Efficient Low Diameter Clustering

with strong diameter in the CONGEST model

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About this Presentation

1. Distributed Algorithms
2. Network Decomposition
3. Low Diameter Clustering

- We want to solve graph problems on **networks**
  - Computers are like nodes in a graph

Distribution $\Rightarrow$ Multiple processors

- We want to solve graph problems on **networks**
  - Computers are like nodes in a graph

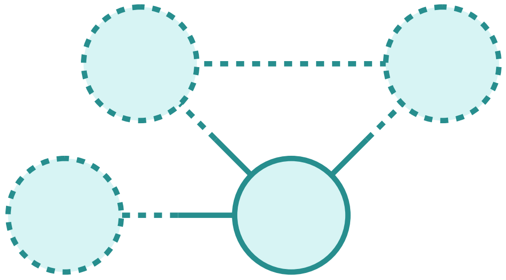
Distribution ⇒ Multiple processors

- Nodes can run code
  - Should be **the same for all nodes**

💡 Each node gives a partial solution

- Arcs are **communication links** between computers

- In the **PN-Network** a node only knows its ***neighbours***
  - And how to “contact” them
- There are no **self loops**
- Connection is two-way
- There is ≤ 1 arc between two nodes



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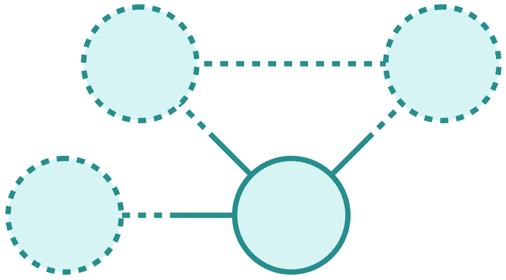

🚨 **A node can't see the whole topology**

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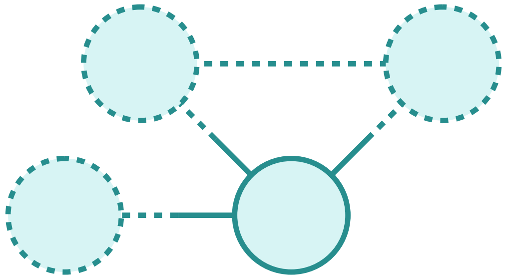

A node can’t see the whole topology

All nodes appear identical

💡 We add **unique identifiers** to the nodes

$$id : V \to \mathbb{N}$$

where $\forall v \in V : id(v) \leq n^c$ for some $c \geq 1$

We choose n^c so we need $O(\log n)$ bits to represent an identifier,
(identifiers are reasonably *"small"*)

- Collaboration requires **exchanging messages** …on a medium that is **slow** and **unreliable**

🚨 $\Rightarrow$ **Communication is the main pitfall**

- Too many messages congest the network
- We **quantify** the number of messages that an algorithm requires
  - An ***"efficient"*** algorithm will need few messages

W.l.o.g.[1] we adopt a model of ***synchronous communication***

Each round, a node $v \in V$ performs these actions:

1. v ***sends*** a message $msg \in \mathbb{N}$ to its neighbours
2. v ***receives*** messages from its neighbours
3. ...

[1] Without loss of generality.

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- (1.) and (2.) establish a ***communication round***
  - **Measure unit of complexity**
  - Few communication rounds $\Rightarrow$ few messages

[2]Without loss of generality.

3. v ***executes locally*** some algorithm (same for each node).
    - A node may ***stop*** in this phase
        - Its local result is **final**

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- When all nodes **stopped** the algorithm terminates

An algorithm is *"efficient"* when it stops in a number of rounds polylogarithmic in $|V|$

