Life Sciences and
Facility Management

Institute of
Computational Life Sciences

Sampling

Image and Signal Processing

Norman Juchler

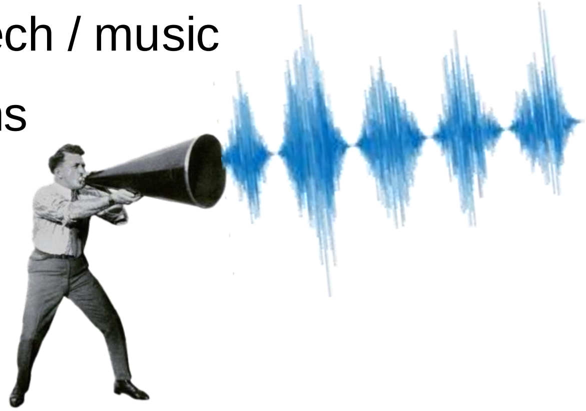
Continuous signals

Mathematical representation and modeling

Occurrence of continuous signals: Examples

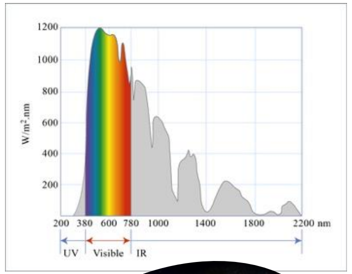
Sound waves

- Speech / music
- Sirens



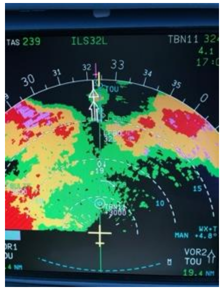
Light waves

- Video projection
- Photography
- Spectroscopy

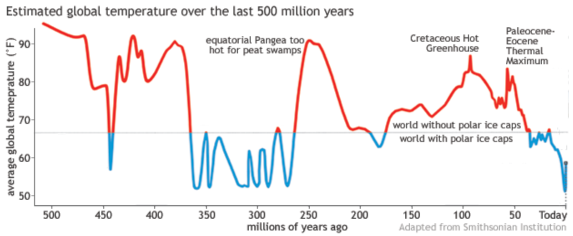


Electromagnetic waves

- Radio transmission
- Radar



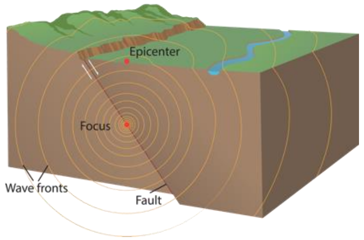
Temperature

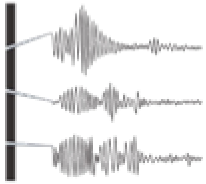


- Global surface temperature
- Body core temperature

Seismic waves

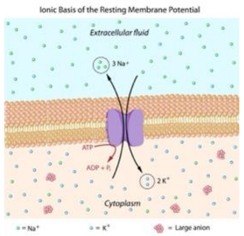
- Earthquake localization
- Volcanic activity

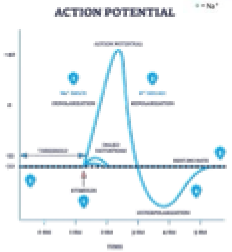




Biochemical signals

- Neural activity
- Hormones
- Cytokines

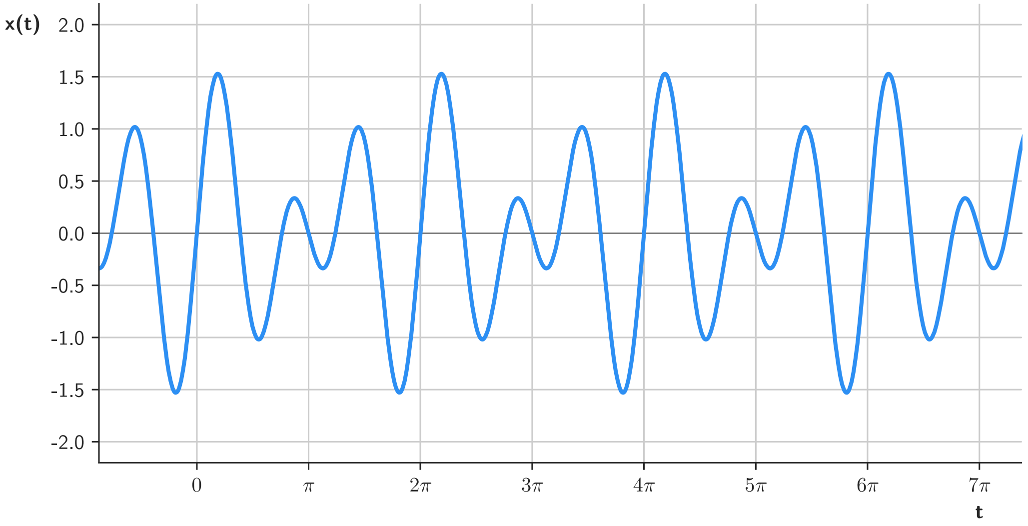




Signals as mathematical functions

Continuous-time: $x := x(t)$

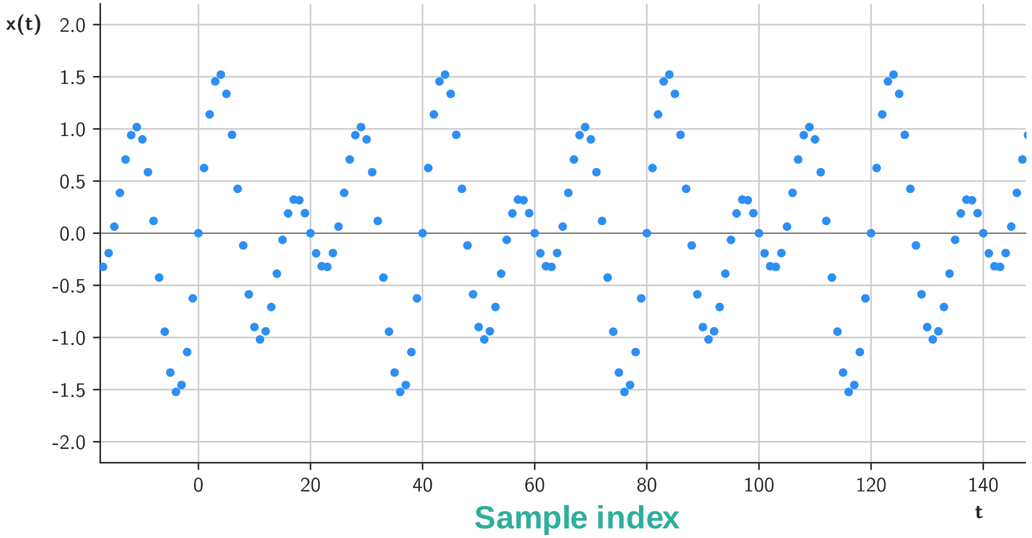
A signal can be represented as a mathematical function that maps a value x=x(t) for each time point t.



- t is the independent variable
- x is the dependent variable

Discrete-time: $x := x[n] = x(n \cdot T_s)$

A signal can also be represented as a discrete 1D sequence which is a function of an index variable n.



Sampling

- Period (s): T_s
- Frequency (Hz): $f_s = \frac{1}{T_s}$

Index equivalent to time:

- $t = n \cdot T_s$

Properties of signals (functions)

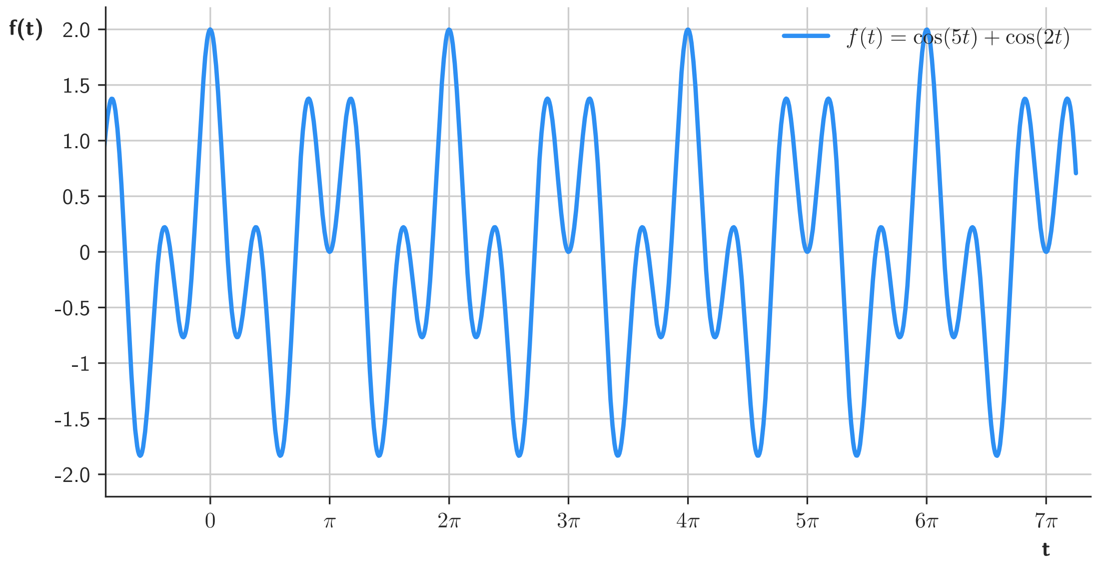
- **Shift**:
  - Shift along the horizontal axis $f_2(t) = f_1(t + \varphi)$
  - Shift along the vertical axis $f_2(t) = f_1(t) + f_0$
- **Scale**:
  - Scale along the horizontal axis: $f_2(t) = f_1(\omega \cdot t)$
  - Scale along the vertical axis: $f_2(t) = A \cdot f_1(\cdot t)$
- **Periodicity**:
  - f(t) is periodic (with period T) if: $f(t) = f(t + T)$
- **Symmetry**:
  - A function is called even if: $f(-t) = f(t)$
  - A function is called odd if: $f(-t) = -f(t)$

