Efficient Low Diameter Clustering

with strong diameter in the CONGEST model

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Distributed Algorithms



Many modern problems apply to **networks of computers**



Distribution ⇒ Parallelism

Idea: we can leverage parallelism to speed up computations

Distributed Algorithms



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Distribution ⇒ Remoteness

Bottleneck: remote communication

Our aim is to study algorithms that execute in a distributed environment

Complexity



W.l.o.g.¹ we adopt a model of execution divided in **communication rounds**

Each round, a node $v \in V$ performs this actions:

- 1. v sends messages to its neighbours;
- 2. v receives messages from its neighbours;

¹Without loss of generality.

Complexity



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3. **v** executes locally some algorithm (same for each node)

W.l.o.g. rounds are synchronized

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A First Example (Wave)



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In Wave a single node starts "waving hello" to its neighbours that, in turn, "wave" to their neighbours

Each communication round can take a significant amount of time to happen

Complexity is measured in rounds

A First Example (Wave)



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The running time of this algorithm on a graph G is O(diam(G))

We say **efficient** meaning O(polylog n), where n = |V|

Centralized Graph Problems



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Example: Maximal Independent Set (MIS)

Solving it in a centralized model is easily done with a greedy algorithm

<u>def</u> "Centralized" ≡ "knowing the graph topology"

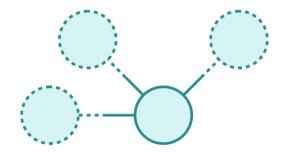


From the perspective of a single node, we don't see the whole topology



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What can a node see?



In the **PN-Network** a node only knows that it has some **ports**, each connected with a **different** node





Each node appears identical to any other

We must break this symmetry





We add unique identifiers to the model

 $id:V\to\mathbb{N}$

where $\forall v \in V : id(v) \le n^c$ for some $c \ge 1$

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We choose n^c so that we need $O(\log n)$ bits to represent an identifier, i.e. identifiers are reasonably **small**

```
1: m \leftarrow m \parallel \perp
2: if m = selected then
3: stop (result: 1)
4: SEND m
5: RECEIVE messages
6: if selected ∈ messages then
7: stop (result: 0)
8: if round = id(v) then
```

9: $m \leftarrow \text{selected}$

This algorithm runs in $O(n^c)$

We can be way smarter than that



Running a centralized algorithm on a single node would take O(1) rounds

The algorithm Gather-All makes all nodes build a local copy of the whole graph

It takes O(diam(G))

Gathering All



Then, we can run a deterministic centralized MIS algorithm on each node and output 1 if the node is in the computed MIS

We aim to bind the message sizes to a reasonably small limit

In the CONGEST model, messages can only be in the size of $O(\log n)$

To send messages bigger than that, more rounds are needed

Examples:

- Sending a single (or a constant amount of) identifier takes O(1) rounds;
- Sending a set of identifiers can take up to O(n) rounds;

For this reason, we can't use Gather-All in the CONGEST model.

There is an algorithm that solves MIS in $O(diam(G) \log^2 n)$ in CONGEST [1]



The diameter can be very large: we can only say that $diam(G) \le n$

A **Network Decomposition** divides a network in colored clusters, where clusters with the same color are not adjacent

Network Decomposition



- It has diameter d if all of its clusters have diameter at almost d;
- It has c colors.



We can run MIS [1] for each color, in parallel in its clusters and remove the neighbours of the newly added nodes

Network Decomposition



This algorithm would have complexity $O(c \cdot d \log^2 n)$

If $c = O(\log n) = d$ then we would have a MIS algorithm in polylogarithmic time

By definition, each color induces a **low diameter clustering**



We can find a low diameter clustering, color them with a color, and repeat on uncolored nodes

How to compute one?