Wave: A First Example

- The node with id(v) = 0 *“waves hello”* to neighbours
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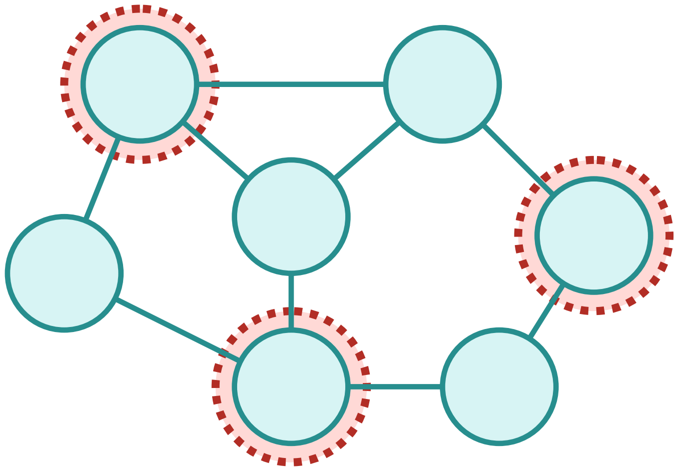
  - ...sending them a message
- When a node receives the message, **forwards** it to its neighbours
  - And then **stops**
- The running time of this algorithm on a graph G is $O(diam(G))$

Example: **Maximal Independent Set** (MIS)



- Solving it ***centralized*** is easy
- How can we solve it ***distributed***?

A Second Example (Naive-MIS)

Let's leverage id(v) to select the next MIS node

- At round #i, node v : id(v) = i executes
  - If no neighbour is in the MIS, add the node
    - And inform the neighbours
  - Otherwise, the node is outside the MIS

A Second Example (Naive-MIS)

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- At round $\#i$, node $v : \text{id}(v) = i$ executes
  - If no neighbour is in the MIS, add the node
    - And inform the neighbours
  - Otherwise, the node is outside the MIS
- It is correct since no node has the same id
- This algorithm runs in $O(n^c)$ (the maximum id)
  - **Very bad**

💡 Running a centralized algorithm on a single node would take O(1) rounds

- We'd like to run a MIS algorithm on each node
  - Each must have a **local copy** of the **entire** graph
  - The algorithm must be deterministic
    - When a node stops it checks if it is included in MIS

- The algorithm GATHER-ALL makes all nodes build a local copy of the whole graph
  - At round i, each node v knows $ball(i, v)$

- All nodes will know the whole graph after $O(diam(G))$ rounds

- Gather-All assumes that messages size is **unbounded**

🚨 **Requires to send the whole graph in one message**

- It is not always possible to send arbitrary large messages
  - Heavy ones may be "sharded"
- We provide an upper bound for message size
  - Messages need to be reasonably ***"small"***
  - Large messages will require more rounds to be sent

In the CONGEST model, messages size has to be $O(\log n)$

- Sending k identifiers takes $O(1)$ rounds
- Sending a *set* of identifiers can take up to $O(n)$
- Sending the whole graph requires $O(n^2)$ rounds:
  - The adjacency matrix suffices...

🚨 **⇒ We can't use** Gather-All **in the CONGEST model**

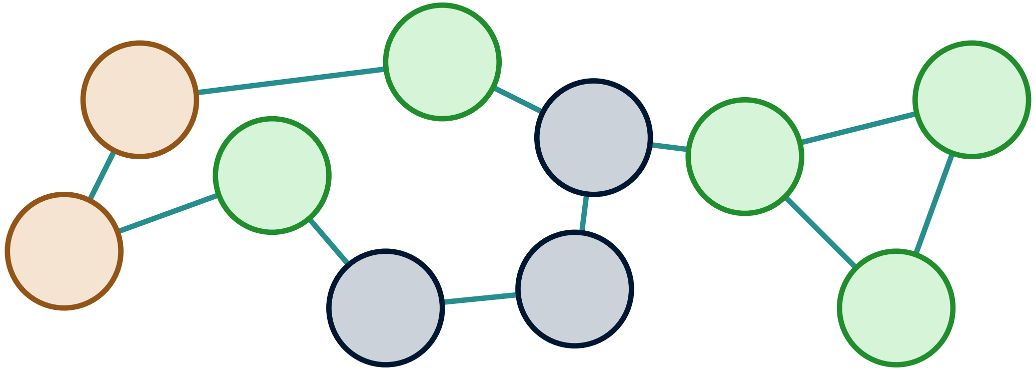
Solving problems in CONGEST

- Censor-Hillel et al. [1] provided an algorithm that solves MIS in $O(diam(G)\log^2 n)$ in CONGEST

🚨 **The diameter can be very large**

- Worst case: $diam(G) = n$
- How can we improve it?

