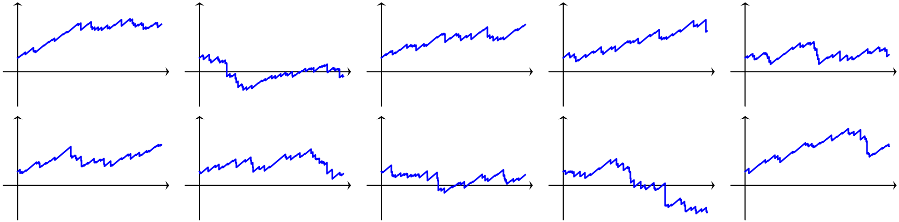

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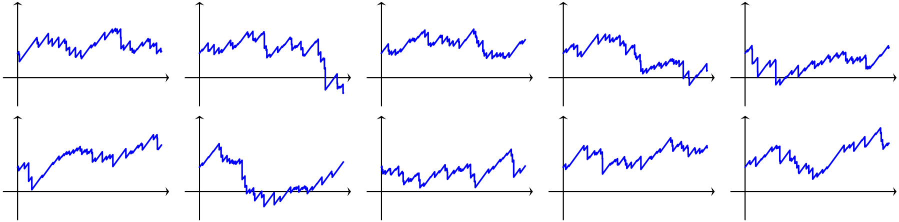

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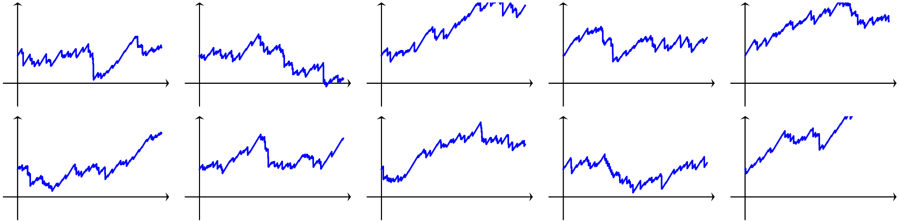

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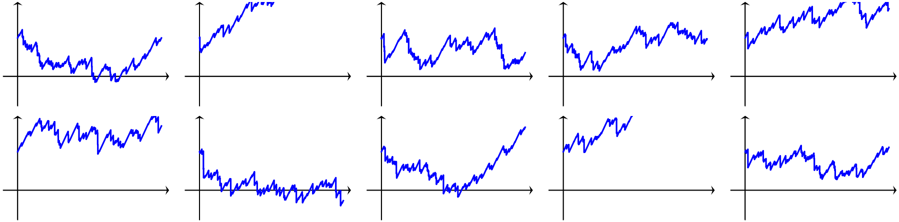

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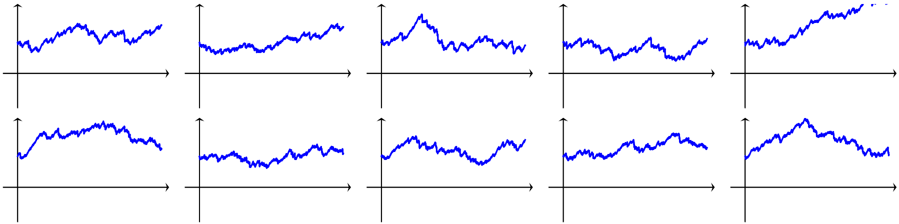

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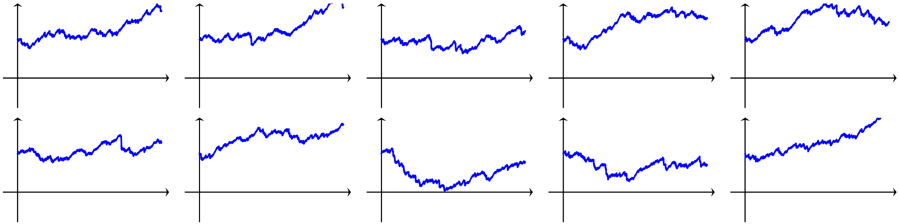

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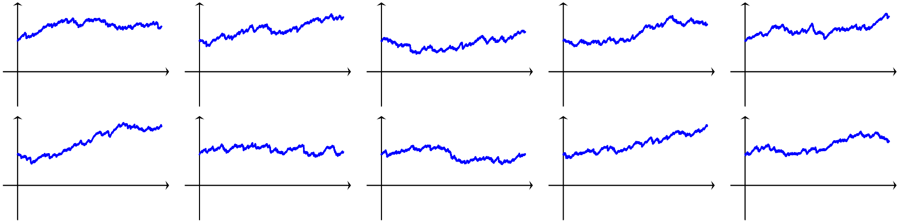

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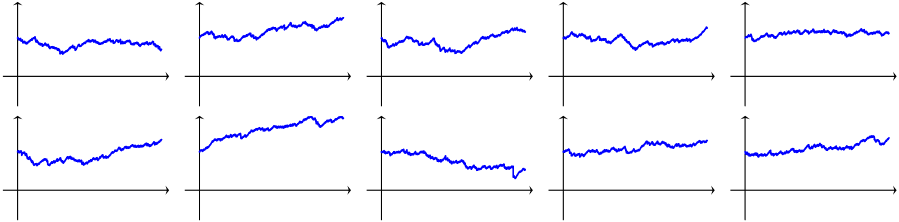

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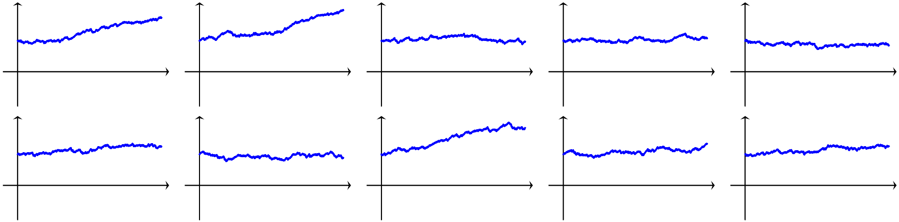

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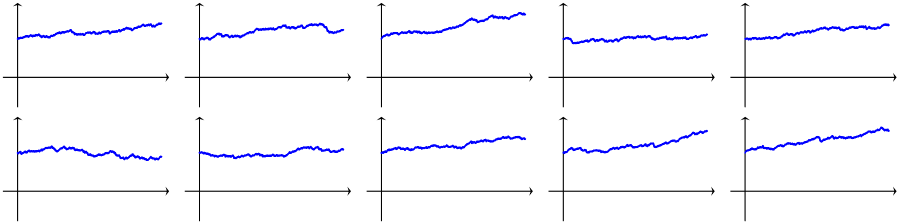

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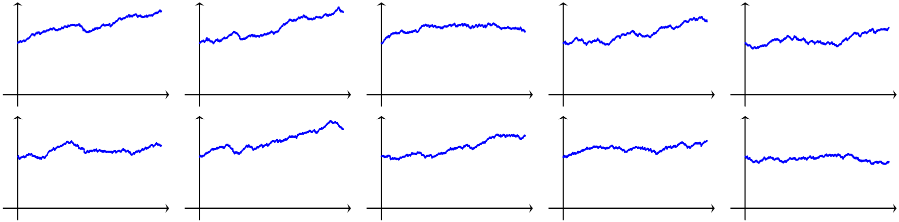

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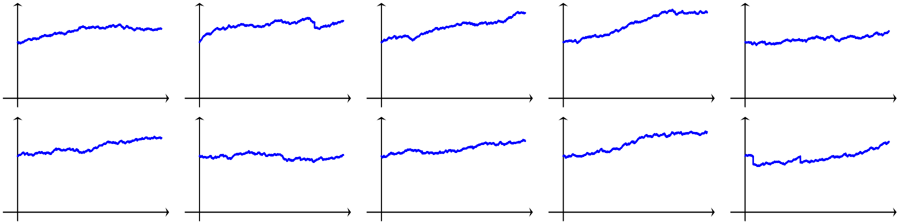

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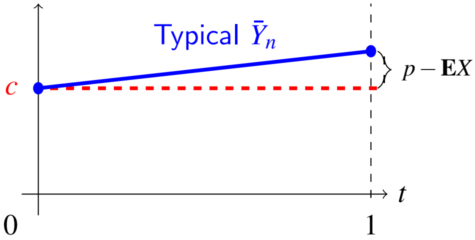

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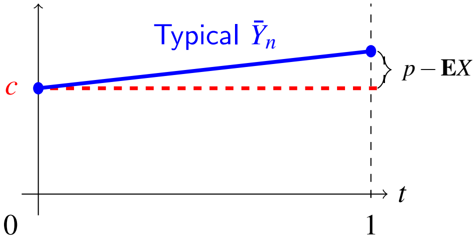


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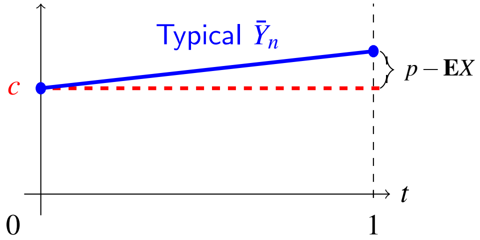
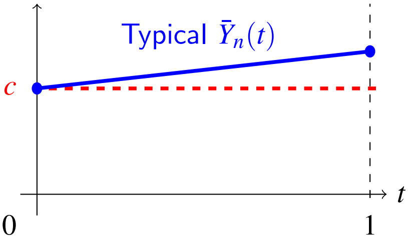



**Typically, your business will flourish**

**regardless of the tail distributions.**

# A Rare Event: Bankruptcy



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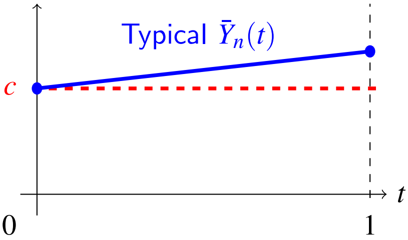

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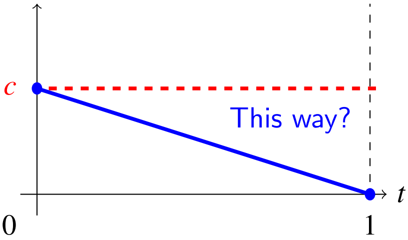

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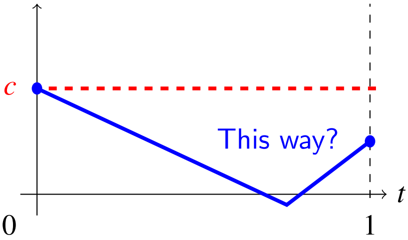

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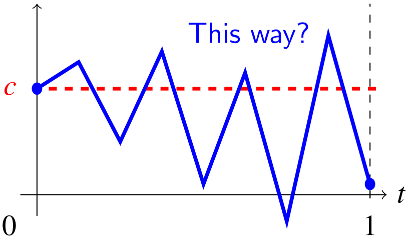

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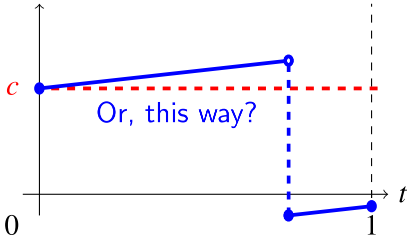

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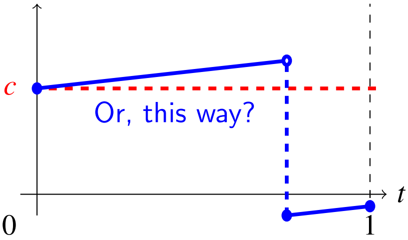


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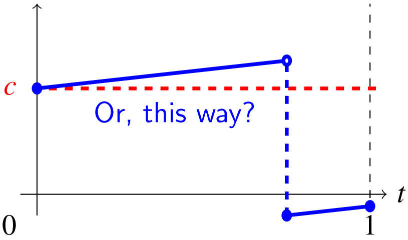


**Are we going to see clear patterns?**

**Do they depend on the tail distributions?**

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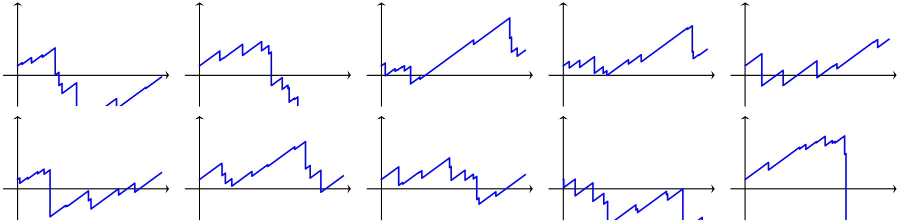
Sample paths of $\bar{Y}_{10}$ conditional on $B$ for **light-tailed** claims:



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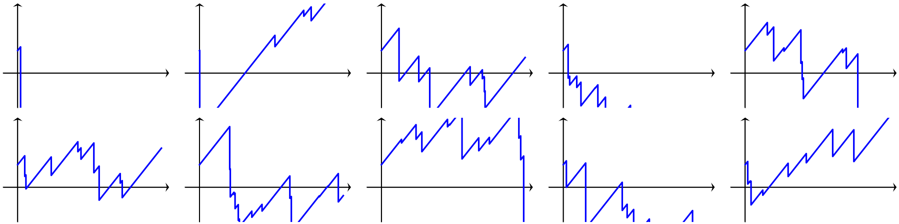

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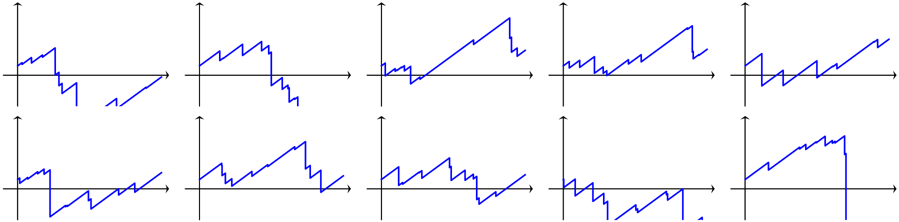

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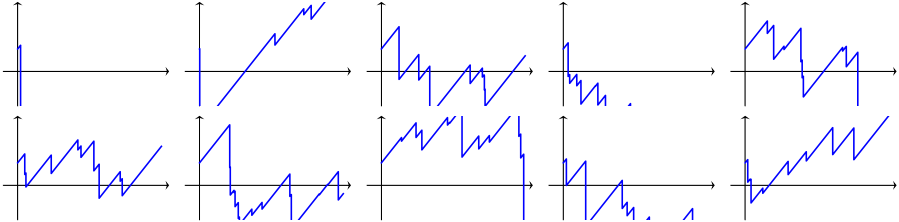

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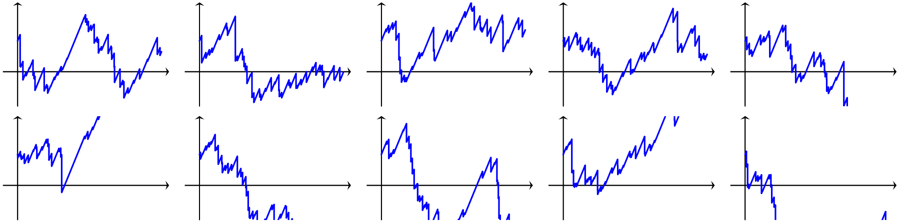

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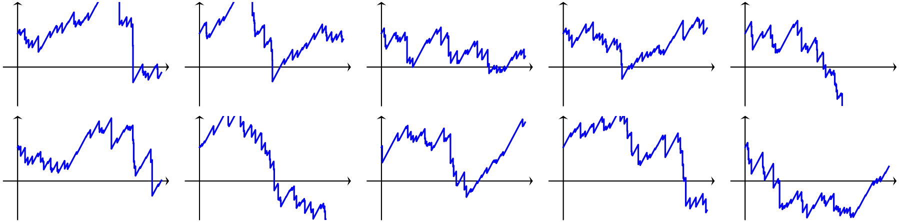

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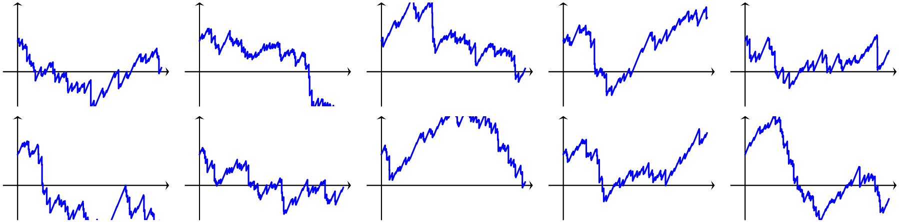

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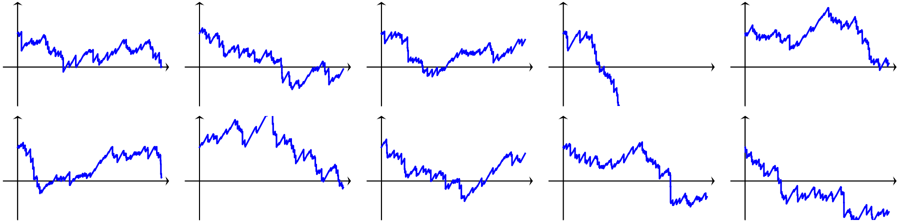

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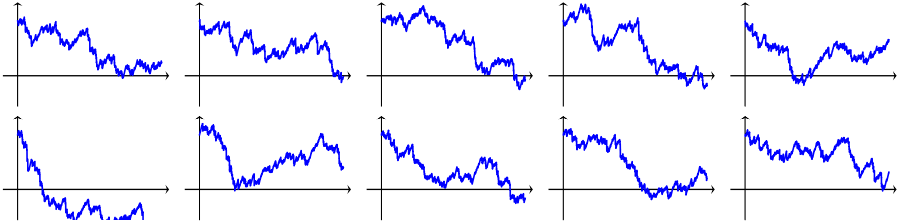

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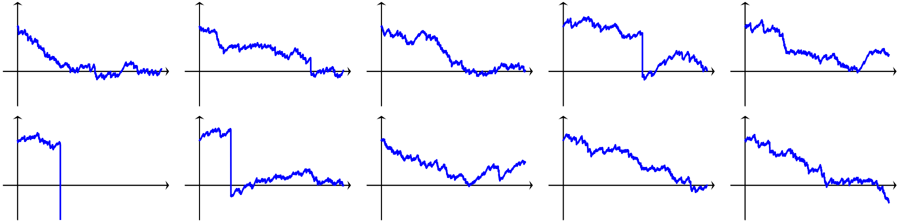

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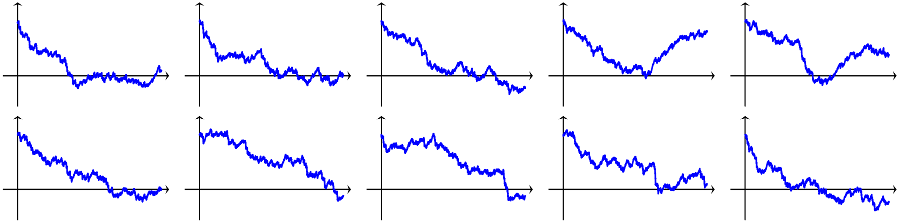

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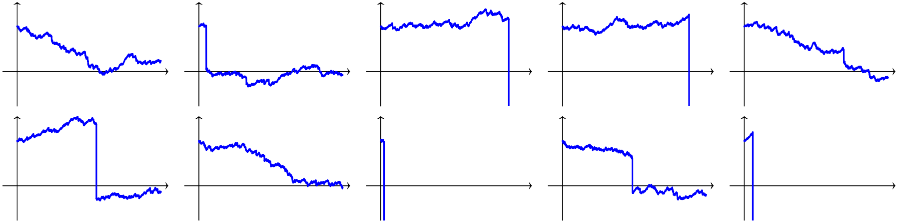

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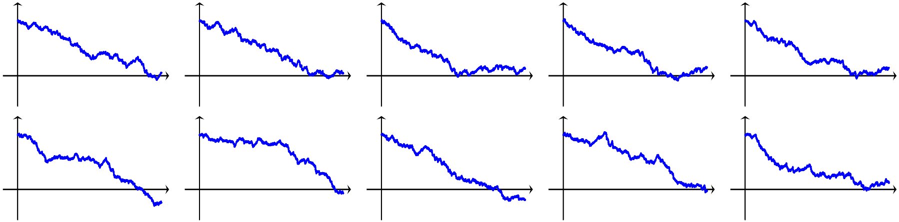

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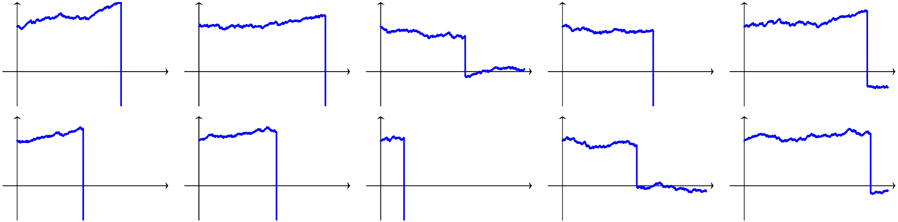

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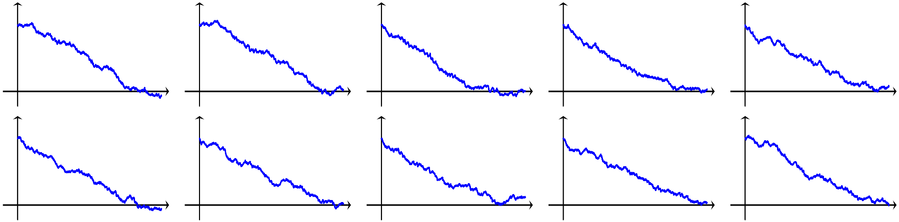

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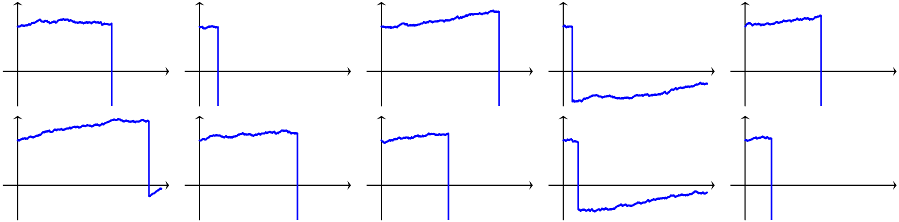
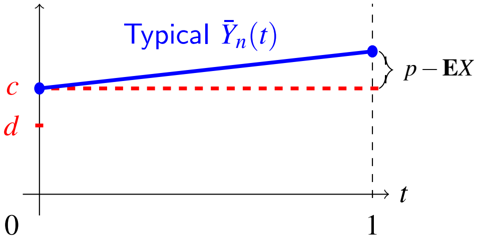


# Bankruptcy Despite of Reinsurance



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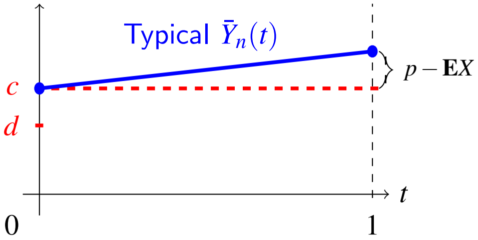

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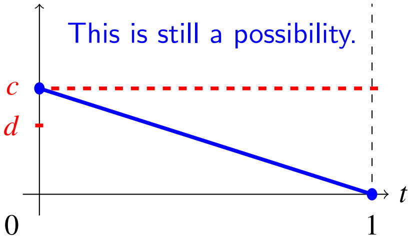

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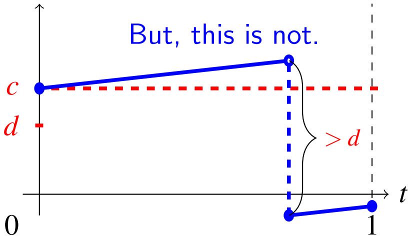
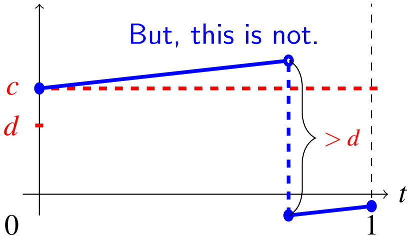


# Bankruptcy Despite of Reinsurance

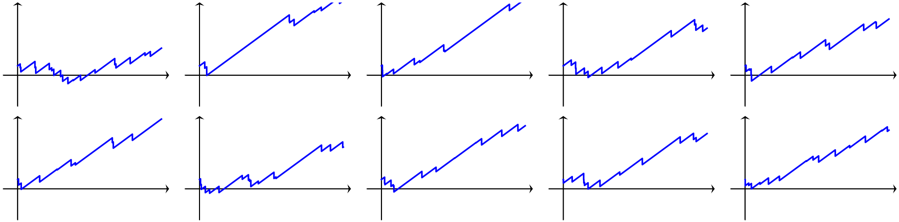
Consider $R \triangleq \{\ \bar{Y}_n$ falls below 0 on [0,1], jump sizes $\leq d\ \}$



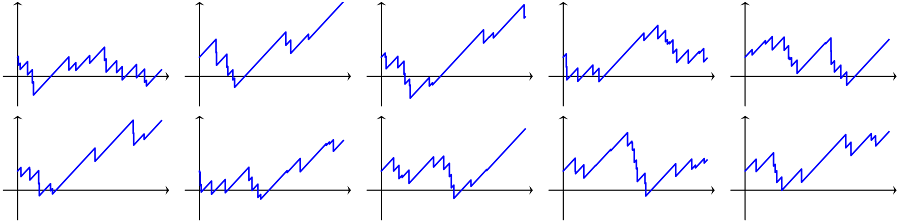

**How does the pattern change in this case?**

# Bankruptcy Despite of Reinsurance

Sample paths of $\bar{Y}_{10}$ conditional on $R$ for **light-tailed** claims:

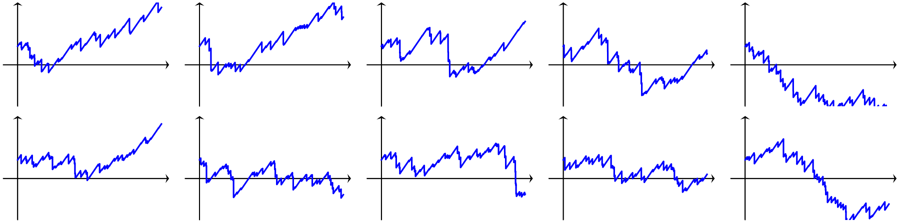


Sample paths of $\bar{Y}_{10}$ conditional on $R$ for **heavy-tailed** claims:

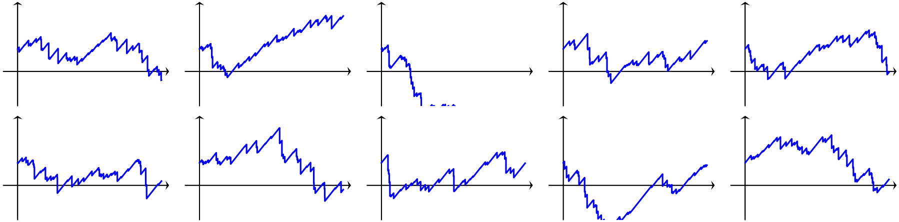


# Bankruptcy Despite of Reinsurance

Sample paths of $\bar{Y}_{50}$ conditional on $R$ for **light-tailed** claims:

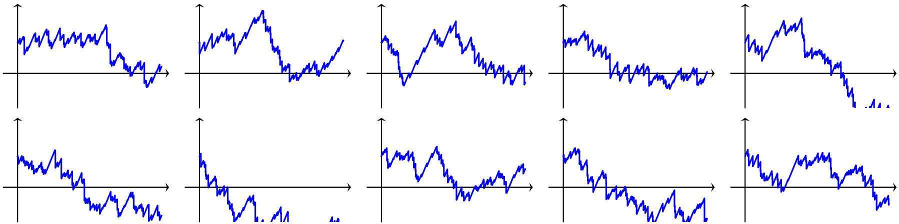


Sample paths of $\bar{Y}_{50}$ conditional on $R$ for **heavy-tailed** claims:

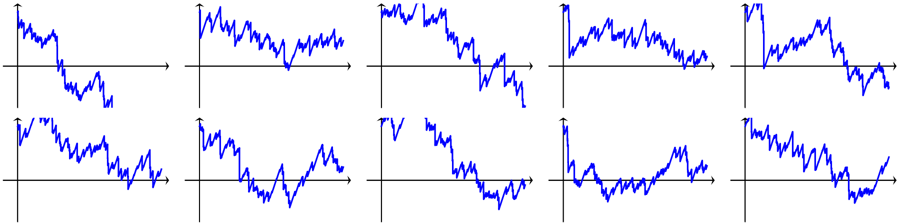


# Bankruptcy Despite of Reinsurance

Sample paths of $\bar{Y}_{100}$ conditional on $R$ for **light-tailed** claims:

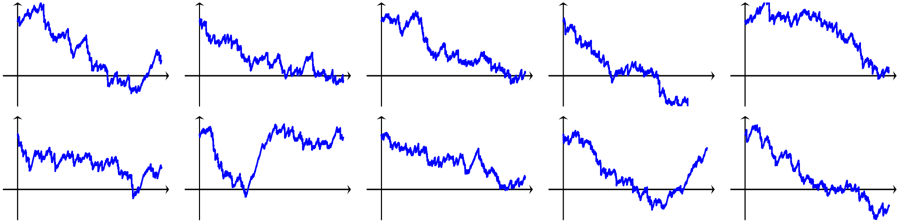


Sample paths of $\bar{Y}_{100}$ conditional on $R$ for **heavy-tailed** claims:

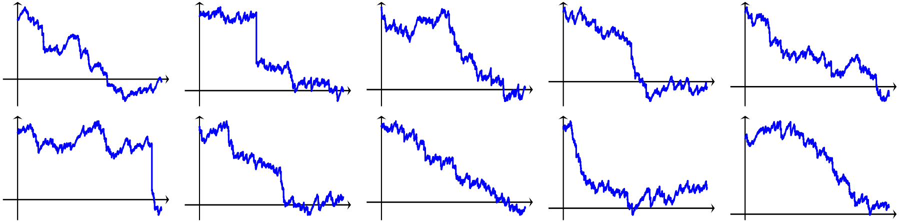


# Bankruptcy Despite of Reinsurance

Sample paths of $\bar{Y}_{500}$ conditional on $R$ for **light-tailed** claims:

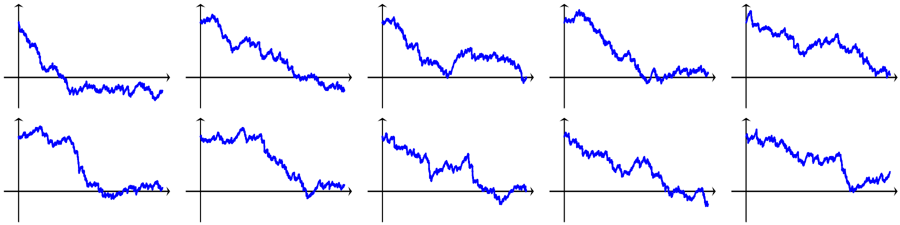


Sample paths of $\bar{Y}_{500}$ conditional on $R$ for **heavy-tailed** claims:

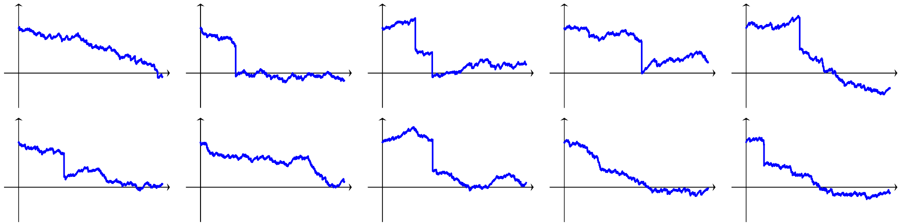


# Bankruptcy Despite of Reinsurance

Sample paths of $\bar{Y}_{1000}$ conditional on $R$ for **light-tailed** claims:

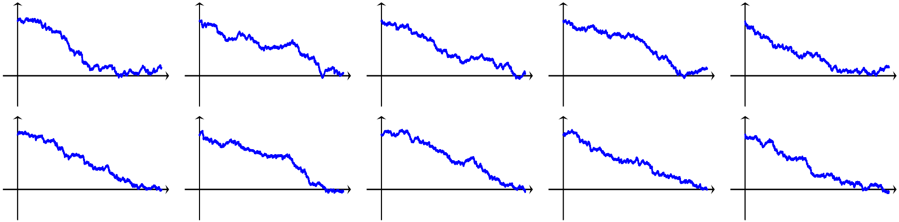


Sample paths of $\bar{Y}_{1000}$ conditional on $R$ for **heavy-tailed** claims:

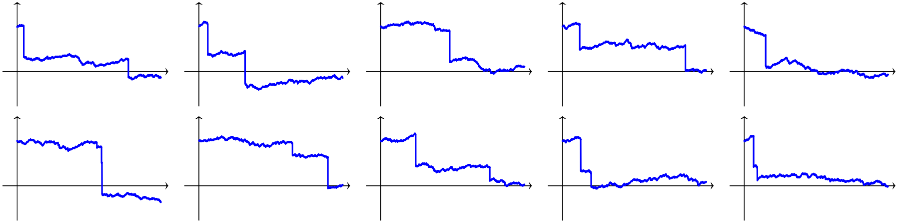


# Bankruptcy Despite of Reinsurance

Sample paths of $\bar{Y}_{2500}$ conditional on $R$ for **light-tailed** claims:

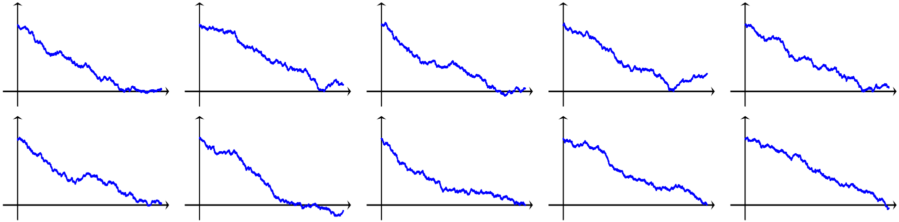


Sample paths of $\bar{Y}_{2500}$ conditional on $R$ for **heavy-tailed** claims:

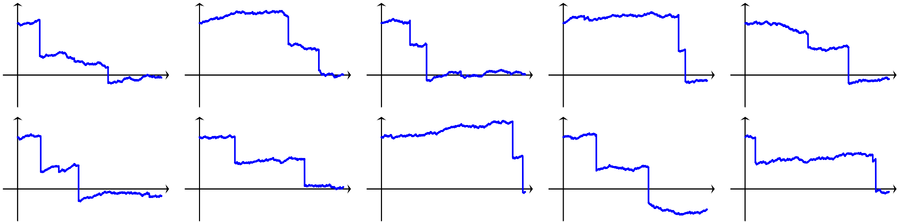


# Bankruptcy Despite of Reinsurance

Sample paths of $\bar{Y}_{5000}$ conditional on $R$ for **light-tailed** claims:

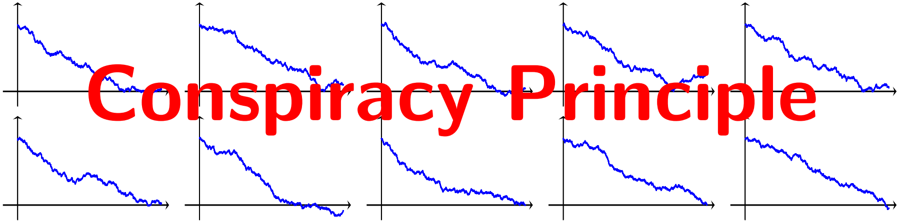


Sample paths of $\bar{Y}_{5000}$ conditional on $R$ for **heavy-tailed** claims:



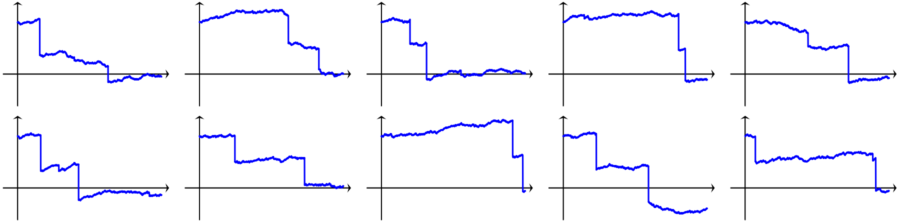
# Bankruptcy Despite of Reinsurance

Sample paths of $\bar{Y}_{5000}$ conditional on $R$ for **light-tailed** claims:



**Conspiracy Principle**

Sample paths of $\bar{Y}_{5000}$ conditional on $R$ for **heavy-tailed** claims:



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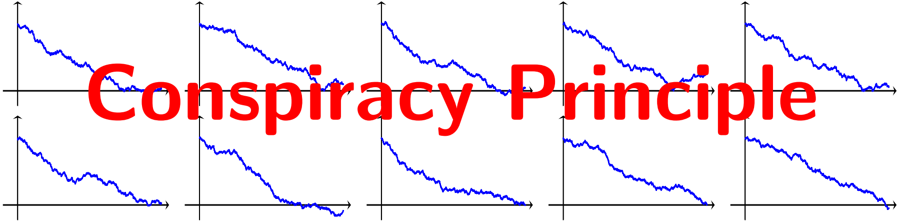
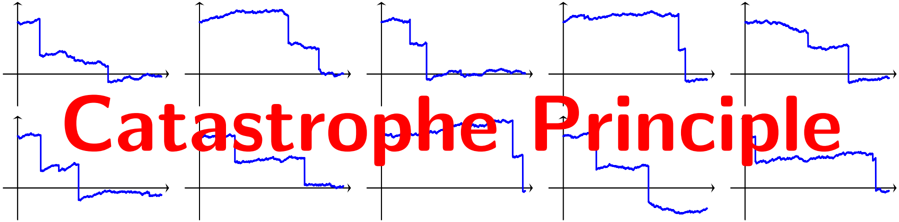
Sample paths of $\bar{Y}_{5000}$ conditional on $R$ for **light-tailed** claims:




Sample paths of $\bar{Y}_{5000}$ conditional on $R$ for **heavy-tailed** claims:



# Heavy-Tailed Large Deviations

$$\bar{S}_n(t) = \frac{1}{n}\sum_{i=1}^{\lfloor nt \rfloor} X_i,$$

$X_i$: centered iid r.v. with $\mathbf{P}(X_i \geq x) = x^{-(\alpha+1)}$, $\alpha > 0$

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**Theorem (R., Blanchet, Zwart, 2019)**

*For "general" $A \subseteq \mathbb{D}$*

$$C(A^\circ) \leq \liminf_{n\to\infty} \frac{\mathbf{P}(\bar{S}_n \in A)}{n^{-\alpha \mathcal{J}(A)}} \leq \limsup_{n\to\infty} \frac{\mathbf{P}(\bar{S}_n \in A)}{n^{-\alpha \mathcal{J}(A)}} \leq C(A^-).$$

- $\mathcal{J}(A)$: *min #jumps for step functions to be inside* $A$
- $C(\cdot)$: *a measure*

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**Theorem (R., Blanchet, Zwart, 2019)**

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$$\mathbf{P}(\bar{S}_n \in A) \sim n^{-\alpha \mathcal{J}(A)}$$

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**Theorem (R., Blanchet, Zwart, 2019)**

*For "general" $A \subseteq \mathbb{D}$*

LD power index

$$\mathbf{P}(\bar{S}_n \in A) \sim n^{-\alpha \mathcal{J}(A)}$$

- $\mathcal{J}(A)$: *min #jumps for step functions to be inside* $A$

# Rigorous Characterization of the Catastrophe Principle

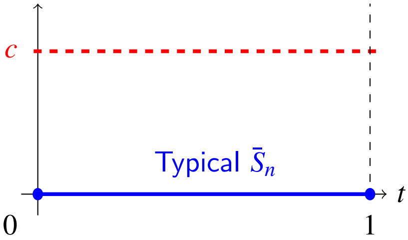
Under certain regularity conditions on $A$,

$$\mathbf{P}(\bar{S}_n \in \cdot | \bar{S}_n \in A) \to \mathbf{P}(\bar{S}_{|A} \in \cdot) = \frac{C_{\mathcal{J}(A)}(\cdot \cap A)}{C_{\mathcal{J}(A)}(A)}$$

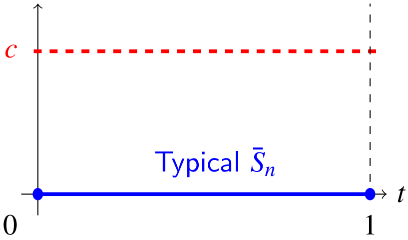
$\bar{S}_{|A}$: a (random) piecewise-constant function with $\mathcal{J}(A)$ jumps.

# Conspiracy vs Catastrophe

$$\bar{S}_n \in \{\, f \in \mathbb{D} :\ f \text{ crosses level } c \text{ on } [0,1] \,\} = A$$




**Light-Tailed Claim Size**

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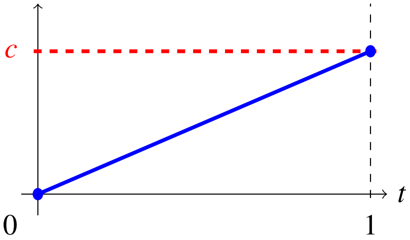


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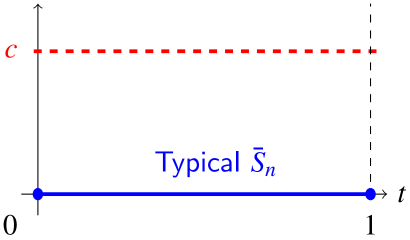


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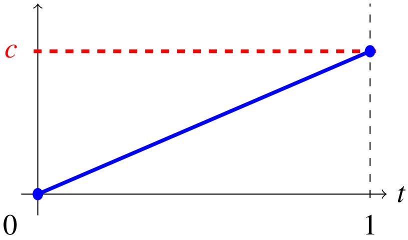


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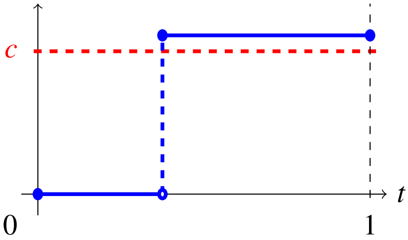


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# Conspiracy vs Catastrophe

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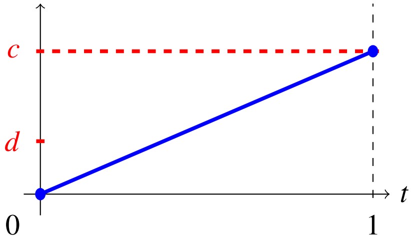


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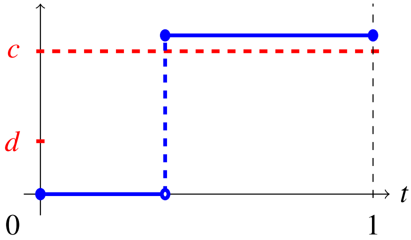


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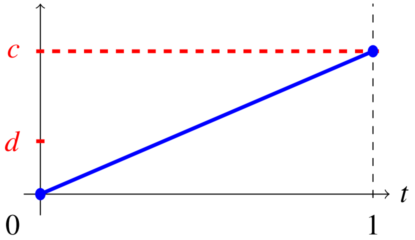


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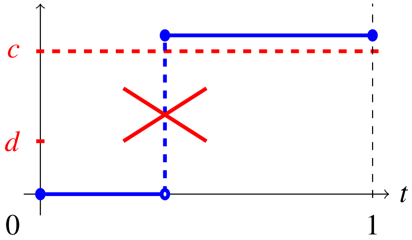


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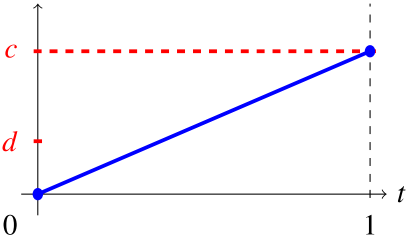


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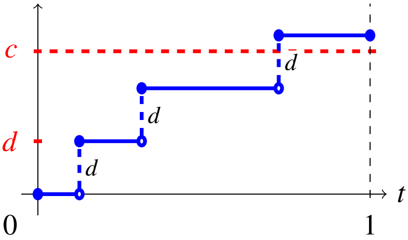


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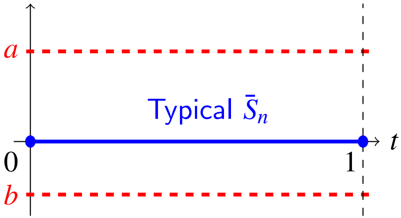
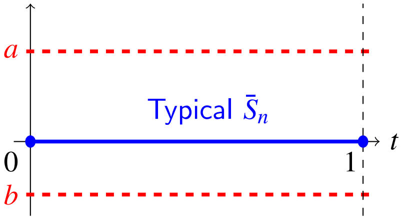
# Conspiracy vs Catastrophe

$$\bar{S}_n \in \{\, f \in \mathbb{D} : \ f \text{ hits below } b \text{ on } [0,1] \text{ and ends up above } a \,\} = A$$




**Light-Tailed Increments**




**Heavy-Tailed Increments**

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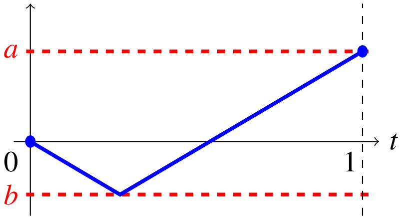


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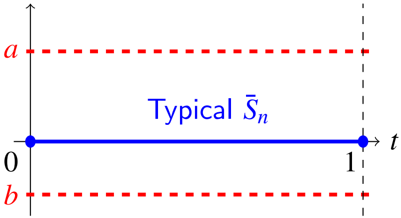
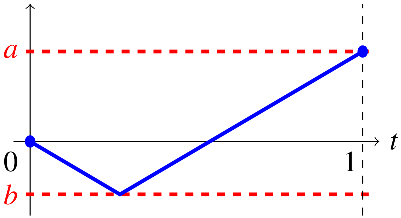


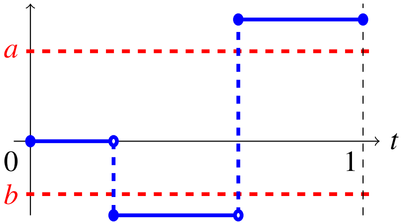

**Heavy-Tailed Increments**

## Conspiracy vs Catastrophe

$\bar{S}_n \in \{ f \in \mathbb{D} : f$ hits below $b$ on $[0,1]$ and ends up above $a \} = A$



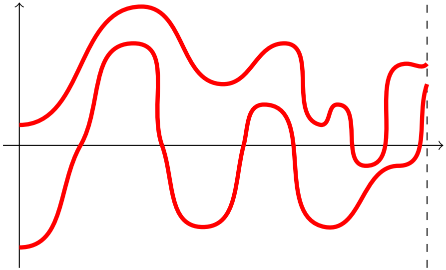

**Light-Tailed Increments**




**Heavy-Tailed Increments**

# Conspiracy vs Catastrophe

$$\bar{S}_n \in \{\, f \in \mathbb{D} : \ f \text{ lies between the two red curves} \,\}$$



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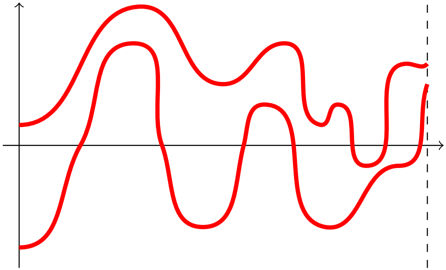

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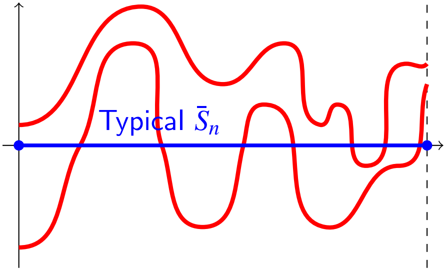


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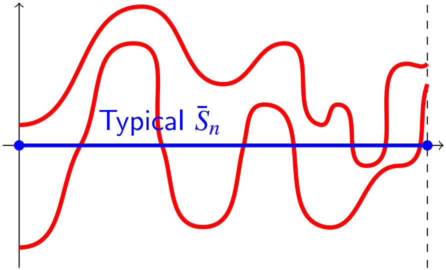


**Heavy-Tailed Increments**

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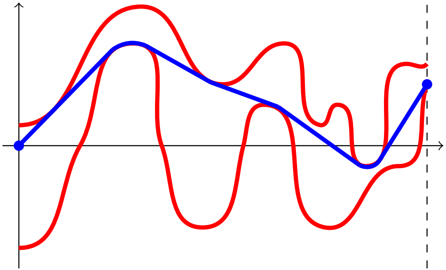
$\bar{S}_n \in \{\, f \in \mathbb{D} : \ f$ lies between the two red curves $\}$



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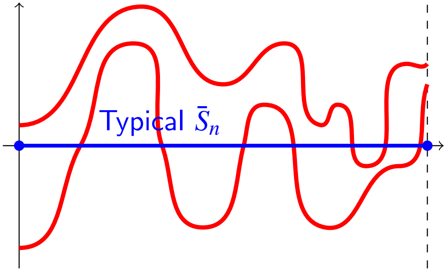


**Heavy-Tailed Increments**

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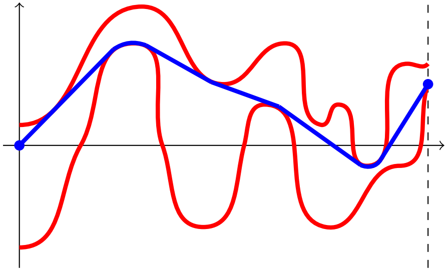
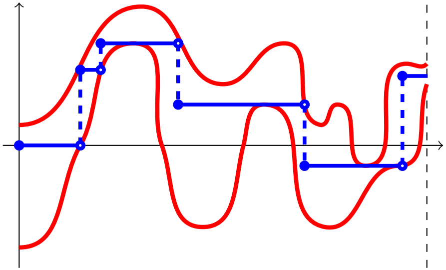
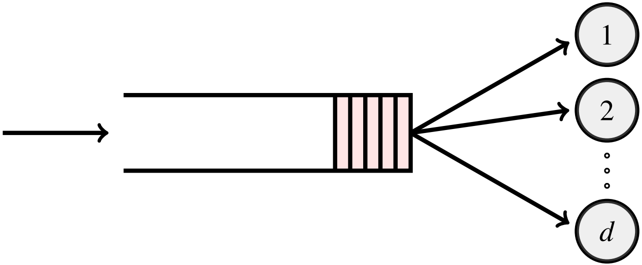


**Light-Tailed Increments**



**Heavy-Tailed Increments**

Congestion of Multiple Server Queue:



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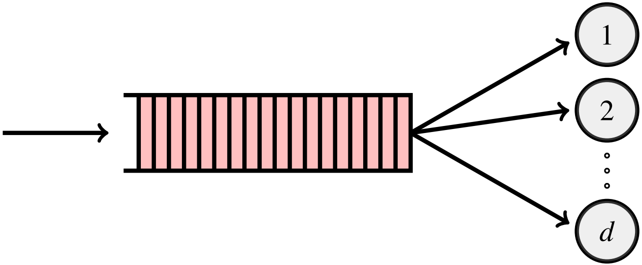
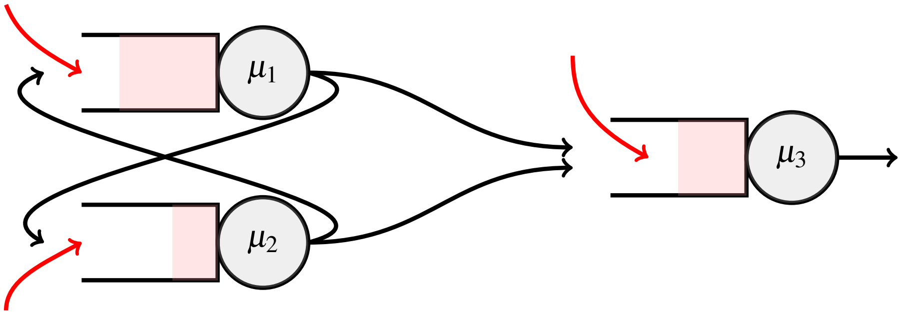


Congestion of Stochastic Fluid Networks:



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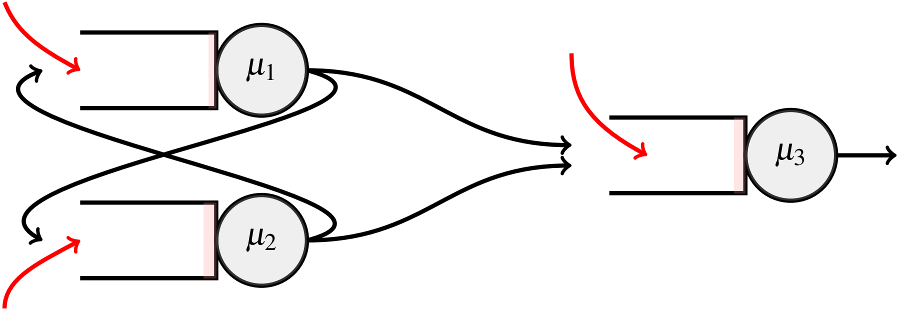

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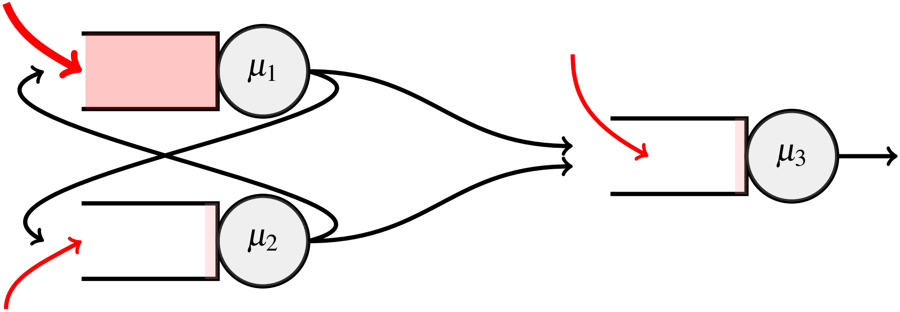

# Conspiracy vs Catastrophe

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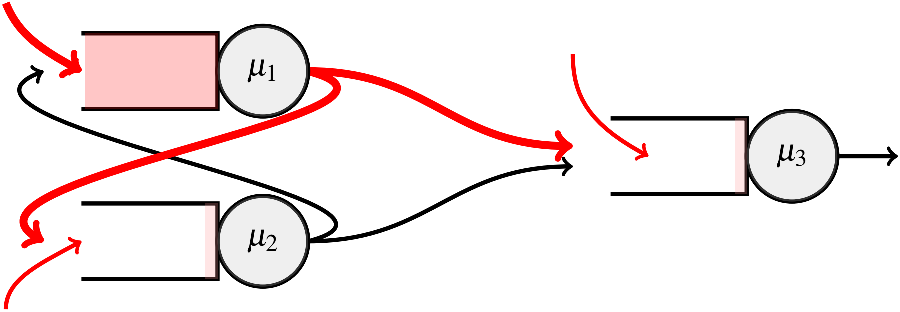

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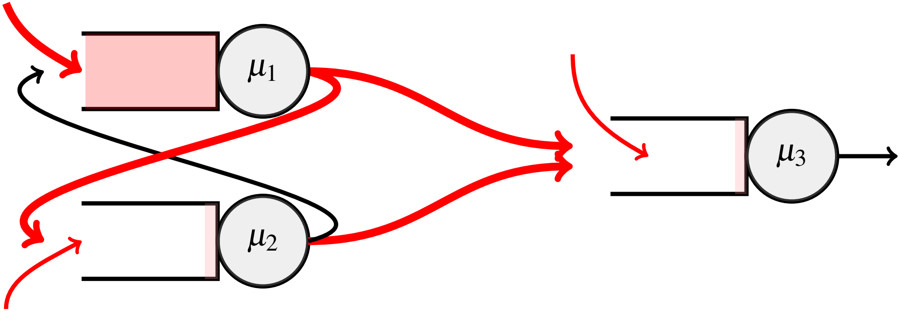

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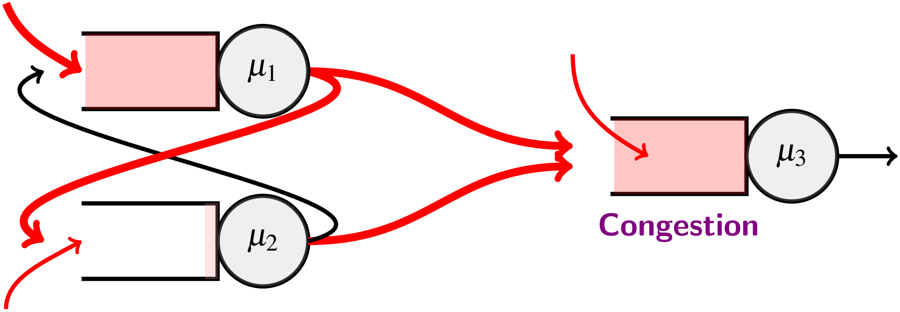

# Lecture 2: Mathematical Machinery for Catastrophe Principle

- $\mathbb{M}(\mathbb{S} \setminus \mathbb{C})$-convergence
  - Formulation
  - Continuous Mapping Theorem
  - Asymptotic Equivalence
  - Product of $\mathbb{M}(\mathbb{S} \setminus \mathbb{C})$-convergence
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# Lecture 2: Mathematical Machinery for Catastrophe Principle

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# Stochastic Gradient Descent

Minimizing loss function $f$:

$$W_{k+1} = W_k - \eta \left( f'(W_k) \right) \qquad k = 0, 1, 2, \ldots$$

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Gradient Flow

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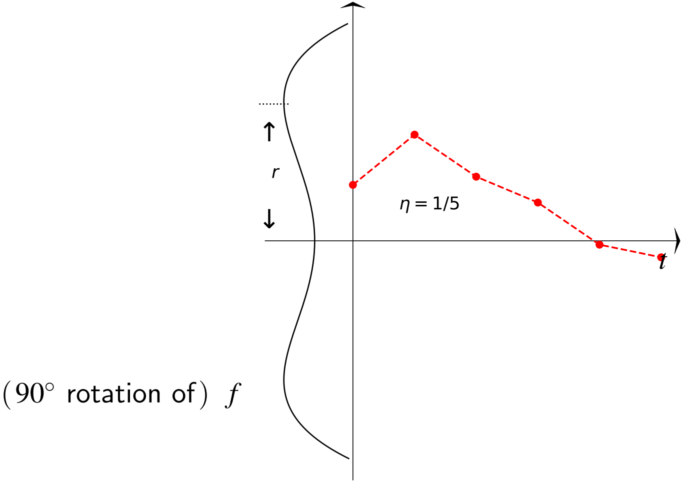
# Gradient Flow: Law of Large Numbers



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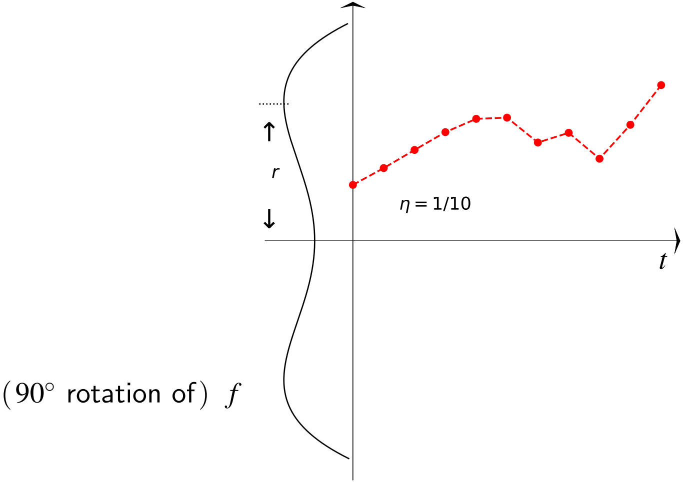

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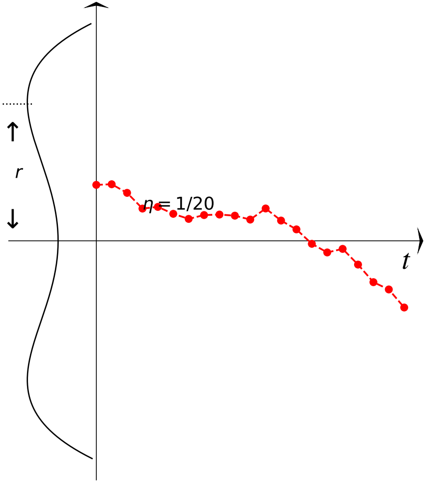


$(90^\circ$ rotation of$)$ $f$

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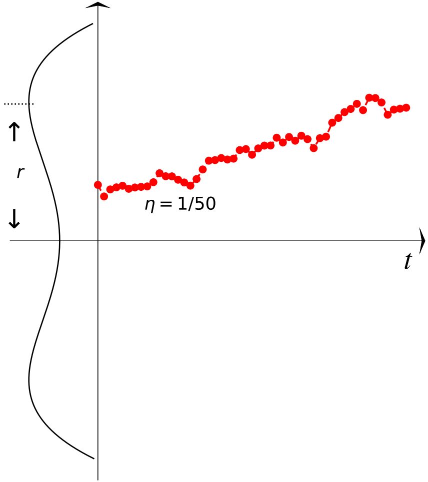


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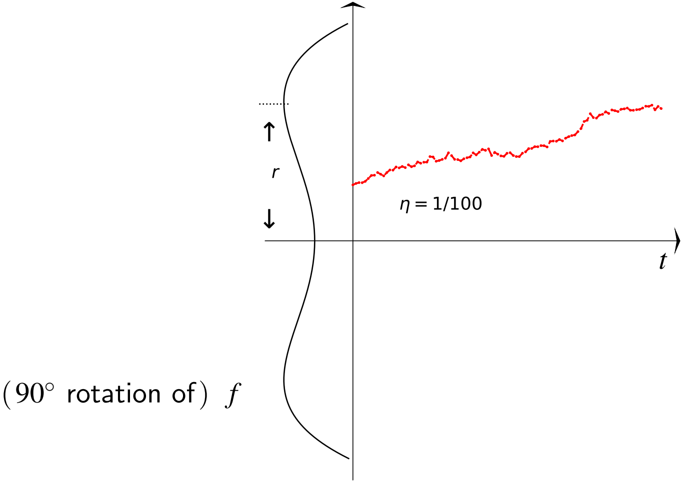

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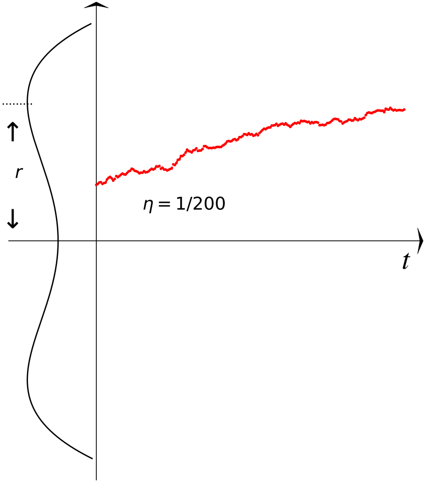


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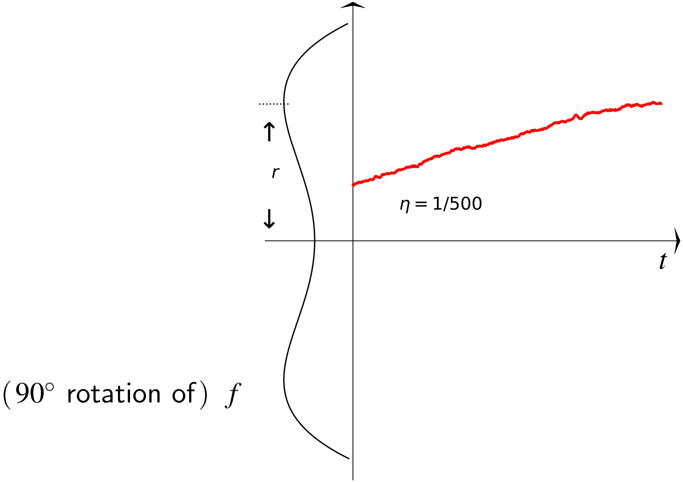

