Large deviations for stochastic fluid networks with Weibullian tails

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Abstract

We consider a stochastic fluid network where the external input processes are compound Poisson with heavy-tailed Weibullian jumps. Our results comprise of large deviations estimates for the buffer content process in the vector-valued Skorokhod space which is endowed with the product J_1 topology. To illustrate our framework, we provide explicit results for a tandem queue. At the heart of our proof is a recent sample-path large deviations result, and a novel continuity result for the Skorokhod reflection map in the product J_1 topology.

Keywords. fluid networks, large deviations, Skorodhod map, heavy tails.

Mathematics Subject Classification: 60K25, 60F10.

1 Introduction

The past 25 years have witnessed a significant research activity on queueing systems with heavy tails, but the important case of queueing networks has received less attention. Early papers focused on generalised Jackson networks (Baccelli et al. (2005)), monotone separable networks (Baccelli and Foss (2004)), and max-plus networks (Baccelli et al. (2004)). Recent work on tail asymptotics of transient cycle times and waiting times for closed tandem queueing networks can be seen in Kim and Ayhan (2015). In two joint papers with Foss, Masakiyo Miyazawa investigated queue lengths in a queueing network with feedback in Foss and Miyazawa (2014) and tandem queueing networks in Foss and Miyazawa (2018). Compared to standard queueing networks tracking movements of discrete customers, fluid networks are somewhat more tractable. In an early paper, it was recognized that tail asymptotics for downstream nodes could be obtained by analyzing the busy period of upstream nodes, under certain assumptions (Boxma and Dumas (1998)). The case of a tandem fluid queue where the input to the first node is a Lévy process with regularly

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varying jump sizes has been investigated in Lieshout and Mandjes (2008) exploiting a Laplace transform expression which is available in that case.

More recently, multidimensional asymptotics for the time-dependent buffer content vector in a fluid queue fed by compound Poisson processes were investigated in Chen et al. (2019). The framework in Chen et al. (2019) allows for the analysis of situations in which a large buffer content may be caused by multiple big jumps in the input process. Such results were established before for multiple server queues and fluid queues fed by on-off sources (see, for example, Zwart et al. (2004), Foss and Korshunov (2012)). The results on fluid networks in Chen et al. (2019) were derived assuming regular variation of the jumps in the arrival processes. Work on fluid networks with light-tailed input is surveyed in Miyazawa (2011). The goal of the present paper is to investigate the case where jumps are semi-exponential (e.g. of Weibull type $\exp\{-x^\alpha\}$ with $\alpha \in (0, 1)$). This case is somewhat more difficult to analyze, especially in the case where rare events of interest are caused by multiple big jumps in the input process, as exhibited in the case of the multiple server queue (Bazhba et al. (2019)).

We focus on a stochastic fluid network comprised of d nodes, with external inputs modeled as compound Poisson processes with semi-exponential increments. We are interested in the event that an arbitrary linear combination of the buffer contents in the network exceeds a large value. We write this functional as a mapping of the input processes using the well-known multidimensional Skorokhod reflection map on the positive orthant (see e.g. Whitt (2002)), and apply a sample-path large deviations principle for the superposition of Poisson processes, which has recently been derived in Bazhba et al. (2020). This sample-path large deviation principle has been established for Poisson processes with semi-exponential jumps, and holds in the product J_1 topology. To apply the contraction principle (the analogue of the continuous mapping argument in a large deviations context), we need to show that the Skorokhod map has suitable continuity properties. The J_1 product topology is not as strong as the standard J_1 topology on $\mathbb{R}^d$, and it turns out that continuity can only be established for input processes with non-negative jumps. However, this result, presented in Theorem 2.1 below, is sufficient for our proof strategy to work.

The contraction principle leads to an expression of the rate function which we analyze in detail. Under some generality, we show that each input process contributes to a large fluid level by at most one big jump, and the computation of the rate function is reduced to a concave optimization problem with $2d$ decision variables. We explicitly solve this optimization problem for the case $d = 2$.

The outline of this paper is as follows: Section 2 contains a description of our model, the topological space in which the input processes are defined, and an introduction to the reflection map. In Sections 3, 4, and 5 we present our main results: upper and lower large deviation bounds for the buffer content process, logarithmic asymptotics for overflow probabilities of the buffer content process over fixed times, and an explicit analysis of the two-node tandem network. Section 6 contains technical proofs. We end this paper with an appendix where we develop several auxiliary large deviations results.

2 Model description and preliminary results

2.1 The Model

In this section, we describe our model and we present some preliminary results that are used in our analysis. We consider a single-class open stochastic fluid network with d nodes. The total amount of external work that arrives at station i is equal to $J^{(i)}(t) \triangleq \sum_{j=1}^{N^{(i)}(t)} J_j^{(i)},\ t \in [0,T]$, which is a compound Poisson process with mean μ_i. If no exogenous input is assigned to node i, then set $J^{(i)}(\cdot) \equiv 0$, and $\mu_i \triangleq 0$. We define $\mathcal{J}$ as the subset of nodes that have an exogenous input. For notational convenience, we assume that the Poisson processes $\{N^{(i)}(t)\}_{t\geq 0}, i \in \mathcal{J}$ have unit rate. We pose an assumption on the distribution of the jump sizes $J_1^{(i)}$, for $i \in \mathcal{J}$, making it semi-exponential:

Assumption 1. *For each $i \in \mathcal{J} \subseteq \{1,\dots,d\}$, $\mathbf{P}\left(J_1^{(i)} \geq x\right) = e^{-c_i L(x) x^\alpha}$ where $\alpha \in (0,1)$, $c_i \in (0,\infty)$, and L is a slowly varying function such that $L(x)/x^{1-\alpha}$ is non-increasing for sufficiently large x.*

Recall that L is slowly varying if $L(ax)/x \to 1$ as $x \to \infty$ for each $a > 0$. At each node $i \in \{1,\dots,d\}$, the fluid is processed and released at a deterministic rate r_i. Fractions of the processed fluid from each node are then routed to other nodes or leave the network. We characterize the stochastic fluid network by a four-tuple $(\mathbf{J}, \mathbf{r}, Q, \mathbf{X}(0))$, where $\mathbf{J}(\cdot) = \left(J^{(1)}(\cdot),\dots,J^{(d)}(\cdot)\right)^\intercal$ is the vector of the assigned input processes at each one of the d nodes, respectively. The vector $\mathbf{r} \triangleq (r_1,\dots,r_d)^\intercal$ is the vector of deterministic output rates at the d nodes, $Q \triangleq [q_{ij}]_{i,j\in\{1,\dots,d\}}$ is a $d \times d$ substochastic routing matrix, and $\mathbf{X}(0) \triangleq (X^{(1)}(0),\dots,X^{(d)}(0))$ is a nonnegative random vector of initial contents at the d nodes. If the buffer at node i and at time t is nonempty, then there is fluid output from node i at a constant rate r_i. On the other hand, if the buffer of node i is empty at time t, the output rate equals the minimum of the combined (i.e., both external and internal) input rates and the output rate r_i.

We now provide more details on the stochastic dynamics of our network. A proportion $q_{i,j}$ of all output from node i is immediately routed to node j, while the remaining proportion $q_i \triangleq 1 - \sum_{j=1}^k q_{ij}$ leaves the network. We assume that $q_{ii} \triangleq 0$, and the routing matrix Q is substochastic, so that $q_{ij} \geq 0$, and $q_i \geq 0$ for all i,j. We also assume that $Q^n \to 0$ as $n \to \infty$ which implies that all input eventually leaves the network. Let $Q^\intercal$ be the transpose matrix of Q. Though we focus on time-dependent behavior, we consider the scenario that the fluid network is stable, ensuring that a high level of fluid is a rare event. We guarantee the stability of the network by posing the following assumption, based on Kella (1996):

Assumption 2. *Let $\boldsymbol{\mu} = (\mu_1,\dots,\mu_d)^\intercal$, and assume that $(\mathrm{I} - Q^\intercal)\mathbf{r} - \boldsymbol{\mu} > 0$.*

Due to our model specifics, the buffer content at station i is processed at a constant rate r_i from the i-th server; and a proportion q_{ij} is routed from the i-th station to the j-th server. Let $\mathcal{Q} = (\mathrm{I} - Q^\intercal)$. To define the buffer content process we first define the potential content vector $\mathbf{X}(t)$, $\mathbf{X}(t) \triangleq \mathbf{X}(0) + \mathbf{J}(t) - \mathcal{Q}rt, t \geq 0$. Let $\mathbf{Z}^{(i)}(t)$ denote the buffer content of the i-th station at time t. We can define the buffer content process by the so-called reflection map. We first provide an intuitive description of this map. It is defined in terms of a pair of processes $(\mathbf{Z}, \mathbf{Y})$ that solve the differential equation

$$d\mathbf{Z}(t) = d\mathbf{X}(t) + \mathcal{Q}d\mathbf{Y}(t),\ t \geq 0. \tag{2.1}$$

Here, $\mathbf{Y}(\cdot)$ is non-decreasing and $\mathbf{Y}_i(t)$ only increases at times where $\mathbf{Z}_i(t) = 0$ for all i and all t. Consequently, as we assume $\mathbf{Z}(0) = 0$, the buffer content is

$$\mathbf{Z}(t) = \mathbf{X}(t) + \mathcal{Q}\mathbf{Y}(t),\ t \geq 0. \tag{2.2}$$

We call the map $\mathbf{X} \mapsto (\mathbf{Y}, \mathbf{Z})$ the reflection map. We now provide a more rigorous definition of this map.

2.2 The reflection map with discontinuities

We start with the definition of the reflection map. Let $\mathbb{D}[0, T]$ denote the Skorokhod space: the space of càdlàg functions over $[0, T]$. Denote with $\mathbb{D}^{\uparrow}[0, T]$ the subspace of the Skorokhod space consisting of non-decreasing functions that are non-negative at the origin. Note that we use the component-wise partial order on $\mathbb{D}[0, T]$ and $\mathbb{R}^k$. That is, we write $x_1 \triangleq (x_1^{(1)}, \ldots, x_1^{(k)}) \leq x_2 \triangleq (x_2^{(1)}, \ldots, x_2^{(k)})$ in $\mathbb{R}^k$ if $x_1^{(i)} \leq x_2^{(i)}$ in $\mathbb{R}$ for all $i \in \{1, \ldots, k\}$, and we write $\xi \leq \zeta$ in $\prod_{i=1}^k \mathbb{D}[0, T]$ if $\xi(t) \leq \zeta(t)$ in $\mathbb{R}^k$ for all $t \in [0, T]$.

Definition 2.1. *(Definition 14.2.1 of Whitt (2002)) For any $\xi \in \prod_{i=1}^k \mathbb{D}[0, T]$ and any reflection matrix $\mathcal{Q} = (\mathrm{I} - Q^\intercal)$, let the feasible regulator set be*

$$\Psi(\xi) \triangleq \left\{ \zeta \in \prod_{i=1}^k \mathbb{D}^{\uparrow}[0, T] : \xi + \mathcal{Q}\zeta \geq 0 \right\},$$

and let the reflection map be

$$\mathbf{R} \triangleq (\psi, \phi) : \prod_{i=1}^k \mathbb{D}[0, T] \to \prod_{i=1}^k \mathbb{D}[0, T] \times \prod_{i=1}^k \mathbb{D}[0, T],$$

with regulator component

$$\psi(\xi) \triangleq \inf \{\Psi(\xi)\} = \inf \left\{ w \in \prod_{i=1}^k \mathbb{D}[0, T] : w \in \Psi(\xi) \right\},$$

and content component

$$\phi(\xi) \triangleq \xi + \mathcal{Q}\psi(\xi).$$

The infimum in the definition of ψ may not exist in general. However, in Theorem 14.2.1 of Whitt (2002), it is proven that the reflection map is properly defined with the component-wise order. That is,

$$\psi^{(i)}(\xi)(t) = \inf\{w_i(t) \in \mathbb{R} : w \in \Psi(\xi)\} \text{ for all } i \in \{1, \ldots, k\} \text{ and } t \in [0, T].$$

In addition, the regulator set $\Psi(\xi)$ is non-empty and its infimum is attained in $\Psi(\xi)$. Now, we state some important results regarding the properties of (ϕ, ψ). The following result gives an explicit representation of the solution of the Skorokhod problem.

