NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF ENGINEERING

EXAMINATION FOR

(Semester I: 2013-2014)

CE5311 - ENVIRONMENTAL FLOWS

Nov/ Dec 2013 - Time allowed: 2.5 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FOUR(4)** questions and comprises **SEVEN(7)** printed pages.

2. Answer all **FOUR(4)** questions.

3. All questions do not carry equal marks.

4. All answers require an explanation or proof

5. This is an "**OPEN BOOK**" examination.

Question 1 (20 marks)

Figure 1 shows a cone shaped volume with an input discharge and an output discharge. At t=0 the cone is assumed to be empty. The water level $h(t)$ is assumed to be strictly horizontal.

a. Give the equation that describes the evolution in time of the water level $h(t)$. This equation will be referred to as eq.(1). [2 marks]

b. Consider the following numerical recipe:

$$h_{k+1} = h_{k-1} + 2\Delta t \frac{10^{-3}}{\pi h_k^2}\left(1 - \sqrt{2g \max(0, h_k - 1)}\right) \qquad \text{eq(2)}$$

Is this a consistent approximation of eq.(1)? Is so, what is the name of the method that is used. [2 marks]

c. If eq(2) is consistent what is the order of the local truncation error? [2 marks]

d. Do you expect eq.(2) to be stable? [2 marks]

e. Is eq(2) mass conservative? [2 marks]

f. If eq(2) is mass conservative, can you give an example of a non-conservative approximation? In addition, if eq(2) is not conservative can you give an example of a conservative approximation? [2 marks]

g. What is the steady state solution of eq.(1) [2 marks]

h. If the steady state solution is to be approximated by a numerical scheme what is the influence of the scheme on the steady state solution? [2 marks]

i. When the system is in steady state we consider input of a dissolved matter: $c_{inp}\,(kg/m^3)$. In the volume we assume a well-mixed concentration $c(t)$. What is the equation for $c(t)$? [2 marks]

j. What is the flushing time of this volume under steady state conditions? [2 marks]

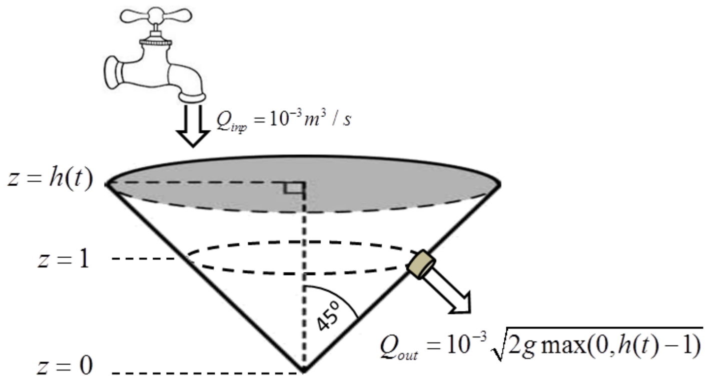


Figure 1, volume with input and output

Question 2 (39 marks)

In Figure 2 a uniform channel is described with a length L=9.81 km. The depth of the channel is given by H=9.81 m. For the frequency ω various values may be chosen. The initial conditions are given by $u(x,0)=0, \zeta(x,0)=0$

a. Transform the equations of figure 2 into the characteristic equations. [3 marks]

b. What is spin-up time? [3 marks]

c. Which combination of boundary conditions has the shortest spin up time? [3 marks]

d. What is the spin up time for the question 2.3? [3 marks]

e. What is the eigen frequency of the channel for the combination of boundary conditions $\left(B_{0,1}, B_{L,1}\right)$? [3 marks]

f. What is the spin up time for this combination? [3 marks]

g. What is the analytical solution of the combination $\left(B_{0,2}, B_{L,1}\right)$ for t>2000 sec? [3 marks]

h. Which combination or combinations of boundary conditions may yield an ill-posed problem? [3 marks]

For the numerical approximation we consider a staggered grid. For this grid the grid size is $\Delta x = 49.05 m$. The numerical approximation is given by:

$$\zeta_m^{k+1} = \zeta_m^k - \Delta t H \frac{u_{m+1/2}^k - u_{m-1/2}^k}{\Delta x}, u_{m+1/2}^{k+1} = u_{m+1/2}^k - \Delta t g \frac{\zeta_{m+1}^k - \zeta_m^k}{\Delta x} \qquad \text{eq(1)}$$

i. Draw the staggered grid and the stencil of eq(1). [3 marks]

j. What is the maximum time step according to the CFL condition? [3 marks]

k. What is the maximum time step according to the Von Neumann condition? [3 marks]

