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Socio-hydrology I: Modelling a socio-economy

The role of humans in shaping our environment can no longer be ignored. One crucial element of the interactions between humans and their environment is how humans take decisions and exploit their environment to their own benefit. This chapter provides fundamentals underlying one way of formulating a society and the actions that it takes. These are based on the principles of mathematical programming and microeconomics. These are convered in the following in a stepwise manner.

1.1. Optimization

Optimization is a means to choose certain variables of a system such that its performance can be optimized. Either in the form of maximization or minimization. Consider two functions in Figure 1.1 that reflects the performance of the system. Both the functions can either be maximized or minimized. However the first function has a clear global maxima and the other one has a clear global minimum. So how can one find that maximum or minimum since these respective points are where the corresponding perfomance is being optimized (either maximized or minimized).

If one looks at the left panel, the slope of the curve is shown in green. As one moves from left to right in the positive x-direction, the slope decreases from being a positive number to a negative number and being 0 at the optimum. For the left panel, the opposite happens. The the slope goes from negative to positive as one goes from left to the right, becoming a 0 at the optimum. This tells us two things: 1) slope of the curve is 0 at the optimum and 2) the slope of the curve changes from positive to negative around a global maximum and from negative to positive in case of a global minimum.

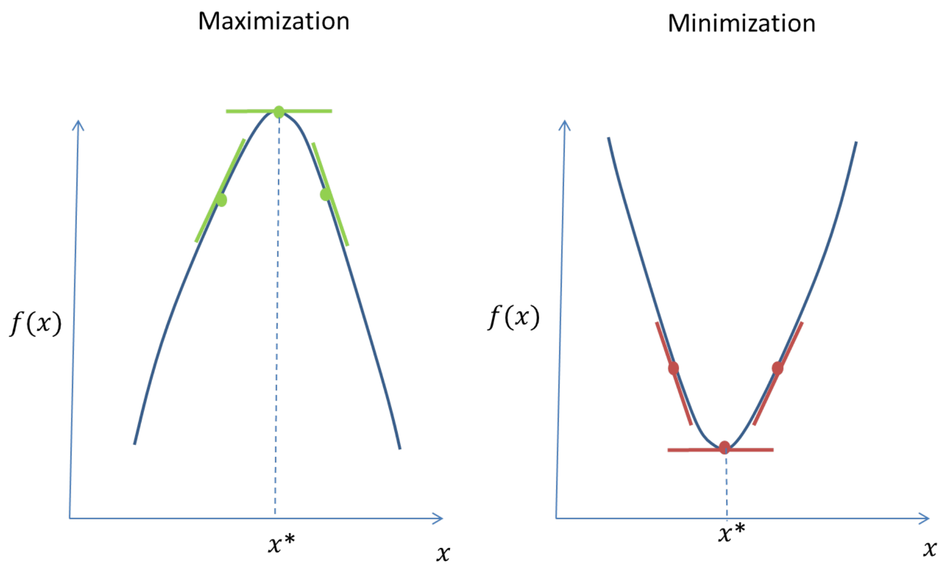

Figure 1.1: Finding the maximum or minimum of a function.

If $f(x)$ is the function that is being optimized and x^* represents the optimum, then these two conditions mean the following (see Figure 1.2):

1. $\frac{\partial f(x)}{\partial x}(x^*) = 0$
2. $\frac{\partial^2 f(x)}{\partial x^2}(x^*) \leq 0$ for a maximum and $\frac{\partial^2 f(x)}{\partial x^2}(x^*) \geq 0$ for a minimum.

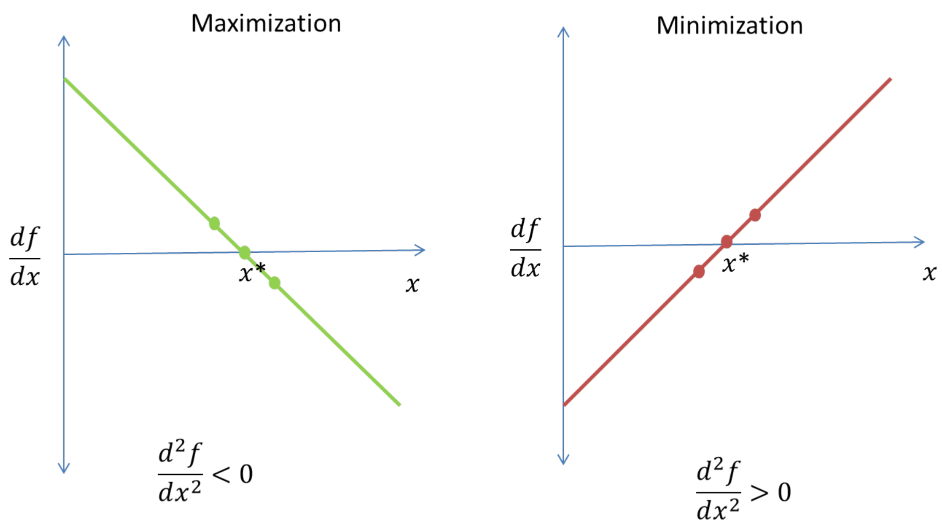

Figure 1.2: Conditions that should be satisfied at the optimum. Note the slope of $\frac{\partial f(x)}{\partial x}$ in the two cases and that it crosses 0 at the optimum

Often systems are not a function of one variable. However, similar conditions apply in higher dimension. For example for a function $z(x, y)$, the conditions for a maximum are:

1. Total variation at the optimum should be 0, i.e $dz(x, y) = 0$. This is the same as $\frac{\partial z(x,y)}{\partial x}dx + \frac{\partial z(x,y)}{\partial y}dy) = 0$, which means that $\frac{\partial z(x,y)}{\partial x} = 0$ and $\frac{\partial z(x,y)}{\partial y} = 0$ should be satisfied at the same time.
2. $d^2 z(x, y) < 0$ for a maximum, which means that $z_{xx}(dx)^2 + 2z_{xy}dxdy + z_{yy}(dy)^2 = 0$ for all possible values for dx and dy. Here $z_{xx} = \frac{\partial^2 z}{\partial x^2}$ and $z_{xy} = \frac{\partial^2 z}{\partial x \partial y}$.

1.2. Constrained optimization

The optimization done has been without constraints. But often performance of a system needs to be optimized under constraints. For example, maximization of production within a water system is subject to constraints such as how the water flows through the system.

For this, consider the problem

$$z_c^*(k) = \max_{x,y} z(x,y) \quad (1.1)$$
$$\text{subject to:}$$
$$g(x,y;k) \geq 0$$

The figure illustrates the constrained optimization problem. Had it been an unconstrained optimization problem, the maximum would have lied on the top of the cone (the z surface). In presence of the g constraint, we need to find an optimum that lies on the slice of the surface z cut by the constraint g.

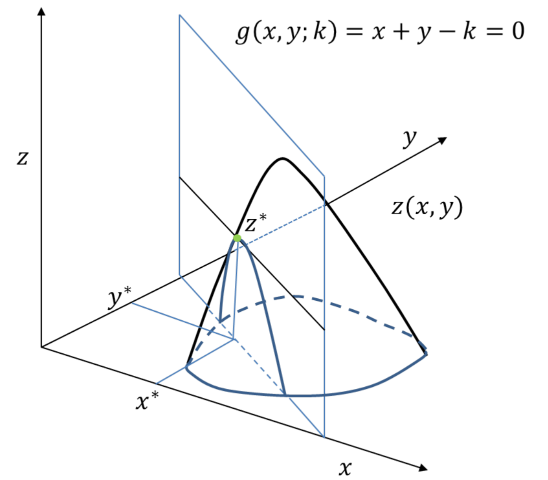

Figure 1.3: Illustration of a constrained optimization problem

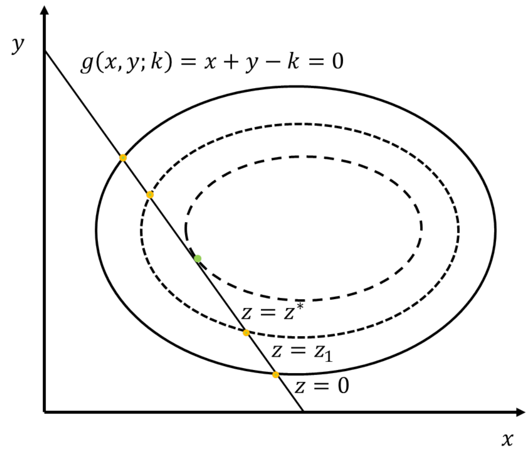

Figure 1.4: Climbing up the slice or contours till it is tangential to the constraints

To find $z^* = z(x^*, y^*)$ illustrated in Figure 1.3, one can think of climbing up the slices cut out of the surface $z(x,y)$ till the slice that barely touches the constraint $g(x,y;k)$. This is the condition when the slice is tangent to the constraint (see Figure 1.2).

To follow the same steps more formally, we follow the Lagrange approach. That is, the constraints are considered as a cost to climb up the surface. We consider a factor λ that scales the cost posed by the constraint. Then, the cost is given by λg, the benefit is given by z and consequently the net benefit is given by $z - \lambda g$. The term λ is called the Lagrange multiplier.

