Life Sciences and
Facility Management

Institute of
Computational Life Sciences

Fourier Transformation

Image and Signal Processing

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Fourier series

Recap

Fourier series – Recap

Limitation: Periodic functions only!

Question: Can we generalize this?

- **Input**:
  - Real valued, periodic signals $x(t)$ with period T
- **Fourier series**:

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,dt$$

$$a_k = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos\left(2\pi\tfrac{k}{T}x\right)\,dt \qquad \text{for } k \geq 1$$

$$b_k = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin\left(2\pi\tfrac{k}{T}x\right)dx, \qquad \text{for } k \geq 1$$

$$x(t) \approx x_N(t) = a_0 + \sum_{k=1}^{N} a_k\cos\left(\frac{2\pi k}{T}t\right) + b_k\sin\left(\frac{2\pi k}{T}\right)$$

- **Output**:
  - Coefficients (e.g., a_k, b_k, and freqs. $\omega_k = 2\pi k/T$)
  - Approximation $x_N(t)$

- **Useful results:**
  - Amplitude and phase spectra
  - Symmetries simplify equations:
    - Odd functions: $a_k = 0$
    - Even functions: $b_k = 0$
  - Superposition principle applies
  - Energy in the time-domain signal is preserved in the frequency domain:
    $$\int_{-T/2}^{T/2} |x(t)|^2\,dt = 2\pi\sum_{k=-\infty}^{\infty}|c_k|^2 = \frac{1}{2}a_0^2 + \sum_{k=1}^{\infty}(a_k^2 + b_k^2)$$
    (Parseval's identity)
  - Lower frequency components are responsible for the general signal characteristics, higher frequencies for the details.

Example: Rectangular wave

Time-domain:

$$x_1(t) = \begin{cases} A, & |t| \leq \frac{r \cdot T}{2} \\ 0, & |t| > \frac{r \cdot T}{2} \end{cases}$$

Frequency-domain:

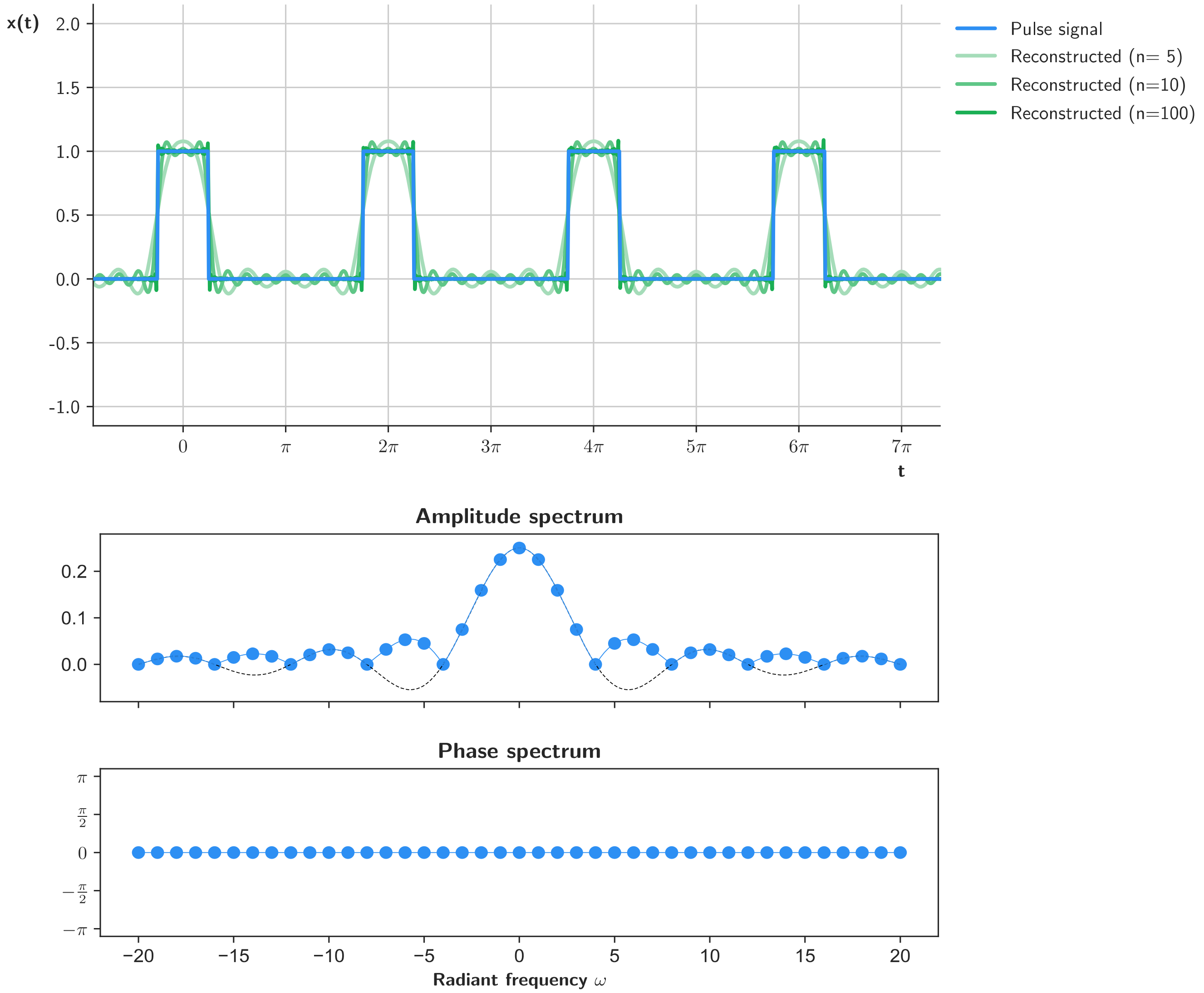
$$a_0 = Ar$$

$$a_k = 2Ar\frac{\sin(\pi nr)}{\pi nr}$$

$$b_k = 0$$

$$\omega_k = \frac{2\pi k}{T}$$

Question:
How to compute the spectra from these coefficients?



