Efficient Low Diameter Clustering

with strong diameter in the CONGEST model

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About this Presentation

1. Distributed Algorithms
2. Distributed Algorithms
3. Network Decomposition
4. Computing a Clustering
5. Computing a Clustering
6. Computing a Clustering

- We want to solve graph problems on **networks**
  - Computers are like nodes in a graph

Distribution $\Rightarrow$ Multiple processors

Intuition

- We want to solve graph problems on **networks**
  - Computers are like nodes in a graph

Distribution ⇒ Multiple processors

💡 Each node gives a partial solution

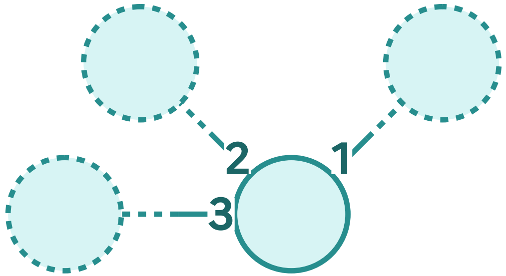
- At the end they are all combined altogether

...how can we combine those partial solutions?

🚨 **Distribution ⇒ Collaboration**

- Arcs are **direct links** between computers
  - Computers have to exchange **messages**

- In the **PN-Network** a node only knows some *Numbered Ports*
  - Connected with a **different** nodes
    - Such nodes are called ***neighbours***
- There are no **self loops**



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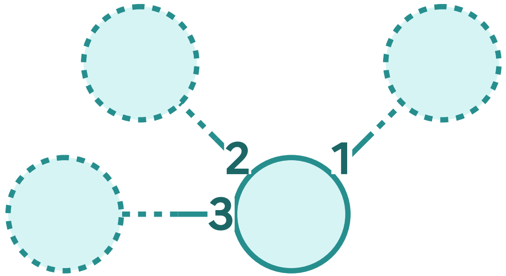

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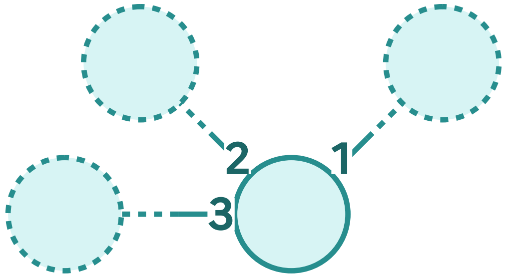

🚨 A node can't see the whole topology

🚨 All nodes appear identical

💡 We add **unique identifiers** to the nodes

$$id : V \to \mathbb{N}$$

where $\forall v \in V : id(v) \leq n^c$ for some $c \geq 1$

We choose n^c so that we need $O(\log n)$ bits to represent an identifier, i.e. identifiers are reasonably ***"small"***

Communication (Intuition)

- Collaboration requires **exchanging messages**
  - ...on a medium that is **slow** and **unreliable**

🚨 ⇒ Communication is the main pitfall

- We **measure** the number of messages that an algorithm requires
  - An ***"efficient"*** algorithm will need few messages

W.l.o.g.[1] we adopt a model of ***synchronous communication***

Each round, a node $v \in V$ performs these actions:

1. v ***sends*** a message $msg \in \mathbb{N}$ to its neighbours;
2. v ***receives*** messages from its neighbours;
3. ...

[1]Without loss of generality.

W.l.o.g.[2] we adopt a model of ***synchronous communication***

Each round, a node $v \in V$ performs these actions:

1. v ***sends*** a message $msg \in \mathbb{N}$ to its neighbours;
2. v ***receives*** messages from its neighbours;
3. ...

- (1.) and (2.) establish a ***communication round***
  - Main measure of complexity
  - Few communication rounds $\Rightarrow$ few messages

[2] Without loss of generality.