Result 2.1. *(Theorem 14.2.1, Theorem 14.2.5 and Theorem 14.2.7 of Whitt (2002)) If $Y(\cdot) = \psi(\mathbf{X})(\cdot)$ and $\mathbf{Z}(\cdot) = \phi(\mathbf{X})(\cdot)$, then $(\mathbf{Y}(\cdot), \mathbf{Z}(\cdot))$ solves the Skorokhod problem associated with the equation (2.1). The mappings ψ and ϕ are Lipschitz continuous maps w.r.t. the uniform metric.*

The next result is a useful property of the Skorokhod map. It allows us to describe the discontinuities of the reflection map under some mild assumptions.

Result 2.2. *(Lemma 14.3.3, Corollary 14.3.4 and Corollary 14.3.5 of Whitt (2002)) Consider $\xi \in \prod_{i=1}^k \mathbb{D}[0,T]$. Let $Disc(\psi(\xi))$ and $Disc(\phi(\xi))$ denote the sets of discontinuity points of $\psi(\xi)$ and $\phi(\xi)$, respectively. Then it holds that $Disc(\psi(\xi)) \cup Disc(\phi(\xi)) = Disc(\xi)$. In addition, if ξ has only positive jumps, then $\psi(\xi)$ is continuous and*

$$\phi(\xi)(t) - \phi(\xi)(t-) = \xi(t) - \xi(t-).$$

Result 2.3. *(Theorem 14.2.6 of Whitt (2002)) If $\xi \leq \zeta$ in $\prod_{i=1}^d \mathbb{D}[0,T]$, $T > 0$, then $\psi(\xi) \geq \psi(\zeta)$.*

2.3 Topologies and large deviations

In this section, we introduce our preliminary results on sample-path large deviations for the input and the content process. For our results we will use the J_1 topology. In order to study networks, we need a multidimensional functional setting. That is, we work on the functional space $(\prod_{i=1}^k \mathbb{D}[0,T], \prod_{i=1}^k \mathcal{T}_{J_1})$ which is a product space equipped with the product J_1 topology. The product topology—which we denote with $\prod_{i=1}^k \mathcal{T}_{J_1}$—is induced by the product metric d_p which in turn is defined in terms of the J_1 metric on $\mathbb{D}[0,T]$. More precisely, for $\xi, \zeta \in \prod_{i=1}^k \mathbb{D}[0,T]$ such that $\xi = (\xi^{(1)}, \ldots, \xi^{(k)})$ and $\zeta = (\zeta^{(1)}, \ldots, \zeta^{(k)})$ we have that $d_p(\xi, \zeta) \triangleq \sum_{i=1}^k d_{J_1}(\xi^{(i)}, \zeta^{(i)})$. Whenever we make a statement on convergence w.r.t. $(\prod_{i=1}^k \mathbb{D}[0,T], \prod_{i=1}^k \mathcal{T}_{J_1})$, we mean convergence w.r.t. d_p. Closed and open sets are understood to be generated by d_p.

2.3.1 Some useful continuous functions

The following two lemmas are elementary. Their proofs are provided in Appendix A.

Lemma 2.2. *For $\kappa \in \mathbb{R}^d$, let $\Upsilon_\kappa : \prod_{i=1}^d \mathbb{D}[0,T] \to \prod_{i=1}^d \mathbb{D}[0,T]$ be such that $\Upsilon_\kappa(\xi)(t) = \xi(t) - \kappa t$. Then,*

i) Υ_κ is Lipschitz continuous w.r.t. d_p,

ii) Υ_κ is a homeomorphism.

Lemma 2.3. *The function $\pi : \prod_{i=1}^d \mathbb{D}[0,T] \mapsto \mathbb{R}^d$, $\pi(\xi) = \xi(T)$ is Lipschitz continuous w.r.t. d_p.*

A key step in our approach is to establish the Lipschitz continuity of the regulator and the buffer content component maps w.r.t. d_p. This is executed in Proposition 2.1 and Theorem 2.1 below. Their proofs are deferred to Section 6. Recall that $\mathbb{D}^{\uparrow}[0,T]$ is the subspace of the Skorokhod space containing non-decreasing paths which are non-negative at the origin. We say that $\xi \in \mathbb{D}[0,T]$ is a pure jump function if $\xi = \sum_{i=1}^{\infty} x_i \mathbb{1}_{[u_i,T]}$ for some x_i's and u_i's such that $x_i \in \mathbb{R}$ and $u_i \in [0,T]$ for each i, and the u_i's are all distinct. Let $\mathbb{D}^{\uparrow}_{\leqslant\infty}[0,T]$ be the subspace of $\mathbb{D}[0,T]$ consisting of non-decreasing pure jump functions that assume non-negative values at the origin. Subsequently, let $\mathbb{D}^{\uparrow}_{\leqslant k}[0,T] \triangleq \{\xi \in \mathbb{D}[0,T] : \xi = \sum_{i=1}^{k} x_i \mathbb{1}_{[u_i,T]}, x_i \geq 0, u_i \in [0,T], i = 1,\ldots,k\}$ be the subset of $\mathbb{D}^{\uparrow}_{\leqslant\infty}[0,T]$ containing pure jump functions of at most k jumps. In addition, for $\beta \in \mathbb{R}$, let $\mathbb{D}^{\beta}_{\leqslant k}[0,T] \triangleq \{\zeta \in \mathbb{D}[0,T] : \zeta(t) = \xi(t) + \beta\cdot t,\ \xi \in \mathbb{D}^{\uparrow}_{\leqslant k}[0,T]\}$ and $\mathbb{D}^{\beta}_{\leqslant\infty}[0,T] \triangleq \{\zeta \in \mathbb{D}[0,T] : \zeta(t) = \xi(t) + \beta\cdot t,\ \xi \in \mathbb{D}^{\uparrow}_{\leqslant\infty}[0,T]\}$. Let $\mathbb{D}_{\leqslant k}[0,T]$ denote the subspace of $\mathbb{D}[0,T]$ consisting of paths with at most k jumps, i.e. $\mathbb{D}_{\leqslant k}[0,T] = \{\xi \in \mathbb{D}[0,T] : |Disc(\xi)| \leq k\}$. Finally, let $\mathbb{D}^{\beta}[0,T] \triangleq \{\zeta \in \mathbb{D}[0,T] : \zeta(t) = \xi(t) + \beta\cdot t,\ \xi \in \mathbb{D}^{\uparrow}[0,T]\}$.

Proposition 2.1. *Let $\beta_i \in \mathbb{R}$ for $i = 1,\ldots,d$. The (restriction of the) regulator map $\psi : \prod_{i=1}^{d} \mathbb{D}^{\beta_i}[0,T] \to \prod_{i=1}^{d} \mathbb{D}[0,T]$ is Lipschitz continuous w.r.t. d_p.*

Theorem 2.1. *Let $\beta_i \in \mathbb{R}$ for $i = 1,\ldots,d$. The (restriction of the) reflection map $\mathbf{R} : \prod_{i=1}^{d} \mathbb{D}^{\beta_i}[0,T] \to \prod_{i=1}^{2d} \mathbb{D}[0,T]$, where $\mathbf{R} = (\phi,\psi)$, is Lipschitz continuous w.r.t. d_p.*

Note that the restriction of the domain to the paths without downward jumps is essential for this type of results to hold. Since the order in which the jumps take place matters for the action of the reflection map, we cannot ensure the continuity of the reflection map without such extra regularity conditions. The main difficulty arises with paths which have jumps with different signs in multiple coordinates appearing almost simultaneously (K. Ramanan, personal communication).

2.3.2 The extended sample-path LDP for the content process

Let $(\mathbb{S}, d)$ be a metric space, and $\mathcal{T}$ denote the topology induced by the metric d. Let $\{X_n\}$ be a sequence of $\mathbb{S}$-valued random variables. Let I be a non-negative lower semi-continuous function on $\mathbb{S}$, and $\{a_n\}$ be a sequence of positive real numbers that tends to infinity as $n \to \infty$.

Definition 2.4. *The probability measures of (X_n) satisfy an* extended *LDP in $(\mathbb{S}, d)$ with speed a_n and rate function I if*

$$-\inf_{x\in A^\circ} I(x) \leq \liminf_{n\to\infty} \frac{\log \mathbf{P}(X_n \in A)}{a_n} \leq \limsup_{n\to\infty} \frac{\log \mathbf{P}(X_n \in A)}{a_n} \leq -\lim_{\epsilon\to 0} \inf_{x\in A^\epsilon} I(x)$$

for any measurable set A.

Here we denote $A^\epsilon \triangleq \{\xi \in \mathbb{S} : d(\xi, A) \leq \epsilon\}$ where $d(\xi, A) = \inf_{\zeta\in A} d(\xi,\zeta)$. The notion of an extended LDP has been introduced in Borovkov and Mogulskii (2010) and is quite useful in the setting of semi-exponential random variables, in which a full LDP is provably impossible, as is shown in Bazhba et al. (2020).

We finish this section by stating an extended large deviations principle for the multidimensional input process of the stochastic fluid network. Recall that $\mathbf{J}$ denotes the input process which is a vector of independent compound Poisson processes with mean vector $\boldsymbol{\mu}$. We consider the scaled version $\bar{\mathbf{J}}_n(\cdot) \triangleq \frac{1}{n}\mathbf{J}(n\cdot)$. For any $\xi \in \prod_{i=1}^d \mathbb{D}[0,T]$, let

$$I(\xi^{(j)}) = \sum_{\{t:\xi^{(j)}(t)\neq\xi^{(j)}(t-)\}} \left(\xi^{(j)}(t) - \xi^{(j)}(t-)\right)^{\alpha}.$$

The next result is an immediate consequence of Theorem 2.3 and Remark 2.2 in Bazhba et al. (2020), combined with Lemma B.1.

Result 2.4. *The probability measures of $\bar{\mathbf{J}}_n$ satisfy the extended LDP in $\left(\prod_{i=1}^d \mathbb{D}^{\uparrow}[0,T], \prod_{i=1}^d \mathcal{T}_{J_1}\right)$ with speed $L(n)n^{\alpha}$, and with rate function $I^{(d)} : \prod_{i=1}^d \mathbb{D}^{\uparrow}[0,T] \to [0,\infty]$, where*

$$I^{(d)}(\xi^{(1)},...,\xi^{(d)}) = \begin{cases} \sum_{j\in\mathcal{J}} c_j I(\xi^{(j)}) & \textit{if } \xi^{(j)} \in \mathbb{D}^{\mu_j}_{\leqslant\infty}[0,T] \textit{ for } j\in\mathcal{J} \\ & \textit{and } \xi^{(j)} \equiv 0 \textit{ for } j\notin\mathcal{J}, \\ \infty & \textit{otherwise.} \end{cases} \tag{2.3}$$

Recall that the potential content vector is a function of the exogenous input plus the internal input; that is, $\mathbf{X}(t) = \mathbf{J}(t) - \mathcal{Q}\mathbf{r}t$. We define the scaled version of the potential content vector $\mathbf{X}_n(\cdot) = \frac{1}{n}\mathbf{X}(n\cdot)$. Obviously, $\mathbf{X}_n$ is the image of $\mathbf{J}_n$ where the map $\Upsilon_{\mathcal{Q}\mathbf{r}}$ is applied. Due to Lemma 2.2, $\Upsilon_{\mathcal{Q}\mathbf{r}}$ is Lipschitz continuous and is a homeomorphism with respect to the product J_1 metric. The following large deviation principle for $\mathbf{X}_n(\cdot)$ is a direct consequence of Result 2.4 and *ii)* of Lemma B.3.