Examples:

- 01-shift-horizontal.gif
- 02-shift-vertical.gif
- 03-scale-horizontal.gif
- 04-scale-vertical
- 05-periodicity.gif
- 06-even.pdf
- 07-odd.pdf

Properties of signals (functions)

- Combinations of functions
  - Given: two functions $f_1(t)$ and $f_2(t)$
  - Calculate new functions by combining them
    - Addition: $f(t) = f_1(t) + f_2(t)$
    - Subtraction: $f(t) = f_1(t) - f_2(t)$
    - Multiplication: $f(t) = f_1(t) \cdot f_2(t)$
    - Division: $f(t) = f_1(t)/f_2(t) \quad f_2(t) \neq 0$
    - Chaining: $f(t) = f_2(f_1(t))$



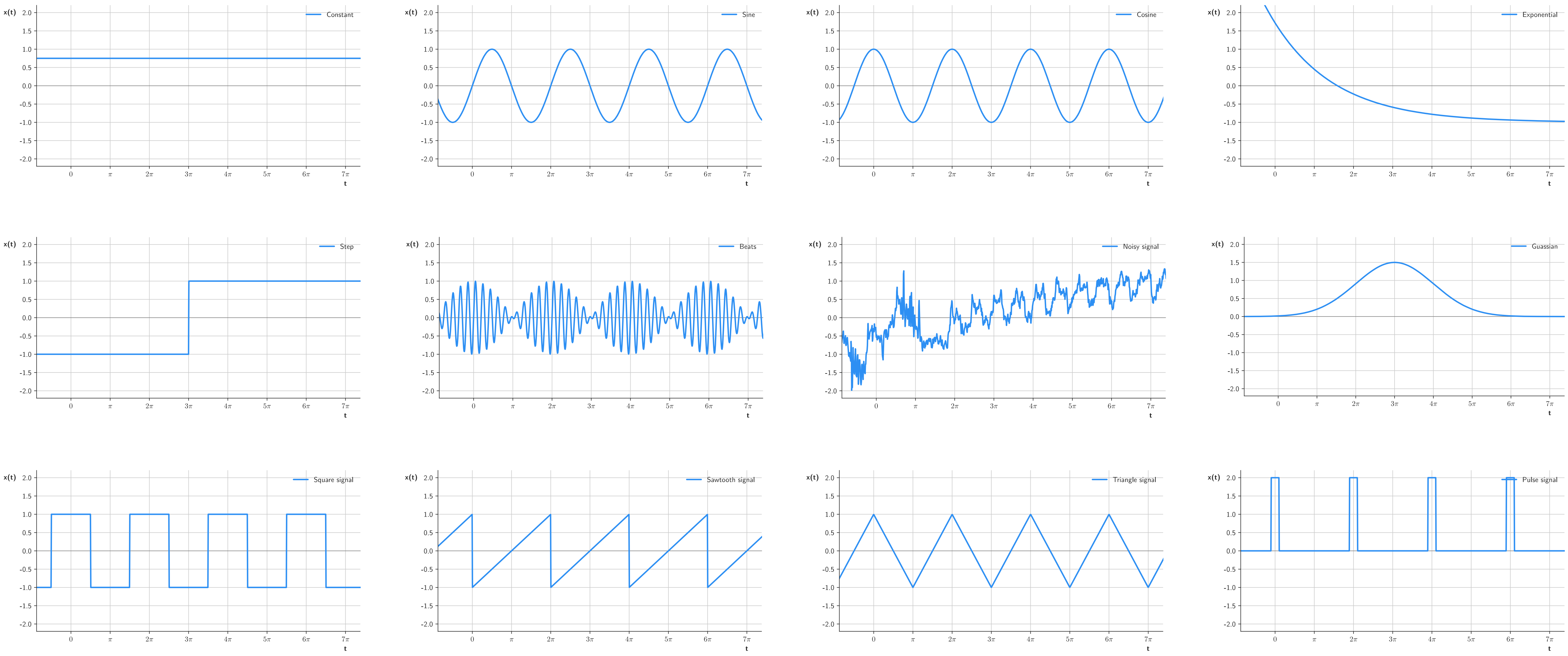
- Signals transmit energy. Using ideas from physics (or other domain knowledge), it is possible to derive the total energy of a signal as follows:

Continuous: $$E = \int_{-\infty}^{\infty} |x(t)|^2 \mathrm{d}t$$

Discrete: $$E = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

Exemplary signals

Task: Can you determine the properties? Symmetry axes, period? Can you even define the function?



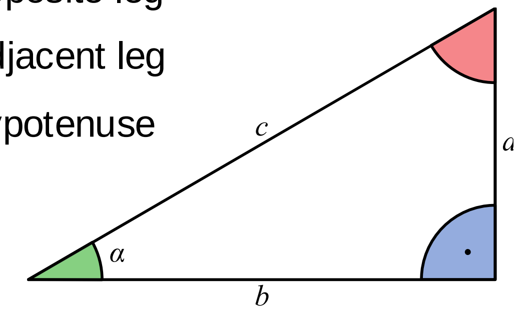
Sinusoidal signals

1: The sine and cosine are defined as ratios of the sides of a right-angled triangle.

a: opposite leg

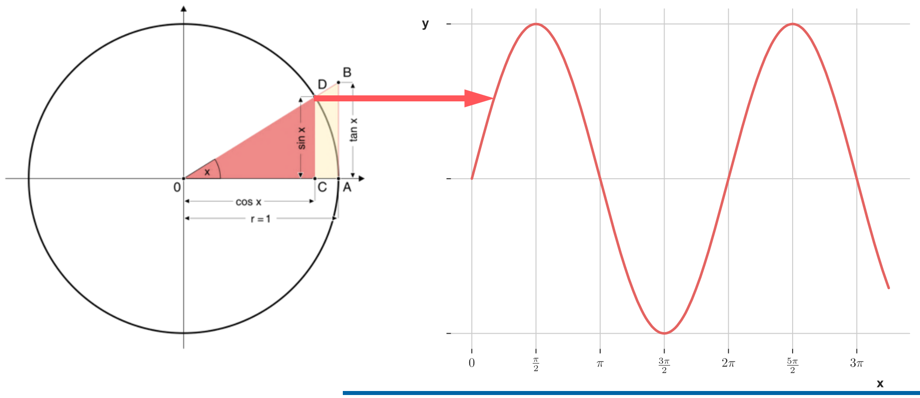
b: adjacent leg

c: hypotenuse



$$\sin(\alpha) = \frac{a}{c} \quad \cos(\alpha) = \frac{b}{c}$$

2: In the unit circle, we find the red right-angled triangle. Because c=1, $a = \sin(\alpha),\, b = \cos(\alpha)$

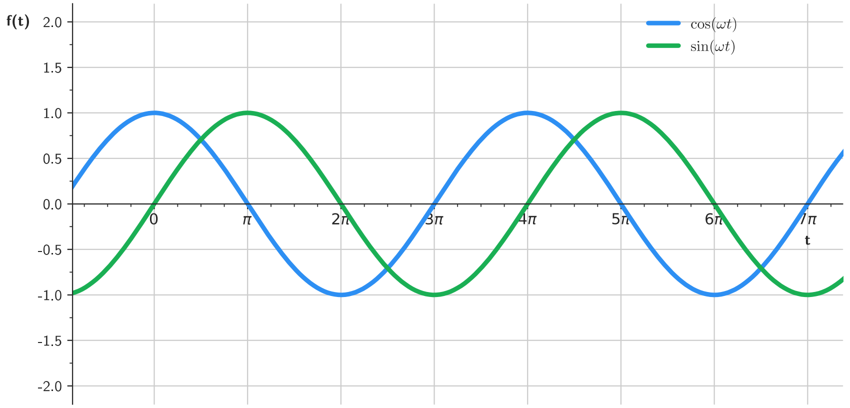


3: By increasing the angle α, we can draw $a = \sin(\alpha)$

4: Introduce params for frequency, phase and amplitude to obtain the general form

$$x(t) = A \cdot \sin(\omega t + \varphi)$$

$$x(t) = A \cdot \cos(\omega t + \varphi)$$



Sinusoidal signals (calculation rules)

- Trigonometric Pythagorean theorem

$$\sin^2(\omega t) + \cos^2(\omega t) = 1$$

- Cosine and sine are shifted by π/2

$$\sin(\omega t + \pi/2) = \cos(\omega t)$$
$$\cos(\omega t - \pi/2) = \sin(\omega t)$$