3. v ***executes locally*** some algorithm (same for each node).
   - A node can ***stop*** in this phase
     - Its local result won't change anymore

🎶 (3.) doesn't affect the algorithm's complexity

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♪♪♩ (3.) doesn't affect the algorithm's complexity

- When all nodes **stopped** the algorithm terminates

We say that an algorithm is ***"efficient"*** when it stops in **polylogarithmic** number of rounds

- The node with id(v) = 1 *"waves hello"* to neighbours
  - ...sending them a message

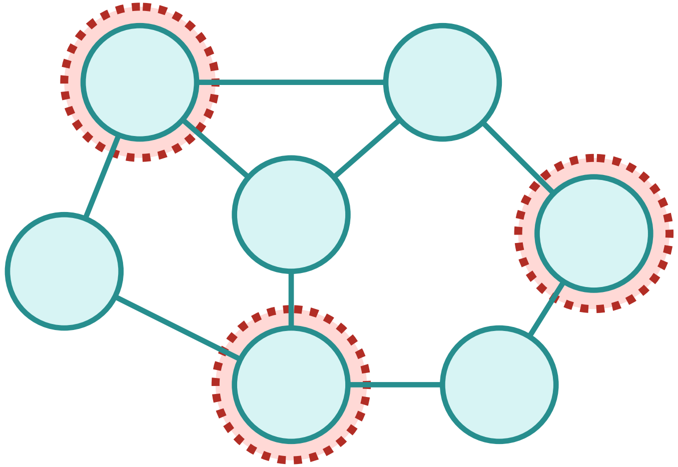
A First Example (Wave)

- The node with id(v) = 1 *"waves hello"* to neighbours
  - ...sending them a message
- When a node receives the message, **forwards** it to its neighbours
  - And then **stops**

A First Example (Wave)

- The node with id(v) = 1 *"waves hello"* to neighbours
  - ...sending them a message
- When a node receives the message, **forwards** it to its neighbours
  - And then **stops**
- The running time of this algorithm on a graph G is $O(diam(G))$

Example: **Maximal Independent Set** (MIS)



- Solving it ***centralized*** is easy
- How can we solve it ***distributed***?

💡 Let's leverage id(v) to select the next MIS node

- At round #i, node v : id(v) = i executes
  - If no neighbour is in the MIS, add the node
    - And inform the neighbours
  - Otherwise, the node is outside the MIS

A Second Example (Naive-MIS)

💡 Let's leverage id(v) to select the next MIS node

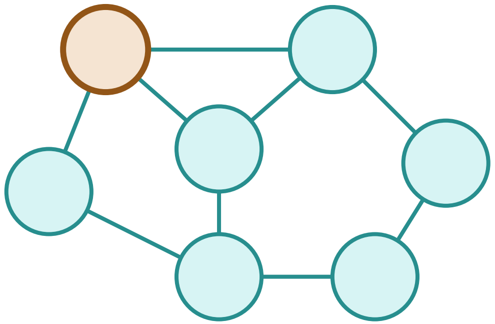
- At round #i, node v : id(v) = i executes
  - If no neighbour is in the MIS, add the node
    - And inform the neighbours
  - Otherwise, the node is outside the MIS
- It is correct since no node has the same id
- This algorithm runs in $O(n^c)$ (the maximum id)
  - **Very bad**

💡 Running a centralized algorithm on a single node would take O(1) rounds

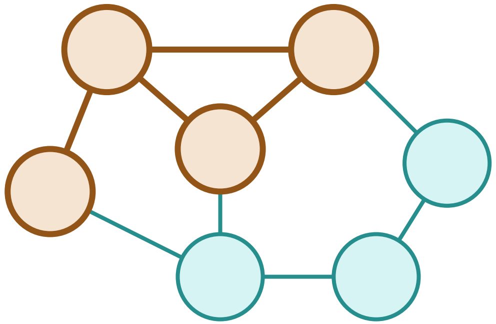
- We'd like to run a MIS algorithm on each node
  - Each must have a **local copy** of the **entire** graph
  - The algorithm must be deterministic
    - When a node stops it checks if it is included in MIS

- The algorithm Gather-All makes all nodes build a local copy of the whole graph
  - At round i, each node v knows $ball(i, v)$

$ball(0, v)$



$ball(1, v)$



- All nodes will know the whole graph after $O(diam(G))$ rounds

Critique to LOCAL

- GATHER-ALL assumes that messages size is **unbounded**

🚨 Requires to send the whole graph in one message

- It is not always possible to send arbitrary large messages
  - Heavy ones may be "sharded"
- We provide an upper bound for message size
  - Messages need to be reasonably ***"small"***
  - Large messages will require more rounds to be sent

In the CONGEST model, messages size has to be $O(\log n)$

- Sending k identifiers takes $O(1)$ rounds
- Sending a *set* of identifiers can take up to $O(n)$
- Sending the whole graph requires $O(n^2)$ rounds:
  - The adjacency matrix suffices...

