Life Sciences and
Facility Management

Institute of
Computational Life Sciences

Fourier Series

Image and Signal Processing

Norman Juchler

Sinusoidal signals

Again, and again!

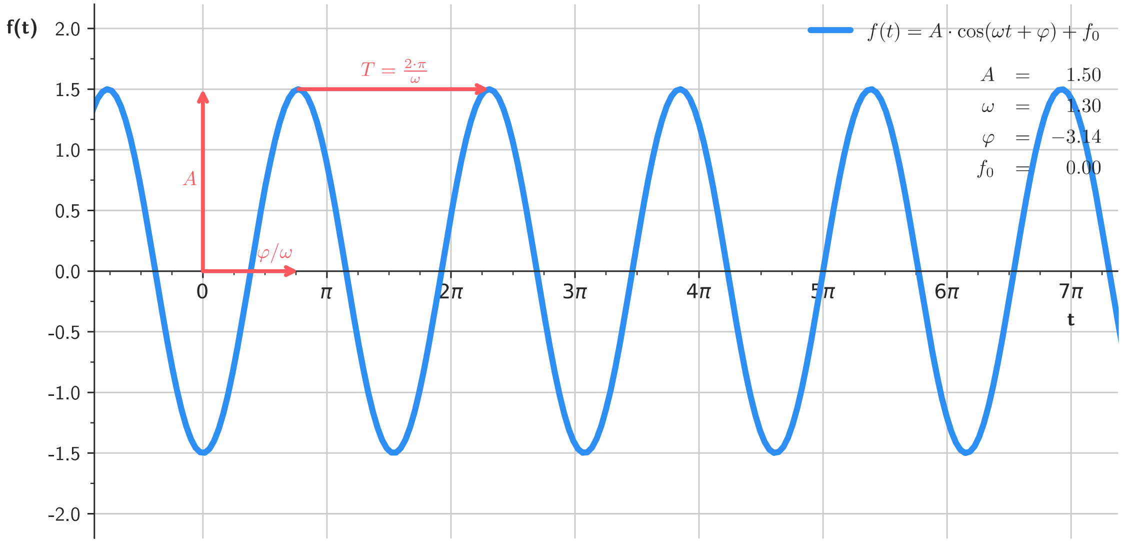
Sinusoidal signals (revisited)

- The cosine takes this general form:

$$\boxed{x(t) = A\cos(\omega t - \varphi)} \qquad \omega = \frac{2\pi}{T} = 2\pi f$$

ω : Angular frequency
f : Frequency
T : Period

A: Amplitude
φ: Phase



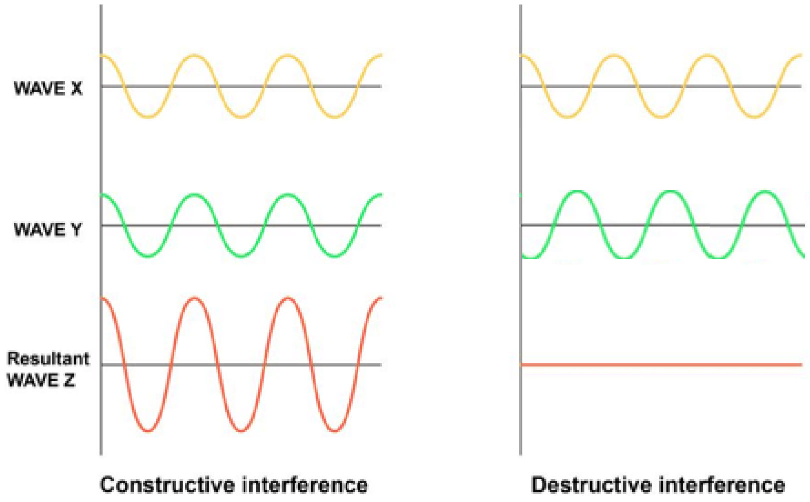
- This is equivalent* to

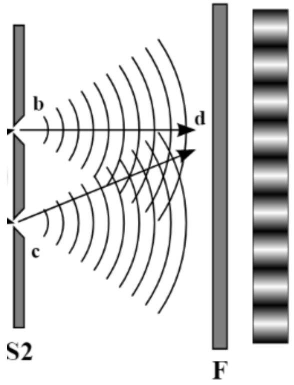
$$\boxed{x(t) = a\cos(\omega t) + b\sin(\omega t)} \qquad \begin{aligned} a &= A\cos(\varphi) \\ b &= A\sin(\varphi) \end{aligned}$$

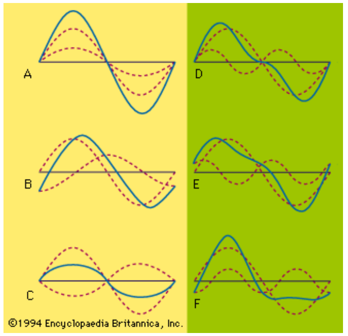
* The expression follows from this trigonometric formula:
$\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$

Interference

- Wave Interference occurs when two or more waves overlap, combining to form a new wave pattern.
- **Constructive interference**: Waves align in phase (peaks with peaks, troughs with troughs), resulting in a wave with greater amplitude.
- **Destructive interference**: Waves are out of phase (peak meets trough), leading to partial or complete cancellation of the wave.
- Videos:
  - Wave interference
  - Wave interference for two sound sources