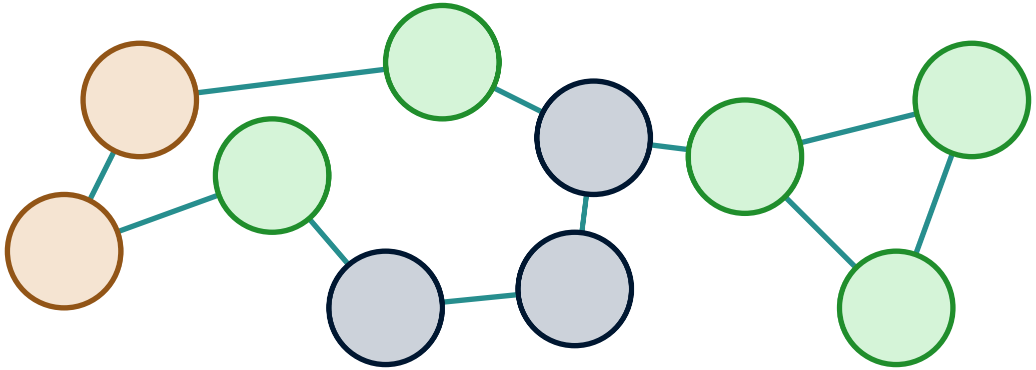
- A ***Network Decomposition*** groups nodes in **colored clusters**
  - Clusters with the same color are not adjacent
  - We say it to ***have diameter*** d if each cluster has diameter at most d
  - It has c colors



💡 Solving MIS in a color gives a partial solution

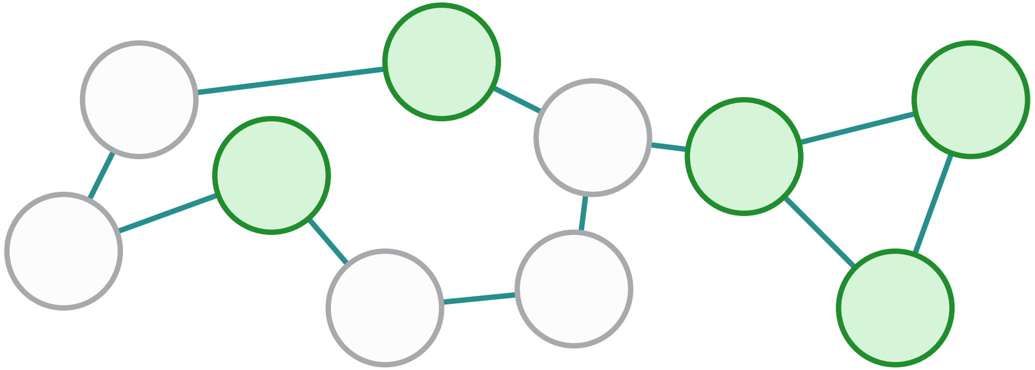
- We can apply [1] for all colors
  - (dropping MIS neighbours after each iter)
  - This has complexity $O\left(c \cdot d \log^2 n\right)$
    - If $c = O(\log n) = d$ it would be ***"efficient"***

How to use it?



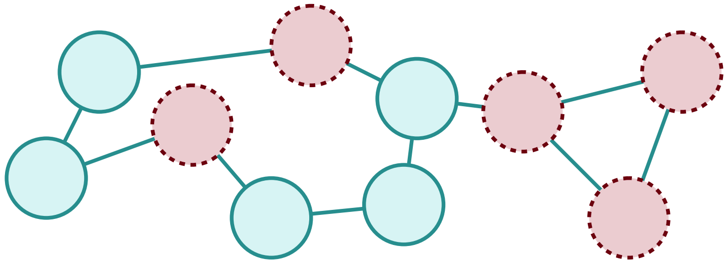
A ***low diameter clustering*** $\mathscr{C} \subseteq 2^V$ for a graph G with diameter d is such:

1. $\forall C_1 \neq C_2 \in \mathscr{C} : dist_G(C_1, C_2) \geq 2$
   - ***"There are no adjacent clusters"***
2. $\forall C \in \mathscr{C} : diam(G[C]) \leq d$
   - ***"Any cluster has diameter at most** d **"***