Example: Rectangular wave

Time-domain:

$$x_2(t) = \begin{cases} A, & 0 \le t < r \cdot T \\ 0, & r \cdot T \le t < T \end{cases}$$

Time-shifted version of $x_1(t)$:

$$x_2(t) = x_1(t - \tfrac{r}{2})$$

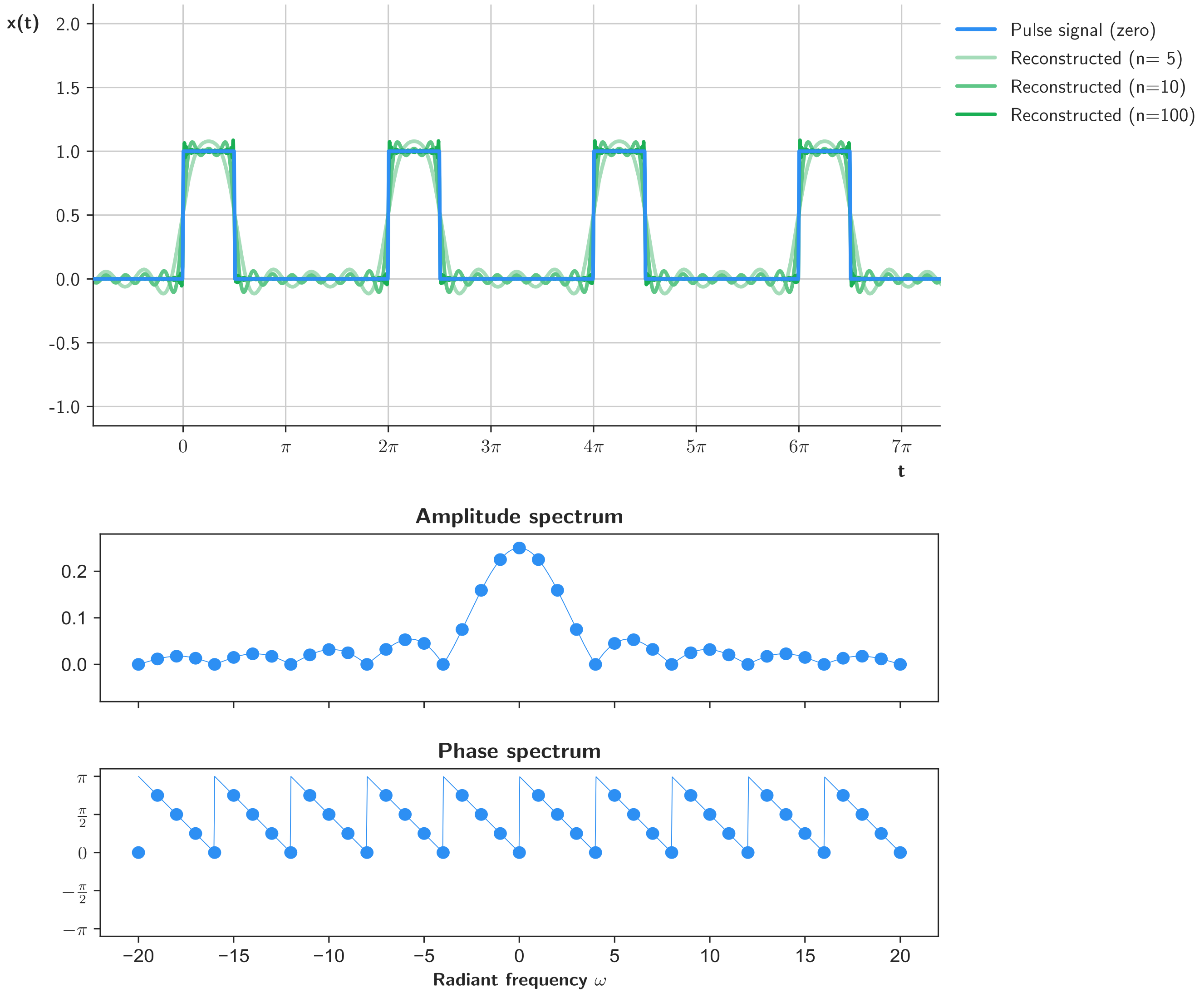
Frequency-domain:

$$a_0 = Ar$$

$$a_k = 2Ar\frac{\sin(2\pi kr)}{2\pi kr}$$

$$b_k = 2Ar\frac{\sin^2(\pi kr)}{\pi kr}$$

$$\omega_k = \frac{2\pi k}{T}$$



Example: Rectangular wave – Observations

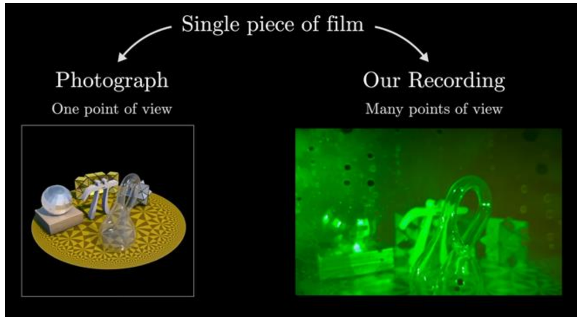
- Symmetry plays a role!
  - a_k are zero for odd functions
  - b_k are zero for even functions
- Time shift affects the phase, not the amplitude
- The phase of a signal encodes information about its timing (shift) and localization of features in the time domain.
- The **sinc function**: We encountered a function so fundamental that it was given a dedicated name.

$$x(t) = \mathrm{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$

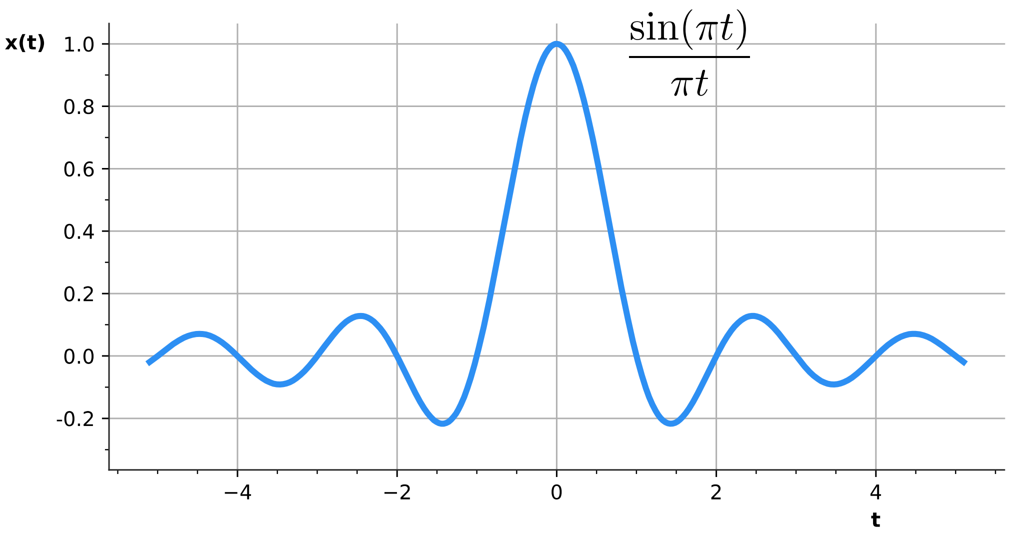
$$x(0) = \mathrm{sinc}(0) = 1$$

Excursion:

Holograms provide an impressive and illustrative example of how crucial phase information is in certain applications.