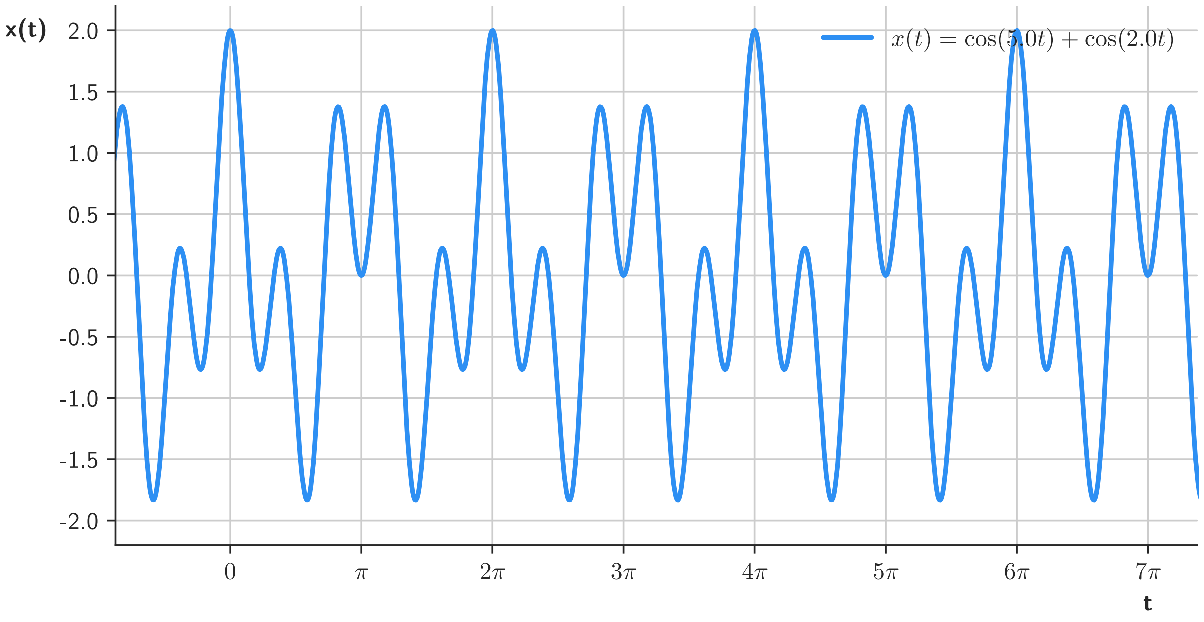
Left: Interference pattern from two light sources. **Right**: Partial interference with differing amplitudes or wavelengths.

Image source: Shanghai Optics and Britannica

Superposition and interference

- We can superimpose an arbitrary number of sinusoidal signals

- **Example 1**: $x(t) = a_1 \cos(\omega_1 t) + a_2 \cos(\omega_2 t)$



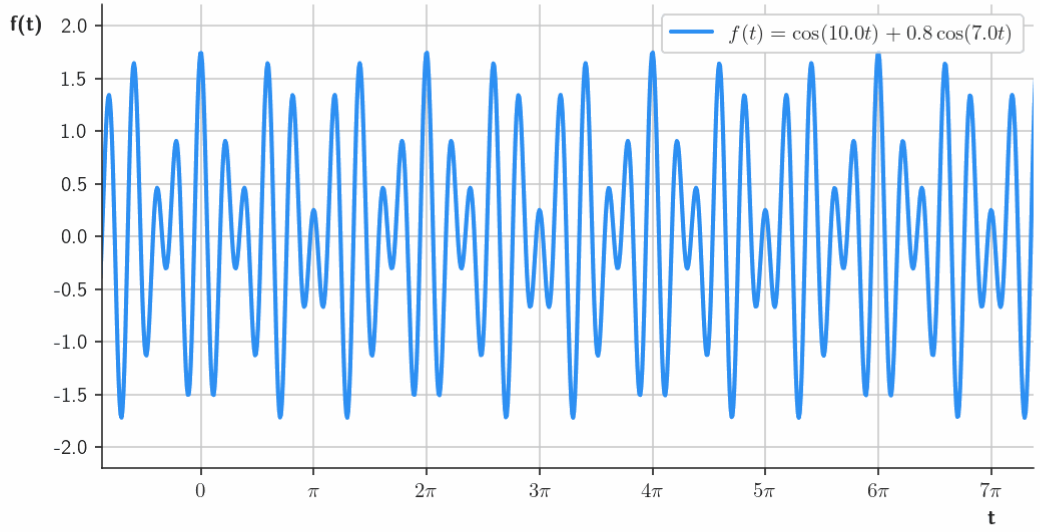
Intermezzo: Beating signals

- A beat is an interference pattern that occurs when two sounds of slightly different frequencies are played together

$$x(t) = a_1 \cos(2\pi f_1 t) + a_2 \cos(2\pi f_2 t)$$

- Beats are perceived as periodic variations in amplitude (pulsating sound)
- One can show that the frequency of the beat is equal to the difference in frequency between the two original sounds.

$$f_{beat} = |f_1 - f_2|$$



- Video: [Beating effect demonstrated with tuning forks]

Two examples from Wikimedia: Link1, Link2

Superposition (revisited)

- We can superimpose an arbitrary number of sinusoidal signals

- **Example 1**: $x(t) = a_1 \cos(\omega_1 t) + a_2 \cos(\omega_2 t)$

- More generally, one can superimpose arbitrary many cosine signals:

$$\boxed{x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(\omega_k t + \varphi_k)}$$

Here we introduced an additional vertical offset A_0

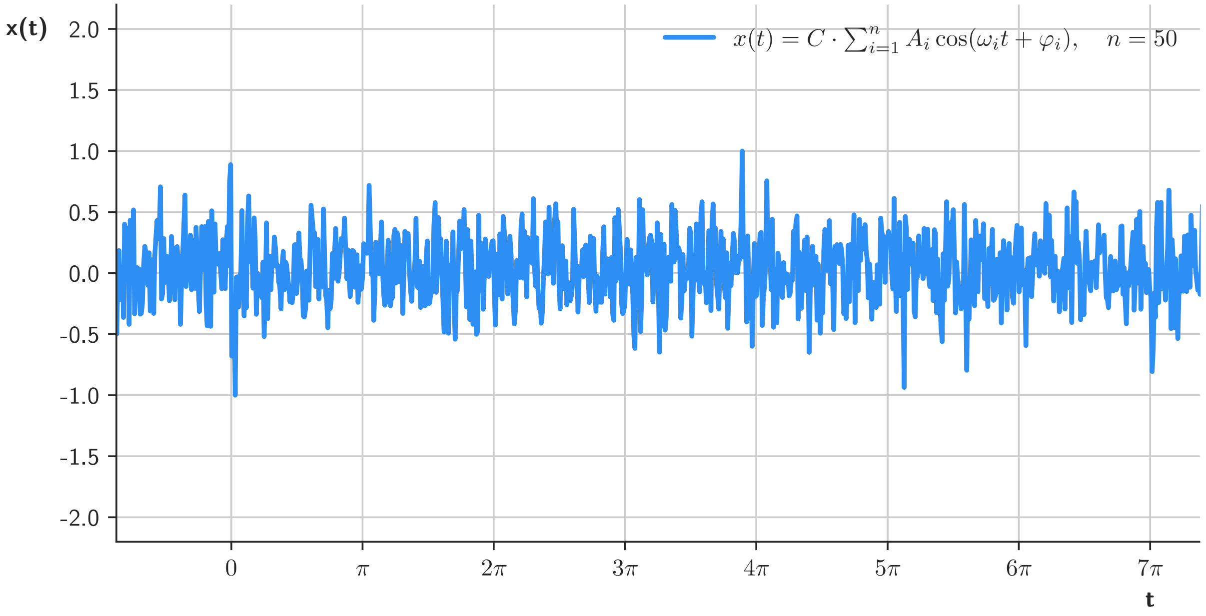
Superposition (revisited)