Main iteration:

1. Find a low diameter clustering
2. Assign a free color to its nodes
3. Repeat to discarded nodes until there are no more left



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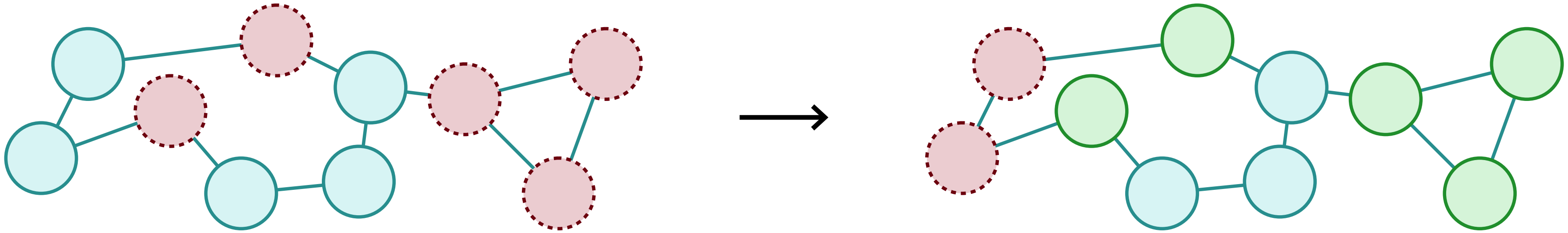

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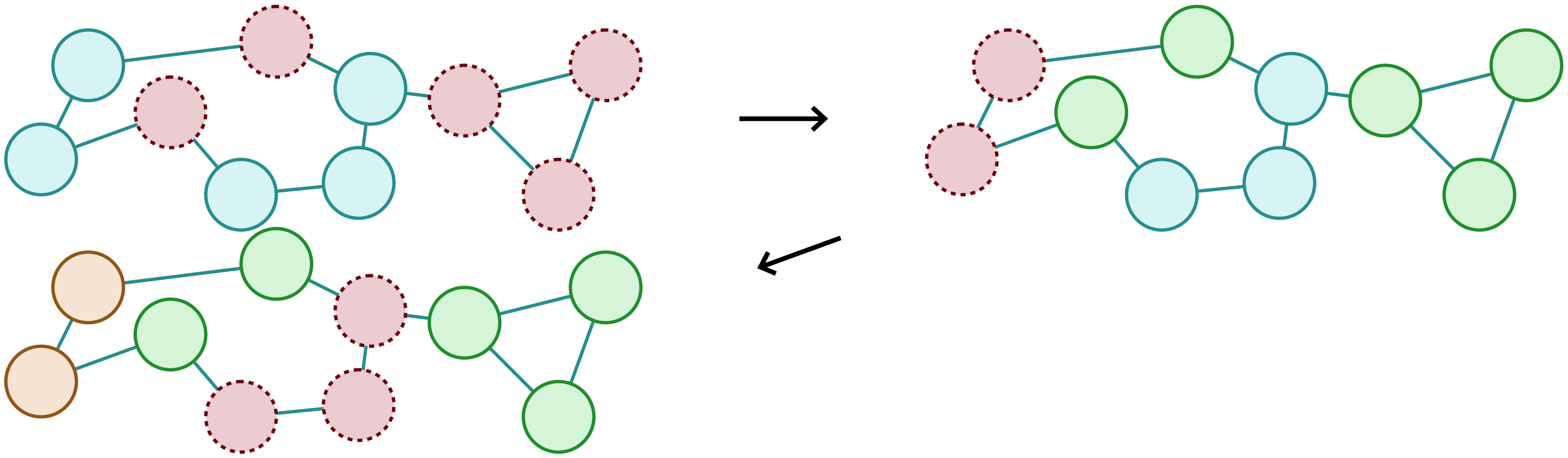

