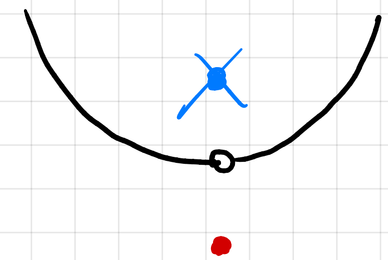
Conspiracy Principle vs Catastrophe Principle via Sample-Path Large Deviations

* Large Deviations Principle

For any measurable $A \subseteq \mathcal{X}$



$\mathbb{P}(X_n \in A) \overset{\log}{\sim} \exp\left(-n \inf_{x \in A^-} I(x)\right)$

$$-\inf_{x \in A^\circ} I(x) \leq \liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}(X_n \in A) \leq \limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}(X_n \in A) \leq -\inf_{x \in A^-} I(x)$$

↳ Interior of A

(Large Deviation Principle)

closure of A

We say that X_n satisfies an LDP

w/ rate function (r.f.) I:

Depends on Topology

Def) $\Psi_I(\alpha) = \{x : I(x) \leq \alpha\}$ is closed $\forall \alpha \geq 0$

Def) $\Psi_I(\alpha)$: compact $\forall \alpha \geq 0$

* If Y_n satisfies LDP w/ a good r.f. I,

then $h(Y_n)$ satisfies LDP w/ a good r.f. I'

where $I'(y) = \inf\{ I(x) ; y = h(x) \}$

"Contraction Principle"

* Light-Tailed Large Deviations (Cramér's Theorem)

$X_1, X_2, \cdots$: iid RVs w/ $\mathbb{E}X_1 = \mu \in \mathbb{R}$, $\hat{X}_n = \dfrac{X_1 + \cdots + X_n}{n}$

$\hat{X}_n$ satisfies an LDP w/ rate function (r.f.) Λ^*

$$\Lambda^*(y) = \sup_{\lambda \in \mathbb{R}} \{ \lambda y - \log \phi(\lambda) \} \qquad \phi(\lambda) = \mathbb{E} \exp(\lambda X_1)$$

i.e.,

$$-\inf_{x \in A^\circ} \Lambda^*(x) \leq \liminf_{n \to \infty} \frac{1}{n} \log \mathbb{P}(\hat{X}_n \in A) \leq \limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}(\hat{X}_n \in A) \leq -\inf_{x \in A^-} \Lambda^*(x)$$

* Heavy-Tailed Large Deviations (Nagaev's Thm)

$X_1, X_2, \cdots$: iid RVs w/ $\mathbb{E}X_1 = \mu \in \mathbb{R}$, $\hat{X}_n = \dfrac{X_1 + \cdots + X_n}{n}$

If X_i are subexponential, (eg. Pareto, Log-normal, Weibull)

$$\mathbb{P}(\hat{X}_n - \mu \geq \varepsilon) = a_n \, n \, \mathbb{P}(X_1 \geq n(\mu + \varepsilon))$$

where $a_n \to 1$ as $n \to \infty$

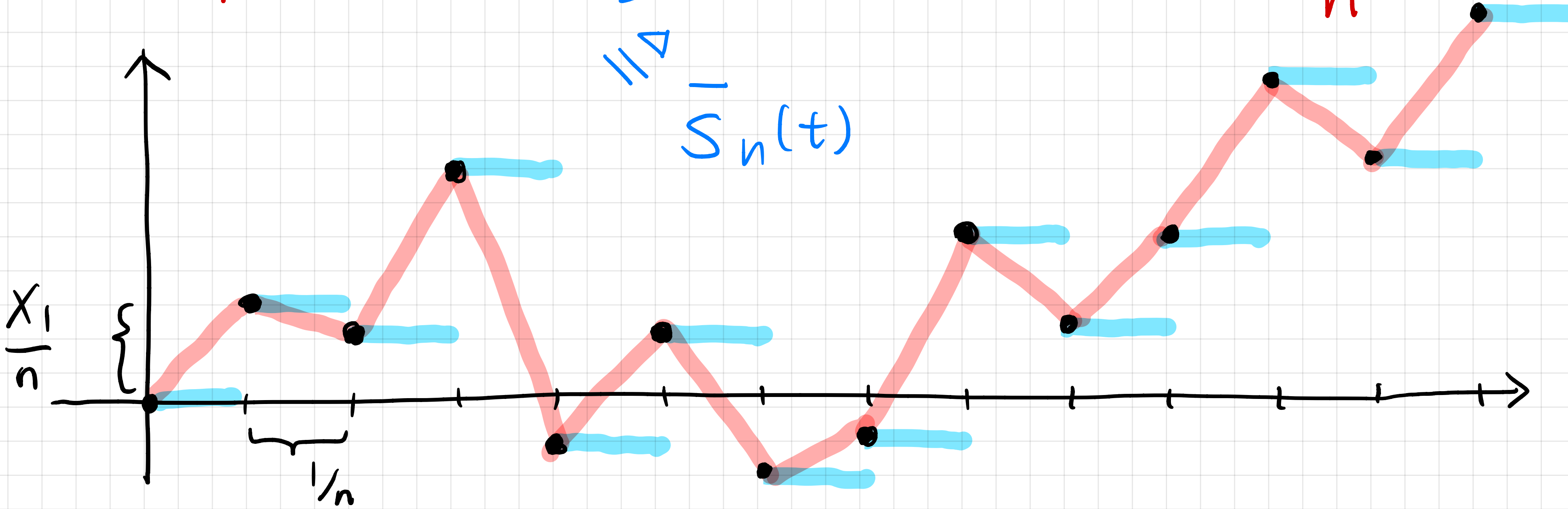
$$\mathbb{P}(X \geq x) \sim \begin{cases} c\, x^{-\alpha} \\ c\, x^{\alpha} e^{-\lambda (\ln x)^{\gamma}} \\ c\, x^{\alpha} e^{-\lambda x^{1/\gamma}} \end{cases}$$