Video: [YouTube / 3blue1brown]



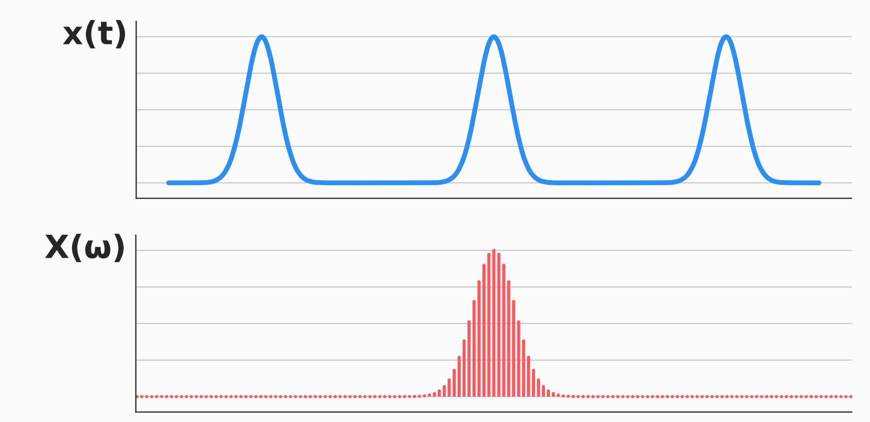
Fourier transformation

Generalization

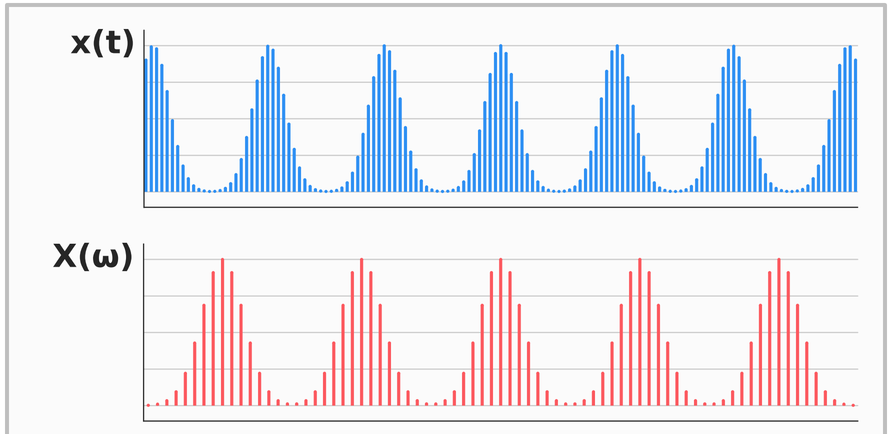
Fourier’s landscape

Time-continuous | *Time-discrete*

Periodic



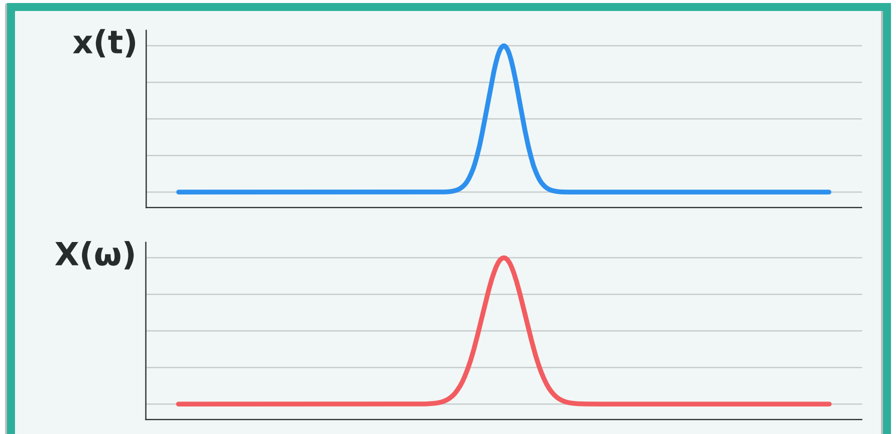
Fourier series (FS)



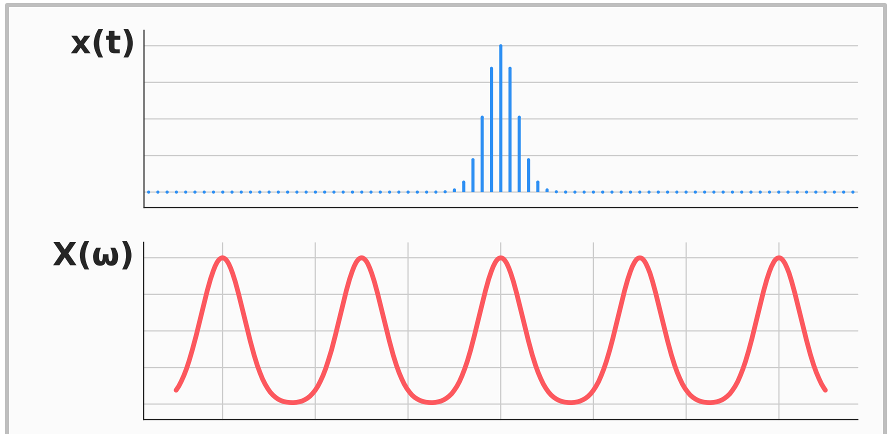
Discrete Fourier transform (DFT)

Related:
- **Fast Fourier transform**
- Discrete cosine transform
- Discrete sine transform

Aperiodic



Continuous-time Fourier transform (CTFT)



Discrete-time Fourier transform (DTFT)

Generalization:
Laplace Transform

Generalization:
z - Transform

Fourier transformation – Outline

- **Idea**: Treat the aperiodic function $x(t)$ like a periodic function with period $T \to \infty$ and then compute Fourier series. In the limit, this results in the Fourier transform.

