# Efficient Low Diameter Clustering

with strong diameter in the CONGEST model

Christian Micheletti





Some graph problems are interesting for networks of computers

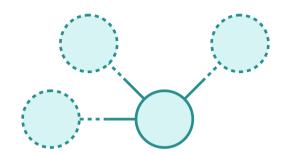
Distribution ⇒ Parallelism



We'd like to leverage parallelism to relieve computation costs

# A First Simple Model





- In the PN-Network a node only knows some Numbered Ports
  - Each connected with a different node
  - def: Those are called <u>neighbours</u>
  - There are no self loops

# A First Simple Model



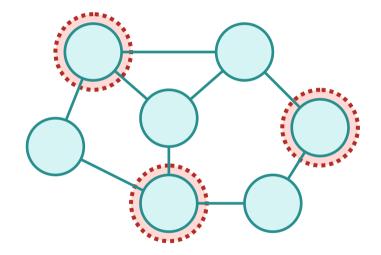


From the perspective of a single node, we don't see the whole topology

# Centralized Graph Problems



Example: Maximal Independent Set (MIS)



Solving it centralized is an easy greedy algorithm





# Each node appears identical to any other

- The only difference could be in the number of ports
  - Not enough!
- We must break this symmetry





# We add unique identifiers to the nodes

 $id:V\to\mathbb{N}$ 

where  $\forall v \in V : id(v) \le n^c$  for some  $c \ge 1$ 

111

We choose  $n^c$  so that we need  $O(\log n)$  bits to represent an identifier, i.e. identifiers are reasonably **small** 

# Distributed Algorithms



- This seems enough for the model
- We must now define:
  - What "distributed" algorithms consist of
  - And the criteria for complexity analysis

# Distributed Algorithms





### Distribution ⇒ Collaboration

- Collaborating requires exchanging messages
  - ...on a medium that is slow and unreliable

⇒ Communication has the most impact on complexity

## Distributed Algorithms



⇒ We are intersted in quantifying the number of messages that travel across the network

#### Communication



W.l.o.g.<sup>1</sup> we adopt a model of **synchronous communication** 

Each round, a node  $v \in V$  performs this actions:

- 1. v sends a message  $msg \in \mathbb{N}$  to its neighbours;
- 2. v receives messages from its neighbours;
- 3. ...

<sup>&</sup>lt;sup>1</sup>Without loss of generality.

#### Communication



3. **v** executes locally some algorithm (same for each node).

<u>def:</u> Any message exchange establishes a communication <u>round</u>

111

Point (3.) doesn't affect the algorithm's complexity

# A First Example (Wave)



In Wave, the node with id(v) = 1 "waves hello"

When a node receives the message, forwards it to its neighbours

The running time of this algorithm on a graph G is O(diam(G))

# A Second Example (Naive MIS)





# Let's leverage **id** to select the first MIS node

- At round #i, node v : id(v) = i executes
  - If no neighbour is in the MIS, add the node
    - And inform the neighbours
  - Otherwise, the node is outside the MIS

# A Second Example (Naive MIS)



```
1: if round = id(v) then
2: m \leftarrow 'want-to-select'
3: SEND m
4: RECEIVE messages
5: if 'want-to-select' ∈ messages then
      stop (result: 'not-in-MIS')
7: if round = id(v) then
      stop (result: 'in-MIS')
8:
```

# A Second Example (Naive MIS)



- It is correct since no node has the same id
- This algorithm runs in  $O(n^c)$  (the maximum id)
  - Very bad

We can be way smarter than that



Running a centralized algorithm on a single node would take O(1) rounds

- We'd like to run a MIS algorithm on each node
  - Centralized ⇒ each node must have a local copy of the entire graph

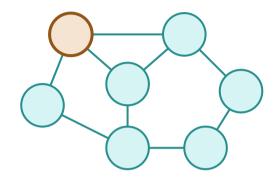
# Gathering All

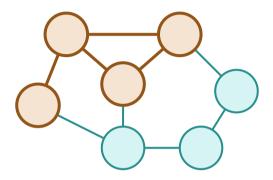


- The algorithm GATHER-ALL makes all nodes build a local copy of the whole graph
  - At round i, each node v knows ball(i, v)

ball(0, v)

ball(1, v)