§ Sample Path Asymptotics

$$\hat{X}_n = \frac{X_1 + X_2 + \cdots + X_n}{n}$$

$$\hat{S}_n(t) = \hat{X}_{\lfloor nt \rfloor} + (nt - \lfloor nt \rfloor)\frac{X_{\lfloor nt \rfloor + 1}}{n}$$

$$\overset{\triangle}{=} \bar{S}_n(t)$$



$\hat{S}_n(\cdot) \in C[0,1]$

$\{ f : [0,1] \to \mathbb{R} : \text{cont}' \}$

* Light-Tailed Sample-Path Large Deviations Theorem (Mogulskii)

If $\Lambda(\lambda) = \log \phi(\lambda) < \infty \quad \forall \lambda \in \mathbb{R}$, then

$\hat{S}_n(\cdot)$ satisfies an LDP in $C[0,1]$ w/ r.f. $I : C[0,1] \to [0,\infty]$

$$I(f) = \begin{cases} \int_0^1 \Lambda^*\left(\frac{d}{dt} f(t)\right) dt & \text{if } f \in AC,\ f(0) = 0 \\ \infty & \text{otherwise} \end{cases}$$

* Rare-event occur in the most likely way (Ganesh et al. 03 Lemma 4.2)

Supp. X_n satisfies an LDP w/ good r.f. I, and C is a closed set s.t. $\inf_{x \in C} I(x) = \min_{x \in C^\circ} I(x) = k < \infty$.

For any nbhd B of $\{x \in C : I(x) = k\}$,

$$\mathbb{P}(X_n \notin B \mid X_n \in C) \longrightarrow 0$$

* Heavy-Tailed Counterpart

If X_i are regularly varying, i.e., $\mathbb{P}(X_i \geq x) = x^{-\alpha} L(x)$

$$\left(\lim_{x \to \infty} \frac{\mathbb{P}(X_i > ax)}{\mathbb{P}(X_i > x)} = g(a) \quad \forall a > 0 \right) \quad \text{where} \quad \lim_{x \to \infty} \frac{L(ax)}{L(x)} = 1 \quad \forall a > 0$$

then, for "general" A,

$$C_{J(A)}(A^\circ) \leq \liminf_{n \to \infty} \frac{\mathbb{P}(\bar{S}_n \in A)}{(n\nu[n,\infty))^{J(A)}} \leq \limsup_{n \to \infty} \frac{\mathbb{P}(\bar{S}_n \in A)}{(n\nu[n,\infty))^{J(A)}} \leq C_{J(A)}(A^-)$$

cf.

$$-\inf_{x \in A^\circ} \Lambda^*(x) \leq \liminf_{n \to \infty} \frac{1}{n} \log \mathbb{P}(\hat{S}_n \in A) \leq \limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}(\hat{S}_n \in A) \leq -\inf_{x \in A^-} \Lambda^*(x)$$