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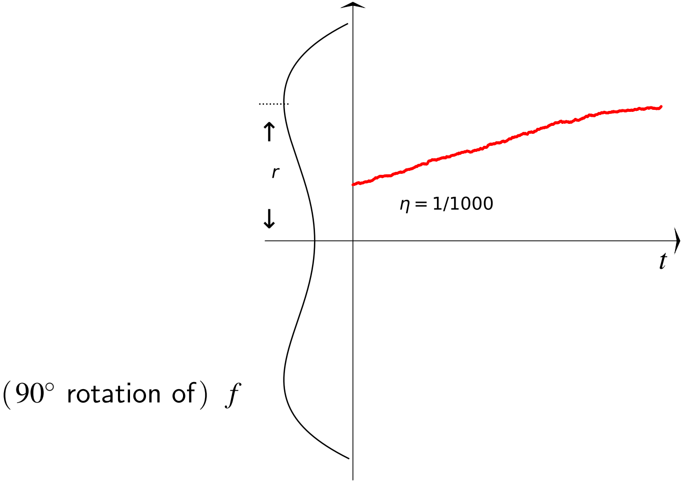

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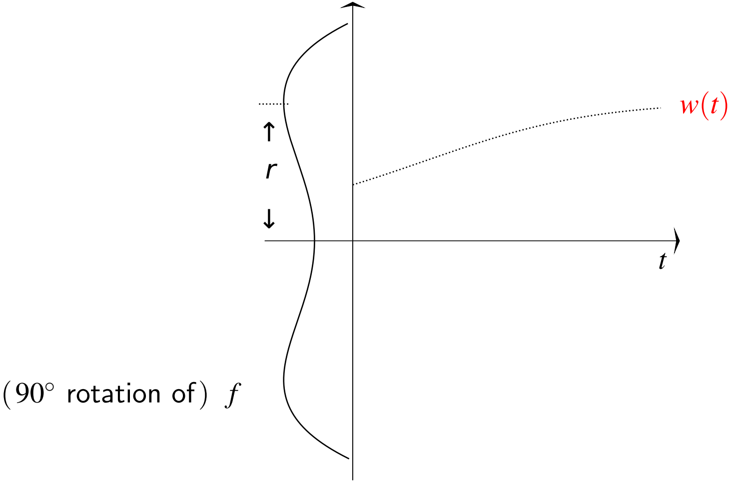

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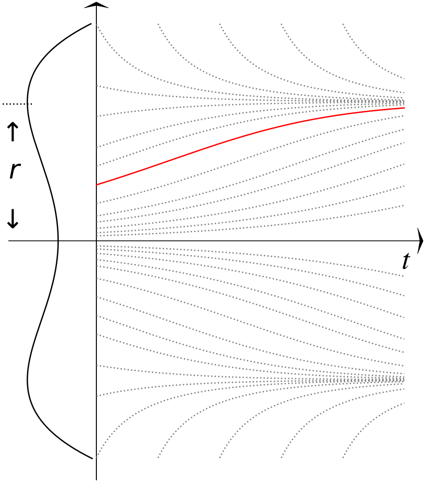


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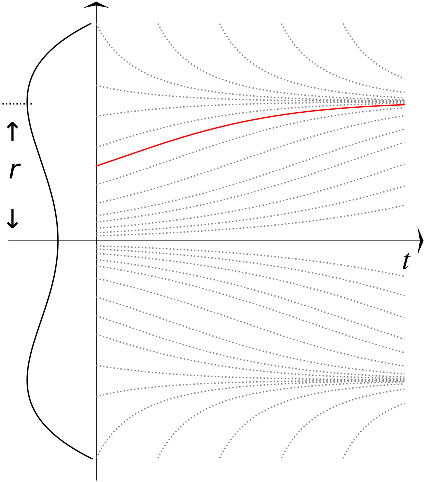


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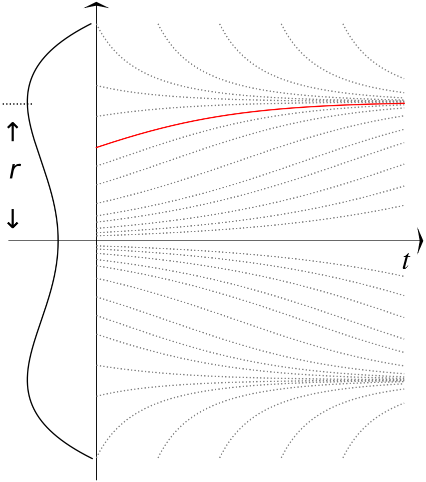


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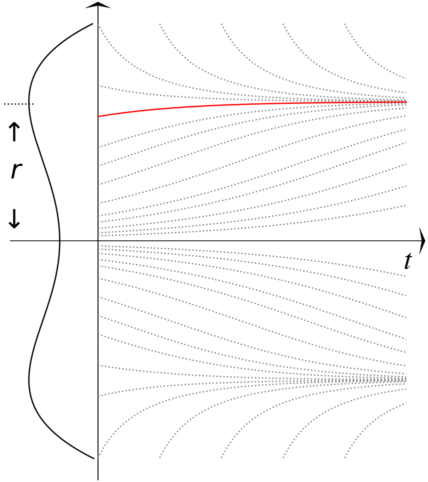


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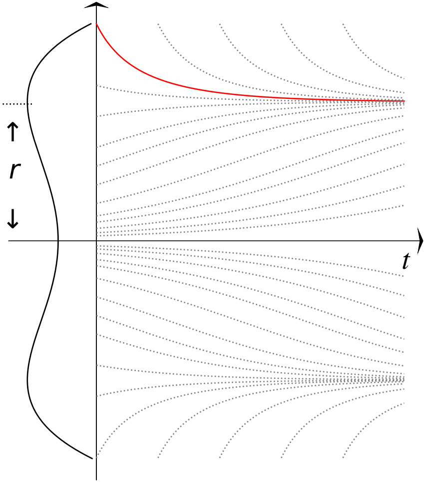


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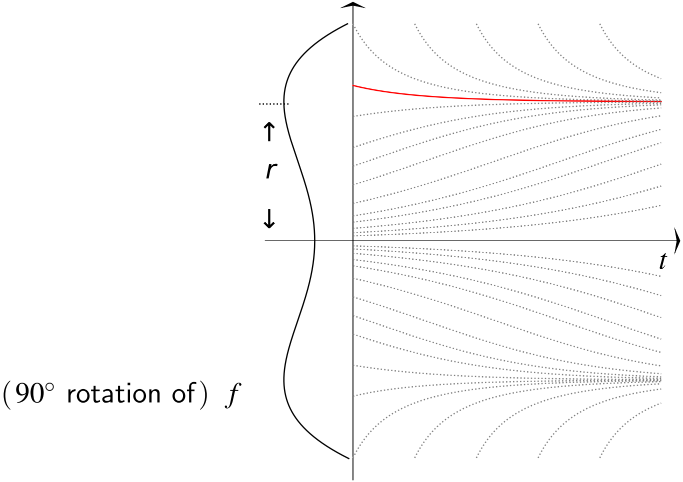

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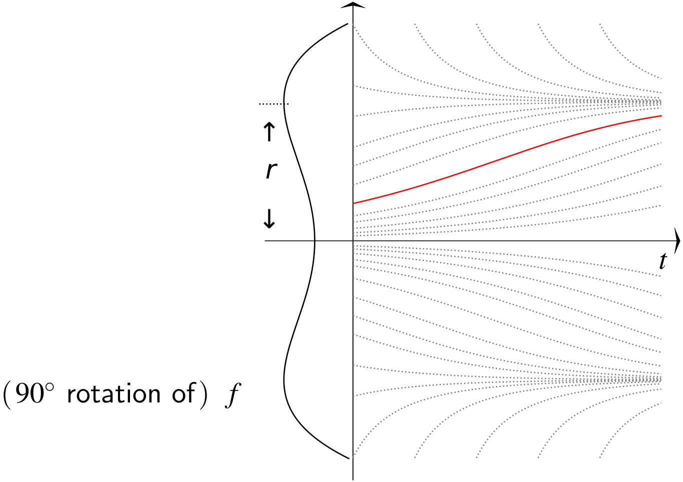

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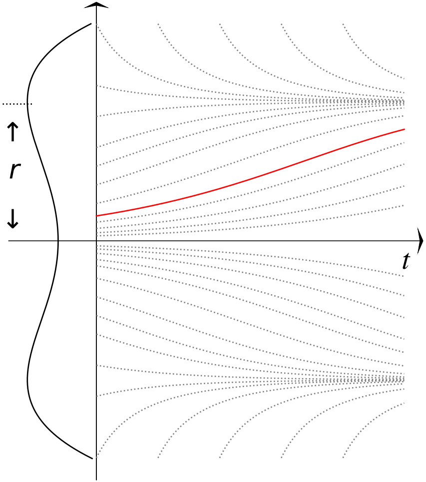


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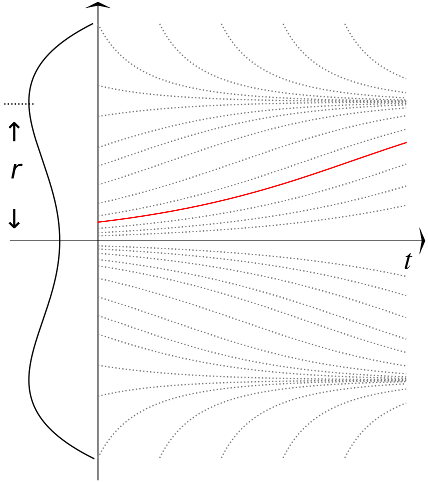


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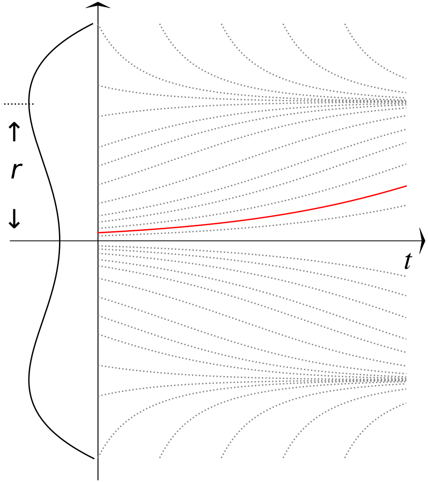


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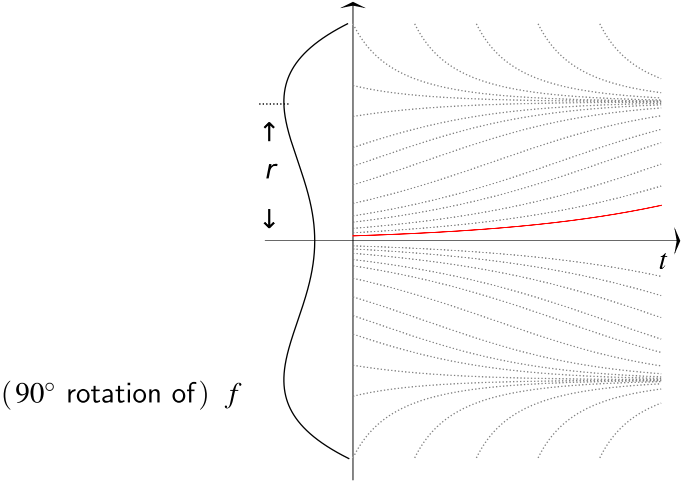

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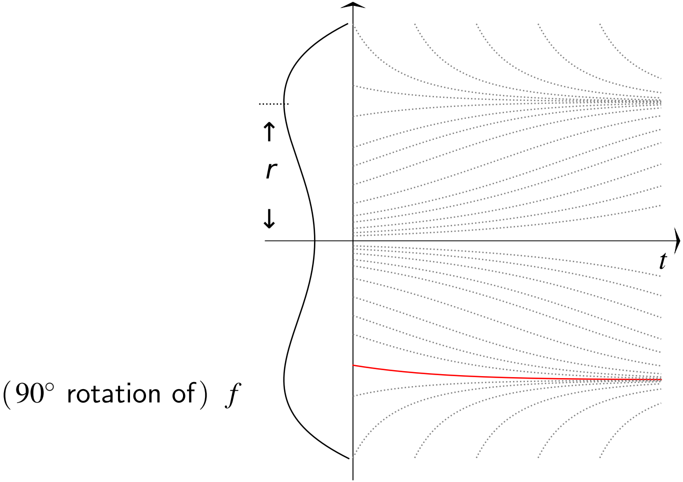

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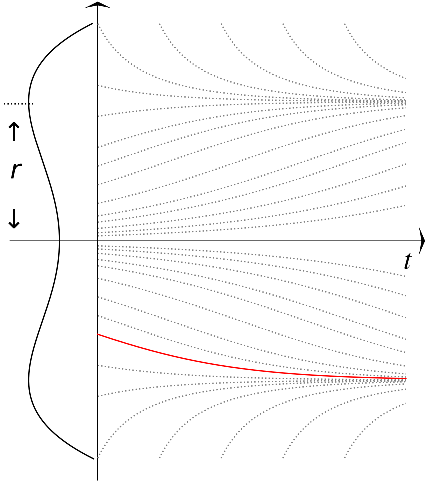


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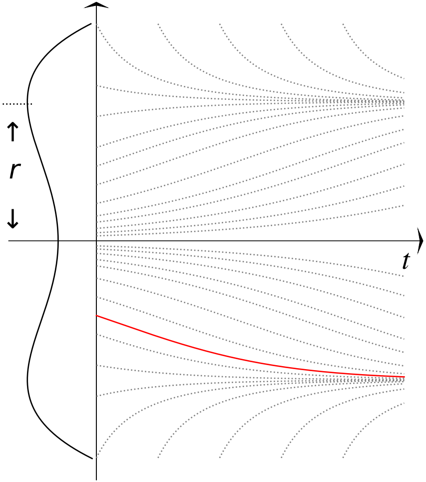


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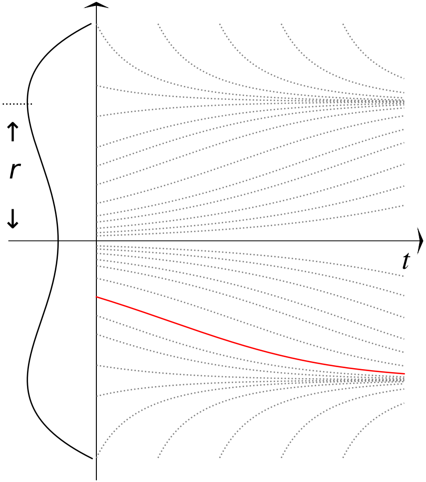


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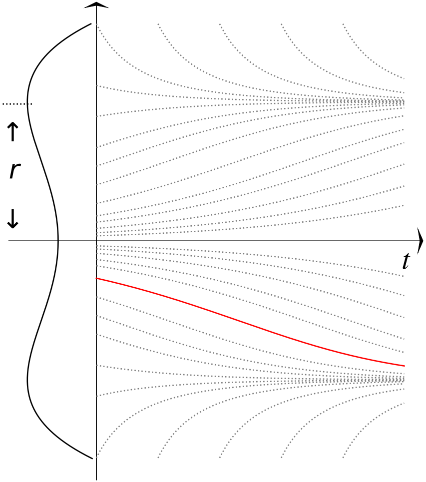


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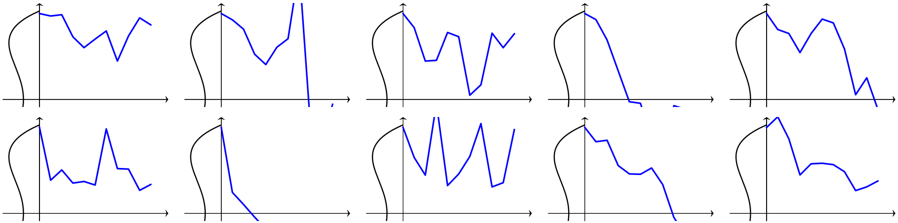
# Typical Scenario

Trajectory of SGD $W^{\eta}$: $\eta = 1/10$ & noises are **light-tailed**



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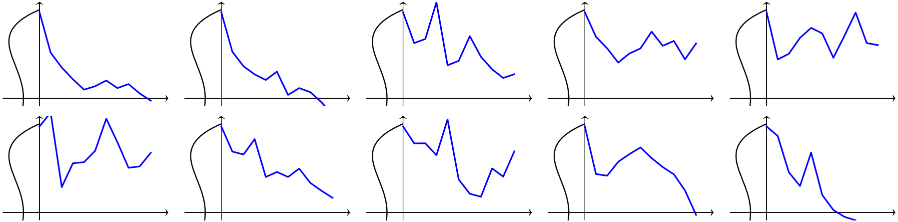

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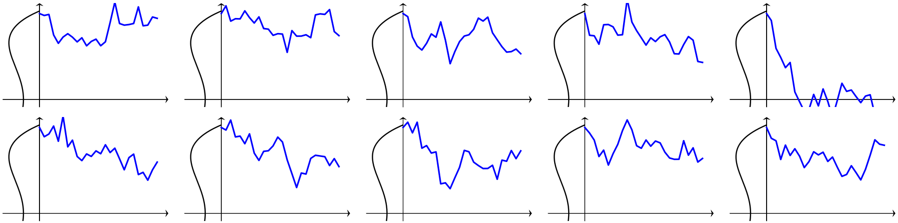
Trajectory of SGD $W^\eta$: $\eta = 1/25$ & noises are **light-tailed**



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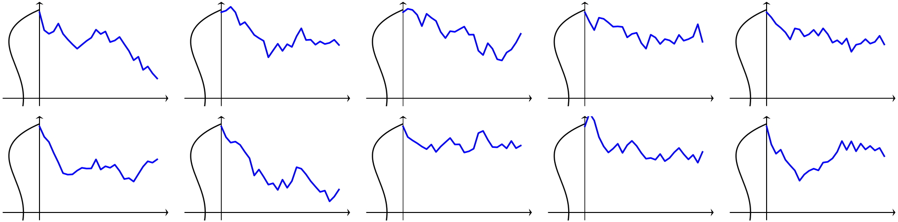

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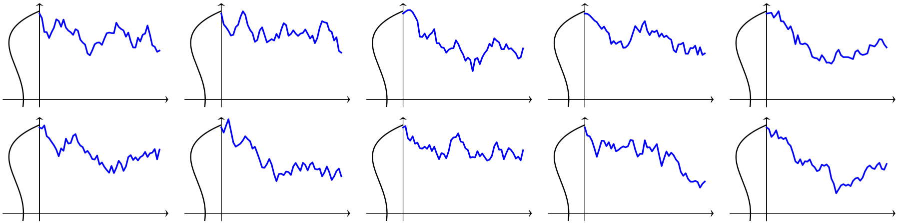
Trajectory of SGD $W^{\eta}$: $\eta = 1/50$ & noises are **light-tailed**



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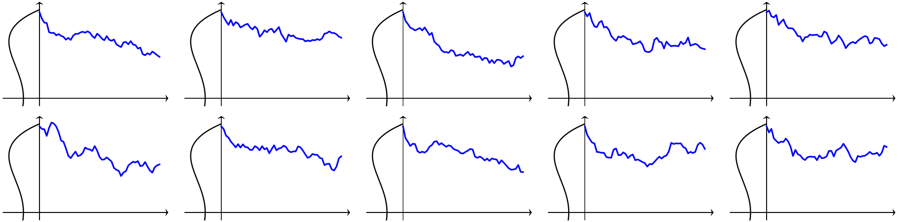

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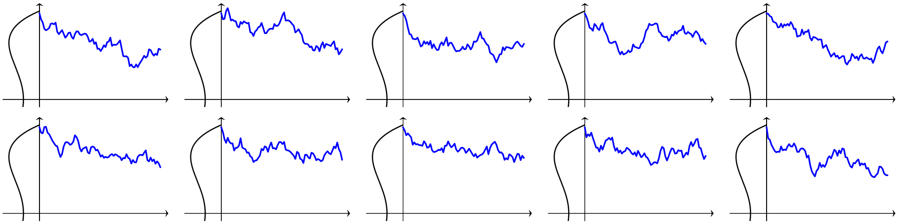
Trajectory of SGD $W^\eta$: $\eta = 1/75$ & noises are **light-tailed**



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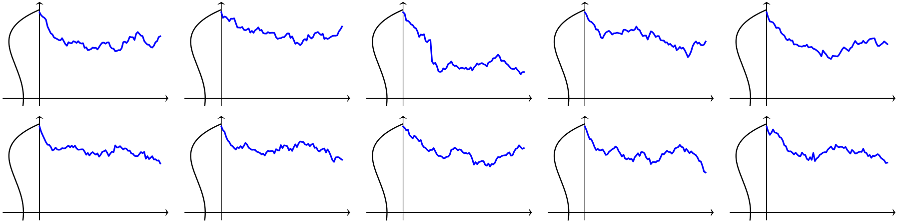

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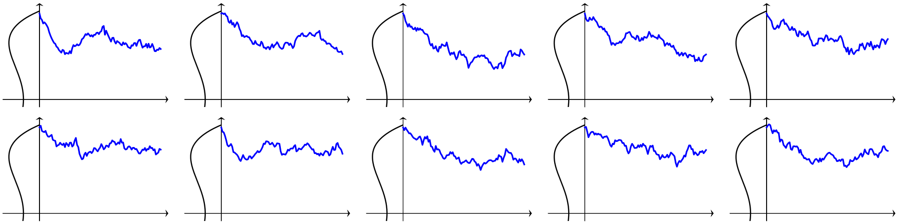
Trajectory of SGD $W^{\eta}$: $\eta = 1/100$ & noises are **light-tailed**



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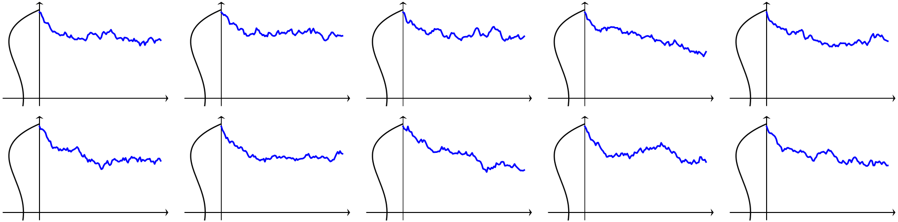

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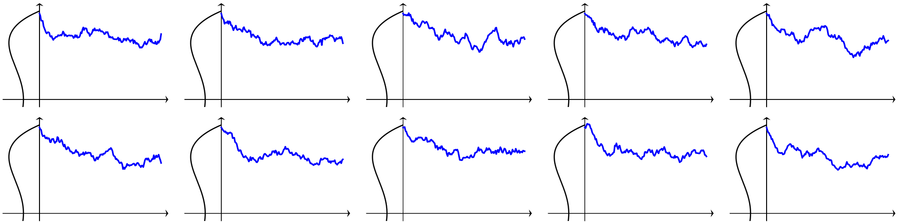
Trajectory of SGD $W^{\eta}$: $\eta = 1/150$ & noises are **light-tailed**



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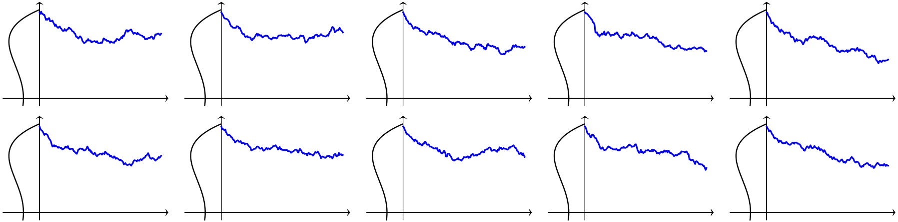

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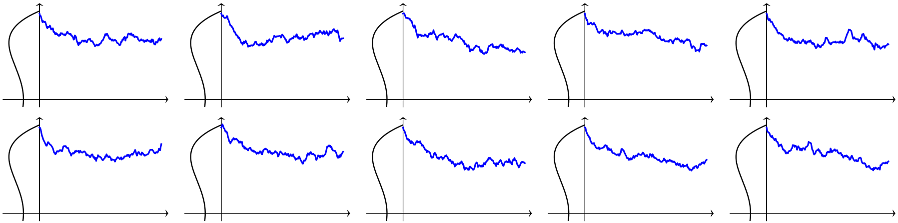
Trajectory of SGD $W^\eta$: $\eta = 1/200$ & noises are **light-tailed**



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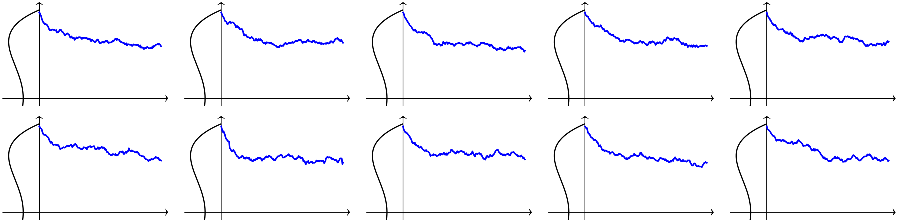

# Heavy-Tailed Large Deviations for SGD

**Theorem (Wang, Su, R., 2022+)**

*For "general" $A \subseteq \mathbb{D}$*

$$C(A^\circ) \leq \liminf_{\eta \to 0} \frac{\mathbf{P}(W^\eta \in A)}{\eta^{\alpha \mathcal{J}(A)}} \leq \limsup_{\eta \to 0} \frac{\mathbf{P}(W^\eta \in A)}{\eta^{\alpha \mathcal{J}(A)}} \leq C(A^-).$$

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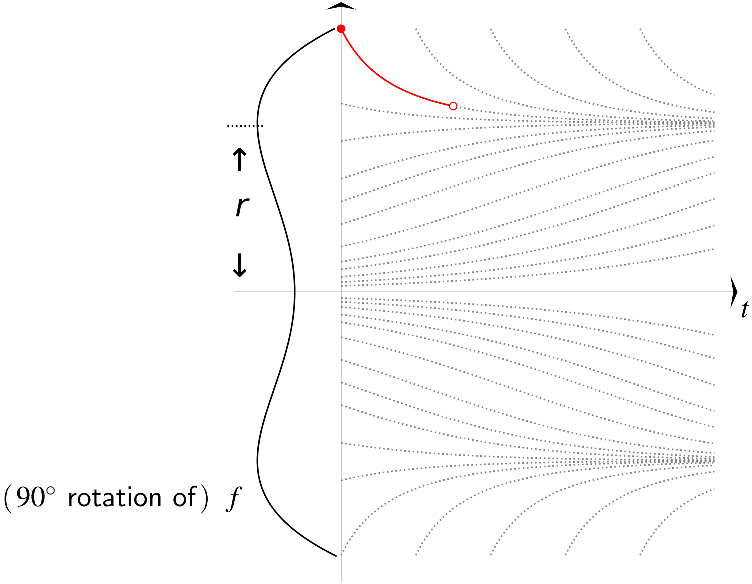
# Adding Jumps to $w(\cdot)$: Piecewise Gradient Flow



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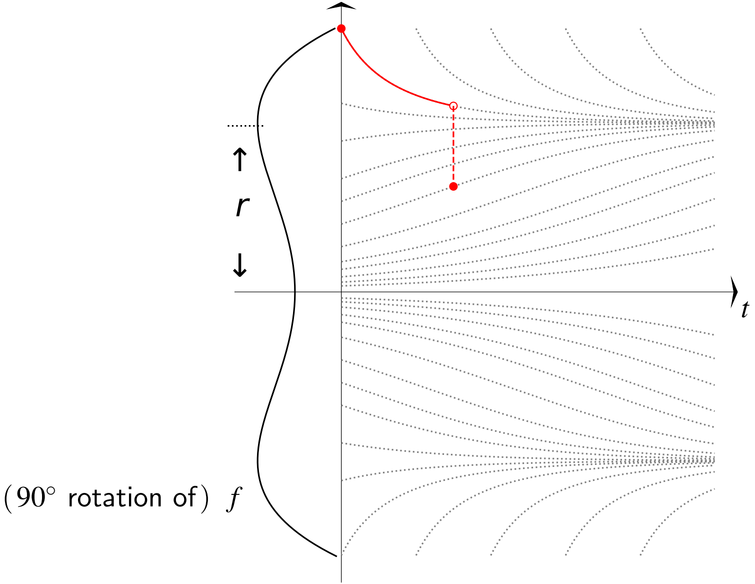

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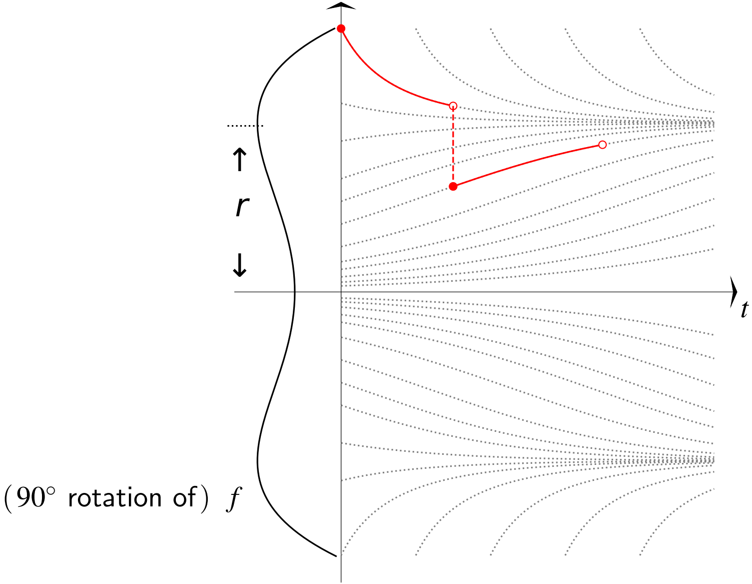

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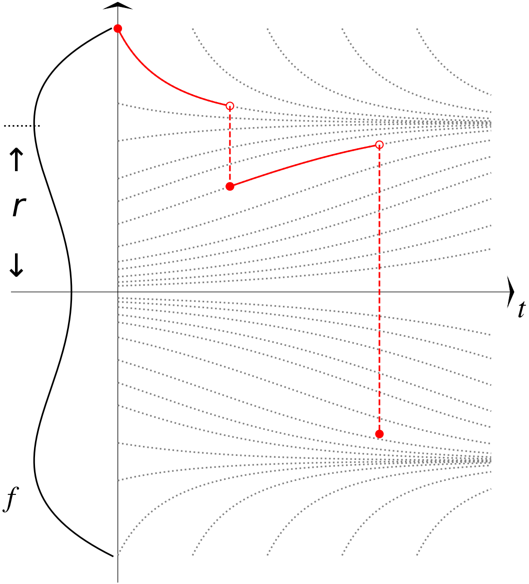


$(90^\circ$ rotation of$)$ $f$

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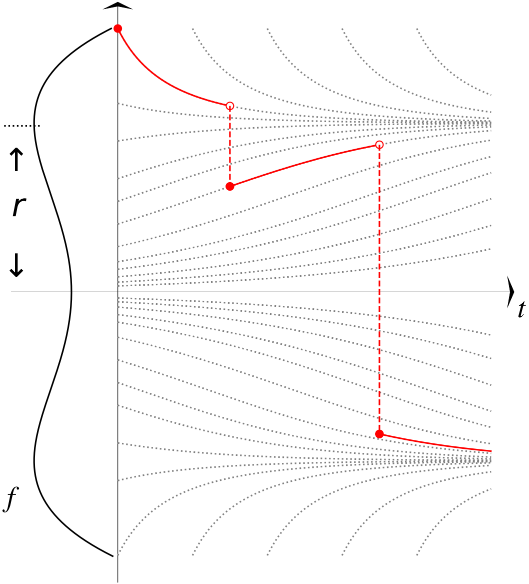