It is however yet not clear how λ should be estimated. For that consider a slight nudge to the constraint by amount Δk. This means that the constraint becomes $g(x,y;k) = \Delta k$. If $\tilde{x}, \tilde{y}$ are the values that maximize the corresponding net benefit function, $z - \lambda(g - \Delta k)$, then we know that $z_c^*(k) = z(x^*, y^*) \geq z(\tilde{x}, \tilde{y}) - \lambda g(\tilde{x}, \tilde{y}; k)$. But note that $g(\tilde{x}, \tilde{y}; k) = \Delta k$. This then means that $\lambda \geq \frac{z(\tilde{x},\tilde{y}) - z(x^*,y^*)}{\Delta k}$. And as we shrink Δk to 0, we obtain $\lambda \geq \frac{\partial z_c^*(k)}{\partial k}$. Or that the Lagrange multiplier or the penalty should be at least as much as the rate with which maximum is reduced by tightening up the constraint.

The above therefore suggests that there is no fixed value for the penalty or the Lagrange multiplier and it has to optimized just as the choice variables x and y. Nonetheless the penalty method helps us to convert a constrained optimization problem to an unconstrained optimization problem. The following provides the optimization solution to a constrained optimization problem in a the form of an unconstrained optimization problem. For this, first a Lagrangian is created, which is the net benefit function defined above.

$$\mathcal{L} = z(x,y) - \lambda g(x,y;k)$$

This Lagrangian function is then maximized with respect to x and y and minimized with respect to λ just as we did in the unconstrained optimization problem to obtain optimal solution.

$$\mathcal{L}^*(x^*, y^*, \lambda^*) = \max_{x,y} \min_{\lambda} [z(x,y) - \lambda g(x,y;k)]$$

We are now ready to dive into the principles of micro-economics.

1.3. Consumer and producer problem

1.3.1. utility maximization

This conceptualizes a consumer problem. We assume that a consumer has a utility function (or a satisfaction index) that it seeks to maximize. It does so by consuming goods that are available in some markets to buy. So, he pays for these good out of his pocket, or more some income that he earns.

We make two assumptions on the type of utility functions, that the consumer gets more utility by consuming more, however the increase in satisfaction declines as more is consumed. If $u(c)$ defines the utility of an agent, derived from consuming c amount of a good, then these two assumptions formally mean that $\frac{\partial u(c)}{\partial c} > 0$ and $\frac{\partial^2 u(c)}{\partial c^2} < 0$. Functions that obey such restrictions are called 'concave' functions. Such 'concavity' assumptions are however often seen as quite restrictive.

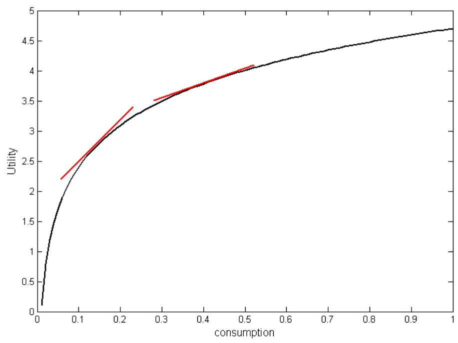

Figure 1.5: A one dimension utility function. Slope of the function is shown by two red lines, which represents how the utility increases with more consumption of the good. As more is consumed, the increase in utility gained reduces.

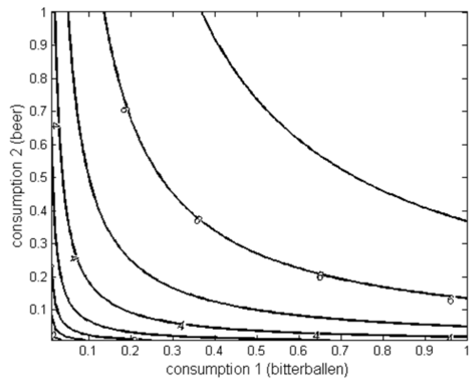

Figure 1.6: A two dimensional utility function

Figure 1.5 shows a one dimensional utility function while Figure 1.3.1 shows a 2 dimensional utility function, where consumption of two goods ('beer' and 'bitterballen') generate utility. In the 2 dimensional case, the level of utility is given by the number on the curve, which increases from the lower left to the upper right corner. In the 2 dimensional utility function case, see for example at level 4, many consumption combination of 'bitterballen' and 'beer' yield similar level of utility. Also, note that the spacing between two utility levels increases as one goes from lower left to upper right corner. This means that more bitterballen and beer need to be consumed to bring about the same change in utility levels, i.e. increase in utility with increasing consumption is smaller at higher levels of consumption.

Any amount of consumption has to be bought, after all beer and bitterballen cost money. If bitterballen amount is denoted by c_1 and beer amount is denoted by c_2 and one can buy it at prices p_1 and p_2 repectively, then total that it costs to consume c_1 bitterballen and c_2 beer is given by $p_1c_1 + p_2c_2$. If further we assume that M is the total income and all is spent on consuming either beer and bitterballen then an individual is constrained by the equation $p_1c_1 + p_2c_2 = M$.

Thus the attempt to maximize utility that one derives from eating bitterballen and drinking beer is constrained by the 'household budget' constraint $p_1c_1 + p_2c_2 = M$, as shown in Figure 1.7. The budget constraint appears as a line in the plot such that if income M increases by ΔM and becomes $M + \Delta M$ then the budget line shifts to the upper right corner. An individual will attempt to reach as high utility level as possible while making sure that total cost of buying beer and bitterballen does not exceed M. Which will be the utility level at which the budget line becomes tangent as shown in Figure **??**. And the point at which the budget line $p_1c_1 + p_2c_2 = M$ touches the utility level (c_1^*, c_2^*) is the amount of beer and bitterballen that satisfied him the most given that he can spent atmost M.

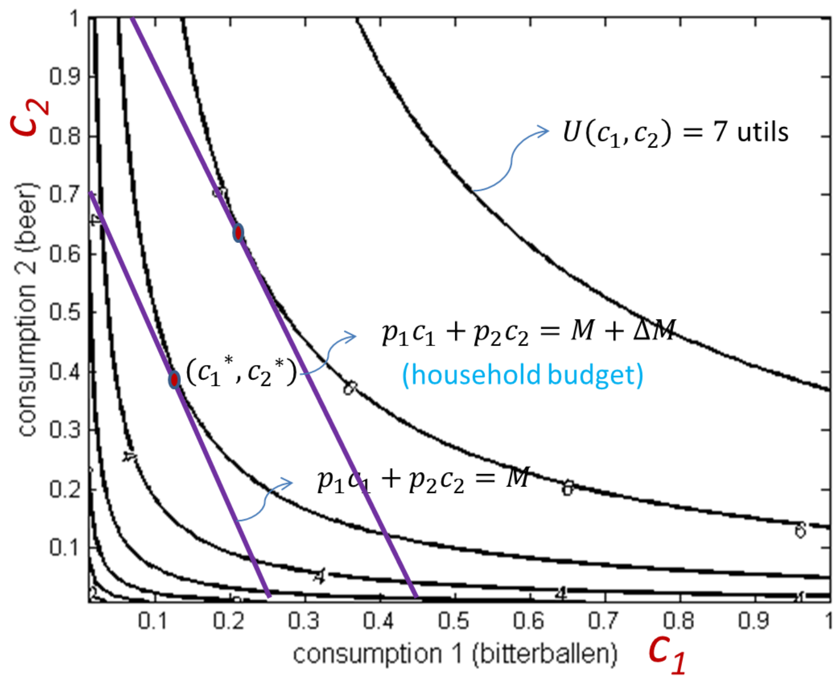

Figure 1.7: Utility maximizaiton subject to household budget constraints

Figure 1.7 graphically shows utility maximization problem subject to a constraint. Note that this is no different from the optimization problem given in Figure 1.2, with the constraint $g(x, y; k) = 0$ given by $g(x, y; k) \equiv p_1c_1 + p_2c_2 - M$ and the objective function $z(x, y)$ given by the utility function, i.e. $z(x, y) \equiv u(c_1, c_2)$. The corresponding optimization problem, following Equation 1.1, can then be given by,

$$U^*(p_1, p_2, M) = \max_{c_1,c_2} u(c_1, c_2) \tag{1.2}$$
$$\text{subject to:}$$
$$p_1c_1 + p_2c_2 = M$$

The Langrangian of the problem given in program 1.2 is given by $\mathcal{L} = u(c_1, c_2) - \lambda(p_1c_1 + p_2c_2 - M)$. Here $\lambda \geq \frac{\partial U^*(p_1,p_2,M)}{\partial M}$. We can now find the utility maximization by solving the following,

$$\mathcal{L}^*(c_1^*, c_2^*, \lambda^*) = \max_{c_1,c_2} \min_{\lambda} [u(c_1, c_2) - \lambda(p_1c_1 - p_2c_2 - M)] \tag{1.3}$$

Note that the maximization of the Lagrangian is an unconstrained maximization problem, i.e. the Lagrangian is not constrained by any constraints. As discussed in the section on optimization, the conditions for maximum are given by $\frac{\partial \mathcal{L}}{\partial c_1} = 0$ and $\frac{\partial \mathcal{L}}{\partial c_2} = 0$. This then means that the optimal consumption levels, (c_1^*, c_2^*), are solutions to the following equations:

$$\frac{\partial \mathcal{L}}{\partial c_1} = \frac{\partial u}{\partial c_1} - \lambda p_1 = 0 \tag{1.4}$$
$$\frac{\partial \mathcal{L}}{\partial c_2} = \frac{\partial u}{\partial c_2} - \lambda p_2 = 0$$
$$p_1c_1^* - p_2c_2^* - M = 0$$

From the first two conditions, one can note an important condition, that $\frac{\partial u}{\partial c_1}/\frac{\partial u}{\partial c_2} = p_1/p_2$. The fraction $\frac{\partial u}{\partial c_1}/\frac{\partial u}{\partial c_2}$ is the slope of the tangent to a utility curve shown in Figure 1.7 at point (c_1^*, c_2^*). The first order conditions tell us that the optimal solution should be such that the slope of this tangent be equal to the ratio of prices p_2/p_1. An interesting interpretation of this condition is that when we maximize our utility from the consumption of two goods, the optimal level of consumption is such that the trade-off between the two is controlled by the ratio of the corresponding prices.

