Complex numbers
A brief primer

ISP / ADLS
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Complex numbers and the complex plane

Complex numbers are numbers of the form

$$z = a + ib, \quad a, b \in \mathbb{R}.$$

The symbol i is the imaginary unit, defined as the solution to the equation $i^2 = -1$. The real number a is called the *real part* and the real number b is called the *imaginary part* of z. The set of all complex numbers is denoted as $\mathbb{C}$. Note that the set of real numbers $\mathbb{R}$ is a subset of $\mathbb{C}$, where all complex numbers z have $b = 0$.

Complex numbers can be visualized in the *complex plane*, which is a two-dimensional coordinate system with the real part a on the horizontal axis and the imaginary part b on the vertical axis. Therefore, z is represented as a point (a, b) in the complex plane.

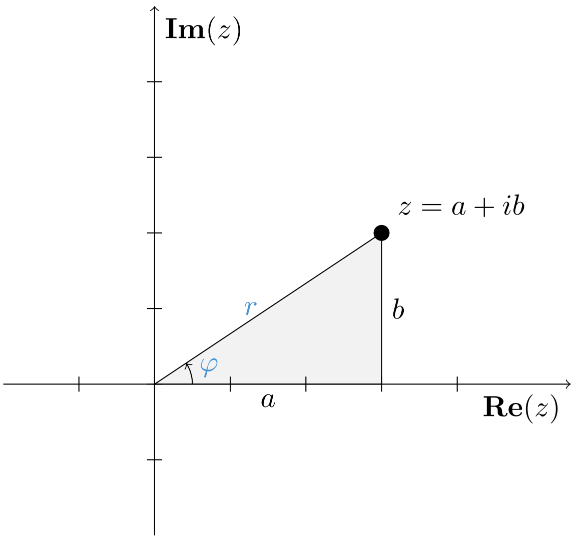

Figure 1: The complex plane with cartesian and polar coordinates

Arithmetics

The fundamental laws of arithmetic apply. We can add, subtract, multiply and divide complex numbers:

$$\begin{aligned} z_1 &= a_1 + ib_1 \\ z_2 &= a_2 + ib_2 \end{aligned}$$

Summation:

$$\begin{aligned} z_1 + z_2 &= (a_1 + ib_1) + (a_2 + ib_2) \\ &= (a_1 + a_2) + i(b_1 + b_2) \end{aligned}$$

Subtraction:

$$\begin{aligned} z_1 - z_2 &= (a_1 + ib_1) - (a_2 + ib_2) \\ &= (a_1 - a_2) + i(b_1 - b_2) \end{aligned}$$

Multiplication:

$$\begin{aligned} z_1 \cdot z_2 &= (a_1 + ib_1) \cdot (a_2 + ib_2) \\ &= a_1a_2 + ia_1b_2 + ia_2b_1 + i^2b_1b_2 \\ &= (a_1a_2 - b_1b_2) + i(a_1b_2 + a_2b_1) \end{aligned}$$

Division:

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{a_1 + ib_1}{a_2 + ib_2} \\ &= \frac{a_1 + ib_1}{a_2 + ib_2} \cdot \frac{a_2 - ib_2}{a_2 - ib_2} \\ &= \frac{(a_1a_2 + b_1b_2) + i(a_2b_1 - a_1b_2)}{a_2^2 + b_2^2} \end{aligned}$$

Complex conjugate: The complex conjugate of $z \in \mathbb{C}$ is denoted as $\bar{z}$:

$$\bar{z} = a - ib.$$

The squared magnitude of a complex number can be computed as $z \cdot \bar{z} = |z|^2$:

$$\begin{aligned} z \cdot \bar{z} &= (a + ib) \cdot (a - ib) \\ &= a^2 + iab - iab - i^2b^2 \\ &= a^2 + b^2 \end{aligned}$$

Polar form

Complex numbers can also be represented in polar form (see Figure 1) using their *magnitude* r and angle φ (also known as *argument*). Magnitude r and argument φ are related to the real and imaginary parts by:

$$r = \sqrt{a^2 + b^2} = |z|$$
$$\varphi = \arctan\left(\frac{b}{a}\right)$$

The polar form of a complex number z is given by:

$$z = r(\cos\varphi + i\sin\varphi) \tag{1}$$

Alternatively, this can be written using *Euler's formula*:

$$\begin{aligned} z &= re^{i\varphi} \\ &= |z|e^{i\varphi} \end{aligned} \tag{2}$$