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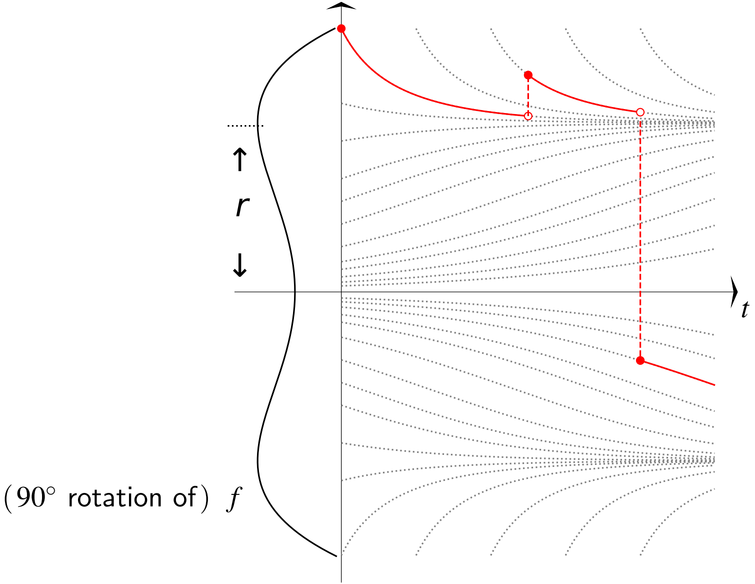

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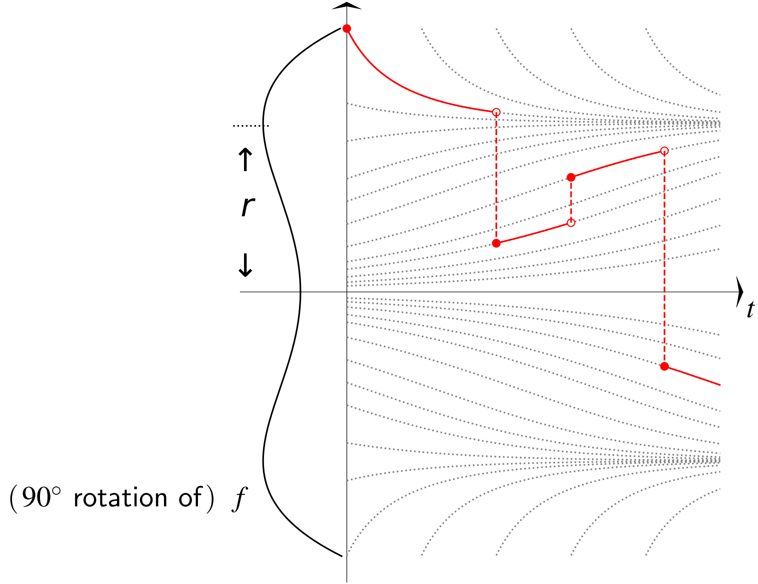

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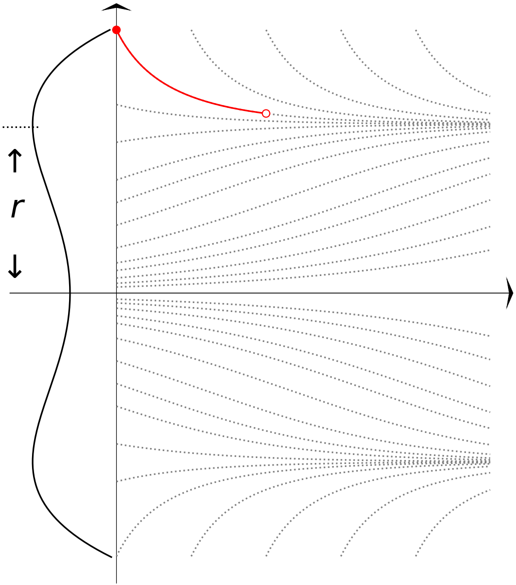
# Catastrophe Principle: Most Likely Escape Route

Most Likely Paths for $\{\, W^{\eta}$ escapes the local minimum $\}$



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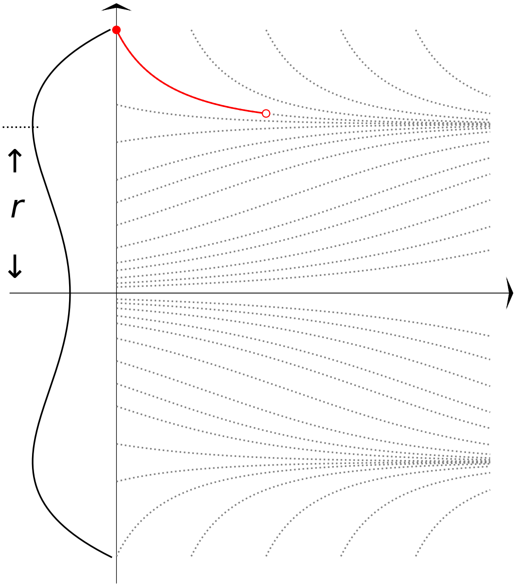


$\mathcal{J}(A) = 1$

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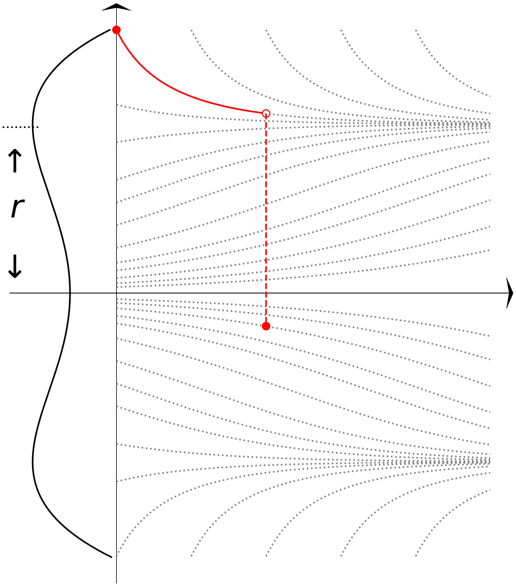


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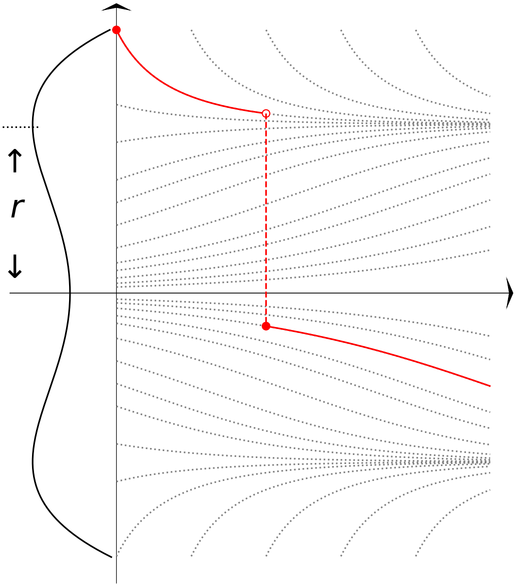


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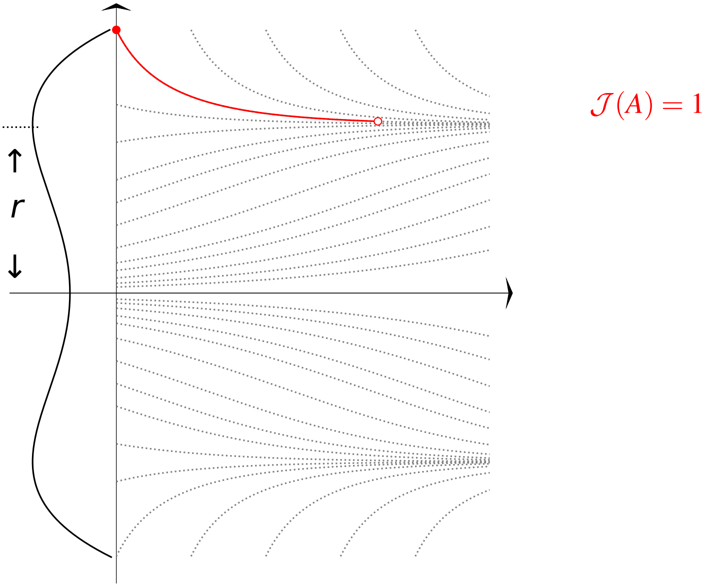

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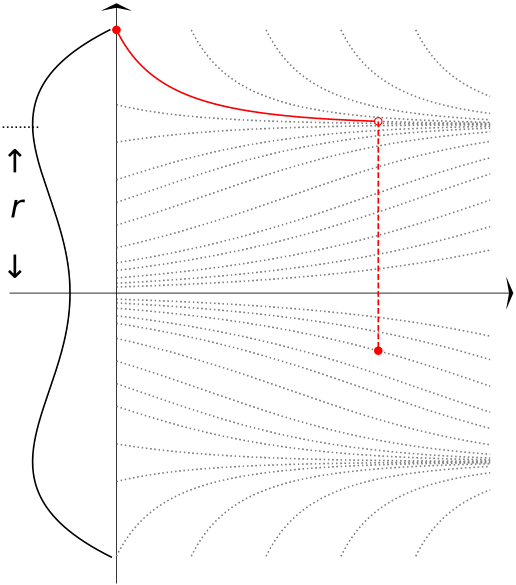


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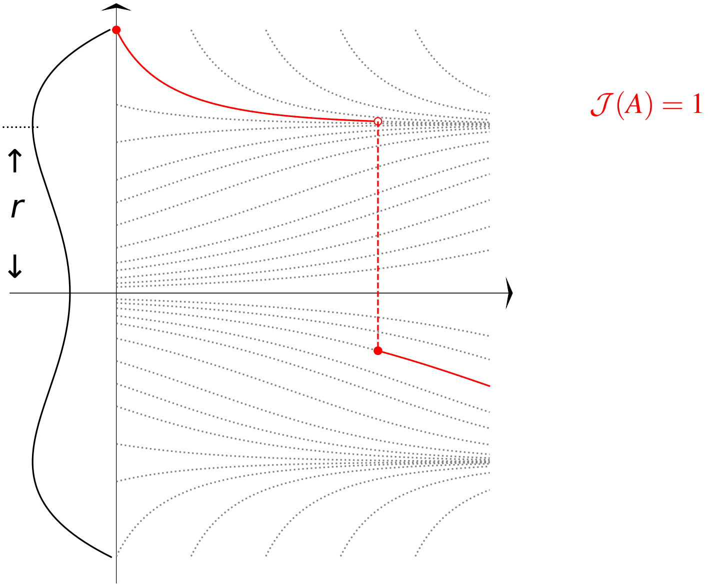

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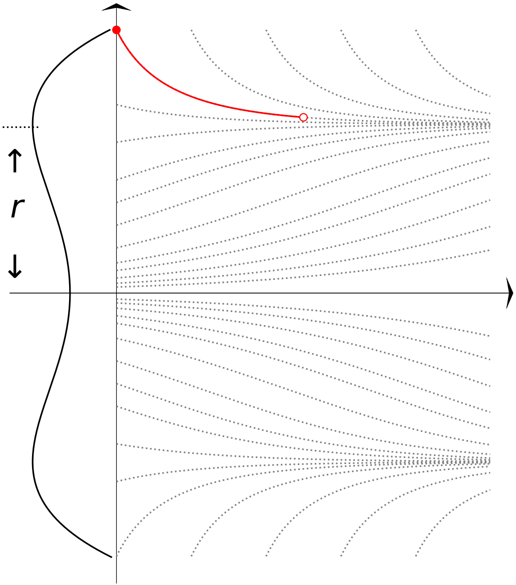


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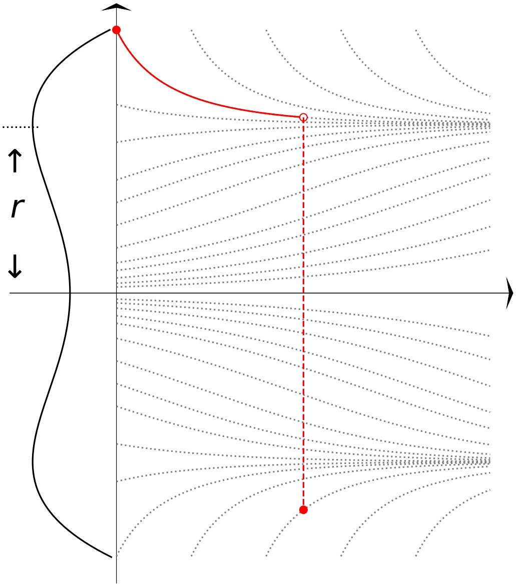


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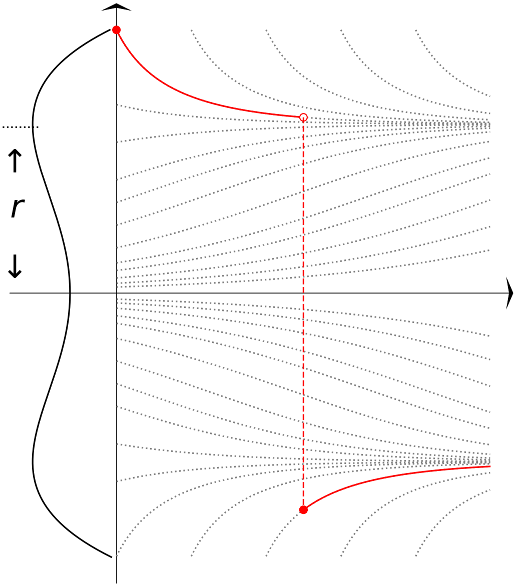


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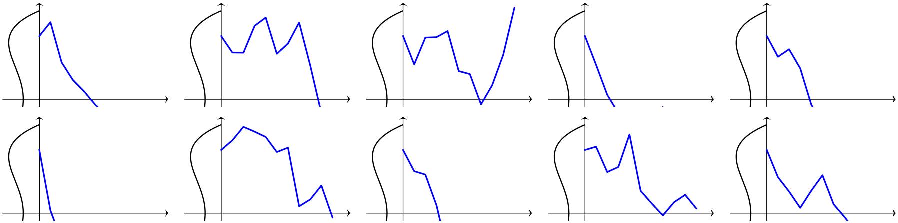
# Catastrophe Principle Dictates SGD's Escape Route

Trajectory of SGD $X^\eta$ conditional on exit: **light-tailed** noises with $\eta = 1/10$



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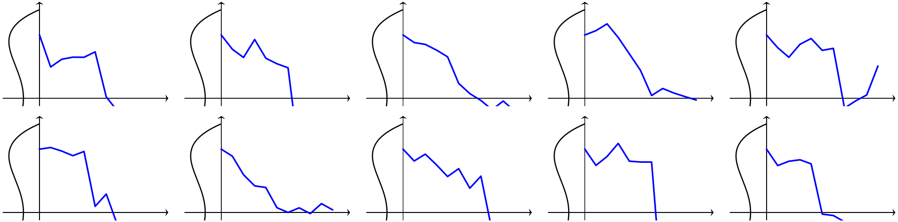

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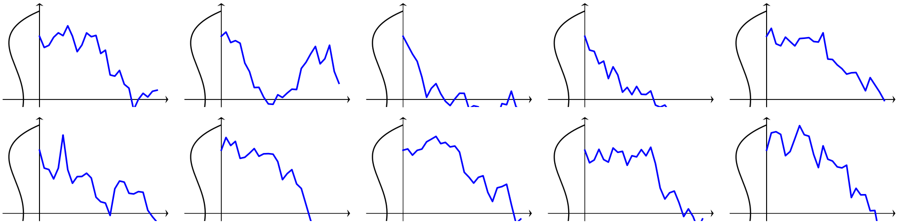
Trajectory of SGD $X^\eta$ conditional on exit: **light-tailed** noises with $\eta = 1/25$



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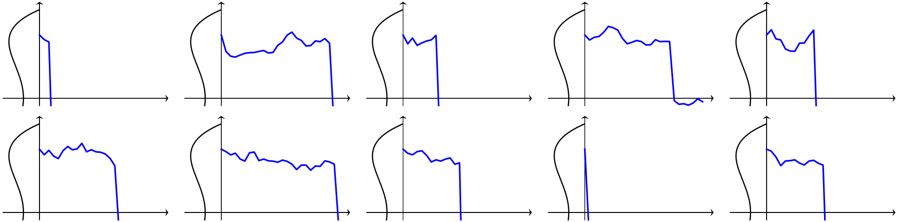

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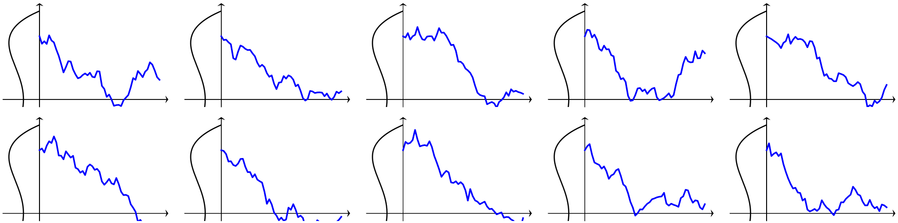

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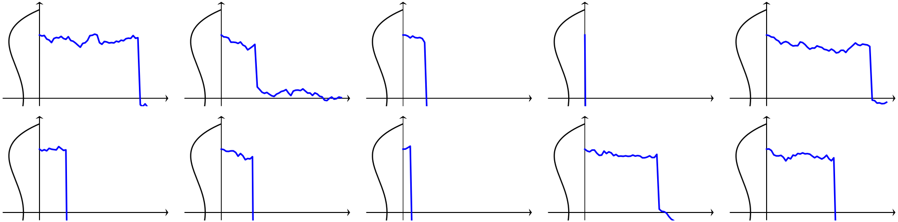

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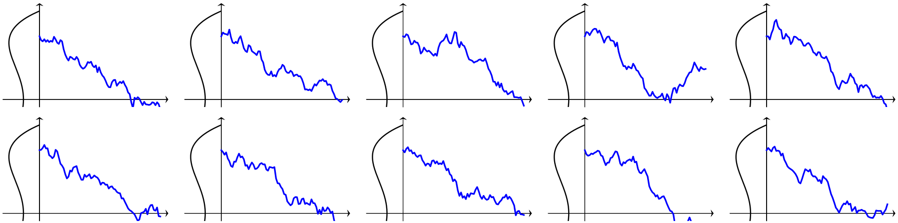

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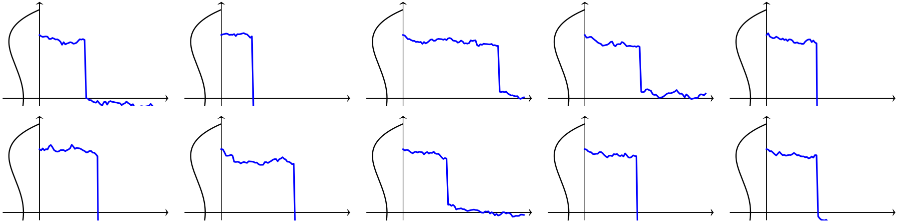

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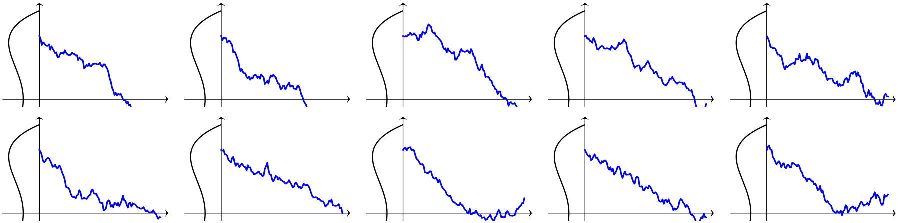

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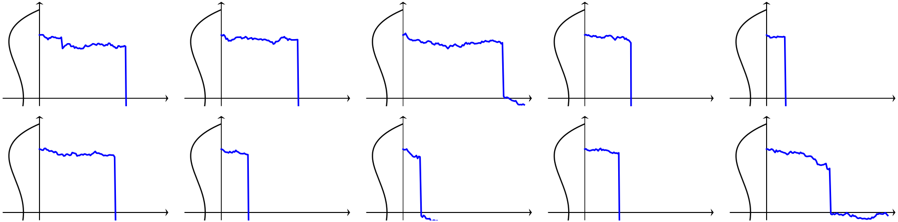

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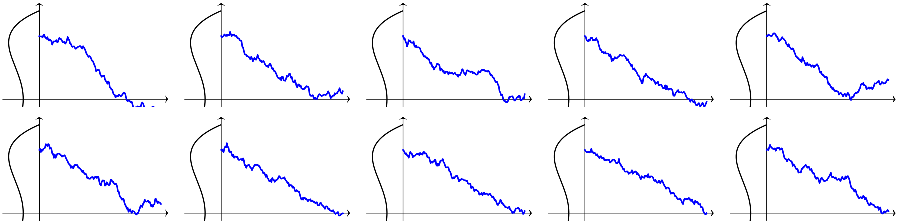
Trajectory of SGD $X^\eta$ conditional on exit: **light-tailed** noises with $\eta = 1/150$



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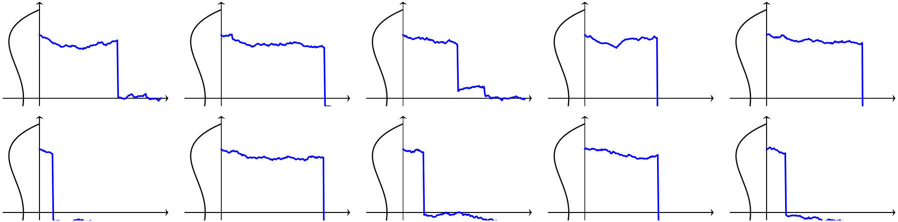
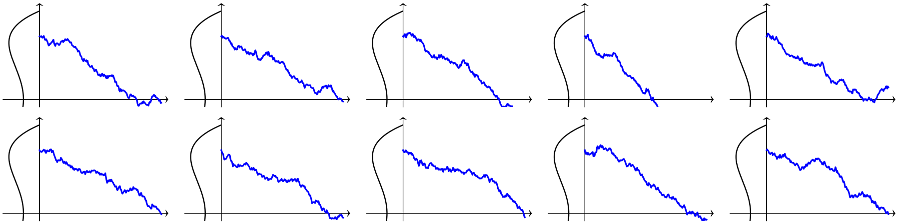


# Catastrophe Principle Dictates SGD's Escape Route

Trajectory of SGD $X^\eta$ conditional on exit: **light-tailed** noises with $\eta = 1/200$



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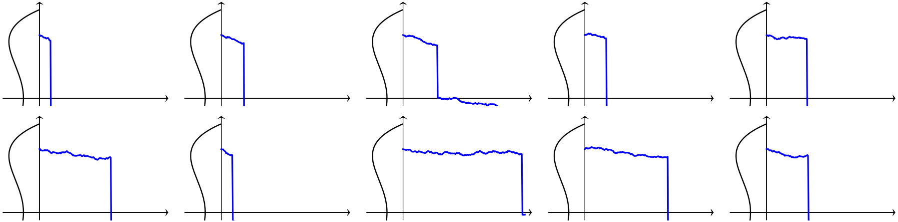

# Truncated Version of Stochastic Gradient Descent

SGD

$$W_{k+1}^{\eta} = W_k^{\eta} - \eta\left(f'(W_k^{\eta}) + Z_k\right) \qquad k = 0, 1, 2, \dots$$

# Truncated Version of Stochastic Gradient Descent

SGD with Gradient Clipping

$$W_{k+1}^{\eta} = W_k^{\eta} - \varphi_c\left(\eta\left(f'(W_k^{\eta}) + Z_k\right)\right) \qquad k = 0, 1, 2, \ldots$$

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where

$$\varphi_c(x) = \frac{x}{|x|}\min\{c, |x|\}.$$

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where

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Then, again,

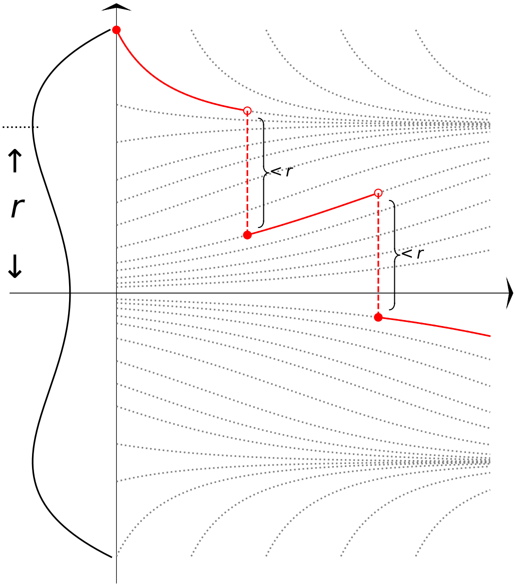
$$W^{\eta}(\cdot) \to w(\cdot) \qquad \text{as} \qquad \eta \to 0$$

where

$$\mathrm{d}w(t) = -f'(w(t))\mathrm{d}t.$$

# How does $\mathcal{J}(A)$ change?

If $r \in (c, 2c)$



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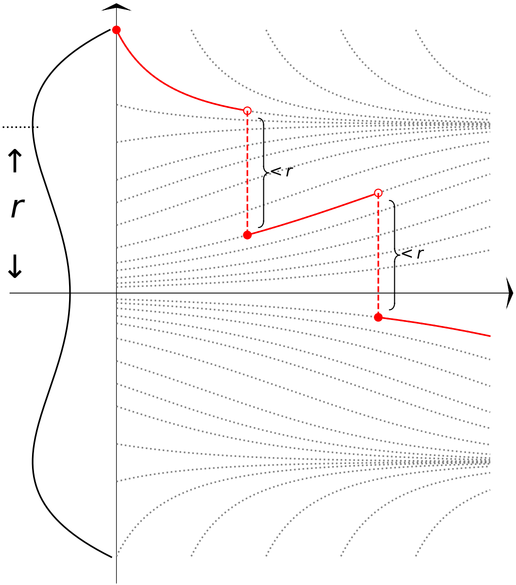


$\mathcal{J}(A) = 2$

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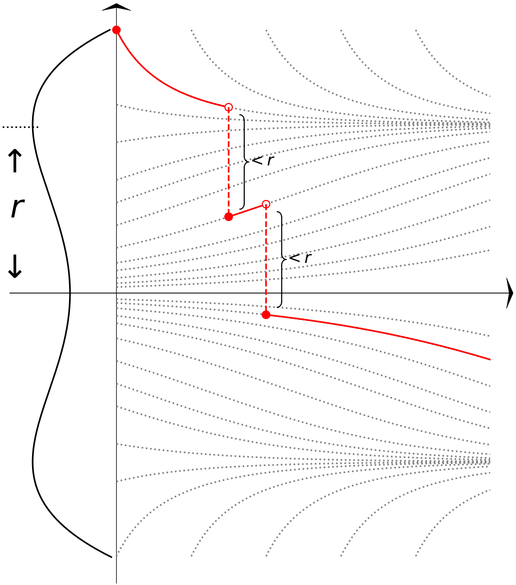


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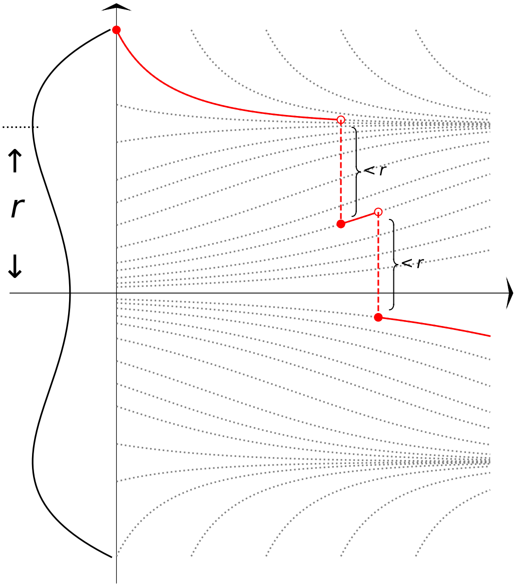


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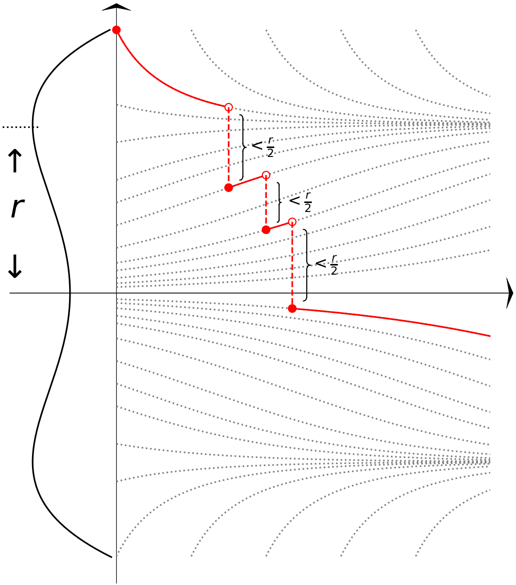

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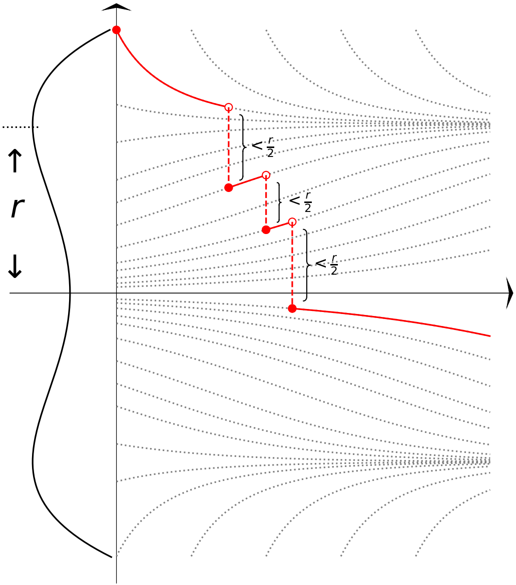
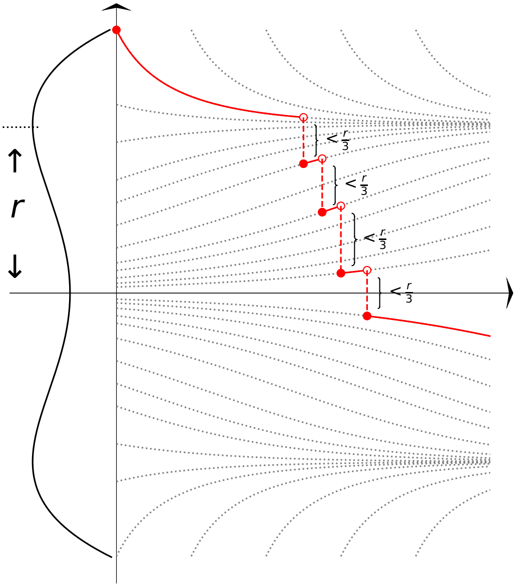



$\mathcal{J}(A) = 3$

# How does $\mathcal{J}(A)$ change?

If $r \in (3c, 4c)$



# How does $\mathcal{J}(A)$ change?

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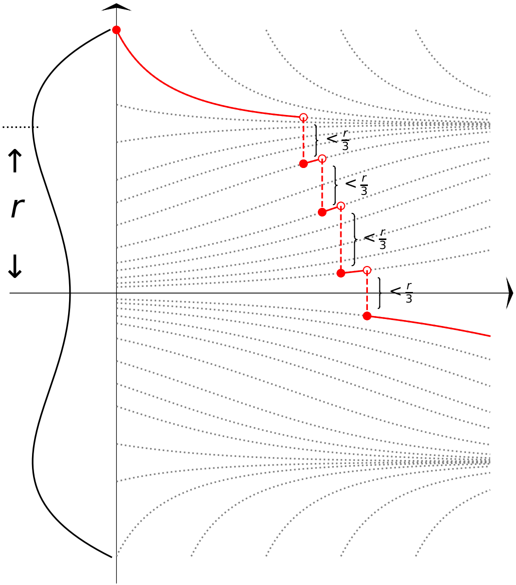
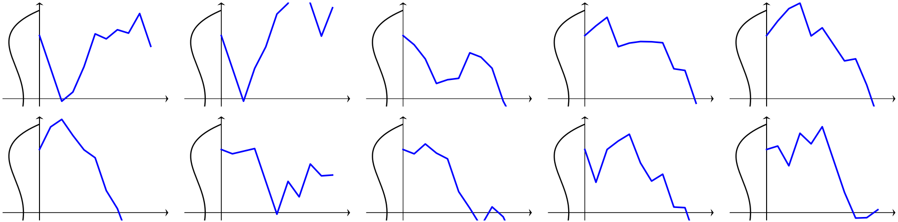