If we consider a utility function of type $u(c_1, c_2) = \ln c_1 + \eta \ln c_2$ then the above condition means that $\eta c_2^*/c_1^* = p_1/p_2$. Consider the consumption of the two goods as consumption of 'all the food put together' and 'consumption' of environment quality. Then η represents how consumption c_1 of all the food put together trades off with environmental quality at a given level of utility. Also, the consumption of food per unit environment quality will depend on how expensive it is to maintain environment quality in comparison to buying food, as represented by the ratio $p_2/\eta p_1$.

The slope of the tangent to utility level sets at the optimum being equal to the ratio of the prices is only one part of the story. The other part is the level of utility or satisfaction that can be achieved. Note from Figure 1.7 that higher income shifts the budget line to the upper right corner. Thus higher income has the role of shifting the budget line while keeping the slope the same, so that higher level of satisfaction can be achieved.

More income here definitely plays an important role in achieving higher levels of satisfaction. So where is this income generated?

1.4. Profit maximization

Income in often generated by participating in income generating activities. This is mostly done either by investing labor (skilled or unskilled hours) or by investing capital in economic activities and hoping to get a return either in the form of wages or rate of return, based on how badly these inputs are needed production. For example, a producer of beer demands ℓ^* amount labor for which he is willing to pay w per unit labor, while demands k^* amount of capital foe which he is willing to pay a return of r. So the income from ℓ^* amount of labor and k^* amount of capital can be given by $w\ell^* + rk^*$.

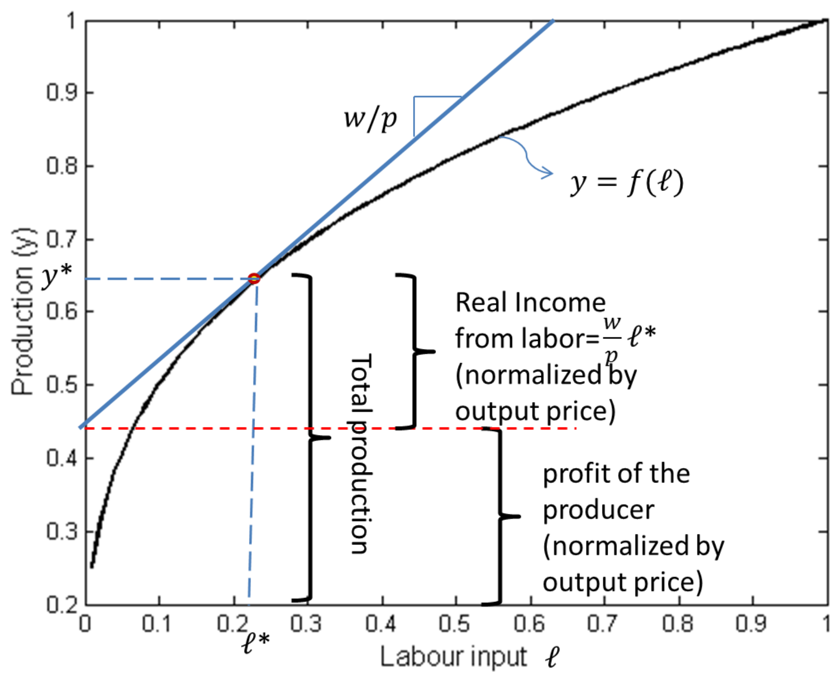

Figure 1.8: Profit maximization when output produced can be sold at price p and labor can be bought at price w.

The demand for labor or capital or both are important considerations in a profit maximization problem because this is what, e.g., determines how many hours an agent can be contribute and how much income can he then generate. Figure 1.8 shows such a profit maximization problem where a producer, e.g. beer, makes decision on how much labor should he buy at price (or wage rate) w to maximize his profit by selling the output y at price p.

If $y = f(\ell)$ defines the production function, which transforms certain level of labor ℓ, e.g. in units of hours, into output y, e.g. number of beer bottles, then profit function of a producer (revenue - cost) can be given by $\pi(y, \ell; p, w) = py - w\ell$. His maximization problem is given by,

$$\Pi(p, w) = \max_{\ell} pf(\ell) - w\ell \tag{1.5}$$

Note that this is an unconstrained optimization problem. The first order condition will the determine the labor demand ℓ^* of a profit maximizing producer given by $p\frac{\partial f}{\partial \ell} - w = 0$. This means that optimal labor demand will be when marginal productivity of labor $\frac{\partial f}{\partial \ell}$is equal to the ratio wage to output price w/p. The latter is also called relative wage. Figure 1.8 also shows the same condition, the slope of the tangent of the production curve $f(\ell)$ at optimal labor demand ℓ has the slope of w/p.

Therefore demand for labor slows down as per unit hour of labor becomes expensive or price of the output becomes cheap. This is because the latter makes the tangent steep, which then can only touch the production function at low values of labor on the production function.

But where is the income that a worker gets by getting involved in this production activity? Note that maximized profit is given by $\Pi(w, p) = pf(\ell^*) - w\ell^*$, which is the amount left to the producer after paying the salaries of the workers given by a total of $w\ell^*$. Consider a world where the output is priced at 1 monetary unit per unit of output. In this world, wage rate per unit hour is given by w/p. Then the total revenue can be given by $y*$ out of which $w\ell^*/p$ goes to labor income and rest is the profit that stays with the producer.

The consumption (c_1, c_2) that maximizes utility depends on prices (p_1, p_2) and income M. Meanwhile income M depends on prices (w, r) and prices of output p, which in an economy that produces two goods, instead of 1 as indicated above, will be indicated by (p_1, p_2). The prices (p_1, p_2, w, r) therefore play roles in both the utility maximization problem (which is a consumer problem) and profit maximization problem (which is a producer problem). Figure 1.9 illustrates the feedback roles that the pries (p_1, p_2) and (w, r) play between a 'representative' consumer and a 'representative' producer of an economy.

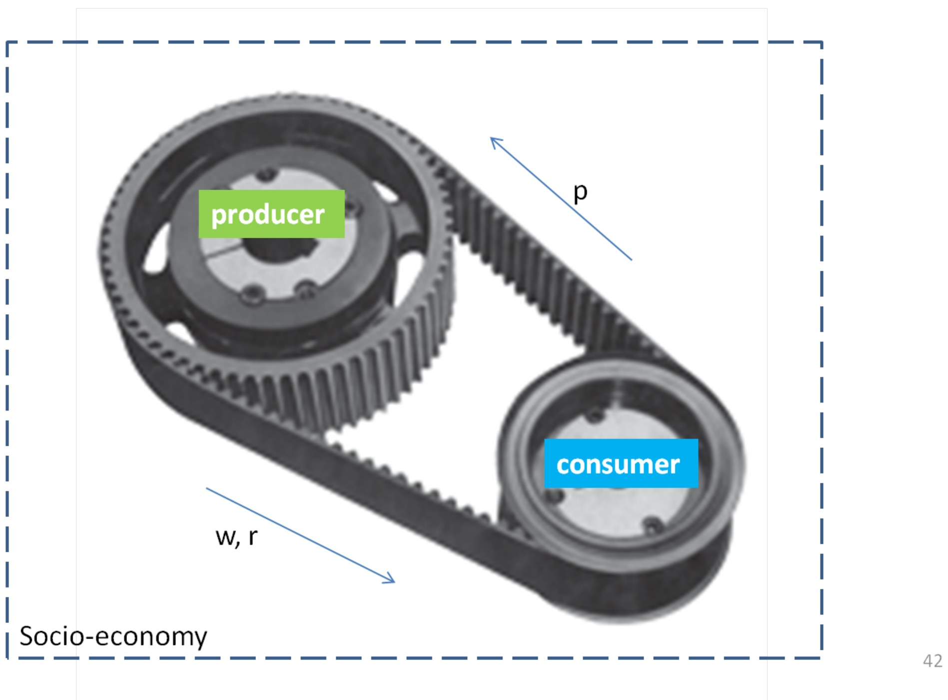

Figure 1.9: The producers and consumers in harmony in a socio-economy due to feedbacks of p and (w, r).