To get a *O*(log *n*) colored decomposition, each clustering has to cluster at least half of the nodes



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<u>def</u> A clustering $\mathscr{C} \subseteq 2^V$ is any set of **disjoint** subsets of V

<u>def</u> We say it has (strong) diameter $d \in \mathbb{N}$ when:

1. No two clusters are adjacent, i.e. $\forall C_1, C_2 \in \mathscr{C}$: $dist_G(C_1, C_2) \ge 2$;

Definitions



This means that a clustering can not be a partitioning: some node has to be left out

2. Each cluster has at most diameter d, i.e. $\forall C \in \mathscr{C}diam_{C}(C) \leq d$.

<u>def</u> We say it has **low** diameter instead when:

1. (unchanged);

Definitions



2. Each cluster has at most diameter in Gd, i.e.

$$\forall C \in \mathscr{C}diam_G(C) \leq d$$
].

Previous works



The state of the art before [2] is [3] and [4]

TODO

The main accomplishment of [2] is to provide an easier algorithm that still runs in polylogarithmic time



Objective: Creating connected components with low diameters

Outline:

- There are $b = O(\log n)$ phases, phase i computes a set of **terminals** Q_i ;
- Q_i is R-ruling, i.e. $dist_G(Q_i, v) \le R$ for all $v \in V$.

Objective:

• Each phase removes some nodes: at most $\frac{|V|}{2b}$;

Outline:

• V_i is the set of living nodes at the beginning of phase i;

$$V_0 = V$$

• V' is the set of living nodes after the last phase;



Objective (formalised):

- Each connected component of G[V'] contains exactly one terminal
- Moreover, it has polylogarithmic diameter

Invariants $\forall i \in [0..b]$



- 1. Q_i is R_i -ruling, i.e. $dist_G(Q_i, v) \le R_i$ for all $v \in V$, with $R_i = i * O(\log n)$
 - Q_0 is 0-ruling, trivially true since $Q_0 = V$;
 - Q_b is $O(\log^3 n)$ -ruling



Each node has polylogarithmic distance from $Q_b \Rightarrow$ each connected component has at least one terminal

Invariants $\forall i \in [0..b]$



- 2. let $q_1, q_2 \in Q_i$ s.t. they are in the same connected component in $G[V_i]$. Then $id(q_1)[0..i] = id(q_2)[0..i]$
 - for i = 0 it's trivially true
 - for i = b there is ≤ 1 terminal in each c.c.



Along with invariant (1.), it means that each c.c. has polylogarithmic diameter!

Invariants $\forall i \in [0..b]$



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3.
$$|V_i| \ge \left(1 - \frac{i}{2b}\right) |V|$$

• $V_0 \ge V$

• $V' \ge \frac{1}{2} |V|$



The algorithm doesn't discard too much vertexes from the graph

Objective: Keeping only terminals from which is possible to build forests whose trees have polylogarithmic diameter

Outline:

- $2b^2$ steps, each computing a forest
- resulting into a sequence of forests $F_0..F_{2b^2}$

Inductive definition

- F_0 is a BFS forest with roots in Q_i
- let T be any tree in F_j , and r its root
 - if id(r)[i] = 0 the whole tree is red, if not blue
 - red vertexes stay red
 - some blue nodes stay blue
 - some others **propose** to join red trees



Proposal

$$v \in V_j^{propose} \Leftrightarrow v \text{ is `blue'}$$

 $\land v$ is the only one in path(v, root(v)) that neighbours a `red` node

Define T_v the (blue) subtree rooted at v

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Step Outline





 ${f v}$ is the only node in ${f T}_{f v}$ that is also in ${f V}_i^{propose}$

Proposal

- Each node in $V_j^{propose}$ proposes to an arbitrary red neighbour
- Each red tree decides to grow or not
 - ▶ If it grows, it accepts all proposing trees
 - ▶ If not, all proposing subtrees are frozen
- Criteria: it decides to grow if it would gain at least $\frac{|V(T)|}{2h}$ nodes

Observations



- If the red tree doesn't decide to grow, it will neighbour red nodes only
- This means it will be able to delete nodes only once in the whole phase (i.e. after $2b^2$ steps)
- Hence at most $\frac{|V|}{2b}$ nodes are lost in each phase After the **b** phases at most $\frac{|V|}{2}$ nodes are
- deleted

