NATIONAL UNIVERSITY OF SINGAPORE

CE5311 – ENVIRONMENTAL MODELLING WITH COMPUTERS

(Semester 1: AY2016/2017)

Time Allowed: 2.5 Hours

INSTRUCTIONS TO CANDIDATES

1. Please write your student number only. **Do not write your name**.
2. This assessment paper contains **FOUR** questions and comprises **FOUR** printed pages.
3. Answer **ALL** questions. All questions carry equal marks.
4. Please start each question on a new page.
5. This is an "OPEN BOOK" assessment.
6. **All class notes and provided reference materials can be brought in.**
7. Electronic calculator is permitted for this exam.

<u>Question 1</u> [25 marks]

The one-dimensional advection equation is

$$\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} = 0$$

where u is the advection velocity.
Discretize with the scheme below,

$$\phi_i^{n+1} = \frac{1}{2}(\phi_{i+1}{}^n + \phi_{i-1}{}^n) - \frac{\Delta t}{2\Delta x}(\phi_{i+1}{}^n - \phi_{i-1}{}^n)$$

(a) Perform stability analysis using von Neumann analysis.

[15 points]

(b) Find the order of accuracy using Hirt's method

[10 points]

<u>Question 2</u> [25 marks]

Consider the second-order PDE with diffusion, advection and reaction terms:

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \phi}{\partial S^2} + rS\frac{\partial \phi}{\partial S} - r\phi = 0$$

where $\phi(S,t)$ is the independent term, S and t are dependent terms, and σ and r are constants.

(a) Identify which are the diffusion, advection and reaction terms.

[5 points]

Discretize the PDE and find the local truncation errors for the following schemes:

(b) Forward in Time, Center in Space (FTCS)

[5 points]

(c) Forward in Time, Backward in Space (FTBS)

[5 points]

(d) Forward in Time, Forward in Space (FTFS)

[5 points]

(e) Crank-Nicholson Method

[5 points]

<u>Note</u>: The Crank-Nicholson Method is given as $\phi_i^{n+1} = \phi_i^n + \frac{1}{2}\left[f\left(\phi_i^{n+1}\right) + f\left(\phi_i^n\right)\right]\Delta t$

<u>Question 3</u> [25 marks]

A reservoir with a volume of 7.5 x 10^6 m^3 has inflows that come from a canal, the only outflow from the reservoir is to water treatment plants. Evaporation occurs over the surface of the reservoir.

(a) Derive the time varying storage budget for the reservoir assuming steady flow. Then simplify it assuming everything is steady-state

[4 marks]

(b) Use the steady-state equation derived in (a) to determine the evaporation rate (cm/year) if reservoir surface area is 1.5 x 10^6 m^2; canal supplies 2.0 x 10^6m m^3/year and outflow to water treatment plants is 9.5 x10^5 m^3/year.

[6 marks]

The water in the canal brings in a pollutant.

(c) Derive the time varying budget for the pollutant in term of its concentration, c, in the water. Then simplify the budget assuming that it is steady-state.

[4 marks]

(d) If the concentration of the pollutant as it exits to the water treatment plants is 8 g/m^3 and the concentration of the pollutant as it enters the canal is 5 g/m^3. Using the equation from (e), and appropriate simplifications, determine if the reservoir is a source or a sink and the amount contributed to (if sink) or comes from the reservoir (if source) in kg/day (assuming 365 days in a year)

[11 marks]

<u>Question 4</u> [25 marks]

You are asked to assess the potential impact of pollutants after a spill in a canal. The width of the canal is 12 m, the depth is assumed to be uniform at 3.0 m. It is assumed that 100 L of a pollutant is accidentally spilled uniformly across the canal and that the pollutant has a relative density of 0.9 kg/L.

For part (a) to (d) you assume that the canal waters are still and that the diffusion coefficient is uniform, D = 2.0 m^2/s.

Thus use the solution to the 1D diffusion equation; $c(x,t) = \frac{M}{\sqrt{4\pi Dt}} \exp\left(-\frac{(x)^2}{4Dt}\right)$

(a) Determine the concentration of the pollutant at x = 0 at T = 2 hours, 3 hours, 4 hours and 8 hours. [6 marks]

(b) Determine the concentration of the pollutant at x = 200m at T= 2 hours, 3 hours, 4 hours and 8 hours. [4 marks]

(c) Show that the time t_{max} for the maximum concentration at any distance x is given by the relationship $t_{max} = \frac{(x)^2}{2D}$ and the corresponding maximum concentration $c_{max} = 0.2420 \frac{M}{x}$ [4 marks]

(d) Find t_{max} and corresponding c_{max} at x = 200 m and at x = 300 m. [2 marks]

(e) If your initial assumption was wrong and the water in the canal was not still but flowing (u = 0.016 m/s), determine the concentration of the pollutant at x = 200 m at T = 1 hours, 2 hours, 3 hours and 8 hours using the solution, $c(x,t) = \frac{M}{\sqrt{4\pi Dt}} \exp\left(-\frac{(x-ut)^2}{4Dt}\right)$ [4 marks]

(f) Comparing your findings in part (b) and (e)
   i) What is the impact of flow (advection) in the canal in terms of magnitude and timing of maximum concentration of pollution at x = 200 m? [2 marks]

   ii) What non-dimensional number can be used to estimate if diffusion or the advection dominates? [2 marks]

   iii) Calculate the non-dimensional number using appropriate scales at x = 200 m and state which, if any of the following transport processes dominate (advection, diffusion, neither). [1 mark]

- END OF PAPER -