Rather than the simplified equations of figure 2 we now consider the following equations:

$$\frac{\partial \zeta}{\partial t} + \frac{\partial \left(hu\right)}{\partial x} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial \zeta}{\partial x} + c_f \frac{u\left|u\right|}{R} = 0$$

l. What is in this case the spin up time for the combination $\left(B_{0,1}, B_{L,1}\right)$? [3 marks]

m. What is the effect on the solution in general if these equations are applied to the channel? [3 marks]

$$\frac{\partial \zeta}{\partial t} + H\frac{\partial u}{\partial x} = 0, \frac{\partial u}{\partial t} + g\frac{\partial \zeta}{\partial x}$$

$B_{0,1} : \zeta(0,t) = \cos(\omega t)$ $\qquad$ $B_{L,1} : u(L,t) = 0$

$B_{0,2} : u(0,t) + \sqrt{\frac{g}{H}}\zeta(0,t) = 2\sqrt{\frac{g}{H}}\cos(\omega t)$ $\qquad$ $B_{L,2} : u(L,t) + \sqrt{\frac{g}{H}}\zeta(L,t) = 0$

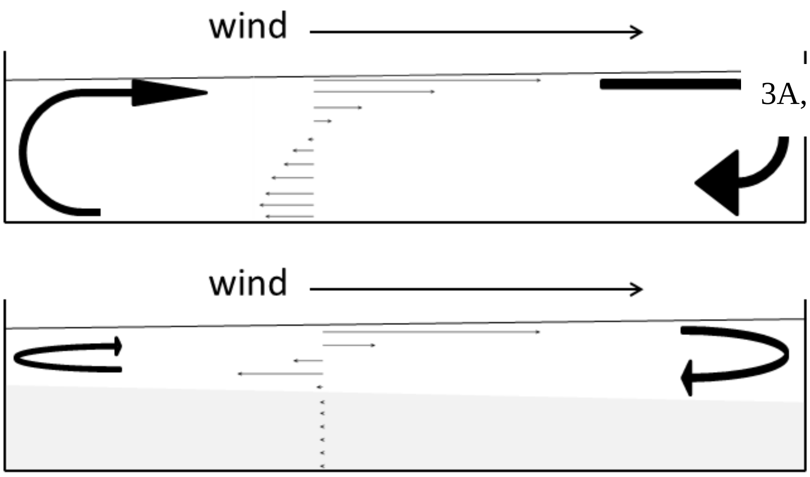
$B_{0,3} : u(0,t) - \sqrt{\frac{g}{H}}\zeta(0,t) = -2\sqrt{\frac{g}{H}}\cos(\omega t)$ $\qquad$ $B_{L,3} : u(L,t) - \sqrt{\frac{g}{H}}\zeta(L,t) = 0$

Figure 2, 1D channel of uniform rectangular cross section based on simplified equations

Question 3 (20 marks)

Figure 3 shows 2 side views of circulation in a lake with a uniform rectangular volume shape of constant depth. The first situation is without stratification while the second is stratified.

a. Explain the wind driven circulation in figure 3A. [2 marks]
b. Explain the wind driven circulation in figure 3B. [2 marks]
c. If 3A or 3B is simulated with a hydrostatic model, such as Delft3D, in which part of the lake are both results probably inaccurate? [2 marks]
d. What is the depth averaged flow in 3A and 3B? [2 marks]
e. Sometimes lakes are modelled with depth averaged models. Like every model, lake models require spin up time. To arrive at situation 3A which model would require more spin up, a depth averaged model or a model that contains the vertical structure? [2 marks]
f. To model 3A would you prefer σ planes or z planes? [2 marks]
g. To model 3B would you prefer σ planes or z planes? [2 marks]
h. Is a steady state as given by 3A possible? [2 marks]
i. Is a steady state as given by 3B possible? [2 marks]
j. If the wind suddenly stops, what will be the response of the lake for 3A and 3B? [2 marks]



3A, uniform temperature

3B, thermocline

Figure 3, lake with wind

Question 4 (21 marks total)

The 2D advection-diffusion equation with constant diffusion coefficients is given by

$$\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} + v\frac{\partial \phi}{\partial y} = \kappa\frac{\partial^2 \phi}{\partial x^2} + \kappa\frac{\partial^2 \phi}{\partial y^2}$$

Assuming steady state, the boundary conditions are:

$$\phi(x=0, y) = 1$$

$$\phi(x, y=0) = 1$$

$$\frac{\partial \phi}{\partial x}(x=1, y) = 0$$

$$\frac{\partial \phi}{\partial x}(x, y=1) = 0$$

The Dirichlet boundary conditions are applied at $x=0$ and $y=0$, while the Neumann boundary conditions are applied at $x=1$ and $y=1$.

(a) Express the 2D advection-diffusion equation in steady state. (5 marks)

(b) Discretize the equation in (a) with second-order schemes. (8 marks)

(c) Explain which of the boundary conditions require ghost points and discretize the boundary conditions. (8 marks)

- END OF PAPER -