- **Fourier transform**
$$\boxed{\hat{x}(k) = \int_{-\infty}^{\infty} x(t)e^{-ikt}\,dt} \quad k \in \mathbb{R}$$

- **Inverse Fourier transform**
$$\boxed{x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{x}(k)e^{ikt}\,dk}$$

$$c_k = \frac{1}{T}\int_0^T x(t) \cdot e^{-i2\pi\frac{k}{T}t}\,dt \quad \forall\, k \in \mathbb{Z}$$

$$x(t) = \sum_{k=-N}^{N} c_k e^{\frac{2\pi i k}{T}t}$$

Compare with Fourier series

- The Fourier transform extends the Fourier series to aperiodic functions. It allows us to analyze signals that don’t repeat periodically.
- Disclaimer: The integration works in many cases (but not all)

Immediate notes

- With the **Fourier transformation**, we bring a continuous-time signal into the frequency domain
- With the **inverse Fourier transformation**, we bring a frequency domain signal back into the time domain
- Alternative notations:
  - Exponential form:

$$\hat{x}(\omega) = X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-i\omega t}\, dt$$

  - Sine-cosine form:

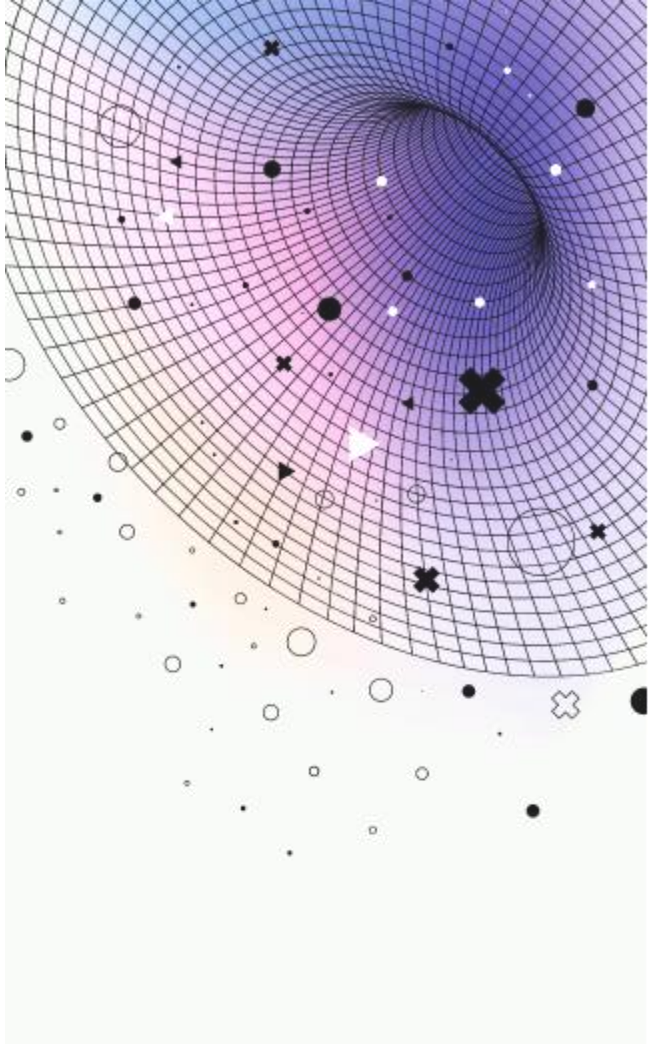
$$\hat{x}(\omega) = X(\omega) = \int_{-\infty}^{\infty} x(t)\cos(\omega t)\, dt - i\int_{-\infty}^{\infty} x(t)\sin(\omega t)\, dt = R(\omega t) + iI(\omega t)$$

If $x(t)$ is real-valued and **even**, the FT is a real-valued function!

$$X(\omega) = R(\omega t)$$

If $x(t)$ is real-valued and **odd**, the FT is imaginary-valued

$$X(\omega) = I(\omega t)$$



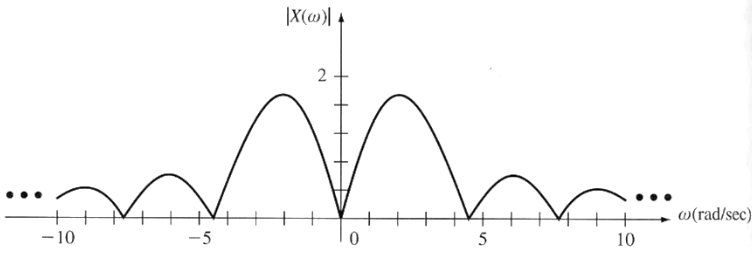
See the Fourier transform as a mathematical wormhole ☺

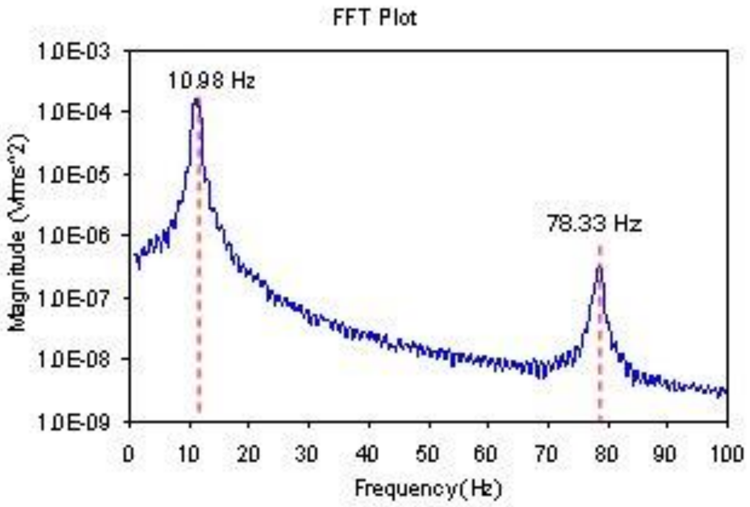
Fourier transform and spectra

- The Fourier transform $X(\omega)$ is generally a continuous, complex-valued function.
- Just as any complex number, the Fourier transform can be written in polar form:

$$X(\omega) = |X(\omega)|e^{i\varphi(\omega)}, \quad \varphi(\omega) = \arg(X(\omega))$$

- Both amplitude spectrum $|X(\omega)|$ and the phase spectrum $\varphi(\omega)$ are continuous functions in ω.
- **Note**: Negative frequencies are a “mathematical artifact” of Fourier theory. Hence, spectra often contain negative ω. However, it can be shown that for real-valued functions $x(t)$, the content of the FT at negative frequencies is redundant.