right continuous with left limit

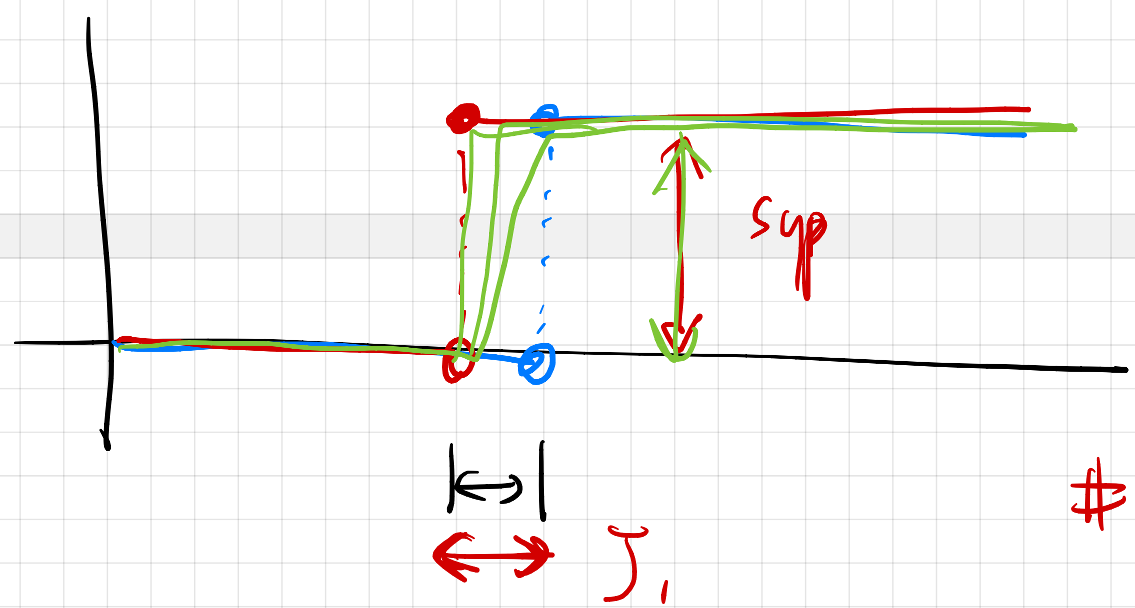
* $(\mathbb{D}, d)$: Skorokhod Space : cadlag

For $\xi, \zeta \in \mathbb{D}$,

$e(t) = t$ $\qquad$ sup norm

J_1 metric : $d(\xi, \zeta) = \inf_{\lambda \in \Lambda} \| \lambda - e \| \vee \| \xi \circ \lambda - \zeta \|$

non-decreasing homeomorphism on $[0,1]$ $\qquad$ $\|f\| = \sup_{t \in [0,1]} |f(t)|$



# of upward jumps in ξ

* $\mathbb{D}_i \triangleq \{ \xi \in D : \xi(0) = 0, \text{ non-decreasing step ftn}, \mathcal{D}_+(\xi) = i \}$

* $\mathbb{D}_{<j} \triangleq \bigcup_{0 \le i < j} \mathbb{D}_i$ $\qquad$ * $\mathbb{D}_{\le j} \triangleq \bigcup_{0 \le i \le j} \mathbb{D}_i$ $\qquad$ * $\mathbb{D}_s^{\uparrow} = \mathbb{D}_{\le \infty}$

* $X(t)$: Levy process

Levy meas.

$$X(s) = sa + B(s) + \int_{|x|\le 1} x\left[N([0,s]\times dx) - s\nu(dx)\right]$$

BM

Poisson R.M. w/ mean Leb $\times \nu$

$$+ \int_{|x|>1} x N([0,s]\times dx)$$

$$\bar{X}_n(s) = \frac{X(sn)}{n} - sa - \frac{1}{\nu[1,\infty)}\int_{[1,\infty)} x\,\nu(dx)$$: centered & scaled

* $\nu_\gamma(x,\infty) = x^{-\gamma}$

$\{x \in \mathbb{R}_+^j : x_1 \ge x_2 \ge \cdots \ge x_j\}$

$\parallel$

* ν_γ^j : restriction of j-fold product measure of ν_γ to $\mathbb{R}_+^{j\downarrow}$

* $C_j(\cdot) \triangleq \mathbb{E}\left[\nu_\gamma^j\left\{y \in (0,\infty)^j : \sum_{i=1}^{j} y_i \mathbb{1}_{[U_i,1]} \in \cdot\right\}\right]$