- Relationship between frequency and period:
  - Since it must hold that $x(t) = x(t+T)$ it follows that
  - f is the frequency
  - ω is the angular frequency
  - T is the period

$$\boxed{T = \frac{2\pi}{\omega} = \frac{2\pi}{2\pi f} = \frac{1}{f}}$$

Sinusoidal signals (differential equations)

- For the derivative of cos(ωt), it holds:

$$x_1(t) = \cos(\omega t)$$

$$\frac{\mathrm{d}x_1}{\mathrm{d}t}(t) = \dot{x}_1(t) = -\omega \sin(\omega t)$$

$$\frac{\mathrm{d}\dot{x}_1}{\mathrm{d}t}(t) = \ddot{x}_1(t) = -\omega^2 \cos(\omega t)$$

- Similar, it holds for sin(ωt):

$$x_2(t) = \sin(\omega t)$$

$$\ddot{x}_2(t) = -\omega^2 \sin(\omega t)$$

- This means that both $x_1(t)$ and $x_2(t)$ satisfy this differential equation:

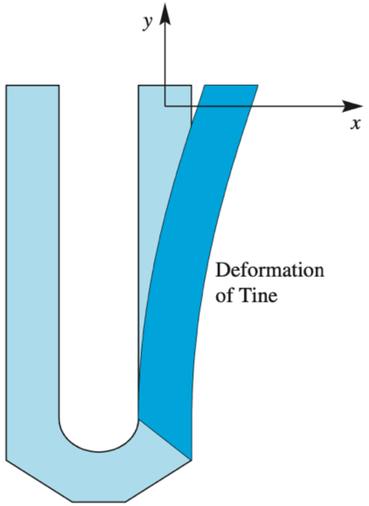
$$\boxed{\frac{1}{\omega^2}\ddot{x}(t) + x(t) = 0}$$

- It can be shown, that the following expression is a solution of the above differential equation

$$x(t) = c_1 x_1(t) + c_2 x_2(t), \quad c_1, c_2 \in \mathbb{R}$$

Sinusoidal signals: Example (Tuning fork)

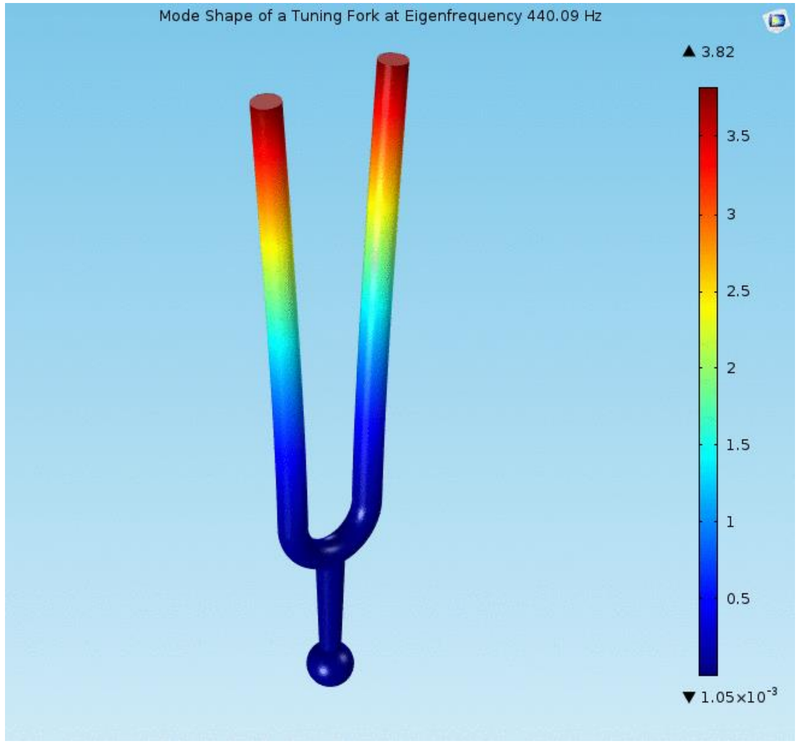
- A lot of natural phenomena can be described with the above differential equation.



$$F = -kx$$

$$F = ma = m\frac{d^2x}{dt^2}$$

$$m\frac{d^2x(t)}{dt^2} = -k\,x(t)$$

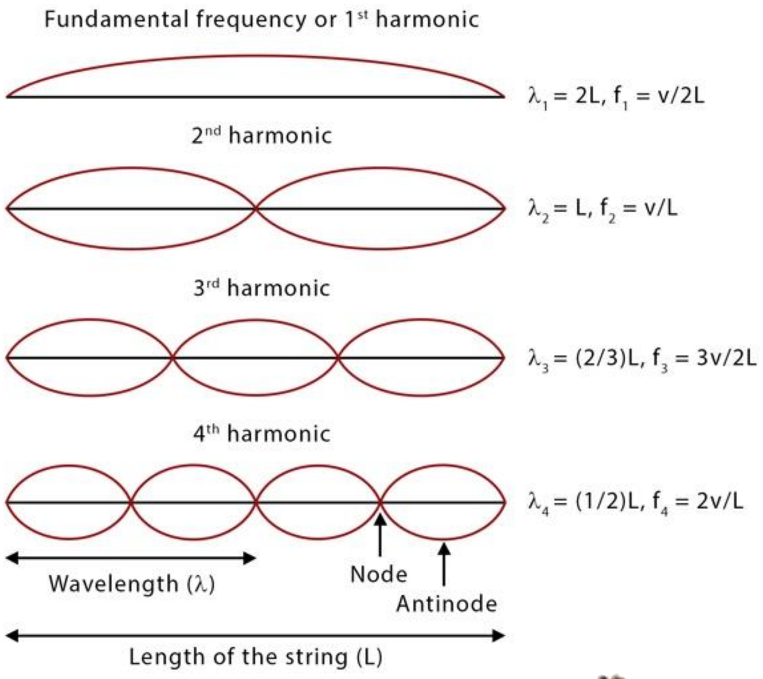


The visualization on the right is the result of an extended analysis of the same system. Here, the resonance coupling was taken into account.

Sinusoidal signals: Example (guitar string)

- A lot of natural phenomena can be described with the above differential equation.

When a guitar string is plucked, it produces standing waves. These waves include the fundamental tone (also known as first harmonic) and overtones, which are multiples of the fundamental frequency. Overtones give every musical instrument its unique sound. [Example on YouTube]



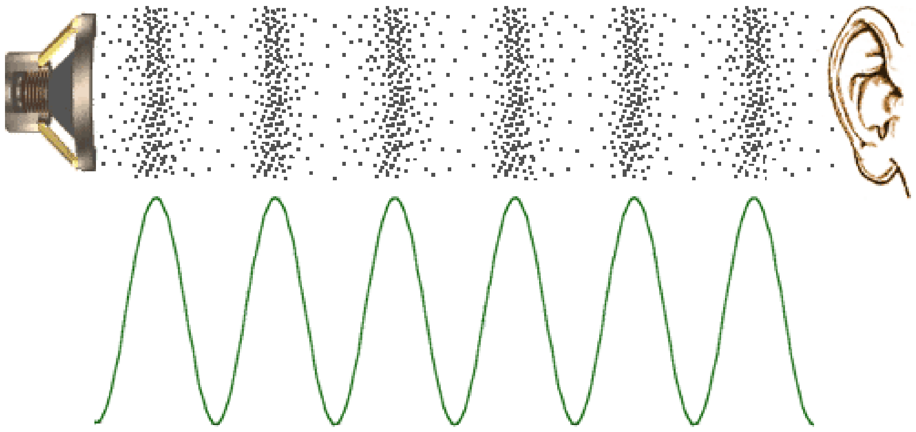
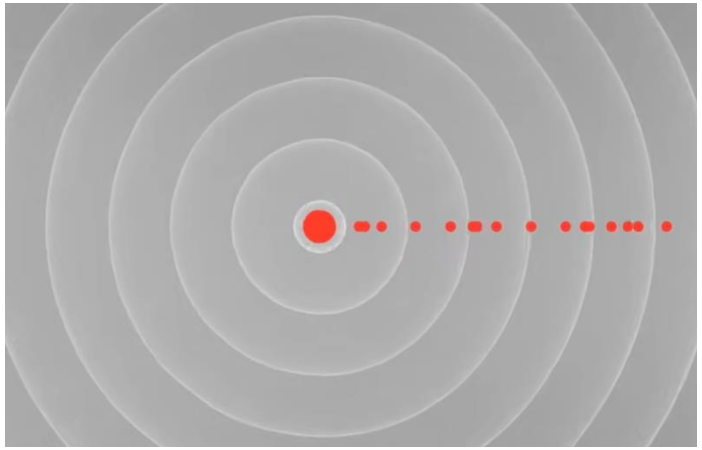


Illustration of the propagation of acoustic waves

Relationship: Frequency, wave speed and -length $\lambda = \frac{v}{f}$

...about acoustic waves

- Understanding sound waves: YouTube



- How to visualize sound: YouTube

The superposition principle

- **Context**:
  - Modelling of (signal generating) systems
  - Dynamic systems can be described using differential equations.
  - We already have seen an example earlier
- **Superposition principle**:
  - A system S is said to obey the superposition principle if its response to a sum of input signals is equal to the sum of the individual responses to each input signal.
  - All systems that obey the superposition principle are called linear systems
- Related: **Linear functions**
  - A function is called linear if it satisfies
  - Additivity: $f(x_1 + x_2) = f(x_1) + f(x_2)$
  - Homogeneity: $f(\alpha x) = \alpha f(x)$