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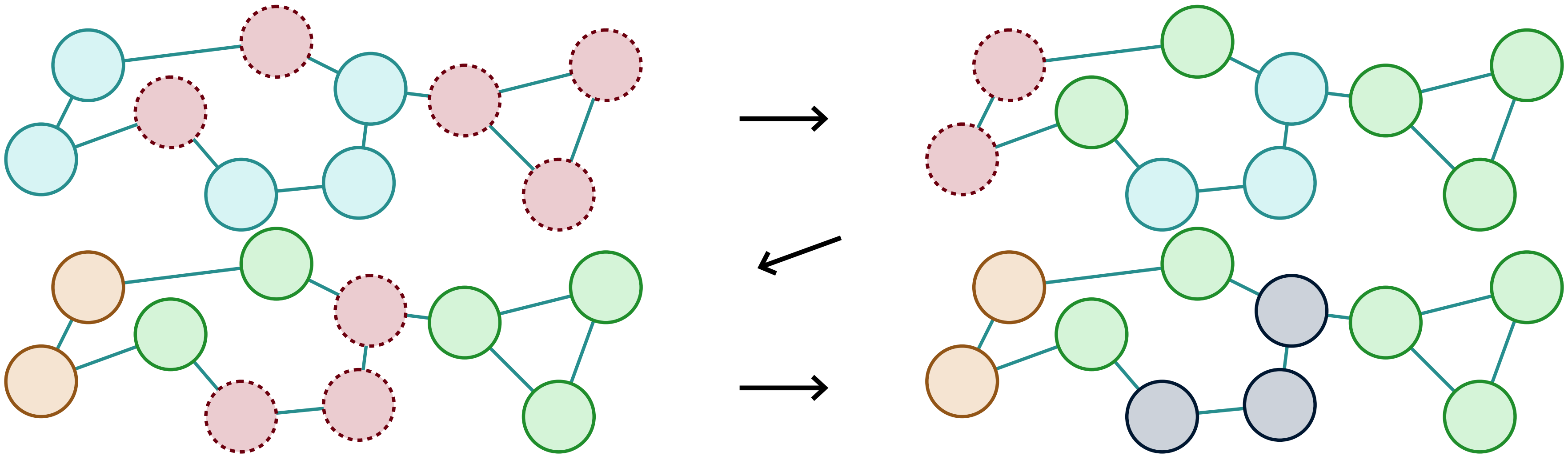
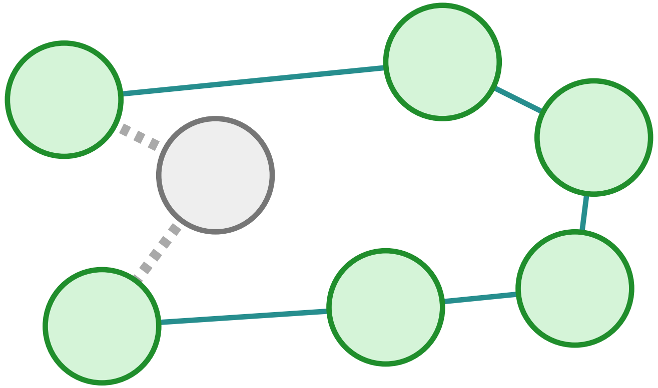


- Our previous definition of diamter is also called ***strong*** diameter

We say a clustering has ***weak*** diameter when:

1. (unchanged) **"There are no adjacent clusters"**
2. Any cluster has ***"diameter in G"*** at most d



How to compute a low diameter clustering

- Main accomplishments of [2]:
  - Terminates in $O\left(\log^6 n\right)$ rounds in the CONGEST model
  - Outputs a clustering with $d = O\left(\log^3 n\right)$
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- [4] did it in $O\left(\log^4 n\right)$ rounds with $d = O\left(\log^3 n\right)$
  - Has to pass by a weak d. intermediate solution

Objectives:

1. Creating connected components with ***“low”*** diameters
   - Keep track of the “center” of the c.c.
     - A.k.a. ***Terminal***
   - We will merge c.c.s
     - Only one terminal is going to be the new center
   - Remove nodes separating c.c.s “too far away”