iid Unif $[0,1]$

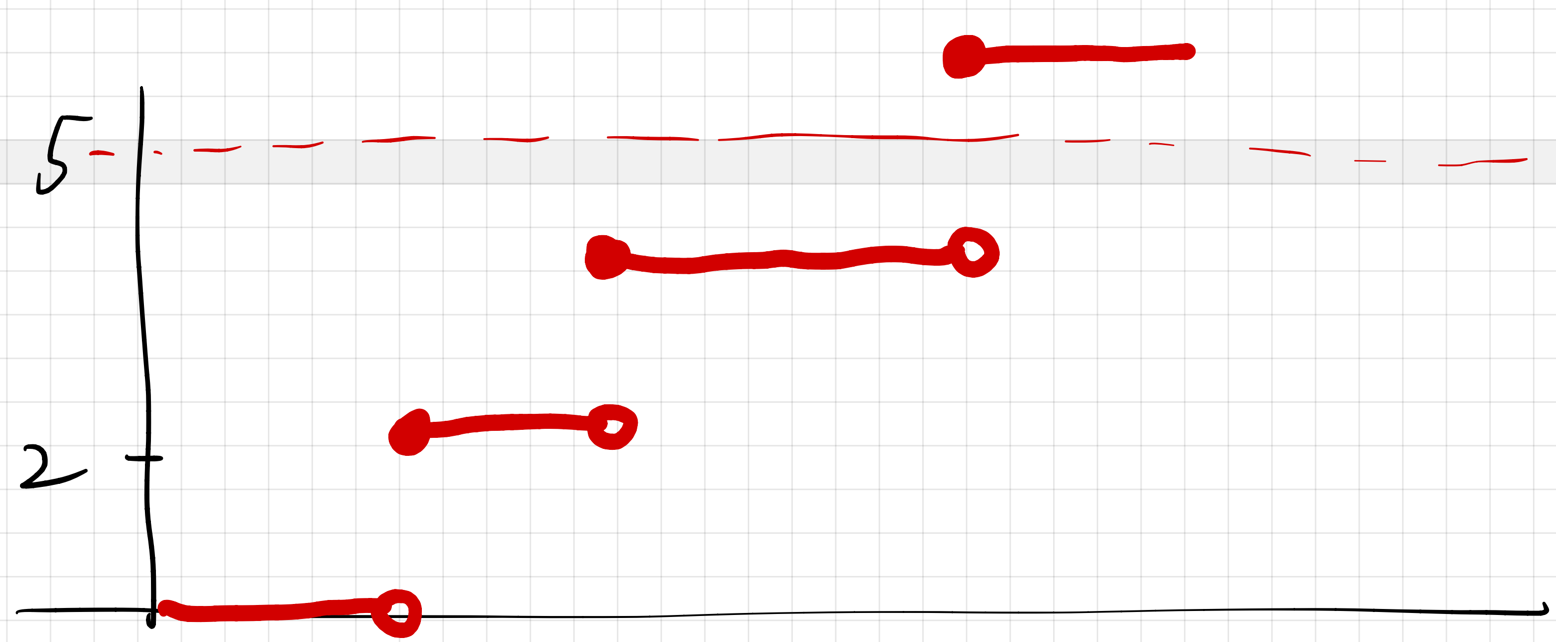
* $J(A) \triangleq \inf_{\xi \in D_s^{\uparrow} \cap A} \mathcal{D}_+(\xi)$

* (Main Result) if A is bdd away from $\mathbb{D}_{<\mathcal{J}(A)}$, can be relaxed

$\mathbb{D}_{<\mathcal{J}(A)} \cap A^{\delta} = \emptyset$

$$C_{\mathcal{J}(A)}(A^{\circ}) \le \liminf_{n\to\infty} \frac{\mathbb{P}(\bar{X}_n \in A)}{(n\nu[n,\infty))^{\mathcal{J}(A)}} \le \limsup_{n\to\infty} \frac{\mathbb{P}(\bar{X}_n \in A)}{(n\nu[n,\infty))^{\mathcal{J}(A)}} \le C_{\mathcal{J}(A)}(A^{-})$$

*



$\mathcal{J}(A) = 3$

$$A = \{ f : \sup_{t\in[0,1]} f(t) \ge 5, \ \sup_{t\in[0,1]} |f(t) - f(t-)| \le 2 \}$$

$\ast$ If $C_{J(B)}(B) = C_{J(B)}(B^-) = C_{J(B)}(B^\circ) > 0$,

$$\mathbb{P}\left(\bar{X}_n \in \cdot \,\middle|\, \bar{X}_n \in B \right) \longrightarrow \mathbb{P}\left(\bar{X}_\infty^{|B} \in \cdot \right) = \frac{C_{J(B)}(\cdot \cap B)}{C_{J(B)}(B)}$$

Recall: $C_{\bar{j}}(\cdot) \triangleq \mathbb{E}\left[\nu_\gamma^{\bar{j}} \left\{ y \in (0,\infty)^{\bar{j}} : \sum_{i=1}^{\bar{j}} y_i \mathbb{1}_{[U_i, 1]} \in \cdot \right\} \right]$

$$* \quad \bar{X}_n(s) = B(ns) + \int_{|x|\le 1} x\,[N([0,ns]\times dx) - ns\,\nu(dx)]$$

$$+ \int_{|x|>1} x\,N([0,ns]\times dx) - \frac{1}{\nu[1,\infty)}\int_{[1,\infty)} x\,\nu(dx)$$

asy. eq. (later)

$$\approx \frac{1}{n}\sum_{\ell=1}^{\tilde{N}_n}\left(Q_n^{\leftarrow}(\Gamma_\ell) - \mu_1^+\right)\mathbb{1}_{[u_\ell,1]}(s) \approx \frac{1}{n}\sum_{\ell=1}^{j} Q_n^{\leftarrow}(\Gamma_\ell)\mathbb{1}_{[u_\ell,1]}$$