🚨 $\Rightarrow$ We can't use Gather-All in the CONGEST model

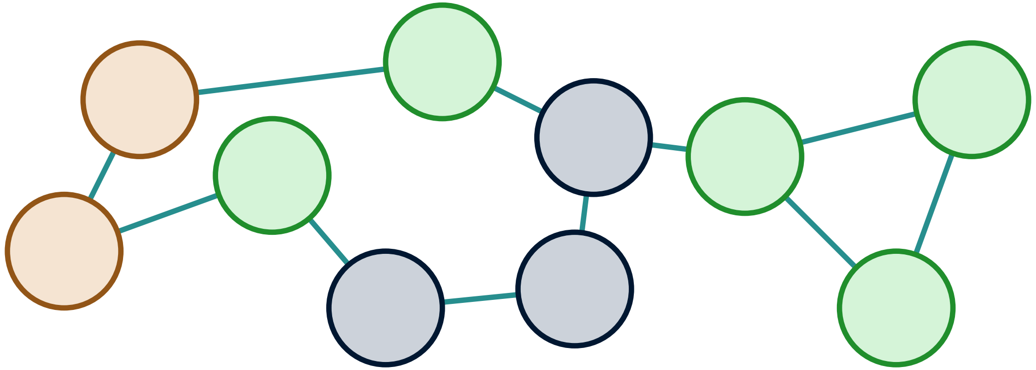
Solving problems in CONGEST

- Censor-Hillel et al. [1] provided an algorithm that solves MIS in $O\left(diam(G)\log^2 n\right)$ in CONGEST

🚨 The diameter can be very large

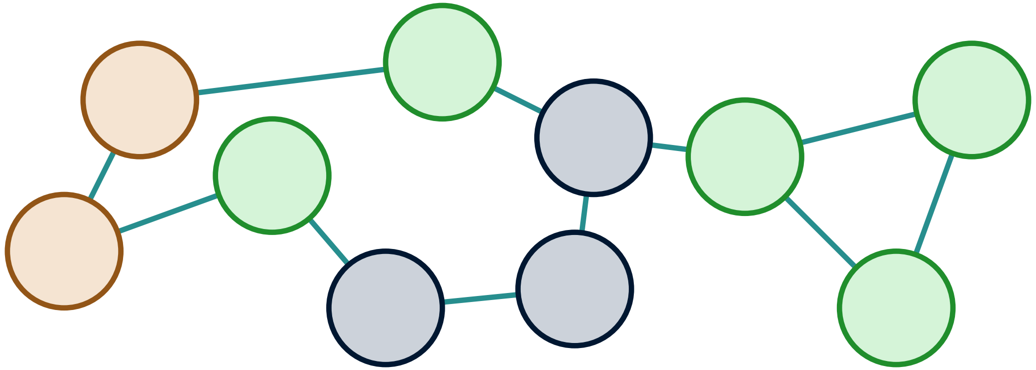
- Worst case: $diam(G) = n$
- How can we improve it?

- A ***Network Decomposition*** groups nodes in **colored clusters**
  - Clusters with the same color are not adjacent
  - We say it to ***have diameter*** d if each cluster has diameter at most d
  - It has c colors



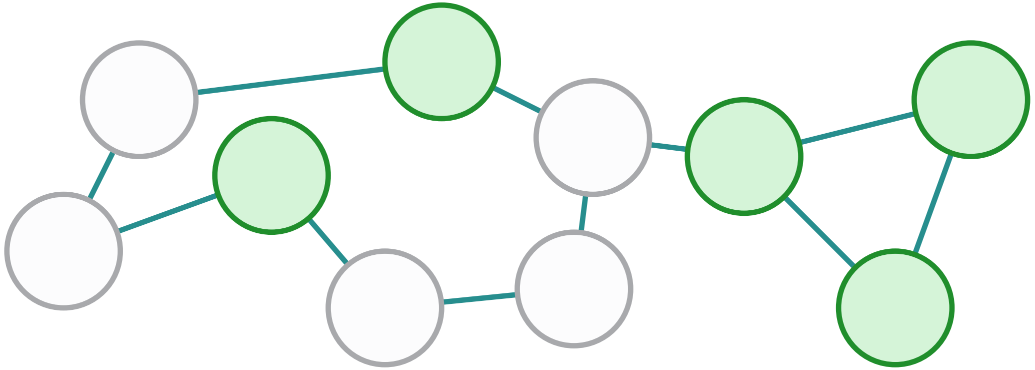
💡 Solving MIS in a color gives a partial solution