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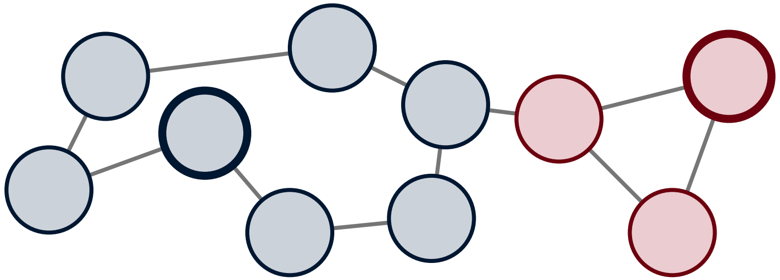
2. Cluster **at least half** of the nodes
   - Required for ***"few"*** colors **network decompositions**

Visual Hint



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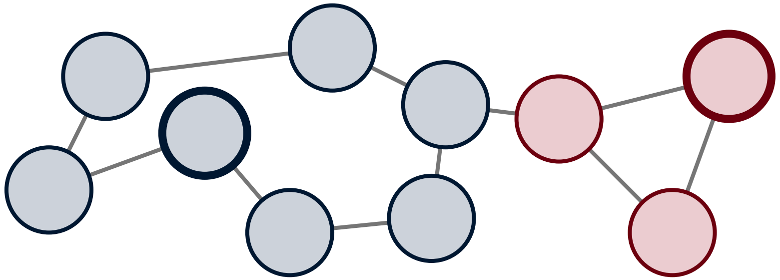

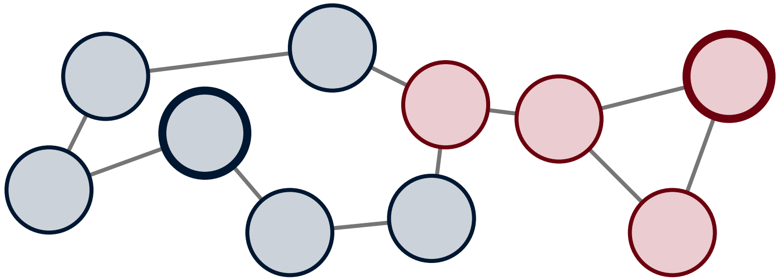

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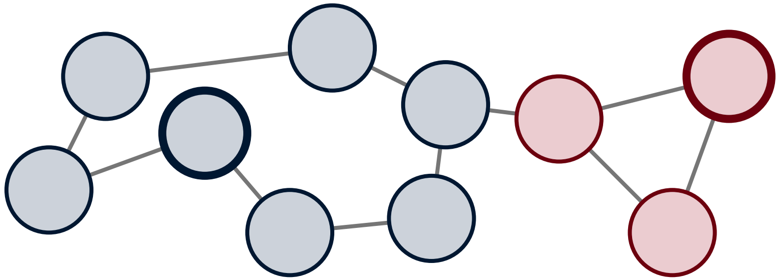