$\mathcal{J}(A) = 4$

# SGD's Escaping Route under Gradient Clipping with $\mathcal{J}(A) = 2$

Trajectory of SGD $X^\eta$ conditional on exit: **light-tailed** noises with $\eta = 1/10$



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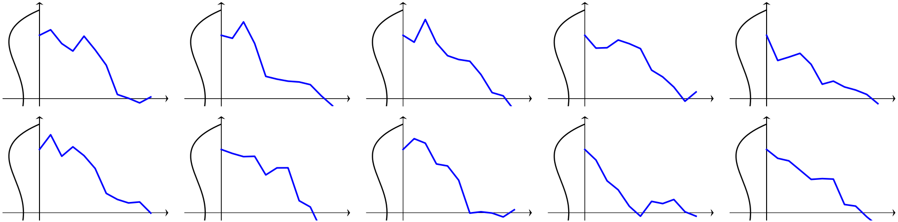
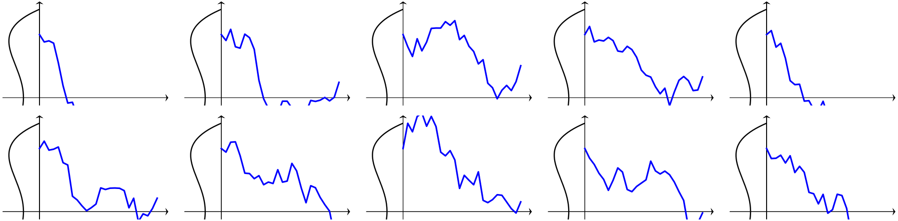


# SGD's Escaping Route under Gradient Clipping with $\mathcal{J}(A) = 2$

Trajectory of SGD $X^\eta$ conditional on exit: **light-tailed** noises with $\eta = 1/25$



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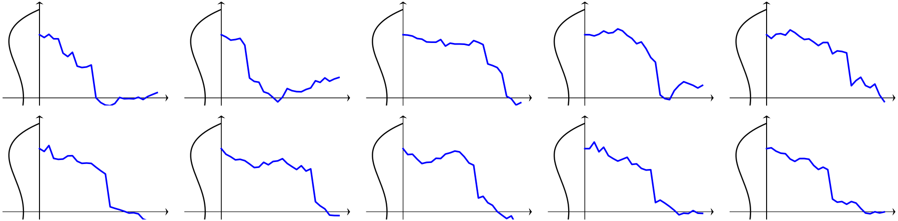

# SGD's Escaping Route under Gradient Clipping with $\mathcal{J}(A) = 2$

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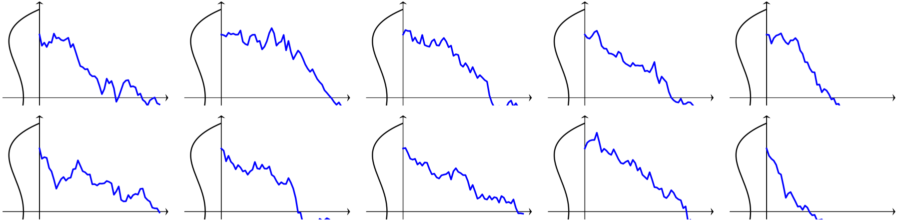

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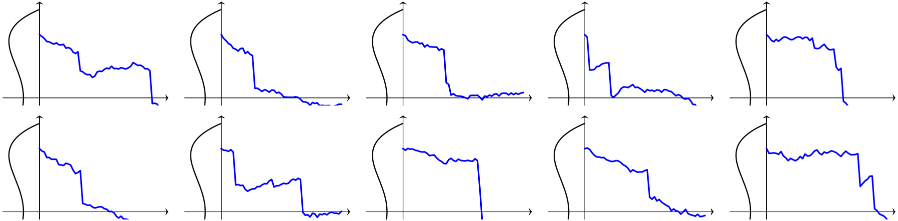

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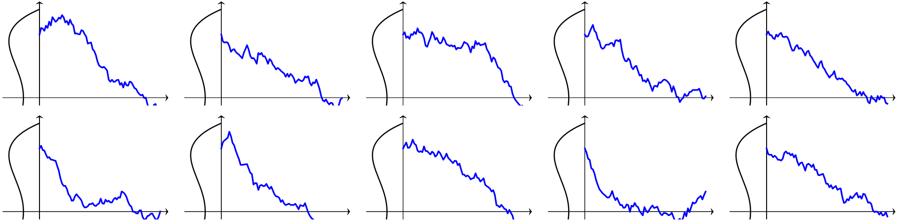

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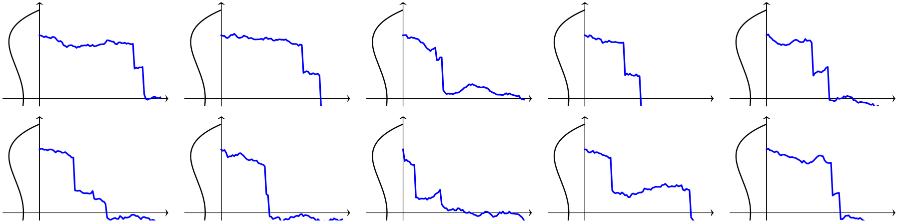

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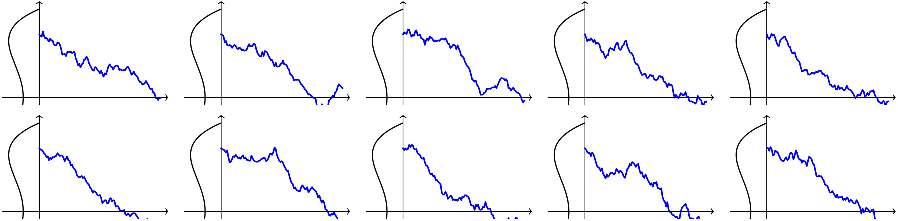

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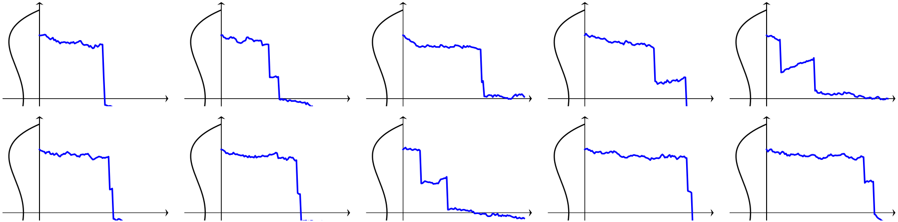

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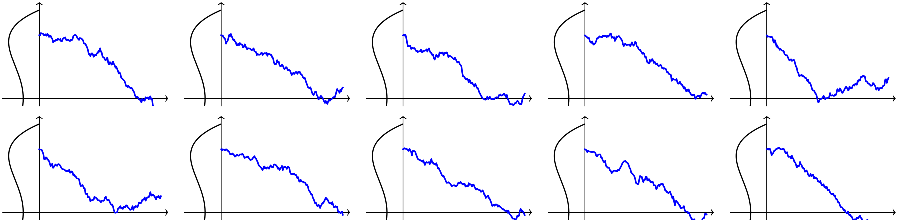

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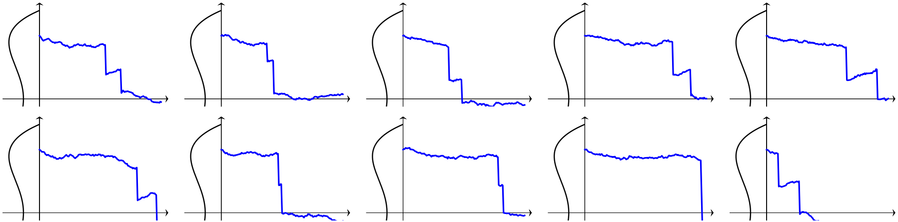

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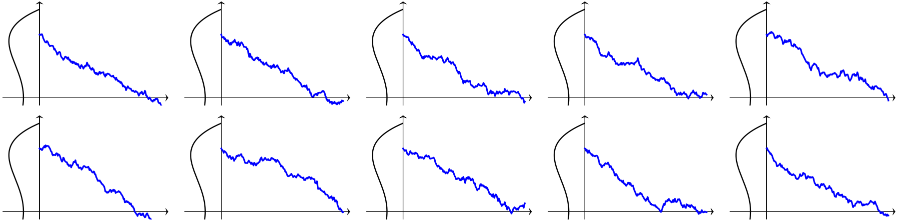

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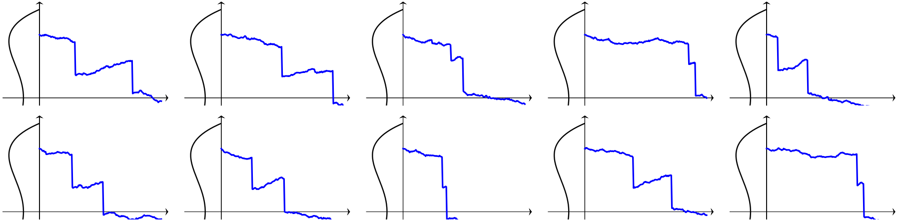

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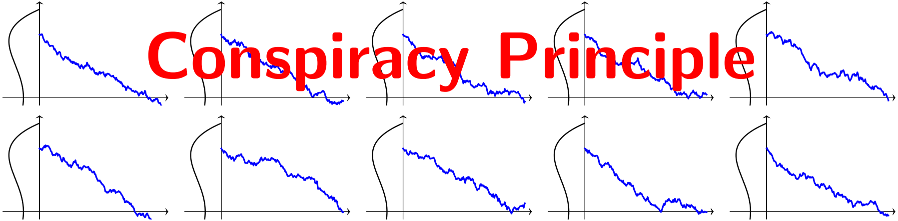
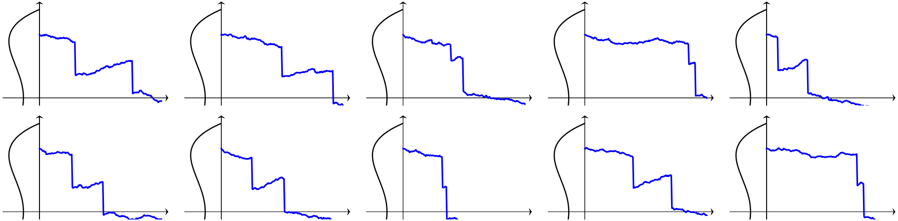


**Conspiracy Principle**

Trajectory of SGD $X^\eta$ conditional on exit: **heavy-tailed** noises with $\eta = 1/200$



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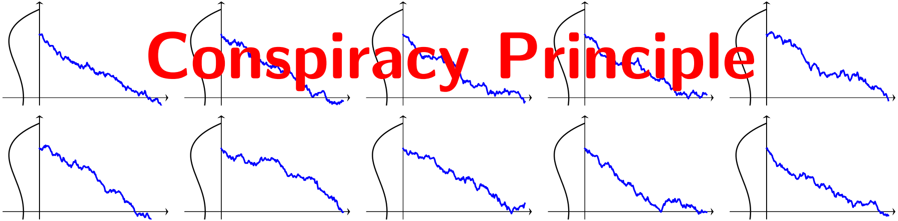


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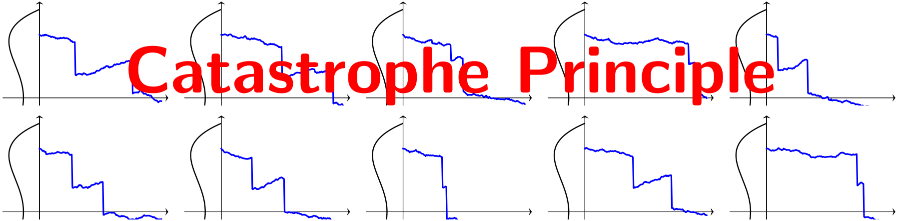

# Metastability of SGD

# Heavy-Tailed Large Deviations for SGD

**Theorem (Wang, R., 2023+)**

*For "general"* $B \subseteq \mathbb{D}[0,T]$

$$C(B^\circ) \leq \liminf_{\eta \to 0} \frac{\mathbf{P}(W^\eta \in B)}{\eta^{\alpha \mathcal{J}(B)}} \leq \limsup_{\eta \to 0} \frac{\mathbf{P}(W^\eta \in B)}{\eta^{\alpha \mathcal{J}(B)}} \leq C(B^-).$$

- $\mathcal{J}(B)$*: min #jumps added to* $w(\cdot)$ *for it to be inside* $B$
- $C(\cdot)$*: a measure*

# Heavy-Tailed Large Deviations for SGD

$$\mathbf{P}(|Z_i| \geq x) = x^{-(\alpha+1)}, \quad \alpha > 0$$

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**Theorem (Wang, R., 2023+)**

*For "general" $B \subseteq \mathbb{D}[0,T]$*

$$C(B^\circ) \leq \lim_{\varepsilon\to 0} \liminf_{\eta\to 0} \frac{\inf_{x\in A(\varepsilon)} \mathbf{P}_x(W^\eta \in B)}{\eta^{\alpha \mathcal{J}(B)}} \leq \lim_{\varepsilon\to 0} \limsup_{\eta\to 0} \frac{\sup_{x\in A(\varepsilon)} \mathbf{P}_x(W^\eta \in B)}{\eta^{\alpha \mathcal{J}(B)}} \leq C(B^-).$$

- $\mathcal{J}(B)$: *min #jumps added to $w(\cdot)$ for it to be inside $B$*
- $C(\cdot)$: *a measure*

# Heavy-Tailed Large Deviations for SGD

$$\mathbf{P}(|Z_i| \geq x) = x^{-(\alpha+1)}, \quad \alpha > 0$$

**Theorem (Wang, R., 2023+)**

*For "general"* $B \subseteq \mathbb{D}[0,T]$

$$C(B^\circ) \leq \lim_{\varepsilon \to 0} \liminf_{\eta \to 0} \frac{\inf_{x \in A(\varepsilon)} \mathbf{P}_x(W^\eta \in B)}{\eta^{\alpha \mathcal{J}(B)}} \leq \lim_{\varepsilon \to 0} \limsup_{\eta \to 0} \frac{\sup_{x \in A(\varepsilon)} \mathbf{P}_x(W^\eta \in B)}{\eta^{\alpha \mathcal{J}(B)}} \leq C(B^-).$$

- $\mathcal{J}(B)$: *min #jumps added to* $w(\cdot)$ *for it to be inside* $B$
- $C(\cdot)$: *a measure*

**Locally Uniform Large Deviations over Asymptotic Atom $\{A(\varepsilon) : \varepsilon > 0\}$**

# $\mathbb{M}$-Convergence

$\mathbb{M}(\mathbb{S}\setminus\mathbb{C}) = \{\nu(\cdot) : \nu(\mathbb{S}\setminus\mathbb{C}^{\varepsilon}) < \infty, \forall\varepsilon > 0; \quad \nu(\cdot)$ Borel measure on $\mathbb{S}\setminus\mathbb{C}\}$

**Definition ($\mathbb{M}$-convergence; Lindskog, Resnick, Roy., 2014)**

Let $\mu^{\eta}, \mu \in \mathbb{M}(\mathbb{S}\setminus\mathbb{C})$ for each $\eta > 0$. We say that $\mu^{\eta}$ converges to $\mu$ in $\mathbb{M}(\mathbb{S}\setminus\mathbb{C})$ as $\eta \to 0$ if

$$\lim_{\eta\downarrow 0} |\mu^{\eta}(f) - \mu(f)| = 0 \qquad \forall f \in \mathcal{C}(\mathbb{S}\setminus\mathbb{C}).$$

# $\mathbb{M}$-Convergence

$$\mathbb{M}(\mathbb{S}\setminus\mathbb{C}) = \{\nu(\cdot) : \nu(\mathbb{S}\setminus\underset{\swarrow\ \varepsilon\text{-fattening of }\mathbb{C}}{\mathbb{C}^{\varepsilon}}) < \infty, \forall\varepsilon > 0; \quad \nu(\cdot) \text{ Borel measure on } \mathbb{S}\setminus\mathbb{C}\}$$

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# $\mathbb{M}$-Convergence

$\varepsilon$-fattening of $\mathbb{C}$

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**Definition ($\mathbb{M}$-convergence; Lindskog, Resnick, Roy., 2014)**

Let $\mu^\eta, \mu \in \mathbb{M}(\mathbb{S}\setminus\mathbb{C})$ for each $\eta > 0$. We say that $\mu^\eta$ converges to $\mu$ in $\mathbb{M}(\mathbb{S}\setminus\mathbb{C})$ as $\eta \to 0$ if

$$\lim_{\eta\downarrow 0} |\mu^\eta(f) - \mu(f)| = 0 \qquad \forall f \in \mathcal{C}(\mathbb{S}\setminus\mathbb{C}).$$

"bounded continuous functions supported on $\mathbb{S}\setminus\mathbb{C}$"

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"bounded continuous functions supported on $\mathbb{S}\setminus\mathbb{C}$"

**Definition (Uniform $\mathbb{M}$-convergence; Wang, R., 2023+)**

Let $\Theta$ be a set of indices. Let $\mu_{\theta}^{\eta}, \mu_{\theta} \in \mathbb{M}(\mathbb{S}\setminus\mathbb{C})$ for each $\eta > 0$ and $\theta \in \Theta$. We say that $\mu_{\theta}^{\eta}$ converges to $\mu_{\theta}$ in $\mathbb{M}(\mathbb{S}\setminus\mathbb{C})$ uniformly in $\theta$ on $\Theta$ as $\eta \to 0$ if

$$\lim_{\eta\downarrow 0} \sup_{\theta\in\Theta} |\mu_{\theta}^{\eta}(f) - \mu_{\theta}(f)| = 0 \qquad \forall f \in \mathcal{C}(\mathbb{S}\setminus\mathbb{C}).$$

# Portmanteau Theorem for Uniform $\mathbb{M}(\mathbb{S}\setminus\mathbb{C})$-Convergence

Theorem (Wang, R., 2023+)

*Let* $\mu_\theta^\eta, \mu_\theta \in \mathbb{M}(\mathbb{S}\setminus\mathbb{C})$ *for each* $\eta > 0$ *and* $\theta \in \Theta$*. Suppose that for any sequence* $(\theta_n)_{n\geq 1}$*, there exist a sub-sequence* $(\theta_{n_k})_{k\geq 1}$ *and* $\theta^* \in \Theta$ *s.t.*

$$\lim_{k\to\infty} \mu_{\theta_{n_k}}(f) = \mu_{\theta^*}(f) \qquad \forall f \in \mathcal{C}(\mathbb{S}\setminus\mathbb{C}).$$

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$$\lim_{k\to\infty} \mu_{\theta_{n_k}}(f) = \mu_{\theta^*}(f) \qquad \forall f \in \mathcal{C}(\mathbb{S}\setminus\mathbb{C}).$$

*Then the following statements are equivalent:*

- $\mu_\theta^\eta \to \mu_\theta$ *in* $\mathbb{M}(\mathbb{S}\setminus\mathbb{C})$ *uniformly in* $\theta$ *on* $\Theta$ *as* $\eta \downarrow 0$;
- *For all* $\varepsilon > 0$, $F, G$ *bounded away from* $\mathbb{C}$,

$$\liminf_{\eta\downarrow 0} \inf_{\theta\in\Theta} \big(\mu_\theta^\eta(G) - \mu_\theta(G_\varepsilon)\big) \geq 0$$

$$\limsup_{\eta\downarrow 0} \sup_{\theta\in\Theta} \big(\mu_\theta^\eta(F) - \mu_\theta(F^\varepsilon)\big) \leq 0$$

# Portmanteau Theorem for Uniform $\mathbb{M}(\mathbb{S}\setminus\mathbb{C})$-Convergence

**Theorem (Wang, R., 2023+)**

*Let $\mu_\theta^\eta, \mu_\theta \in \mathbb{M}(\mathbb{S}\setminus\mathbb{C})$ for each $\eta > 0$ and $\theta \in \Theta$. Suppose that for any sequence $(\theta_n)_{n\geq 1}$, there exist a sub-sequence $(\theta_{n_k})_{k\geq 1}$ and $\theta^* \in \Theta$ s.t.*

$$\lim_{k\to\infty} \mu_{\theta_{n_k}}(f) = \mu_{\theta^*}(f) \qquad \forall f \in \mathcal{C}(\mathbb{S}\setminus\mathbb{C}).$$

*Then the following statements are equivalent:*

- $\mu_\theta^\eta \to \mu_\theta$ *in* $\mathbb{M}(\mathbb{S}\setminus\mathbb{C})$ *uniformly in* $\theta$ *on* $\Theta$ *as* $\eta \downarrow 0$;
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$$\limsup_{\eta\downarrow 0} \sup_{\theta\in\Theta} \big(\mu_\theta^\eta(F) - \mu_\theta(F^\varepsilon)\big) \leq 0$$

*Furthermore, they both imply*

- *For all open* $G$ *and closed* $F$ *that are bounded away from* $\mathbb{C}$,

$$\inf_{\theta\in\Theta} \mu_\theta(G) \leq \liminf_{\eta\downarrow 0} \inf_{\theta\in\Theta} \mu_\theta^\eta(G)$$
$$\limsup_{\eta\downarrow 0} \sup_{\theta\in\Theta} \mu_\theta^\eta(F) \leq \sup_{\theta\in\Theta} \mu_\theta(F).$$

# Asymptotic Atom $\{A(\varepsilon) \subseteq \mathbb{S} : \varepsilon > 0\}$ of Markov Chain $\{V_j^\eta(x) : j \geq 0\}$

For measurable $B \subseteq \mathbb{S}$, there exist $\delta_B : (0,\infty) \times (0,\infty) \to (0,\infty)$, $\varepsilon_B > 0$, and $T_B > 0$ s.t.

$$C(B^\circ) - \delta_B(\varepsilon, T) \leq \liminf_{\eta \downarrow 0} \frac{\inf_{x \in A(\varepsilon)} \mathbf{P}\big(\tau^\eta_{I(\varepsilon)^\complement}(x) \leq T/\eta;\ V^\eta_{\tau_\varepsilon}(x) \in B\big)}{\gamma(\eta) T/\eta}$$

$$\leq \limsup_{\eta \downarrow 0} \frac{\sup_{x \in A(\varepsilon)} \mathbf{P}\big(\tau^\eta_{I(\varepsilon)^\complement}(x) \leq T/\eta;\ V^\eta_{\tau_\varepsilon}(x) \in B\big)}{\gamma(\eta) T/\eta} \leq C(B^-) + \delta_B(\varepsilon, T)$$

$$\limsup_{\eta \downarrow 0} \frac{\sup_{x \in I(\varepsilon)} \mathbf{P}\big(\tau^\eta_{(I(\varepsilon)\setminus A(\varepsilon))^\complement}(x) > T/\eta\big)}{\gamma(\eta) T/\eta} = 0$$

$$\liminf_{\eta \downarrow 0} \inf_{x \in I(\varepsilon)} \mathbf{P}\big(\tau^\eta_{A(\varepsilon)}(x) \leq T/\eta\big) = 1$$

($\{I(\varepsilon) \subseteq I : \varepsilon > 0\}$: covering of $I$)

for any $\varepsilon \leq \varepsilon_B$ and $T \geq T_B$, where $\gamma(\eta)/\eta \to 0$ as $\eta \downarrow 0$ and $\delta_B$'s are such that

$$\lim_{\varepsilon \to 0} \lim_{T \to \infty} \delta_B(\varepsilon, T) = 0.$$

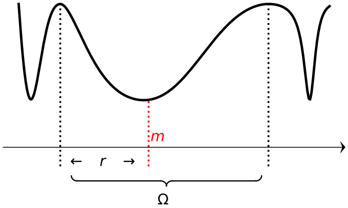
# Exit Time and Location under the Presence of Asymptotic Atom

**Theorem (Wang, R., 2023+)**

*If Markov chain* $\{V_j^\eta(x) : j \geq 0\}$ *possesses an asymptotic atom* $\{A(\varepsilon) \subseteq \mathbb{S} : \varepsilon > 0\}$, *then*

$$\begin{aligned} C(B^\circ) \cdot e^{-t} &\leq \liminf_{\eta \downarrow 0} \inf_{x \in I(\varepsilon)} \mathbf{P}\Big(\gamma(\eta)\tau^\eta_{I^\complement}(x) > t,\, V^\eta_\tau(x) \in B\Big) \\ &\leq \limsup_{\eta \downarrow 0} \sup_{x \in I(\varepsilon)} \mathbf{P}\Big(\gamma(\eta)\tau^\eta_{I^\complement}(x) > t,\, V^\eta_\tau(x) \in B\Big) \leq C(B^-) \cdot e^{-t}. \end{aligned}$$

# First Exit Time Analysis for SGD




gradient noise

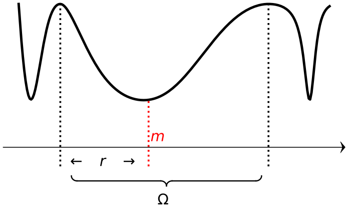
$l = \lceil r/c \rceil$: "width" of the attraction fields, $\mathbf{P}(|Z_i| \geq x) = x^{-(\alpha+1)}, \quad \alpha > 0$

**Theorem (Wang, Oh, R., 2022)**

*Let* $\sigma(\eta) = \min\{j \geq 0 : W_j^\eta \notin \Omega\}$ *and* $\lambda(\eta) \sim \eta^{1+\alpha \cdot l}$

$$\sigma(n)\lambda(n) \Rightarrow \mathrm{Exp}(1)$$

# First Exit Time Analysis for SGD

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First Exit Time

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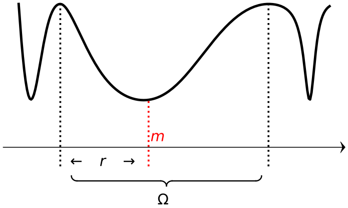
# First Exit Time Analysis for SGD

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First Exit Time

$$\sigma(n)\lambda(n) \Rightarrow \mathrm{Exp}(1)$$

$\sim (1/\eta)^{1+\alpha \cdot l}$

# Eliminating Sharp Local Minima with Truncated Heavy-Tails

gradient noise

$l^*$: "width" of the widest attraction fields, $\mathbf{P}(|Z_i| \geq x) = x^{-(\alpha+1)}, \quad \alpha > 0$

**Theorem (Wang, Oh, R., 2022)**

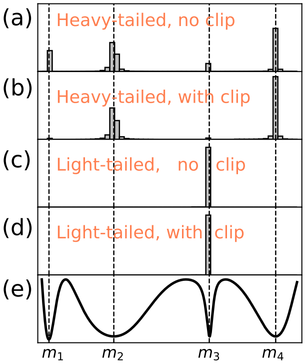
*Under certain structural conditions, for any $t > 0$ and $\beta > 1 + \alpha \cdot l^*$,*

$$\frac{1}{t/\eta^\beta} \int_0^{\lfloor t/\eta^\beta \rfloor} \mathbb{I}\{W^\eta_{\lfloor u \rfloor} \in \textit{sharp minima}\} du \xrightarrow{p} 0$$

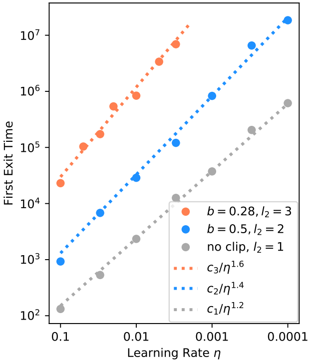
**In fact, $W^\eta_{\lfloor t/\eta^{1+\alpha \cdot l^*} \rfloor}$ converges to a Markov jump processes**

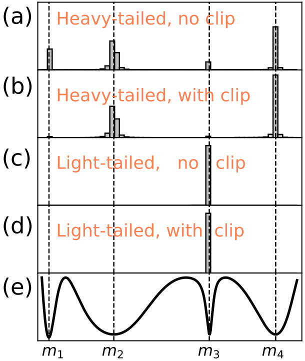
**whose state space consists of wide local minima only.**

# Eliminating Sharp Local Minima with Truncated Heavy-Tails

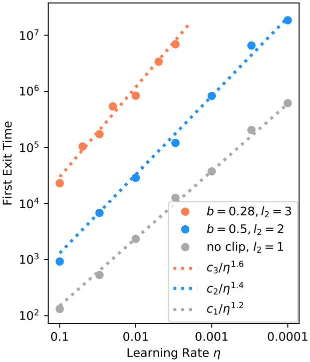


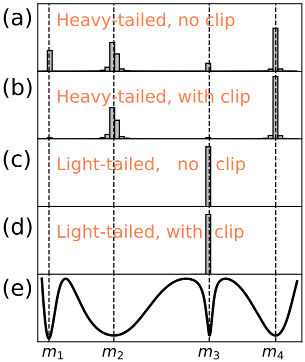
# Eliminating Sharp Local Minima with Truncated Heavy-Tails

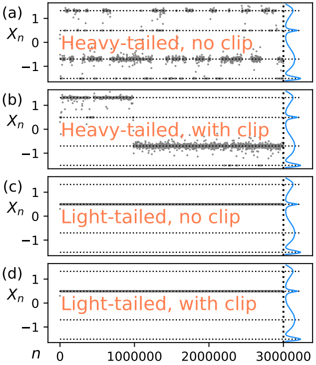


# Eliminating Sharp Local Minima with Truncated Heavy-Tails



# New Training Strategy: Tail-INflation-Truncation

# Tail-Inflation-Truncation Scheme

$$\nabla\tilde{f} \quad = \nabla f_{\text{small batch}}$$

$$W_{k+1} = W_k - \varphi_b\big(\eta \cdot \nabla\tilde{f}(W_k)\big)$$

# Tail-Inflation-Truncation Scheme

Stochastic Gradient

$$\searrow$$

$$\nabla\tilde{f} \quad = \nabla f_{\text{small batch}}$$

$$W_{k+1} = W_k - \varphi_b\big(\eta \cdot \nabla\tilde{f}(W_k)\big)$$

# Tail-Inflation-Truncation Scheme

Stochastic Gradient

$$\nabla\tilde{f}_{\text{our}} = \nabla f_{\text{small batch}} + R \cdot (\nabla f_{\text{large batch}} - \nabla f_{\text{small batch}})$$

$$W_{k+1} = W_k - \varphi_b\big(\eta \cdot \nabla\tilde{f}(W_k)\big)$$

# Tail-Inflation-Truncation Scheme

$$\underset{\text{Stochastic Gradient}}{\nabla\tilde{f}_{\text{our}}} = \nabla f_{\text{small batch}} + \underset{\text{Pareto RV}}{R} \cdot (\nabla f_{\text{large batch}} - \nabla f_{\text{small batch}})$$

$$W_{k+1} = W_k - \varphi_b\big(\eta \cdot \nabla\tilde{f}(W_k)\big)$$

# Tail-Inflation-Truncation Scheme

Heavy-Tailed Stochastic Gradient Pareto RV

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# Tail-Inflation-Truncation Scheme

Heavy-Tailed Stochastic Gradient          Pareto RV

$$\nabla\tilde{f}_{\text{our}} = \nabla f_{\text{small batch}} + R\cdot(\nabla f_{\text{large batch}} - \nabla f_{\text{small batch}})$$

| Test accuracy | LB | SB | SB+Clip | SB+Noise | Our 1 | Our 2 |
|---|---|---|---|---|---|---|
| FashionMNIST, LeNet | 68.66% | 69.20% | 68.77% | 64.43% | 69.47% | **70.06%** |
| SVHN, VGG11 | 82.87% | 85.92% | 85.95% | 38.85% | **88.42%** | 88.37% |
| CIFAR10, VGG11 | 69.39% | 74.42% | 74.38% | 40.50% | 75.69% | **75.87%** |
| Expected Sharpness | LB | SB | SB+Clip | SB+Noise | Our 1 | Our 2 |
| FashionMNIST, LeNet | 0.032 | 0.008 | 0.009 | 0.047 | 0.003 | **0.002** |
| SVHN, VGG11 | 0.694 | 0.037 | 0.041 | 0.012 | **0.002** | 0.005 |
| CIFAR10, VGG11 | 2.043 | 0.050 | 0.039 | 2.046 | **0.024** | 0.037 |

# Tail-Inflation-Truncation Scheme

| CIFAR10-VGG11 | SB + Clip | Our 1 | Our 2 |
|---|---|---|---|
| Test Accuracy | 89.54% | **90.67%** | 90.45% |
| Expected Sharpness | 0.167 | **0.085** | 0.096 |
| PAC-Bayes Sharpness | $1.31 \times 10^4$ | $\mathbf{9 \times 10^3}$ | $10^4$ |
| Maximal Sharpness | $1.66 \times 10^4$ | $1.29 \times 10^4$ | $\mathbf{1.22 \times 10^4}$ |
| CIFAR100-VGG16 | SB + Clip | Our 1 | Our 2 |
| Test Accuracy | 56.32% | **65.44%** | 62.99% |
| Expected Sharpness | 0.857 | **0.441** | 0.479 |
| PAC-Bayes Sharpness | $2.49 \times 10^4$ | $\mathbf{1.9 \times 10^4}$ | $1.98 \times 10^4$ |
| Maximal Sharpness | $2.75 \times 10^4$ | $\mathbf{2.12 \times 10^4}$ | $2.16 \times 10^4$ |

# Choice of Clipping Threshold $b$

- Recall

$$\tau(\eta) \approx O(1/\eta^{1+\lceil r/b \rceil \alpha})$$

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- Recall

width of attraction field

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clipping threshold

- Performance under different $b$

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- Recall

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clipping threshold

- Performance under different $b$

| b | 0.005 | 0.025 | 0.1 | 0.25 | 0.5 | 1 | 2.5 | 5 |
|---|---|---|---|---|---|---|---|---|
| Test Accuracy (%) | 70.2 | 71.1 | 74.6 | 76.9 | 76.9 | 76.3 | 52.5 | 30.4 |

CIFAR10-VGG11. $\alpha = 1.4$. $\eta = 0.05$. Training iterations 30,000.