All nodes will know the whole graph after O(diam(G)) rounds



# Graph can be really large

- In a real world setting, it is not always possible to send arbitrary large messages
- We'd like to lift this assumption
  - Messages need to be reasonably <u>small</u>

- We provide an upper bound on messages size
  - Messages larger than will require more rounds to be fully sent

In the CONGEST model, messages can only be in the size of *O*(log *n*)

# **Examples:**

- Sending a single (or a constant amount of) identifier takes O(1) rounds
- Sending a set of identifiers can take up to O(n)
- Sending the whole graph requires  $O(n^2)$  rounds:
  - The adjacency matrix alone reaches that

⇒ We can't use Gather-All in the CONGEST model • Censor-Hillel et al. provided an algorithm that solves MIS in  $O(diam(G) \log^2 n)$  in CONGEST [1]



The diameter can be very large Worst case: diam(G) = n

- A <u>Network Decomposition</u> groups nodes in colored clusters
  - Clusters with the same color are not adjacent
  - We say it to <u>"have diameter"</u> d if all of its clusters have diameter at most d
  - ► It has c colors



# Solving MIS in a color will give a correct partial solution

- We can iterate this action with [1] for all colors
  - (dropping neighbours of different colors)
- This algorithm has complexity  $O(c \cdot d \log^2 n)$ 
  - ▶ If  $c = O(\log n) = d$  then we would have a MIS algorithm in polylogarithmic time

Each color induces a low diameter clustering

<u>def:</u> A <u>low diameter clustering</u>  $\mathscr{C} \subseteq 2^V$  for a graph G with diameter d is such:

- 1.  $\forall C_1 \neq C_2 \in \mathscr{C} : dist_G(C_1, C_2) \geq 2$ 
  - "There are no adjacent clusters"
- 2.  $\forall C \in \mathscr{C} : diam(G[C]) \leq d$ 
  - "Any cluster has diameter at most" d



# A clustering can not be a partitioning: some nodes have to be left out

#### Main iteration:

- 1. Find a low diameter clustering
- 2. Assign a free color to its nodes
- Repeat to discarded nodes until there are no more left

### How to compute one?





To get a *O*(log *n*)-colors decomposition, each color has to cluster at least half of the uncolored nodes

#### **Definitions**



 Our previous definition of diamter is also called <u>strong</u> diameter

<u>def:</u> We say a clustering has <u>weak</u> diameter when:

- 1. (unchanged) "There are no adjacent clusters"
- 2. Each cluster has at most diameter in Gd, i.e.  $\forall C \in \mathscr{C}diam_G(C) \leq d$ .



# We now want to present [2]

- Previously [3] provided an algorithm for low diameter clustering with weak diameter
  - $O(\log^7 n)$  rounds with  $O(\log^3 n)$  colors
  - It's possible to turn it into strong diameter
- [4] provides trong diameter in  $O(\log^4 n)$  rounds with  $O(\log^3 n)$  colors

#### Previous works



- Still has to turn weak diameter into strong diameter
- The main accomplishment of [2] is to provide a straightforward algorithm
  - $\rightarrow O(\log^6 n)$

**Objective:** Creating connected components with low diameters

#### **Outline:**

- There are  $b = O(\log n)$  phases, phase i computes a set of **terminals**  $Q_i$ ;
- $Q_i$  is R-ruling, i.e.  $dist_G(Q_i, v) \le R$  for all  $v \in V$ .

# **Objective:**

• Each phase removes some nodes: at most  $\frac{|V|}{2b}$ ;

#### **Outline:**

•  $V_i$  is the set of living nodes at the beginning of phase i;

$$V_0 = V$$

• V' is the set of living nodes after the last phase;



# Objective (formalised):

- Each connected component of G[V'] contains exactly one terminal
- Moreover, it has polylogarithmic diameter

## Invariants $\forall i \in [0..b]$



- 1.  $Q_i$  is  $R_i$ -ruling, i.e.  $dist_G(Q_i, v) \le R_i$  for all  $v \in V$ , with  $R_i = i * O(\log n)$ 
  - $Q_0$  is 0-ruling, trivially true since  $Q_0 = V$ ;
  - $Q_b$  is  $O(\log^3 n)$ -ruling



Each node has polylogarithmic distance from  $Q_b \Rightarrow$  each connected component has at least one terminal

## Invariants $\forall i \in [0..b]$



- 2. let  $q_1, q_2 \in Q_i$  s.t. they are in the same connected component in  $G[V_i]$ . Then  $id(q_1)[0..i] = id(q_2)[0..i]$ 
  - for i = 0 it's trivially true
  - for i = b there is  $\leq 1$  terminal in each c.c.