The famous Euler formula $e^{i\varphi} = \cos\varphi + i\sin\varphi$ can be derived using Taylor series, which you may have already learned about in a math course. Note what happens if we compute two complex numbers z_1 and z_2:

$$\begin{aligned} z_1 \cdot z_2 &= |z_1| \cdot e^{i\varphi_1} \cdot |z_2| \cdot e^{i\varphi_2} \\ &= |z_1| \cdot |z_2| \cdot e^{i(\varphi_1+\varphi_2)} \\ &= r_1 r_2 \cdot e^{i(\varphi_1+\varphi_2)} \end{aligned}$$

In the product of two complex numbers, the result is again a complex number whose *magnitude* is the *product* of the two magnitudes r_1, r_2, and whose *argument* is the *sum* of the two arguments φ_1, φ_2.

By using Euler's formula, we can derive the following expressions for $\cos\varphi$ and $\sin\varphi$:

$$\cos\varphi = \frac{1}{2}(e^{i\varphi} + e^{-i\varphi})$$
$$\sin\varphi = \frac{1}{2i}(e^{i\varphi} - e^{-i\varphi}) \tag{3}$$

Complex functions

In the context of signal processing and Fourier theory, we will encounter complex functions. Two types of complex functions are relevant:

- The function $f(t)$ maps real valued arguments onto the complex plane:
$$t \to z : z = f(t), \quad t \in \mathbb{R}, z \in \mathbb{C}$$
- The function $f(z)$ maps complex valued arguments onto the complex plane:
$$z \to w : w = f(z), \quad z, w \in \mathbb{C}$$

Example: the function $f(t) = \cos t + i\sin t$ maps a complex value z to every real t.

Exercises

1. What are the solutions to the following quadratic equation?

$$z^2 + 6z + 25 = 0, z \in \mathbb{C}$$

2. Give the following complex numbers in polar form:
   a) $z = 4 + 3i$
   b) $z = 2i$
   c) $z = 4$
   d) $z = -2$
   e) $z = -1 - 2i$

3. Convert the to normal (rectangular) form:
   a) $5e^{i\pi}$
   b) $3e^{i\pi/2}$
   c) $2e^{i\pi/4}$
   d) $6e^{-i\pi/3}$
   e) $9e^{i0}$

4. Compute the inverse of $z = 3 + 4i$

5. Compute $z_1 = (\frac{1}{2} - 2i)^3$ and $z_2 = \sqrt{5 - 12i}$

Solutions

1. Using the quadratic formula (Mitternachtsformel), $z_{1/2} = -3 \pm 4i$.

2. Using equation (2), one can compute following results. Note that we can compute the solutions in Python using the ‘complex‘ keyword and the function ‘cmath.polar()‘.

```
from cmath import polar
print(polar(complex(4,3)))
```

   a) $r = 5,\ \varphi = 0.64 = 36.87°$
   b) $r = 2,\ \varphi = \pi/2 = 90°$
   c) $r = 4,\ \varphi = 0 = 0°$
   d) $r = 2,\ \varphi = \pi = 180°$
   e) $r = 2.24,\ \varphi = -2.03 = -116.57°$

3. This time, we use equation (1), or, alternatively, the following Python code.

```
from cmath import rect
print(rect(r=5, phi=np.pi))
```

   a) $-5(\cos(\pi) + i\sin(\pi)) = -5 + 0i = -5$
   b) $3i$
   c) $\sqrt{2} + \sqrt{2}i \qquad$ Note: $\cos(\pi/4) = \sin(\pi/4) = \frac{\sqrt{2}}{2}$
   d) $3 - 5.19j$
   e) 9

4. $\frac{1}{3+4i} = \frac{3-4i}{9+16} = 0.12 + 0.16i$

5. It is easy to take the power in polar form, because

$$(e^a)^b = e^{a\cdot b} \quad \Rightarrow \quad (re^{i\varphi})^p = r^p \cdot (e^{i\varphi})^p = r^p \cdot e^{i\varphi\cdot p}$$

Therefore, we can write

$$\begin{aligned} z_1 &= \left(\sqrt{17/4}\right)^3 e^{i\cdot 3\cdot \arctan(4)} \\ &= -5.875 - 6.5i \end{aligned}$$

$$\begin{aligned} z_2 &= (5-12i)^{1/2} = 13^{0.5} \cdot e^{i\cdot 0.5\arctan(-5/12)} \\ &= 3 - 2i \end{aligned}$$

Again, we can use Python to solve this problem:

```
z_1 = pow(complex(1/2,2),3)
z_2 = pow(complex(5,-12),0.5)
```