Numerical Methods in Mechanics and <u>Environmental Flows</u>

NOV 10, 2017

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Group and Term Project (Reminder)

Assignment	Group	Group Members
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Outline for Environmental Flows

Oct 13
- Introduction
- Delft3D Introduction and Assignment 1: Spin-up?

Oct 20
- Box models and solution methods
- Delft3D Assignment 2 - Boundary conditions; initial conditions

Oct 27
- Solution methods: vertical layers / transport processes
- Delft3D Assignment 3 – stratification (wind-driven flows)

Nov 3
- Transport processes in flows (2)
- Delft3d Project

Nov 10
- Transport processes in flows (3) – Practical issues
- Term project
- Delft3d assignment 4 – model extents (estuarine stratification as an example)

Nov 17
- Presentation of term assignment (4 groups)
- Revision / past exam questions

Reservoir with pollutant

Reservoir with following conditions:

- $V = 2x10^5 m^3$; $Q_{u/s} = 9x10^4 m^3/yr$; $Q_{evap} = 1x10^4 m^3/yr$; Assume steady state; upstream c = 6 mg/l; c decays at K = 0.12/year

Find c

- What is budget?
- What is then c?

What if now upstream c = 0 due to changes in management?

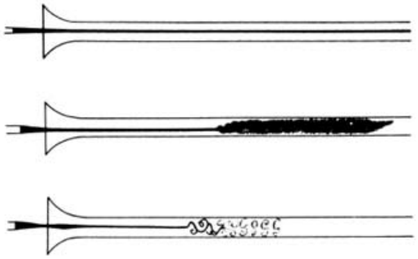
- What is budget?
- How long does it take to drop to 50%
- How long does it take to reach 0.1 mg/l?
- What is the main cause of the improvement in c?

Turbulent Flows

Reviewed 5 basic characteristics

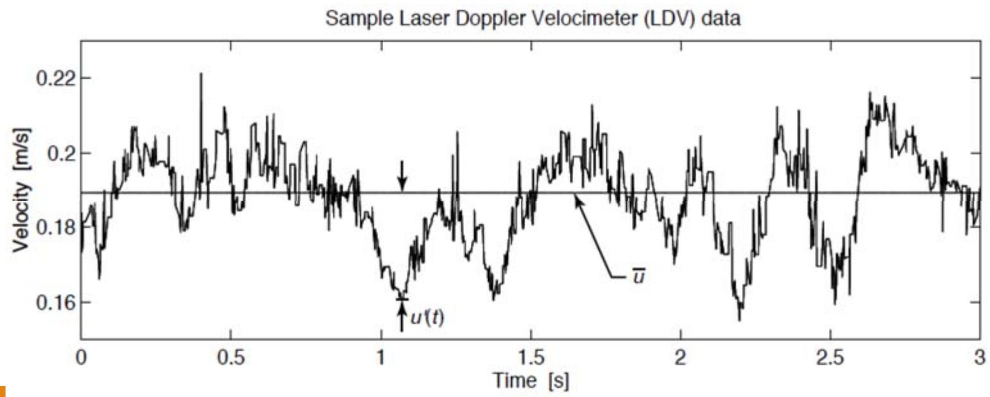
- Impact on mixing → More effective mixing as there is more agitation→ Larger D!

Classification



Description and concept

- Energy cascade
- Length scale → dependent on rate of dissipation
- Measurement
- Implications



Remember Reynolds Decomposition?

Decompose velocity into mean and fluctuating components → $u_i(x_i, t) = \overline{u_i}(x_i) + u_i'(x_i, t)$.

- Essentially Integral Time Scale becomes the time for $\overline{u}_i$ to become steady/constant

Which allows us to have a 3rd important quantity

$$u_{rms} = \sqrt{\overline{u'u'}}$$

- Why is this important?

Turbulence and mixing

Reynolds Decomposition $C(x,t) = \bar{C}(x) + C'(x,t)$

And Time Averaging

$$q_x = \overline{uC}$$

$$\overline{uC} = \frac{1}{t_I}\int_{t_I}^{t+t_I} uC\,d\tau = \overline{(\bar{u}_i + u'_i)(\bar{C} + C')} = \overline{u'C'} + \overline{uC}$$

Results in

$$\frac{\partial \bar{c}}{\partial t} + \bar{u}\frac{\partial \bar{c}}{\partial x} = -\frac{\partial \overline{u'c'}}{\partial x} + \frac{\partial}{\partial x}\left(D\frac{\partial \bar{c}}{\partial x}\right)$$

Taking the Fickian relationship $\overline{u'c'} = D_t\frac{\partial \bar{c}}{\partial x}; \quad where \quad D_t = u_I L_I$

Results in

$$\frac{\partial \bar{c}}{\partial t} + \bar{u}\frac{\partial \bar{c}}{\partial x} = \frac{\partial}{\partial x}\left(D_t\frac{\partial \bar{c}}{\partial x}\right) + \frac{\partial}{\partial x}\left(D\frac{\partial \bar{c}}{\partial x}\right)$$

Turbulent Closure Models

Why do we have them?

