AS4

1.Manual solution

(a)solve out the temperature among a to h

We're going to use FDM method to solve this problem.

The steady state heat conduction is given by

$$\frac{\partial^2 T}{\partial^2 x}+\frac{\partial^2 T}{\partial^2 y}=0$$

Central difference equation

$$h^2 y_n'' = y_{n-1} - 2y_n + y_{n+1}$$

Based on this, we can get

$$T_{i-1,j}+T_{i+1,j}+T_{i,j-1}+T_{i,j+1}-4T_{i,j}=0$$

Boundary condition:

In up and down side of rectangular, temperatures are 0 and 100 degrees respectively. To the left and right side, energy flux is 0.

$$\frac{\partial^2 T}{\partial^2 x}\Big|_{(0,j)}=aT_{(0,j)}+bT_{(1,j)}+c\frac{\partial T}{\partial x}\Big|_{(0,j)}$$

$$\frac{\partial^2 T}{\partial x^2}\Big|_{(0,j)}=aT_{(0,j)}+b(T_{(0,j)}+\Delta x\frac{\partial T}{\partial x}\Big|_{(0,j)}+\frac{\Delta x^2}{2}\frac{\partial^2 T}{\partial x^2}\Big|_{(0,j)})+c\frac{\partial T}{\partial x}\Big|_{(0,j)}$$

To solve this equation,

$$a+b=0$$
$$b\Delta x+c=0$$
$$b\frac{\Delta x^2}{2}=1$$

For Δx=0.5, a=-8,b=8,c=-4.

Thus, in two sides of this plate, temperature doesn't change between boundary points and first point near them in horizontal direction.

Also, in the central opening area, this is applicable. In horizontal side, Temperature at boundary point is the same as the first point nearby in vertical direction.While in vertical side, temperature at boundary point is identical to the first point nearby in horizontal direction.

Hence, we divided this plate into small pieces according to what we desire to calculate the temperature at specific points.And we use symmetric characteristic to simplify this question. And point a to h is shown as point (1,3,4,9,11,1,5) respectively.

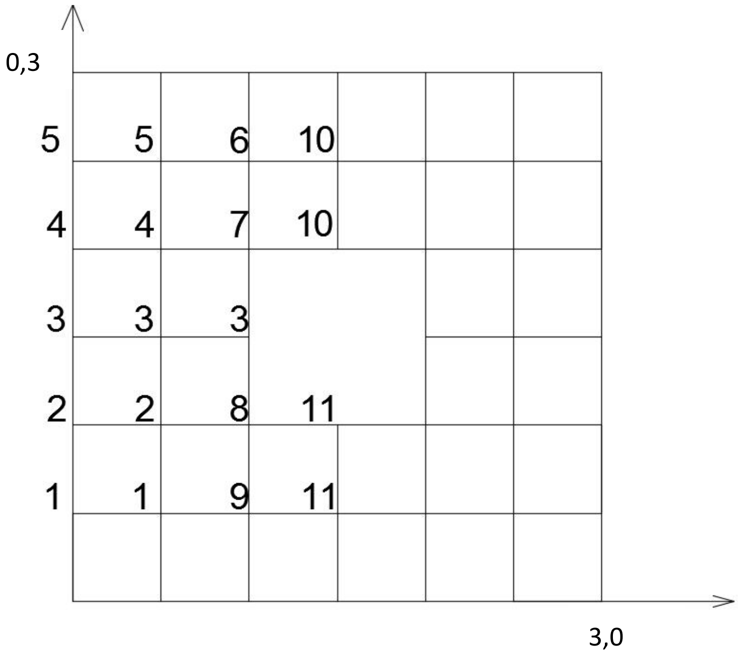


Fig1. division of the plate

Then we use FDM method to form a matrix. For example, at point 1 we have following equation,

$$T_1 + T_9 + T_2 + 100 - 4T_1 = 0$$

Finally we get a matrix,

$$\begin{pmatrix} -3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -3 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & -4 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & -4 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -4 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & -3 & 0 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \\ T_{10} \\ T_{11} \end{pmatrix} + \begin{pmatrix} 100 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 100 \\ 100 \\ 0 \end{pmatrix} = 0$$

Solve it by matlab and results are,

$$\begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \\ T_{10} \\ T_{11} \end{pmatrix} = \begin{pmatrix} 86.0364 \\ 69.8680 \\ 48.8001 \\ 27.7322 \\ 12.6837 \\ 10.3190 \\ 21.7128 \\ 74.7675 \\ 88.2412 \\ 6.8793 \\ 92.1608 \end{pmatrix} {}^\circ C$$