High Level Pseudocode



```
1: V_0 \leftarrow V
2: Q_0 \leftarrow V
3: for i \in 0..b - 1 do
    init F_0
                  BUILD V_j^{propose} \left. \begin{cases} O(\log diam(T_v)) \end{cases} 2b^2 = O(\log^2 n) \end{cases} b = O(\log n)
5: for j \in 0...2b^2 - 1 do
6:
           V_{i+1} \leftarrow V(F_{2b^2})
Q_{i+1} \leftarrow roots(F_{2b^2})
8:
9:
```

Step Complexity



Recall invariant (1.): $\forall v \in V : dist_G(Q_i, v) = O(\log^3 n)$, for all $i \in 0...b$

Hence, $diam(T_v) = O(\log^3 n)$, for all $v \in V$

The algorithm runs in $O(\log^6 n)$ communication steps

Bibliography



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Bibliography

- K. Censor-Hillel, M. Parter, and G. Schwartzman, "Derandomizing Local Distributed Algorithms under Bandwidth Restrictions." [Online]. Available: https://arxiv. org/abs/1608.01689
- [2] V. Rozhoň, B. Haeupler, and C. Grunau, "A Simple Deterministic Distributed Low-

Bibliography



Diameter Clustering." [Online]. Available: https://arxiv.org/abs/2210.11784

[3] V. Rozhoň and M. Ghaffari, "Polylogarithmic-Time Deterministic Network Decomposition and Distributed Derandomization." [Online]. Available: https://arxiv.org/abs/1907.10937

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Bibliography



[4] V. Rozhoň, M. Elkin, C. Grunau, and B. Haeupler, "Deterministic Low-Diameter Decompositions for Weighted Graphs and Distributed and Parallel Applications." [Online]. Available: https://arxiv.org/abs/2204.08254

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About this presentation



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This presentation is supposed to briefly showcase what you can do with this package.

For a full documentation, read the online book.



Let's explore what we have here.

On the top of this slide, you can see the slide title.

We used the title argument of the #slide function for that:



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```
#slide(title: "First slide")[
    ...
]
(This works because we utilise the clean theme;
more on that later.)
```

Titles are not mandatory, this slide doesn't have one.

But did you notice that the current section name is displayed above that top line?

We defined it using #new-section-slide("Introduction").

This helps our audience with not getting lost after a microsleep.

You can also spot a short title above that.

The bottom of the slide



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Now, look down!

There we have some general info for the audience about what talk they are actually attending right now.

You can also see the slide number there.

Lorem ipsum dolor sit amet, consectetur adipiscing elit, sed do eiusmod tempor incididunt ut labore et dolore magnam aliquam quaerat voluptatem. Ut enim aeque doleamus animo, cum corpore dolemus, fieri tamen permagna accessio potest, si aliquod aeternum et infinitum impendere malum nobis opinemur. Quod idem licet transferre in voluptatem, ut postea variari

voluptas distinguique possit, augeri amplificarique non possit. At etiam Athenis, ut e.

A dynamic slide with pauses



Sometimes we don't want to display everything at once.

A dynamic slide with pauses



Sometimes we don't want to display everything at once.

That's what the **#pause** function is there for!

A dynamic slide with pauses



Sometimes we don't want to display everything at once.

That's what the **#pause** function is there for!

It makes everything after it appear at the next subslide.

(Also note that the slide number does not change while we are here.)

Fine-grained control



When **#pause** does not suffice, you can use more advanced commands to show or hide content.

These are some of your options: - #uncover

- #only
- #alternatives
- #one-by-one
- #line-by-line

Let's explore them in more detail!

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With #uncover, content still occupies space, even when it is not displayed.

For example, are only visible on the second "subslide".

In () behind #uncover, you specify when to show the content, and in [] you then say what to show:



#uncover(3)[Only visible on the third "subslide"]



With #uncover, content still occupies space, even when it is not displayed.

For example, these words are only visible on the second "subslide".

In () behind #uncover, you specify when to show the content, and in [] you then say what to show:



#uncover(3)[Only visible on the third "subslide"]



With #uncover, content still occupies space, even when it is not displayed.

For example, are only visible on the second "subslide".

In () behind #uncover, you specify when to show the content, and in [] you then say what to show:



#uncover(3)[Only visible on the third "subslide"]

Only visible on the third"subslide"



So far, we only used single subslide indices to define when to show something.

We can also use arrays of numbers ...