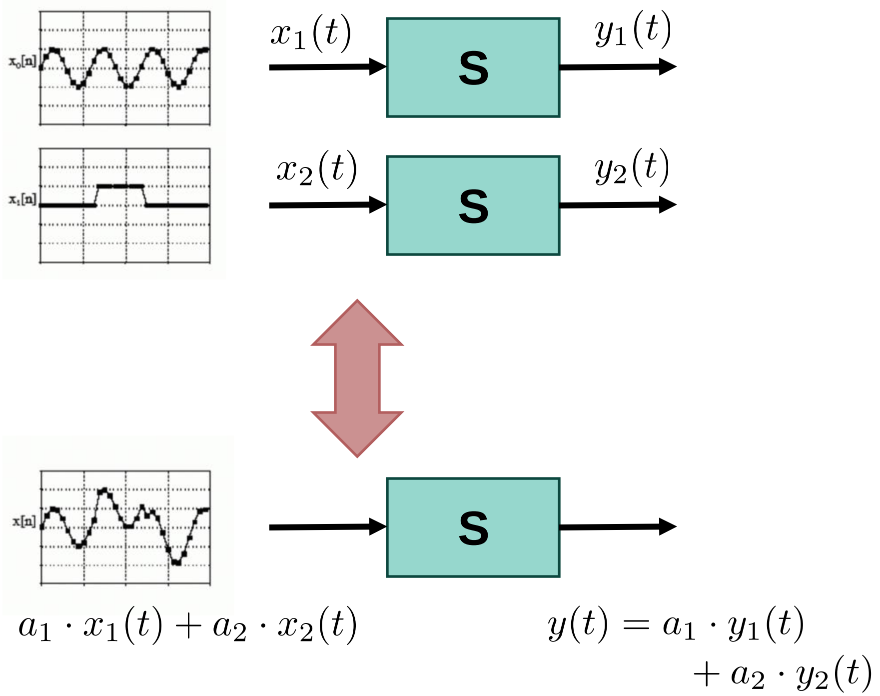


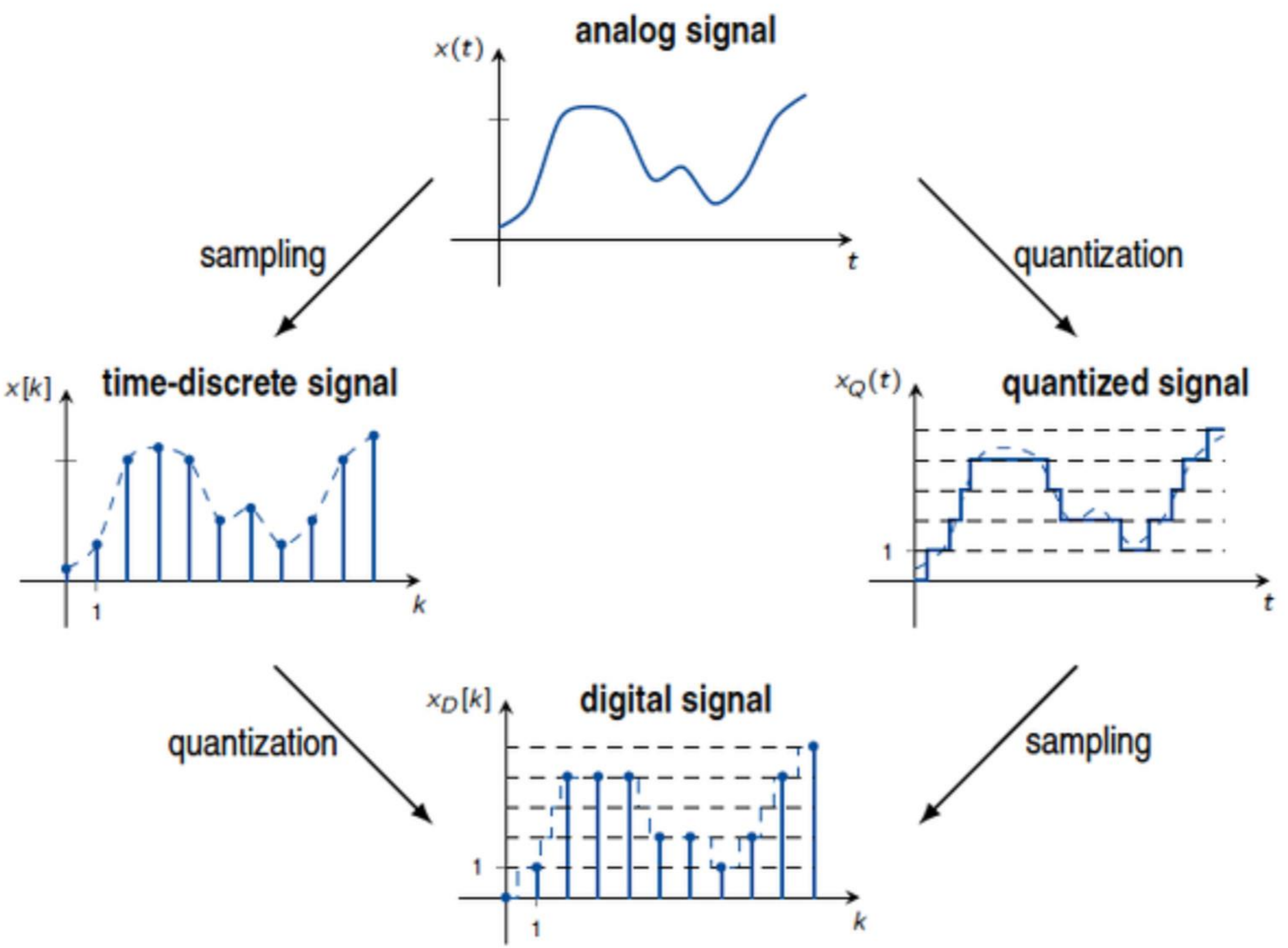
Illustration based on this source

Analog to digital conversion

From a continuous to a digital signal

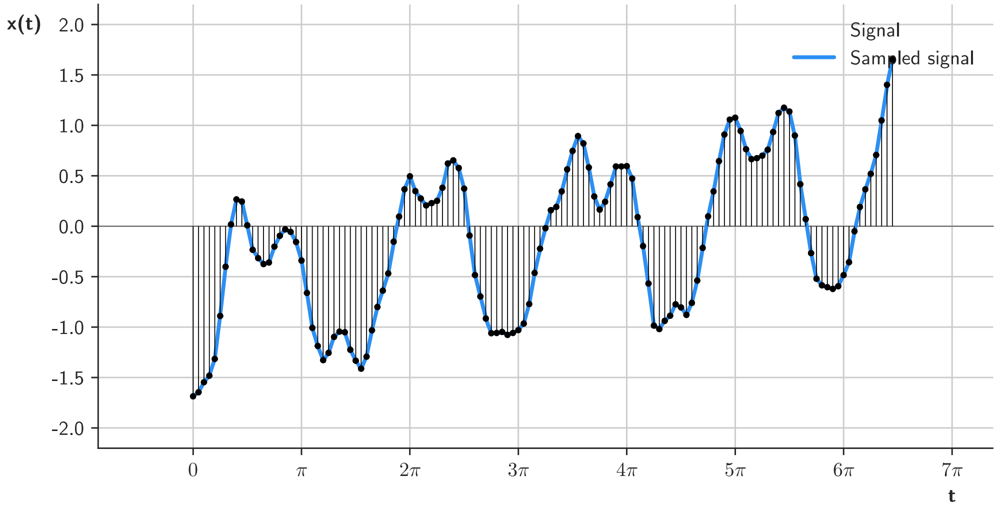
Discretization vs. quantization

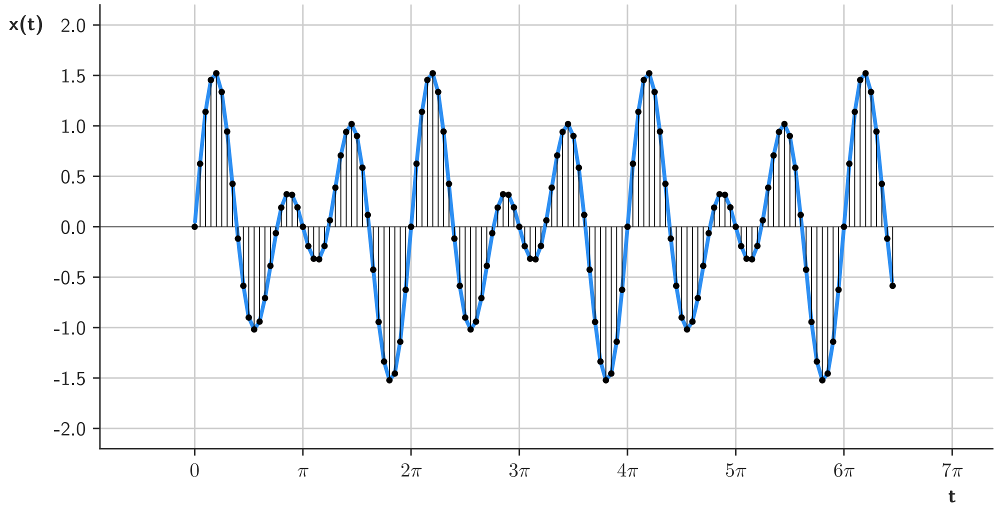
- **Discretization** divides signals into discrete intervals
- **Quantization** approximates continuous amplitudes using a finite set of values.
- Together, they enable efficient digital signal representation and processing
- Quantization error: The difference between the original signal and the quantized signal (rounding error)



Over- and undersampling

- **Sampling objective**: Capture the fundamental features of a signal without significantly compromising its characteristics due to the sampling process.
- The adjustable parameter is the **sampling rate** $f_s = 1/T_s$. We must strike a balance between capturing sufficient signal information and minimizing data storage and processing requirements.
- Situation on the right: **oversampling**. Everything is great.