$$\tilde{N}_n = N_n([0,1]\times[1,\infty)) \sim \mathrm{Poisson}(n\,\nu[1,\infty))$$

$$(\#\,\ell:\ \Gamma_\ell \le n\nu_1^+)$$

$$N_n = \sum_{\ell=1}^{\infty}\delta_{(u_\ell,\,Q_n^{\leftarrow}(\Gamma_\ell))}$$

$$Q_n(x) = n\nu[x,\infty)$$

$$\Gamma_\ell = E_1 + \cdots + E_\ell$$

$$Q_n^{\leftarrow}(y) = \inf\{s>0:\ n\nu[s,\infty) < y\}$$

$$\nwarrow \ \mathrm{Exp}(1)$$

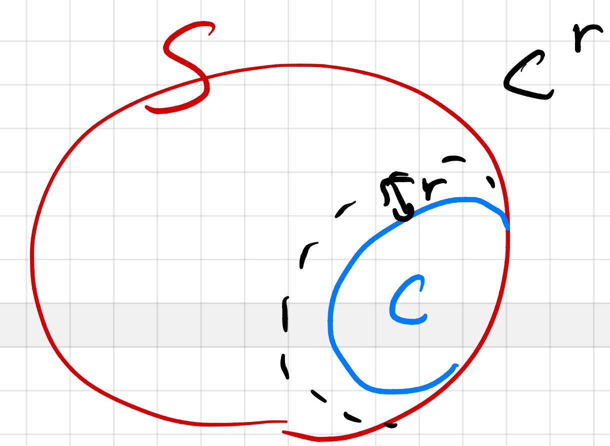
§ $M(S \setminus C)$ – Convergence

* (S, d) : Complete Separable Metric Space

* $C \subseteq S$: closed

* **Def** $M(S\setminus C)$: space of Borel measures μ s.t. $\mu(S \setminus C^r) < \infty \quad \forall r > 0$

$$C^r = \{x \in S : d(x, C) < r\}$$



* **Def** $M(S\setminus C)$ – convergence

$\mu_n \to \mu$ in $M(S\setminus C)$ if

$$\begin{cases} \limsup_{n\to\infty} \mu_n(F) \le \mu(F) & \forall F : \text{closed, bdd away from } C \\ \liminf_{n\to\infty} \mu_n(G) \ge \mu(G) & \forall G : \text{open, } \quad '' \end{cases}$$

* (key Thm) $\forall j \ge 0$

$$(n\nu[n,\infty))^{-j}\, \mathbb{P}(\bar{X}_n \in \cdot) \to C_j(\cdot) \quad \text{in } M(\mathbb{D} \setminus \mathbb{D}_{<j})$$

Recall : if A is bdd away from $\mathbb{D}_{<J(A)}$,

$$C_{J(A)}(A^\circ) \leq \liminf_{n\to\infty} \frac{\mathbb{P}(\bar{X}_n \in A)}{(n\nu[n,\infty))^{J(A)}} \leq \limsup_{n\to\infty} \frac{\mathbb{P}(\bar{X}_n \in A)}{(n\nu[n,\infty))^{J(A)}} \leq C_{J(A)}(A^-)$$

* <u>Def</u> (S, d) : metric sp.

X_n, Y_n : S- valued r.v.s

X_n, Y_n are asymptotically equivalent w.r.t. ε_n if $\limsup_{n\to\infty} \varepsilon_n^{-1} \mathbb{P}(d(X_n, Y_n) \geq \delta) = 0$ for all $\delta > 0$

* <u>Lemma</u>

If i) X_n, Y_n are asymptotically equivalent w.r.t. ε_n

ii) $\varepsilon_n^{-1} \mathbb{P}(X_n \in \cdot) \to \mu(\cdot)$ in $\mathbb{M}(S \setminus C)$

$\mu(S \setminus S_0) = 0$

$\mathbb{P}(X_n \in S_0) = 1$

Then, $\varepsilon_n^{-1} \mathbb{P}(Y_n \in \cdot) \to \mu(\cdot)$ in $\mathbb{M}(S \setminus C)$

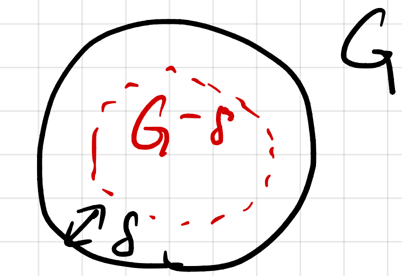
upper & lower bound hold for F, G s.t. $F_\delta \cap S_0$, $G \cap S_0$ bdd away from C

pf) $\liminf\limits_{n\to\infty} \varepsilon_n^{-1}\, \mathbb{P}(Y_n \in G) \geq \mu(G)$ $\forall\, G$: open, bdd away from C

$$\geq \liminf \varepsilon_n^{-1}\, \mathbb{P}(X_n \in G^{-\delta},\ d(X_n, Y_n) < \delta)$$