1.5. General equilibrium

So where do the prices emerge from that bring producers and consumers of an economy in harmony? It is clear that these prices are such that 1) the representative 'consumer' maximizes his utility and 2) the representative 'producer' maximizes his profit. But what consumers produce should be what producers produce. Since what consumers pay for consumption goods are the prices at which producers sell their product, p should be the price at which the market of the consumption good 'clears'. Similarly, wage rate w is the price at which demand of labor clear the supply and the interest rate r is the price at which demand for capital is cleared by its supply.

This leads us to the definition of a socio-economy at equilibrium. It is the level of consumption, production and prices such that 1) consumers maximize their utilities, 2) producers maximize their profits and 3) prices clear the markets for consumption goods, labor and capital. This definition and corresponding interpretation can be generalized to any economy with any number of goods being consumed and produced, skilled and unskilled labor, capital and services of various types.

1.5.1. Production possibility frontier

The consumer problem of utility maximization dealt with the maximization of utility as a function of two goods (c_1, c_2), while the producer problem of maximizing profit dealt with only 1 good, i.e. y. An extension of the producer's problem from 1 output to 2 outputs is therefore needed to fully understand the general equilibrium concept in 2-dimension (with corresponding market clearing prices of p_1 and p_2).

For this consider production of two good x and y in an economy, both as a function of labor and capital, i.e. $x = x(\ell_x, k_x)$ and $y = y(\ell_y, k_y)$, in a socio-economy with finite endowment of capital $\bar{K}$ and labor $\bar{L}$. The production functions $x(.,.)$ and $y(.,.)$ incorporate the technologies in producing the two goods. A socio-economy that is driven by efficiency will try to allocate (k_x, ℓ_x) and $(\bar{K} - k_x, \bar{L} - \ell_x)$ such that the production of the two goods, x and y, is maximized.

The so-called 'Edgworth' box in Figure 1.10 illustrates how the total amount of labor, ℓ, and capital, k, can be allocated between the production of the two goods.

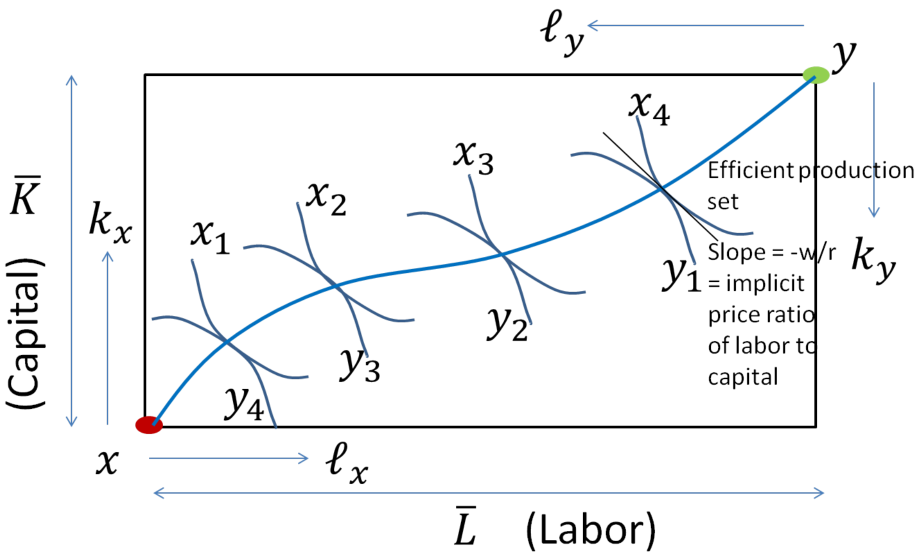

Figure 1.10: The Edgeworth box that is obtained by joining the plot for the production function of x with the inverted plot for the production of y. This makes the length of x-axis to be $\bar{L} = \ell_x + \ell_y$ and total length of y-axis to be $\bar{K} = k_x + k_y$.

Essentially what Edgeworth box does is that it first fixes a certain production level of x, say at x_1 and then maximize the production of the second good y subject to the constraints that the socio-economy has $\bar{L}$ total labor and $\bar{K}$ total capital to allocate between the production of the two goods. As the Figure 1.10 suggests, this leads to production level y_4 for the second good such that it barely touches the production of first good x at level x_1. The corresponding input points (ℓ_x, k_x) and (ℓ_y, k_y) 'efficiently' produce the outputs at production levels x_1 and y_4. By repeating this process of 'efficient allocation of capital and labor for all possible level of x (from 0 to $x(\bar{L},\bar{K})$), we obtain the efficient production set which is the locus of all allocation pair that efficiently produce x and y.

Mathematically, it can be given by the following maximization problem,

$$\begin{aligned}\max_{k_y,\ell_y,k_x,\ell_x} & \; y(k_y,\ell_y) \\ \text{subject to} & \\ x(k_x,\ell_x) &= x_c \\ k_x + k_y &= \bar{K} \\ \ell_x + \ell_y &= \bar{L}\end{aligned} \tag{1.6}$$

Here x_c represents a certain constant value. In order to solve this maximization problem, we first create a Lagrangian $\mathcal{L} = y(k_y,\ell_y) - \lambda(x(k_x,\ell_x) - x_c) - r(k_x + k_y - \bar{K}) - w(\ell_x + \ell_y - \bar{L})$ and then maximize it with respect to (k_y,ℓ_y,k_x,ℓ_x). Here (λ,w,r). Corresponding first order conditions are (excluding the constraints),

$$\frac{\partial y(k_y,\ell_y)}{\partial k_y} - r = 0 \tag{1.7}$$

$$\frac{\partial y(k_y,\ell_y)}{\partial \ell_y} - w = 0 \tag{1.8}$$

$$\lambda\frac{\partial x(k_x,\ell_x)}{\partial k_x} - r = 0$$

$$\lambda\frac{\partial x(k_x,\ell_x)}{\partial \ell_x} - w = 0$$

As also shown in Figure 1.10, the above mean that along the efficient production set of inputs, $\frac{\partial y(k_y,\ell_y)}{\partial k_y} / \frac{\partial y(k_y,\ell_y)}{\partial \ell_y} = \frac{\partial x(k_x,\ell_x)}{\partial k_x} / \frac{\partial x(k_x,\ell_x)}{\partial \ell_x} = w/r$. In other words, at any point on the efficient production set, the slope of the line through the point where x and y production function touch each other is equal to the ratio of wage rate to the interest rate (note here that this is just a statement on the ratio, w and r are not necessarily corresponding prices - this is clarified in the next chapter). w is the Lagrange multiplier at which the demand for labor in the production of the two goods, i.e. $\ell_x + \ell_y$ is 'cleared' by the supply $\bar{L}$. Similarly, r is the Lagrange multiplier where supply $\bar{K}$ clears the demand for capital.

In summary, 'efficient allocation of inputs' is an allocation of inputs to the production of outputs such that it is not possible to produce more of one good without producing less of another. Meanwhile, 'efficient production set' is the set of efficient allocation of inputs for all possible level of outputs. The corresponding possible combinations of output levels x and y that can be efficiently produced for given total amounts of capital $\bar{K}$, labor $\bar{L}$ and technology (as implicit in the production functions) yields gives the production possibility frontier. In other words it is the maximum levels of y that can be produced by the program 1.6 for varying levels of x lying between 0 and $x(\bar{L},\bar{K})$. The production possibility frontier is a function of type $F(x,y) = 0$, which may also be represented as y being a function of x. This is also illustrated in Figure 1.11,

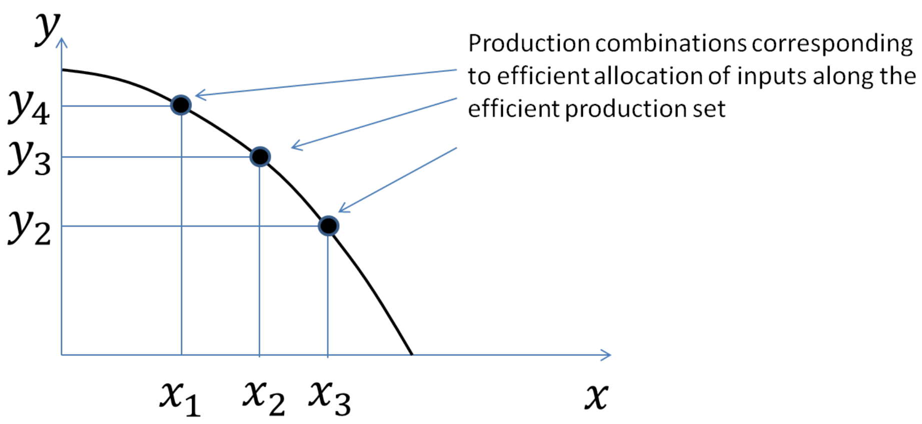

Figure 1.11: The production possibility frontier, which is the maximum levels of y that can be produced by the program 1.6 for varying levels of x lying between 0 and $x(\bar{L}, \bar{K})$.Note that that y and x combinations are the same as shown in Figure 1.10.