- Momentum $$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} = -\frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 \overline{u_i}}{\partial x_i x_j} - \frac{\partial \overline{u_i' u_j'}}{\partial x_j}$$
- Scalar $$\frac{\partial \bar{c}}{\partial t} + \bar{u}_j \frac{\partial \bar{c}}{\partial x_j} = \frac{\partial}{\partial x_j}\left(D \frac{\partial \bar{c}}{\partial x_j} \right) - \frac{\partial \overline{u_j' c'}}{\partial x_j}$$

How do we solve for the extra unknowns?

$$\overline{u_i' u_j'} = \frac{2}{3} k \delta_{ij} - \nu_t \left(\frac{\partial \overline{u_i}}{\partial x_j} + \frac{\partial \overline{u_j}}{\partial x_i} \right) \quad ; \quad k = \frac{1}{2} \overline{u_i' u_i'}$$

EDDY VISCOSITY

- And the mixing length concept

$$\nu_t = C_\mu \sqrt{k} L$$

Turbulent Closure Models (2)

Zero Equation

- *k* calculated from the flow and L from $\sqrt{k} = L\frac{\partial u}{\partial y}$

One equation

- Solve for k using a transport equation $\frac{\partial k}{\partial t} + \overline{u}_j \frac{\partial k}{\partial x_j} = +D_t \frac{\partial^2 k}{\partial x_i x_j} + P_k + P_{kw} + B_k - \varepsilon$
- Prescribe *L* as above

Two equation

- Solve for *k* and the dissipation rate of *k*, ε.
- From *k* and ε calculate mixing length and viscosity

$$\frac{\partial k}{\partial t} + \frac{u}{\sqrt{G_{\xi\xi}}}\frac{\partial k}{\partial \xi} + \frac{v}{\sqrt{G_{\eta\eta}}}\frac{\partial k}{\partial \eta} + \frac{\omega}{d+\zeta}\frac{\partial k}{\partial \sigma} = + \\ + \frac{1}{(d+\zeta)^2}\frac{\partial}{\partial \sigma}\left[\left(\nu_{mol} + \frac{\nu_{3D}}{\sigma_k}\right)\frac{\partial k}{\partial \sigma}\right] + P_k + P_{kw} + B_k - \varepsilon,$$

$$\nu_{3D} = c'_\mu L\sqrt{k} = c_\mu \frac{k^2}{\varepsilon}$$

$$\frac{\partial \varepsilon}{\partial t} + \frac{u}{\sqrt{G_{\xi\xi}}}\frac{\partial \varepsilon}{\partial \xi} + \frac{v}{\sqrt{G_{\eta\eta}}}\frac{\partial \varepsilon}{\partial \eta} + \frac{\omega}{d+\zeta}\frac{\partial \varepsilon}{\partial \sigma} = \\ \frac{1}{(d+\zeta)^2}\frac{\partial}{\partial \sigma}\left[\left(\nu_{mol} + \frac{\nu_{3D}}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial \sigma}\right] + P_\varepsilon + P_{\varepsilon w} + B_\varepsilon - c_{2\varepsilon}\frac{\varepsilon^2}{k}.$$

Mixing and Turbulence?

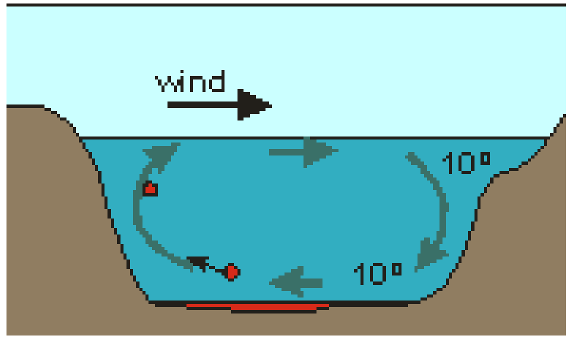
Simplest method $\overline{u_j'c'} = -D_t \frac{\partial \overline{c'}}{\partial x_j}$; $D_t = \frac{\nu_t}{Sc_t}$