Result 2.5. *The probability measures of $\mathbf{X}_n$ satisfy an extended LDP in $\left(\prod_{i=1}^d \mathbb{D}^{-(\mathcal{Q}\mathbf{r})_i}[0,T], \prod_{i=1}^d \mathcal{T}_{J_1}\right)$ with speed $L(n)n^{\alpha}$ and with rate function*

$$\tilde{I}^{(d)}(\xi^{(1)},...,\xi^{(d)}) = \begin{cases} \sum_{j\in\mathcal{J}} c_j I(\xi^{(j)}) & \textit{if } \xi^{(j)} \in \mathbb{D}^{(\boldsymbol{\mu}-\mathcal{Q}\mathbf{r})_i}_{\leqslant\infty}[0,T] \textit{ for } j\in\mathcal{J} \\ & \textit{and } \xi^{(j)}(t) = -(\mathcal{Q}\mathbf{r})_j(t) \textit{ for } j\notin\mathcal{J}, \\ \infty & \textit{otherwise.} \end{cases} \tag{2.4}$$

In the next sections, we state our main results along with some additional useful lemmas.

3 Large deviations for the buffer content process

In this section, we state large deviation bounds for the scaled buffer content process $\mathbf{Z}_n(\cdot) \triangleq \frac{1}{n}\mathbf{Z}(n\cdot)$ with $\mathbf{Z}$ defined in (2.2). The reflection map enables us to represent the buffer content process in terms of the potential content process $\mathbf{X}_n(\cdot) \triangleq \frac{1}{n}\mathbf{X}(n\cdot)$ and the map ϕ by $\mathbf{Z}_n = \phi(\mathbf{X}_n)$. We apply an analogue of the contraction principle for extended LDP's (Lemma B.3) to obtain asymptotic estimates for the probability measures of $(\mathbf{Z}_n)$:

Theorem 3.1. *The probability measures of* $\mathbf{Z}_n$ *satisfy:*

i) For any set F *that is closed in* $\left(\prod_{i=1}^d \mathbb{D}[0,T], \prod_{i=1}^d \mathcal{T}_{J_1}\right)$,

$$\limsup_{n\to\infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P}\left(\mathbf{Z}_n \in F\right) \le -\lim_{\epsilon\to 0} \inf_{\xi\in F^\epsilon} I_S(\xi).$$

ii) For set G *that is open in* $\left(\prod_{i=1}^d \mathbb{D}[0,T], \prod_{i=1}^d \mathcal{T}_{J_1}\right)$,

$$\liminf_{n\to\infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P}\left(\mathbf{Z}_n \in G\right) \ge -\inf_{\xi\in G} I_S(\xi),$$

where

$$I_S(\zeta) = \inf\left\{ \tilde{I}^{(d)}(\xi) : \zeta = \phi(\xi),\ \xi \in \prod_{i=1}^d \mathbb{D}^{-(\mathcal{Q}\mathbf{r})_i}[0,T] \right\}.$$

Note that I_S may not be lower semi-continuous, because $\tilde{I}$ is not a good rate function; see Bazhba et al. (2020) for details.

Proof. Theorem 2.1 ensures that ϕ is Lipschitz continuous w.r.t. d_p. The bounds therefore follows immediately from Lemma B.3 and Result 2.5. □

The function I_S is the solution of a constrained minimization problem over step functions, with a concave objective function, and a constraint that depends on the solution of the Skorokhod problem displayed in Theorem 3.1. Though this Skorokhod problem only needs to be evaluated for step functions, this minimization problem is in general not tractable. To get more concrete results we look at more specific functionals of the buffer content process in subsequent sections.

4 Asymptotics for overflow probabilities

This section examines the probability that the buffer content associated with a subset of nodes in the system exceeds a high level. In particular, we fix $\mathbf{b} = (b_1, \ldots, b_d) \in \mathbb{R}_+^d$ and consider a functional $\mathscr{B} : \mathbb{D}[0,T] \to \mathbb{R}_+$ defined as $\mathscr{B}(\eta) \triangleq \mathbf{b}^\intercal \pi(\phi(\eta)) = \mathbf{b}^\intercal \phi(\eta)(T)$ for $\eta \in \mathbb{D}[0,T]$. We study the linear combination of the buffer content at fixed time given by $\mathbf{b}^\intercal \mathbf{Z}_n(T) = \mathscr{B}(\mathbf{X}_n)$. Let

$$I'(x) \triangleq \inf\left\{ \tilde{I}^{(d)}(\xi_1, \ldots, \xi_d) : \mathscr{B}(\xi) = x,\ \xi \in \prod_{i=1}^d \mathbb{D}^{-(\mathcal{Q}\mathbf{r})_i}[0,T] \right\},$$

define the set $V_{\geqslant}(y) \triangleq \{\xi \in \prod_{i=1}^d \mathbb{D}_{\leqslant 1}^{(\boldsymbol{\mu}-\mathcal{Q}\mathbf{r})_i}[0,T] : \mathscr{B}(\xi) \ge y\}$, and let $V_{\geqslant}^*(y)$ be the optimal value of $\tilde{I}^{(d)}$ over the set $V_{\geqslant}(y)$, i.e. $V_{\geqslant}^*(y) \triangleq \inf_{\xi\in V_{\geqslant}(y)} \tilde{I}^{(d)}(\xi)$. Similarly, let $V_{>}(y) \triangleq \{\xi \in \prod_{i=1}^d \mathbb{D}_{\leqslant 1}^{(\boldsymbol{\mu}-\mathcal{Q}\mathbf{r})_i}[0,T] : \mathscr{B}(\xi) > y\}$ and set $V_{>}^*(y) \triangleq \inf_{\xi\in V_{>}(y)} \tilde{I}^{(d)}(\xi)$. Note that $V_{\geqslant}^*(y)$ and $V_{>}^*(y)$ depend on T, but we suppress the dependence for notational simplicity.

The next lemma, which is proven in Section 6, enables us to reduce the feasible region of the optimization problem $\inf_{x\geq y} I'(x)$ from $\prod_{i=1}^d \mathbb{D}_{\leqslant\infty}^{(\boldsymbol{\mu}-\mathcal{Q}\mathbf{r})_i}[0,T]$ to $\prod_{i=1}^d \mathbb{D}_{\leqslant 1}^{(\boldsymbol{\mu}-\mathcal{Q}\mathbf{r})_i}[0,T]$, i.e. we can restrict our class of functions to those that have at most one discontinuity in each coordinate.

Lemma 4.1. *Let* $\xi \in \prod_{i=1}^d \mathbb{D}_{\leqslant\infty}^{(\boldsymbol{\mu}-\mathcal{Q}\mathbf{r})_i}[0,T]$*. Then, there exists a path* $\tilde{\xi} \in \prod_{i=1}^d \mathbb{D}_{\leqslant 1}^{(\boldsymbol{\mu}-\mathcal{Q}\mathbf{r})_i}[0,T]$ *such that*

i) $\tilde{I}^{(d)}(\tilde{\xi}) \leq \tilde{I}^{(d)}(\xi)$,

ii) $\phi(\tilde{\xi})(T) \geq \phi(\xi)(T)$,

Since $\tilde{I}^{(d)}(\xi) = \infty$ for $\xi \notin \prod_{i=1}^d \mathbb{D}_{\leqslant\infty}^{(\boldsymbol{\mu}-\mathcal{Q}\mathbf{r})_i}[0,T]$, an immediate consequence of Lemma 4.1 is that

$$V_{\geqslant}^*(y) = \inf_{x\in[y,\infty)} I'(x) \qquad \text{and} \qquad V_{>}^*(y) = \inf_{x\in(y,\infty)} I'(x). \tag{4.1}$$

Recall that $\mathcal{J}$ is the set of nodes with exogenous input. Next, let $I^+ = \{j \in \{1,\dots,d\} : b_j > 0\}$. The following lemmas, proven in Section 6, ensure the continuity of $V_{\geqslant}^*(\cdot)$.

Lemma 4.2. *Assume that* $\mathcal{J} \cap I^+ \neq \emptyset$*. The map* $x \mapsto V_{\geqslant}^*(x)$ *is* α*-Hölder continuous:*

$$|V_{\geqslant}^*(y) - V_{\geqslant}^*(x)| \leq \max_{\{1\leq i\leq d: b_i>0\}} \frac{c_i}{b_i^\alpha} \cdot |y-x|^\alpha.$$

Lemma 4.3. *Assume that* $\mathcal{J} \cap I^+ \neq \emptyset$*. It holds that* $V_{\geqslant}^*(y) = V_{>}^*(y)$*.*

For a given $y > 0$, let $P_{\mathbf{b},y}^*$ denote the optimal value of the following optimization problem ($P_{\mathbf{b},y}$):

$$\begin{aligned} &\inf_{(x_1,\dots,x_d)\in\mathbb{R}^d} \sum_{i=1}^d c_i x_i^\alpha \\ &\quad s.t. \quad \mathscr{B}\left(x_1 \mathbb{1}_{[u_1,T]}(\cdot) + (\boldsymbol{\mu}-\mathcal{Q}\mathbf{r})_1(\cdot),\ \dots,\ x_d \mathbb{1}_{[u_d,T]}(\cdot) + (\boldsymbol{\mu}-\mathcal{Q}\mathbf{r})_d(\cdot)\right) \geq y, \\ &\qquad\quad x_i \geq 0 \text{ for } i \in \mathcal{J},\ x_i = 0 \text{ for } i \notin \mathcal{J}, \\ &\qquad\quad u_i \in [0,T]\ for\ i = 1,\dots,d. \end{aligned} \tag{$P_{\mathbf{b},y}$}$$

For a general stochastic network with routing matrix Q and reflection matrix $\mathcal{Q} = (\mathrm{I} - Q^\intercal)$, the following logarithmic asymptotics hold:

Theorem 4.1. *For a fixed* $\mathbf{b} = (b_1,\dots,b_d) \in \mathbb{R}_+^d$ *assume that* $\mathcal{J} \cap I^+ \neq \emptyset$*. The overflow probabilities* $\mathbf{P}\left(\mathscr{B}(\mathbf{Z}_n) \geq y\right)$ *satisfy the following logarithmic asymptotics:*

$$\lim_{n\to\infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P}(\mathscr{B}(\mathbf{Z}_n) \geq y) = -V_{\geqslant}^*(y) \tag{4.2}$$

Moreover, $V_{\geqslant}^*(y)$ *is equal to the optimal value* $P_{\mathbf{b},y}^*$ *of the optimization problem* ($P_{\mathbf{b},y}$) *above.*

Proof. Note first that $\mathscr{B}$ is Lipschitz continuous w.r.t. d_p, since it is a linear functional of the composition of π and ϕ, which are Lipschitz continuous (from Lemma 2.3 and Theorem 2.1). This allows us to proceed with applying Lemma B.3 *i)* to deduce asymptotic upper and lower bounds for $\frac{1}{L(n)n^\alpha} \log \mathbf{P}(\mathscr{B}(\mathbf{Z}_n) \geq y)$.