The temperature distribution is ,

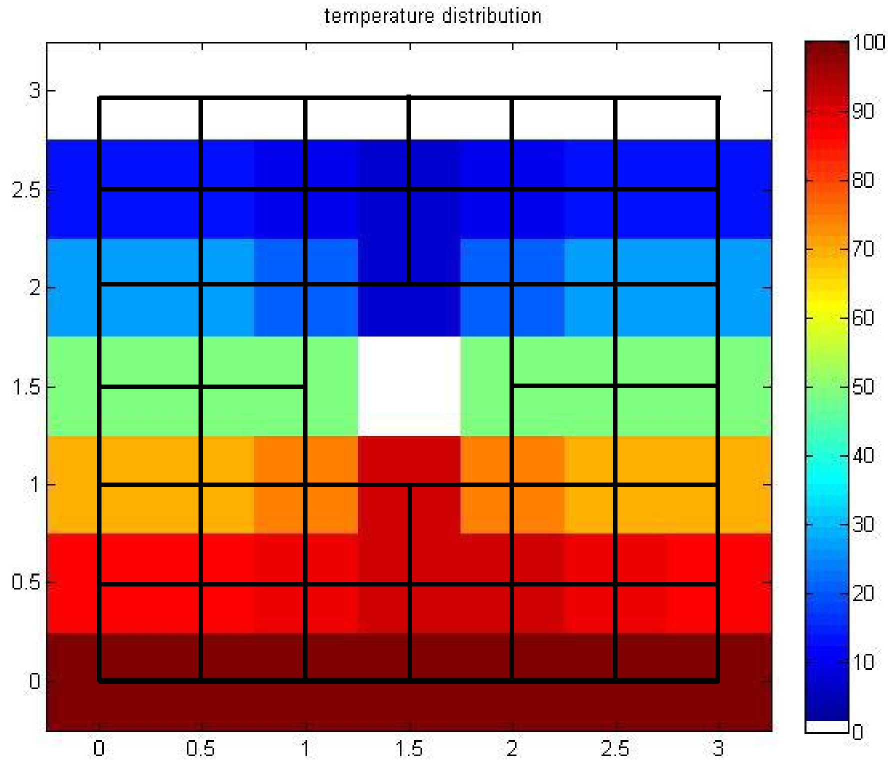


Fig2. temperature distribution in a plate

Tab.1 results of different points

point	Temperature(°C)
a	86.0364
b	48.8001
c	27.7322
d	88.2412
e	92.1608

f	86.0364
h	12.6837

(b)Discuss the accuracy

According to heat conduction governing equation

$$\frac{\partial^2 f(x_n,y_n)}{\partial x^2}=\frac{f(x_n-\Delta x,y_n)-2f(x_n,y_n)+f(x_n+\Delta x,y_n)}{\Delta x^2}+O(\Delta x^2)$$

$$\frac{\partial^2 f(x_n,y_n)}{\partial y^2}=\frac{f(x_n,y_n-\Delta y)-2f(x_n,y_n)+f(x_n,y_n+\Delta y)}{\Delta y^2}+O(\Delta y^2)$$

$$\frac{\partial^2 f(x_n,y_n)}{\partial x^2}+\frac{\partial^2 f(x_n,y_n)}{\partial y^2}=\frac{1}{0.25}(f(x_n-0.5,y_n)+f(x_n,y_n-0.5)-4f(x_n,y_n)+f(x_n+0.5,y_n)+f(x_n,y_n+0.5))+O(\Delta x^2+\Delta y^2)$$

For this heat conduction governing equation, it is second-order accurate approximation.

And due to temperature flux, we used Euler's forward method.

$$\frac{\partial T_n}{\partial x}=\frac{T_{n+1}-T_n}{\Delta x}+O(\Delta x)$$

This is first order accurate approximation and it becomes the domain of all process. Hence, this solution should be first-order accurate approximation.

2.General solution

According to heat conduction problem, we can form a matrix by matlab as a general solution to such question and could make it more accurate than manual solution