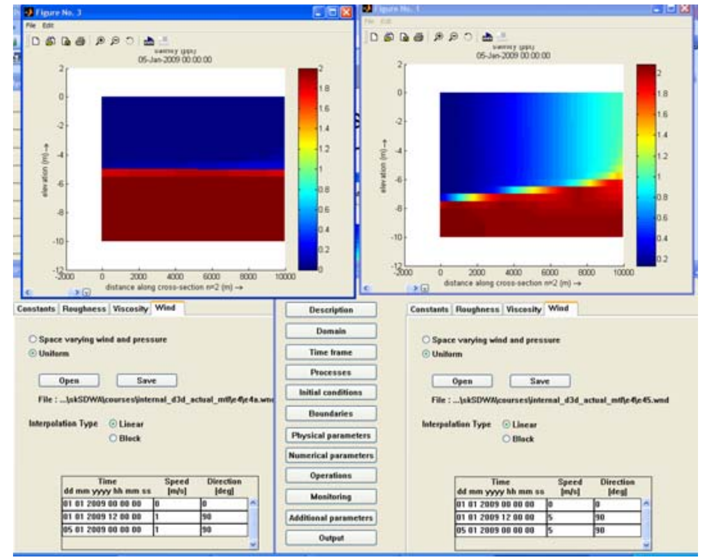
Practical Issues: Stratification

A QUICK LOOK AT

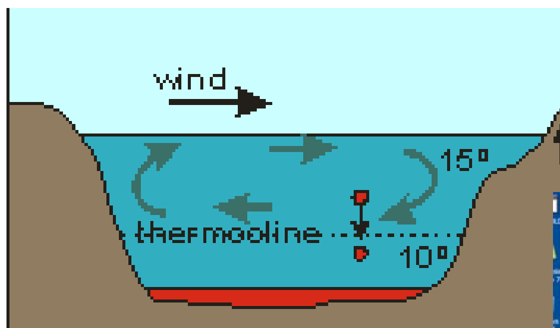
The issue with stratification

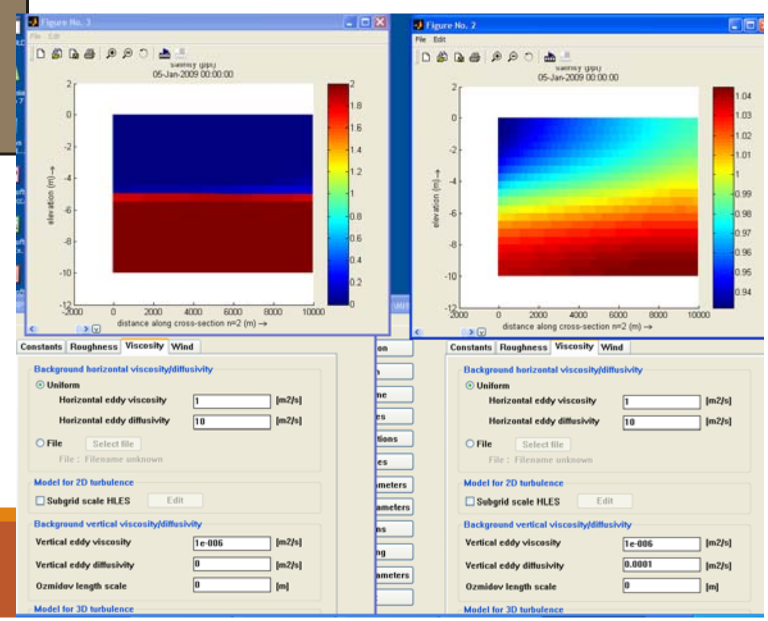
Without





With





What is the difference?

What aids stratification?

What does heat do?

Heat affects fluid densities

- How?
  - Temperature ; Water expands → Density
- This difference in density typically creates an additional body force → ___________ force
- The issue of thermal expansion

Linear Relationship

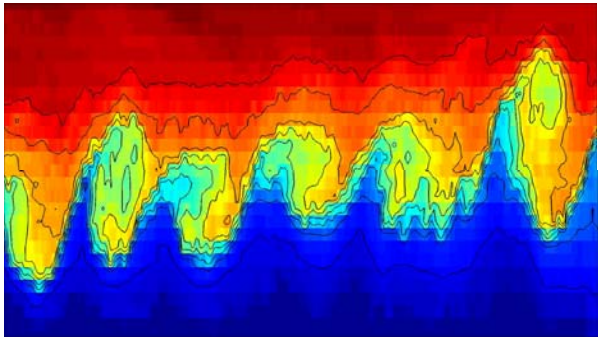
$$\rho = \rho_0[1 - \alpha(T - T_0)]$$

How do stratified fluids mix?

Conduction → typically if source is available

More generally through an instability→ the Kelvin-Helmholtz Instability

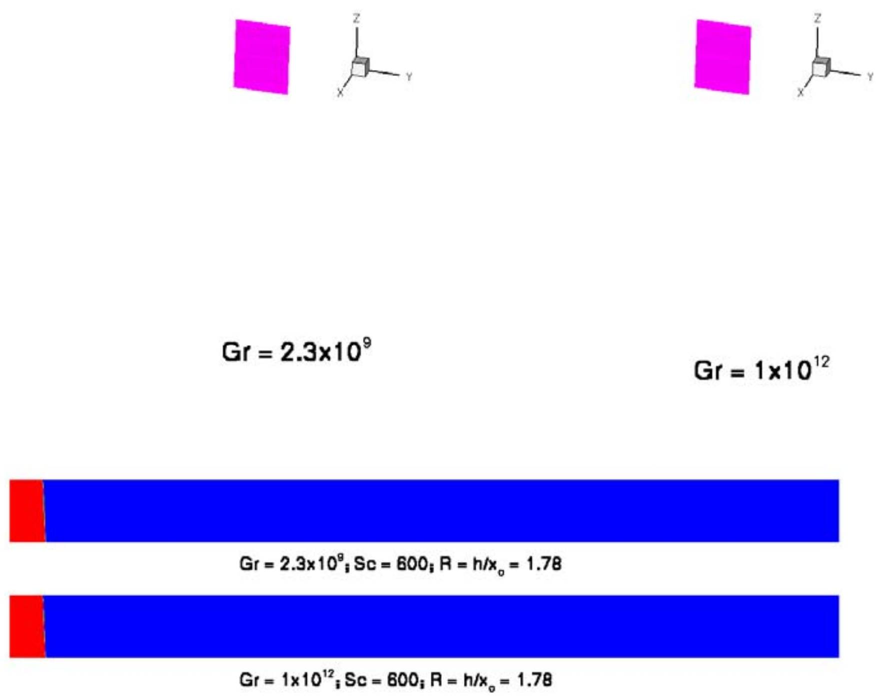
- Deep ocean mixing:

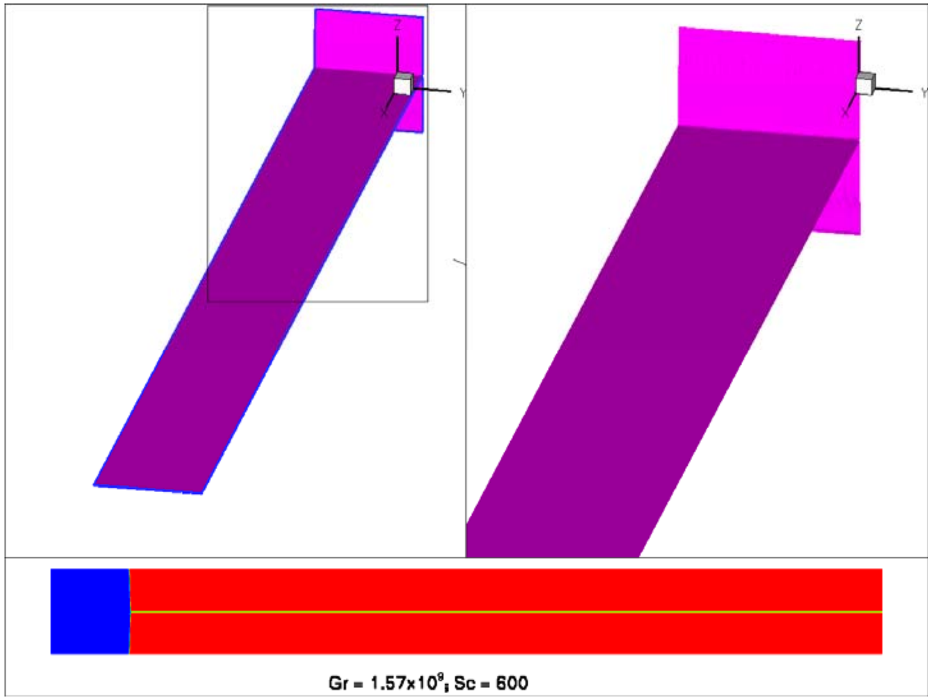


- Atmosphere:

Mixing of Stratified Flows

Lock Exchange

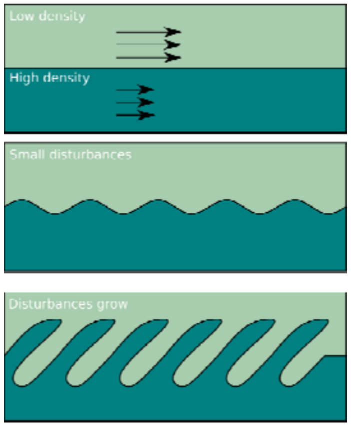




The K-H Theory

How do we set up the problem?

- We need:
  - 2 fluids with different densities
  - Differential velocity between the two
  - Buoyancy and viscosity ratio
- Assume infinitely high, deep and wide domain



This will result in

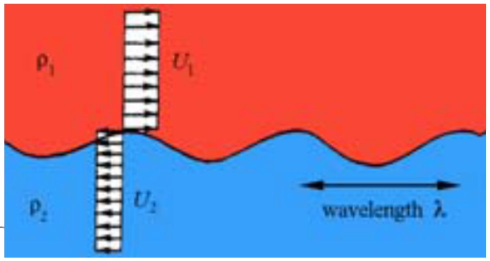
- Waves with wavelength, λ, arising from pertubations
- Shorter waves which satisfy

$$\boxed{g\lambda(\rho_2^2 - \rho_1^2) < 2\pi\rho_1\rho_2(u_1 - u_2)^2}$$

will grow in time → unstable

Derived from instability analysis

Given the result…



There must be a critical wavelength that separates shorter unstable waves from longer stable waves

→ $$\boxed{\lambda_{crit} \approx \frac{\pi \Delta u^2 \rho_0}{g \Delta \rho} = \frac{\pi \Delta u^2}{\alpha g \Delta T}}$$

- Assuming that $$\Delta u = u_2 - u_1$$ $$\Delta \rho = \rho_2 - \rho_1$$ $$\rho_0 \approx \rho_1 \approx \rho_2$$

Thus vertical mixing must occur over a height of approximately

$$\boxed{h = Const. \frac{\Delta u^2}{\alpha g \Delta T}}$$

In turbulent flows...

We can take $\boxed{d = \dfrac{\Delta u^2}{\alpha g \Delta T}}$ and $u_* = \Delta u$

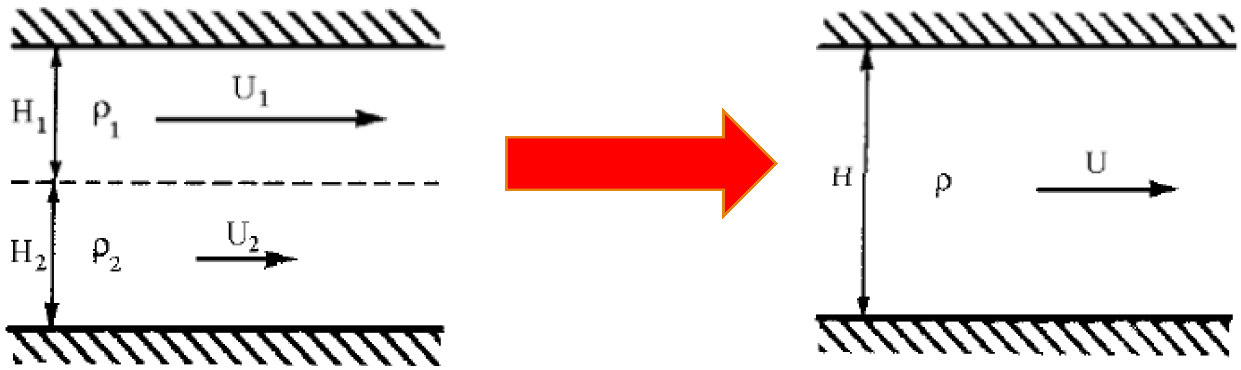
This allows us to estimate the time of vertical mixing $\tau = d / u_*$

And typically horizontal diffusion,

$$K \propto d u_*$$

From energy perspective..