# Other Related Topic I: Heavy Tails from Multiplicative Dynamics

# Markov Modulated Process

We want sample path large deviations of

$$\bar{S}_n(t) = \frac{1}{n}\sum_{i=0}^{\lfloor nt \rfloor} X_i \qquad \text{where} \qquad X_{n+1} = A_n X_n + B_n$$

**Light-Tailed Components**

# Markov Modulated Process

We want sample path large deviations of

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# Markov Modulated Process

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$$\mathbf{E} A_n^{\alpha} = 1$$

**for some** $\alpha > 1$

# Markov Modulated Process

We want sample path large deviations of

$\mathbf{E}|B_n|^{\alpha+\varepsilon} < \infty$

**for some** $\varepsilon > 0$

$$\bar{S}_n(t) = \frac{1}{n}\sum_{i=0}^{\lfloor nt \rfloor} X_i \qquad \text{where} \qquad X_{n+1} = A_n X_n + B_n$$

$\mathbf{E}A_n^{\alpha} = 1$

**for some** $\alpha > 1$

# Large Deviations for Markov Modulated Processes

$$\bar{S}_n(t) = \frac{1}{n}\sum_{i=1}^{\lfloor nt \rfloor} X_i, \qquad X_{n+1} = A_n X_n + B_n, \qquad \mathbf{E} A_1^{\alpha} = 1$$

**Theorem (Chen, R., Zwart, 2020+)**

*For "general" $A \subseteq \mathbb{D}$*

$$C(A^\circ) \le \liminf_{n\to\infty} \frac{\mathbf{P}(\bar{S}_n \in A)}{n^{-\alpha \mathcal{J}(A)}} \le \limsup_{n\to\infty} \frac{\mathbf{P}(\bar{S}_n \in A)}{n^{-\alpha \mathcal{J}(A)}} \le C(A^-).$$

- $\mathcal{J}(A)$: *min #jumps for step functions to be inside* $A$
- $C(\cdot)$: *a measure*

# Large Deviations for Markov Modulated Processes

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**Theorem (Chen, R., Zwart, 2020+)**

*For "general" $A \subseteq \mathbb{D}$*

$$\mathbf{P}(\bar{S}_n \in A) \sim n^{-\alpha \mathcal{J}(A)}$$

- $\mathcal{J}(A)$: *min #jumps for step functions to be inside* $A$

# Large Deviations for Markov Modulated Processes

$$\bar{S}_n(t) = \frac{1}{n}\sum_{i=1}^{\lfloor nt \rfloor} X_i, \qquad X_{n+1} = A_n X_n + B_n, \qquad \mathbf{E} A_1^{\alpha} = 1$$

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*For "general" $A \subseteq \mathbb{D}$*

$$\mathbf{P}(\bar{S}_n \in A) \sim n^{-\alpha \mathcal{J}(A)}$$

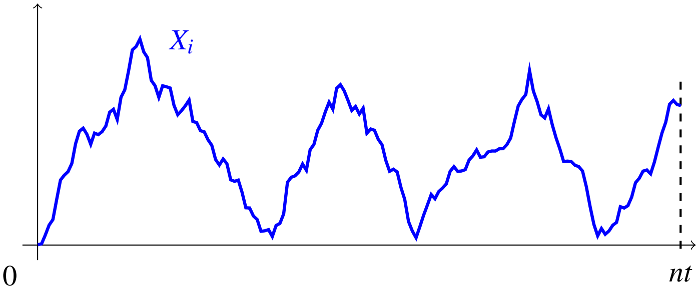
- $\mathcal{J}(A)$: *min #jumps for step functions to be inside* $A$

**Catastrophe Principle:** $\mathcal{L}(\bar{S}_n | \bar{S}_n \in A) \to \mathcal{L}(\text{step functions with } \mathcal{J}(A) \text{ jumps})$

# View Regeneration Cycle as Heavy-Tailed Increment

- $\bar{S}_n(t) = (X_0 + X_1 + \cdots + X_{r_1-1} + X_{r_1} + X_{r_1+1} + \cdots + X_{r_2-1}$

$$+ \cdots + X_{r_{N(nt)-1}} + \cdots + X_{r_{N(nt)}-1} + X_{r_{N(nt)}} + \cdots + X_{\lfloor nt \rfloor})/n$$



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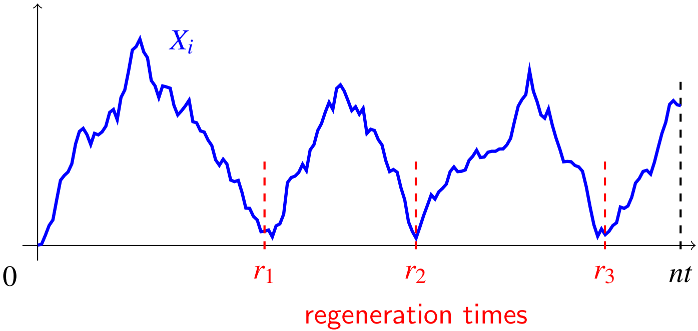
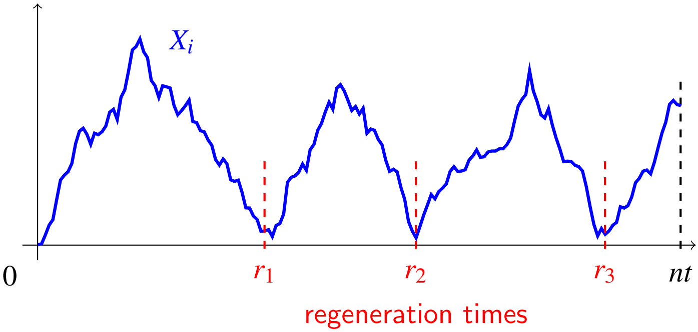


# View Regeneration Cycle as Heavy-Tailed Increment

- $\bar{S}_n(t) = (X_0 + X_1 + \cdots + X_{r_1-1} + X_{r_1} + X_{r_1+1} + \cdots + X_{r_2-1}$

$$+ \cdots + X_{r_{N(nt)-1}} + \cdots + X_{r_{N(nt)}-1} + X_{r_{N(nt)}} + \cdots + X_{\lfloor nt \rfloor})/n$$




$N(nt)$
# regenerations

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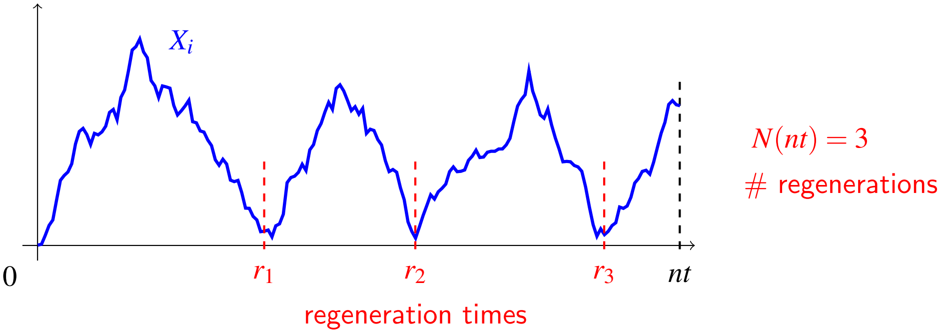

# View Regeneration Cycle as Heavy-Tailed Increment

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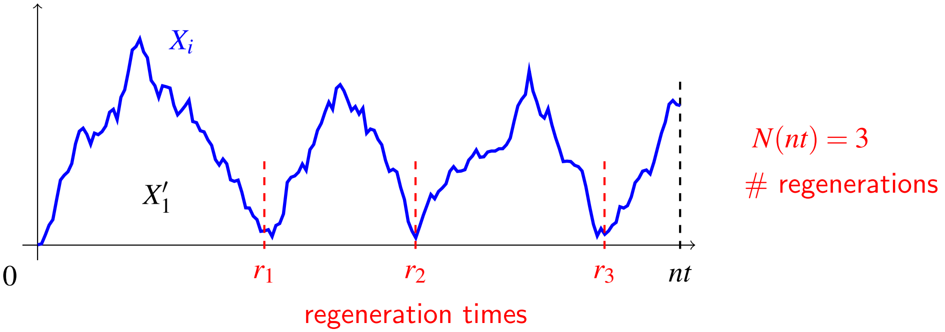

# View Regeneration Cycle as Heavy-Tailed Increment

- $\bar{S}_n(t) = (\underbrace{X_0 + X_1 + \cdots + X_{r_1-1}}_{\triangleq X'_1} + \underbrace{X_{r_1} + X_{r_1+1} + \cdots + X_{r_2-1}}_{\triangleq X'_2}$
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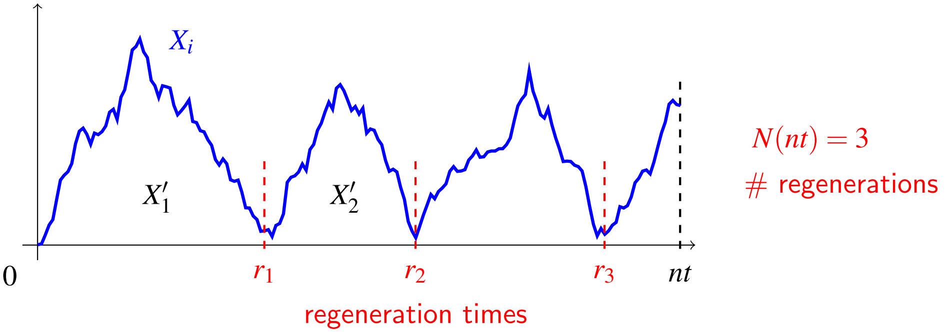

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- $\bar{S}_n(t) = (\underbrace{X_0 + X_1 + \cdots + X_{r_1-1}}_{\triangleq X_1'} + \underbrace{X_{r_1} + X_{r_1+1} + \cdots + X_{r_2-1}}_{\triangleq X_2'}$
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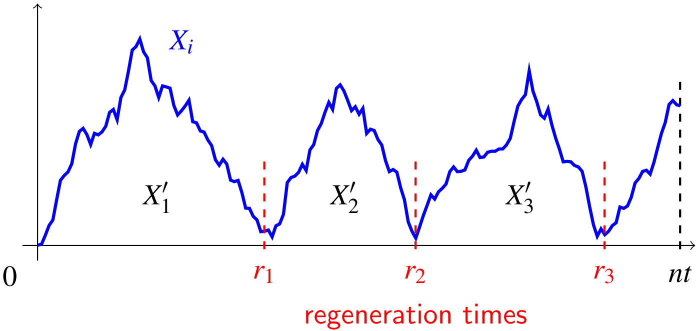


$N(nt) = 3$

# regenerations

# View Regeneration Cycle as Heavy-Tailed Increment

- $\bar{S}_n(t) = (\underbrace{X_0 + X_1 + \cdots + X_{r_1-1}}_{\triangleq X_1'} + \underbrace{X_{r_1} + X_{r_1+1} + \cdots + X_{r_2-1}}_{\triangleq X_2'}$
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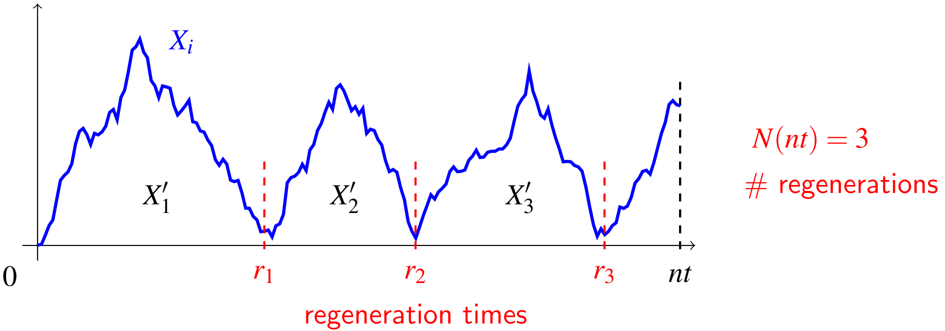

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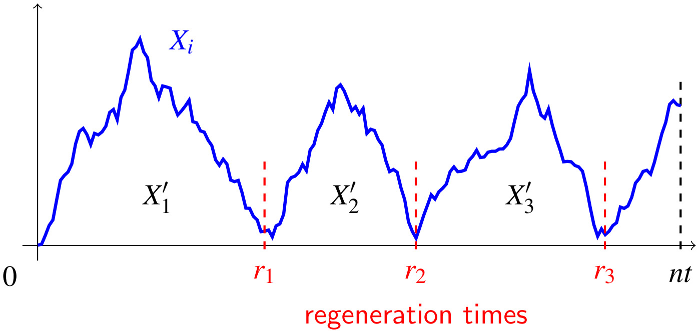
$X'_i$'s are iid RV($\alpha$)




$N(nt) = 3$

\# regenerations

# View Regeneration Cycle as Heavy-Tailed Increment

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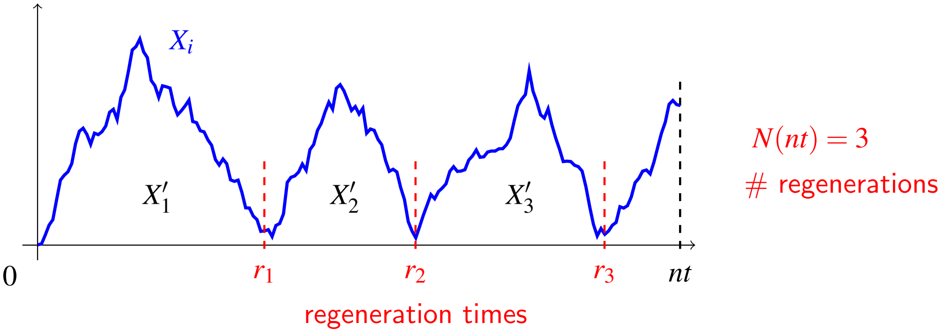
$X_i'$'s are iid RV($\alpha$)
- **A Key Step:** Sample Path LD for $\bar{S}_n'(t) = (X_1' + \cdots + X'_{N(nt)})/n$



# View Regeneration Cycle as Heavy-Tailed Increment

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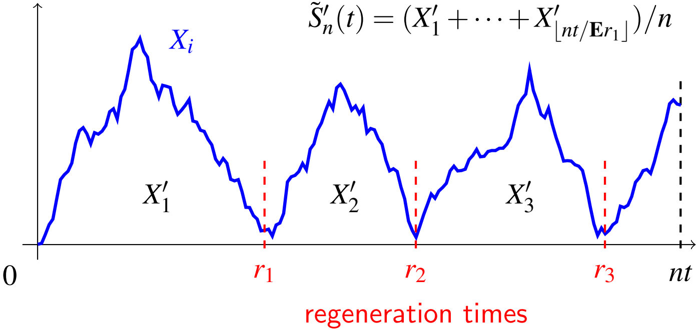
$X_i'$'s are iid RV($\alpha$)

- **A Key Step:** Sample Path LD for $\bar{S}_n'(t) = (X_1' + \cdots + X_{N(nt)}')/n$

via "Asymptotic Equivalence" with

$\tilde{S}_n'(t) = (X_1' + \cdots + X_{\lfloor nt/\mathbf{E}r_1 \rfloor}')/n$




$N(nt) = 3$

\# regenerations

# Asymptotic Equivalence between $\bar{S}'_n$ and $\tilde{S}'_n$

- Our original definition of asymptotic equivalence:

$$\limsup_{n\to\infty} n^{(\alpha-1)j}\mathbf{P}\big(d(\bar{S}'_n,\tilde{S}'_n) > \varepsilon\big) \;=\; 0, \quad \forall j \geq 0,$$

# Asymptotic Equivalence between $\bar{S}'_n$ and $\tilde{S}'_n$

- On the previous page:

$$\bar{S}'_n(t) = (X'_1 + \cdots + X'_{N(nt)})/n$$

and

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- No hope for this since

$$\mathbf{P}(X'_i > n) \sim n^{-\alpha}.$$

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and

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- Generalized asymptotic equivalence:

$$\limsup_{n\to\infty} n^{(\alpha-1)j}\mathbf{P}\big(\tilde{S}'_n \in (\mathbb{D}\setminus\mathbb{D}_{<j})^{-\gamma}, d(\bar{S}'_n,\tilde{S}'_n) > \varepsilon\big) \ = \ 0, \quad \forall j \geq 0,$$

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"$j$ big jumps"

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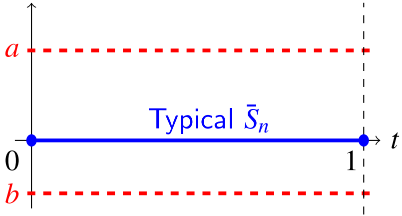
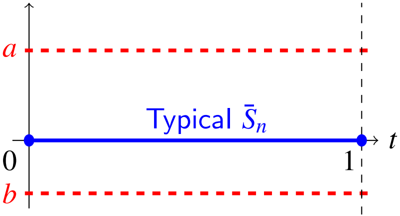
- Then $\tilde{S}'_n$ and $\bar{S}'_n$ are asymptotically equivalent

# Example: Barrier Option



# Example: Barrier Option

$$\bar{S}_n \in \{\, f \in \mathbb{D} : f \text{ hits below } b \text{ on } [0,1] \text{ and ends up above } a \,\} \triangleq E$$



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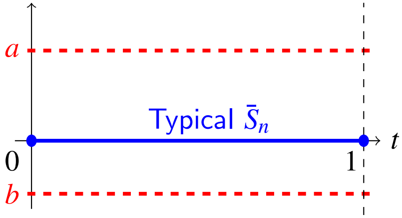


**How do we expect this rare event arises?**

# Example: Barrier Option

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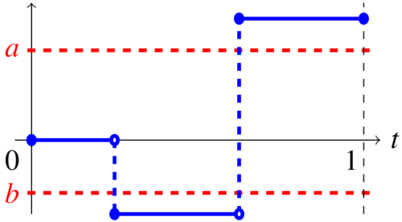
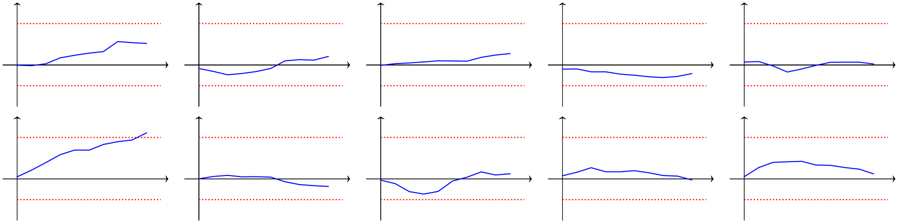



This way!

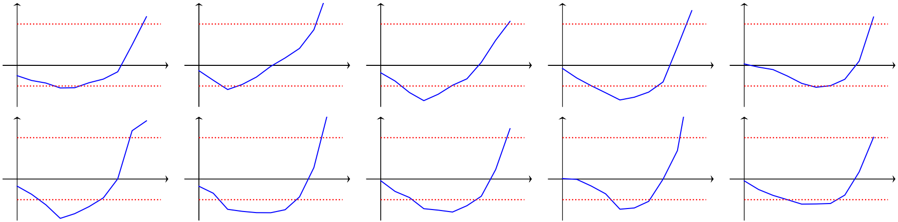
**How do we expect this rare event arises?**

# Example: Barrier Option

Typical sample paths of $\bar{S}_{10}$

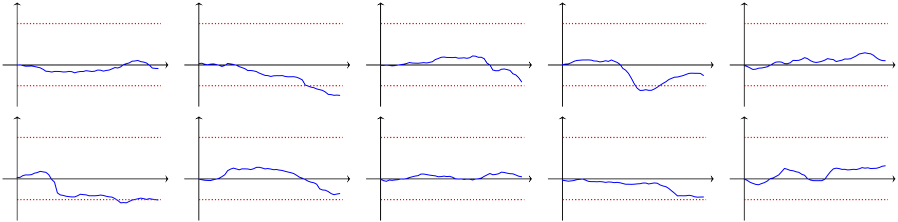


Sample paths of $\bar{S}_{10}$ conditional on $\bar{S}_n \in E$

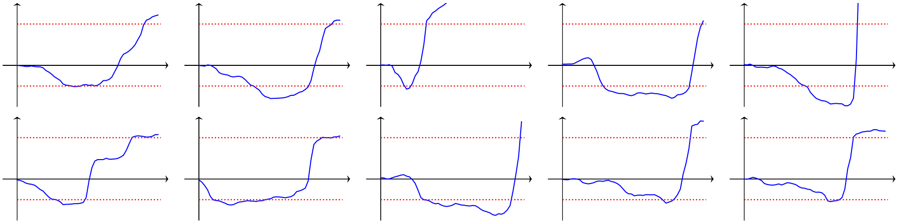


# Example: Barrier Option

Typical sample paths of $\bar{S}_{50}$

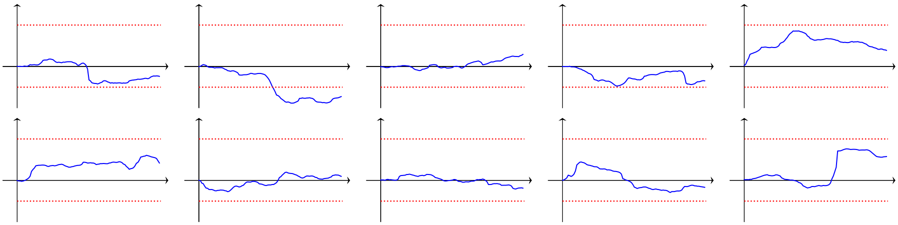


Sample paths of $\bar{S}_{50}$ conditional on $\bar{S}_n \in E$

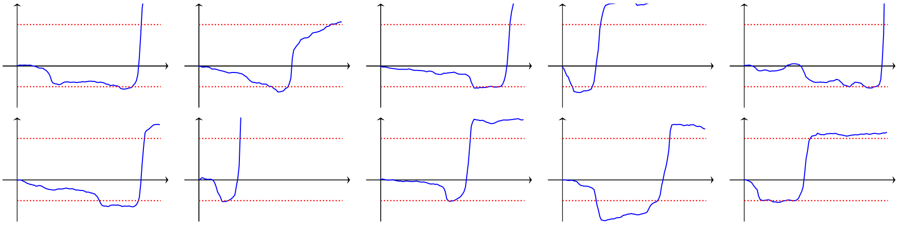


# Example: Barrier Option

Typical sample paths of $\bar{S}_{100}$

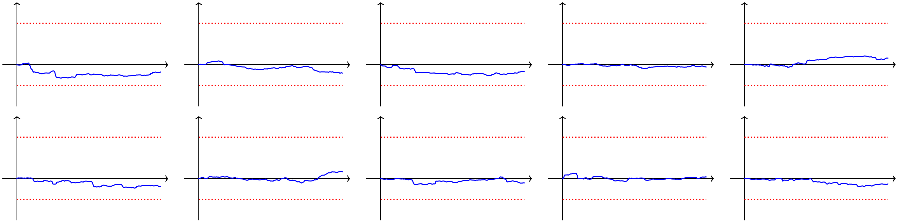


Sample paths of $\bar{S}_{100}$ conditional on $\bar{S}_n \in E$

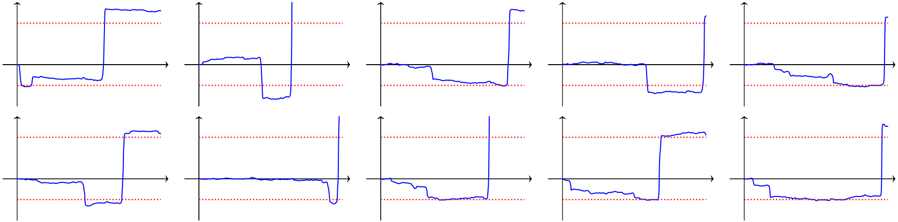


# Example: Barrier Option

Typical sample paths of $\bar{S}_{500}$

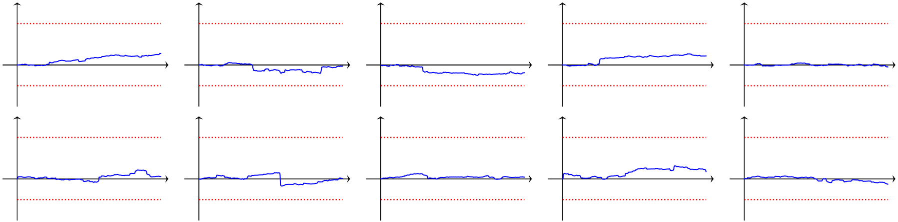


Sample paths of $\bar{S}_{500}$ conditional on $\bar{S}_n \in E$

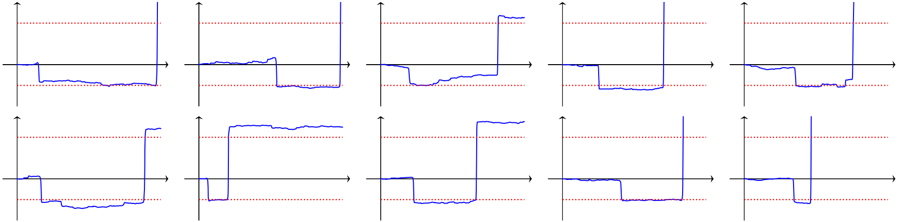


# Example: Barrier Option

Typical sample paths of $\bar{S}_{1000}$

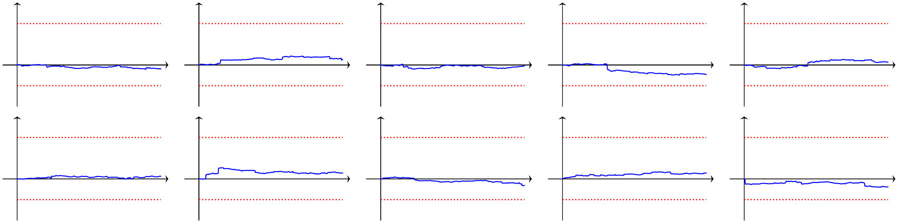


Sample paths of $\bar{S}_{1000}$ conditional on $\bar{S}_n \in E$

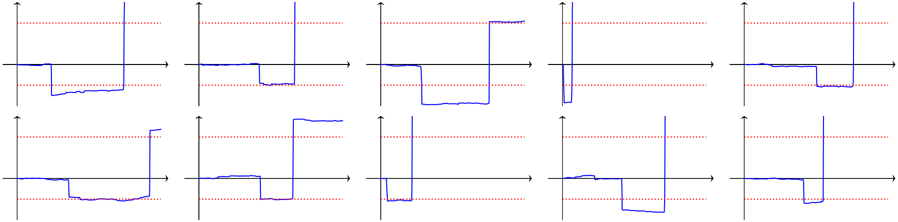


# Example: Barrier Option

Typical sample paths of $\bar{S}_{2500}$

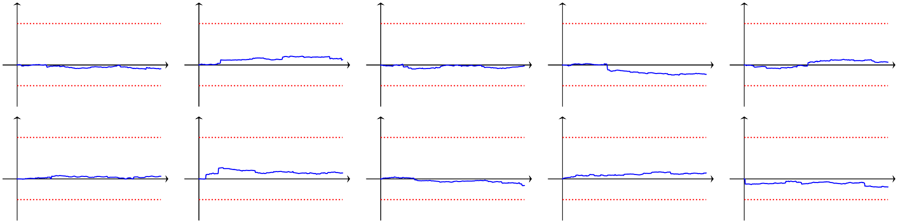


Sample paths of $\bar{S}_{2500}$ conditional on $\bar{S}_n \in E$

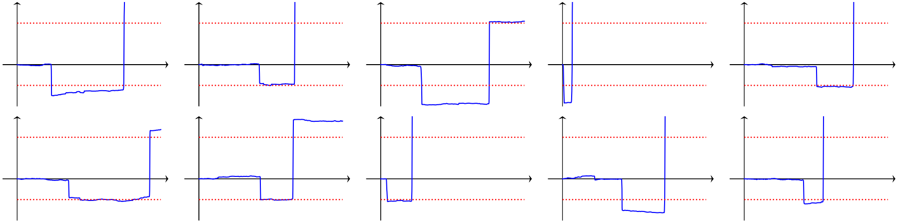


# Example: Barrier Option

Typical sample paths of $\bar{S}_{2500}$



Sample paths of $\bar{S}_{2500}$ conditional on $\bar{S}_n \in E$



# Other Topic II: Large Deviations for Weibull Tails

$$\mathbf{P}(X_i \geq x) = \exp(-x^{\alpha}),\ \ \alpha \in (0,1)$$

# What's already known: LDP w.r.t. $L_1$ topology

Nina Gantert (1998)

- $\limsup_{n\to\infty} \frac{1}{L(n)n^{\alpha}} \log \mathbf{P}\left(\bar{S}_n \in A\right) \leq -\inf_{\xi \in A^-} I(\xi)$, $A^-$: closure of $A$
- $\liminf_{n\to\infty} \frac{1}{L(n)n^{\alpha}} \log \mathbf{P}\left(\bar{S}_n \in A\right) \geq -\inf_{\xi \in A^\circ} I(\xi)$, $A^\circ$ : interior of $A$

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- $\liminf_{n\to\infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P}\left(\bar{S}_n \in A\right) \ge -\inf_{\xi\in A^\circ} I(\xi)$, $\qquad$ $A^\circ$: interior of $A$
- $I(\xi) = \begin{cases} \sum_{\{t:\xi(t)\neq\xi(t-)\}} \left(\xi(t)-\xi(t-)\right)^\alpha & \text{if } \xi \text{ is a nondecreasing step function} \\ \infty & \text{otherwise} \end{cases}$

# What's already known: LDP w.r.t. $L_1$ topology

Nina Gantert (1998)

- $\mathbf{P}\left(\bar{S}_n \in A\right) \lesssim \exp\left(-L(n)n^{\alpha}I(A^{-})\right)$, $\quad I(A^{-}) = \inf_{\xi \in A^{-}} I(\xi)$
- $\mathbf{P}\left(\bar{S}_n \in A\right) \gtrsim \exp\left(-L(n)n^{\alpha}I(A^{\circ})\right)$, $\quad I(A^{\circ}) = \inf_{\xi \in A^{\circ}} I(\xi)$
- $I(\xi) = \begin{cases} \sum_{\{t:\xi(t)\neq\xi(t-)\}} \left(\xi(t) - \xi(t-)\right)^{\alpha} & \text{if } \xi \text{ is a nondecreasing step function} \\ \infty & \text{otherwise} \end{cases}$

