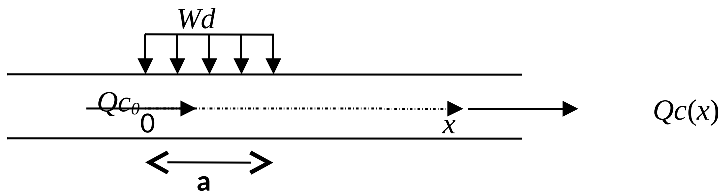
# Assignment 2 Solution 2

Environmental modelling (Technische Universiteit Delft)

Assignment 2: Solution
CIE4400

1. Consider an advection only polluted water system. This means that there is no dispersion but advection due to presence of flow velocity U [$LT^{-1}$] (=$Q/A_c$) in the positive $x$ [L] direction. The pollutant loading remains the same, which is a continuous loading $W_d$ [$ML^{-3}T^{-1}$] distributed uniformly over length a [L] from the origin and the pollutant decays with a first order kinetic rate of k [$T^{-1}$]. The concentration at the origin is $c_0$ [$ML^{-3}$]. 1) Formulate the mass balance equation for the system. 2) Solve its <u>steady state</u> concentration profile. 3) Plot the steady state concentration profile for the following characteristics: $c_0$ = 16 mg/L, Q = 12 x $10^6$ $m^3d^{-1}$, $A_c$ (cross sectional area) = 2000 $m^2$, k = 0.8 $d^{-1}$, $S_d$ = 15 $gm^{-3}d^{-1}$ and a = 8km.




**Solution:**

Conservation of mass equation for $0 \le x \le a$ is given by:

$$\frac{\partial c}{\partial x}+\frac{k}{U}c=\frac{W_d}{U}.$$

Since loading only starts at x = a and this is an advective system, concentration is 0 for x<0. We therefore only solve for x≥0.

We first solve for $0 \le x \le a$, followed by solution for x ≥a.

***For $0 \le x \le a$:***

<u>Complementary solution</u>:-

$$\frac{\partial c}{\partial x}+\frac{k}{U}c=0$$

Let

$$f=\frac{\partial}{\partial x}$$

$$\therefore f+\frac{k}{U}=0$$

$$f=-\frac{k}{U}$$

$$\Rightarrow c_c=c_1 e^{-\frac{k}{U}x}$$

Particular Solution:-

$$y=c_1 y_1(x)$$

Where,

$$y_1(x)=e^{-\frac{k}{U}x}$$

Wronskian,

$$\overline{W}=|y_1|$$

$$\overline{W}=|e^{-\frac{k}{U}x}|$$

$$\overline{W}=e^{-\frac{k}{U}x}$$

$\Rightarrow D_1=1$ (For 1 x 1 matrix)

And let $g(x')=\frac{W_d}{U}$

$$\therefore c_p=y_1\int_{-\infty}^{x}\frac{g(x')}{\overline{W}}D_1\,dx'$$

$$c_p=e^{-\frac{k}{U}x}\int_{0}^{x}\frac{W_d}{U}e^{-\frac{k}{U}x'}\,dx'$$

$$c_p=\frac{W_d}{k}$$

Therefore, general solution is

$$c=c_c+c_p$$

$$\underline{\underline{c=c_1 e^{-\frac{k}{U}x}+\frac{W_d}{k}}}$$

When $x=0$, $c=c_0$. Thus for $0 \le x \le a$:

$$c_0 = c_1 + \frac{W_d}{k}$$

$$c_1 = c_0 - \frac{W_d}{k}$$

$$\therefore c = \underline{\underline{\left(c_0 - \frac{W_d}{k}\right) e^{-\frac{k}{U}x} + \frac{W_d}{k}}}. \qquad (1)$$

***For $x \ge a$:***

Note that there is no loading from this point on. Thus, the particular solution is given by the following.