For the upper bound, recall that $I'(y) = \inf\{\tilde{I}^{(d)}(\xi) : \mathscr{B}(\xi) = y\}$. Thanks to Result 2.5, the upper bound in Lemma B.3 *i)*, and (4.1), we have that

$$\limsup_{n\to\infty} \frac{1}{L(n)n^{\alpha}} \log \mathbf{P}(\mathscr{B}(\mathbf{Z}_n) \geq y) \leq -\lim_{\epsilon\to 0} \inf_{x\in[y-\epsilon,\infty)} I'(x) = -\lim_{\epsilon\to 0} V^*_{\geqslant}(y-\epsilon) = -V^*_{\geqslant}(y)\,,$$

where the last equality is from the continuity of $V^*_{\geqslant}(\cdot)$ in Lemma 4.2.

For the lower bound, we apply Result 2.5, the lower bound in Lemma B.3 *i)*, (4.1), and Lemma 4.3 to obtain

$$\begin{aligned}\liminf_{n\to\infty} \frac{1}{L(n)n^{\alpha}} \log \mathbf{P}(\mathscr{B}(\mathbf{Z}_n) \geq y) &\geq \liminf_{n\to\infty} \frac{1}{L(n)n^{\alpha}} \log \mathbf{P}(\mathscr{B}(\mathbf{Z}_n) > y) \\ &\geq -\inf_{x\in(y,\infty)} I'(x) = -V^*_{>}(y) = -V^*_{\geqslant}(y).\end{aligned}$$

To conclude that $V^*_{\geqslant}(y) = P^*_{\mathbf{b},y}$, we apply Lemma 4.1 to show that their feasible regions are identical. We observe that for $\xi(\cdot) = \big(x_1\mathbb{1}_{[u_1,T]}(\cdot) + (\boldsymbol{\mu} - \mathcal{Q}\mathbf{r})_1(\cdot),\ \ldots,\ x_d\mathbb{1}_{[u_d,T]}(\cdot) + (\boldsymbol{\mu} - \mathcal{Q}\mathbf{r})_d(\cdot)\big)$, we obtain $\tilde{I}^{(d)}(\xi) = \sum_{i=1}^{d} c_i x_i^{\alpha}$, which equals the cost function of (P_y). □

Note that large deviations behavior is a consequence of at most one jump in the external input process to each node. Consequently, the rate function is the solution of a $2d$-dimensional optimization problem where one has to optimize the size as well as the time of the jumps. This provides a significant reduction in complexity compared to the setting of Section 3. Nevertheless, ($P_{\mathbf{b},y}$) is still rather intricate in general: it is an L^{α}-norm minimization problem with $\alpha \in (0,1)$. In general, such problems are strongly NP-hard, cf. Ge et al. (2011). In addition, checking whether a solution to ($P_{\mathbf{b},y}$) is feasible requires one to compute values of the Skorokhod map for step functions, which is nontrivial. To get more explicit results and gain some physical insights, we therefore consider a two-node tandem network in the next section.

5 A two-node example

We consider a two-node tandem network where content from node 1 flows into node 2, and content from node 2 leaves the system, i.e. $q_{12} = 1$, and $q_{ij} = 0$ otherwise. We assume that each node has an exogenous input process (i.e. $\mathcal{J} = \{1, 2\}$). We consider the problem of estimating $\mathbf{P}\left(\mathbf{Z}_n^{(2)}(T) \geq y\right)$. Theorem 4.1 implies that this is equivalent to solving ($P_{\mathbf{b},y}$) with $\mathbf{b} = (0, 1)$, which is the subject of this section. To keep the presentation concise, we give an outline of the key steps and focus on physical insight.

We first develop an explicit expression for the buffer content at time T for input processes of the form $\xi_i(t) = \mu_i t + x_i\mathbb{1}(u_i \geq t), t \in [0, T],\ x_i \geq 0,\ u_i \in [0, T], i = 1, 2$. To develop physical intuition is it instructive to write the buffer content process at node 2 as the solution of a one-dimensional reflection mapping, fed by the superposition of ξ_2 and the output process of node 1, which in turn is governed by a one-dimensional reflection mapping as well. To this end, observe that $z_1(t) = \xi_1(t) - \inf_{s\leq t} \xi_1(s)$. The amount of fluid flowing from node 1 to node 2 in $[0, t]$ is given by $\xi_1(t) - z_1(t) = -\inf_{s\leq t} \xi_1(s)$.

Consequently, we can write

$$z_2(T) = \xi_2(T) - \inf_{s\le T} \xi_1(s) - \inf_{u\le T}(\xi_2(u) - \inf_{s\le u} \xi_1(s)). \tag{5.1}$$

Our goal is to minimize the cost $c_1x_1^\alpha + c_2x_2^\alpha$ subject to the constraint $z_2(T) \ge y$, over $x_1 \ge 0,\ x_2 \ge 0,\ u_1 \in [0,T],\ u_2 \in [0,T]$. We simplify this problem by identifying convenient choices of u_1 and u_2 which do not lose optimality.

To this end, observe that a jump of size x_2 at time u_2 can instead take place at time $u_2 = T$ without decreasing $z_2(T)$. To determine a convenient choice of u_1, note that a jump of size x_1 in node 1 at time u_1 causes an outflow of rate r_1 from node 1 to node 2 in the interval $[u_1, u_1 + x_1/(r_1 - \mu_1)]$, and rate μ_1 after time $u_1 + x_1/(r_1 - \mu_1)$. Therefore, we can take u_1 such that $u_1 + x_1/(r_1 - \mu_1) = T$, without decreasing $z_2(T)$. This choice is feasible as long as u_1 remains non-negative, i.e. we require that $x_1/(r_1 - \mu_1) \le T$. Observe that choosing $x_1/(r_1 - \mu_1) > T$ would not be optimal, as it would increase the cost term involving x_1^α without increasing $z_2(T)$.

We proceed by solving (5.1) by taking $\xi_1(t) = \mu_1 t + x_1 \mathbb{1}(t \ge T - x_1/(r_1-\mu_1))$ and $\xi_2(t) = \mu_2 t + x_2\mathbb{1}(t = T)$. Straightforward manipulations show that

$$z_2(T) = x_2 + (r_1 + \mu_2 - r_2)^+ \frac{x_1}{r_1 - \mu_1}. \tag{5.2}$$

We see that a jump at node 1 has no effect on the buffer content in node 2 if $r_2 \ge r_1 + \mu_1$, which is intuitively obvious since node 2 is still rate stable when the output of node 1 equals r_1. Therefore, our first conclusion is that

$$\lim_{n\to\infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P}\left(\mathbf{Z}_n^{(2)}(T) \ge y\right) = -c_2 y^\alpha, \qquad r_2 \ge r_1 + \mu_2. \tag{5.3}$$

We now turn to the more interesting case $r_2 < r_1 + \mu_2$. We do not lose optimality if the constraint on $z_2(T)$ is tight, so we can impose the constraints

$$x_2 + \frac{r_1 + \mu_2 - r_2}{r_1 - \mu_1} x_1 = y, \qquad x_1 \in [0, (r_1 - \mu_1)T], \qquad x_2 \ge 0. \tag{5.4}$$

From convex optimization theory, see Corollary 32.3.2 in Rockafellar (1970), the minimum of the concave objective function $c_1x_1^\alpha + c_2x_2^\alpha$ subject to the constraints (5.4) is achieved over the extreme points of (5.4). In our particular situation, this implies that an optimal solution should correspond to one of the following 3 cases: (i) $x_1 = 0$, (ii) $x_2 = 0$, (iii) $x_1 = (r_1 - \mu_1)T$. In case (iii) we would have $x_2 = y - (r_1 + \mu_2 - r_2)T$, which is only feasible if $y \ge (r_1 + \mu_2 - r_2)T$.

If $y \le (r_1 + \mu_2 - r_2)T$, we can conclude that either case (i) holds with $x_1 = 0, x_2 = y$, and cost $c_2 y^\alpha$, or case (ii) holds with $x_2 = 0$, $x_1 = y\frac{r_1-\mu_1}{r_1+\mu_2-r_2}$, and cost $c_1\left(y\frac{r_1-\mu_1}{r_1+\mu_2-r_2}\right)^\alpha$. We conclude that

$$\lim_{n\to\infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P}\left(\mathbf{Z}_n^{(2)}(T) \ge y\right) = -\min\left\{c_1\left(\frac{r_1 - \mu_1}{r_1 + \mu_2 - r_2}\right)^\alpha, c_2\right\} y^\alpha, \quad y \le (r_1 + \mu_2 - r_2)T. \tag{5.5}$$

We now turn to the case $y > (r_1 + \mu_2 - r_2)T$. In this case, the time horizon T is small w.r.t. y: the output of node 1 alone is never enough to cause the buffer content of node 2 to reach level y at time T. Thus, we must compare case (i) and case (iii). Case (i) has solution $x_2 = y$ with cost $c_2 y^\alpha$. Case (iii) has solution $x_1 = (r_1 - \mu_1)T$, $x_2 = y - (r_1 + \mu_2 - r_2)T$, with cost $c_1((r_1 - \mu_1)T)^\alpha + c_2(y - (r_1 + \mu_2 - r_2)T)^\alpha$. We conclude that, if $y > (r_1 + \mu_2 - r_2)T$,

$$\lim_{n\to\infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P}\left(\mathbf{Z}_n^{(2)}(T) \geq y\right) = -\min\{c_2 y^\alpha, c_1((r_1 - \mu_1)T)^\alpha + c_2(y - (r_1 + \mu_2 - r_2)T)^\alpha\}. \quad (5.6)$$

To give a numerical example, take $y = 2, T = 1, r_1 = r_2 = 3, \mu_1 = \mu_2 = 1$. In this case, the inequality $y > (r_1 + \mu_2 - r_2)T$ holds. To evaluate (5.6), note that the cost of case (i) equals $c_2 2^\alpha$ and the cost of case (iii) equals $c_1 2^\alpha + c_2$. So we conclude that case (iii) is the most likely way for the event $\{\mathbf{Z}_n^{(2)}(1) \geq 2\}$ to occur if $c_1 \leq c_2(1 - 2^{-\alpha})$, corresponding to a most likely behavior of two big jumps: $x_1 = 2$, occuring at node 1 at time 0, and $x_2 = 1$, occuring at node 2 at time 1.

6 Complementary proofs

6.1 Proofs of Lemma 4.1, 4.2, and 4.3

Proof of Lemma 4.1. Recall that $\mathcal{Q}$ is the reflection matrix, which is invertible with

$$\mathcal{Q}^{-1} = (\mathrm{I} - Q^\intercal)^{-1} = \mathrm{I} + Q + Q^2 + \ldots.$$

Consequently, $\mathcal{Q}^{-1}$ is a matrix with non-negative entries. Next, we continue with an observation, which we call (O1): if $u, v \in \mathbb{R}^k$, and A is such that $A \in \mathbb{R}_+^{k\times k}$, then $u \geq v$ implies $Au \geq Av$. Let $\xi = (\xi^{(1)}, \ldots, \xi^{(d)}) \in \prod_{i=1}^d \mathbb{D}_{\leqslant\infty}^{(\boldsymbol{\mu}-\mathcal{Q}\mathbf{r})_i}[0,T]$. Since $\xi^{(i)} \in \mathbb{D}_{\leqslant\infty}^{(\boldsymbol{\mu}-\mathcal{Q}\mathbf{r})_i}[0,T]$, we have the following representation:

$$\xi^{(i)}(t) = (\boldsymbol{\mu} - \mathcal{Q}\mathbf{r})_i \cdot t + \sum_{j=1}^\infty x_j^{(i)} \mathbb{1}_{\left[u_j^{(i)},T\right]}(t),\ t \in [0,T].$$

This representation plays a key role in our proof, which consists of the following steps.