(a)subroutines

```
% grid size is N assume delta x= delta y
N=6;
dx=3/N;
u=zeros(N);
A_diag=eye(N-1)*(-4/(dx^2));
A_diag=A_diag+diag(ones(N-2,1),1)*1/(dx^2);
A_diag=A_diag+diag(ones(N-2,1),-1)*1/(dx^2);
A_off=eye(N-1)*(1/(dx^2));
A=eye((N-1)*(N-1),(N-1)*(N-1));
for i=1:N-1
  A((i-1)*(N-1)+1:(i-1)*(N-1)+(N-1),(i-1)*(N-1)+1:(i-1)*(N-1)+(N-1))=A_diag;
end
for i=2:N-1
```

```
    A((i-2)*(N-1)+1:(i-2)*(N-1)+(N-1),(i-1)*(N-1)+1:(i-1)*(N-1)+(N-1))=A_off;%upper
diagnal
    A((i-1)*(N-1)+1:(i-1)*(N-1)+(N-1),(i-2)*(N-1)+1:(i-2)*(N-1)+(N-1))=A_off;%lower
diagnol
end
% boundary condition
% left side and right side condition
A_new=eye(N*N-1,N*N-1);
A_new(N:N*(N-1),N:N*(N-1))=A;
A_new(1:N-1,1:N-1)=-eye(N-1)*1/dx^2;
A_new(N*(N-1)+1:end,N*(N-1)+1:end)=-eye(N-1)/dx^2;
A_new(N:2*N-2,1:N-1)=A_off;
A_new(1:N-1,N:2*N-2)=A_off;
A_new(N*(N-1)+1:end,(N-1)^2+1:N*(N-1))=A_off;
A_new((N-1)^2+1:N*(N-1),N*(N-1)+1:end)=A_off;
%openning condition
  % donwside of opening area
A_new(N*N/2-1,:)=0;
A_new(N*N/2-1,N*N/2-1)=-1/dx^2;
A_new(N*N/2-1,N*N/2-2)=1/dx^2;
  % left side of opening area
A_new(N*N/2-(N-1),:)=0;
A_new(N*N/2-(N-1),N*N/2-(N-1))=-1/dx^2;
A_new(N*N/2-(N-1),N*N/2-(N-1)*2)=1/dx^2;
 %right side of opening area
A_new(N*N/2+(N-1),:)=0;
A_new(N*N/2+(N-1),N*N/2+(N-1))=-1/dx^2;
A_new(N*N/2+(N-1),N*N/2+(N-1)*2)=1/dx^2;
 %upside of opening area
A_new(N*N/2+1,:)=0;
A_new(N*N/2+1,N*N/2+1)=-1/dx^2;
A_new(N*N/2+1,N*N/2+2)=1/dx^2;
%boundary condition in up and down side
f=zeros(N*N-1,1);
f(N:N-1:(N-1)*(N-1)+1)=100/dx^2;
B=-A_new\f;
B=reshape(B,N-1,N+1);
surf(B)
```

(b)Result analysis

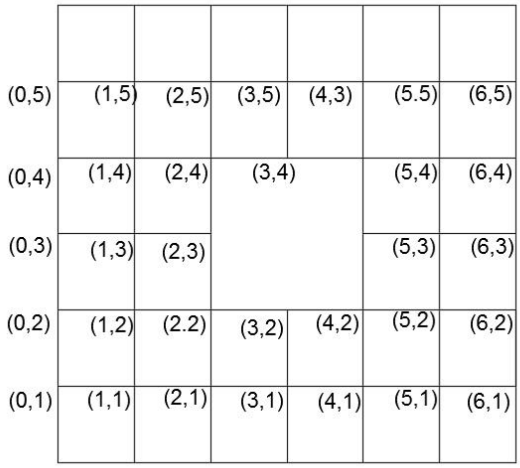
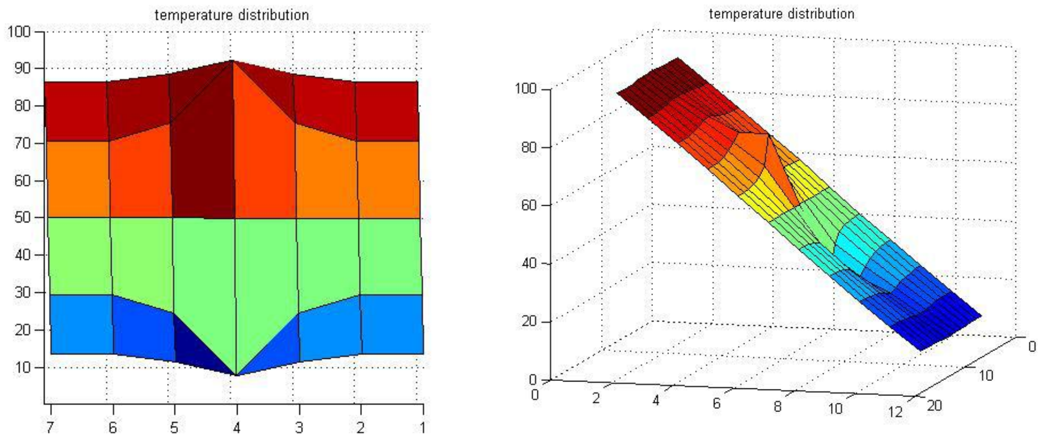


Fig.3

For N=6 which is the same as manual solution,

Tab.2 temperature at different point when N=6

	0	1	2	3	4	5	6
1	86.36	86.36	88.52	92.34	88.52	86.36	86.36
2	70.57	70.57	75.36	92.34	75.36	70.57	70.57
3	50.00	50.00	50.00	50.00	50.00	50.00	50.00
4	29.43	29.43	24.64	7.66	24.64	29.43	29.43
5	13.64	13.64	11.48	7.66	11.48	13.64	13.64