Along with invariant (1.), it means that each c.c. has polylogarithmic diameter!

#### Invariants $\forall i \in [0..b]$



3. 
$$|V_i| \ge \left(1 - \frac{i}{2b}\right) |V|$$

•  $V_0 \ge V$ 

•  $V' \ge \frac{1}{2} |V|$ 



The algorithm doesn't discard too much vertexes from the graph

**Objective:** Keeping only terminals from which is possible to build forests whose trees have polylogarithmic diameter

#### **Outline:**

- $2b^2$  steps, each computing a forest
- resulting into a sequence of forests  $F_0..F_{2b^2}$

#### Inductive definition

- $F_0$  is a BFS forest with roots in  $Q_i$
- let T be any tree in  $F_j$ , and r its root
  - if id(r)[i] = 0 the whole tree is red, if not blue
    - red vertexes stay red
    - some blue nodes stay blue
    - some others **propose** to join red trees



# **Proposal**

$$v \in V_j^{propose} \Leftrightarrow v \text{ is `blue'}$$

A v is the only one in path(v, root(v)) that neighbours a `red` node

Define  $T_v$  the (blue) subtree rooted at v

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#### Step Outline





 ${f v}$  is the only node in  ${f T}_{f v}$  that is also in  ${f V}_i^{propose}$ 

# **Proposal**

- Each node in  $V_j^{propose}$  proposes to an arbitrary red neighbour
- Each red tree decides to grow or not
  - ► If it grows, it accepts all proposing trees
  - ▶ If not, all proposing subtrees are frozen
- Criteria: it decides to grow if it would gain at least  $\frac{|V(T)|}{2h}$  nodes

#### Observations



- If the red tree doesn't decide to grow, it will neighbour red nodes only
- This means it will be able to delete nodes only once in the whole phase (i.e. after  $2b^2$  steps)
- Hence at most  $\frac{|V|}{2b}$  nodes are lost in each phase After the **b** phases at most  $\frac{|V|}{2}$  nodes are
- deleted

#### High Level Pseudocode



```
1: V_0 \leftarrow V
2: Q_0 \leftarrow V
3: for i \in 0..b - 1 do
    init F_0
                BUILD V_j^{propose}
F_{j+1} \leftarrow STEP
O(\log diam(T_v))
b = O(\log n)
5: for j \in 0...2b^2 - 1 do
6:
         V_{i+1} \leftarrow V(F_{2b^2})
Q_{i+1} \leftarrow roots(F_{2b^2})
8:
9:
```

## Step Complexity



Recall invariant (1.):  $\forall v \in V : dist_G(Q_i, v) = O(\log^3 n)$ , for all  $i \in 0...b$ 

Hence,  $diam(T_v) = O(\log^3 n)$ , for all  $v \in V$ 

The algorithm runs in  $O(\log^6 n)$  communication steps

## Bibliography



# **Bibliography**

- [1] K. Censor-Hillel, M. Parter, and G. Schwartzman, "Derandomizing Local Distributed Algorithms under Bandwidth Restrictions." [Online]. Available: https://arxiv.org/abs/1608.01689
- [2] V. Rozhoň, B. Haeupler, and C. Grunau, "A Simple Deterministic Distributed Low-

# Bibliography



Diameter Clustering." [Online]. Available: https://arxiv.org/abs/2210.11784

[3] V. Rozhoň and M. Ghaffari, "Polylogarithmic-Time Deterministic Network Decomposition and Distributed Derandomization." [Online]. Available: https://arxiv.org/abs/1907.10937

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## Bibliography



[4] V. Rozhoň, M. Elkin, C. Grunau, and B. Haeupler, "Deterministic Low-Diameter Decompositions for Weighted Graphs and Distributed and Parallel Applications." [Online]. Available: https://arxiv.org/abs/2204.08254

#### About this presentation



This presentation is supposed to briefly showcase what you can do with this package.