We can start with



- Complete mixing
- H1=H2=H/2

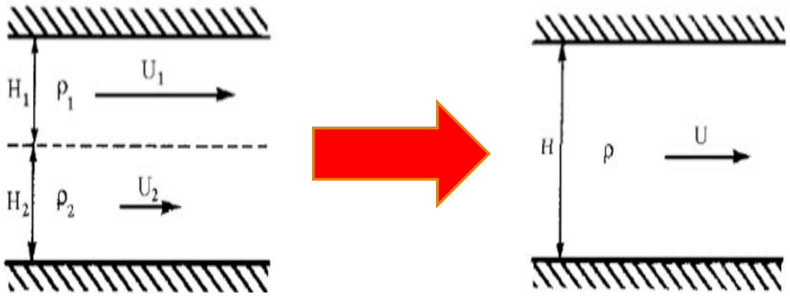
We can now compare PE and KE before and after mixing

Before looking at energy we need to look at our balances

Since we assume complete mixing, we can use simple mass and momentum conservation

- Mass conservation

$$\rho H = \rho_1 H_1 + \rho_2 H_2$$



- Momentum conservation

$$\rho U H = \rho_1 U_1 H_1 + \rho_2 U_2 H_2;$$

$$Assume \quad \rho\ UH = \rho\ U_1 H_1 + \rho\ U_2 H_2$$

$$U = \frac{U_1 H_1 + U_2 H_2}{H} = \frac{U_1 + U_2}{2}$$

Energy…

Before mixing

$$KE_{start} = \frac{1}{2}\rho_0 U_1^2 H_1 + \frac{1}{2}\rho_0 U_2^2 H_2$$
$$= \frac{1}{4}\rho_0 (U_1^2 + U_2^2) H$$

$$PE_{start} = \int_0^{H_2} \rho_2 gz\, dz = \int_{H_2}^{H_1+H_2} \rho_1 gz\, dz$$
$$= \frac{1}{8}\rho_2 gH^2 + \frac{3}{8}\rho_1 gH^2$$

If mixing occurs

$$KE_{after} = \frac{1}{2}\rho_0 U^2 H = \frac{1}{8}\rho_0 (U_1^2 + U_2^2) H$$

$$PE_{after} = \int_0^{H} \rho gz\, dz = \frac{1}{4}(\rho_2 + \rho_1) gH^2$$

Gain/Loss

$$KE_{loss} = \frac{1}{8}\rho_0 (U_1 - U_2)^2 H$$

$$PE_{gain} = \frac{1}{8}(\rho_2 - \rho_1) gH^2$$

What do you see in this derivation?

The resultant

So then to have mixing, KE loss > PE gain

$$\frac{1}{8}\rho_0(U_1 - U_2)^2 H > \frac{1}{8}(\rho_2 - \rho_1)gH^2$$

However in real life we only see a fraction of the KE loss resulting in PE gain

$$Const.\frac{1}{8}\rho_0(U_1 - U_2)^2 H > \frac{1}{8}(\rho_2 - \rho_1)gH^2$$

Where *Const.* is approximately 0.3, resulting in

$$H < 0.3\frac{\rho_0(U_1 - U_2)^2}{(\rho_2 - \rho_1)g}$$

What if?

Inequality is not satisfied? $$H < 0.3\frac{\rho_0(U_1 - U_2)^2}{(\rho_2 - \rho_1)g}$$

Then mixing will be confined to a depth of

$$h \approx 0.3\frac{\rho_0(U_1 - U_2)^2}{(\rho_2 - \rho_1)g} = 0.3\frac{\Delta u^2}{\alpha g \Delta T}$$

This allows us to develop a criteria for stable mixing

Remember: $$H < 0.3\frac{\Delta U^2}{\alpha g \Delta T}$$

If we define a dimensionless term, *Ri* where

$$Ri = \frac{\alpha g h \Delta T}{\Delta U^2}$$

Then we can infer that 2 conditions

- Ri > 0.3
- Ri < 0.3

Application: Estuaries

Estuaries

What are estuaries?

- Rivers meet the sea or ocean
  - Transition zone from river to ocean
  - Biologically and ecologically important

What drives the flows?

Issues

Many ways to classify estuaries

Classification for mixing!

There are different estuaries

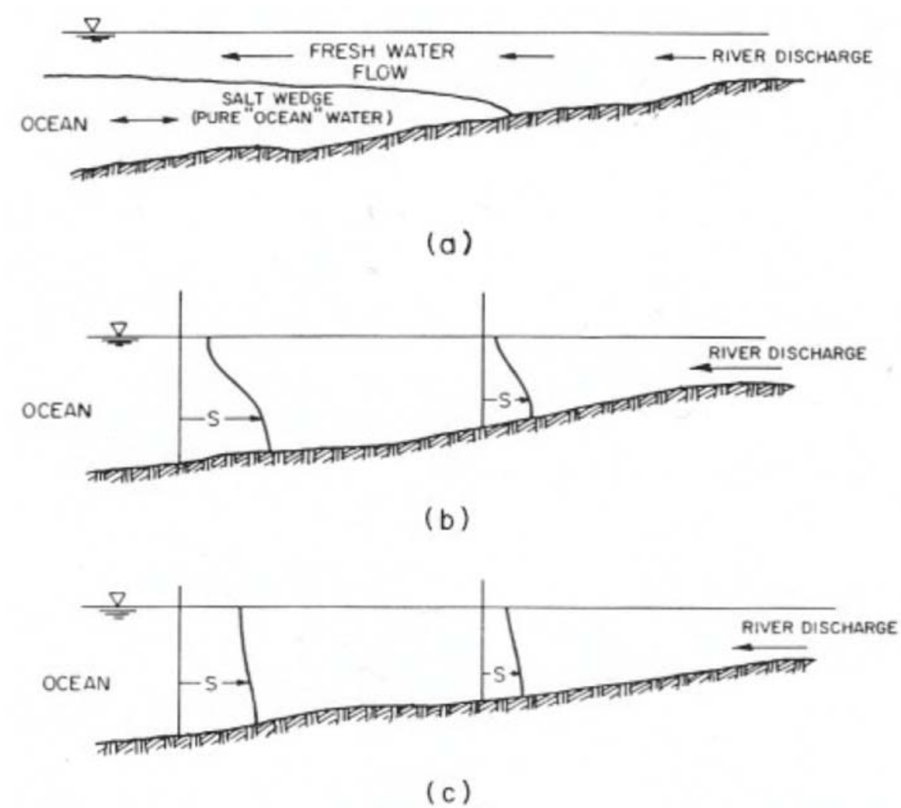
- River strong enough to keep seawater out
- Rivers weak (Singapore) or deep enough (Johor) that salt wedge can develop and move upstream

