

Contents

1 Question Bank	1
Bibliography	11

Question Bank

Question: Non-steady Feed forward systems: Consider a feedforward system composed of two completely mixed systems with their characteristics identified by subscripts 1 and 2 in Figure 1.1 below.

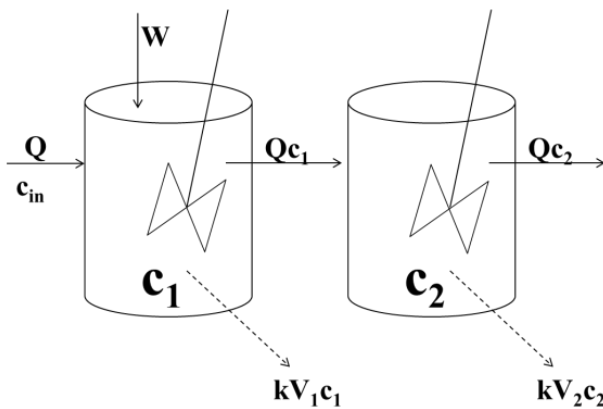


Figure 1.1: A feedforward system of two completely mixed systems.

Solution: We have the following conservation of mass for lake 1:

$$V_1 \frac{dc}{dt} = Qc_{in} - kV_1c_1 - Qc_1$$

At steady state we have $dc/dt = 0$.

This implies that $Qc_{in} - kV_1c_1 - Qc_1 = 0$, or $c_1(Q + kV_1) = Qc_{in}$. Thus, the steady state concentration of lake 1 is $c_1 = Qc_{in}/\lambda_1$ where $\lambda_1 = Q + kV_1$. Similarly for lake 2 we have the following steady state condition: $Qc_1 - kV_2c_2 - Qc_2 = 0$. This implies that $c_2(Q + kV_2) = Qc_1$. From the expression of steady state concentration, c_1 in lake 1, we have $c_2(Q + kV_2) = Q^2c_{in}/\lambda_1$. Thus the steady state concentration in lake 2 is: $c_2 = Q^2c_{in}/(\lambda_1\lambda_2)$, where $\lambda_2 = Q + kV_2$.

Question: Consider figure 1.2 where each of the two lakes are conceptualized by two completely mixed systems as in figure 1.1. Let $W = m\delta(t)$ be impulse load of mass $m[M]$ at time $t = 0$ and let $c_1(0) = c_2(0) = 0$ (Initial concentrations are 0).

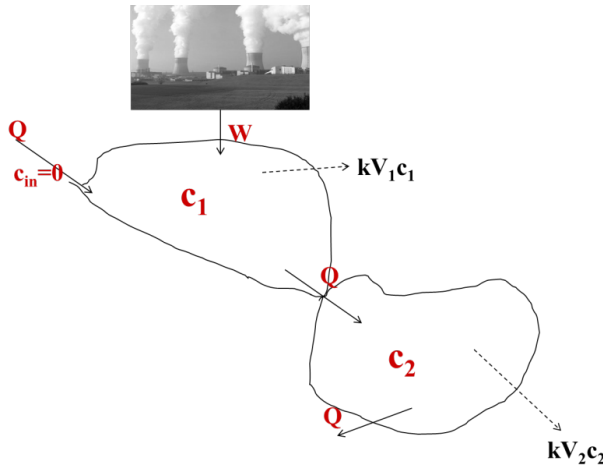


Figure 1.2: A toy example with two connected completely mixed lakes for impulse radioactive loading and system's recovery.

Solution: 2.1. Can a steady state model for such a case make physical sense?

No, because the system would be in a transient state due to “pulse” (or instantaneous) type of loading. We have therefore implicitly assumed a loading that is continuous rather than a pulse in our solution to question 1

2.2. Solve for the non-steady state solution of the system? (Hint: solution for $\partial y/\partial x + \mu y = w \exp(-\theta x)$ is $y = \frac{w}{(\mu-\theta)} [\exp(-\theta x) - \exp(-\mu x)] + A \exp(-\mu x)$, where A is a constant of integration).