Then the general solution (here only complementary solution is needed) can be given by:

$$c = c_1 e^{-\frac{k}{U}x}.$$

By continuity at x = a and using (1) for concentration at x= a from the left hand side,

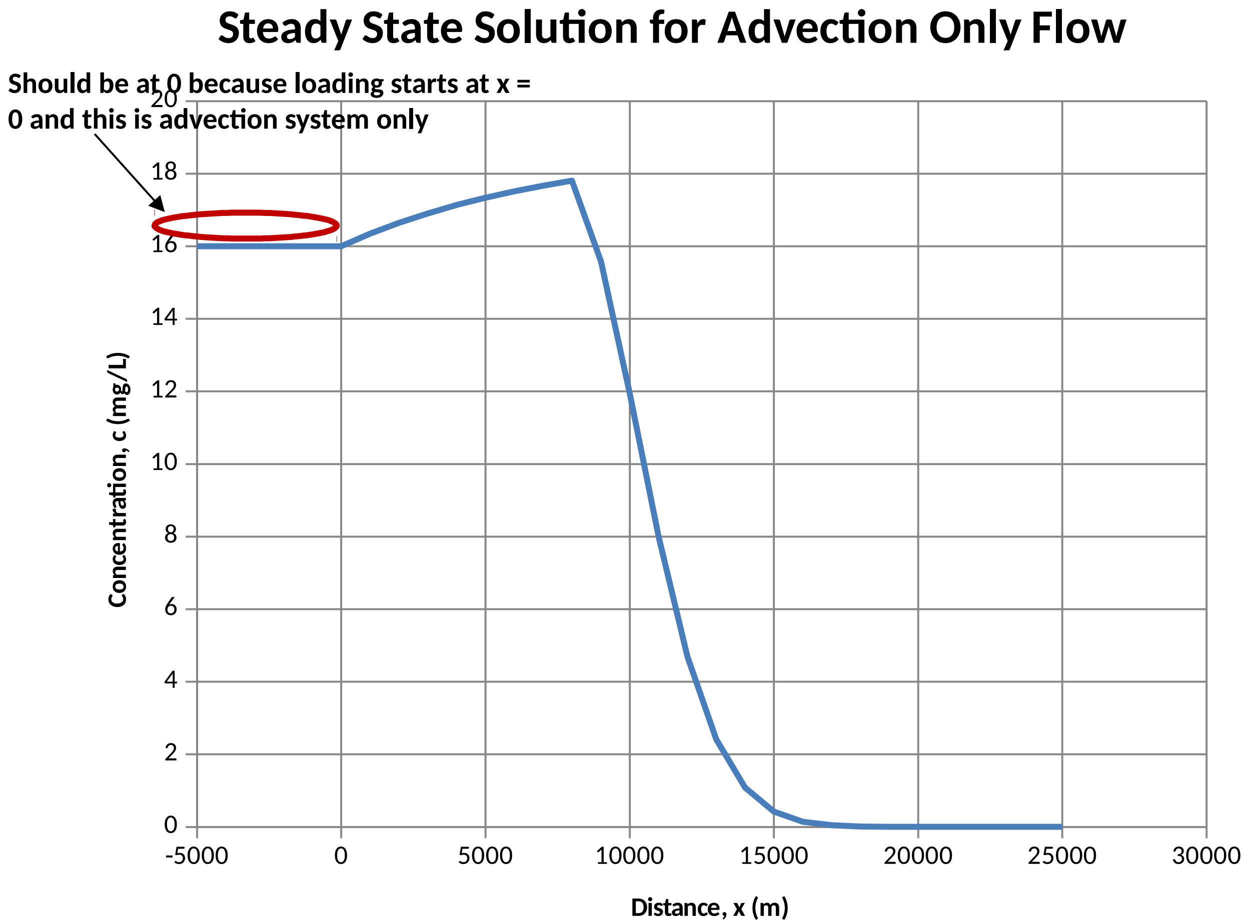
$$c_1 e^{-\frac{k}{U}a} = \left(c_0 - \frac{W_d}{k}\right) e^{-\frac{k}{U}a} + \frac{W_d}{k}$$

$$\Rightarrow c_1 = \left(c_0 - \frac{W_d}{k}\right) + \frac{W_d}{k} e^{\frac{k}{U}a}.$$

Thus, we obtain the solution as:

$$c = \left(c_0 - \frac{W_d}{k}\right) e^{-\frac{k}{U}x} + \frac{W_d}{k} e^{-\frac{k}{U}(x-a)}.$$

The following is the plot for the above concentration profiles with given parameters. Note that concentration should be 0 for x<0 for the reasons stated previously.


Steady State Solution for Advection Only Flow
Should be at 0 because loading starts at x =
0 and this is advection system only
Concentration, c (mg/L)
Distance, x (m)
0
2
4
6
8
10
12
14
16
18
20
-5000
0
5000
10000
15000
20000
25000
30000