For a full documentation, read the online book.



Let's explore what we have here.

On the top of this slide, you can see the slide title.

We used the title argument of the #slide function for that:

```
#slide(title: "First slide")[
    ...
]
(This works because we utilise the clean theme;
more on that later.)
```

Titles are not mandatory, this slide doesn't have one.

But did you notice that the current section name is displayed above that top line?

We defined it using #new-sectionslide("Introduction"). This helps our audience with not getting lost after a microsleep.

You can also spot a short title above that.

#### The bottom of the slide



Now, look down!

There we have some general info for the audience about what talk they are actually attending right now.

You can also see the slide number there.

Lorem ipsum dolor sit amet, consectetur adipiscing elit, sed do eiusmod tempor incididunt ut labore et dolore magnam aliquam quaerat voluptatem. Ut enim aeque doleamus animo, cum corpore dolemus, fieri tamen permagna accessio potest, si aliquod aeternum et infinitum impendere malum nobis opinemur. Quod idem licet transferre in voluptatem, ut postea variari

voluptas distinguique possit, augeri amplificarique non possit. At etiam Athenis, ut e.

#### A dynamic slide with pauses



Sometimes we don't want to display everything at once.

## A dynamic slide with pauses



Sometimes we don't want to display everything at once.

That's what the **#pause** function is there for!

## A dynamic slide with pauses



Sometimes we don't want to display everything at once.

That's what the **#pause** function is there for!

It makes everything after it appear at the next subslide.

(Also note that the slide number does not change while we are here.)

## Fine-grained control



When **#pause** does not suffice, you can use more advanced commands to show or hide content.

These are some of your options: - #uncover

- #only
- #alternatives
- #one-by-one
- #line-by-line

Let's explore them in more detail!

#### #uncover: Reserving space



With #uncover, content still occupies space, even when it is not displayed.

For example, are only visible on the second "subslide".

In () behind #uncover, you specify when to show the content, and in [] you then say what to show:

#uncover(3)[Only visible on the third "subslide"]

#### #uncover: Reserving space



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#### **#uncover**: Reserving space



#uncover(3)[Only visible on the third "subslide"]

Only visible on the third"subslide"

So far, we only used single subslide indices to define when to show something.

We can also use arrays of numbers ...

```
\#uncover((1, 3, 4))[Visible on subslides 1, 3, and 4]
```

Visible on subslides 1, 3, and 4

...or a dictionary with beginning and/or until keys:

```
#uncover((beginning: 2, until: 4))[Visible on
subslides 2, 3, and 4]
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```

Visible on subslides 1, 3, and 4

...or a dictionary with beginning and/or until keys:

### Complex display rules

```
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subslides 2, 3, and 4]
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### Complex display rules

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subslides 2, 3, and 4]
Visible on subslides 2, 3, and 4
```

### Convenient rules as strings



As as short hand option, you can also specify rules as strings in a special syntax.

Comma separated, you can use rules of the form

- 1-3 from subslide 1 to 3 (inclusive)
  - -4 all the time until subslide 4 (inclusive)
  - 2- from subslide 2 onwards
    - 3 only on subslide 3

Everything that works with #uncover also works with #only.

However, content is completely gone when it is not displayed.

For example, the rest of this sentence moves.

Again, you can use complex string rules, if you want.

#only("2-4, 6")[Visible on subslides 2, 3, 4, and 6]

Everything that works with #uncover also works with #only.

However, content is completely gone when it is not displayed.

For example, see how the rest of this sentence moves.

Again, you can use complex string rules, if you want.

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## You might be tempted to try

```
#only(1)[Ann] #only(2)[Bob] #only(3)[Christopher]
likes #only(1)[chocolate] #only(2)[strawberry]
#only(3)[vanilla] ice cream.