$G^{-\delta} = \{x \in S : d(x, G^c) > \delta\}$

$$= \liminf \varepsilon_n^{-1}\left(\mathbb{P}(X_n \in G^{-\delta}) - \mathbb{P}(X_n \in G^{-\delta},\ d(X_n, Y_n) \geq \delta)\right)$$



$$= \liminf \varepsilon_n^{-1}\, \mathbb{P}(X_n \in G^{-\delta})$$

$$\geq \mu(G^{-\delta}) \to \mu(G) \quad \text{as} \quad \delta \to 0$$

$$\limsup \varepsilon_n^{-1}\, \mathbb{P}(Y_n \in F)$$

$$= \limsup \varepsilon_n^{-1}\left(\mathbb{P}(Y_n \in F,\ d(X_n, Y_n) < \delta) + \mathbb{P}(Y_n \in F,\ d(X_n, Y_n) \geq \delta)\right)$$

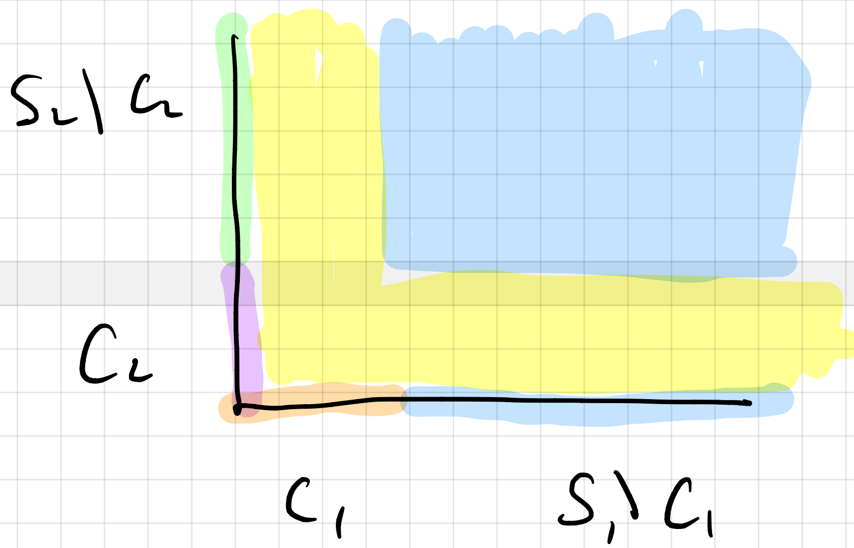
$$\leq \limsup \varepsilon_n^{-1}\, \mathbb{P}(X_n \in F_\delta) \leq \mu(F_\delta) \to \mu(F) \quad \delta \to 0$$

$F_\delta = \{x \in S : d(x, F) \leq \delta\}$

* Lemma $(S_i, C_i) : i = 1, 2, \cdots, d$

$$\mu_n^{(i)}(\cdot) \to \mu^{(i)}(\cdot) \quad \text{in} \quad M(S_i \setminus C_i) \quad i = 1, 2$$

Then, $\mu_n^{(1)} \times \mu_n^{(2)}(\cdot) \to \mu^{(1)} \times \mu^{(2)}(\cdot)$ in $M\Big((S_1 \times S_2) \setminus \big((S_1 \times C_2) \cup (C_1 \times S_2)\big)\Big)$



$$\mu_n^{(1)} \times \cdots \times \mu_n^{(d)} \to \mu^{(1)} \times \cdots \times \mu^{(d)} \quad \text{in} \quad M\left(\prod_{i=1}^{d} S_i \setminus \left(\bigcup_{i=1}^{d} \left(\prod_{j=1}^{i-1} S_j\right) \times C_i \times \left(\prod_{j=i+1}^{d} S_j\right)\right)\right)$$

pf) $\mu_n \to \mu$ in $M(S \setminus C) \iff \mu_n|_{S \setminus C^r} \xrightarrow{d} \mu|_{S \setminus C^r}$

for all but countably many $r > 0$

* <u>Lemma</u> $\mu^{(i)}$, $C(i) \subseteq S$ $i = 0, 1, 2, \cdots, m$

$\varepsilon_n(i)^{-1} \mathbb{P}(X_n \in \cdot) \to \mu^{(i)}(\cdot)$ in $M(S \setminus C(i))$

$\bar{\mu}^{(0)}(\cdot) \triangleq \mu^{(0)}(\cdot \setminus C(0))$

if $\bar{\mu}^{(0)} \in M(S \setminus \bigcap_{i=0}^{m} C(i))$ and $\limsup_{n\to\infty} \dfrac{\varepsilon_n(i)}{\varepsilon_n(0)} = 0$

and for each $r > 0$, $\exists\, r_0, r_1, \cdots, r_m$ s.t. $\bigcap_{i=0}^{m} C(i)^{r_i} \subseteq \left(\bigcap_{i=0}^{m} C(i)\right)^{r}$

then, $\varepsilon_n(0)^{-1} \mathbb{P}(X_n \in \cdot) \to \bar{\mu}^{(0)}$ in $M(S \setminus \bigcap_{i=0}^{m} C(i))$