(a) front view (b)side view

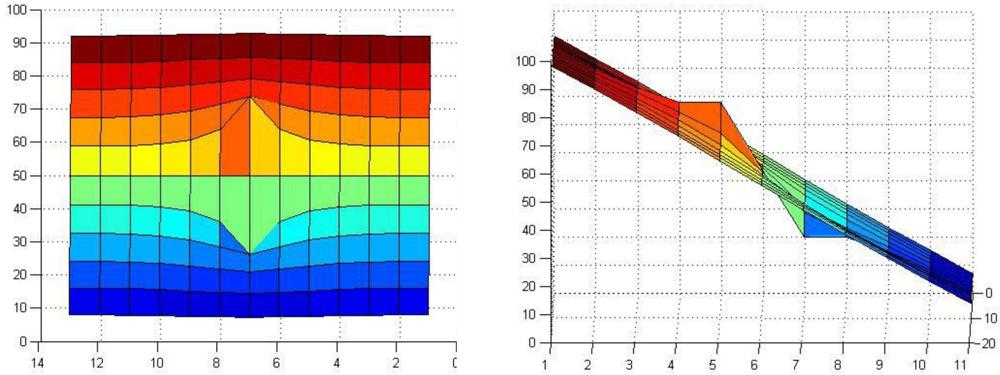
Fig.4 temperature distribution when N=6

For N=12 grid size is half of manual solution.

Tab.3 temperature at different point when N=12

	0	1	2	3	4	5	6
1	84.01	84.16	84.87	85.60	84.87	84.16	84.01
2	67.45	67.70	69.43	73.95	69.43	67.70	67.45
3	50.00	50.00	50.00	50.00	50.00	50.00	50.00
4	32.55	32.30	30.57	26.05	30.57	32.30	32.55

5	15.99	15.84	15.13	14.40	15.13	15.84	15.99



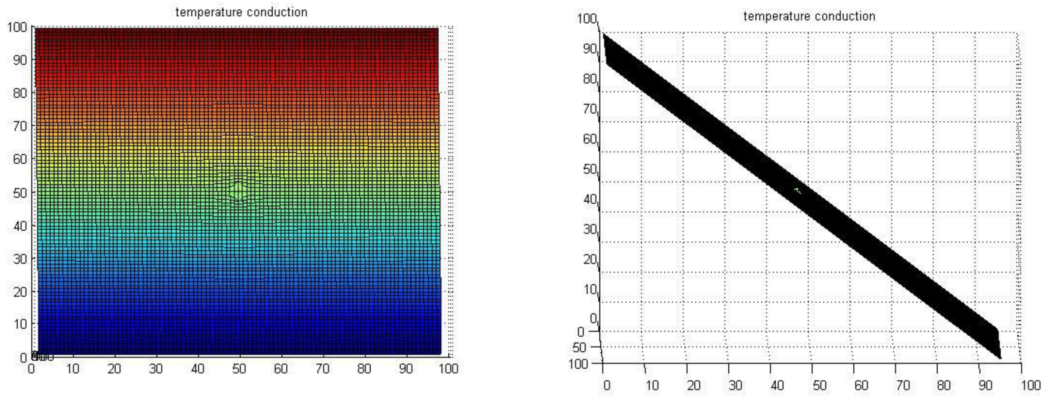
(a) front view (b) side view

Fig.5 temperature distribution when N=12

For N=96,

Tab.4 temperature at different point when N=96

	0	1	2	3	4	5	6
1	84.38	84.39	84.40	84.41	84.40	84.39	84.38
2	67.72	67.72	67.75	67.80	67.75	67.72	67.72
3	51.04	51.04	51.05	53.11	51.05	51.04	51.04
4	34.37	34.36	34.33	34.26	34.33	34.36	34.37
5	17.70	17.69	17.68	17.67	17.68	17.69	17.70



(a) front view (b) side view

Fig.6 temperature distribution when N=96

Tab.5 comparison of different methods and grid size

Method	Manual solution	Computational solution		
Point	N=6	N=6	N=12	N=96
a	86.04	86.36	84.16	84.39
b	48.80	50	50	51.04
c	27.73	29.43	32.30	34.37
d	88.24	88.52	84.87	84.40
e	92.16	92.34	85.60	84.41
f	86.04	86.34	84.16	84.39

h	12.68	13.64	15.84	17.69

Generally speaking, the smaller the grid size, more precise the result. We can observe this conclusion by comparing the difference between N=6 to N=96 and N=12 to N=96. Find out N=12 is closer to the result of N=96.

Also, we found a little difference between manual solution and computational solution when N=6, that may be because in the opening area we assume the heat does not conduct in such area, while in whole matrix, we can see it does.