Ann
likes chocolate
ice cream.
```

But it is hard to see what piece of text actually changes because everything moves around. Better:

```
#alternatives[Ann][Bob][Christopher] likes
#alternatives[chocolate][strawberry][vanilla] ice
cream.
```

Ann likes chocolate ice cream.

## You might be tempted to try

```
#only(1)[Ann] #only(2)[Bob] #only(3)[Christopher]
likes #only(1)[chocolate] #only(2)[strawberry]
#only(3)[vanilla] ice cream.

Bob
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ice cream.
```

But it is hard to see what piece of text actually changes because everything moves around. Better:

```
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#alternatives[chocolate][strawberry][vanilla] ice
cream.
```

Bob likes strawberry ice cream.

## You might be tempted to try

```
#only(1)[Ann] #only(2)[Bob] #only(3)[Christopher]
likes #only(1)[chocolate] #only(2)[strawberry]
#only(3)[vanilla] ice cream.

Christopher
likes vanilla
ice cream.
```

But it is hard to see what piece of text actually changes because everything moves around. Better:

```
#alternatives[Ann][Bob][Christopher] likes
#alternatives[chocolate][strawberry][vanilla] ice
cream.
```

Christopher likes vanilla ice cream.

#alternatives is to #only what #one-by-one is to
#uncover.

#one-by-one behaves similar to using #pause but you can additionally state when uncovering should start.

#one-by-one(start: 2)[one ][by ][one]

start can also be omitted, then it starts with the first subside:

```
#one-by-one[one ][by ][one]
```

one

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#one-by-one[one ][by ][one]
oneby
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#one-by-one[one ][by ][one]

onebyone

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onebyone
```

start can also be omitted, then it starts with the first subside:

#one-by-one[one ][by ][one]

onebyone

Sometimes it is convenient to write the different contents to uncover one at a time in subsequent lines.

This comes in especially handy for bullet lists, enumerations, and term lists.

```
#line-by-line(start: 2)[
    - first
.
```

# #line-by-line: syntactic sugar for #one-by-one



```
- second
- third
]
```

start is again optional and defaults to 1.

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    - first
```

- first
- second

# #line-by-line: syntactic sugar for #one-by-one



```
secondthirdthird
```

While #line-by-line is very convenient syntax-wise, it fails to produce more sophisticated bullet lists, enumerations or term lists. For example, non-tight lists are out of reach.

For that reason, there are #list-one-by-one, #enum-one-by-one, and #terms-one-by-one, respectively.

Note that, for technical reasons, the bullet points, numbers, or terms are never covered.

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#### How a slide looks...



... is defined by the *theme* of the presentation.

This demo uses the unipd theme.

Because of it, the title slide and the decoration on each slide (with section name, short title, slide number etc.) look the way they do.

Themes can also provide variants, for example ...

## ... this one!

It's very minimalist and helps the audience focus on an important point.

#### Your own theme?



If you want to create your own design for slides, you can define custom themes!

The book explains how to do so.

#### The utils module



Polylux ships a utils module with solutions for common tasks in slide building.

## Fit to height



You can scale content such that it has a certain height using #fit-to-height(height, content):

## Fill remaining space



This function also allows you to fill the remaining space by using fractions as heights, i.e. fit-to-height(1fr)[...]:

## Side by side content



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Often you want to put different content next to each other. We have the function #side-by-side for that:

Lorem ipsum dolor sit amet, consectetur adipiscing elit, sed do.

Lorem ipsum dolor sit amet, consectetur adipiscing elit, sed do

Lorem ipsum dolor sit amet, consectetur adipiscing elit, sed do

## Side by side content



eiusmod tempor incididunt ut labore. eiusmod tempor.



## Why not include an outline?

- 1. Overview
- 2. Models
- 3. Models
- 4. LOCAL Algorithms



- 5. CONGEST Algorithms
- 6. CONGEST Algorithms
- 7. Clusterings
- 8. The Algorithm
- 9.



- 10.
- 11.
- 12. Introduction
- 13. Dynamic content
- 14. Dynamic content



- 15. Themes
- 16. Utilities
- 17. Typst features
- 18. Conclusion

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Typst gives us so many cool things<sup>2</sup>. Use them!

<sup>&</sup>lt;sup>2</sup>For example footnotes!

Hopefully you now have some kind of idea what you can do with this template.

Consider giving it a GitHub star or open an issue if you run into bugs or have feature requests.