- We can superimpose an arbitrary number of sinusoidal signals

- **Example 2**:

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(\omega_k t + \varphi_k)$$

  - Randomly sampled parameters
    $A_k, \omega_k, \varphi_k > 0$
  - Observation: For large N, the signal resembles a noisy signal



Superposition (revisited)

- We can superimpose an arbitrary number of sinusoidal signals

- **Example 3**:

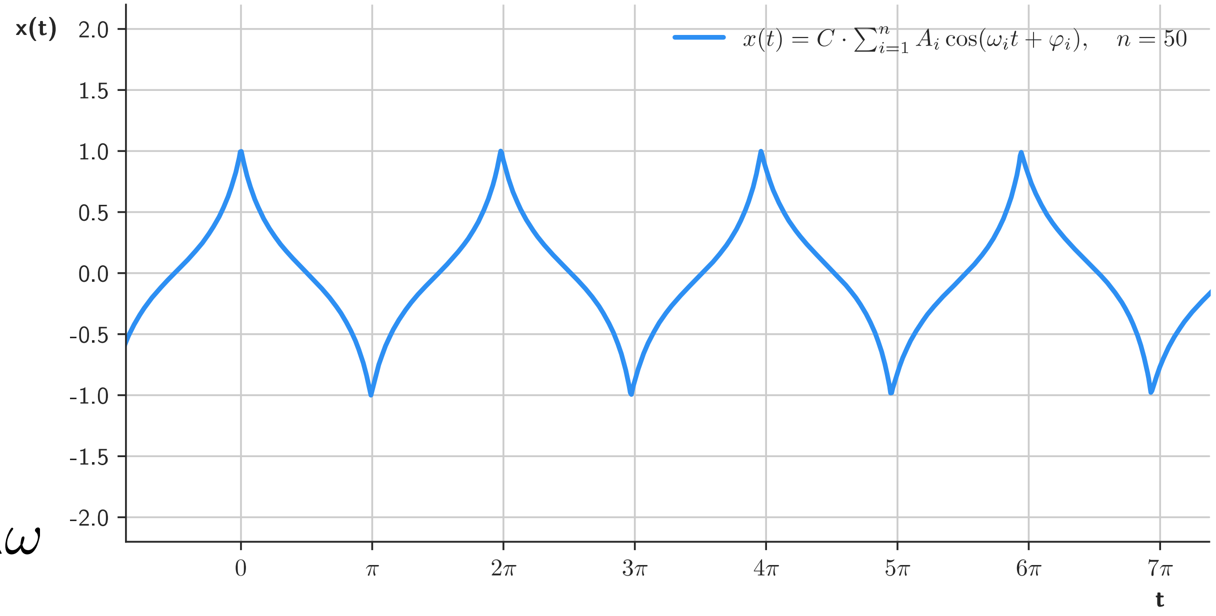
$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(\omega_k t + \varphi_k)$$

  - This time, the parameters were sampled according to a logic:

$$A_k = \frac{c}{k^2}$$

$$\omega_k = \omega_0 + k \cdot \Delta\omega$$

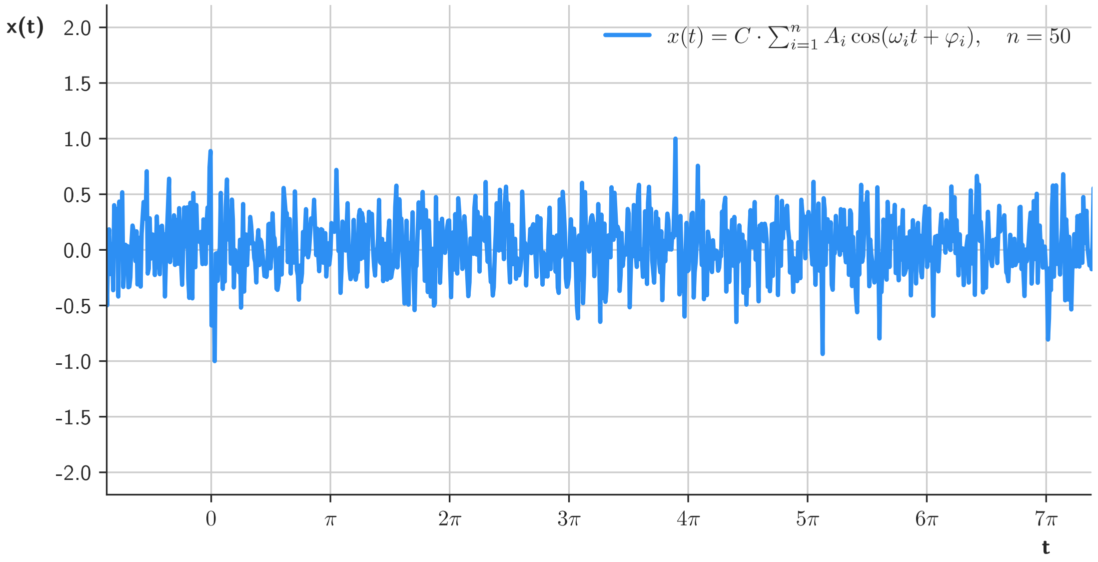
$$\varphi_k = 0$$

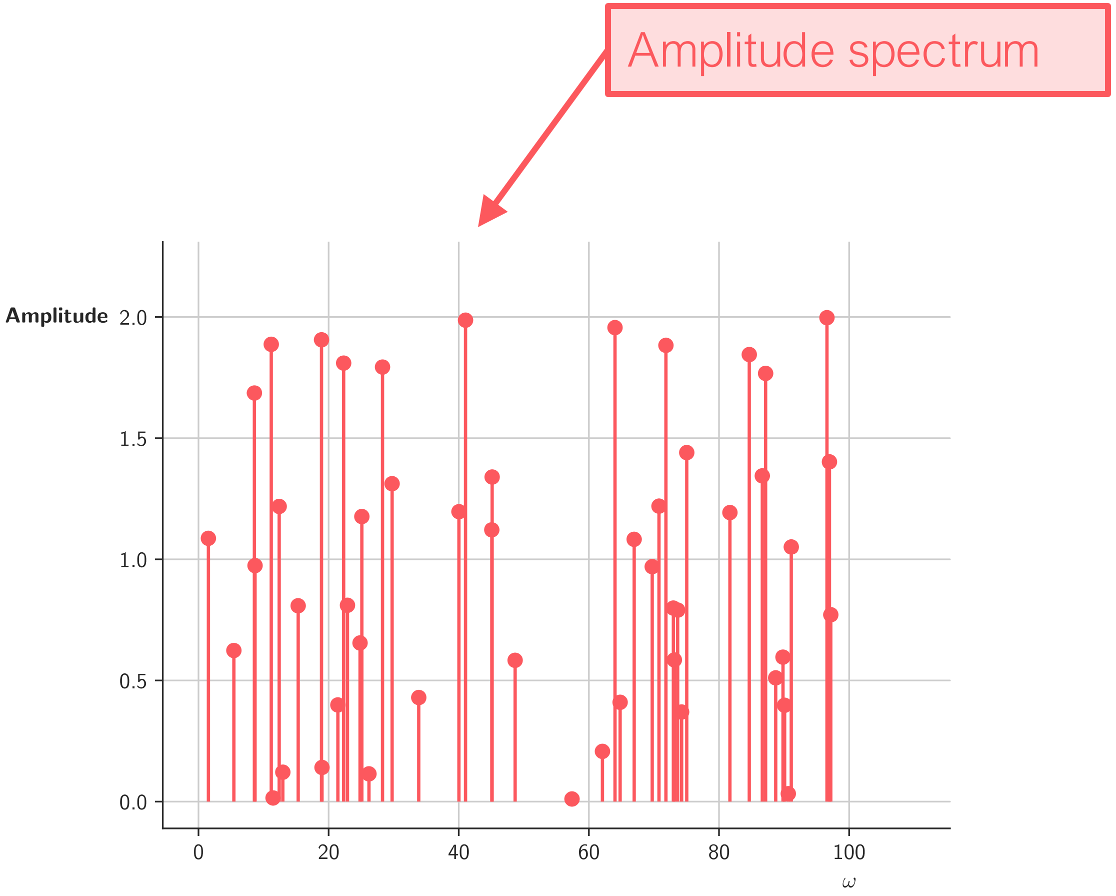


  - Observation: For growing N, the signal quickly converges to periodic signal

The amplitude spectrum

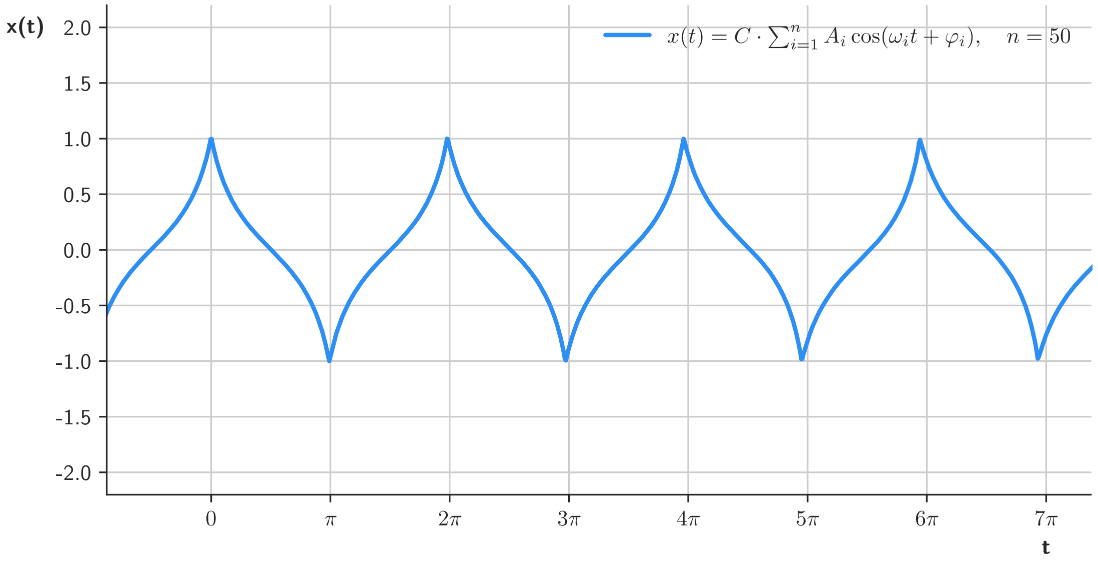
- The amplitude spectrum is a representation of the amplitude of the various frequency components of a signal.
- Plot: frequencies (f or ω) vs. amplitude
- Spectrum for **Example 2**:

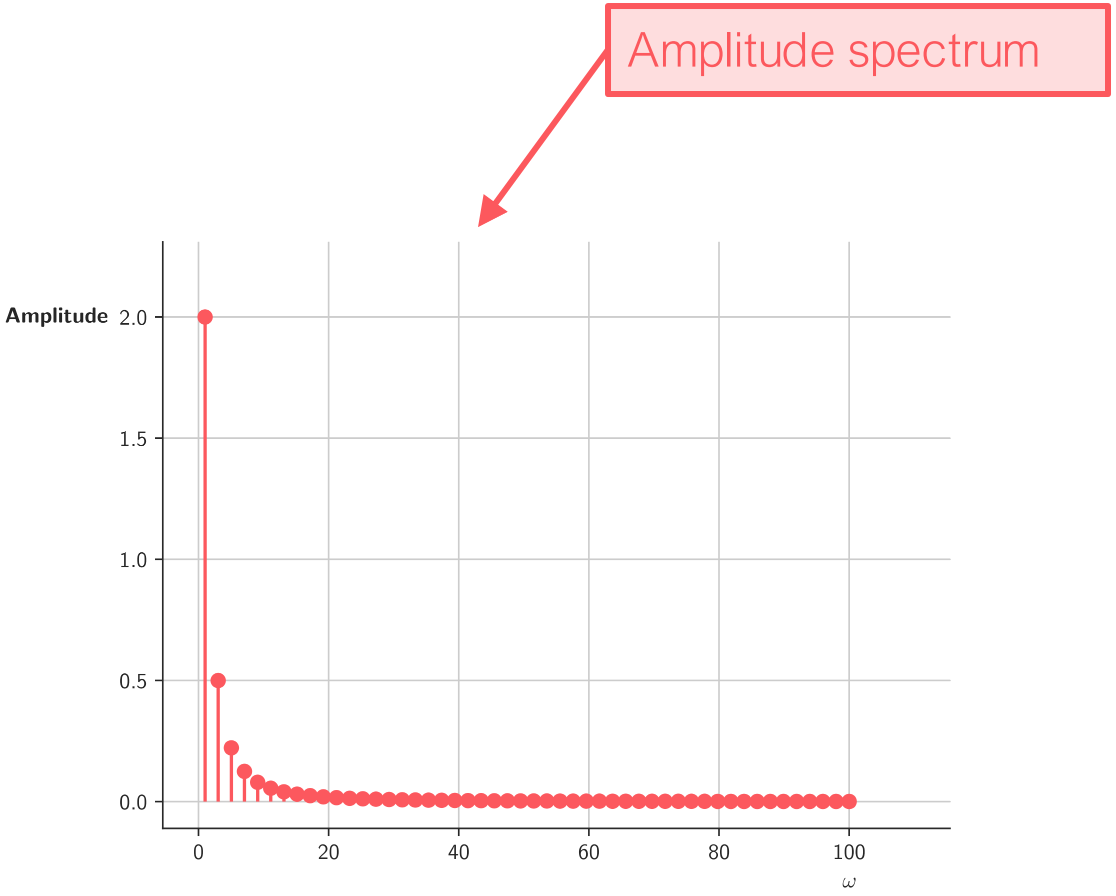




The amplitude spectrum

- The amplitude spectrum is a representation of the amplitude of the various frequency components of a signal.
- Plot: amplitude vs frequencies (f or ω)
- Spectrum for **Example 3**:





Fourier Series

Concepts

Fourier series

- The previous examples suggest that we can decompose signals into sinusoidal components.
- Fourier theory examines whether a (periodic) function or signal $x(t)$ can be expressed as a **series of trigonometric functions**:

$$\boxed{x(t) = A_0 + \sum_{k=1}^{N} A_k \cos\left(\omega_k t - \varphi_k\right)}$$

- How to determine the coefficients $A_k, k \geq 0$?
- For the beginning, we restrict ourselves to **periodic, continuous-time signals** $x(t)$ with period $T > 0$:

$$x(t+T) = x(t)$$

Joseph Fourier, 1768-1830, French mathematician and physicist. See video

Image source: Britannica

Fourier series: Notation

- The following representations of the Fourier series are equivalent.

- **Fourier series in amplitude-phase form**

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos\left(\omega_k t - \varphi_k\right)$$

- **Fourier series in sine-cosine form**

$$x(t) = a_0 + \sum_{k=1}^{N} a_k \cos\left(\omega_k t\right) + b_k \sin\left(\omega_k t\right)$$

- **Fourier series, exponential form**

$$x(t) = \sum_{k=-N}^{N} c_k e^{i\omega_k t}$$

The last expression follows with some mathematical creativity using complex numbers.

Fourier series: Notation

- Frequency of the base functions increases with this pattern:

$$\omega_k = 2\pi f_k = \frac{2\pi k}{T}$$

- **Fourier series in amplitude-phase form**

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos\left(\frac{2\pi k}{T} t - \varphi_k\right)$$

- **Fourier series in sine-cosine form**

$$x(t) = a_0 + \sum_{k=1}^{N} a_k \cos\left(\frac{2\pi k}{T} t\right) + b_k \sin\left(\frac{2\pi k}{T}\right)$$

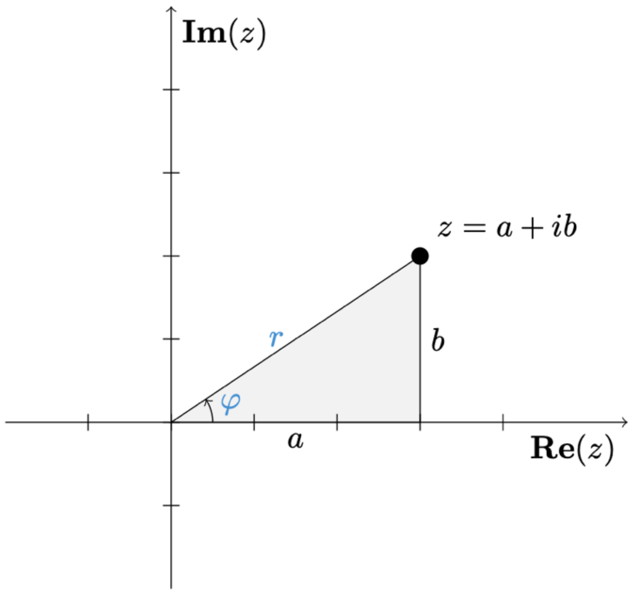
- **Fourier series, exponential form**

$$x(t) = \sum_{k=-N}^{N} c_k e^{\frac{2\pi i k}{T} t}$$

The last expression follows with some mathematical creativity using complex numbers.