- We can apply [1] for all colors
  - (dropping MIS neighbours after each iter)
  - This has complexity $O(c \cdot d \log^2 n)$
    - If $c = O(\log n) = d$ it would be ***"efficient"***



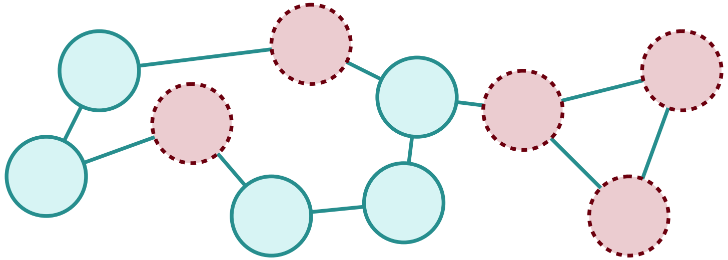
A ***low diameter clustering*** $\mathscr{C} \subseteq 2^V$ for a graph G with diameter d is such:

1. $\forall C_1 \neq C_2 \in \mathscr{C} : dist_G(C_1, C_2) \geq 2$
   - ***"There are no adjacent clusters"***
2. $\forall C \in \mathscr{C} : diam(G[C]) \leq d$
   - ***"Any cluster has diameter at most d"***



Main iteration:

1. Find a low diameter clustering
2. Assign a free color to its nodes
3. Repeat to discarded nodes until there are no more left



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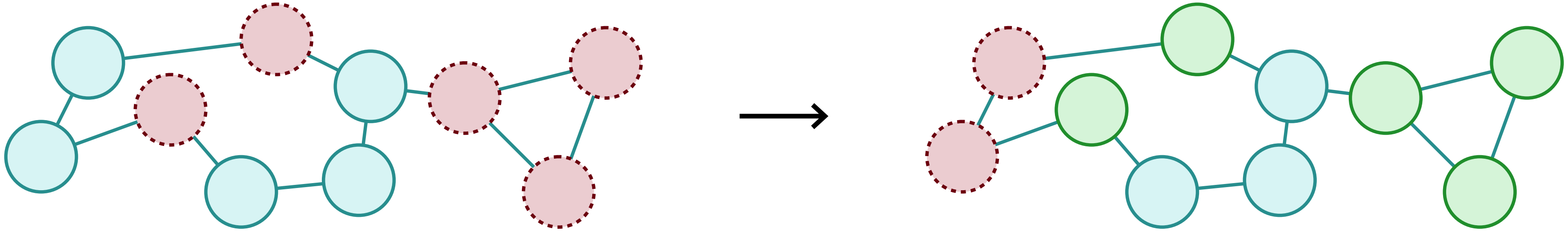

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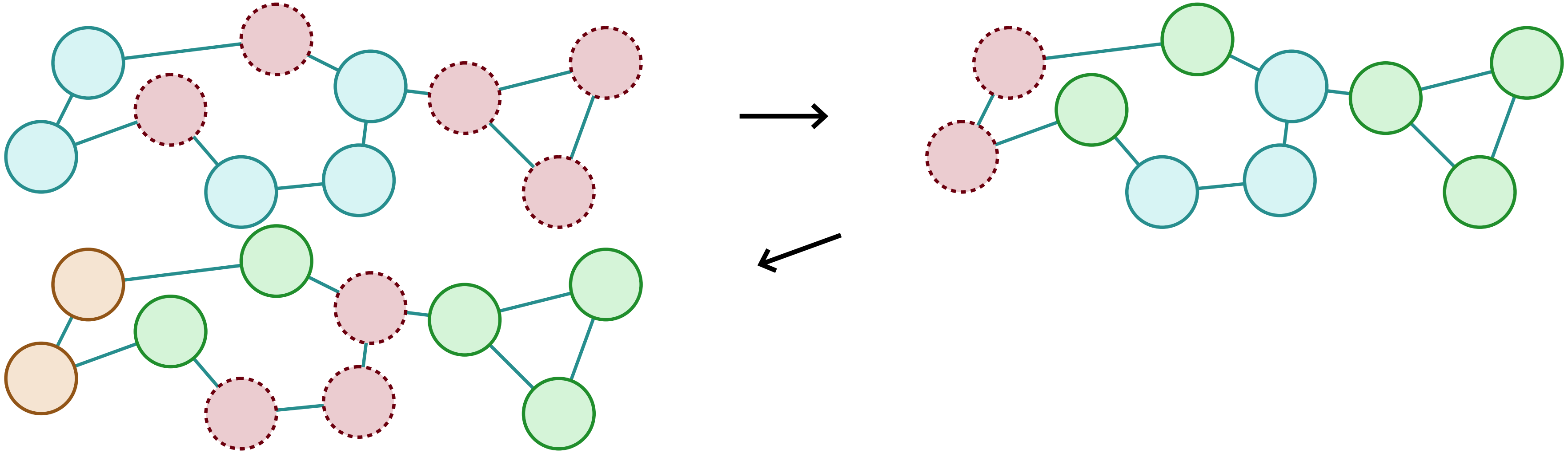