Let us now take a simple example for given functional forms for x and y. Let $x(k_x, \ell_x) = k_x^{1/2}\ell_x^{1/2}$ and $y(k_y, \ell_y) = k_y^{1/2}\ell_y^{1/2}$. Then from the tangency first order condition for efficient production set above, we note that $\ell_x/k_x = \ell_y/k_y$. But since $\ell_x/k_x = (\bar{L} - \ell_x)/(\bar{K} - k_x)$, we obtain the following relationship between k_x and ℓ_x that plots the efficient production set in the Edgeworth box, $\ell_x\bar{K} - k_x\bar{L} = 0$ or $\ell_x/k_x = \bar{K}/\bar{L}$. The efficient production set is a straight line with slope $\bar{K}/\bar{L}$.

The corresponding production possibility frontier can now be obtained by substitutions. For a given level x and knowing that $\ell_x = \bar{K}k_x/\bar{L}$, we obtain $k_x = \left(\bar{L}/\bar{K}\right)^{1/2} x$ and $\ell_x = \left(\bar{K}/\bar{L}\right)^{1/2} x$. Plugging it in $y = k_y^{1/2}\ell_y^{1/2}$, we obtain

$$y = \left(\bar{K} - \left(\bar{L}/\bar{K}\right)^{1/2} x\right)^{1/2} \left(\bar{L} - \left(\bar{K}/\bar{L}\right)^{1/2} x\right)^{1/2} \tag{1.9}$$

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1.5.2. General equilibrium 1.0

Now steps towards a general equilibrium of this two-good socio-economy is straight forward. We have a production possibility frontier, which gives the best possible set of production combinations that producers in a socio-economy can come up with. Let this be represented by $F(x, y) = 0$. Let the utility of consumers be given by $u(x, y)$ (note here the change in notation for the consumption of two goods, e.g. coconuts and fish in a Robinson Crusoe economy, which in the utility maximization section was given by c_1 and c_2; however this change in notation does not harms the set of conclusions drawn). The maximum utility that is achievable in a socio-economy is limited by the constraint posed by the production possibility frontier, i.e. highest level of utility that is achievable will be the one that is tangent to the production possibility frontier. Further, the slope of the tangent, as shown in Figure 1.12 will be the ratio of prices p_1/p_2, which are the prices at which demand for x and y by utility maximizing consumers is exhausted by the corresponding supply provided by the producers.

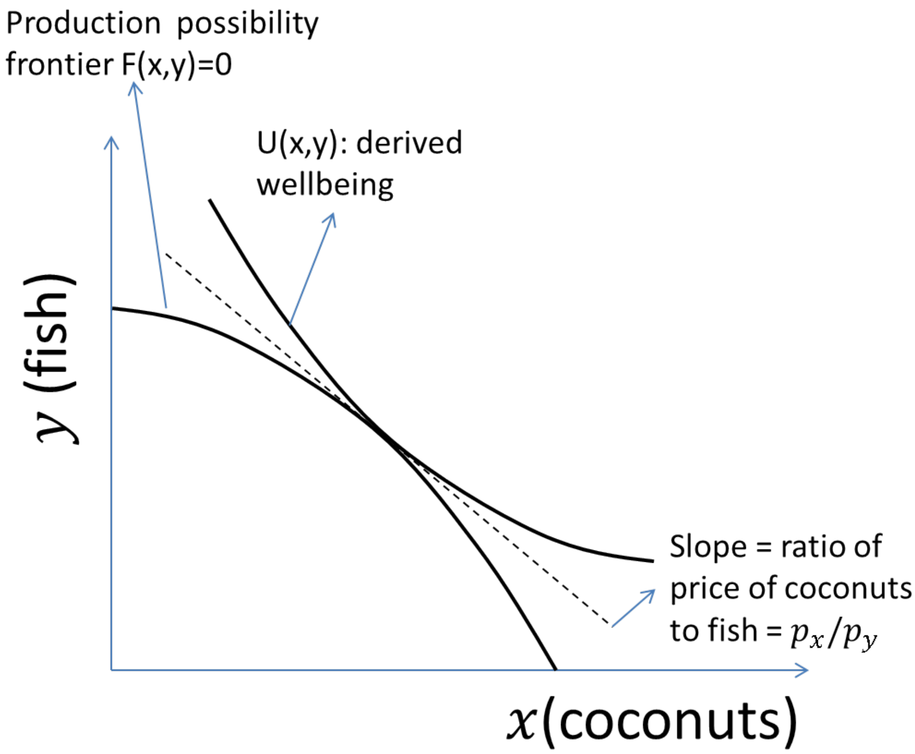

Figure 1.12: General equilibrium: utility u(x,y) is maximized when it is tangent to the production possibility frontier F(x,y) = 0

Mathematically, the optimization program is

$$\begin{aligned} \max_{c_1,c_2,x,y} u(c_1,c_2) \\ \text{subject to} \\ c_1 \leq x \\ c_2 \leq y \\ F(x,y) = 0 \end{aligned} \tag{1.10}$$

Now think of a log utility function, e.g. of form $u(c_1, c_2) = \ln c_1 + \eta \ln c_1$ and a functional form for $F(x, y) = 0$ for the case when x represents food and y represents the environmental good. Assume that the production of both the goods requires 'water' and capital as input but based on different functional forms for its production function.

2

Socio-hydrology II: Coupled quality-growth dynamics over generations

2.1. Solving a Robinson-Crusoe type socio-economy

The previous chapter on socio-hydrology dealt with general equilibrium modelling of a socio-economy. For two goods economy, where x and y goods are produced using production functions (which implicitly considers technology) $x(k_x, \ell_x) = k_x^{1/2}\ell_x^{1/2}$ and $y(k_y, \ell_y) = k_y^{1/2}\ell_y^{1/2}$, we came up with the efficient production set given by $\ell_x/k_x = \bar{K}/\bar{L}$. Think of x as the production of all the food that the economy is driven by and y as the production of the environment quality. We assume that both compete for labor and capital. The efficient production set in the Edgeworth box is a straight line with slope $\bar{K}/\bar{L}$.

If we assume that the economy is endowed with 100 units of capital and 400 units of labor, i.e. $\bar{K} = 100$ and $\bar{L} = 400$, then the efficient production set is $\ell_x/k_x = 1/4$. Plugging this in Equation 1.9 of the Production Possibility Frontier derived in the previous chapter, we obtain

$$
\begin{aligned}
y^2 &= \left(100 - (4)^{1/2}\, x\right)\left(400 - (1/4)^{1/2}\, x\right) \\
\Rightarrow F(x,y) &= y^2 - x^2 + 850x - 40000 = 0
\end{aligned}
\tag{2.1}
$$

The Robinson Crusoe's general equilibrium type problem can now be solved if we assume that his PPF is given by equation given in Equation 2.1. Let the utiity function as mentioned in the previous section be given $u(c_1, c_2) = \ln c_1 + \eta \ln c_2$. Here c_1 is the consumption of food that is linked with economic growth while c_2 is the consumption of environment quality. Then mathematical program (following Equation 1.10) corresponding to the general equilibrium of this socio-economy is given by,

$$
\begin{aligned}
\max_{c_1,c_2,x,y} u(c_1,c_2) = \max_{c_1,c_2,x,y} & \ln c_1 + \eta \ln c_2 \\
\text{subject to} & \\
c_1 \le x & \quad (p_1) \\
c_2 \le y & \quad (p_2) \\
F(x,y) = 0 & \quad (\lambda)
\end{aligned}
\tag{2.2}
$$

Here the Lagrange multipliers are given in the parenthesis alongside each of the constraints. Forming the Lagrangian, we solve the following unconstrained optimization,

$$
\max_{c_1,c_2,x,y}\ \min_{p_1,p2,\lambda} \mathcal{L}(c_1,c_2,x,y,p_1,p2,\lambda), \tag{2.3}
$$

where,

$\mathcal{L}(c_1, c_2, x, y, p_1, p2, \lambda) = \ln c_1 + \eta \ln c_2 - p_1(c_1 - x) - p_2(c_2 - y) - \lambda(y^2 - x^2 + 850x - 40000).$

The first order conditions of the above unconstrained optimization problem gives,

$$\frac{1}{c_1} - p_1 = 0 \tag{2.4}$$
$$\frac{\eta}{c_2} - p_1 = 0 \tag{2.5}$$
$$p_1 + 2\lambda x - 850\lambda = 0 \tag{2.6}$$
$$p_2 - 2\lambda y = 0 \tag{2.7}$$
$$c_1 = x \tag{2.8}$$
$$c_2 = y \tag{2.9}$$
$$y^2 - x^2 + 850x - 40000 = 0 \tag{2.10}$$

The general equilibrium solution, $(c_1^*, c_2^*, x^*, y^*, p_1^*, p_2^*, \lambda^*)$, solves the above set of equations. By substituting out λ in equations 2.6 and 2.7, we obtain

$$\frac{p_1^*}{p_2^*} = \frac{850 - 2x^*}{2y^*}$$

Since from equations 2.4 and 2.5 we have $p_1/p_2 = c_2^*/\eta c_1^*$, this means that

$$\frac{p_1^*}{p_2^*} = \frac{850 - 2x^*}{2y^*} = \frac{c_2^*}{\eta c_1^*} \tag{2.11}$$

From equations 2.8,2.9 and 2.11, we obtain a relationship between $(850 - 2c_1^*)c_1^*\eta = 2(c_2^*)^2$, which implies $(425 - c_1^*)c_1^*\eta = (c_2^*)^2$. Substituting this in Equation 2.10, we can now solve for c_1 as,

$$425c_1^*\eta - (c_1^*)^2\eta - (c_1^*)^2 + 850c_1^* - 40000 = 0$$
$$\Rightarrow -(1+\eta)(c_1^*)^2 + (850 + 425\eta)c_1^* - 40000 = 0 \tag{2.12}$$

Solving for c_1^*, we get the general solution as $c_1^* = -\frac{-(850+425\eta)\pm\sqrt{(850+425\eta)^2-160000(1+\eta)}}{2(1+\eta)}$. This shows the possibility that two solutions are possible. Does that meant that a non-unique solution is possible?