Due to this we can classify by

- Flow ratio = u_{tide}/u_{river}

Type of flow ratios

- Flow ratio = u_{tide}/u_{river}
  - Flow ratio < 0.1
  - 0.1 < Flow ratio < 10
  - Flow ratio > 10

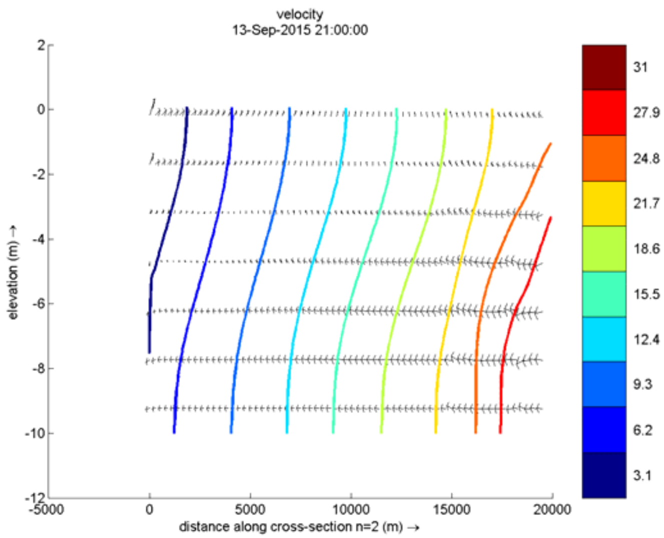


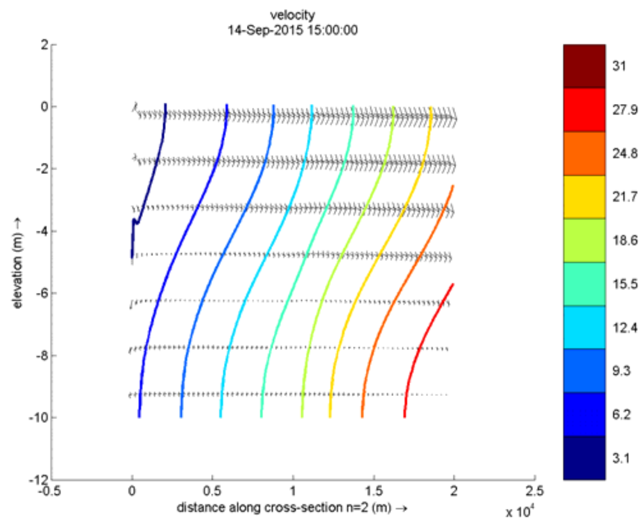
What causes mixing in estuaries?
- Rivers

How?

- Turbulent kinetic energy due to tides sloshing the river

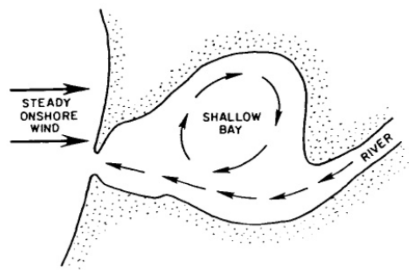
- In addition lateral mixing due to longitudinal dispersion compounded by density gradients





What causes mixing in estuaries?
- Wind

Typically important for shallow wide estuaries



Along-channel wind enhances estuarine exchange in different ways depending on direction

- Up-wind

- Down-wind

What causes mixing in estuaries?
- Tides

Shear Flow Dispersion

$$K = K_0 f(T'),$$

$$K = 0.1\overline{u'^2}T[(1/T')f(T')].$$

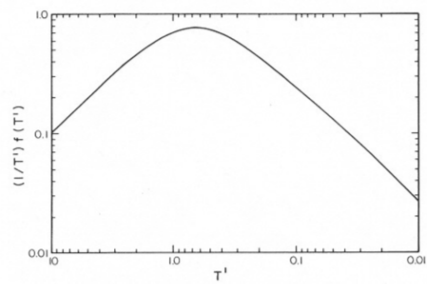


Figure 7.4 The quantity $T'^{-1}f(T')$ used in Eq. (7.2).

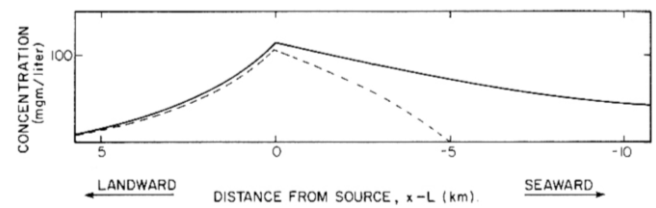


Figure 7.24 Longitudinal concentration distributions for Example 7.5: ——— discharge 30 km from the mouth; – – – discharge 5 km from the mouth.