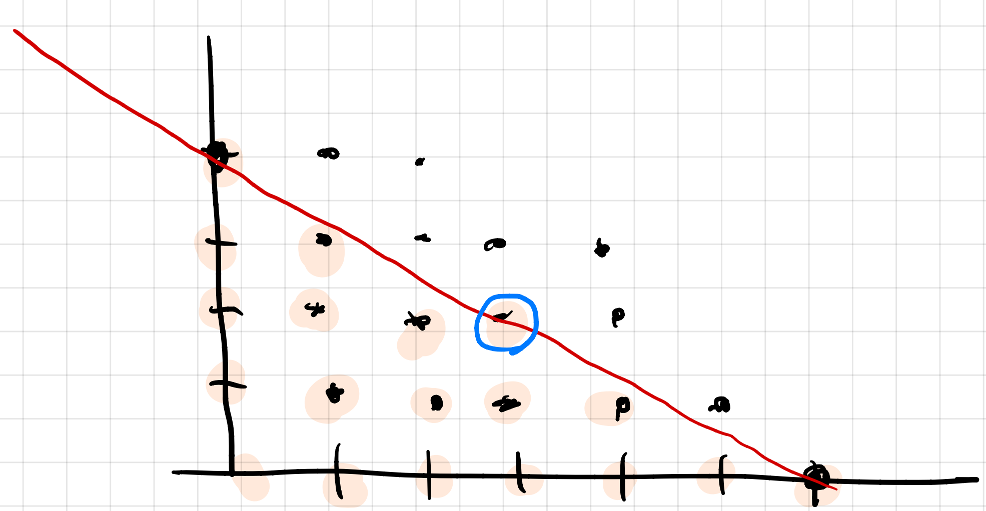
* (key thm) $X^{(1)}, X^{(2)}$ Lévy proc's w/ Lévy meas. ν_1, ν_2

w/ reg. var. index α_1, α_2

$$\frac{\mathbb{P}\left((\bar{X}_n^{(1)}, \bar{X}_n^{(2)}) \in \cdot \right)}{(n\nu_1[n,\infty))^{\mathcal{J}_1} \, (n\nu_2[n,\infty))^{\mathcal{J}_2}} \to C_{\mathcal{J}_1}^{(1)} \times C_{\mathcal{J}_2}^{(2)}(\cdot) \quad \text{in } M\left(\mathbb{D}^2 \setminus \mathbb{D}_{<(\mathcal{J}_1, \mathcal{J}_2)}\right)$$

$$\mathbb{D}_{<(\mathcal{J}_1,\mathcal{J}_2)} = \bigcup_{(l_1,l_2) \in \mathcal{I}_{<(\mathcal{J}_1,\mathcal{J}_2)}} \mathbb{D}_{l_1} \times \mathbb{D}_{l_2}$$

$$\mathcal{I}_{<(\mathcal{J}_1,\mathcal{J}_2)} = \left\{ (l_1,l_2) \in \mathbb{Z}_+^2 \setminus \{(\mathcal{J}_1,\mathcal{J}_2)\} : \underbrace{(\alpha_1 - 1) l_1 + (\alpha_2 - 1) l_2}_{= \mathcal{I}(l_1,l_2)} \leq (\alpha_1 - 1)\mathcal{J}_1 + (\alpha_2 - 1)\mathcal{J}_2 \right\}$$



$$* \quad \bar{X}_n(s) = B(ns) + \int_{|x| \le 1} x \left[N([0,ns] \times dx) - ns\, \nu(dx) \right]$$

$$+ \int_{x>1} x\, N([0,ns] \times dx) - \frac{1}{\nu[1,\infty)} \int_{[1,\infty)} x\, \nu(dx)$$

$\bar{X}_n^{(+)}(s)$

$$+ \int_{x<-1} x\, N([0,ns] \times dx) - \frac{1}{\nu(-\infty,-1]} \int_{[1,\infty)} x\, \nu(dx)$$

$-\bar{X}_n^{(-)}(s)$

M-conv. of $\left(\bar{X}_n^{(+)}, \bar{X}_n^{(-)} \right)$

* **Lemma** (Continuous mapping / Contraction Principle)

$h : (S \setminus C, \mathscr{S}_{S \setminus C}) \to (S' \setminus C', \mathscr{S}_{S' \setminus C'})$: measurable