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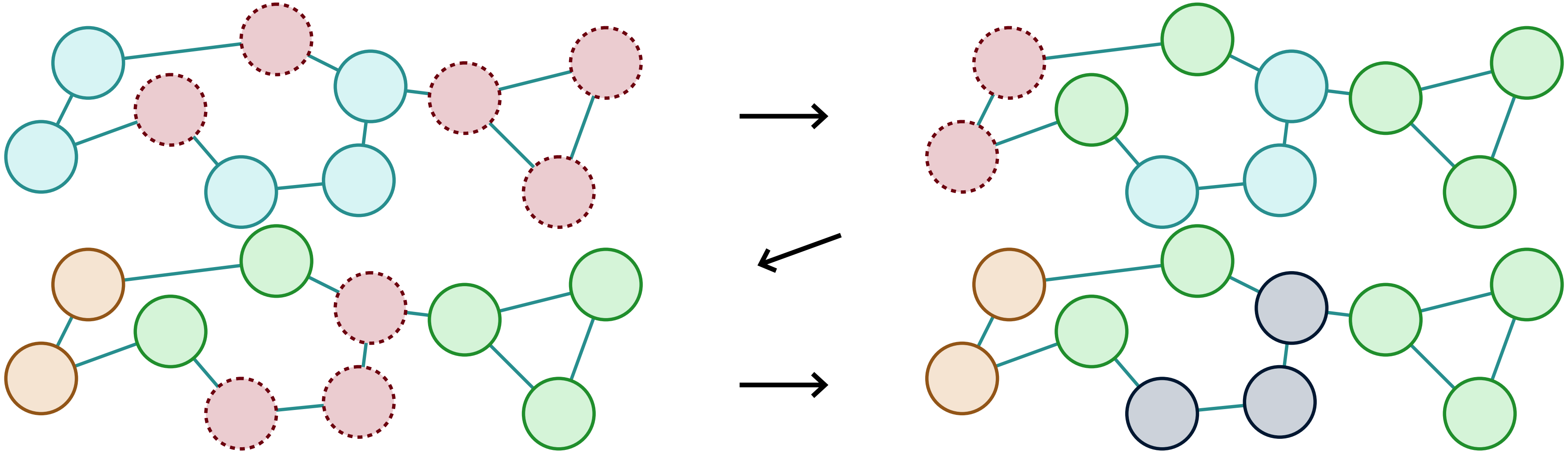

Definitions

- Our previous definition of diamter is also called ***strong*** diameter

We say a clustering has ***weak*** diameter when:

1. (unchanged) **"There are no adjacent clusters"**
2. Any cluster has ***"diameter in** G**"*** at most d

How to compute a low diameter clustering

- Main accomplishments of [2]:
  - Terminates in $O\left(\log^6 n\right)$ rounds in the CONGEST model
  - Outputs a clustering with $O\left(\log^3 n\right)$ colors
  - Directly strong diameter

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  - $O(\log^7 n)$ rounds with $O(\log^3 n)$ colors
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- Main accomplishments of [2]:
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- Previously [3] provided a l.d.c with **weak** diameter
  - $O(\log^7 n)$ rounds with $O(\log^3 n)$ colors
  - It's possible to turn it into strong diameter
- [4] did it in $O(\log^4 n)$ rounds with $O(\log^3 n)$ colors
  - Has to pass by a weak d. intermediate solution