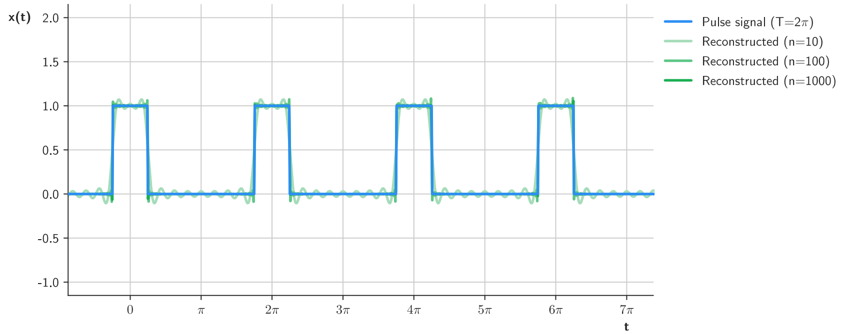


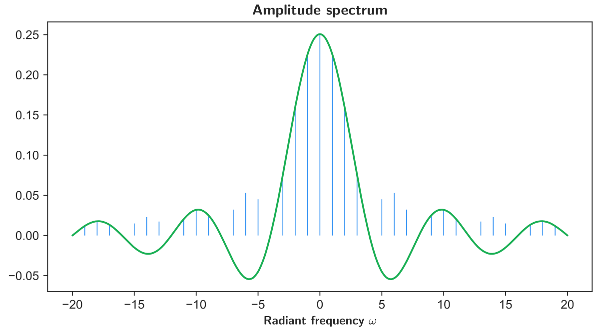
Examples of an amplitude spectrum of a continuous-time signal (top) and a power spectrum of the so-called impulse response of a system (bottom)

Further reading: here, and here

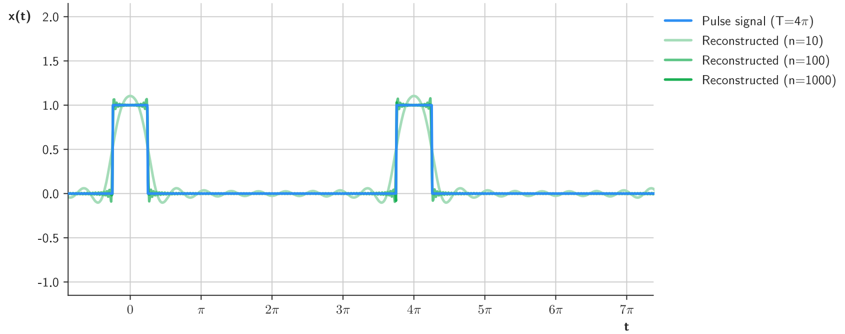
Illustration: Fourier series of the pulse wave for increasing T

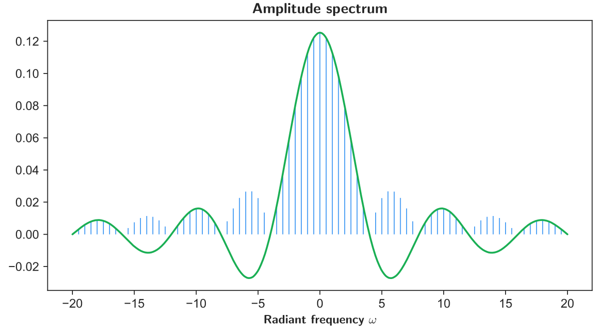
A = 1
T = 2π
r = ¼



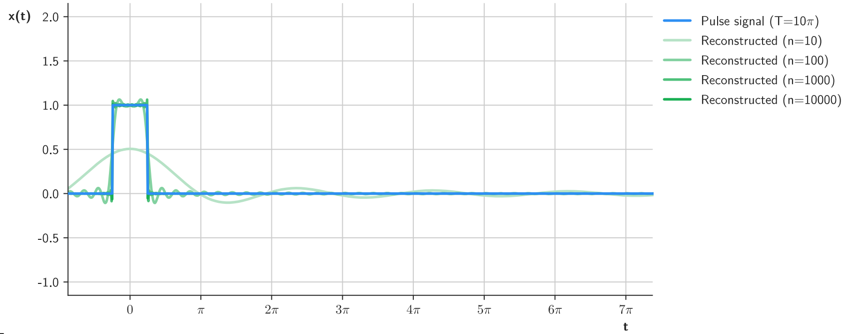


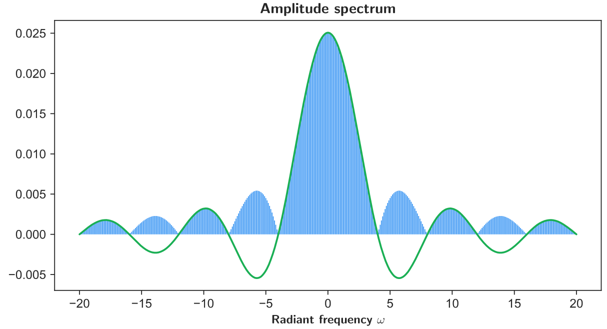
A = 1
T = 2·2π
r = ¼ / 2





A = 1
T = 10·2π
r = ¼ / 10





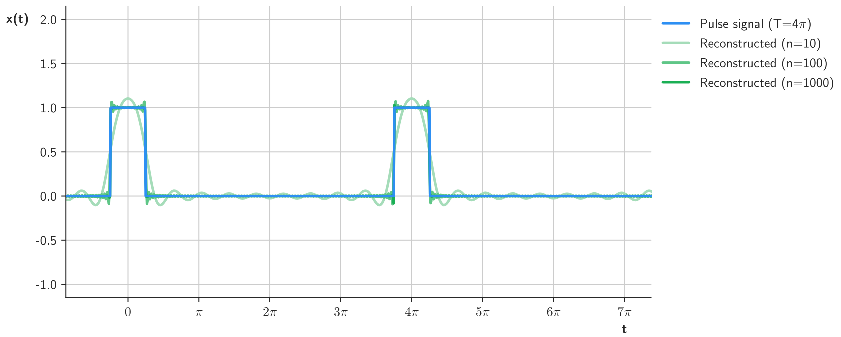
$$x(t) = \begin{cases} A, & |t| \le \frac{r \cdot T}{2} \\ 0, & |t| > \frac{r \cdot T}{2} \end{cases}$$

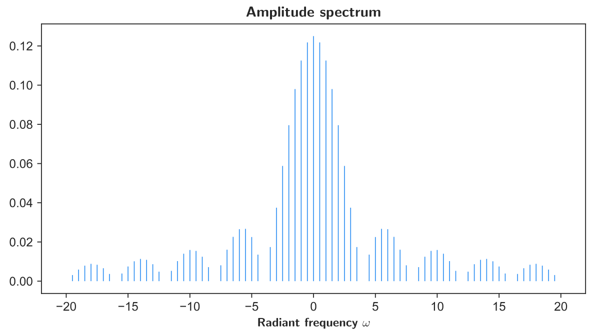
Observations:

- The larger the period, the denser the samples in the frequency domain
- The envelope is linked to the **Fourier transform** of the aperiodic pulse at $t = 0$.
- The discrete series coefficients form a continuity in the extreme case $t \to \infty$