Let us solve this for the case when the society weighs the consumption of the two goods equally. That is, when $\eta = 1$ or that the society is willing to give up one unit of economic growth to gain one unit of environmental quality. For this, we obtain two solutions of c_1^*, 604.4 and 33.1 units of the first good. However, when we plug it into the relationship between c_1^* and c_2^*, which is $(425 - c_1^*)c_1^*\eta = (c_2^*)^2$, we obtain complex number solution when c_1^* = 604.4. The other real number, and acceptable, solution for c_1^* = 33.1 is c_2^* = 113.8 units of c_2. The price solutions can be consequently calculated using Equations 2.4 and 2.5 and are given by p_1^* = 0.030 units and p_2^* = 0.008 units. So

Till now we have dealing with static general equilibrium. That is, we get only one solution for all times. It is also not clear how production of water quality works on environmental degradation. Both socio-economy and the environment are not static. Hence a more thoughtful model is needed for coupled economic growth and environment quality system.

Following questions remain unanswered.

1. How can we build low order models of economies or societies realistically?

2. Do we need models that determine how environment and economic growth shape each other as societies evolve over generations?

3. Can we make optimal investments in environment and economic growth?

2.2. Coupled growth-environment quality Model

The basic ingredient of this model is modeling the socio-economy over time. For this we conceptualize a society as composed to two 'overlapping' generations that think alike. Each such generation is composed of 2 types of agents, one young and the other old. Each such generation is therefore assumed to live 2 periods, first they are born and become 'young' and then they grow old and die. A society is assumed to be composed of two such overlapping generations such that at any time the young of one generation overlaps with the old of the other. Figure 2.1 below illustrates such a conceptualization.

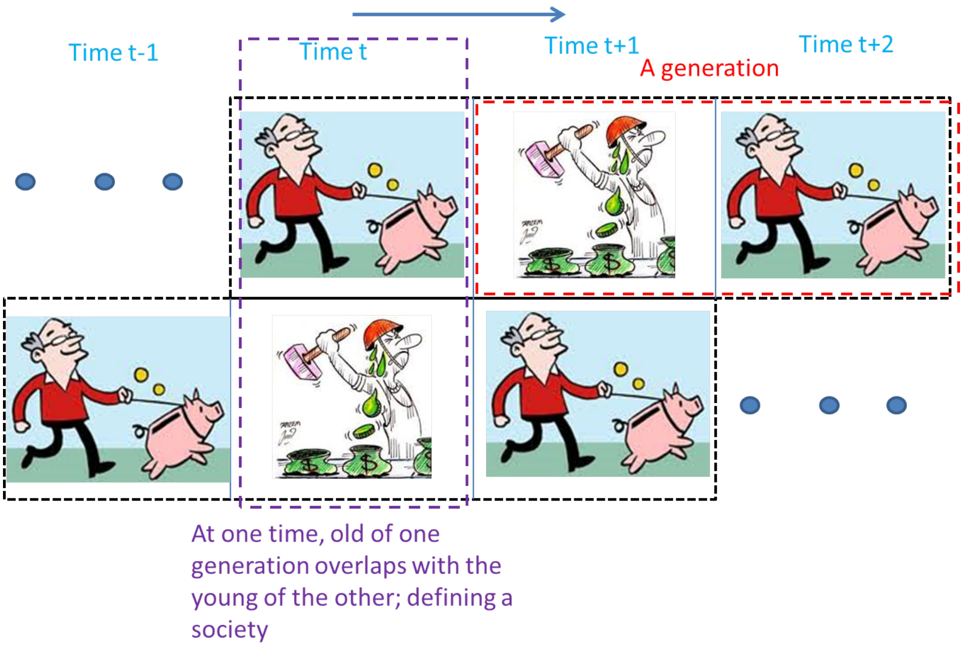

Figure 2.1: Conceptualizing societies in time: overlapping generations model.

An infinite horizon is considered in discrete time. The two types of agents at any point in time play complementary roles. Young agents only work and invest both for their retirement in the next time step as well as for their environment. In the same time period, the old agent of the other generation do not work but live off the savings or investment made by himself when young (in the previous time step). The old use this saving to consume what is being produced by the young agent of the first generation in the current time step.

The amount that is consumed and savings that is made is such that a generation maximizes its utility. The production is efficient (i.e. it is profit maximizing). The wage rate and interest rates are such that all the available labor and capital is used. All these conditions correspond to general equilibrium conditions. Thus consumption, production and prices are such that the society is at equilibrium. The coupled dynamics of environment quality and economic growth emerges as a result of this equilibrium - essentially from the first order condition of the mathematical program corresponding to the general equilibrium problem.

This is further clarified in the below.

2.2.1. Young agent

As mentioned earlier, the young agents role in this constructed society is solely to work and plan for the future old self, either by investing in saving or in the environment. However note that he is part of the generation and it is not his young self that decides how he invests in the saving and the environment. It is the generation as a whole which makes such a decision (to be clarified later). And since the society is composed of two similar minded generations, the decision problem for both the generations is the same.

The 'partial' picture that the young agent sees is that at time t he earns a wage w_t from 1 unit of labor that is split between saving s_t for the old age and m_t that is an investment to 'produce' environment quality (or reduce environmental degradation.

So his budget is given by,

$$s_t + m_t = w_t \tag{2.13}$$

Note that the young agent does not consume but only slaves himself for when he gets old.

2.2.2. Old agent

The young agent within a generation gets old in the next time step $t+1$. It is now time to enjoy all the hard work done when he was young, hoping that the 'markets' will sufficiently value his savings. The old agent consumes c_{t+1} amount of all-the-food-put-together but pays a consumption tax τc_{t+1} since the production of this good pollutes the environment and this is collected and used to produce environment quality. If the old agent is able to consume $c_t + 1$ amount, he surely has some income to buy this. This income comes from the saving that he did when he was young. He reaps an interest of r_{t+1}, set by the financial markets (i.e. endogenous and emerge from general equilibrium condition). So his income is $(1 + rt + 1)s_t$. So his budget is given by,

$$(1+\tau)c_{t+1} = (1 + r_{t+1})s_t \tag{2.14}$$

Note that the old agent is also in charge of producing off-spring for the next generation so that humanity can continue for the next time step.

2.2.3. Environment quality

So the production of food pollutes the environment. Let the state of the environment be given by E_t. The production of amount c_t, degrades the environment by βc_t. The environment quality is assumed to degrade on its own due to factors not considered here at a rate b. Finally investment in the environment, $m_t + \tau c_{t+1}$ improves environment quality by a factor of γ.

The above improvements lead to an updated state for environmetal quality given by the following equation,

$$E_{t+1} = E_t - bE_t - \beta E_t + \gamma(m_t + \tau c_t) \tag{2.15}$$

2.2.4. Preference structure of the society

As highlighted earlier, it is neither the young or the old agent who independently decides how to use his income. It is the entire generation as a whole that take a collective decision for what is best for it in terms of food (i.e. economic growth) and the environment. This is based on how it prefers food versus the environment.

A log utility function is assumed that is not just a function of consumption versus the environment but also considers habit formation of a generation of good environment quality (i.e. a generation get used to good environment quality). A generation at any time is assumed to maximize this utility function.

$$u_t(c_{t+1}, E_{t+1}, E_t) = \ln c_{t+1} + \eta \ln(E_{t+1} - \phi E_t) \tag{2.16}$$

Here η is a trade-off parameter that measure the relative preference of a generation for the environment.

Another important parameter is ϕ that quantifies the intensity of habit formation. Note that a generation would want to keep $E_{t+1} - \phi E_t$ sufficiently positive since units of consumption per unit improvement in environment quality goes to infinity (in order to sustain a certain level of utility) as $E_{t+1} - \phi E_t$ goes to 0. Thus ϕ measures how demanding a generation is with regards to environment quality. A value of ϕ close to 1 will mean that every successive generation will never like to see degrading environment quality. However a ϕ close to 0 is more relaxed about environmental degradation. Note that $0 \leq \phi < 1$.

