Symmetry properties
...in Fourier series integrals

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Introduction

When calculating the Fourier series coefficients, the integrals involved can often be simplified by utilizing the symmetry properties of the input function.

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t) \right),$$

where the coefficients are given by the following definite integrals:

$$a_0 = \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, dt, \tag{1}$$

$$a_k = \frac{2}{T} \int_{-T/2}^{T/2} x(t) \cos(k\omega_0 t)\, dt, \tag{2}$$

$$b_k = \frac{2}{T} \int_{-T/2}^{T/2} x(t) \sin(k\omega_0 t)\, dt. \tag{3}$$

Even and odd functions

A function $f(t)$ is classified as even or odd based on the following properties:

$$\begin{aligned} \text{Even:} \quad f(-t) &= f(t) \\ \text{Odd:} \quad f(-t) &= -f(t) \end{aligned}$$

Figure 1 illustrates examples of odd and even functions: $f_{\text{even}}(t)$ and $f_{\text{odd}}(t)$.

A function can exhibit additional symmetries with respect to other reference points. For example, the cosine function is even around $t = 0$ but also possesses odd symmetry around $t = \frac{\pi}{2}$, meaning that:

$$\cos\left(\frac{\pi}{2} - t\right) = -\cos\left(\frac{\pi}{2} + t\right)$$

In the remainder of this text, we focus on even and odd symmetry <u>around zero</u>. However, the concepts discussed can be adapted to functions with other symmetry properties as well.

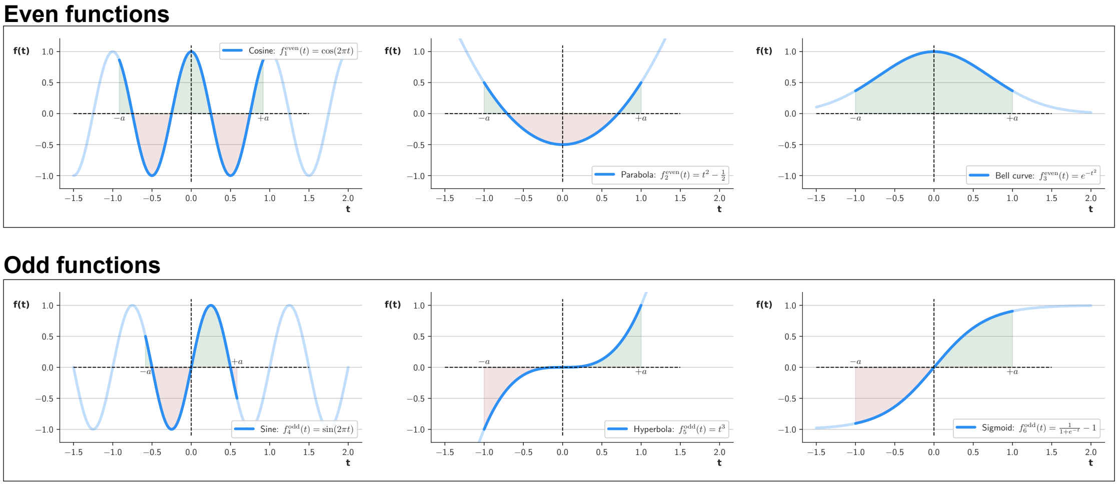

Figure 1: Examples of even and odd functions. The areas under the curves, within symmetric integration limits around zero, are shaded in green (for positive values) and red (for negative values).

You might wonder: Can a function be both even and odd? The answer is yes, but only in the trivial case where $f(t) = 0$. To see why, we set the defining equations of even and odd functions equal to each other:

$$
\begin{aligned}
f(-t) &= f(t) \\
f(-t) &= -f(t)
\end{aligned}
$$

$$
\begin{aligned}
f(t) &= -f(t) \\
2f(t) &= 0 \qquad \Rightarrow f(t) = 0
\end{aligned}
$$

Thus, the only function that is both even and odd is the trivial function $f(t) = 0$.