# What's already known: LDP w.r.t. $L_1$ topology

Nina Gantert (1998)

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- $\mathbf{P}\left(\bar{S}_n \in A\right) \gtrsim \exp\left(-L(n)n^{\alpha} I(A^{\circ})\right)$, $\qquad I(A^{\circ}) = \inf_{\xi \in A^{\circ}} I(\xi)$
- $I(\xi) = \begin{cases} \sum_{\{t:\xi(t)\neq\xi(t-)\}} \left(\xi(t) - \xi(t-)\right)^{\alpha} & \text{if } \xi \text{ is a nondecreasing step function} \\ \infty & \text{otherwise} \end{cases}$
- $I$ is a good rate function w.r.t. $L_1$ topology

  i.e., $d(\xi,\zeta) = \int_0^1 |\xi(s) - \zeta(s)| ds$

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$\Rightarrow$ **Classical Tools in LD Theory Suffices**

# What's already known: LDP w.r.t. $L_1$ topology

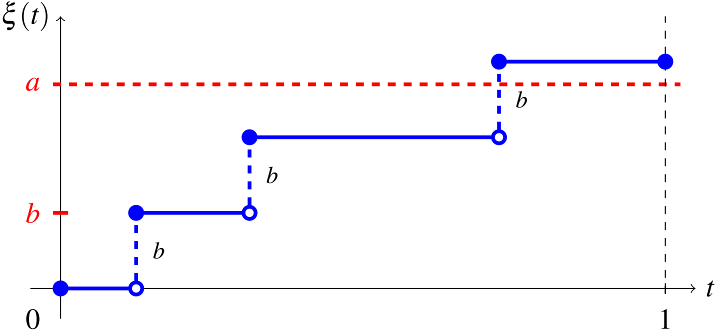
Nina Gantert (1998)

- $\mathbf{P}\left(\bar{S}_n \in A\right) \lesssim \exp\left(-L(n)n^{\alpha}I(A^{-})\right)$, $\quad I(A^{-}) = \inf_{\xi \in A^{-}} I(\xi)$
- $\mathbf{P}\left(\bar{S}_n \in A\right) \gtrsim \exp\left(-L(n)n^{\alpha}I(A^{\circ})\right)$, $\quad I(A^{\circ}) = \inf_{\xi \in A^{\circ}} I(\xi)$
- $I(\xi) = \begin{cases} \sum_{\{t:\xi(t)\neq\xi(t-)\}} \left(\xi(t)-\xi(t-)\right)^{\alpha} & \text{if } \xi \text{ is a nondecreasing step function} \\ \infty & \text{otherwise} \end{cases}$
- $I$ is a good rate function w.r.t. $L_1$ topology

  i.e., $d(\xi,\zeta) = \int_0^1 |\xi(s)-\zeta(s)|ds$ $\quad\Rightarrow\quad$ **Classical Tools in LD Theory Suffices**

**BUT**

# $L_1$ topology is weak

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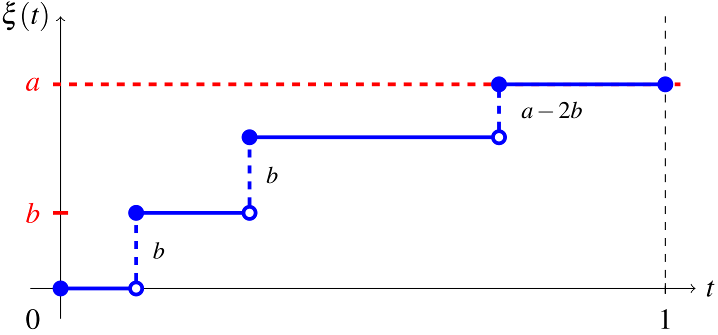


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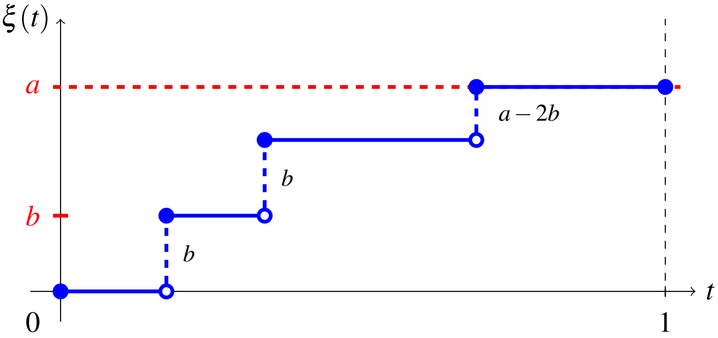


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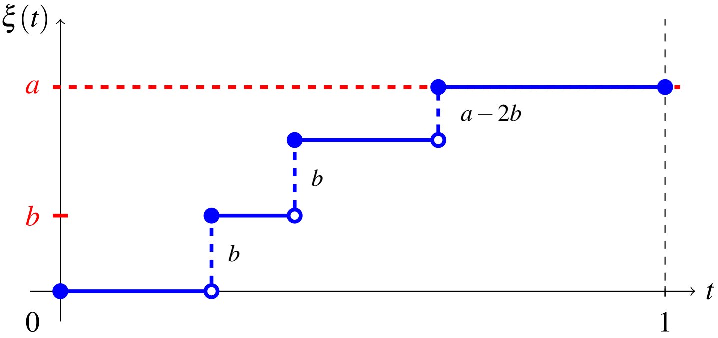


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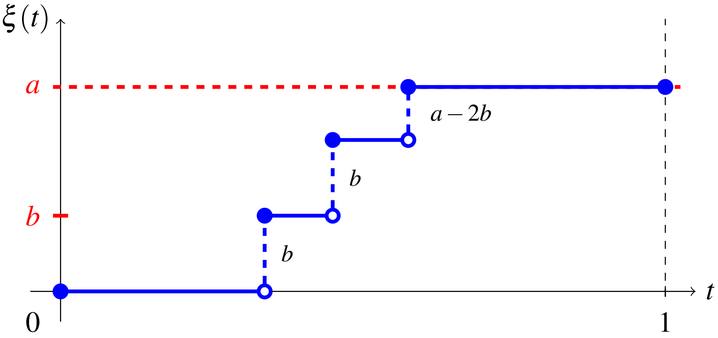


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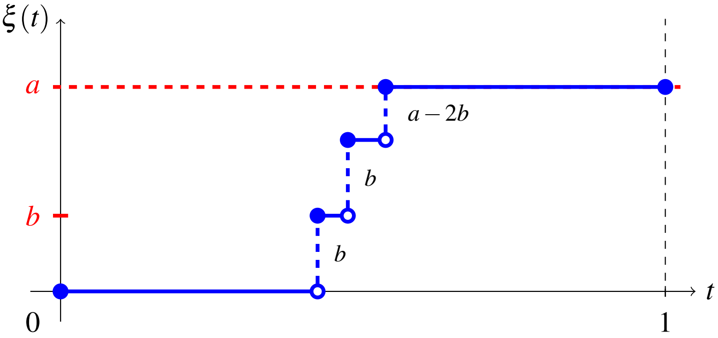


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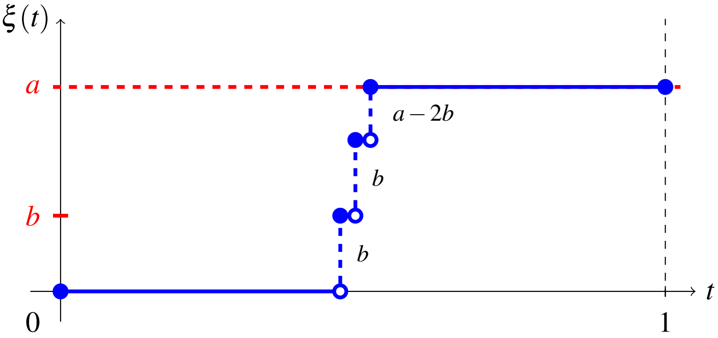


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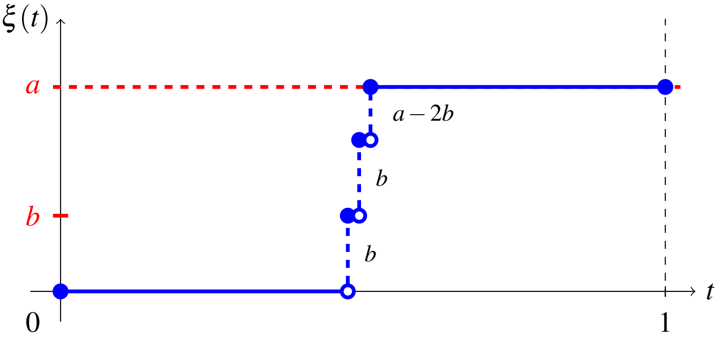


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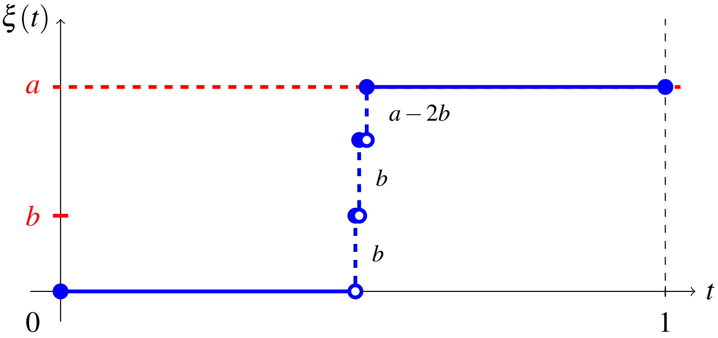


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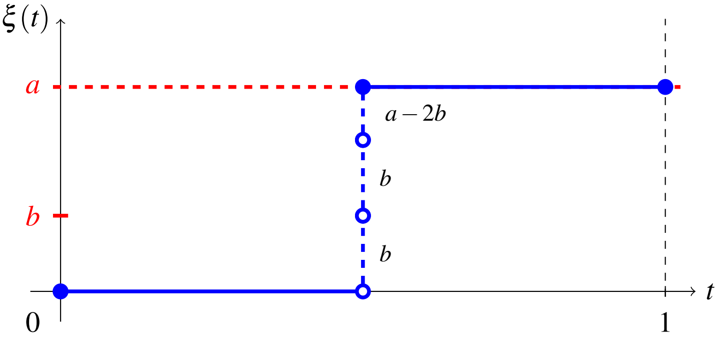


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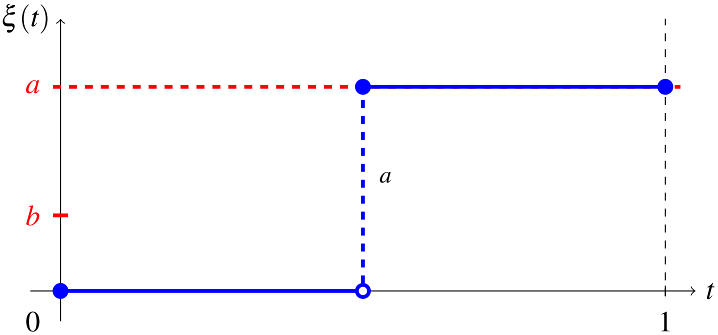


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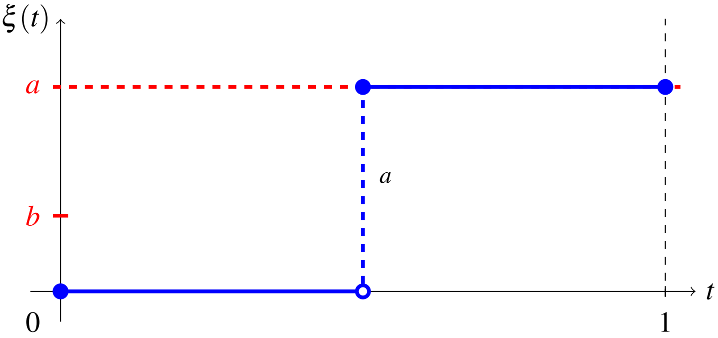


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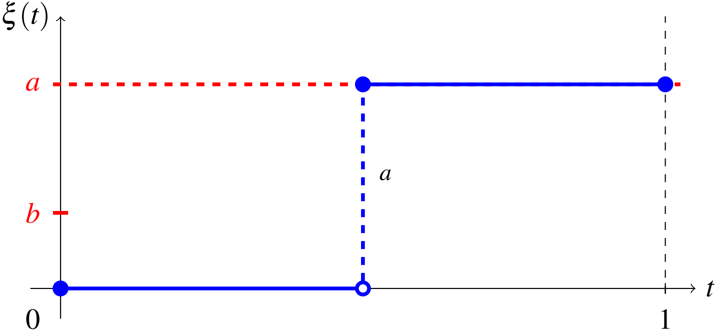


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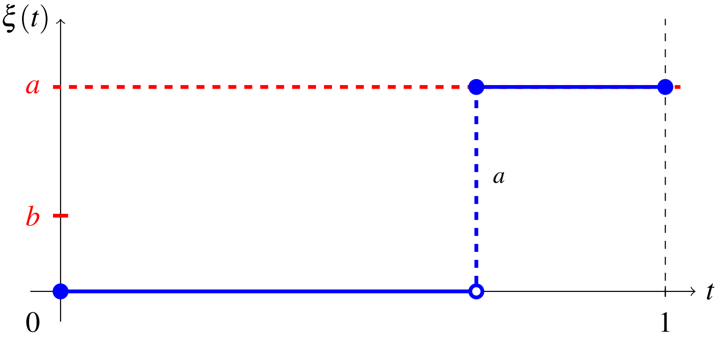


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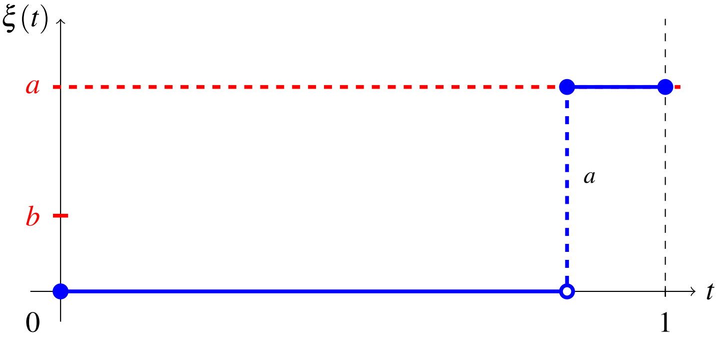


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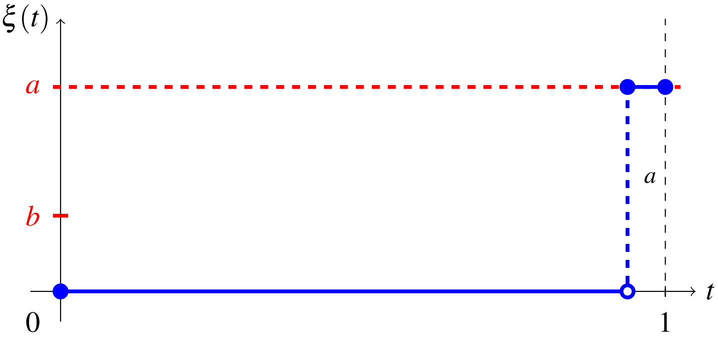


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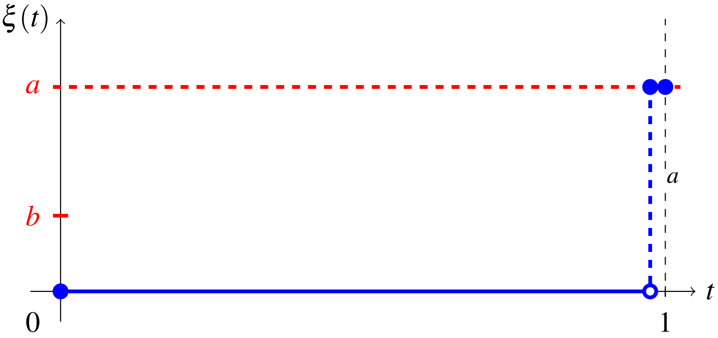


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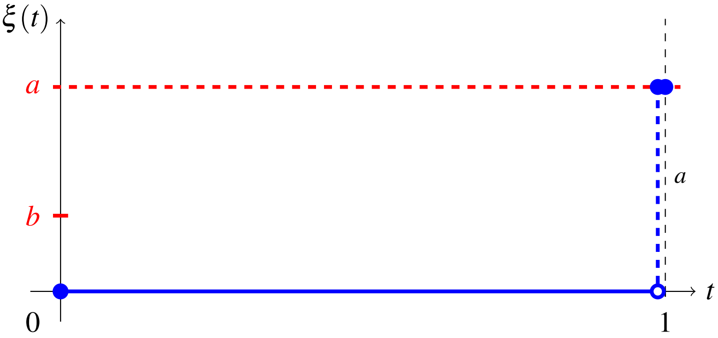


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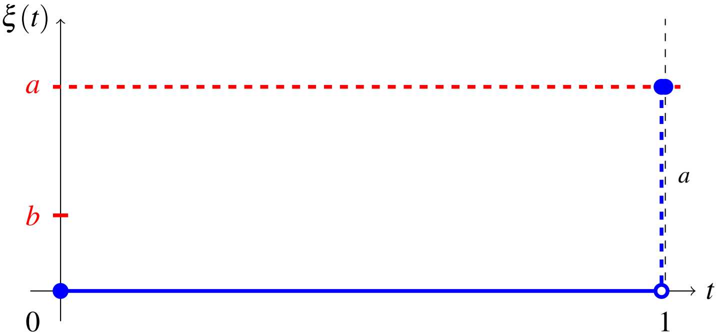


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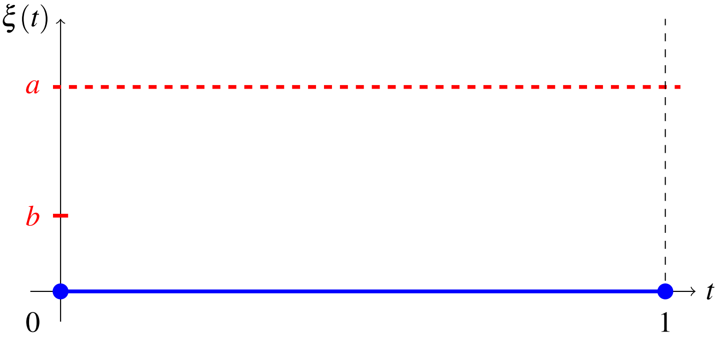


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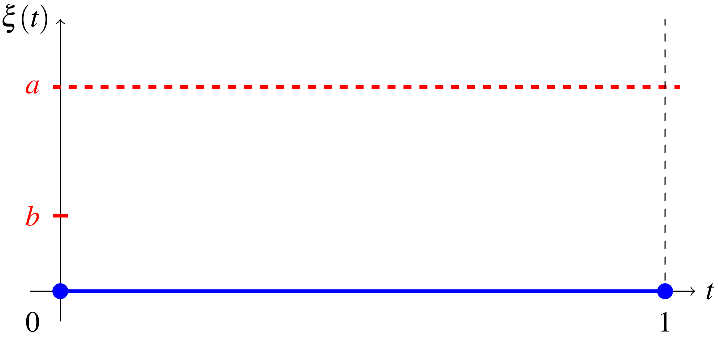


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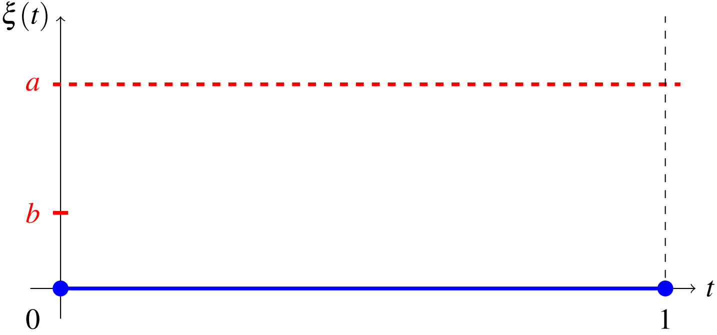


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**No information!**

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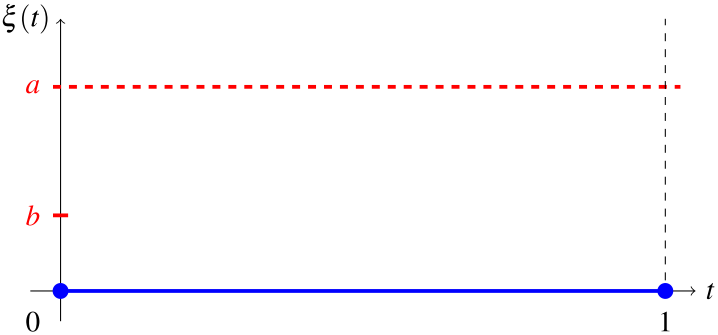


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**Want a stronger topology, ideally $J_1$ topology!**

# LDP w.r.t. $J_1$ Topology is Impossible

**Counterexample (Bazhba, Blanchet, R., Zwart 2020)**

*There exists a closed set $A \subseteq \mathbb{D}$ such that*

$$\limsup_{n\to\infty} \frac{1}{\sqrt{n}} \log \mathbf{P}\left(\bar{S}_n \in A\right) \not\leq -\inf_{\xi\in A} I(\xi)$$

- $L(n) = 1$ and $\alpha = \frac{1}{2}$ so that $\mathbf{P}(X_1 > x) = e^{-\sqrt{x}}$
- Paths in $A$ have $m$ increases of size $O\left(\frac{1}{m^2}\right)$, for some $m$

# “Extended” Large Deviation Principle

**Theorem (Bazhba, Blanchet, R., Zwart 2020)**

$\bar{X}_n$ *satisfies an “extended LDP” w.r.t. $J_1$ topology, i.e.,*

$$\limsup_{n\to\infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P}\left(\bar{X}_n(t) \in A\right) \le -\lim_{\varepsilon\to 0} \inf_{\xi\in A^\varepsilon} I(\xi)$$

$$\liminf_{n\to\infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P}\left(\bar{X}_n(t) \in A\right) \ge -\inf_{\xi\in A^\circ} I(\xi)$$

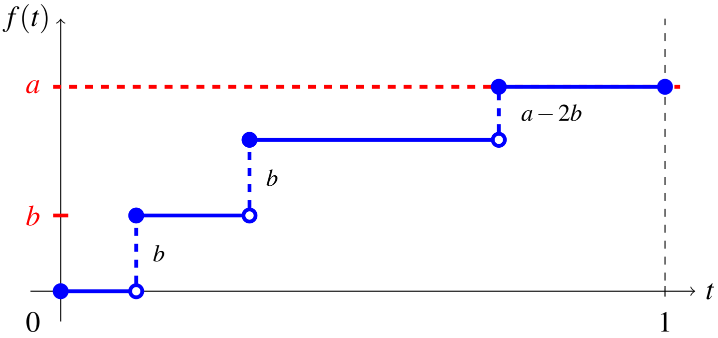
*where*

$$I(\xi) = \begin{cases} \sum_{\{t:\xi(t)\neq\xi(t^-)\}} \left(\xi(t)-\xi(t^-)\right)^\alpha, & \text{if } \xi \text{ is a nondecreasing pure jump function} \\ \infty, & o.w. \end{cases}$$

**Corollary**

*If $\phi$ is Lipschitz, $\phi\left(\bar{X}_n\right)$ satisfies a LDP, if the resulting rate function is good.*

# Back to our reinsurance example




- $A = \{f \in \mathbb{D} : f$ crosses level $a$ on $[0,1]$ & jump sizes $\leq b\}$
- $\mathbf{P}\left(\bar{S}_n \in A\right) \;\sim\; \exp\left(-n^{\alpha} I^{*}\right)$

  where $I^{*} = b^{\alpha} + b^{\alpha} + (a-2b)^{\alpha}$.

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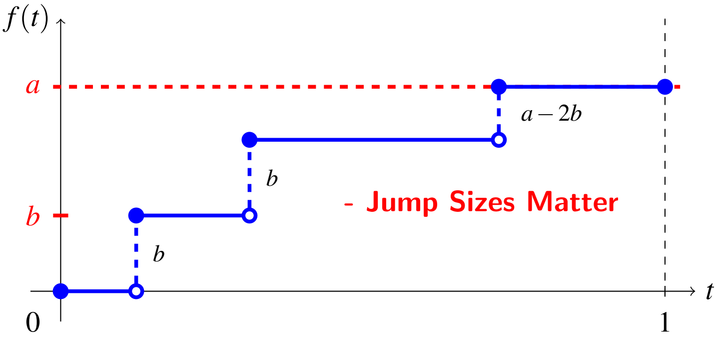


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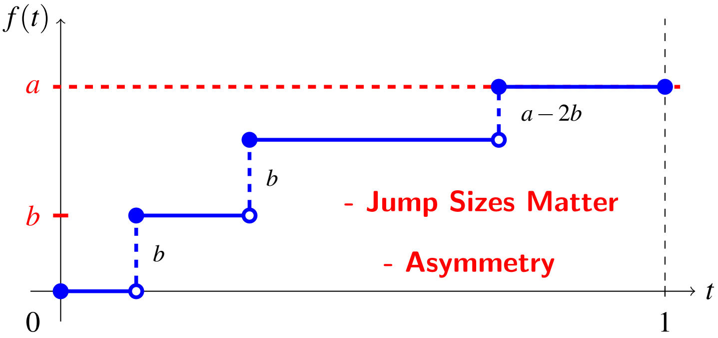


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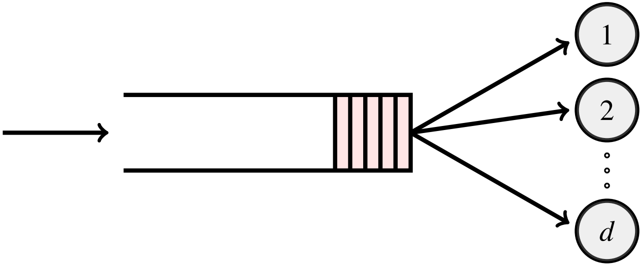
# Queue Length Asymptotics for the Weibull GI/GI/d Queue

In case service time distribution is Weibull, i.e., $\mathbf{P}(S > x) \sim \exp(-x^{\alpha})$,

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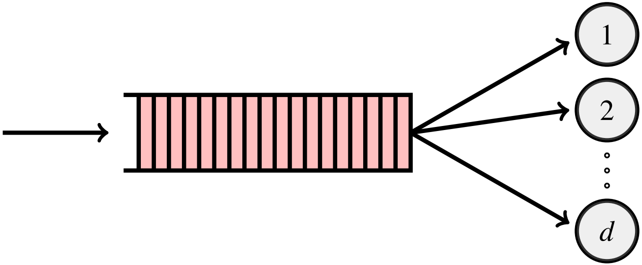


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**Solution to Open Question Posed by Whitt (2000) and Foss (2009)**

# More Specifically,

If $\gamma > \frac{1}{\lambda - \lfloor \lambda \rfloor}$ : job/jump sizes (of the most likely scenario) are **symmetric**

- If $\lfloor \lambda \rfloor \leq \frac{\lambda - \alpha d}{1 - \alpha}$: smallest possible number of big jobs to block enough servers
  $\Rightarrow$ **same as the power law case**

- If $\lfloor \lambda \rfloor > \frac{\lambda - \alpha d}{1 - \alpha}$: larger number of moderately big jobs might be more likely
  $\Rightarrow$ **qualitatively different from the power law case**

If $\gamma < \frac{1}{\lambda - \lfloor \lambda \rfloor}$ : job/jump sizes may be **asymmetric** (upto 3 different sizes)

# Other Topic III: Rare Event Simulation

# Challenge of Rare Event Simulation

Monte Carlo simulations as repetitive random experiments:

e.g. Coin flip: want to estimate $\mathbf{P}(\text{Head})$

- Flip the coin 100 times
- Count the number of head
- Divide by 100 and report the number

Should be reasonably close to $1/2$

# Challenge of Rare Event Simulation

Monte Carlo simulations as repetitive random experiments:

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**Is 0 a useful answer?**

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**Is 0 a useful answer? No.**

**e.g., Nuclear Meltdown, Large-Scale Blackout, Large Financial Loss**

# Challenge of Rare Event Simulation

Monte Carlo simulations as repetitive random experiments:

e.g. Coin flip: want to estimate $\mathbf{P}(\text{Edge}) \overset{\textit{Suppose}}{\approx} 10^{-6}$

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- Flip the coin ~~100~~ a few million times
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**Much harder than $\mathbf{P}$(Head)**

# Importance Sampling

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  - i.e., Report $\frac{1}{m}\sum_{i=1}^{m} \mathbb{I}^{(i)}_{\text{Edge}} \left(\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}\mathbf{Q}}\right)^{(i)}$ as an estimate of $\mathbf{P}(\text{Edge})$

# Importance Sampling for $\mathbf{P}(\bar{S}_n \in A)$

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**Finding a good alternative universe $\mathbf{Q}_n$ is crucial.**

# What is a good alternate universe $\mathbf{Q}_n$ for $\mathbf{P}(\bar{S}_n \in A)$?

General principle for making $\overbrace{\mathbb{I}_{\{\bar{S}_n \in A\}} \frac{\mathrm{d}\mathbf{P}}{\mathrm{d}\mathbf{Q}_n}}^{\text{IS estimator}}$ an efficient estimator:

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- **Make sure that $\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}\mathbf{Q}_n}$ does not blow up.**

# Goal: Strongly Efficient IS Estimator

$\underbrace{\mathbb{I}_{\{\bar{S}_n \in A\}} \frac{\mathrm{d}\mathbf{P}}{\mathrm{d}\mathbf{Q}_n}}^{\text{IS estimator}}$ is a **strongly efficient** estimator for $\mathbf{P}(\bar{S}_n \in A)$, if

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$\Rightarrow$ Number of simulation runs required remains bounded.