A' : bdd away from C' $\Longrightarrow$ $h^{-1}(A')$ bdd away from C

Then, $\hat{h} : M(S \setminus C) \to M(S' \setminus C')$ is continuous at μ

$\nu \mapsto \nu \circ h^{-1}$

if $\mu_n \to \mu$ in $M(S \setminus C)$

then $\mu_n \circ h^{-1} \to \mu \circ h^{-1}$ in $M(S' \setminus C')$

provided that $\mu(D_h) = 0$

D_h: discontinuity pts of h

$S_0 \subseteq S$

$h : (S_0 \setminus C, \mathscr{S}_{S_0 \setminus C}) \to (S' \setminus C', \mathscr{S}_{S' \setminus C'})$: measurable

A' : bdd away from C' $\Longrightarrow$ $h^{-1}(A')$ bdd away from C

Then, $\hat{h} : M(S \setminus C) \to M(S' \setminus C')$ is continuous at μ

$\nu \mapsto \nu \circ h^{-1}$

provided that $\mu(D_h \setminus C^r) = 0$ and $\mu(\partial S_0 \setminus C^r) = 0$ $\forall r > 0$

$$h(x,y) = x - y$$

M-conv. of $(\bar{X}_n^{(+)}, \bar{X}_n^{(-)}) \Longrightarrow$ M-conv. of $\bar{X}_n$

* (Main Thm)

$$(\mathcal{J}(A), \mathcal{K}(A)) = \underset{\substack{(j,k) \in \mathbb{Z}_+^2 \\ \mathbb{D}_{j,k} \cap A \neq \emptyset}}{\arg\min} \mathcal{I}(j,k) = (\alpha - 1) j + (\beta - 1) k$$

$\mathbb{D}_{j,k}$: sp. of step ftn's w/ exactly j upward jumps & k downward jumps.

$$\liminf_{n \to \infty} \frac{P(\bar{X}_n \in A)}{(n\nu[n,\infty))^{\mathcal{J}(A)} (n\nu(-\infty,-n])^{\mathcal{K}(A)}} \geq C_{\mathcal{J}(A),\mathcal{K}(A)}(A^\circ)$$

$$\limsup_{n \to \infty} \frac{P(\bar{X}_n \in A)}{(n\nu[n,\infty))^{\mathcal{J}(A)} (n\nu(-\infty,-n])^{\mathcal{K}(A)}} \leq C_{\mathcal{J}(A),\mathcal{K}(A)}(A^-)$$

* $C_{j,k}(\cdot) = \mathbb{E}\left[\nu_\alpha^{j} \times \nu_\beta^{k}\left\{(x,y) \in (0,\infty)^j \times (0,\infty)^k : \sum_{i=1}^{j} x_i \mathbb{1}_{[U_i,1]} - \sum_{i=1}^{k} y_i \mathbb{1}_{[V_i,1]} \in \cdot \right\}\right]$

$U_i, V_i \sim \mathrm{Unif}[0,1]$

* $\nu_\gamma(x,\infty) = x^{-\gamma}$

* ν_γ^{j} : restriction of j-fold product measure of ν_γ to $\mathbb{R}_+^{j\downarrow}$

* (key thm) $j,k \in \mathbb{Z}_+^2$

$$(n\nu[n,\infty))^{-j} (n\nu(-\infty,-n])^{-k} \mathbb{P}(\bar{X}_n \in \cdot) \to C_{j,k}(\cdot) \text{ in } \mathbb{M}(\mathbb{D} \setminus \mathbb{D}_{<j,k})$$

* Recall : $\hat{S}_n(t) = \hat{X}_{\lfloor nt \rfloor}$ $\hat{X}_n = \dfrac{X_1 + X_2 + \cdots + X_n}{n}$

$$\left(\mathcal{J}(A), K(A)\right) = \underset{\substack{(j,k) \in \mathbb{Z}_+^2 \\ \mathbb{D}_{j,k} \cap A \neq \emptyset}}{\arg\min}\, \mathcal{I}(j,k) = (\alpha-1)j + (\beta-1)k$$

$\mathbb{D}_{j,k}$: sp. of step ftn's w/ exactly j upward jumps & k downward jumps.

$$\liminf_{n\to\infty} \frac{\mathbb{P}(\hat{S}_n \in A)}{\left(n\,\mathbb{P}(S_1 \geq n)\right)^{\mathcal{J}(A)} \left(n\,\mathbb{P}(S_1 \leq -n)\right)^{K(A)}} \geq C_{\mathcal{J}(A),K(A)}(A^\circ)$$

$$\limsup_{n\to\infty} \frac{\mathbb{P}(\hat{S}_n \in A)}{\left(n\,\mathbb{P}(S_1 \geq n)\right)^{\mathcal{J}(A)} \left(n\,\mathbb{P}(S_1 \leq -n)\right)^{K(A)}} \leq C_{\mathcal{J}(A),K(A)}(A^-)$$

* Classical LDP doesn't hold for R.V. Levy processes in $\mathbb{J}_1$