Product of functions

Based on the above definitions for odd and even functions, it can be shown that the following rules apply to products of functions:

Even $\times$ **Even** $=$ **Even**: The product of even functions $f_{\text{even}}(t)$ and $g_{\text{even}}(t)$ will be even.

Odd $\times$ **Odd** $=$ **Even**: The product of odd functions $f_{\text{odd}}(t)$ and $g_{\text{odd}}(t)$ will be even.

Odd $\times$ **Even** $=$ **Odd**: The product of an even function $f_{\text{even}}(t)$ and an odd function $g_{\text{odd}}(t)$ will be odd.

Figure 2 illustrates these different cases using the functions in Figure 1.

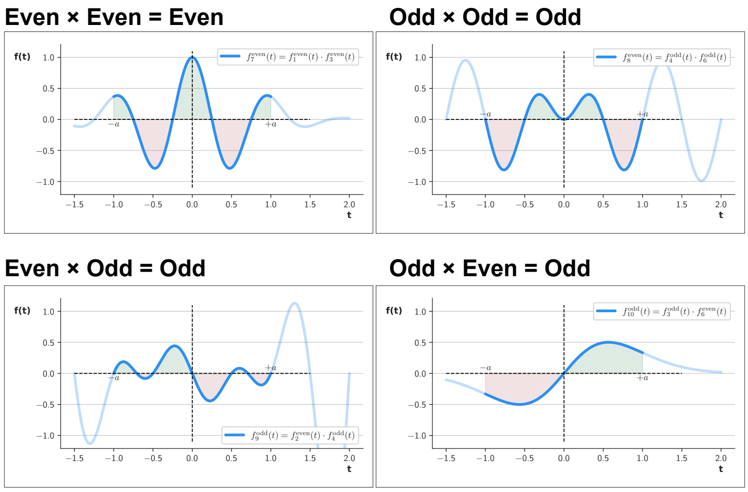

Figure 2: Examples of products of even and odd functions, demonstrating how the resulting functions retain symmetry properties. (Compare with Figure 1).)

Even and odd functions in integrals

The above formulas (1-3) for computing Fourier coefficients involve solving definite integrals. Recall that a definite integral represents the signed area enclosed by the graph of the integrand and the integration interval. Areas below the horizontal axis contribute negatively to the integral.

The symmetry properties of a function influence its integral over symmetric limits. The following statements hold and can be verified using the examples in Figure 1 and 2. Compare the differently shaded areas above and below the horizontal axis.

The integral of an **odd** function over a symmetric interval around zero is always zero:

$$\int_{-a}^{a} f_{\text{odd}}(t)\, dt = 0.$$

The integral of an **even** function over a symmetric interval simplifies to:

$$\int_{-a}^{a} f_{\text{even}}(t)\, dt = 2 \int_{0}^{a} f_{\text{even}}(t)\, dt.$$

Note that the integration boundary a is arbitrary in all these examples.

Effect on Fourier Coefficients

Let's connect these ideas to Fourier coefficients. The definite integrals in equations (1-3) involve the product of two functions: the periodic function $x(t)$ and the basis functions $\cos(k\omega_0 t)$ and $\sin(k\omega_0 t)$, which are even and odd, respectively. If $x(t)$ exhibits symmetry, we can use these properties to simplify the Fourier coefficients.

- If $x(t)$ is **even**:
  - $x(t)\cos(k\omega_0 t)$ is even $\Rightarrow a_k \neq 0$.
  - $x(t)\sin(k\omega_0 t)$ is odd $\Rightarrow b_k = 0$.
- If $x(t)$ is **odd**:
  - $x(t)\cos(k\omega_0 t)$ is odd $\Rightarrow a_k = 0$.
  - $x(t)\sin(k\omega_0 t)$ is even $\Rightarrow b_k \neq 0$.
- If $x(t)$ has **no particular symmetry**, both a_k and b_k may be nonzero.

Identifying whether $x(t)$ is even or odd allows us to immediately determine which Fourier coefficients vanish, significantly simplifying the computation of the Fourier series.