Step 1: There exists a path $\tilde{\xi} \in \prod_{i=1}^d \mathbb{D}_{\leqslant 1}^{(\boldsymbol{\mu}-\mathcal{Q}\mathbf{r})_i}[0,T]$ such that $\tilde{\xi} \leq \xi$, and $\tilde{\xi}(T) = \xi(T)$. To see this, let $s^{(i)} = \sup\{t : \xi^{(i)}(t) - \xi^{(i)}(t-) > 0\}$ for each $i = 1, \ldots, d$, and let

$$\tilde{\xi}^{(i)}(t) = (\mu - \mathcal{Q}r)_i \cdot t + \left(\sum_{j=1}^\infty x_j^{(i)}\right) \mathbb{1}_{[s^{(i)},T]}(t),\ t \in [0,T].$$

By construction, $\tilde{\xi} \leq \xi$ component-wise, and $\tilde{\xi}(T) = \xi(T)$.

Step 2: It holds that $\psi(\tilde{\xi})(T) \geq \psi(\xi)(T)$, which follows immediately from Result 2.3.

Step 3: Since $\mathcal{Q}$ consists of non-negative entries, it holds that $\mathcal{Q}\psi(\tilde{\xi})(T) \geq \mathcal{Q}\psi(\xi)(T)$.

Now, we conclude the proof of our statement. Recall that $\phi(\xi)(T) = \xi(T) + \mathcal{Q}\psi(\xi)(T)$. From **Step**

3 and that $\tilde{\xi}(T) = \xi(T)$ in **Step 1**, *ii)* of our lemma is obvious. For *i)*, due to the sub-additivity of the function $x \mapsto x^\alpha$ for $\alpha \in (0,1)$, we notice that

$$\begin{aligned}\tilde{I}^{(d)}(\tilde{\xi}) &= \sum_{i=1}^{d} c_i \cdot \left[\sum_{\{t:\tilde{\xi}^{(i)}(t)-\tilde{\xi}^{(i)}(t-)>0\}} (\tilde{\xi}^{(i)}(t) - \tilde{\xi}^{(i)}(t-))^\alpha \right] \\ &= \sum_{i=1}^{d} c_i \cdot \left(\sum_{j=1}^{\infty} x_j^{(i)} \right)^\alpha \le \sum_{i=1}^{d} c_i \cdot \sum_{j=1}^{\infty} \left(x_j^{(i)} \right)^\alpha \\ &= \sum_{i=1}^{d} c_i \cdot \left[\sum_{\{t:\xi^{(i)}(t)-\xi^{(i)}(t-)>0\}} (\xi^{(i)}(t) - \xi^{(i)}(t-))^\alpha \right] = \tilde{I}^{(d)}(\xi).\end{aligned}$$

□

Next, we focus on the continuity of $\mathrm{V}^*_{\geqslant}(\cdot)$. Let $\mathbb{D}_+[0,T]$ be the subspace of $\mathbb{D}[0,T]$ that contains paths with only positive discontinuities: $\mathbb{D}_+[0,T] = \{\xi \in \mathbb{D}[0,T] : \xi(t) - \xi(t-) \ge 0,\ \forall t \in [0,T]\}$. Recall that $\mathbb{D}_{\leqslant k}[0,T] = \{\xi \in \mathbb{D}[0,T] : |Disc(\xi)| \le k\}$.

Lemma 6.1. *If $a = (a_1, \dots, a_d) \in \mathbb{R}^d_+$ and $\xi \in \prod_{i=1}^{d}(\mathbb{D}_+[0,T] \cap \mathbb{D}_{\leqslant 1}[0,T])$, let $\zeta = \xi + a\mathbb{1}_{\{T\}}$. Then*

i) $\psi(\zeta) = \psi(\xi)$,

ii) $\phi(\zeta)(T) = \phi(\xi)(T) + a$, *and*

iii) $\tilde{I}^{(d)}(\zeta) \le \tilde{I}^{(d)}(\xi) + \sum_{i=1}^{d} c_i a_i^\alpha$.

Proof. For *i)*, from the proof of the Theorem 14.2.2 in Whitt (2002), we see that for any $\iota \in \prod_{i=1}^{k} \mathbb{D}[0,T]$ the regulator component $\psi(\iota)$ is the limit (w.r.t. the uniform metric) of $\rho_\iota^n(\mathbf{0})$ where $\mathbf{0}$ is the zero function and ρ_ι^n is the n fold composition of $\rho_\iota : \prod_{i=1}^{d} \mathbb{D}^\uparrow[0,T] \to \prod_{i=1}^{d} \mathbb{D}^\uparrow[0,T]$ such that $\rho_\iota(\eta)(t) = 0 \vee \sup_{s\in[0,t]}\{Q\eta(s) - \iota(s)\}$. We first claim that $\psi(\zeta)(t) = \psi(\xi)(t)$ for $t \in [0, T-\epsilon]$ for any fixed $\epsilon > 0$. Obviously, $\rho^1_\zeta(\mathbf{0})(t) = 0 \vee \sup_{s\in[0,t]}\{-\zeta(s)\}, 0\} = 0 \vee \sup_{s\in[0,t]}\{-\xi(s)\}, 0\} = \rho^1_\xi(\mathbf{0})(t)$ for $t \le T - \epsilon$. To proceed with induction, suppose that $\rho^k_\zeta(\mathbf{0})(t) = \rho^k_\xi(\mathbf{0})(t)$ for $t \in [0, T-\epsilon]$. Then, for any $t \in [0, T-\epsilon]$ we have

$$\rho_\zeta^{k+1}(\mathbf{0})(t) = 0 \vee \sup_{s\in[0,t]} \left\{ Q\rho_\zeta^k(\mathbf{0})(s) - \zeta(s) \right\} = 0 \vee \sup_{s\in[0,t]} \left\{ Q\rho_\xi^k(\mathbf{0})(s) - \xi(s) \right\} = \rho_\xi^{k+1}(\mathbf{0})(t).$$

Since the equality holds for every $k \in \mathbb{N}$, their (uniform) limits $\psi(\zeta)$ and $\psi(\xi)$ should also coincide on $[0, T-\epsilon]$, proving the claim. Since ϵ was arbitrary, this means that $\psi(\zeta)(t) = \psi(\xi)(t)$, $t \in [0,T)$. Furthermore, since $\psi(\zeta)$ and $\psi(\xi)$ are continuous from Result 2.2, we conclude that they coincide on $[0,T]$, i.e., $\psi(\xi) = \psi(\zeta)$.

For *ii)*, observe that $\phi(\zeta)(T) = \zeta(T) + \psi(\zeta)(T) = \xi(T) + a + \psi(\xi)(T) = \phi(\xi)(T) + a$.

For *iii)*, we assume that $\xi^{(j)} \in \mathbb{D}_{\leqslant\infty}^{(\boldsymbol{\mu}-\mathcal{Q}\mathbf{r})_i}[0,T]$ for $j \in \mathcal{J}$ and $\xi^{(j)}(t) = -(\mathcal{Q}\mathbf{r})_j(t)$ for $j \notin \mathcal{J}$ since if not $\tilde{I}^{(d)}(\zeta) = \tilde{I}^{(d)}(\xi) = \infty$, and the inequality holds trivially otherwise. Let $\zeta = (\zeta^{(1)}, \dots, \zeta^{(d)})$, and

$\xi = (\xi^{(1)}, \ldots, \xi^{(d)})$. Recall that the function $x \mapsto x^\alpha$, $\alpha \in (0,1)$, is sub-additive, and

$$I(\zeta^{(i)}) = \sum_{t \in [0,T]: \zeta^{(i)}(t) \neq \zeta^{(i)}(t-)} \left(\zeta^{(i)}(t) - \zeta^{(i)}(t-)\right)^\alpha.$$

For any $i \in \{1, \ldots, d\}$, we have

$$\begin{aligned} I(\zeta^{(i)}) &= \sum_{t \in [0,T): \xi^{(i)}(t) \neq \xi^{(i)}(t-)} \left(\xi^{(i)}(t) - \xi^{(i)}(t-)\right)^\alpha + \left(\xi^{(i)}(T) - \xi^{(i)}(T-) + a_i\right)^\alpha \\ &\leq \sum_{t \in [0,T): \xi^{(i)}(t) \neq \xi^{(i)}(t-)} \left(\xi^{(i)}(t) - \xi^{(i)}(t-)\right)^\alpha + \left(\xi^{(i)}(T) - \xi^{(i)}(T-)\right)^\alpha + a_i^\alpha \\ &= I(\xi^{(i)}) + a_i^\alpha. \end{aligned}$$

Therefore, $\tilde{I}^{(d)}(\zeta) = \sum_{j \in \mathcal{J}} c_j I(\zeta^{(j)}) \leq \sum_{j \in \mathcal{J}} c_j I(\xi^{(j)}) + \sum_{j \in \mathcal{J}} c_j a_j^\alpha \leq \tilde{I}^{(d)}(\xi) + \sum_{j=1}^d c_j a_j^\alpha$. $\square$

Proof of Lemma 4.2. Let $y \geq x \geq 0$. It is obvious that $V_{\geqslant}^*(y) \geq V_{\geqslant}^*(x) \geq 0$. By the definition of the infimum, for any $\epsilon > 0$, there exists a $\zeta \in V_{\geqslant}(x)$ so that $\tilde{I}^{(d)}(\zeta) < V_{\geqslant}^*(x) + \epsilon$. Next, let $i \in I^+$ where $I^+ = \{j \in \{1, \ldots, d\} : b_j > 0\}$. Subsequently, let $\xi = \zeta + \mathbf{v}\mathbb{1}_{\{T\}}$ where $\mathbf{v} = (0, \ldots, \frac{y-x}{b_i}, \ldots, 0)$. Due to *ii)* of Lemma 6.1,

$$\mathbf{b}^\intercal \phi(\xi)(T) = \mathbf{b}^\intercal(\phi(\zeta)(T) + \mathbf{v}) \geq \mathbf{b}^\intercal \phi(\zeta)(T) + b_i \frac{(y-x)}{b_i} \geq x + y - x = y.$$

Hence, $\xi \in V_{\geqslant}(y)$. Due to *iii)* of Lemma 6.1, $\tilde{I}^{(d)}(\xi) \leq \tilde{I}^{(d)}(\zeta) + \max_{\{1 \leq i \leq d: b_i > 0\}} \frac{c_i}{b_i^\alpha}(y-x)^\alpha$. We see that

$$V_{\geqslant}^*(y) \leq \tilde{I}^{(d)}(\xi) \leq \tilde{I}^{(d)}(\zeta) + \max_{1 \leq i \leq d: b_i > 0} \frac{c_i}{b_i^\alpha}(y-x)^\alpha < V_{\geqslant}^*(x) + \max_{\{1 \leq i \leq d: b_i > 0\}} \frac{c_i}{b_i^\alpha}(y-x)^\alpha + \epsilon.$$

This leads to $V_{\geqslant}^*(y) - V_{\geqslant}^*(x) \leq \max_{\{1 \leq i \leq d: b_i > 0\}} \frac{c_i}{b_i^\alpha}(y-x)^\alpha + \epsilon$. We obtain the desired result by letting ϵ tend to 0. Thus, $|V_{\geqslant}^*(y) - V_{\geqslant}^*(x)| \leq \max_{\{1 \leq i \leq d: b_i > 0\}} \frac{c_i}{b_i^\alpha} \cdot |y - x|^\alpha$. $\square$

We end this section with the proof of $V_{\geqslant}^*(y) = V_{>}^*(y)$.