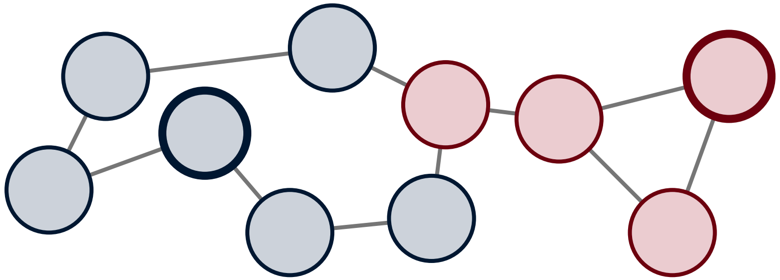

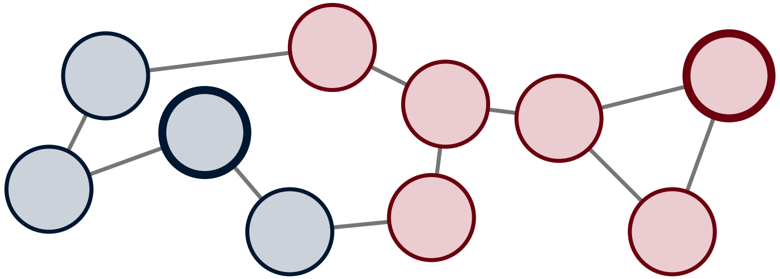

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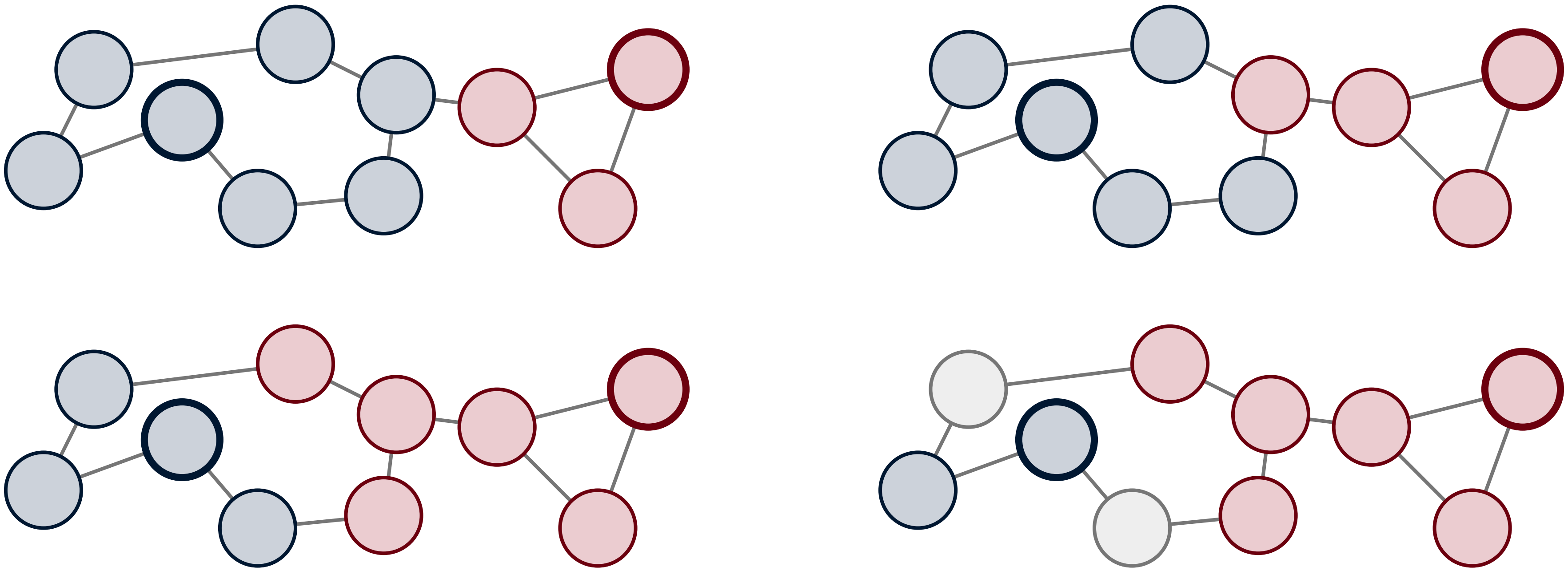

Phases

- There are $b = \log\left(\max_{v \in V} \mathrm{id}(v)\right) = O(\log n)$ **phases**
- "One phase for each bit in index"
  - Phase $i \in [0, b-1]$ computes ***terminals set*** Q_i

Notation:

- Q_i is the terminals set built *before* phase i
- Q_b is the terminals set built *after* phase $b - 1$
- V_i is the set of **living nodes** at the beginning of phase i
- $V' = V_b$ is the set of **living nodes** after the last phase

Phase Invariants $\forall i \in [0..b]$

1. Q_i is R_i-ruling, i.e. $dist_G(Q_i, v) \leq R_i$ for all $v \in V_i$
   - **We set** $R_i = i * O(\log^2 n)$
     - Q_0 is 0-ruling, trivially true with $Q_0 = V$
       - ***"All nodes are terminals at the beginning"***
     - Q_b is $O(\log^3 n)$-ruling

"Each node has polylog distance from Q_b"

$\Rightarrow$ **Each c.c. has at least one terminal**

2. Let $q_1, q_2 \in Q_i$ s.t. they are in the same c.c. in $G[V_i]$.
   Then $\mathrm{id}(q_1)[0..i] = \mathrm{id}(q_2)[0..i]$
   - For $i = 0$ it's trivially true
   - For $i = b$ there is ≤ 1 terminal in each c.c.