Over- and undersampling

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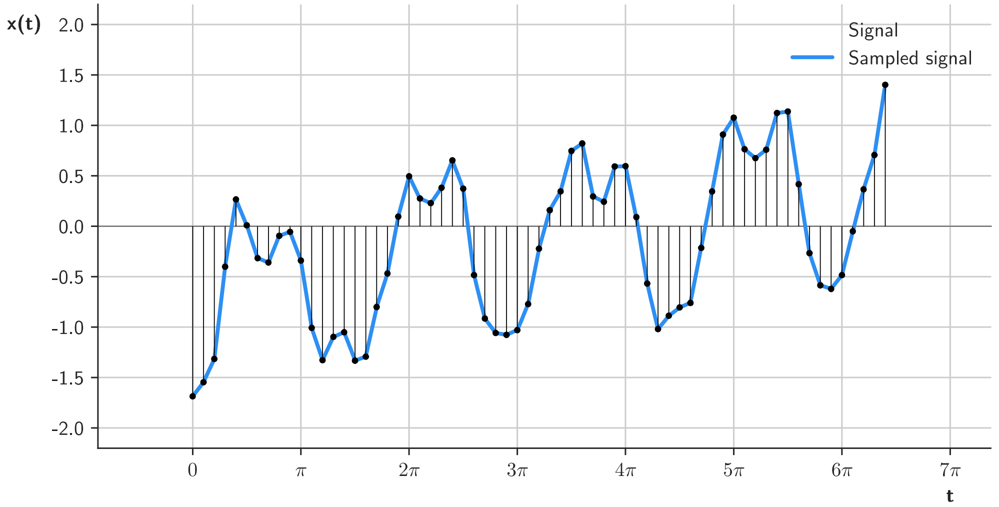
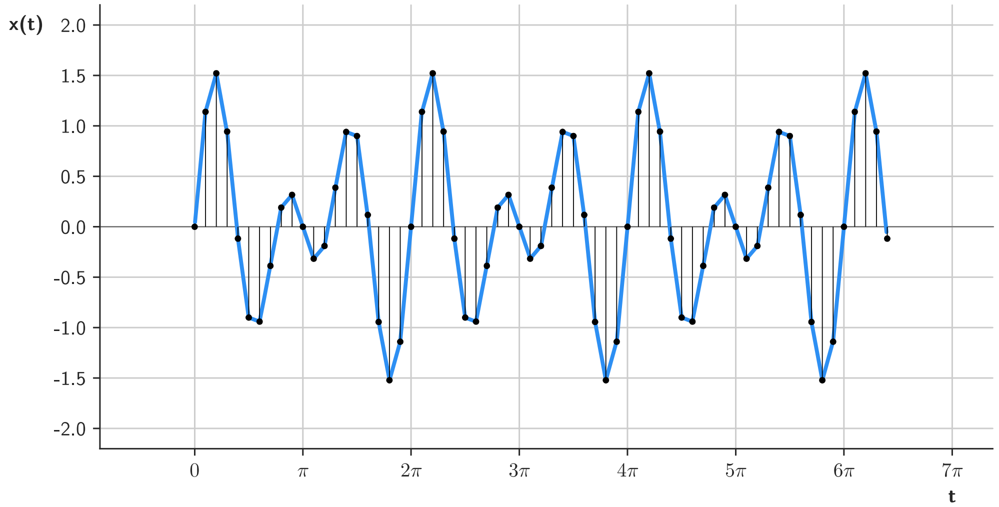
- The adjustable parameter is the **sampling rate** $f_s = 1/T_s$. We must strike a balance between capturing sufficient signal information and minimizing data storage and processing requirements.
- Situation on the right: **oversampling**. Everything looks okay.





Over- and undersampling

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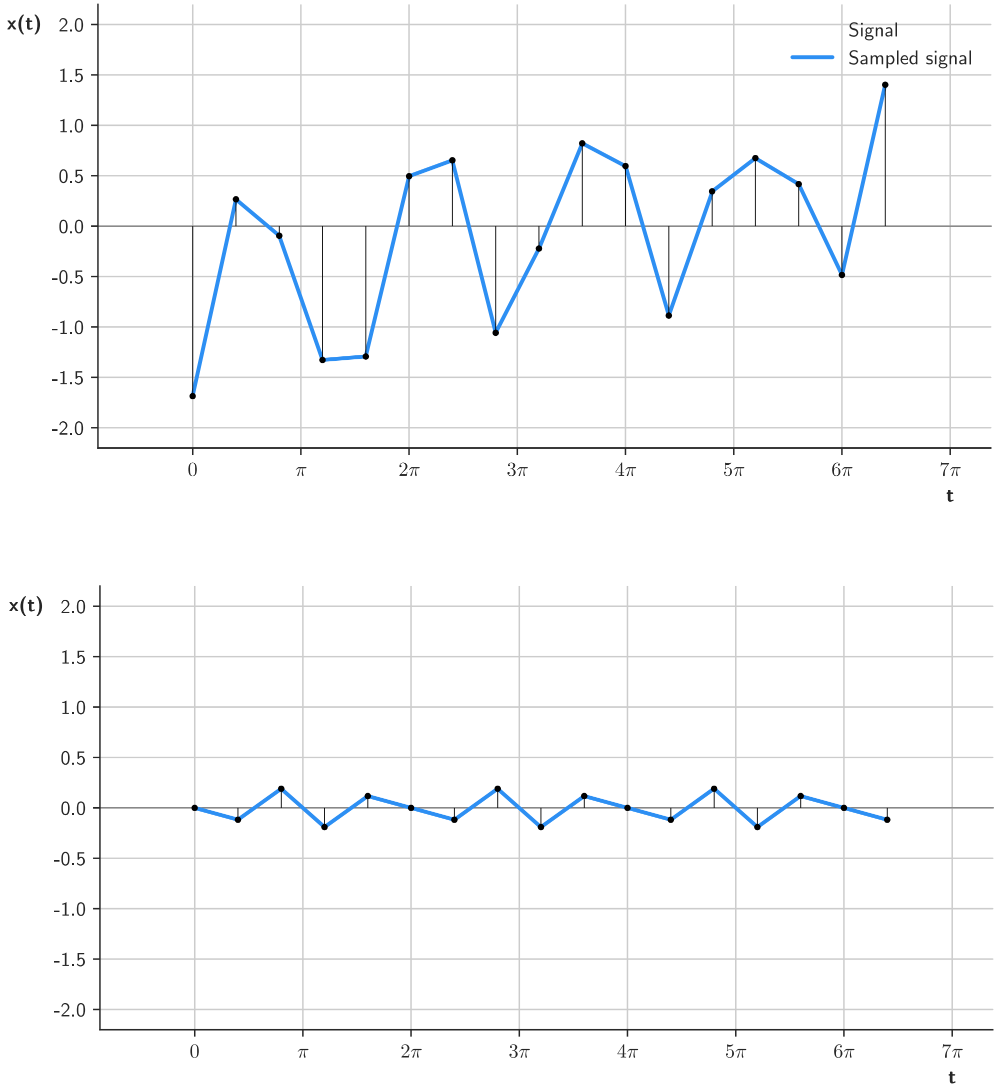
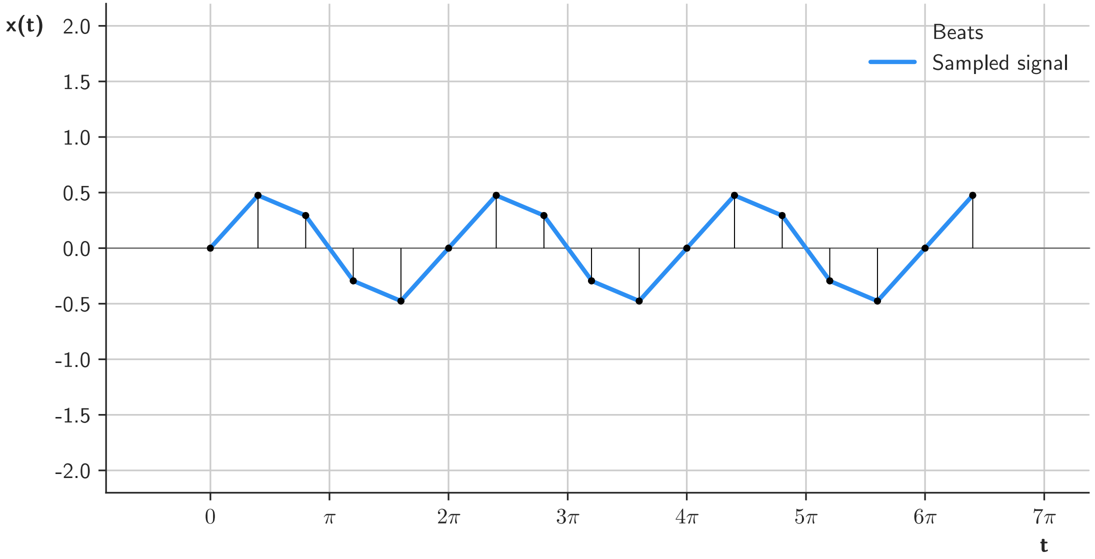
- The adjustable parameter is the **sampling rate** $f_s = 1/T_s$. We must strike a balance between capturing sufficient signal information and minimizing data storage and processing requirements.
- Situation on the right: **undersampling**. We have lost substantial information because the sampling rate is too small for our data.