Proof of Lemma 4.3. For any $\epsilon > 0$, we have that $V_{\geqslant}^*(y + \epsilon) \geq V_{>}^*(y)$. Hence, in view of Lemma 4.2,

$$|V_{>}^*(y) - V_{\geqslant}^*(y)| = V_{>}^*(y) - V_{\geqslant}^*(y) \leq V_{\geqslant}^*(y+\epsilon) - V_{\geqslant}^*(y) \leq \max_{\{1 \leq i \leq d: b_i > 0\}} \frac{c_i}{b_i^\alpha}|\epsilon|^\alpha.$$

Now, we let ϵ go to 0 to obtain the desired result. $\square$

6.2 Proof of Theorem 2.1

Recall that $\prod_{i=1}^d \mathbb{D}[0,T]$ is the Skorokhod space equipped with the product J_1 topology and $\mathbb{D}^\uparrow[0,T] \triangleq \{\xi \in \mathbb{D}[0,T] : \xi$ is non-decreasing on $[0,T]$ and $\xi(0) \geq 0\}$. $\mathbb{D}^\uparrow[0,T]$ is a closed subspace of $\mathbb{D}[0,T]$ w.r.t. the J_1 topology. Hence, $\prod_{i=1}^d \mathbb{D}^\uparrow[0,T]$ is a closed subspace of $\prod_{i=1}^d \mathbb{D}[0,T]$ w.r.t. the product J_1 topology.

Since $\mathbb{D}^{\beta}[0,T]$ is the image of $\mathbb{D}^{\uparrow}[0,T]$ under the homeomorphism Υ_{β}, we have that $\prod_{i=1}^{d}\mathbb{D}^{\beta_i}[0,T]$ is a closed subset of $\prod_{i=1}^{d}\mathbb{D}[0,T]$. Let $\boldsymbol{\beta}=(\beta_1,\ldots,\beta_d)$, and let $\|\boldsymbol{\beta}\|_\infty=\max_{1\le i\le d}|\beta_i|$.

6.2.1 Some supporting lemmas

Lemma 6.2. *Suppose that λ,κ are strictly increasing functions such that*

i) $\lambda(0)=\kappa(0)=0$,

ii) $\lambda(T)=\kappa(T)=T$,

iii) $\|\lambda-e\|_\infty<\delta$, *and* $\|\kappa-e\|_\infty<\delta$.

Then, $\|\lambda\circ\kappa-e\|_\infty<2\delta$.

Proof. $\|\lambda\circ\kappa-e\|_\infty=\|\lambda-\kappa^{-1}\|_\infty\le\|\lambda-e\|_\infty+\|\kappa^{-1}-e\|_\infty=\|\lambda-e\|_\infty+\|e-\kappa\|_\infty\le 2\delta.$ □

We now consider properties of continuous and increasing time deformations $w^{(i)},\ i=1,\ldots,d$.

Lemma 6.3. *If $w^{(i)}$ is increasing and continuous so that $w^{(i)}(0)=0$, and $w^{(i)}(T)=T$ for each $i=1,\ldots,d$, then $\check{w}(s)=\min\{w^{(1)}(s),\ldots,w^{(d)}(s)\}$, and $\hat{w}(s)=\max\{w^{(1)}(s),\ldots,w^{(d)}(s)\}$ are increasing, continuous such that $\check{w}(0)=\hat{w}(0)=0$, and $\check{w}(T)=\hat{w}(T)=T$.*

Proof. The min and max of continuous and increasing functions is increasing and continuous. The other properties are easily verified. □

Recall that ψ is Lipschitz continuous w.r.t. uniform norm (Theorem 14.2.5 of Whitt (2002)). Let K denote the Lipschitz constant of ψ w.r.t. the uniform norm.

Lemma 6.4. *Let $\zeta\in\prod_{i=1}^{d}\mathbb{D}^{\beta_i}[0,T]$ and w be an increasing function such that $w(0)=0,\ w(T)=T$. Then, it holds that*

$$\|\psi(\zeta)(w)-\psi(\zeta)(e)\|_\infty<K\|\boldsymbol{\beta}\|_\infty\|w-e\|_\infty.$$

Proof. The proof is a consequence of the following two claims:

i) for every $s\in[0,T]$ there exists a $U(s)\in(0,\infty)$ so that $|\psi(\zeta)(w(s))-\psi(\zeta)(s)|\le U(s)$, and

ii) it holds that $\sup_{s\in[0,T]}U(s)\le K\|\boldsymbol{\beta}\|_\infty\|w-\epsilon\|_\infty$.

For claim i), pick an $s\in[0,T]$. If $w(s)\ge s$, since $\psi(\zeta)$ is an increasing function, $\psi(\zeta)(w(s))\ge\psi(\zeta)(s)$. Hence, we only need to bound $\psi(\zeta)(w(s))-\psi(\zeta)(s)$. Moreover, since $\zeta\in\prod_{i=1}^{d}\mathbb{D}^{\beta_i}[0,T]$, we have that

$$\zeta(w(s))=\zeta((w(s)-s)+s)\ge\zeta(s)+\boldsymbol{\beta}(w(s)-s). \tag{6.1}$$

Next, consider the path $\tilde{\zeta}_1$ where

$$\tilde{\zeta}_1=\begin{cases}\zeta(t), & t\in[0,s],\\ \zeta(s)+\boldsymbol{\beta}(t-s), & t\in[s,w(s)].\end{cases}$$

Since $\tilde{\zeta}_1 \leq \zeta$ over $[0, w(s)]$ and Result 2.3, we have $\psi(\tilde{\zeta}_1)(w(s)) \geq \psi(\zeta)(w(s))$. Furthermore, let

$$\tilde{\zeta}_2 = \begin{cases} \zeta(t), & t \in [0, s], \\ \zeta(s), & t \in [s, w(s)]. \end{cases}$$

By the definition of $\tilde{\zeta}_2$ we have that $\psi(\tilde{\zeta}_2)(w(s)) = \psi(\zeta)(s)$. Moreover, due to the construction of $\tilde{\zeta}_1$,

$$\psi(\zeta)(w(s)) - \psi(\zeta)(s) \leq \psi(\tilde{\zeta}_1)(w(s)) - \psi(\tilde{\zeta}_2)(w(s)) \leq K \sup_{t \in [0, w(s)]} |\tilde{\zeta}_1(s) - \tilde{\zeta}_2(s)| \leq K\|\boldsymbol{\beta}\|_\infty |w(s) - s|.$$

Conversely, if $w(s) \leq s$, since $\psi(\zeta)$ is an increasing function, $\psi(\zeta)(s) \geq \psi(\zeta)(w(s))$. Furthermore, since $\zeta \in \prod_{i=1}^d \mathbb{D}^{\beta_i}[0, T]$, we have that

$$\zeta(s) = \zeta((s - w(s)) + w(s)) \geq \zeta(w(s)) + \boldsymbol{\beta}(s - w(s)). \tag{6.2}$$

Next, consider the path $\tilde{\zeta}_1$, where

$$\tilde{\zeta}_1 = \begin{cases} \zeta(t), & t \in [0, w(s)], \\ \zeta(s) + \boldsymbol{\beta}(t - w(s)), & t \in [w(s), s]. \end{cases}$$

Since $\tilde{\zeta}_1 \leq \zeta$ over $[0, s]$, due to Result 2.3, we have that $\psi(\tilde{\zeta}_1)(s) \geq \psi(\zeta)(s)$. On the other hand, let

$$\tilde{\zeta}_2(t) = \begin{cases} \zeta(t), & t \in [0, w(s)], \\ \zeta(s), & t \in [w(s), s]. \end{cases}$$

Due to the construction of $\tilde{\zeta}_2$, we have that $\psi(\tilde{\zeta}_2)(w(s)) = \psi(\zeta)(s)$. Moreover,

$$\psi(\zeta)(w(s)) - \psi(\zeta)(s) \leq \psi(\tilde{\zeta}_1)(w(s)) - \psi(\tilde{\zeta}_2)(w(s)) \leq K \sup_{t \in [0, w(s)]} |\tilde{\zeta}_1(s) - \tilde{\zeta}_2(s)| \leq K\|\boldsymbol{\beta}\|_\infty |w(s) - s|.$$

For claim ii), observe that for every $s \in [0, T]$, $K\|\boldsymbol{\beta}\|_\infty |w(s) - s| \leq K\|\boldsymbol{\beta}\|_\infty \|w - e\|_\infty$. Hence, $\|\psi(\zeta)(w) - \psi(\zeta)\|_\infty \leq K\|\boldsymbol{\beta}\|_\infty \|w - e\|_\infty$. ☐

Note that, if $\boldsymbol{\beta} = \mathbf{0}$ and $\zeta \in \prod_{i=1}^d \mathbb{D}^{\beta_i}[0, T]$, then ζ belongs to $\prod_{i=1}^d \mathbb{D}^{\uparrow}[0, T]$ and is non-negative at the origin. This implies $\psi(\zeta) = 0$ and the upper bound in Lemma 6.4 holds trivially. Next, we state two more lemmas which are needed in our proof for the Lipschitz continuity of the regulator map in $\prod_{i=1}^d \mathbb{D}^{\beta_i}[0, T]$.

Lemma 6.5. *Consider $w = (w^{(1)}, \ldots, w^{(d)})$, each component of which is a time deformation. Recall $\hat{w}$ and $\check{w}$ in Lemma 6.3. That is, $\hat{w}(t) = \max\{w^{(1)}(t), \ldots, w^{(d)}(t)\}$, and $\check{w}(t) = \min\{w^{(1)}(t), \ldots, w^{(d)}(t)\}$. For any $\xi \in \prod_{i=1}^d \mathbb{D}^{\beta_i}[0, T]$,*

i) $\psi(\xi(w^{(1)}), \ldots, \xi(w^{(d)})) \leq \psi(\xi)(\check{w}) + 2K\|\boldsymbol{\beta}\|_\infty(\|w - e\|_\infty)$, *and*

ii) $\psi(\xi(w^{(1)}), \ldots, \xi(w^{(d)})) + 2K\|\boldsymbol{\beta}\|_\infty(\|w - e\|_\infty) \geq \psi(\xi)(\hat{w})$.

Proof. We start with *i)*. Since $\xi \in \prod_{i=1}^d \mathbb{D}^{\beta_i}[0,T]$ and $\check{w}(s) \leq \min\{w^{(1)}(s), \ldots, w^{(d)}(s)\}$, we have that for each $i = 1, \ldots, d$,

$$\xi^{(i)}(w^{(i)}(s)) \geq \xi^{(i)}(\check{w}(s)) - \|\boldsymbol{\beta}\|_\infty (w^{(i)}(s) - \check{w}(s)), \ s \in [0,T].$$

Therefore, due to Result 2.3, and the Lipschitz continuity of ψ w.r.t. the uniform metric,

$$\begin{aligned}
\psi(\xi^{(1)}(w^{(1)}), \ldots, \xi^{(d)}(w^{(d)})) &\leq \psi(\xi^{(1)}(\check{w}) - \|\boldsymbol{\beta}\|_\infty (w^{(1)} - \check{w}), \ldots, \xi^{(d)}(\check{w}) - \|\boldsymbol{\beta}\|_\infty (w^{(d)} - \check{w})) \\
&\leq \psi(\xi)(\check{w}) + K\|\boldsymbol{\beta}\|_\infty \max_{1\leq i\leq d}(\|\check{w} - w^{(i)}\|_\infty) \\
&\leq \psi(\xi)(\check{w}) + 2K\|\boldsymbol{\beta}\|_\infty \max_{1\leq i\leq d}(\|w^{(i)} - e\|_\infty) \\
&= \psi(\xi)(\check{w}) + 2K\|\boldsymbol{\beta}\|_\infty (\|w - e\|_\infty).
\end{aligned}$$