```
\#uncover((1, 3, 4))[Visible on subslides 1, 3, and 4]
```

Visible on subslides 1, 3, and 4

...or a dictionary with beginning and/or until keys:

```
#uncover((beginning: 2, until: 4))[Visible on
subslides 2, 3, and 4]
```



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#uncover((beginning: 2, until: 4))[Visible on
subslides 2, 3, and 4]
Visible on subslides 2, 3, and 4
```



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```

Visible on subslides 1, 3, and 4

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subslides 2, 3, and 4]
Visible on subslides 2, 3, and 4
```



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Visible on subslides 1, 3, and 4

...or a dictionary with beginning and/or until keys:



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#uncover((beginning: 2, until: 4))[Visible on
subslides 2, 3, and 4]
Visible on subslides 2, 3, and 4
```

Convenient rules as strings



As as short hand option, you can also specify rules as strings in a special syntax.

Comma separated, you can use rules of the form

- 1-3 from subslide 1 to 3 (inclusive)
 - -4 all the time until subslide 4 (inclusive)
 - 2- from subslide 2 onwards
 - 3 only on subslide 3

Everything that works with #uncover also works with #only.

However, content is completely gone when it is not displayed.

For example, the rest of this sentence moves.

Again, you can use complex string rules, if you want.

#only("2-4, 6")[Visible on subslides 2, 3, 4, and 6]

Everything that works with #uncover also works with #only.

However, content is completely gone when it is not displayed.

For example, see how the rest of this sentence moves.

Again, you can use complex string rules, if you want.

#only("2-4, 6")[Visible on subslides 2, 3, 4, and 6]

Visible on subslides 2, 3, 4, and 6

Everything that works with #uncover also works with #only.

However, content is completely gone when it is not displayed.

For example, the rest of this sentence moves.

Again, you can use complex string rules, if you want.

#only: Reserving no space



#only("2-4, 6")[Visible on subslides 2, 3, 4, and 6]

Visible on subslides 2, 3, 4, and 6

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#only: Reserving no space



#only("2-4, 6")[Visible on subslides 2, 3, 4, and 6]

Visible on subslides 2, 3, 4, and 6

You might be tempted to try

```
#only(1)[Ann] #only(2)[Bob] #only(3)[Christopher]
likes #only(1)[chocolate] #only(2)[strawberry]
#only(3)[vanilla] ice cream.

Ann
likes chocolate
ice cream.
```

But it is hard to see what piece of text actually changes because everything moves around. Better:

```
#alternatives[Ann][Bob][Christopher] likes
#alternatives[chocolate][strawberry][vanilla] ice
cream.
```

Ann likes chocolate ice cream.

You might be tempted to try

```
#only(1)[Ann] #only(2)[Bob] #only(3)[Christopher]
likes #only(1)[chocolate] #only(2)[strawberry]
#only(3)[vanilla] ice cream.

Bob
likes strawberry
ice cream.
```

But it is hard to see what piece of text actually changes because everything moves around. Better:

```
#alternatives[Ann][Bob][Christopher] likes
#alternatives[chocolate][strawberry][vanilla] ice
cream.
```

Bob likes strawberry ice cream.

You might be tempted to try

```
#only(1)[Ann] #only(2)[Bob] #only(3)[Christopher]
likes #only(1)[chocolate] #only(2)[strawberry]
#only(3)[vanilla] ice cream.
Christopher
likes vanilla
ice cream.
```

But it is hard to see what piece of text actually changes because everything moves around. Better:

```
#alternatives[Ann][Bob][Christopher] likes
#alternatives[chocolate][strawberry][vanilla] ice
cream.
```

Christopher likes vanilla ice cream.

#alternatives is to #only what #one-by-one is to
#uncover.

#one-by-one behaves similar to using #pause but you can additionally state when uncovering should start.

#one-by-one(start: 2)[one][by][one]

start can also be omitted, then it starts with the first subside:

```
#one-by-one[one ][by ][one]
```

one

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#alternatives is to #only what #one-by-one is to
#uncover.