2.2.5. Technology based production function

The young agents participates in the economic activity of producing the 'food' that is consumed by the old agent of the other overlapping generation. In lieu of this participation, the agent receives a wage. The following production function gives the process of transforming capital that is saved over time by agents and 1 unit of labor to produce the consumption good.

$$y_t = f(k_t, \ell_t) = Ak_t^{\alpha}\ell_t^{1-\alpha} - \delta k_t \tag{2.17}$$

Here $0 < \alpha < 1$ is the capital share of output and $0 < \delta < 1$ is called the depreciation rate of capital (the rate at which capital such as chair degrades over time on its own). A is a technology related factor, improved technology linearly scales up production level.

One important consideration is that the economy produces only one good. So its price may as well be measured in terms of capital. This is equivalent to suggesting that the price of the produced good is always set to 1.

2.3. Dynamic general equilibrium between economic growth and environmental quality

We are modeling a 'closed' society over an infinite horizon that produces a commodity (food) that it consumes at the cost of environment quality. Thus consumption influences the society's environment, which may lead to suppressed consumption in the future via its relative preference for the environment if it degrades too much.

The temporal dynamics of how society consumes in time and how its environment quality fares emerges from the interaction between utility maximizing generational behavior and technology dependent profit maximizing behavior of consumers who produce the consumption good such that the commodity balances of capital and labor hold at each time step.

2.3.1. Utility maximization

The 'partial problem ' of utility maximization of a generation is given by below. A generation needs to act, i.e. maximize utlity given in Equation 2.16 based on interest rate r_{t+1} and wage rate w_t, its budgets for when he is young and when he is old (Equations 2.13 and 2.14) and the state of the environment (Equation 2.15). It is for this reason that it is called a 'partial problem'.

$$\max_{c_t, E_t, E_{t+1}, s_t, m_t} u(c_t, E_t, E_{t+1} \tag{2.18}$$

$$\begin{aligned} &\text{subject to} \\ s_t + m_t &= w_t \\ (1+\tau)c_{t+1} &= (1+r_{t+1})s_t \\ E_{t+1} &= E_t - bE_t - \beta E_t + \gamma(m_t + \tau c_t) \end{aligned} \tag{2.19}$$

2.3.2. Profit maximization

Profit maximization will dictate how capital is allocated to the production of the consumption good. Again for given interest rate r_{t+1} and wage rate w_t, the objective of the producers within the society will be to maximize the profit,

$$max_{k_t, \ell_t} f(k_t, \ell_t) - r_t k_t - w_t \ell_t, \tag{2.20}$$

where the production function, $f(k_t, \ell_t)$, is given by Equation 2.17.

2.3.3. Balance of commodities and emergence of prices

The third set of conditions are about the commodity balance and the emergence of prices. There are two commodity balances, one for capital k_t and the other one is for labor ℓ_t.

$$\begin{aligned} k_t &= w_{t-1} \\ \ell_t &= 1 \end{aligned} \tag{2.21}$$

Essentially the above balances suggest that level of capital at any point in time is equal to the savings made by a young agents in the last time step. The labor supply is 1 unit, supplied by the young agent.

Since prices should be such that these commodity balances hold and that efficiency conditions given in program 2.20 hold, this means that the following conditions set the prices.

$$k_t = \frac{\partial f(k_t, 1)}{\partial k_t} = A(\alpha)k_t^{\alpha-1}\ell_t^{1-\alpha} - \delta \tag{2.22}$$

$$w_t = \frac{\partial f(k_t, 1)}{\partial \ell_t} = A(1-\alpha)k_t^{\alpha} = (1-\alpha)f(k_t, 1) \tag{2.23}$$

In the above, wage w_t is set equal to marginal productivity of labor for amount of labour ℓ_t set to 1. Similarly the price of capital or the rate of return on capital, r_t, is set to marginal productivity of capital with labour input level set to 1.

2.4. Emergence of coupled economic growth - environment quality dynamics

The coupled dynamic is revealed by solving the first order conditions of the mathematical program corresponding to the above general equilibrium (i.e. the problems specified in the above 3 subsections considered jointly). In this section we will solve for the coupled dynamic equations.

The Lagrangian corresponding to the utility maximization problem of a generation given in the program 2.18 is given by

$$\begin{aligned} \mathcal{L}(c_t, E_t, E_{t+1}, s_t, m_t, \lambda_1, \lambda_2, \lambda_3) = \\ \ln c_{t+1} + \eta \ln(E_{t+1} - \phi E_t) \\ +\lambda_1[E_{t+1} - (1-b)E_t + \beta c_t - \gamma(m_t + \tau c_t)] \\ +\lambda_2[s_t + m_t - w_t] \\ +\lambda_3[(1+\tau)c_{t+1} - (1+r_{t+1})s_t] \end{aligned} \tag{2.24}$$

The first order conditions of the program 2.18 can be obtain from the first order condition of the following unconstrained optimization,

$$\max_{c_t, E_t, E_{t+1}, s_t, m_t} min \lambda_1, \lambda_2, \lambda_3 \mathcal{L}(c_t, E_t, E_{t+1}, s_t, m_t, \lambda_1, \lambda_2, \lambda_3)$$

The corresponding first order conditions are given below where $\mathcal{L}_z$ means $\frac{\partial \mathcal{L}}{\partial z}$,

$$\mathcal{L}_{c_{t+1}} = 1/c_{t+1} + \lambda_3(1+\tau) = 0 \tag{2.25}$$
$$\mathcal{L}_{E_{t+1}} = \eta/(E_{t+1} - \phi E_t) + \lambda_1 = 0 \tag{2.26}$$
$$\mathcal{L}_{E_t} = -\eta\phi/(E_{t+1} - \phi E_t) - \lambda_1(1-b) = 0 \tag{2.27}$$
$$\mathcal{L}_{s_t} = -\lambda_2 + \lambda_3(1 + r_{t+1}) = 0 \tag{2.28}$$
$$\mathcal{L}_{m_t} = -\lambda_1\gamma + \lambda_2 = 0 \tag{2.29}$$
$$s_t + m_t = w_t \tag{2.30}$$
$$(1+\tau)c_{t+1} = (1+r_{t+1})s_t \tag{2.31}$$
$$E_{t+1} = E_t - bE_t - \beta E_t + \gamma(m_t + \tau c_t) \tag{2.32}$$

Based on the above conditions, we first solve for optimal consumption of a generation, c_{t+1}. Note from here on we suppress the * in the superscript, which otherwise is used to indicate optimal value. From Equations 2.28 and 2.29 we note that $\lambda_3 = \lambda_1\gamma/(1+r_{t+1})$. From equation 2.25 this means that $c_{t+1} = -(1+r_{t+1})/[\lambda_1\gamma(1+\tau)]$.

From equation 2.26 we note that $\lambda_1 = \eta/(E_{t+1} - \phi E_t)$. Substituting this, we obtain $c_{t+1} = (1 + r_{t+1})(E_{t+1} - \phi E_t)/[\eta\gamma(1+\tau)]$. This equation shows that consumption is linearly related to improvement in environmental quality.

Since $c_{t+1} = (1 + r_{t+1})(E_{t+1} - \phi E_t)/[\eta\gamma(1+\tau)]$, this means that for $t = t - 1$, $c_t = (1 + r_t)(E_t - \phi E_{t-1})/[\eta\gamma(1+\tau)]$. Plugging this in Equation 2.30, we obtain,

$$\begin{aligned} E_{t+1} &= (1-b)E_t - (\beta - \gamma\tau)c_t + \gamma m_t \\ &= (1-b)E_t + \frac{(\beta-\gamma\tau)(1+r_t)}{\eta\gamma(1+\tau)}(E_t - \phi E_{t-1}) + \gamma m_t \end{aligned} \tag{2.33}$$

There are still variables in the equation above that can be substituted out, especially in terms of capita k_t, leading us towards set of coupled equations. Based on commodity balance equations given in 2.21, the next time period capital is equal to current period savings. That is $k_{t+1} = s_t$. Also from Equation 2.32, $s_t = (1+\tau)/(1+r_{t+1})c_{t+1} = (E_{t+1} - \phi E_t)/(\eta\gamma)$. The last equality appears because $c_{t+1} = (1+r_t)(E_{t+1} - \phi E_t)/[\eta\gamma(1+\tau)]$. This gives us two things,

1. Equation for capital

$$k_{t+1} = s_t = \frac{1}{\eta\gamma}(E_{t+1} - \phi E_t) \tag{2.34}$$

2. Equation for investment in the environment, m_t, since it is what is left from income, w_t, after saving s_t (see equation 2.31),

$$m_t = w_t - \frac{1}{\eta\gamma}(E_{t+1} - \phi E_t) \tag{2.35}$$

We now plug in Equation 2.35 in Equation 2.33 and collect the terms so that E_{t+1} is on the left hand side, we obtain the evolution equation for environment quality as,

$$E_t = (1 + \frac{1}{\eta})^{-1}\left[(1-b)E_{t-1} - \frac{(\beta-\gamma\tau)(1+r_{t-1})}{\eta\gamma(1+\tau)}(E_{t-1} - \phi E_{t-2}) + \gamma(w_{t-1} + \frac{\phi E_{t-1}}{\eta\gamma})\right] \tag{2.36}$$

Here, $w_t = (1-\delta)f(k_t, 1)$ and $r_t = \frac{\partial f(k_t,1)}{\partial k_t} - \delta$ are functions of capital k_t. The second of the coupled economic growth environmental quality dynamics is the capital equation from Equation 2.34

$$k_{t+1} = s_t = \frac{1}{\eta\gamma}(E_{t+1} - \phi E_t) \tag{2.37}$$

Equations 2.36 and 2.37 describe the co-evolutionary dynamics of coupled economic growth and environmental quality that emerges out of the close overlapping generations society described above.