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**Considered Notoriously Hard for Heavy-Tailed Processes.**

# First Universal Rare-Event Simulation Scheme for Heavy-Tails

- Fix $w \in (0,1)$ and define

$$\mathbf{Q}_n(\,\cdot\,) \triangleq w\mathbf{P}(\,\cdot\,) + (1-w)\mathbf{P}(\,\cdot\,|\,\bar{X}_n \in B^{\gamma})$$

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- Fix $w \in (0,1)$ and define

$$\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}\mathbf{Q}_n} \text{ not too big}$$

$$\uparrow$$

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# First Universal Rare-Event Simulation Scheme for Heavy-Tails

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$$\sim \big(\mathbf{P}(\bar{X}_n \in A)\big)^2$$

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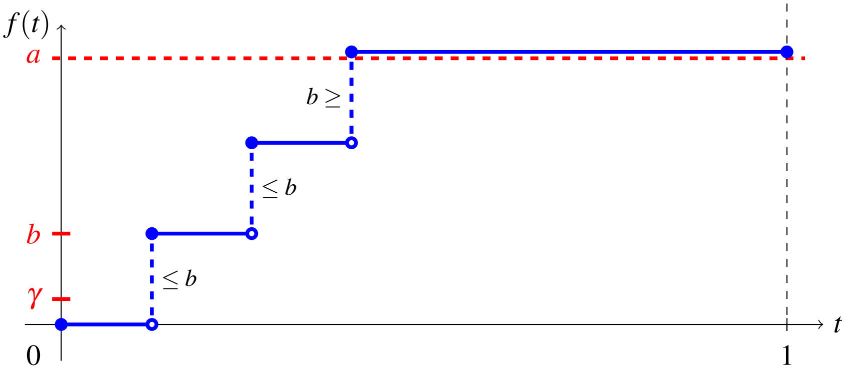
$Z_n \triangleq \mathbb{1}_{\{\bar{X}_n \in A\}} \frac{d\mathbf{P}}{d\mathbf{Q}_n}$ **is strongly efficient for** $\mathbf{P}(\bar{X}_n \in A)$**!**

Chen, Blanchet, R., and Zwart (2019)

# How to ensure $\mathcal{J}(A \setminus B^\gamma) \geq 2\mathcal{J}(A)$: Reinsurance Example




- $A$ $= \{$paths that cross level $a$ on $[0,1]$ & jump sizes $\leq b\}$
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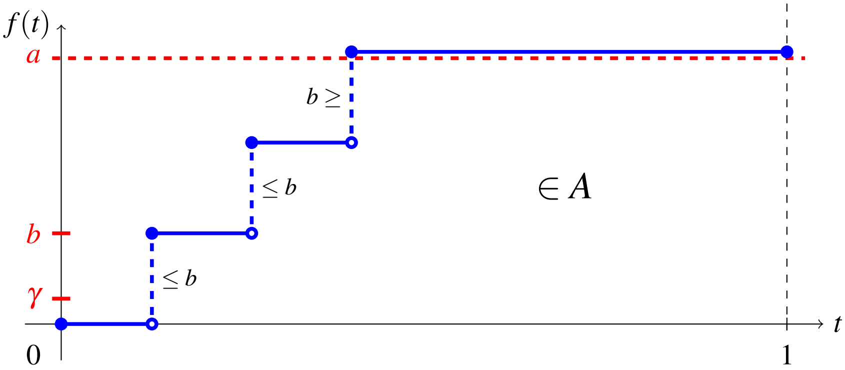


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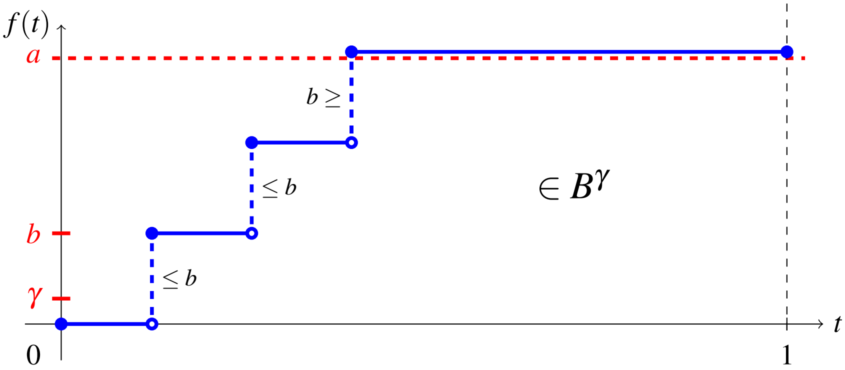


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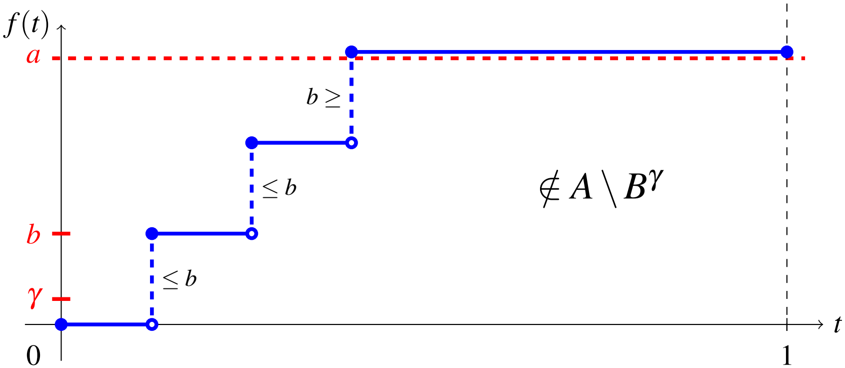


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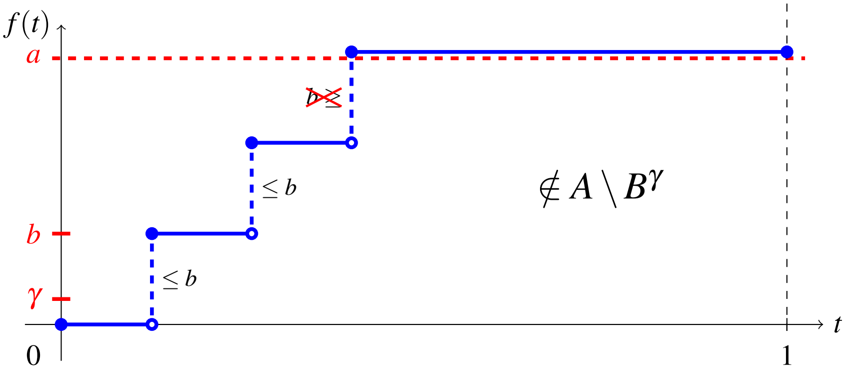


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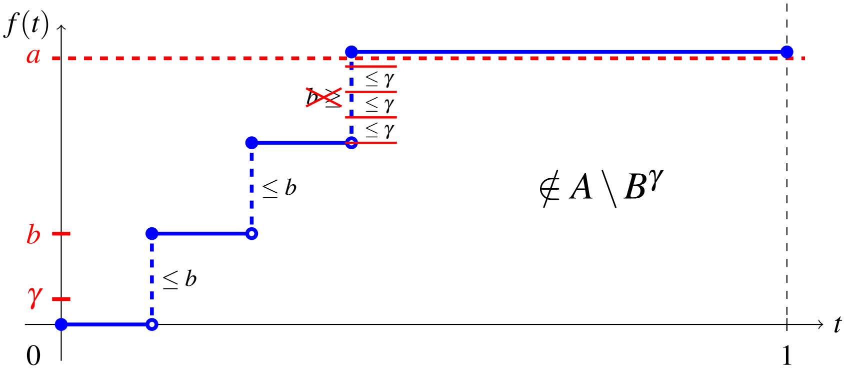


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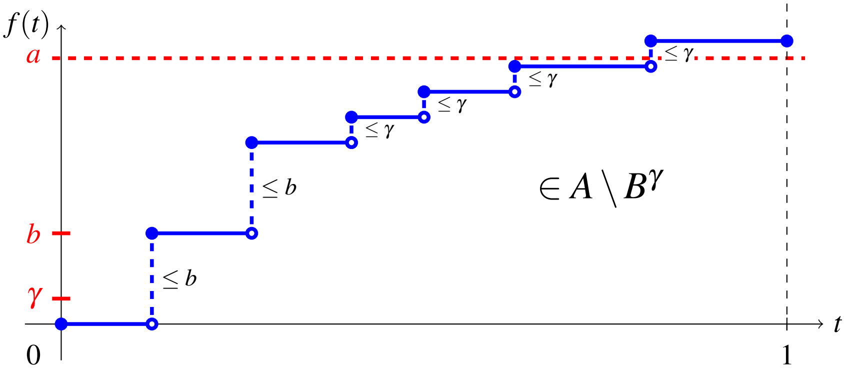


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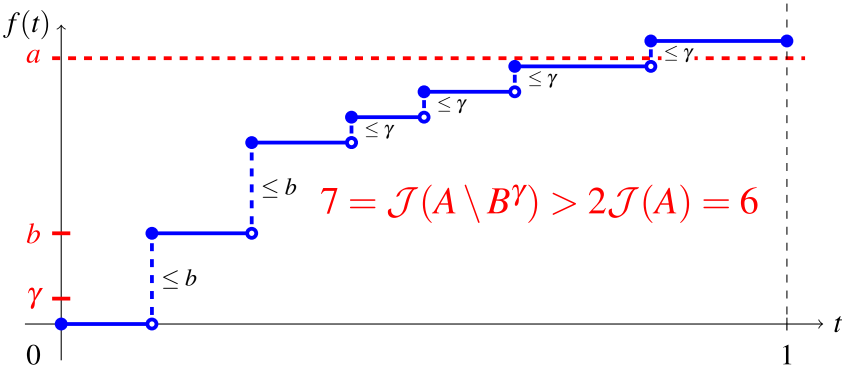


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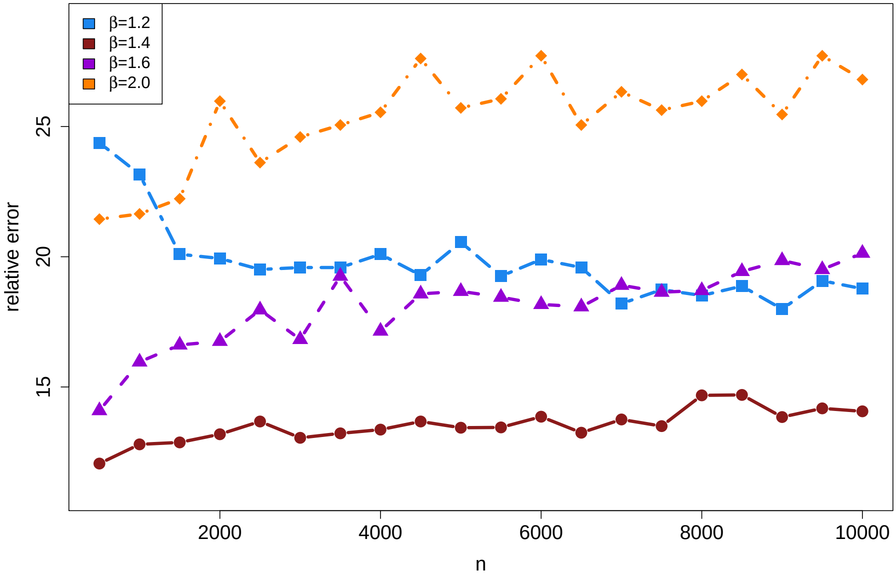
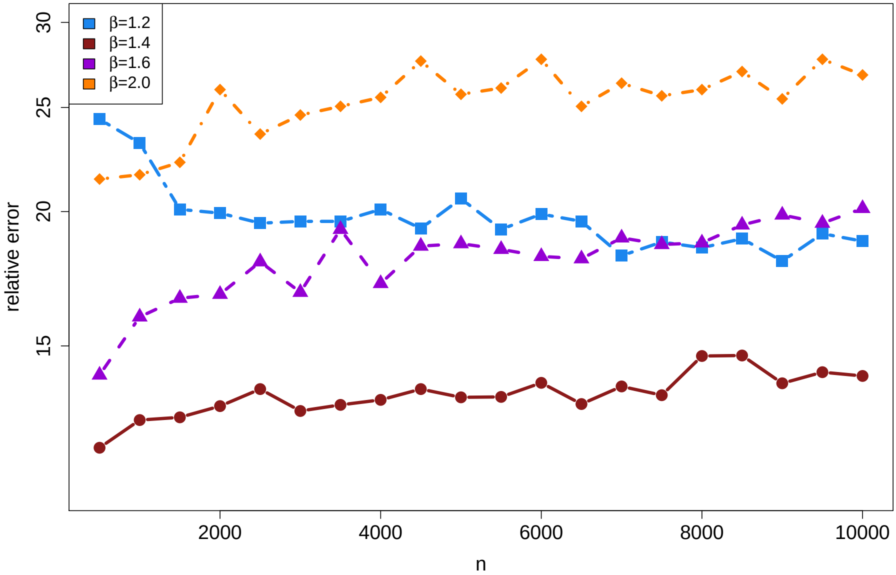
# Numerical Experiments for Reinsurance Example



# Numerical Results for Reinsurance Example



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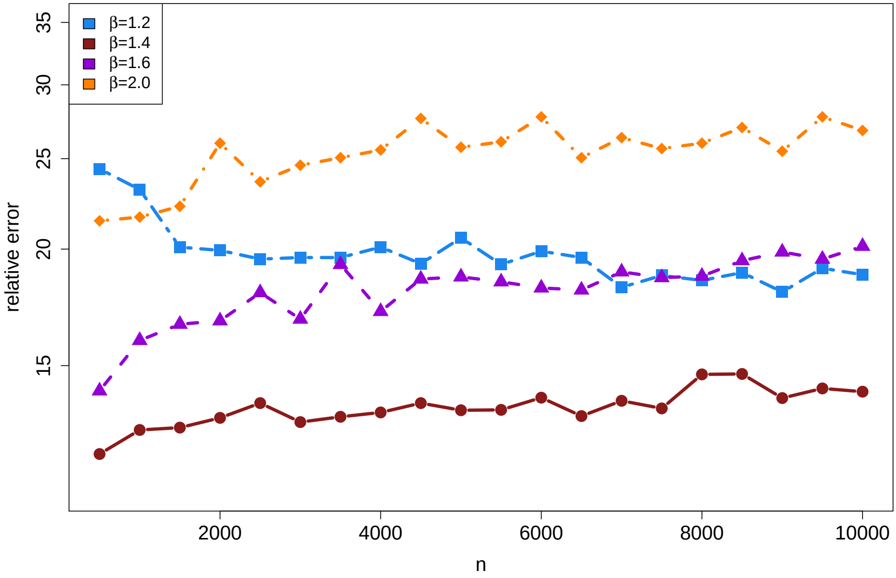

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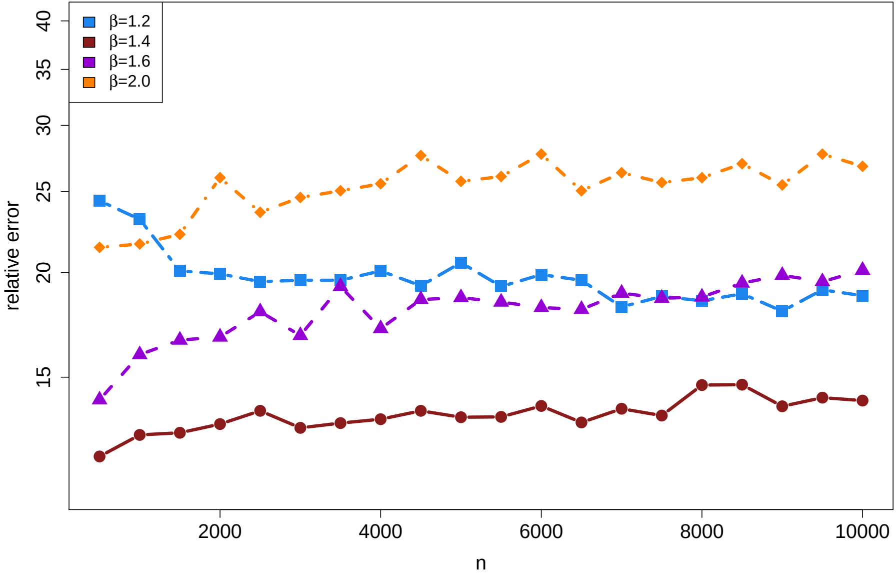

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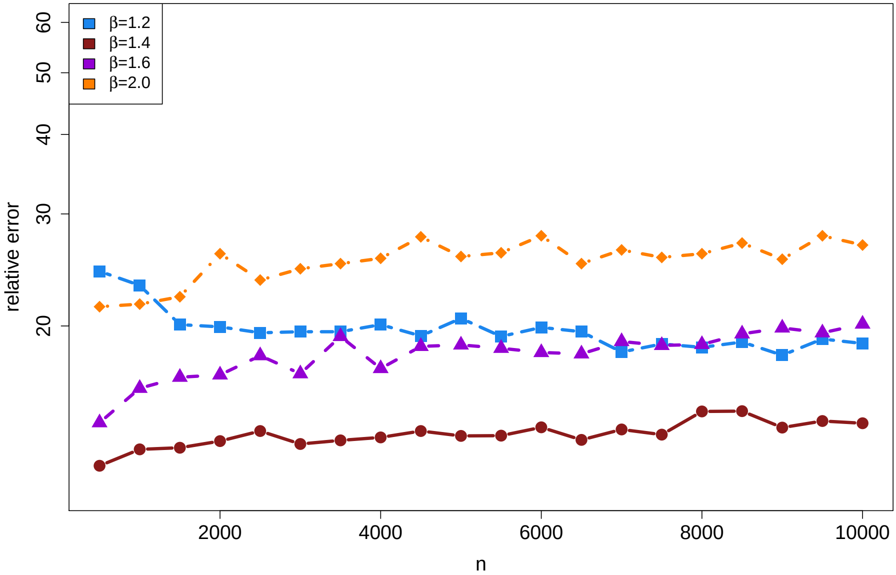

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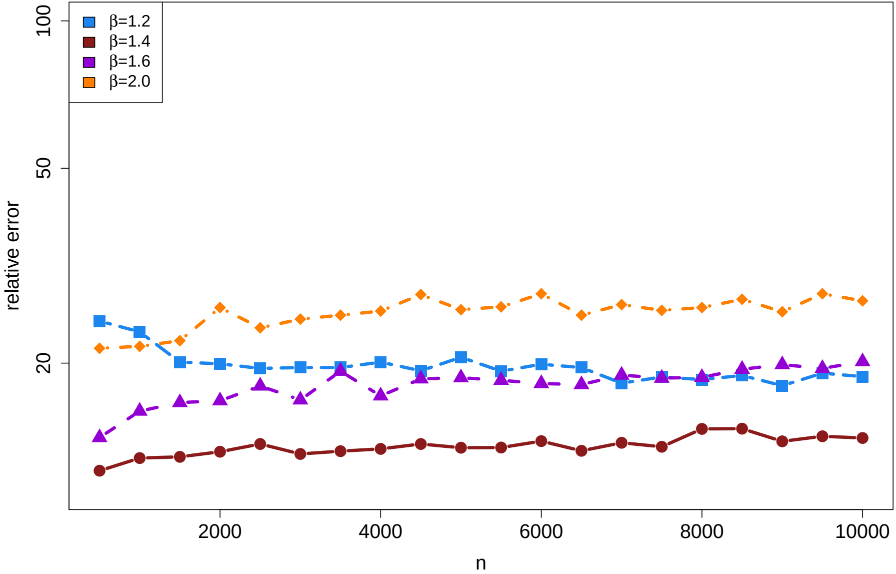

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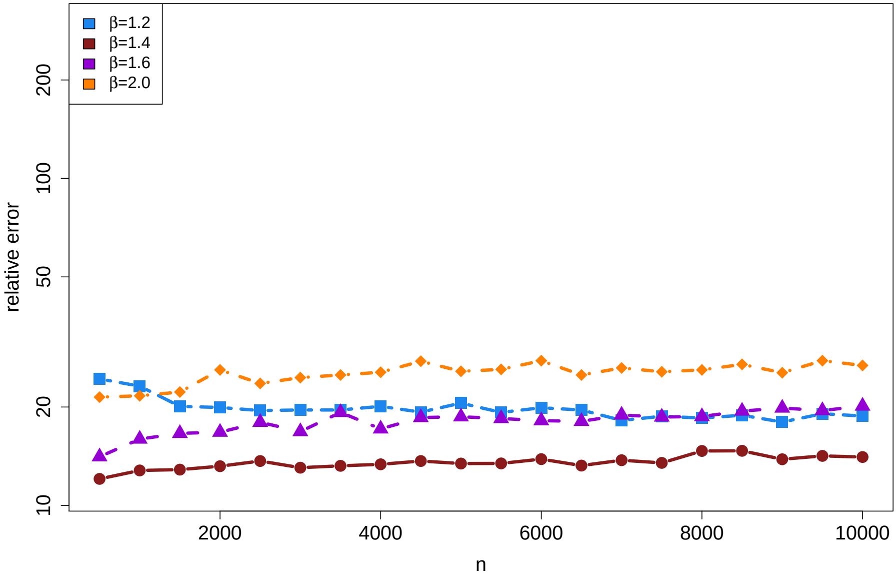

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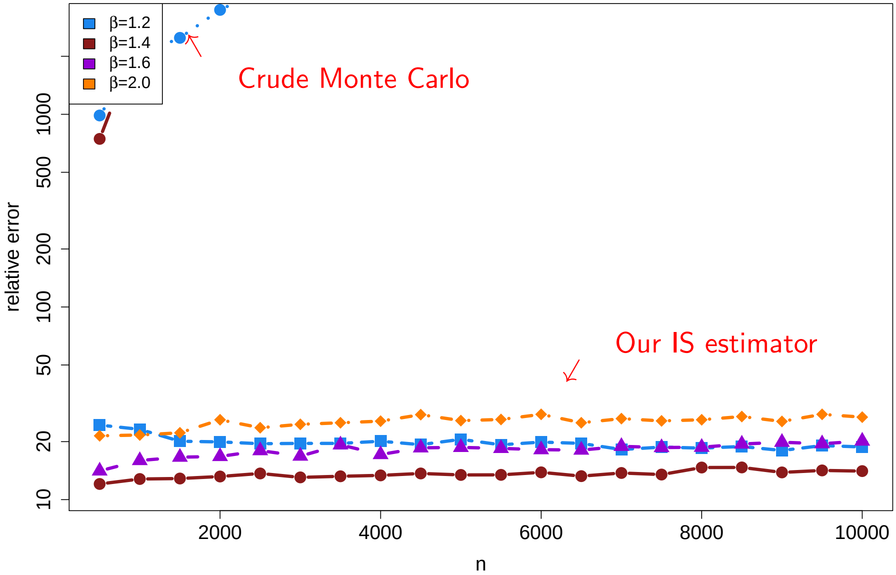

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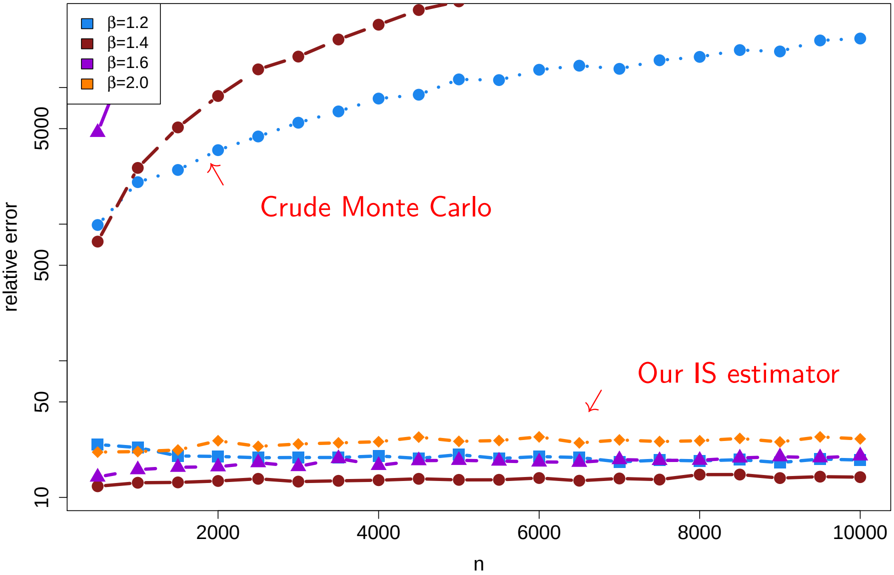

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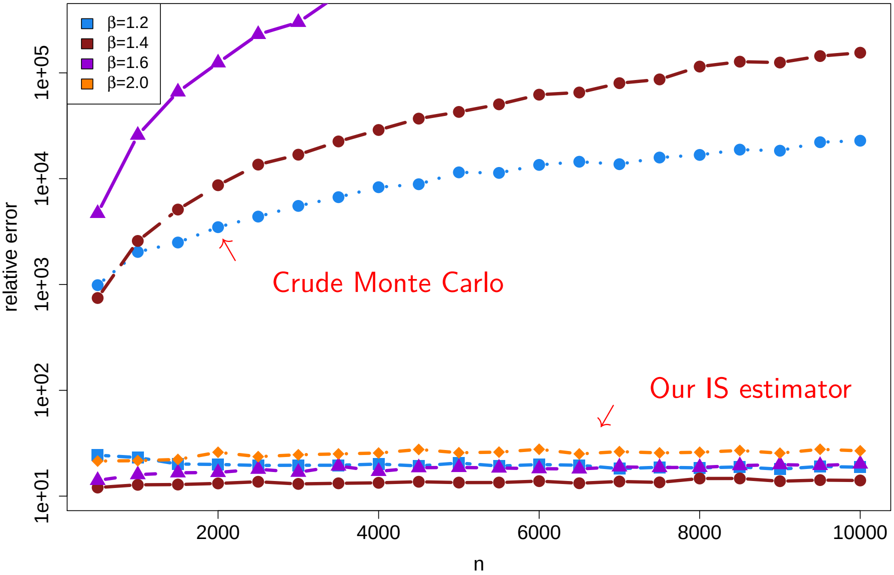

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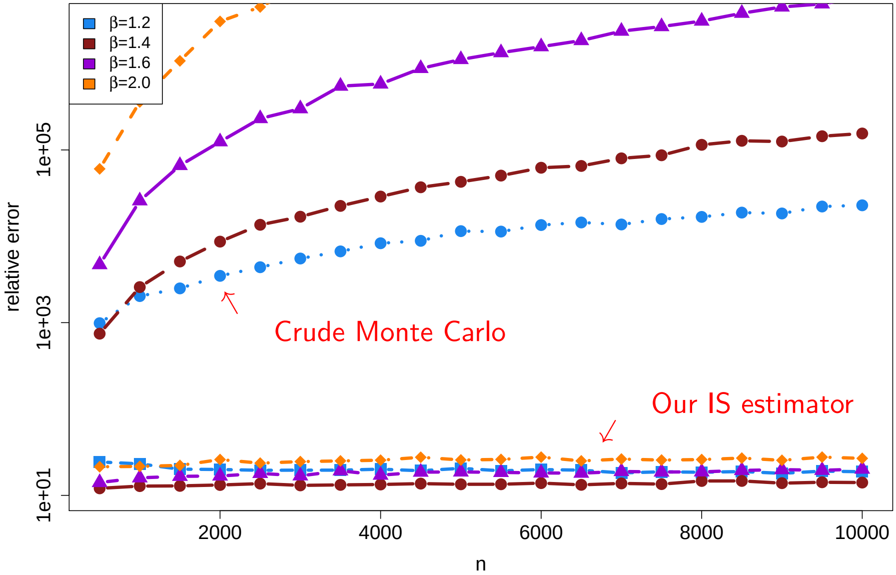

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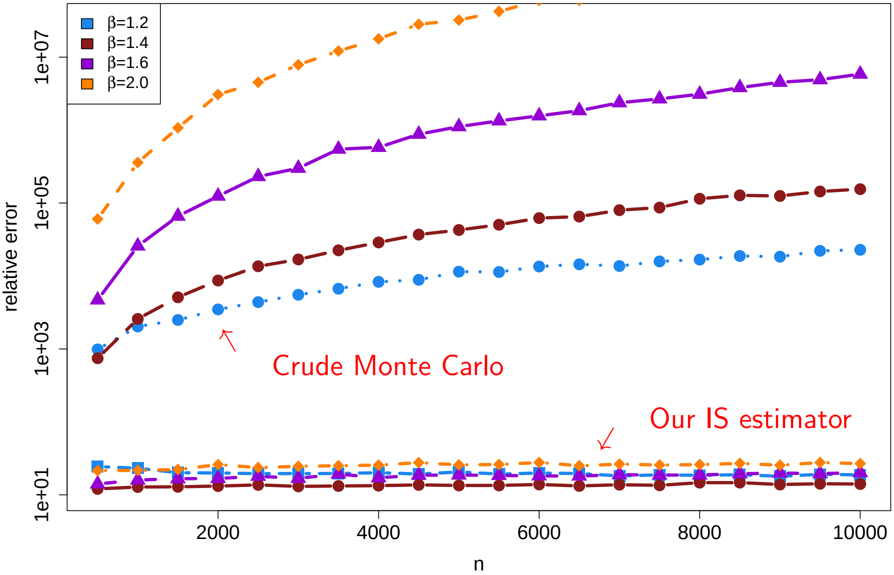

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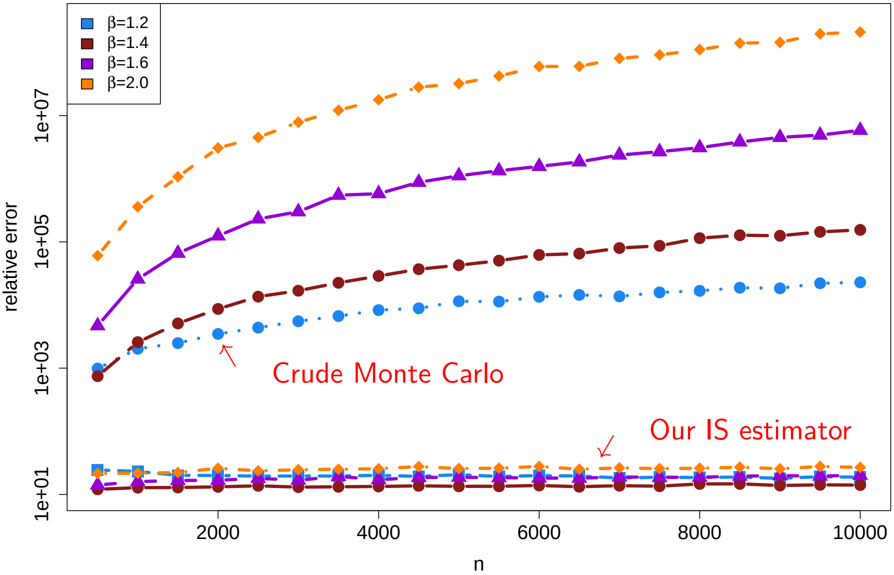

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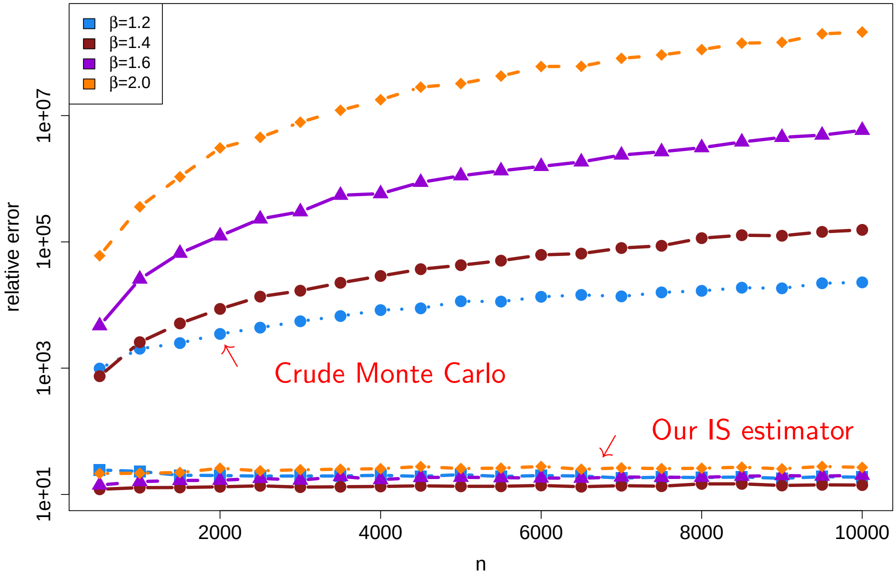
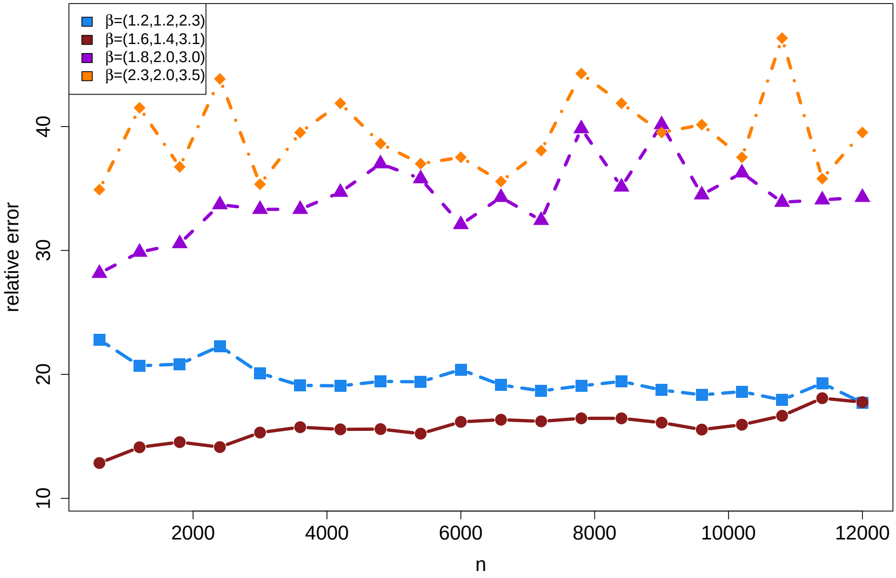


# Numerical Results for Fluid Network Example



# Summary

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- Catastrophe via Heavy-Tailed Large Deviation

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