(Fischer *et al.*, 1979, page 273

Tidal trapping

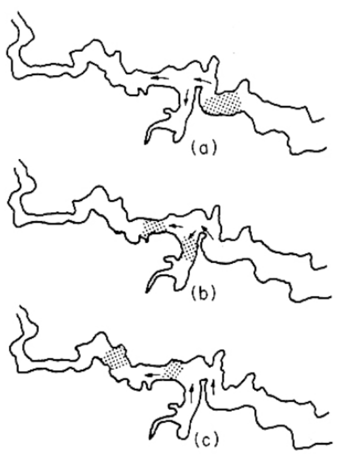


Figure 7.9 The phase effect in a branching channel. (a) A cloud of tracer being carried upstream on a flooding tide. (b) At high water some of the particles are trapped in the branch. (c) During the early stages of the receding tide the flow in the main channel is still upstream. The particles trapped in the branch reenter the main channel, but are separated from their previous neighbors.

(Fischer *et al.*, 1979, page 242)

$$K = \frac{K'}{1 + r} + \frac{r u_0^{\,2}}{2k(1 + r)^2(1 + r + \sigma/k)},$$

Okubo (1973)

Application: Lakes

What are lakes?

Body of water impounded by an obstacle

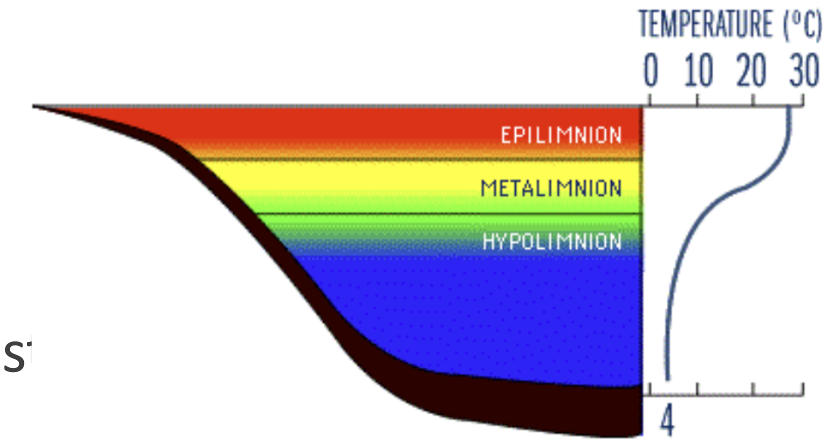
- So reservoirs are included
- Relatively still

What drives the flows?

Layers and stratification in a lake

Epilimnion
- Warm; less dense

Metalimnion
- Thin layer where there is a dist



Hypolimnion
- Bottom layer; cold, denser

Mixing in Lakes

Variable

Dependent on size, climate, time

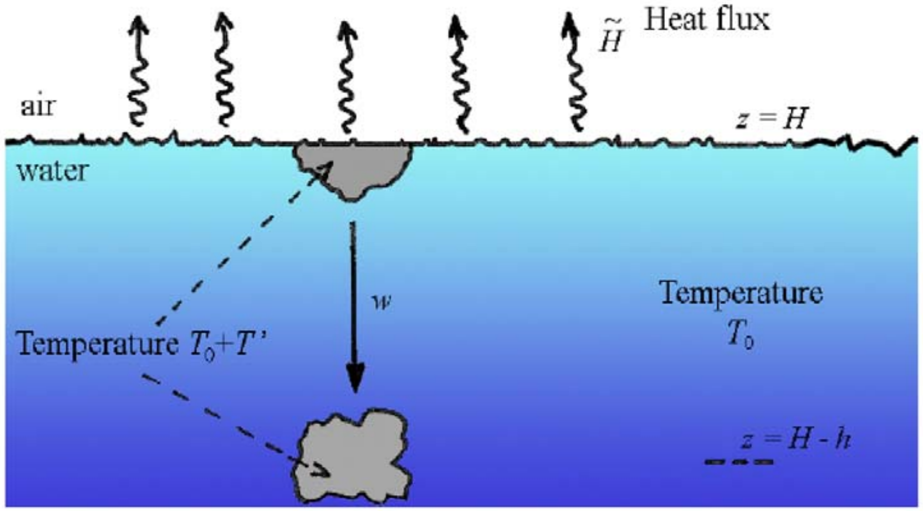
Residence Time

- Characteristic of lakes
- Due partly to:
  - Weak velocities
  - Larger Depths
- Average time spent in the lake/reservoir due to inflows/outflows
- Why is it important?

Energy Budget

Let's look into more detail

What is the energy budget of a lake?



Convection in Lakes

First we have to quantify vertical stratification:

- Remember?

$$\rho = \rho_0[1-\alpha(T-T_0)]$$

- Let's decompose density

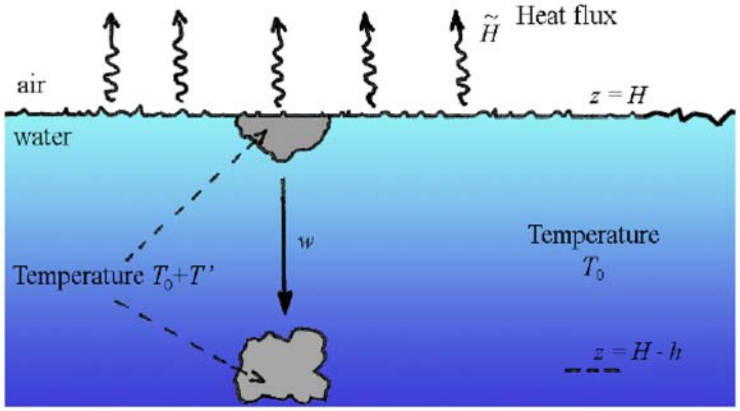
$$\rho = \rho_0 + \rho_e(z) + \rho'(x,y,z,t)$$