#one-by-one behaves similar to using #pause but you can additionally state when uncovering should start.

```
#one-by-one(start: 2)[one ][by ][one]
one
```

start can also be omitted, then it starts with the first subside:

```
#one-by-one[one ][by ][one]
oneby
```

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onebyone

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```
#one-by-one(start: 2)[one ][by ][one]
onebyone
```

start can also be omitted, then it starts with the first subside:

```
#one-by-one[one ][by ][one]
```

onebyone

Sometimes it is convenient to write the different contents to uncover one at a time in subsequent lines.

This comes in especially handy for bullet lists, enumerations, and term lists.

```
#line-by-line(start: 2)[
    - first
    .
```

#line-by-line: syntactic sugar for #one-by-one



```
- second
- third
]
```

start is again optional and defaults to 1.

Sometimes it is convenient to write the different contents to uncover one at a time in subsequent lines.

This comes in especially handy for bullet lists, enumerations, and term lists.

#line-by-line: syntactic sugar for #one-by-one



```
- second
- third
]
```

start is again optional and defaults to 1.

Sometimes it is convenient to write the different contents to uncover one at a time in subsequent lines.

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```
#line-by-line(start: 2)[
    - first
```

- first
- second

#line-by-line: syntactic sugar for #one-by-one



```
- second
- third
]
```

start is again optional and defaults to 1.

Sometimes it is convenient to write the different contents to uncover one at a time in subsequent lines.

This comes in especially handy for bullet lists, enumerations, and term lists.

```
#line-by-line(start: 2)[
    - first
```

- first
- second

#line-by-line: syntactic sugar for #one-by-one



```
secondthirdthird
```

start is again optional and defaults to 1.

While #line-by-line is very convenient syntax-wise, it fails to produce more sophisticated bullet lists, enumerations or term lists. For example, non-tight lists are out of reach.

For that reason, there are #list-one-by-one, #enum-one-by-one, and #terms-one-by-one, respectively.

Note that, for technical reasons, the bullet points, numbers, or terms are never covered.

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Note that, for technical reasons, the bullet points, numbers, or terms are never covered.

start is again optional and defaults to 1.

How a slide looks...



... is defined by the *theme* of the presentation.

This demo uses the unipd theme.

Because of it, the title slide and the decoration on each slide (with section name, short title, slide number etc.) look the way they do.

Themes can also provide variants, for example ...

... this one!

It's very minimalist and helps the audience focus on an important point.

Your own theme?



If you want to create your own design for slides, you can define custom themes!

The book explains how to do so.

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The utils module



Polylux ships a utils module with solutions for common tasks in slide building.

Fit to height



You can scale content such that it has a certain height using #fit-to-height(height, content):

Fill remaining space



This function also allows you to fill the remaining space by using fractions as heights, i.e. fit-to-height(1fr)[...]:

Side by side content



Often you want to put different content next to each other. We have the function #side-by-side for that:

Lorem ipsum dolor sit amet, consectetur adipiscing elit, sed do.

Lorem ipsum dolor sit amet, consectetur adipiscing elit, sed do

Lorem ipsum dolor sit amet, consectetur adipiscing elit, sed do

Side by side content



eiusmod tempor incididunt ut labore. eiusmod tempor.



Why not include an outline?

- 1. Overview
- 2. Overview
- 3. Models
- 4. Models



- 5. LOCAL Algorithms
- 6. LOCAL Algorithms
- 7. CONGEST Algorithms
- 8. CONGEST Algorithms
- 9. Clusterings



- 10. Clusterings
- 11. Clusterings
- 12. The Algorithm
- 13.
- 14.

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- 15.
- 16. Introduction
- 17. Dynamic content
- 18. Dynamic content
- 19. Themes



- 20. Utilities
- 21. Typst features
- 22. Conclusion

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Typst gives us so many cool things². Use them!

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²For example footnotes!



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Hopefully you now have some kind of idea what you can do with this template.

Consider giving it a GitHub star or open an issue if you run into bugs or have feature requests.