Objectives:

1. Creating connected components with ***"low"*** diameters
   - Keep track of the "center" of the c.c.
     - A.k.a. ***Terminal***
   - We will merge c.c.s
     - Only one terminal is going to be the new center
   - Remove c.c.s with nodes "too far away"

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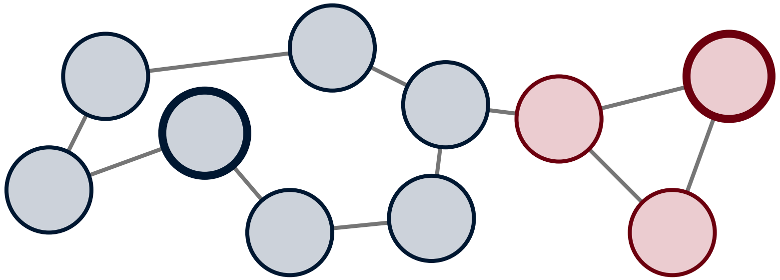
   - Remove c.c.s with nodes "too far away"
2. Cluster **at least half** of the nodes
   - Required for ***"few"*** colors **network decompositions**

Visual Hint



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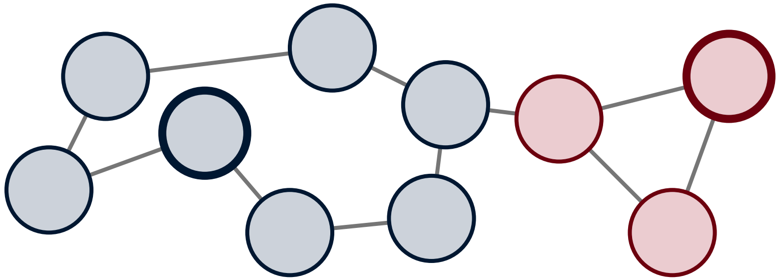

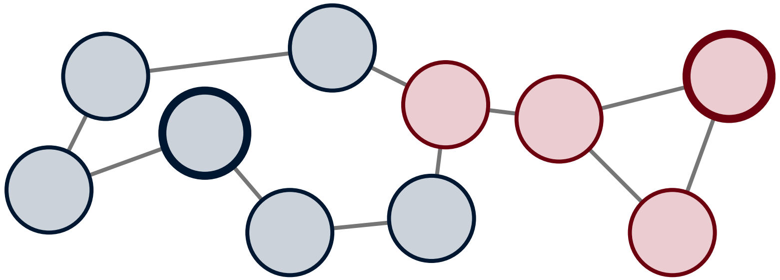

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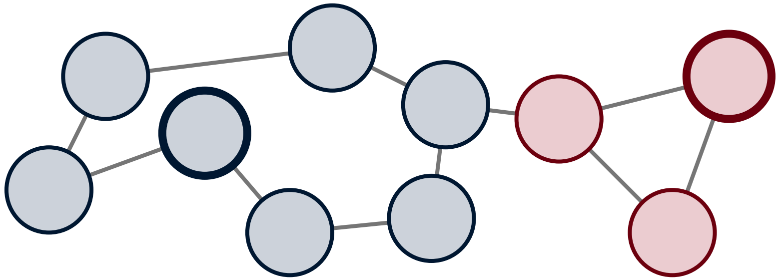