Effects of undersampling: Loss of information and aliasing

- Undersampling occurs when the sampling rate is too low for a given signal or when the signal contains frequencies too high for the chosen sampling rate.
- This not only leads to **information loss** but can also introduce artificial frequencies (**aliasing**) that were not present in the original signal.
- Avoidance strategy: Low-pass filtering



- Videos illustrating aliasing:
  - Illustration of aliasing: YouTube (by Gergely)
  - A bit more technical: YouTube (by Zach Star)
    (No worries if you don't understand everything)

Nyquist–Shannon sampling theorem

- What is the natural boundary between over- and undersampling?
- The **Nyquist-Shannon sampling theorem** provides the answer:

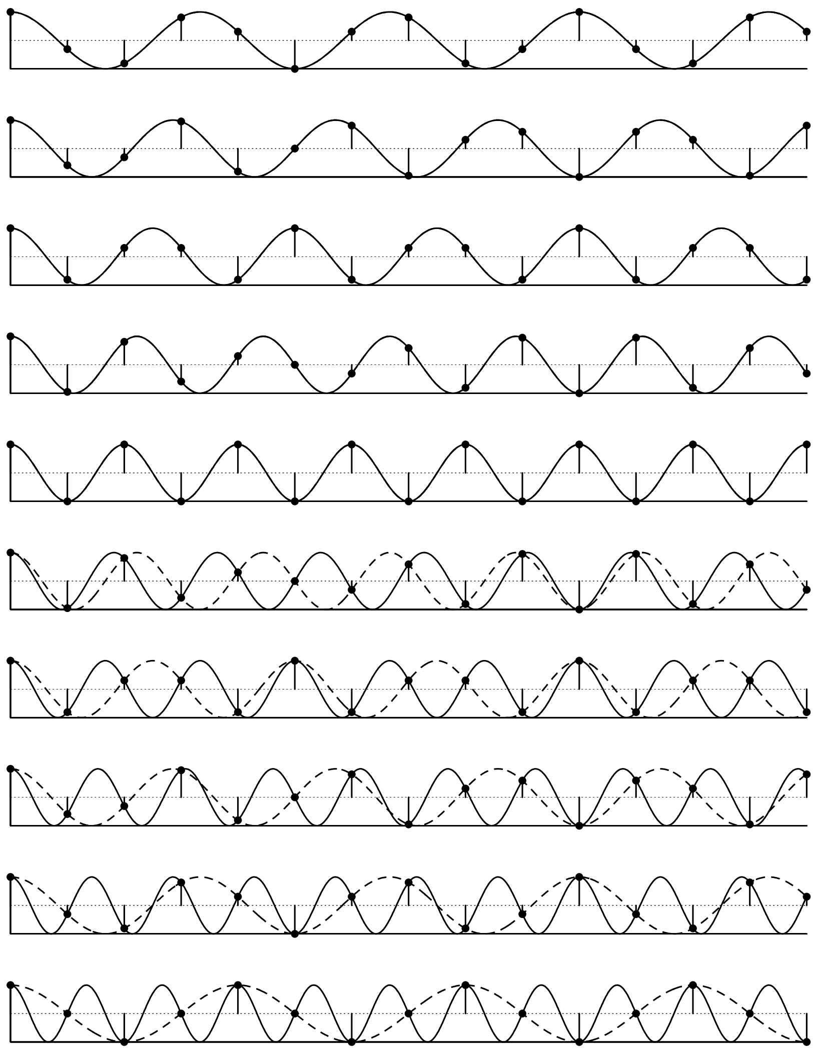
> The sample rate must be at least twice the maximal frequency of the signal to avoid the effects of undersampling: $f_s > 2f_{\max}$

- The theorem is a fundamental principle in digital signal processing!

- Alternative formulation: If a function $x(t)$ contains no frequencies higher than $f_{\max}$ Hertz, then it can be completely determined from sampling points spaced (strictly) less than $T_s < \frac{1}{2f_{\max}}$.

Sampling across the critical frequency

- The frequency of a sampled signal is increased from top to bottom, while the sampling frequency is constant.
- The dashed lines are possible aliases that would have the same samples.



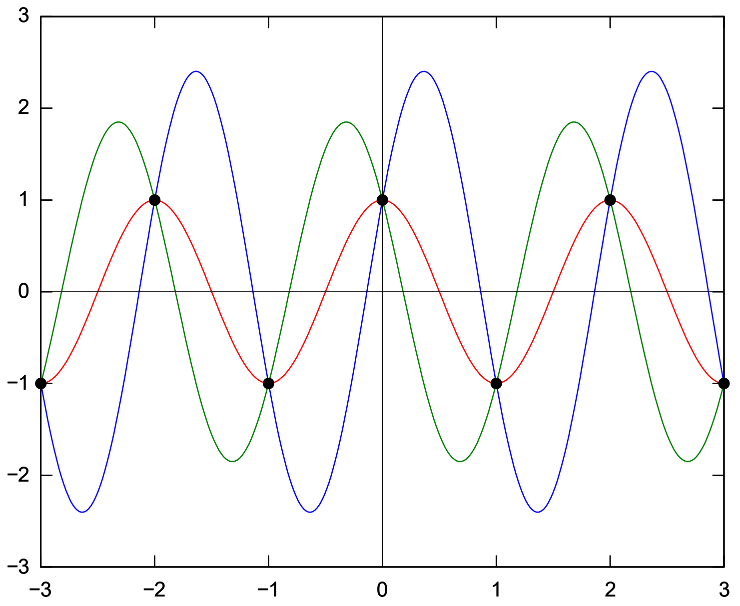
Tasks:

- Describe in your own words what happens.
- Which illustration shows the situation at the critical frequency?

Source: Wikimedia

Aliasing and the critical frequency

- The plot on the right shows
  - A sequence of sampling points (alternating +1 and -1)
  - Three sinusoids with identical frequency $f = \frac{1}{2} f_s$
- Observations:
  - All sinusoids match the sampling points (they are *aliases*)
  - We cannot uniquely fit a sinusoid at frequency $f = \frac{1}{2} f_s$
  - To determine the sine uniquely, we must sample with f strictly larger than $2f$: $\boxed{f_s > 2f}$



Source: Wikimedia

Tasks:

- What is the frequency f of the three sine curves?
- What is the relationship between f and ω in $\sin(\omega\, t)$?
- If f is the same for all three sines, then what changes?

Answers:

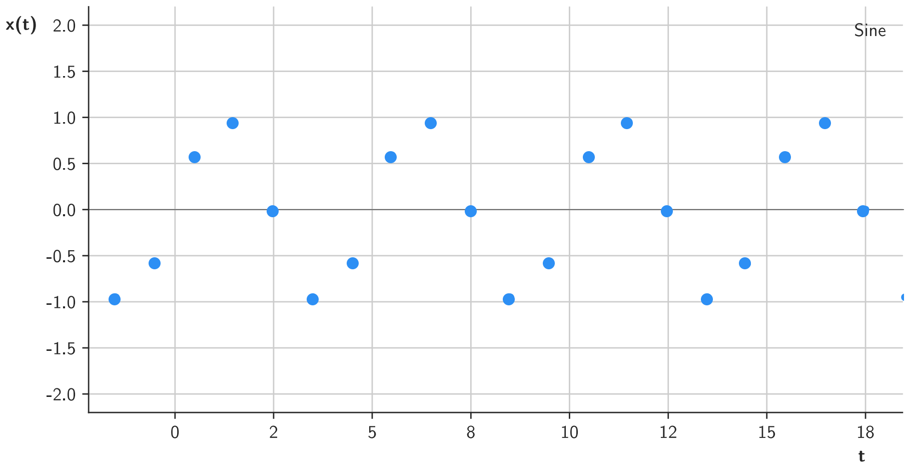
- The period is $T = 2$. Therefore $f = \frac{1}{T} = 0.5$
- $f = 2\pi\omega$
- The amplitude and the phase