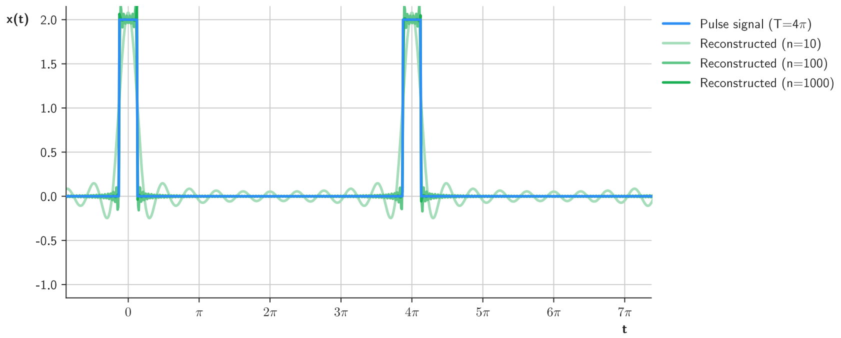
Illustration: Fourier series of the pulse wave for decreasing width

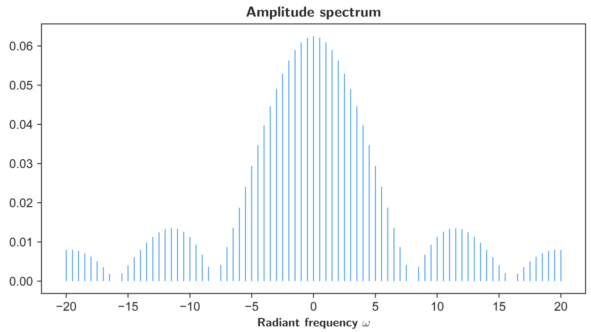
A = 2
T = 2·2π
r = ¼ / 2



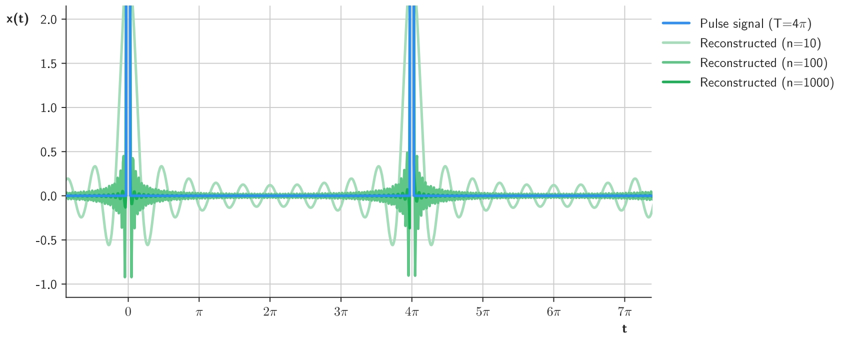


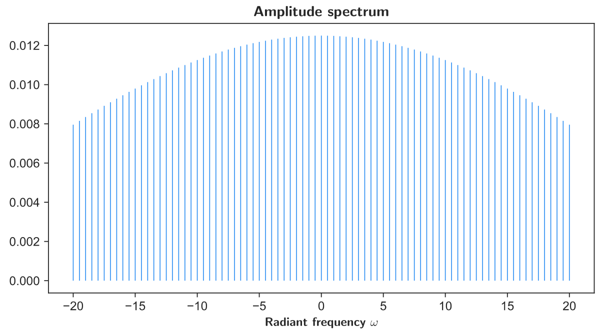
A = 4
T = 2·2π
r = ¼ / 4





A = 10
T = 2·2π
r = ¼ /10





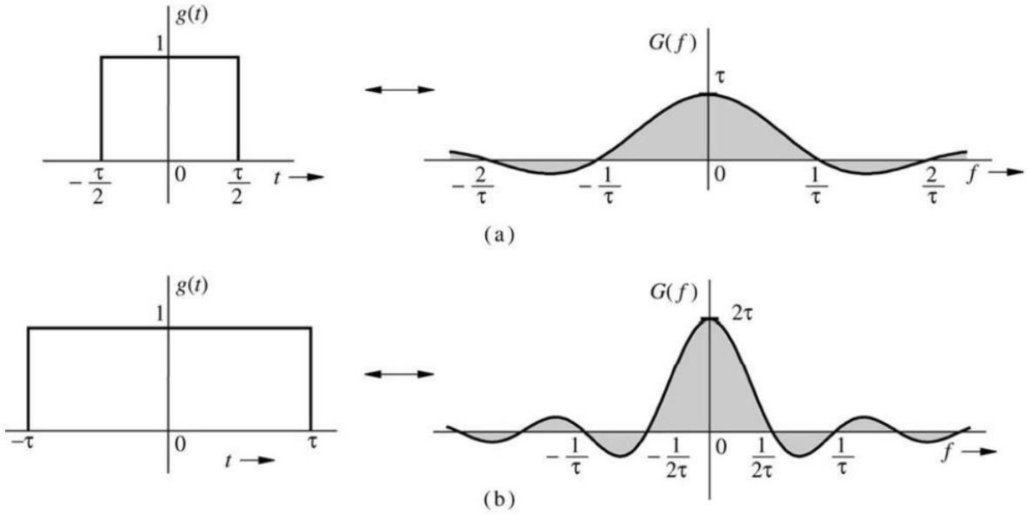
Observations:

- The narrower the pulses in the time-domain, the wider they are in the frequency domain
- In the extreme case, with infinitely thin pulses, the spectral coefficients are constant.

Example: Rectangular function

$$x(t) = \begin{cases} A, & -\tau/2 \leq t \leq \tau/2 \\ 0, & otherwise \end{cases}$$

$$\hat{x}(k) = \int_{-\infty}^{\infty} x(t) e^{-ikt}\, dt \quad k \in \mathbb{R}$$

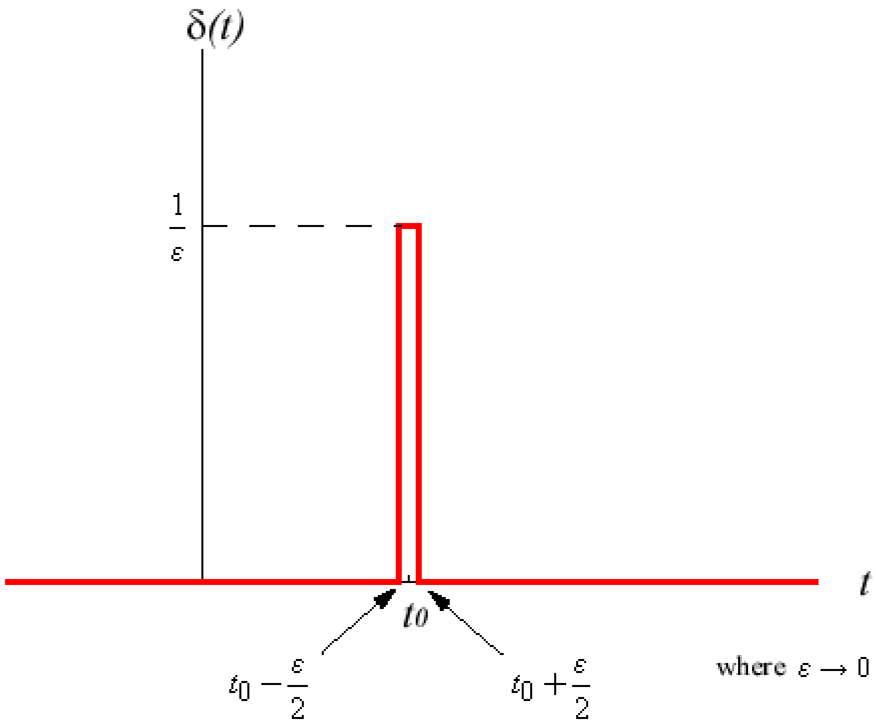


Example: Impulse function

- Idea:
  - Begin with a rectangular pulse of width 1 and area 1
  - Reduce the width, while keeping the area
  - Take this process to the extreme: $\epsilon \to 0$
  - Such a function is called impulse function $\delta\left(t - t_0\right)$