For *ii)*, observe that $\xi^{(i)}(\hat{w}(s)) \geq \xi^{(i)}(w^{(i)}(s)) - \|\boldsymbol{\beta}\|_\infty(\hat{w}(s) - w^{(i)}(s))$ for each $i = 1, \ldots, d$ and $s \in [0,T]$, since $\xi \in \prod_{i=1}^d \mathbb{D}^{\beta_i}[0,T]$, and $\hat{w}(s) \geq \max\{w^{(1)}(s), \ldots, w^{(d)}(s)\}$. Therefore, due to Result 2.3, and the Lipschitz continuity of ψ w.r.t. the uniform metric,

$$\begin{aligned}
\psi(\xi^{(1)}, \ldots, \xi^{(d)})(\hat{w}) &\leq \psi(\xi^{(1)}(\hat{w}) - \|\boldsymbol{\beta}\|_\infty (\hat{w} - w^{(1)}), \ldots, \xi^{(d)}(\hat{w}) - \|\boldsymbol{\beta}\|_\infty (\hat{w} - w^{(d)})) \\
&\leq \psi(\xi)(\hat{w}) + K\|\boldsymbol{\beta}\|_\infty \max_{1\leq i\leq d}(\|\hat{w} - w^{(i)}\|_\infty) \\
&\leq \psi(\xi)(\hat{w}) + 2K\|\boldsymbol{\beta}\|_\infty \max_{1\leq i\leq d}(\|w^{(i)} - e\|_\infty) \\
&= \psi(\xi)(\hat{w}) + 2K\|\boldsymbol{\beta}\|_\infty (\|w - e\|_\infty).
\end{aligned}$$

□

Lemma 6.6. *For any* $\xi \in \prod_{i=1}^d \mathbb{D}^{\beta_i}[0,T]$,

$$\|\psi(\xi(w^{(1)}), \ldots, \xi(w^{(d)})) - \psi(\xi^{(1)}, \ldots, \xi^{(d)})\|_\infty \leq 3K\|\boldsymbol{\beta}\|_\infty \|w - e\|_\infty.$$

Proof. Due to Lemma 6.4 and Lemma 6.5,

$$\begin{aligned}
\psi(\xi(w^{(1)}), \ldots, \xi(w^{(d)})) - \psi(\xi^{(1)}, \ldots, \xi^{(d)}) &\leq \psi(\xi)(\check{w}) - \psi(\xi^{(1)}, \ldots, \xi^{(d)}) + 2K\|\boldsymbol{\beta}\|_\infty (\|w - e\|_\infty) \\
&\leq \psi(\xi) + K\|\boldsymbol{\beta}\|_\infty (\|w - e\|_\infty) - \psi(\xi) + 2K\|\boldsymbol{\beta}\|_\infty (\|w - e\|_\infty) \\
&= 3K\|\boldsymbol{\beta}\|_\infty (\|w - e\|_\infty).
\end{aligned}$$

For the other inequality, notice that

$$\psi(\xi^{(1)}, \ldots, \xi^{(d)}) - \psi(\xi(w^{(1)}), \ldots, \xi(w^{(d)})) \leq \psi(\xi) - \psi(\xi)(\hat{w}) + 2K\|\boldsymbol{\beta}\|_\infty \|w - e\|_\infty \leq 3K\|\boldsymbol{\beta}\|_\infty \|w - e\|_\infty.$$

□

6.2.2 Lipschitz continuity of the reflection map

In this section, we prove the Lipschitz continuity of the regulator map and the buffer content component map in the product J_1 topology. We start with the Lipschitz continuity of the regulator map ψ.

Proof of Proposition 2.1. Given ξ and ζ, let δ be such that $d_p(\xi,\zeta) < \delta$. Then, there exists $\lambda^{(i)}$ s.t. $\|\xi^{(i)} \circ \lambda^{(i)} - \zeta^{(i)}\|_\infty \vee \|\lambda^{(i)} - e\|_\infty < \delta$ for each $i = 1, \dots, d$. Notice that

$$\begin{aligned}
d_p(\psi(\xi), \psi(\zeta)) &\le \inf_{w^{(1)},\dots,w^{(d)} \in \Lambda} \left\{ \sum_{i=1}^d \|\psi^{(i)}(\xi) \circ w^{(i)} - \psi^{(i)}(\zeta)\|_\infty \right\} \le \sum_{i=1}^d \|\psi^{(i)}(\xi) \circ \lambda^{(1)} - \psi^{(i)}(\zeta)\|_\infty \\
&\le \sum_{i=1}^d \|\psi^{(i)}(\xi) \circ \lambda^{(1)} - \psi^{(i)}(\xi^1 \circ \lambda^{(1)}, \dots, \xi^{(d)} \circ \lambda^{(d)})\|_\infty \\
&\qquad + \sum_{i=1}^d \|\psi^{(i)}(\xi^1 \circ \lambda^{(1)}, \dots, \xi^{(d)} \circ \lambda^{(d)}) - \psi^{(i)}(\zeta^{(1)}, \dots, \zeta^{(d)})\|_\infty \\
&\le \sum_{i=1}^d \|\psi^{(i)}(\xi) \circ \lambda^{(1)} - \psi^{(i)}(\xi^1 \circ \lambda^{(1)}, \dots, \xi^{(d)} \circ \lambda^{(d)})\|_\infty + d \cdot K \max_{1 \le i \le d} \|\xi^{(i)} \circ \lambda^{(i)} - \zeta^{(i)}\|_\infty \\
&\le \sum_{i=1}^d \|\psi^{(i)}(\xi) \circ \lambda^{(1)} - \psi^{(i)}(\xi^1 \circ \lambda^{(1)}, \dots, \xi^{(d)} \circ \lambda^{(d)})\|_\infty + d \cdot K \cdot \delta/2 \\
&= \sum_{i=1}^d \|\psi^{(i)}(\xi) - \psi^{(i)}(\xi^1 \circ \lambda^{(1)}, \dots, \xi^{(d)} \circ \lambda^{(d)}) \circ {\lambda^{(1)}}^{-1}\|_\infty + d \cdot K \cdot \delta/2 \\
&= \sum_{i=1}^d \|\psi^{(i)}(\xi) - \psi^{(i)}(\xi^1 \circ \lambda^{(1)} \circ {\lambda^{(1)}}^{-1}, \dots, \xi^{(d)} \circ \lambda^{(d)} \circ {\lambda^{(1)}}^{-1})\|_\infty + d \cdot K \cdot \delta/2. \qquad (6.3)
\end{aligned}$$

Since $\|\lambda^{(i)} \circ {\lambda^{(1)}}^{-1} - e\| < 2\delta$ for each $i = 1, \dots, d$, and by invoking Lemma 6.6, we have that

$$\|\psi(\xi) - \psi(\xi^1 \circ \lambda^{(1)} \circ {\lambda^{(1)}}^{-1}, \dots, \xi^{(d)} \circ \lambda^{(d)} \circ {\lambda^{(1)}}^{-1})\|_\infty \le d6K\|\boldsymbol{\beta}\|_\infty \delta. \qquad (6.4)$$

Combining (6.3) and (6.4), we have that $d_p(\psi(\xi), \psi(\zeta)) \le Kd(6\|\boldsymbol{\beta}\|_\infty + 1)\delta$. Letting $\delta \downarrow d_p(\xi,\zeta)$ we obtain Lipschitz continuity of ψ. ☐

Proof of Theorem 2.1. The Lipschitz continuity of the regulator map has been proven in Proposition 2.1. We only need to verify the Lipschitz continuity of the buffer content component map ϕ. Let δ be such that $d_p(\xi,\zeta) < \delta$. Then, $d_{J_1}(\xi^{(i)}, \zeta^{(i)}) < \delta$ for each $i = 1 \dots, d$. Note that $\phi^{(i)}(\xi) = \xi^{(i)} + \psi^{(i)}(\xi) - \sum_{\{j \in \{1,\dots,d\} \setminus i\}} q_{ji} \psi^{(j)}(\xi)$. Hence,

$$\begin{aligned}
d_{J_1}(\phi^{(i)}(\xi), \phi^{(i)}(\zeta)) &\le d_{J_1}(\xi^{(i)}, \zeta^{(i)}) + \sum_{i=1}^d d_{J_1}(\psi^{(i)}(\xi), \psi^{(i)}(\zeta)) \\
&< \delta + Kd(6\|\boldsymbol{\beta}\|_\infty + 1)\delta = \delta(1 + Kd(6\|\boldsymbol{\beta}\|_\infty + 1)).
\end{aligned}$$

Since $d_p(\phi(\xi),\phi(\zeta)) \le \sum_{i=1}^d d_{J_1}(\phi^{(i)}(\xi),\phi^{(i)}(\zeta)) \le \delta d(1+Kd(6\|\boldsymbol{\beta}\|_\infty+1))$, we have that ϕ is Lipschitz continuous in $\prod_{i=1}^d \mathbb{D}^{\beta_i}[0,T]$ by letting $\delta \downarrow d_p(\xi,\zeta)$. $\square$

A Continuity of some useful functions

In this appendix, we include the proofs of continuity of some functions in the product J_1 topology. Recall the function $\Upsilon_\kappa : \prod_{i=1}^d \mathbb{D}[0,T] \to \prod_{i=1}^d \mathbb{D}[0,T]$ where $\Upsilon_\kappa(\xi)(t) = \xi(t) - \kappa \cdot t$ for $t \in [0,T]$ and $\pi : \prod_{i=1}^d \mathbb{D}[0,T] \to \mathbb{R}^d$ where $\pi(\xi) = \xi(T)$.

Proof of Lemma 2.2. For *i)*, suppose that $d_p(\xi,\zeta) < \epsilon$ w.r.t. the product J_1 topology. Then, there exists a homeomorphism $\lambda^{(i)}$, $i \in \{1,\dots,d\}$, so that $\|\xi - \zeta \circ \lambda^{(i)}\|_\infty + \|\lambda^{(i)} - e(\cdot)\|_\infty < 2\epsilon$. Therefore,

$$\begin{aligned} d_{J_1}\left(\Upsilon_\kappa^{(i)}(\xi),\Upsilon_\kappa^{(i)}(\zeta)\right) &\le \|\xi^{(i)} - \zeta^{(i)} \circ \lambda^{(i)} + \kappa^{(i)}(\lambda^{(i)} - e(\cdot))\|_\infty + \|\lambda^{(i)} - e(\cdot)\|_\infty \\ &\le \|\xi^{(i)} - \zeta^{(i)} \circ \lambda^{(i)}\|_\infty + (1+|\kappa^{(i)}|)\|\lambda^{(i)} - e(\cdot)\|_\infty \\ &\le 2(1+|\kappa^{(i)}|)\epsilon. \end{aligned} \tag{A.1}$$

Consequently,

$$d_p(\Upsilon_\kappa(\zeta),\Upsilon_\kappa(\xi)) \le \sum_{i=1}^d d_p(\Upsilon_\kappa^{(i)}(\zeta),\Upsilon_\kappa^{(i)}(\xi)) \le 2\sum_{i=1}^d (1+|\kappa^{(i)}|)\epsilon = 2(d+\sum_{i=1}^d |\kappa^{(i)}|)\epsilon.$$

ii). Note that $\Upsilon_\kappa^{-1}(\zeta)(\cdot) = \zeta(\cdot) + \kappa(\cdot) = \Upsilon_{-\kappa}(\zeta)$. Hence, Υ_κ is injective and surjective. Furthermore, the continuity of Υ_κ^{-1} is obtained by applying *i)* to $\Upsilon_{-\kappa}$. $\square$

Finally, we prove that the projection map is Lipschitz continuous in the product J_1 topology.