- And define a stratification frequency, N

$$N^2 = -\frac{g}{\rho_0}\frac{d\rho_e}{dz} = +\alpha g\frac{dT}{dz} > 0$$

Convection (1)

Penetrative Convection



- First define:
  - Sinking velocity
  - Heat flux at the surface
  - Which can be rewritten as:

$$w = -\sqrt{\alpha g h |T'|}$$

$$\tilde{H} = \rho_0 C_V |wT'|$$

$$w = -\left(\frac{\alpha g h \tilde{H}}{\rho_0 C_V}\right)^{1/3}$$

$$T' = -\left(\frac{\tilde{H}^2}{(\rho_0 C_V)^2 \alpha g h}\right)^{1/3}$$

Convection (2)

Define budgets

- Heat $$\int_{H-h}^{H} \rho_0 C_V T(z)dz = \int_{t=0}^{t=T} \tilde{H} dt$$
- Energy $$\int_{H-h}^{H} \rho_0 \alpha g T(z) z dz = Unchange$$

Solution → $$h = \sqrt{\frac{6\alpha g \tilde{H} t}{\rho_0 C_V N^2}}; \quad \Delta T = \sqrt{\frac{2N^2 \tilde{H} t}{3\rho_0 C_V \alpha g}}$$

Diffusivity = 0.1 hw $$= 0.33 \frac{\alpha g \tilde{H} t}{\rho_0 C_V N^2} (Nt)^{2/3}$$

Convection (3)

Full-depth

- Vertically mixed over depth
- Heat continually removed
- In this case:
  - Fall Velocity $$w = -\left(\frac{\alpha g H \tilde{H}}{\rho_0 C_V}\right)^{1/3}$$
  - Diffusivity $$D_z = 0.1 H w = 0.1\left(\frac{\alpha g H^4 \tilde{H}}{\rho_0 C_V}\right)^{1/3}$$
  - Time to mix $$T = 0.536\frac{H^2}{D_z} = 5.36\left(\frac{\rho_0 C_V H^2}{\alpha g \tilde{H}}\right)^{1/3}$$

What does the wind do?

Erodes stratification

Consider irregular winds (not what you did for Assignment)

How do we transfer wind momentum to water?

$$\tau_{wind} = C_D \rho_{air} U^2 = \rho_{water} u_*^2$$

How do we calculate wind mixing?

Turbulent velocity

$$u_* = \sqrt{\frac{\tau_{wind}}{\rho_0}}$$

Energy budget results in

$$h = \left(\frac{12(1.25)u_*^3 t}{N^2} \right)^{1/3}$$

Vertical diffusivity = 0.1 hu_*

$$= 0.25 \frac{u_*^2}{N} (Nt)^{1/3}$$

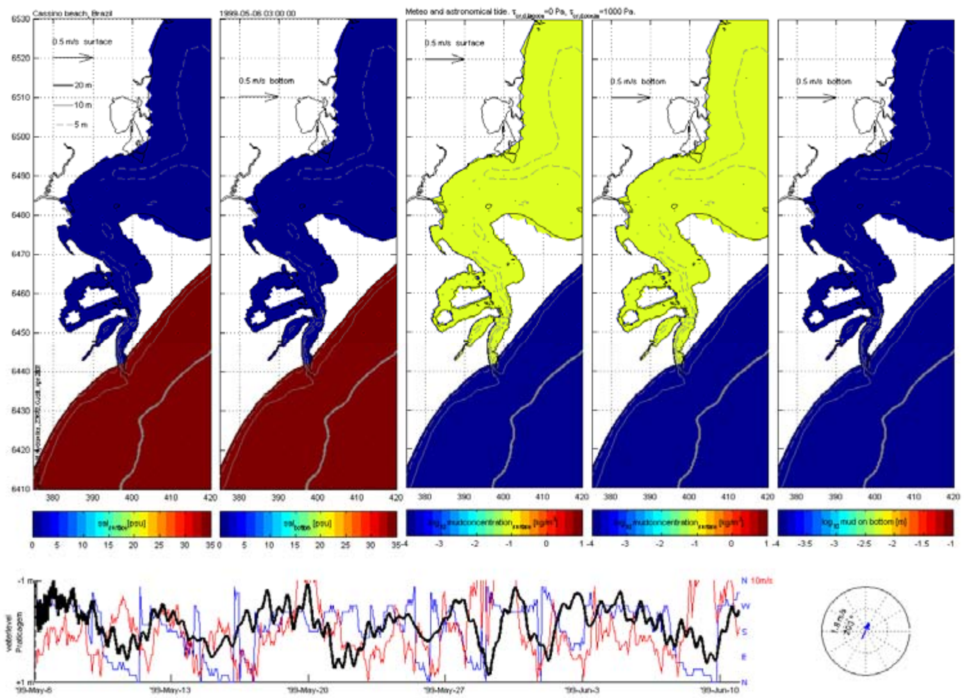
Mixing by wind

Wind stress from?

From Kelvin Helmholtz → $h = 0.3U^2/(\alpha g \Delta T)$

From heat budget → $h = 1.4\ U/N$

Diffusivity → $D_z=0.1hU=0.077U^2/N$



WE ARE AT THE END…

NOW YOU SHOULD BE ABLE TO

- ASSESS THE SITUATION USING FIRST PRINCIPLES
- ASSESS THE LIMITATIONS/ISSUES OF MODELS (NUMERICALLY AND PHYSICALLY)