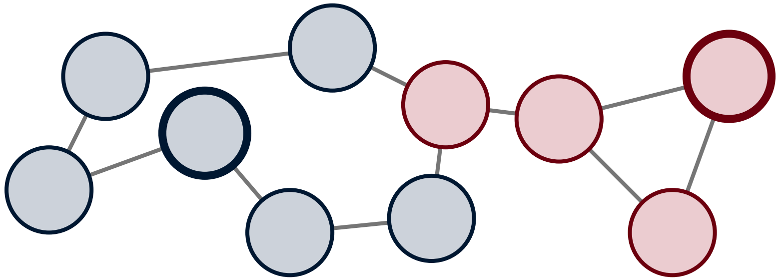

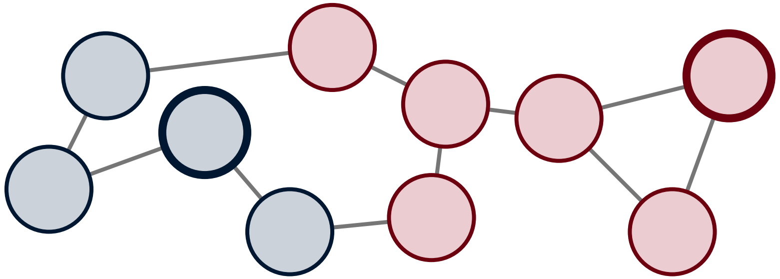

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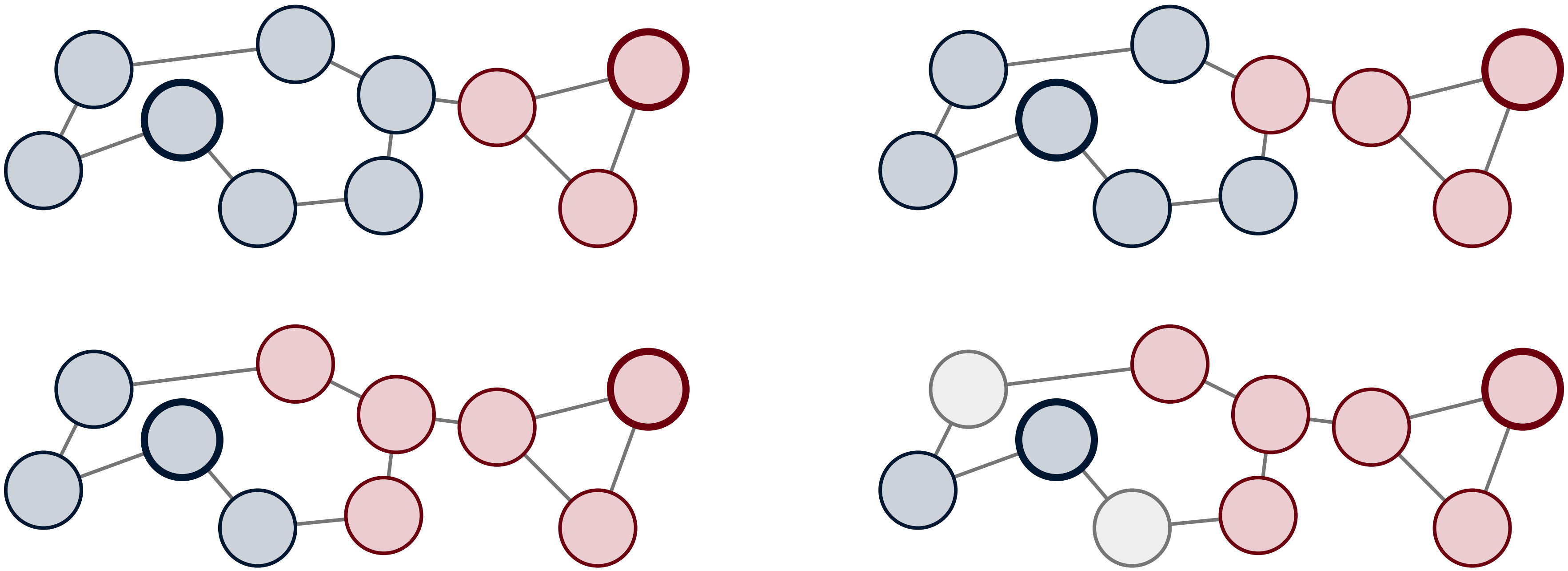

Phases

- There are $b = \log\left(\max_{v \in V} \mathrm{id}(v)\right) = O(\log n)$ **phases**
- "One phase for each bit in index"
  - Phase $i \in [0, b-1]$ computes ***terminals set*** Q_i

Notation:

- Q_i is the terminals set built *before* phase i
- Q_b is the terminals set built *after* phase $b-1$
- V_i is the set of **living nodes** at the beginning of phase i
- $V' = V_b$ is the set of **living nodes** after the last phase

Phase Invariants $\forall i \in [0..b]$

1. Q_i is R_i-ruling, i.e. $dist_G(Q_i, v) \leq R_i$ for all $v \in V$
   - **We set** $R_i = i * O(\log^2 n)$
     - Q_0 is 0-ruling, trivially true with $Q_0 = V$
       - ***"All nodes are terminals at the beginning"***
     - Q_b is $O(\log^3 n)$-ruling

"Each node has polylog distance from Q_b"
$\Rightarrow$ Each c.c. has **at least one terminal**

2. Let $q_1, q_2 \in Q_i$ s.t. they are in the same c.c in $G[V_i]$.
   Then $\mathrm{id}(q_1)[0..i] = \mathrm{id}(q_2)[0..i]$
   - For $i = 0$ it's trivially true
   - For $i = b$ there is ≤ 1 terminal in each c.c.