Proof of Lemma 2.3. Suppose that $d_p(\xi,\zeta) \le \epsilon$. Then, we have that $d_{J_1}(\xi^{(i)},\zeta^{(i)}) \le \epsilon$ for every $i = 1,\dots,d$. For any homeomorphism $\lambda : [0,T] \to [0,T]$ that satisfies $\lambda(0) = 0$ and $\lambda(T) = T$, we have that

$$|\pi^{(i)}(\xi) - \pi^{(i)}(\zeta)| = |\xi^{(i)}(T) - \zeta^{(i)}(T)| = |\xi^{(i)}(T) - \zeta^{(i)}(\lambda(T))| \le \|\xi^{(i)} - \zeta^{(i)} \circ \lambda\|_\infty.$$

Since this is true for any λ, $|\pi^{(i)}(\xi) - \pi^{(i)}(\zeta)| \le d_{J_1}(\xi^{(i)},\zeta^{(i)}) \le \epsilon$. Since this holds for every $i = 1,\dots,d$, we have that $\|\pi(\xi) - \pi(\zeta)\|_\infty \le \epsilon$. $\square$

B Some useful tools on large deviations

In this appendix, we include results that facilitate the use of the extended LDP. Given that the probability measures of (X_n) satisfy the extended LDP in a metric space $(\mathcal{X}, d)$, our results include the derivation of the extended LDP in closed subspaces of $\mathcal{X}$, and a variation of the contraction principle for Lipschitz continuous maps. Let $\mathcal{D}_I \triangleq \{x \in X : I(x) < \infty\}$ denote the effective domain of I.

Lemma B.1. *Let E be a closed subset of $\mathcal{X}$ and let X_n be such that $\mathbf{P}(X_n \in E) = 1$ for all $n \geq 1$. Suppose that E is equipped with the topology induced by $\mathcal{X}$. Then, if the probability measures of (X_n) satisfy the extended LDP in $(\mathcal{X}, d)$ with speed a_n, and with rate function I so that $\mathcal{D}_I \subset E$, then the same extended LDP holds in E.*

Proof. In the topology induced on E by $\mathcal{X}$, the open sets are sets of the form $G \cap E$ with $G \subseteq \mathcal{X}$ open. Similarly, the closed sets in this topology are the sets of the form $F \cap E$ with $F \subseteq \mathcal{X}$ closed. Furthermore, $\mathbf{P}(X_n \in \Gamma) = \mathbf{P}(X_n \in \Gamma \cap E)$ for any $\Gamma \in \mathcal{B}$, where $\mathcal{B}$ is the Borel sigma-algebra. Suppose that an extended LDP holds in $\mathcal{X}$. Now, for the upper bound, let F be a closed subset of E. Then, F is a closed subset of $\mathcal{X}$. Hence, $\limsup_{n\to\infty} \frac{1}{a_n} \log \mathbf{P}\left(X_n \in F\right) \leq -\inf_{x \in F^\epsilon} I(x) = -\inf_{x\in F^\epsilon \cap E} I(x)$. Next, let G be an open subset of E. That is, $G = G' \cap E$, where G' is an open subset of $\mathcal{X}$. Then,

$$\begin{aligned}&\liminf_{n\to\infty} \frac{1}{a_n} \log \mathbf{P}\left(X_n \in G\right) = \liminf_{n\to\infty} \frac{1}{a_n} \log \mathbf{P}\left(X_n \in G' \cap E\right) \\ &= \liminf_{n\to\infty} \frac{1}{a_n} \log \mathbf{P}\left(X_n \in G'\right) \geq -\inf_{x\in G'\cap E} I(x) = -\inf_{x\in G} I(x).\end{aligned}$$

The level sets $\Psi_I(\alpha) \subseteq \mathcal{X}$ are closed, so I restricted to E remains lower semicontinuous. $\square$

We continue with a useful lemma on pre-images of Lipschitz continuous maps on metric spaces. For a closed subset of the metric space $(\mathcal{X}, d)$, recall that $A^\epsilon \triangleq \{\xi \in \mathcal{X} : d(\xi, A) \leq \epsilon\}$, where $d(\xi, A) = \inf_{\zeta\in A} d(\xi, \zeta)$.

Lemma B.2. *Let $(\mathbb{S}, \sigma)$ and $(\mathbb{X}, d)$ be metric spaces. Suppose that $\Phi : (\mathbb{X}, d) \to (\mathbb{S}, \sigma)$ is a Lipschitz continuous mapping with Lipschitz constant $\|\Phi\|_{\mathrm{Lip}}$. Then, for any set $F \subset \mathbb{X}$, it holds that*

$$\left(\Phi^{-1}(F)\right)^\epsilon \subseteq \Phi^{-1}\left(F^{\|\Phi\|_{\mathrm{Lip}}\epsilon}\right).$$

Proof. Let $\zeta \in \left(\Phi^{-1}(F)\right)^\epsilon$. For each $n \geq 1$, since $\Phi^{-1}(F)$ is a closed set, there exists a $\xi_n \in \Phi^{-1}(F)$ so that $d(\xi_n, \zeta) \leq \epsilon + 1/n$. Since $\xi_n \in \Phi^{-1}(F)$ we have that $\Phi(\xi_n) \in F$. Furthermore, $\sigma(\Phi(\xi_n), \Phi(\zeta)) \leq \|\Phi\|_{\mathrm{Lip}}(\epsilon + 1/n)$. Hence, $d\big(\Phi(\zeta), F\big) \leq \|\Phi\|_{\mathrm{Lip}}(\epsilon + 1/n)$. Letting $n \to \infty$, we get $d\big(\Phi(\zeta), F\big) \leq \|\Phi\|_{\mathrm{Lip}}\epsilon$, that is, $\zeta \in \Phi^{-1}(F^{\|\Phi\|\epsilon})$. Since this holds for any $\zeta \in \left(\Phi^{-1}(F)\right)^\epsilon$, the statement holds true. $\square$

The following lemma is a version of the contraction principle adapted to the setting of extended LDP's.

Lemma B.3. *Let $(\mathbb{S}, \sigma)$ and $(\mathbb{X}, d)$ be metric spaces. Suppose that the sequence of probability measures of $(\mathbf{X}_n)$ satisfies the extended LDP in $(\mathbb{X}, d)$ with speed a_n and rate function I. Moreover, let $\Phi : (\mathbb{X}, d) \to (\mathbb{S}, \sigma)$ be a Lipschitz continuous mapping and set $I'(y) \triangleq \inf_{\Phi(x)=y} I(x)$. Then,*

i) $\mathbf{S}_n = \Phi(\mathbf{X}_n)$ *satisfies the following bounds: for any open set $G \subseteq \mathbb{S}$,*

$$\liminf_{n\to\infty} \frac{1}{a_n} \log \mathbf{P}\left(\mathbf{S}_n \in G\right) \geq -\inf_{x\in G} I'(x),$$

and for any closed set $F \subseteq \mathbb{S}$,

$$\limsup_{n\to\infty} \frac{1}{a_n} \log \mathbf{P}\left(\mathbf{S}_n \in F\right) \le - \lim_{\epsilon\to 0} \inf_{x\in F^\epsilon} I'(x).$$

ii) Suppose in addition that Φ is a homeomorphism. Then, $\mathbf{S}_n$ satisfies the extended LDP in $(\mathbb{S},\sigma)$ with speed a_n and rate function I'.

iii) If I' is a good rate function—i.e., $\Psi_{I'}(M) \triangleq \{y \in \mathbb{S} : I'(y) \le M\}$ is compact for each $M \in [0,\infty)$, then $\mathbf{S}_n$ satisfies the large deviation principle in $(\mathbb{S}, d)$ with speed a_n and good rate function I'.

Proof. i). For the upper bound, let F be a closed subset of $(\mathbb{S},\sigma)$. Thanks to Lemma B.2, for any $\epsilon > 0$, we have that $\left(\Phi^{-1}(F)\right)^\epsilon \subseteq \Phi^{-1}\left(F^{\|\Phi\|_{\mathrm{Lip}}\epsilon}\right)$. Hence,

$$- \inf_{x\in\left(\Phi^{-1}(F)\right)^\epsilon} I(x) \le - \inf_{x\in\Phi^{-1}\left(F^{\epsilon\|\Phi\|_{\mathrm{Lip}}}\right)} I(x). \tag{B.1}$$

Furthermore, by the extended LDP of $\mathbf{X}_n$, for $\delta > 0$ there exists an $n(\delta)$ such that for any $n \ge n(\delta)$,

$$\mathbf{P}(\mathbf{X}_n \in \Phi^{-1}(F)) \le \exp\left[a_n\left(- \inf_{x\in\left(\Phi^{-1}(F)\right)^\epsilon} I(x) + \delta\right)\right] \text{ for any } \epsilon > 0. \tag{B.2}$$

Consequently, (B.1), and (B.2) lead to

$$\mathbf{P}(\mathbf{X}_n \in \Phi^{-1}(F)) \le \exp\left[a_n\left(- \inf_{x\in\Phi^{-1}\left(F^{\epsilon\|\Phi\|_{\mathrm{Lip}}}\right)} I(x) + \delta\right)\right], \tag{B.3}$$

for any $n \ge n(\delta)$ and $\epsilon > 0$. Next, for $n \ge n(\delta)$,

$$\mathbf{P}\left(\mathbf{S}_n \in F\right) = \mathbf{P}\left(\Phi\left(\mathbf{X}_n\right) \in F\right) = \mathbf{P}\left(\mathbf{X}_n \in \Phi^{-1}\big(F\big)\right) \le \exp\left[a_n\left(- \inf_{x\in\Phi^{-1}\left(F^{\epsilon\|\Phi\|_{\mathrm{Lip}}}\right)} I(x) + \delta\right)\right].$$

Therefore,

$$\limsup_{n\to\infty} \frac{1}{a_n} \log \mathbf{P}\left(\mathbf{S}_n \in F\right) \le - \inf_{x\in\Phi^{-1}\left(F^{\epsilon\|\Phi\|_{\mathrm{Lip}}}\right)} I(x) + \delta = - \inf_{y\in F^{\epsilon\|\Phi\|_{\mathrm{Lip}}}} I'(y) + \delta. \tag{B.4}$$

Letting $\delta \to 0$ and $\epsilon \to 0$ in (B.4), we arrive at the desired large deviation upper bound.

Turning to the lower bound, consider an open set G. Since $\Phi^{-1}(G)$ is open,

$$\liminf_{n\to\infty} \frac{1}{a_n} \log \mathbf{P}\left(\mathbf{S}_n\right) \in G) = \liminf_{n\to\infty} \frac{1}{a_n} \log \mathbf{P}\left(\mathbf{X}_n \in \Phi^{-1}(G)\right) \ge - \inf_{y\in\Phi^{-1}(G)} I(\xi) = - \inf_{x\in G} I'(x).$$

ii). The upper and lower bounds for the extended large deviation principle have been proved in *i*). Since I is a rate function, its level sets $\Psi_I(M) \triangleq \{x \in \mathbb{X} : I(x) \le M\}$ are closed for every $M > 0$. Now, we verify that I' is lower-semicontinuous. The level sets of I' are $\Psi_{I'}(M) \triangleq \{y \in \mathbb{S} : I'(y) \le M\}$, for every

$M > 0$. Note that

$$\{y \in \mathbb{S} : I'(y) \leq M\} = \{\Phi(x) : I(x) \leq M\} = \Phi(\Psi_I(M)).$$

Since Φ is a homeomorphism the r.h.s. is closed. Hence, $\mathbf{S}_n$ satisfies the extended LDP.

iii). Due to our assumption, I' is a good rate function. Applying i), we obtain $\lim_{\epsilon \to 0} \inf_{y \in F^{\|\Phi\|_{\mathrm{Lip}}\epsilon}} I'(y) = \inf_{y \in F} I'(y)$. Consequently,

$$\limsup_{n \to \infty} \frac{\log \mathbf{P}\left(\mathbf{S}_n \in F\right)}{a_n} \leq - \lim_{\epsilon \to 0} \inf_{y \in F^{\epsilon\|\Phi\|_{\mathrm{Lip}}}} I'(y) = - \inf_{y \in F} I'(y).$$

$\square$

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