Along with invariant (1.), it means that each c.c. has polylog diameter!

3. $|V_i| \geq \left(1 - \frac{i}{2b}\right)|V|$
   - $|V_0| \geq |V|$
   - $|V'| \geq \frac{1}{2}|V|$

- ***"The algorithm clusters at least half of the nodes"***

Objective: In a c.c. remove from Q_i all terminals with different id(v) prefix except one

- Keep c.c. ***"small"***
- Divide it if not possible by removing nodes

Outline:

- $2b^2$ ***steps***, each computing a forest
- Resulting into a sequence of forests $F_0..F_{2b^2}$

Inductive definition:

- F_0 is a BFS forest with roots set Q_i
- Let T be any tree in F_j and r its root
  - If id(r)[i] = 0 the whole tree is `red`, otherwise `blue`
    - `red` vertexes stay `red`
    - Some `blue` nodes stay `blue`
    - Some others ***propose*** `red` trees to join
      - If accepted, the become `red`
      - Otherwise, they are **deleted**

Proposal:

$v \in V_j^{propose} \Leftrightarrow v$ is **blue**

$\wedge\, v$ is the only one in $path(v, root(v))$

that neighbours a **red** node

- Define T_v the (**blue**) subtree rooted at v

v is the only node in T_v that is also in $V_j^{propose}$

Proposal:

- Each node in $V_j^{propose}$ proposes to a `red` neighbour
- Each `red` tree decides to grow or not
  - If it grows, it accepts all proposing subtrees
    - `blue` nodes become `red`
  - If not, all proposing subtrees are **deleted**
- **Criteria**: it decides to grow if gains at least $\frac{|V(T)|}{2b}$ nodes

Observations

♪♪ If a `red` tree doesn't decide to grow, it will neighbour `red` nodes only

- This means it will be able to delete nodes only once in the whole phase
  $\Rightarrow$ At most $\frac{|V|}{2b}$ nodes are lost in each phase
  $\Rightarrow$ After the b phases at most $\frac{|V|}{2}$ nodes are removed

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- `red` trees grow at most of $(i+1) \cdot O(\log^2 n)$

$V_0 \leftarrow V$

$Q_0 \leftarrow V$

for $i \in 0..b-1$ **do**

  INIT F_0

  for $j \in 0..2b^2-1$ **do**

    BUILD $V_j^{propose}$

    $F_{j+1} \leftarrow$ STEP

  $V_{i+1} \leftarrow V\left(F_{2b^2}\right)$

  $Q_{i+1} \leftarrow roots\left(F_{2b^2}\right)$

$O(diam(T_v))$

$2b^2 = O\left(\log^2 n\right)$

$b = O(\log n)$

Complexity

- Recall invariant (1.)
  - $\forall v \in V : dist_G(Q_i, v) = O(\log^3 n)$, for all $i \in 0..b$
  - Hence, $diam(T_v) = O(\log^3 n)$, for all $v \in V$

- Complexity is $\#steps \cdot \#phases \cdot O(diam(T_v))$
  $= O(\log n) \cdot O(\log^2 n) \cdot O(\log^3 n)$

$\Rightarrow$ The algorithm runs in $O(\log^6 n)$ communication steps

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$\Rightarrow$ We end up solving MIS in $O\left(\log^7 n\right)$ rounds

Bibliography

[1] K. Censor-Hillel, M. Parter, and G. Schwartzman, “Derandomizing Local Distributed Algorithms under Bandwidth Restrictions.” [Online]. Available: https://arxiv.org/abs/1608.01689

[2] V. Rozhoň, B. Haeupler, and C. Grunau, “A Simple Deterministic Distributed Low-Diameter Clustering.” [Online]. Available: https://arxiv.org/abs/2210.11784

[3] V. Rozhoň and M. Ghaffari, “Polylogarithmic-Time Deterministic Network Decomposition and Distributed Derandomization.” [Online]. Available: https://arxiv.org/abs/1907.10937

[4] V. Rozhoň, M. Elkin, C. Grunau, and B. Haeupler, “Deterministic Low-Diameter Decompositions for Weighted Graphs and Distributed and Parallel Applications.” [Online]. Available: https://arxiv.org/abs/2204.08254