- Compute the Fourier transform of

$$\hat{x}(k) = \int_{-\infty}^{\infty} x(t)e^{-ikt}\,dt \quad k \in \mathbb{R}$$



Properties of Fourier Transforms

Properties

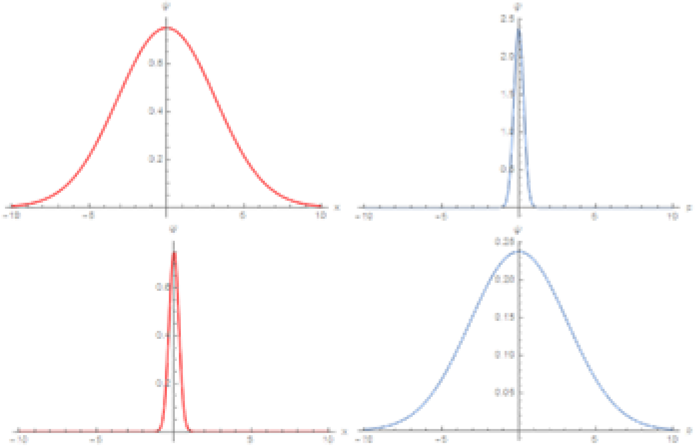
The Fourier transform offers many unexpected and beautiful properties. Here is a selection.

- **Uncertainty principle**: “Narrow” functions in the time domain appear as expanded in the frequency domain, and vice versa.

$$x(at) \quad \Leftrightarrow \quad \frac{1}{a} X\left(\frac{\omega}{a}\right)$$

- **Linearity**: The linear combination of two functions in the time domain is equivalent to the linear combination of their Fourier transforms in the frequency domain:

$$a \cdot x(t) + b \cdot y(t) \quad \Leftrightarrow \quad a \cdot X(\omega) + b \cdot Y(\omega)$$



Time domain

Frequency domain

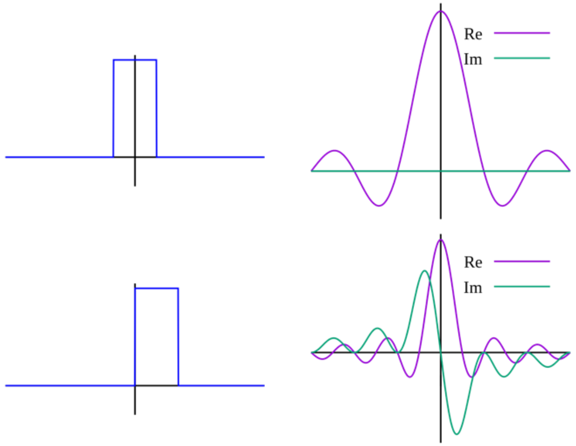
Properties

- **Time shift**: If a signal $x(t)$ is shifted by a time amount t_0 its Fourier transform is modified by a linear phase shift. The magnitude is not affected.

$$x(t - t_0) \quad \Leftrightarrow \quad X(\omega) \cdot e^{-i\omega t_0}$$

- **Differentiation** (in time domain): The differentiation of a function in time domain is equivalent to the multiplication of its Fourier transform by a factor $i\omega$ in frequency domain

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) \quad \Leftrightarrow \quad (i\omega) \cdot X(\omega)$$

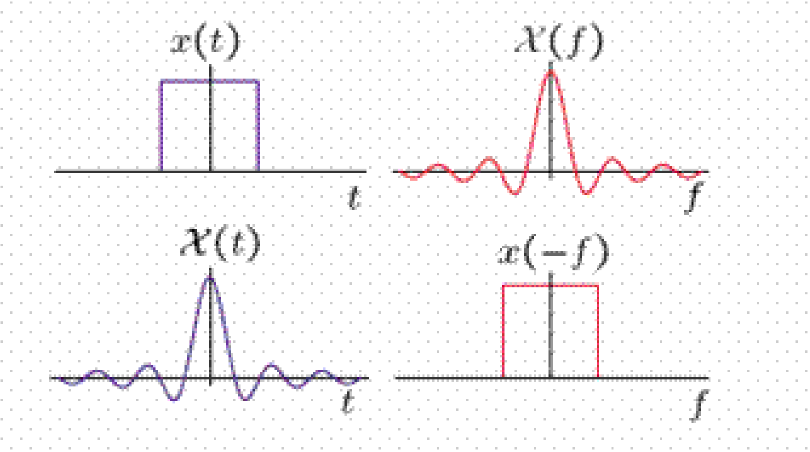


We've seen this effect already for the Fourier series of a rectangular wave

Properties

- **Duality**: If $x(t)$ has a Fourier transform $X(\omega)$, and we form a new function of time that has the functional form of the transform $X(\omega)$, then it will have a Fourier transform $x(\omega)$ that has the functional form of the original time function but is a function of frequency.

$$x(t) \leftrightarrow X(t) \quad \Leftrightarrow \quad X(t) \leftrightarrow 2\pi x(-\omega)$$



- **Energy preserving**: The total energy in a signal across all time is equal to the total energy in the transform across all frequencies (Parseval's theorem)

$$\int_{\mathbb{R}} |x(t)|^2 \, \mathrm{d}t = \int_{\mathbb{R}} |X(\omega)|^2 \, \mathrm{d}\omega$$