Along with invariant (1.), it means that each c.c. has polylog diameter!

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Phase Invariants $\forall i \in [0..b]$

3. $|V_i| \geq \left(1 - \frac{i}{2b}\right) |V|$
    - $V_0 \geq V$
    - $V' \geq \frac{1}{2} |V|$

- ***"The algorithm clusters at least half of the nodes"***

Objective: In a c.c. remove from Q_i all terminals with same id(v) prefix except one

- Keep c.c. ***"small"***
- Divide it if not possible by removing nodes

Outline:

- $2b^2$ ***steps***, each computing a forest
- Resulting into a sequence of forests $F_0..F_{2b^2}$

Inductive definition:

- F_0 is a BFS forest with roots set Q_i
- Let T be any tree in F_j and r its root
  - If id(r)[i] = 0 the whole tree is `red`, otherwise `blue`
    - `red` vertexes stay `red`
    - Some `blue` nodes stay `blue`
    - Some others ***propose*** `red` trees to join
      - If accepted, the become `red`
      - Otherwise, they are **deleted**

Proposal:

$v \in V_j^{propose} \Leftrightarrow v$ is `blue`

$\wedge\ v$ is the only one in $path(v, root(v))$

that neighbours a `red` node

- Define T_v the (`blue`) subtree rooted at v

v is the only node in T_v that is also in $V_j^{propose}$

Proposal:

- Each node in $V_j^{propose}$ proposes to a `red` neighbour
- Each `red` tree decides to grow or not
  - If it grows, it accepts all proposing subtrees
    - `blue` nodes become `red`
  - If not, all proposing subtrees are **deleted**
- **Criteria**: it decides to grow if gains at least $\frac{|V(T)|}{2b}$ nodes

♫ If a **red** tree doesn't decide to grow, it will neighbour **red** nodes only

- This means it will be able to delete nodes only once in the whole phase

  $\Rightarrow$ At most $\frac{|V|}{2b}$ nodes are lost in each phase

  $\Rightarrow$ After the b phases at most $\frac{|V|}{2}$ nodes are removed

$V_0 \leftarrow V$

$Q_0 \leftarrow V$

for $i \in 0..b-1$ **do**

    INIT F_0

    for $j \in 0..2b^2-1$ **do**

        BUILD $V_j^{propose}$

        $F_{j+1} \leftarrow$ STEP

    $V_{i+1} \leftarrow V\left(F_{2b^2}\right)$

    $Q_{i+1} \leftarrow roots\left(F_{2b^2}\right)$

$O\left(diam\left(T_v\right)\right)$

$2b^2 = O\left(\log^2 n\right)$

$b = O(\log n)$

- Recall invariant (1.)
  - $\forall v \in V : dist_G(Q_i, v) = O(\log^3 n)$, for all $i \in 0..b$
  - Hence, $diam(T_v) = O(\log^3 n)$, for all $v \in V$
- Complexity is $\#steps \cdot \#phases \cdot O(diam(T_v))$
  $= O(\log n) \cdot O(\log^2 n) \cdot O(\log^3 n)$

$\Rightarrow$ The algorithm runs in $O(\log^6 n)$ communication steps

- We've seen how to build a **low diameter clustering**
  - In $O(\log^6 n)$ communication steps
  - It clusters at least $\frac{n}{2}$ nodes

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- We solve MIS [1] for each color $(O(\log n) \cdot \ldots)$
  - In parallel in the clusters $(\ldots \cdot O(\log^3 n \cdot \log^2 n))$

- We've seen how to build a **low diameter clustering**
  - In $O\left(\log^6 n\right)$ communication steps
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$\Rightarrow$ We end up solving MIS in $O\left(\log^7 n\right)$ rounds

Bibliography

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