Excursion: Complex numbers

- **Definition**: Complex numbers z are numbers of the form $\boxed{z = a + ib, \quad a, b \in \mathbb{R}}$
- Can be represented in the complex plane



- The fundamental laws of arithmetic apply:
  - We can add, subtract, multiply and divide complex numbers!
- It holds the famous Euler formula $\boxed{e^{i\varphi} = \cos\varphi + i\sin\varphi}$
- Complex numbers can be represented in polar form:

$$\boxed{\begin{aligned} z &= r(\cos\varphi + i\sin\varphi) \\ &= re^{i\varphi} \end{aligned}}$$

$$r = \sqrt{a^2 + b^2} = |z|$$

$$\varphi = \arctan\left(\frac{b}{a}\right)$$

Fourier series: Notation

- The coefficients of the different Fourier series can be converted into each other

- **Amplitude-phase form ↔ sine-cosine form**

$$a_k = A_k \cos\varphi_k \quad \text{and} \quad b_k = A_k \sin\varphi_k$$

$$A_k = \sqrt{a_k^2 + b_k^2} \quad \text{and} \quad \varphi_k = \arctan\left(\frac{b_k}{a_k}\right)$$

- **Exponential form ↔ sine-cosine form**

$$\begin{aligned} a_0 &= a_0 \\ a_k &= c_k + c_{-k} && \text{for } k > 0 \\ b_k &= i(c_k - c_{-k}) && \text{for } k > 0 \end{aligned}$$

$$c_k = \left\{ \begin{array}{ll} a_0, & k = 0 \\ \frac{1}{2}(a_k - ib_k), & k > 0 \\ \frac{1}{2}(a_k + ib_k), & k < 0 \end{array} \right\}$$

Fourier series: How to find the coefficients?

- After one or two good digs into the mathematical bag of tricks, you will find the following expressions for the coefficients
- **Sine-cosine form**:

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, dt$$

$$a_k = \frac{2}{T}\int_{-T/2}^{T/2} x(t) \cos\left(2\pi \tfrac{k}{T} x\right)\, dt \qquad \text{for } k \geq 1$$

$$b_k = \frac{2}{T}\int_{-T/2}^{T/2} x(t) \sin\left(2\pi \tfrac{k}{T} x\right) dx, \qquad \text{for } k \geq 1$$

- **Exponential form**:

$$c_k = \frac{1}{T}\int_{0}^{T} e^{-i2\pi \frac{k}{T} t} x(t)\, dt, \quad \forall\, k \in \mathbb{Z}$$

Fourier Series

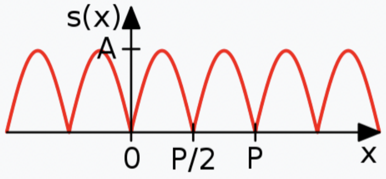
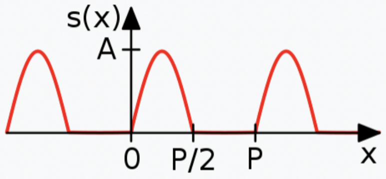
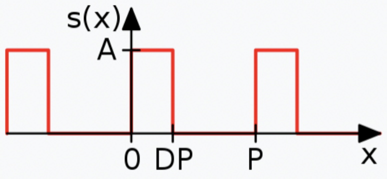
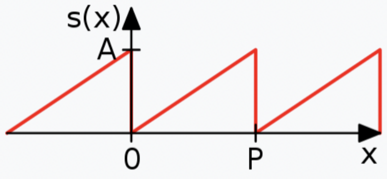
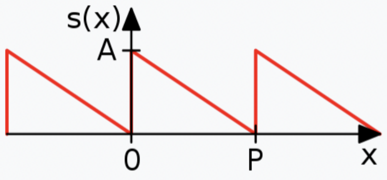
Examples

How to apply Fourier series?

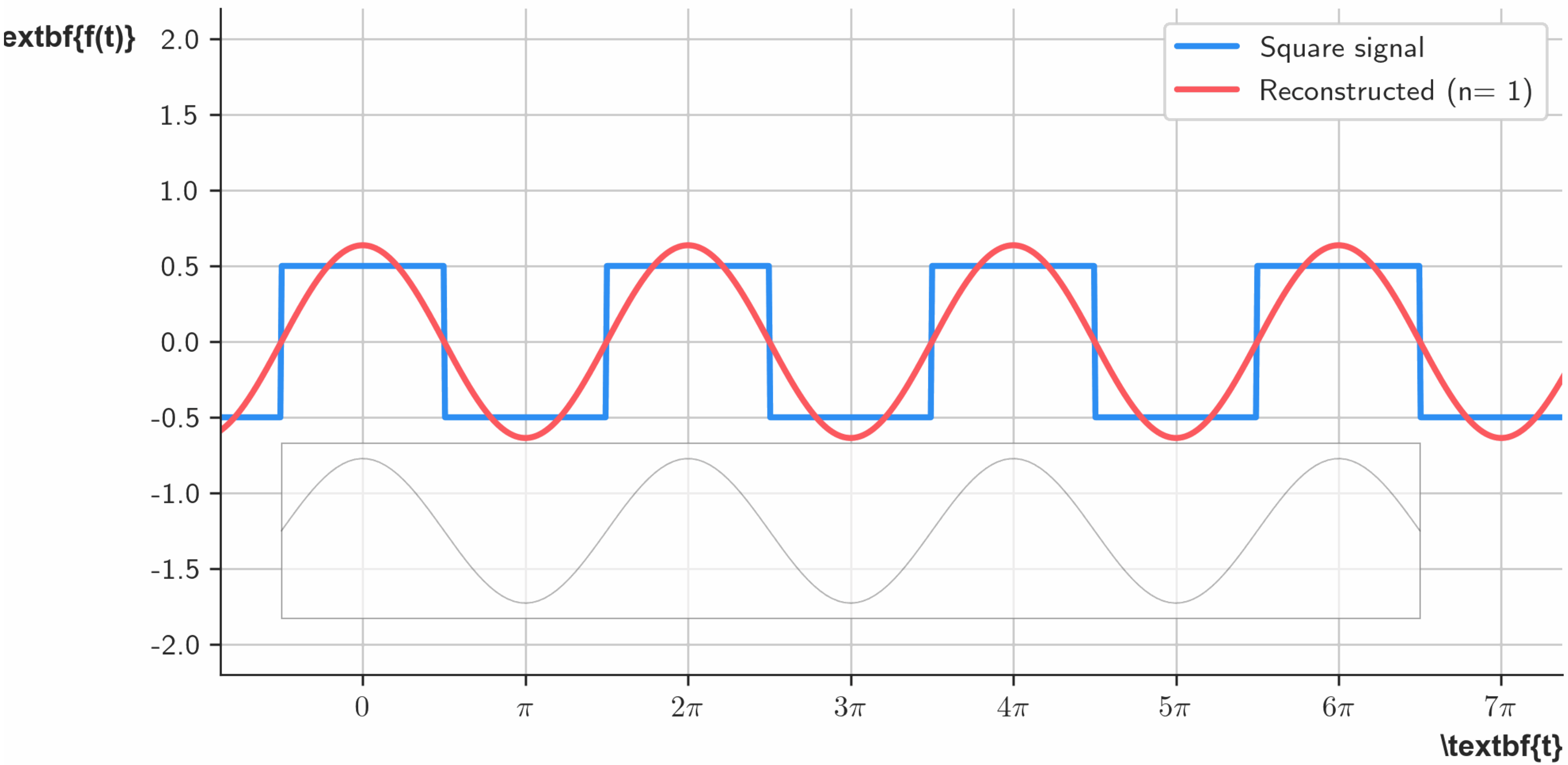
- Identify a periodic, continuous-time function $x(t)$
- Determine its period T
- Compute the Fourier series coefficients:
  - by solving the integral
  - by using a formulary
- Notes:
  - The coefficients a_k, b_k, c_k correspond to specific frequencies ω_k
  - These coefficients can be visualized in a (amplitude) spectrum.
  - For even and odd functions, certain terms simplify.