Interpolation

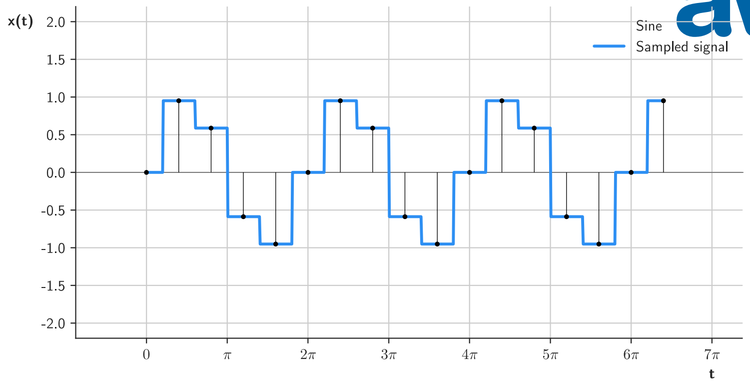
...undo sampling

Interpolation

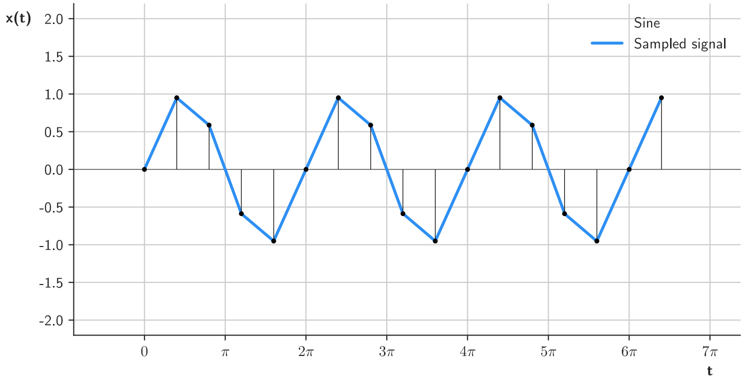
- **Definition**: Interpolation is the process of reconstructing a continuous-time signal x(t) from its discrete samples $x[n] = x(nT_s)$, where T_s represents the sampling interval.
- **Purpose**: Interpolation allows us to estimate the signal values at points between the original samples.
- **Input**:



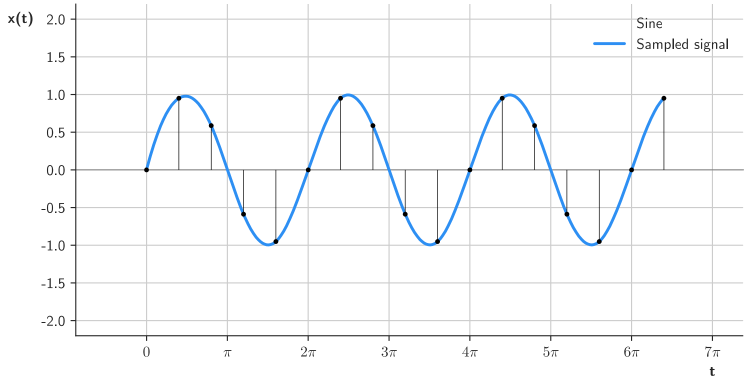
Nearest neighbor interpolation



Piecewise linear interpolation



Piecewise cubic interpolation



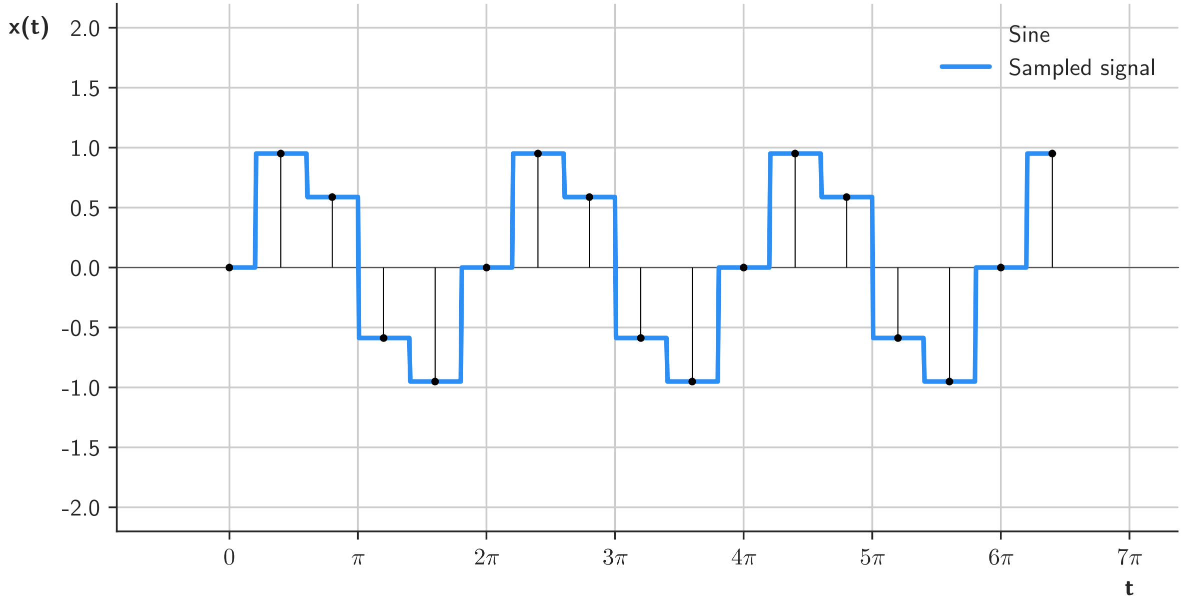
Nearest neighbor interpolation

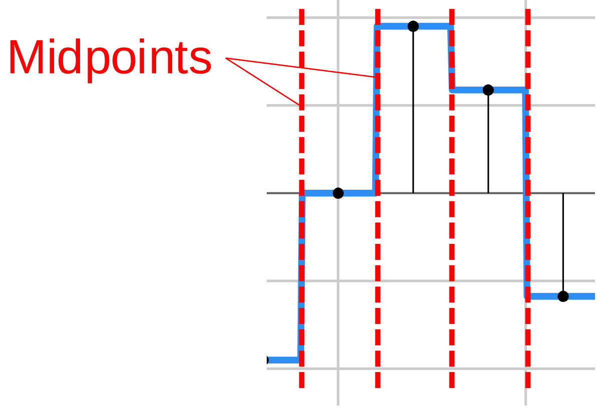
- Given data points (t[i],x[i]), the function approximates x(t) by assigning the value of the nearest known data point.

- The interpolated function is piecewise constant between the midpoints:

$$\hat{x}(t) = x[i], \quad \text{where } t \in \left[\frac{t[i-1]+t[i]}{2}, \frac{t[i]+t[i+1]}{2}\right)$$

- **Properties**: The function is discontinuous and not smooth at the midpoints





Piecewise linear interpolation

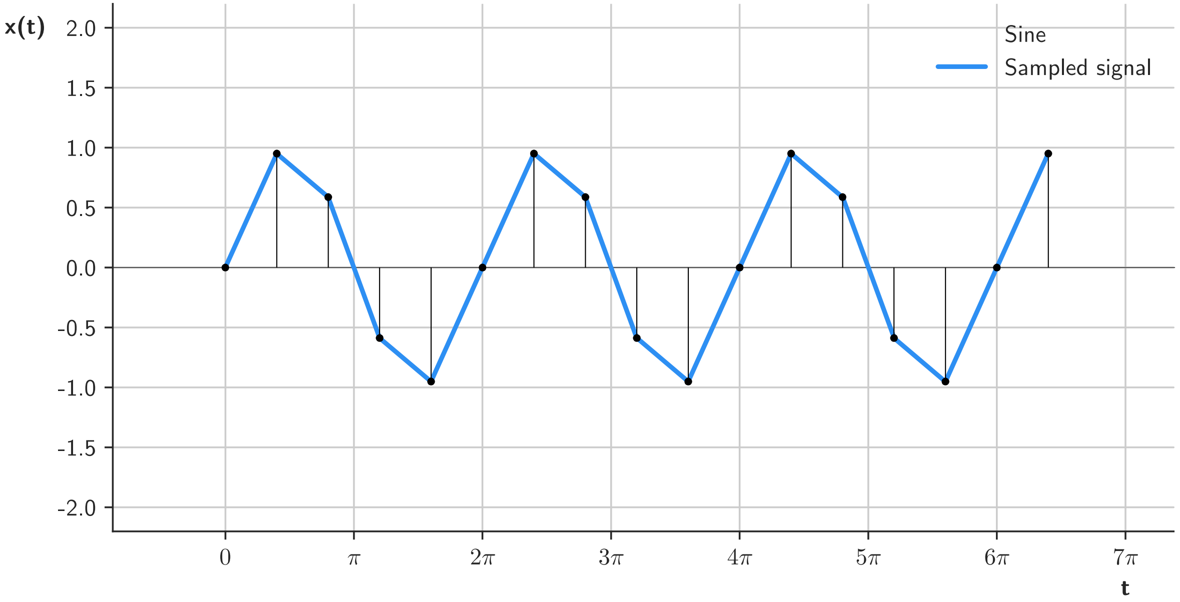
- Given data points (t[i],x[i]), the function linearly interpolates between adjacent points.