2.5. Simulating the coupled dynamics

The Equations 2.36 and 2.37 that provide the coupled dynamics of growth and economic quality are parameterized by several parameters that define the preference structure of the society and the production technology that the society has. The following describes these characteristics.

Parameter	Desription
η	preference for environment quality
ϕ	degree of habit formation for quality
b	environment self rate of degradation
γ	environmental maintenance efficiency
τ	tax rate to conserve environment per unit consumption
δ	capital rate of depreciation
β	degradation of environment cause by 1 unit of consumption
α	capital share of output
A	total factor of productivity

Simulating such a system is straightforward. The two state variables of the coupled system, E_t and k_t, can be initialized based on a modelers perception of the initial state of a society under consideration, depending on the parameter values assigned. For example, consider a society that is environmentally conscious and heavily habituated to environmental quality. For this, we may specify the following parameter values.

Parameter	Desription
η	0.5
ϕ	0.9
b	0.2
γ	0.5
τ	0.1
δ	0.025
β	0.1
α	0.1
A	1

2.5.1. Steady state functions of environmental quality and capital

The steady state equations are obtained from equations 2.36 and 2.37 by setting $E_{t_1} = E_t = \bar{E}$ and $K_{t-1} = k_t = \bar{k}$, meaning that neither capital nor environment quality change over time. Steady state equations for quality and capital are then obtained as,

$$\bar{E} = \frac{\gamma \bar{w}(\frac{\eta+1}{\eta})}{B} = \varphi(\bar{k}) \tag{2.38}$$

$$\bar{k} = \frac{1-\phi}{\eta\gamma}\bar{E} = \phi(\bar{k}) \tag{2.39}$$

Where in Equation 2.38,

$$\begin{aligned} B = 1 - (\frac{\eta+1}{\eta})\left[(1-b) - \frac{(\beta-\gamma\tau)(1+\bar{r})}{\eta\gamma(1+\tau)}(1-\phi) + \frac{\phi}{\eta}\right] \\ \bar{w} = (1-\delta)f(\bar{k},1) \\ \bar{r} = \frac{\partial f(\bar{k},1)}{\partial k_t} - \delta \end{aligned} \tag{2.40}$$

Figure 2.2 shows the plot of the two steady state equations for the heavily habituated highly environmentally conscious society in so called 'state space' of the two states, E(environment quality) and k (capital). For this case, it shows that the two curves intersect at two points, meaning that these are two possible steady state points. This means that the temporal dynamics will converge to either of these two points irrespective of where the society starts from in terms of its initial environment quality and capital. One of these steady states (0,0) can be thought of as total system collapse while the other steady state has positive state values or a 'sustainable' steady state.

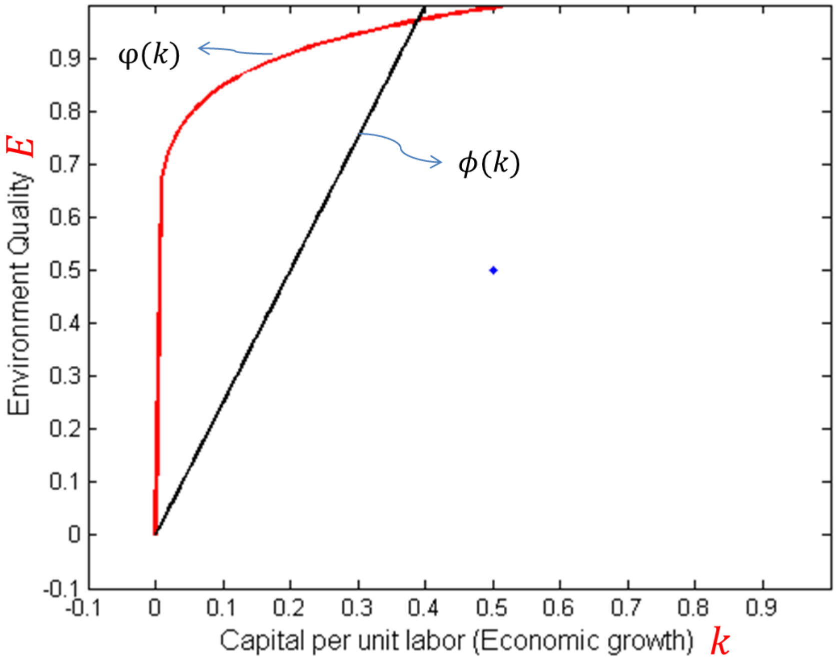

Figure 2.2: Steady state evolution equations for the parameters given above.

2.5.2. Simulating the temporal dynamics

Let us now see what happens in the state space when we initialize the society to start from 'medium' initial environmental quality and capital ($E_0 = 0.5$ and $k_0 = 0.5$). Figure **??** plots the 'co-evolution' of the two state variables in time.

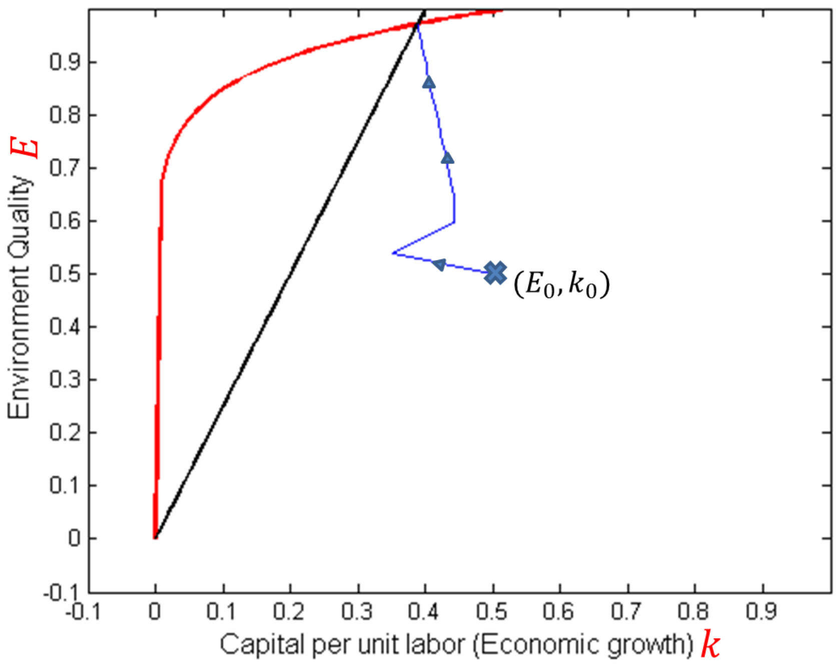

Figure 2.3: Co-evolution of environmental quality

or medium intial conditionsand capital NSteadyState

The co-evolution is attracted to the top-right 'sustainable' steady state, instead of collapse. This co-evolution naturally has partial evolutionary stories of capital and environment quality. This is shown in the following two plots.

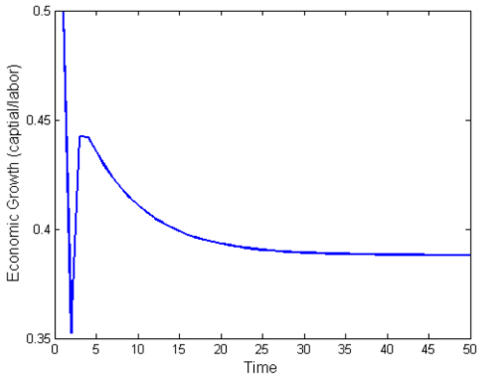

Figure 2.4: Partial evolution of capital in the coupled dynamics

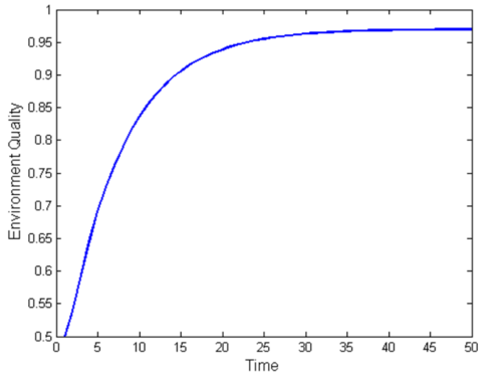

Figure 2.5: Partial evolution of environmental quality in the coupled dynamics

Figure 2.4 shows that capital temporarily dips in the beginning but picks up later on towards the sustainable level. Meanwhile environmental quality consistently increases and approaches a sustainable level (Figure 2.5.2).

Naturally different dynamics will emerge under different parameterization of societies. How about under different tax regimes for moderately rich society with high initial environment quality? Can they achieve a sustainable future? Plug and play with the model to come up with a future scenario for its sustainable development!

Bibliography