- Explore examples in this week's Jupyter notebook!

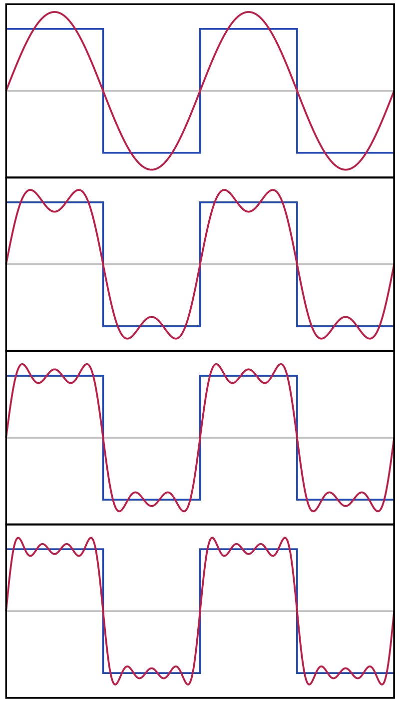
Fourier series: Extract of a formulary

$s(x) = A\left\|\sin\left(\frac{2\pi}{P}x\right)\right\|$ for $0 \le x < P$		$A_0 = \frac{2A}{\pi}$ $A_n = \begin{cases} \frac{-4A}{\pi}\frac{1}{n^2-1} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$ $B_n = 0$
$s(x) = \begin{cases} A\sin\left(\frac{2\pi}{P}x\right) & \text{for } 0 \le x < P/2 \\ 0 & \text{for } P/2 \le x < P \end{cases}$		$A_0 = \frac{A}{\pi}$ $A_n = \begin{cases} \frac{-2A}{\pi}\frac{1}{n^2-1} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$ $B_n = \begin{cases} \frac{A}{2} & n = 1 \\ 0 & n > 1 \end{cases}$
$s(x) = \begin{cases} A & \text{for } 0 \le x < D \cdot P \\ 0 & \text{for } D \cdot P \le x < P \end{cases}$		$A_0 = AD$ $A_n = \frac{A}{n\pi}\sin(2\pi nD)$ $B_n = \frac{2A}{n\pi}(\sin(\pi nD))^2$
$s(x) = \frac{Ax}{P}$ for $0 \le x < P$		$A_0 = \frac{A}{2}$ $A_n = 0$ $B_n = \frac{-A}{n\pi}$
$s(x) = A - \frac{Ax}{P}$ for $0 \le x < P$		$A_0 = \frac{A}{2}$ $A_n = 0$ $B_n = \frac{A}{n\pi}$

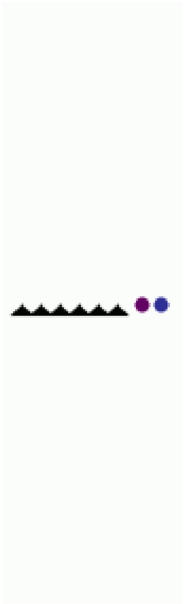
Example: Pulse function



Example: Pulse function



The first four partial sums of the Fourier series for a square wave. As more harmonics are added, the partial sums converge to the square wave. Source



Visualization of the sixth partial sum (that is N=6) of the Fourier series for the square wave. The perfect square wave is represented by the blue dot on the right, while the approximated signal is shown by the purple dot. Source

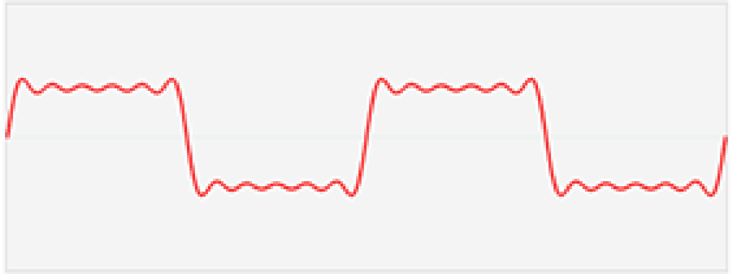


Illustration of the relationship between the Fourier series approximation (with N=6) of a square signal and its amplitude spectrum. Source