Fourier transform pairs

- Like with Fourier series, formularies exist for common Fourier transformations

Table of Common Functions and their Fourier Transforms

Function name	Function in the time domain	Fourier Transform (in the frequency domain)
	$\mathrm{w}(t)$	$\hat{\mathrm{W}}(\mathrm{f})$
Dirac delta	$\delta(t)$	1
Constant	1	$\delta(f)$
Cosine	$\cos(2\pi f_0 t)$	$\frac{\delta(f-f_0)+\delta(f+f_0)}{2}$
Sine	$\sin(2\pi f_0 t)$	$\frac{\delta(f-f_0)-\delta(f+f_0)}{2\mathrm{j}}$
Unit step function	$u(t)=\begin{cases}0, \text{if } t<0\\1, \text{if } t\geq 0\end{cases}$	$\frac{1}{j\omega}$ (for $\omega=2\pi f$)
Decaying exponential (for t > 0)	$e^{-\alpha t}u(t)$,	$\frac{1}{\alpha+j2\pi f}$, $\alpha>0$
Box or rectangle function	$\mathrm{rect}(at)=\begin{cases}0, \text{if } \lvert at\rvert>\frac{1}{2}\\1, \text{if } \lvert at\rvert\leq\frac{1}{2}\end{cases}$	$\frac{1}{\lvert a\rvert}\mathrm{sinc}\left(\frac{f}{a}\right)=\frac{\sin(\pi f/a)}{\pi f/a}$
Sinc function	$\mathrm{sinc}(at)=\frac{\sin(\pi at)}{\pi at}$	$\frac{1}{\lvert a\rvert}\mathrm{rect}\left(\frac{f}{a}\right)$
Comb function	$\sum_{n=-\infty}^{\infty}\delta(\mathrm{t}-nT)$	$\frac{1}{T}\sum_{\mathrm{k}=-\infty}^{\infty}\delta(f-\frac{k}{T})$
Gaussian	$e^{-\alpha t^2}$	$\sqrt{\frac{\pi}{\alpha}}e^{-\frac{(\pi f)^2}{\alpha}}$

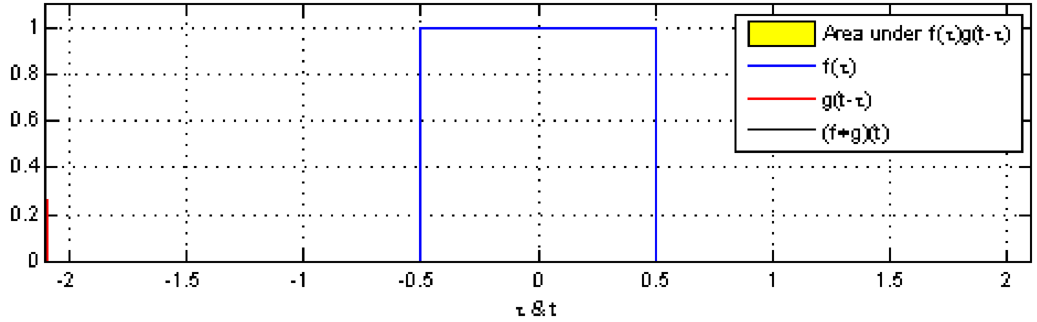
Convolution

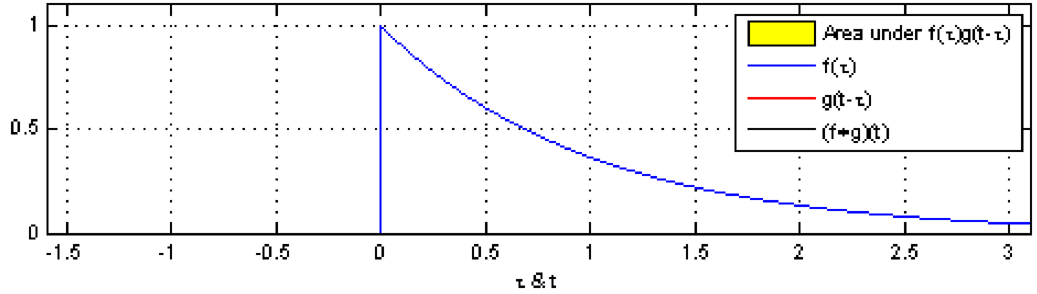
Convolution

- We still miss one important property of Fourier transformations.
- For this, we need to introduce first a new concept: The **convolution of two signals**:

$$y(t) = (h * x)(t) := \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\, d\tau$$

- The convolution is an operation that combines two functions to produce a third function.
- In words, integral computes the area under the product of $h(\tau)$ and $x(t-\tau)$ as τ varies.



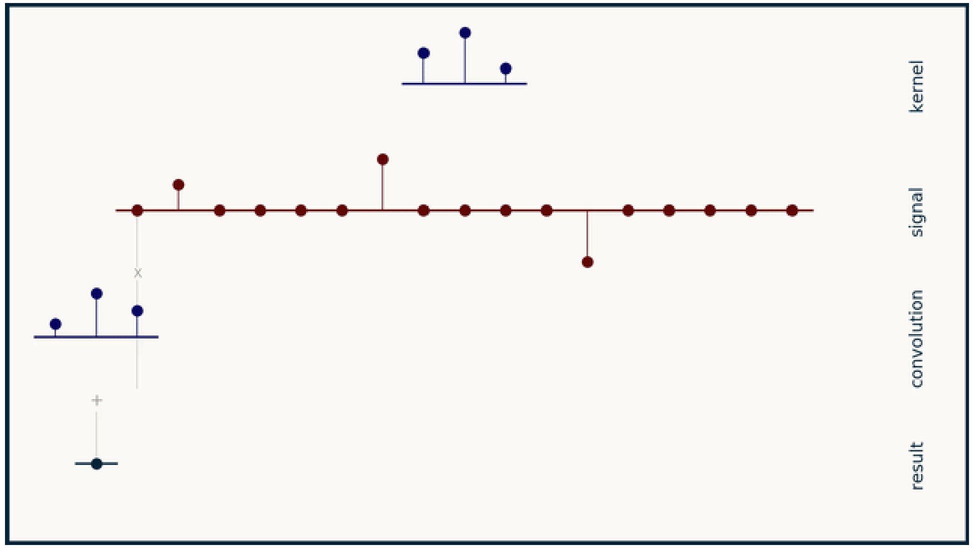


Convolution

- There exists also a discrete-time definition for convolution:

$$y[n] = (h * g)[n] = \sum_{m=-\infty}^{\infty} h[m]x[n-m]$$

- In the animation right is $x[n]$ the input signal, $h[n]$ the kernel, and $y[n]$ the output signal.



Animation: Link

Convolution and Fourier transform

- **Result**: The Fourier transform simplifies the convolution operation to a simple multiplication in the frequency domain!

$$y(t) = h(t) * x(t) \quad \Leftrightarrow \quad H(\omega) \cdot X(\omega)$$

- Since the convolution operation is relatively complicated to calculate the Fourier transform, the multiplication in the frequency domain and subsequent application of the inverse Fourier transform is much easier to achieve!
- Especially in connection with discrete-time signals and the Fast Fourier Transform (FFT), this property is extremely useful.
- Furthermore, convolution is the basis for many digital signal filters. More about this in the next lectures!