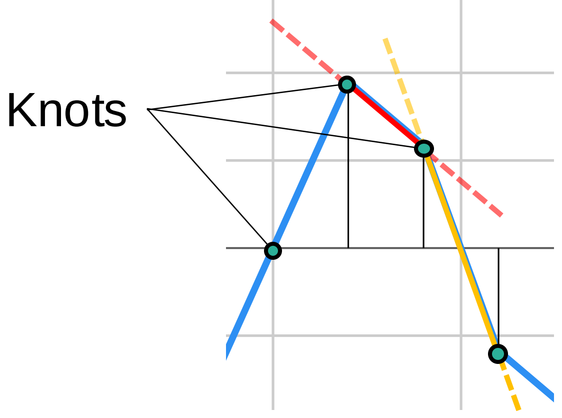
- Linear segments:

$$\hat{x}(t) = a_1(t - t[i]) + a_0, \quad t[i] \le t \le t[i+1]$$
$$x(t[i]) = x[i]$$
$$x(t[i+1]) = x[i+1]$$

- **Properties**: The function is continuous, but not differentiable at the knots x[i].

- Knots: Points at which the segments are "glued" together





Piecewise cubic interpolation

- Given data points (t[i],x[i]), the function interpolates using a cubic polynomial

- Cubic segments:

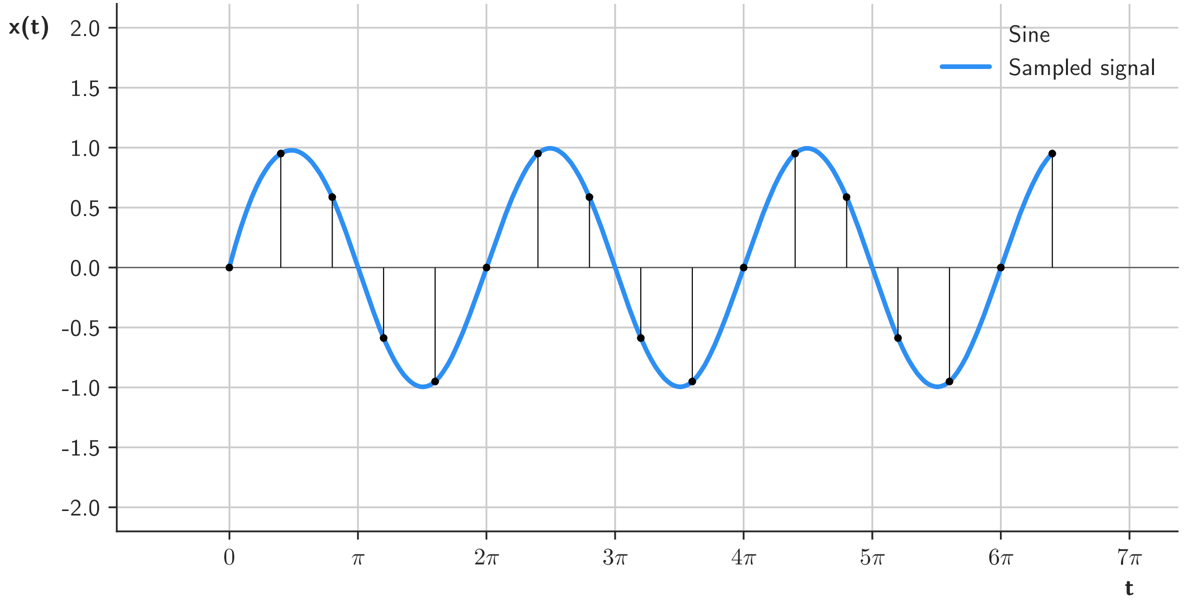
$$\hat{x}(t) = a_3(t - t[i])^3 + a_2(t - t[i])^2 + a_1(t - t[i]) + a_0,$$

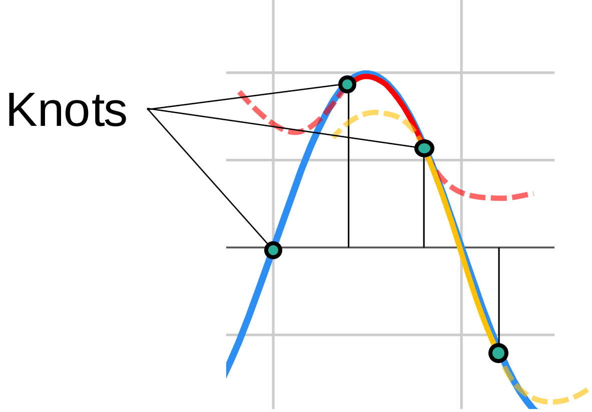
$$t[i] \leq t \leq x[i+1]$$

$$x(t[i]) = x[i], \quad x(t[i+1]) = x[i+1],$$

$$x'(t[i]) = \dots, \quad x'(t[i+1]) = \dots$$

- The coefficients are determined by ensuring continuity and smoothness (e.g., Hermite or spline conditions)





Whittaker–Shannon interpolation (sinc interpolation)

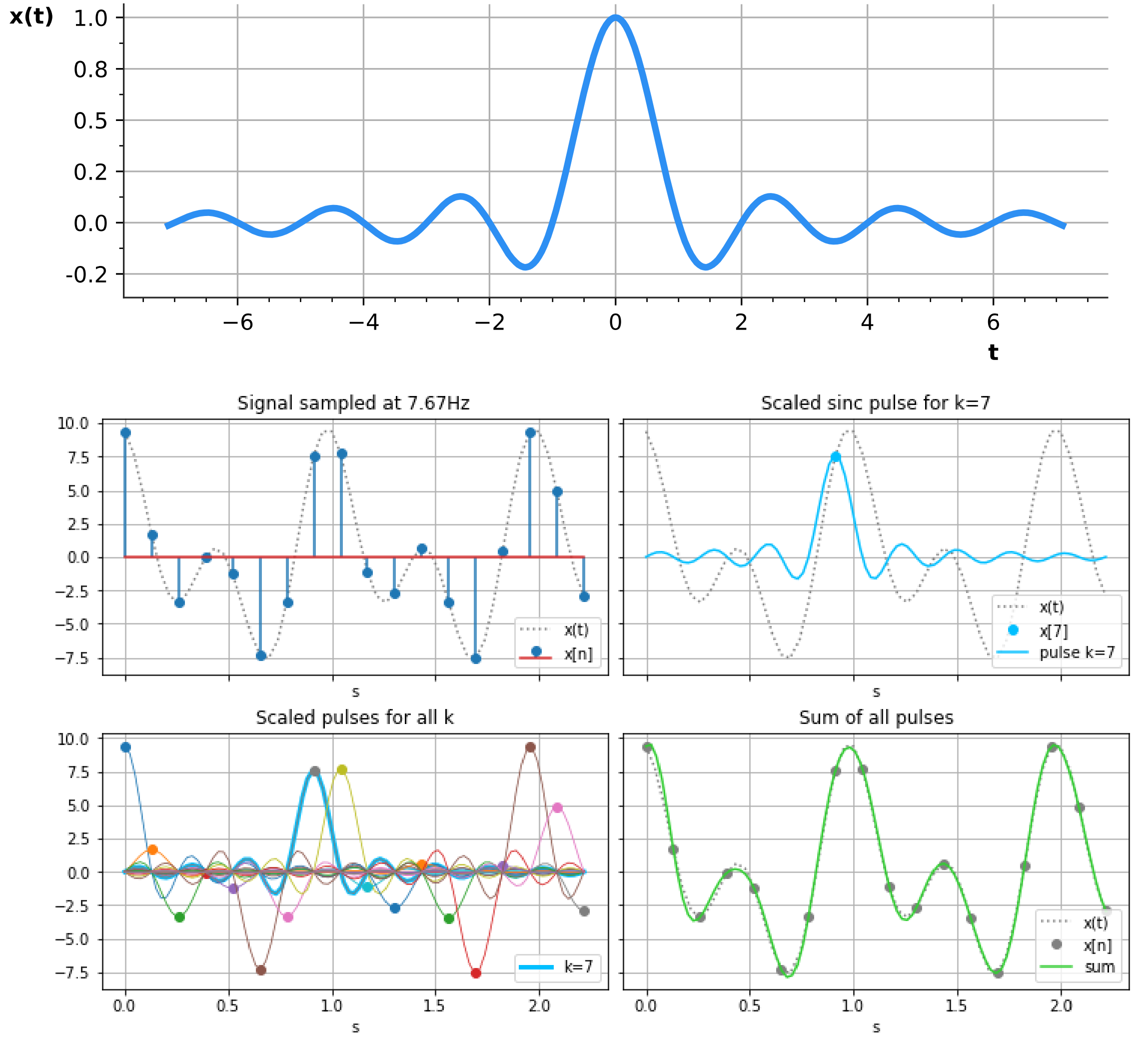
- Given N data points (t[i],x[i]) sampled at fixed time intervals T, approximate the function x(t) using

$$\hat{x}(t) = \sum_{i=0}^{N-1} x[i] \,\mathrm{sinc}\left(\frac{t - t[i]}{T}\right)$$

$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

- The function is differentiable and reconstructs the original signal if sampled correctly (no aliasing).

- Useful for band-limited signals (i.e., if the signal does not contain freq. larger than the Nyquist frequency)



Source: StackOverflow