Outlook

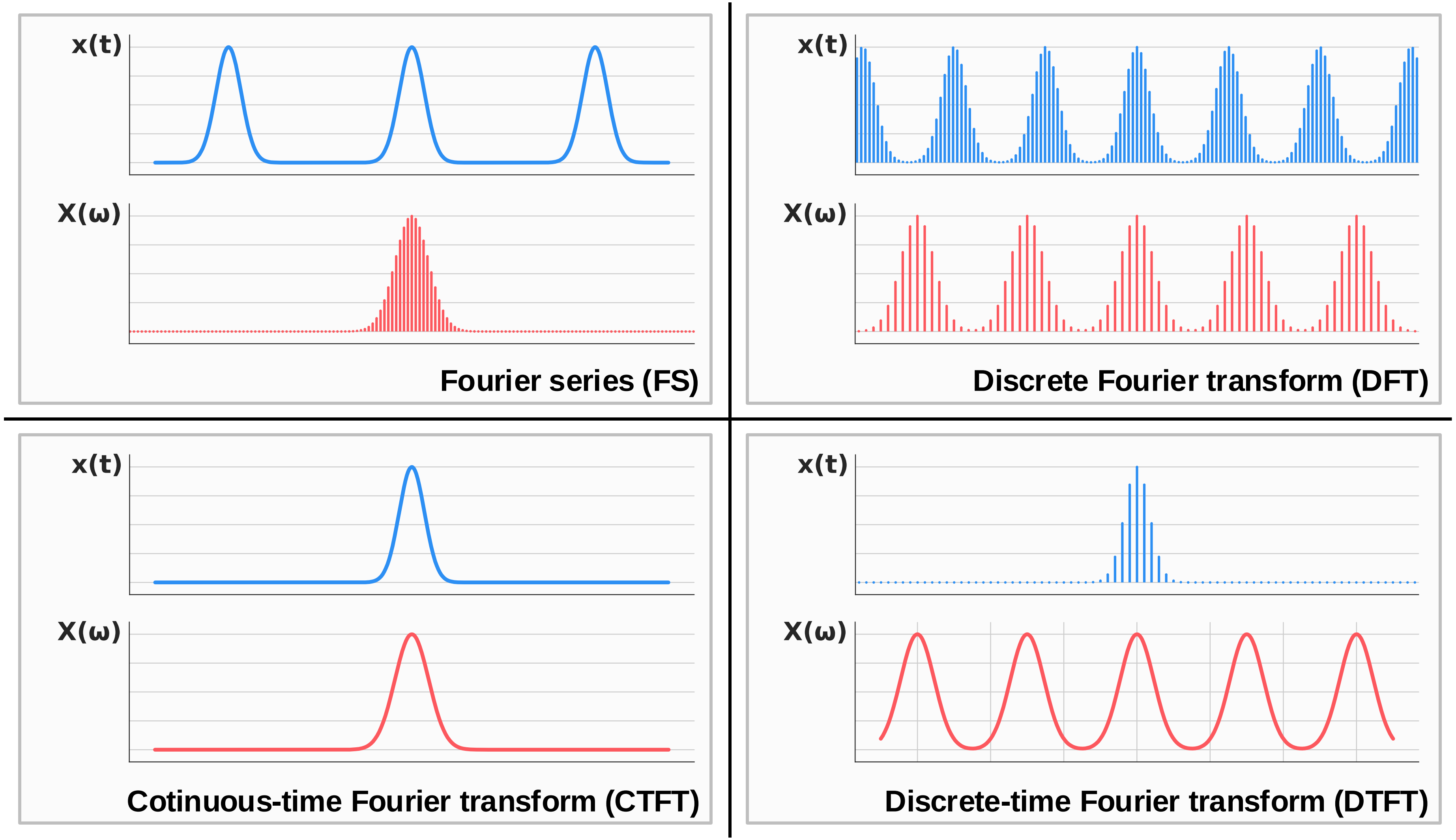
- Generalize to general, non-periodic functions $x(t)$:
  Fourier transformation (FT)
- Study the properties of Fourier transformations
- Examine the discrete-time Fourier transformation (DTFT)
- Understand the implications of sampling using FT
- Have a quick look at an implementation: FFT
- Introduce signal covariance and auto-correlation

Fourier’s landscape

Time-continuous | *Time-discrete*

Periodic

Aperiodic



Related:
- **Fast Fourier transform**
- Discrete cosine transform
- Discrete sine transform

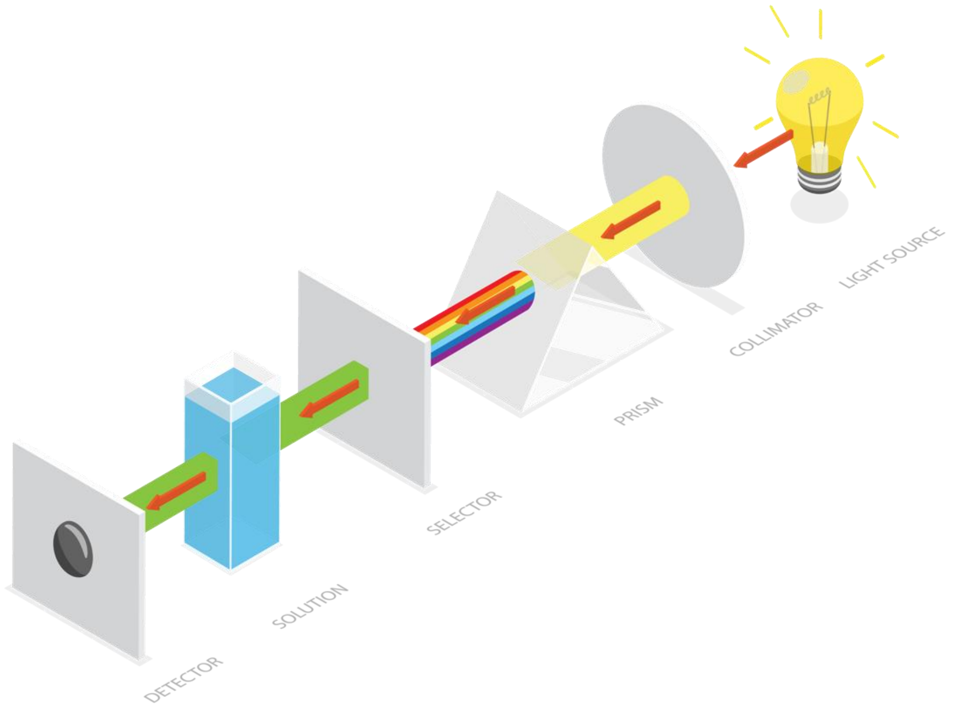
Generalization:
z - Transform

Generalization:
Laplace Transform

Spectrum analysis vs. synthesis

- **Analysis**:
  - Time domain → Frequency domain
  - Break down a signal into its individual frequency components to study its spectral content.
  - Example: **Spectroscopic methods** are used to identify or describe chemical compounds by analyzing their absorption or emission spectra.

- **Synthesis**:
  - Frequency domain → Time domain
  - Create new signals by combining individual frequency components or modifying existing ones.
  - Example: **Audio synthesis** (sound design, speech synthesis)



Example for spectral analysis: **Spectrography**. The method permits to analyze molecules resolved in the test solution.

Illustration based on Adobe Stock / 657900155