Mass balance for lake 1 yields:

$$V_1 \partial c_1 / \partial t = W - kV_1 c_1 - Qc_1$$

We simplify the above by substituting $\lambda_1 = Q + kV_1$ in the above to obtain:

$$\begin{aligned} V_1 \partial c_1 / \partial t + c_1 \lambda_1 &= W, \text{ or} \\ \partial c_1 / \partial t + c_1 \lambda_1 / V_1 &= W / V_1 \end{aligned}$$

Recall from Week1 lecture slides that solution to the above ordinary differential equation is:

$$c(t) = \int_{-\infty}^t \frac{W}{V_1} \exp(-\lambda_1/V_1(t-\tau)) d\tau + A \exp(-\lambda_1 t/V_1),$$

where A is a constant of integration.

Since for $t = 0$, $c = 0$ and $W = m\delta(0)$, we have

$$c_1(t) = m/V_1 \exp(-\lambda_1 t/V_1)$$

Similarly we solve for the mass balance equation in lake 2,

$$V_2 \partial c_2 / \partial t = Qc_1 - kV_2 c_2 - Qc_2$$

With $\lambda_2 = Q + kV_2$, we have $V_2 \partial c_2 / \partial t + c_2 \lambda_2 = Qc_1$.

Substituting transient solution for lake 1 in the above:

$$\partial c_2 / \partial t + c_2 \lambda_2 / V_2 = Qm/(V_1 V_2) \exp(-\lambda_1 t/V_1)$$

We note that the mass balance equation for lake 2 of form $\partial y/\partial x + \mu y = w \exp(-\theta x)$ with $y = c_2$, $\mu = \lambda_2/V_2$, $W = Qm/(V_1 V_2)$, $\theta = \lambda_1/V_1$ and $x = t$.

Thus (from Hint), we have transient concentration solution for lake 2,

$$c_2(t) = \frac{Qm}{[V_1 V_2 (\lambda_2/V_2 - \lambda_1/V_1)]} (\exp(-\lambda_1 t/V_1) - \exp(-\lambda_2 t/V_2)) + A \exp(-\lambda_2 t/V_2)$$

where A is a constant of integration.

Finally we note that $A = 0$, as initial condition for lake 2 is $c_2(0) = 0$. Thus,

$$c_2(t) = Qm/(V_1V_2)/(\lambda_2/V_2 - \lambda_1/V_1)(\exp(-\lambda_1 t/V_1) - \exp(-\lambda_2 t/V_2)). \quad (1.1)$$

2.3. Discuss the conditions under which the concentration in the second reservoir will never rise?

From the expression provided above for the second lake, its concentration will always rise (and then fall) for positive loading in the first lake. Thus the condition under which the concentration in the second reservoir never rises is that loading in the first lake is 0.

2.4. what is the time scale of the recovery for the second lake?

We define time scale of recovery of the second lake as the time it takes for the lake to reach 10% (or another percentage, for e.g. use of 50% is equivalent to half-life estimation of pollutant in the lake) of its maximum concentration (after attaining its maximum). Let time to achieve maximum concentration be given by t_{\max} , which can be obtained by solving the following: $\partial c_2/\partial t = 0$.

Solving for the above maximum concentration condition yields,

$$c_{\max} = \hat{c} \left[\left(\frac{\lambda_2/V_2}{\lambda_1/V_1} \right)^{-(\lambda_1/V_1)/(\lambda_2/V_2 - \lambda_1/V_1)} - \left(\frac{\lambda_2/V_2}{\lambda_1/V_1} \right)^{-(\lambda_2/V_2)/(\lambda_2/V_2 - \lambda_1/V_1)} \right]$$

where $\hat{c} = (Qm/(V_1V_2))/(\lambda_2/V_2 - \lambda_1/V_1)$, and $t_{\max} = \log(\lambda_2/V_2)/(\lambda_1/V_1) / (\lambda_2/V_2 - \lambda_1/V_1)$.

Let t_{10} be the time when concentration in the second lake reaches 10% of c_{\max} , defined as c_{10} . Then t_{10} can be estimated by solving the following equation:

$$\begin{aligned} c_{10} &= c(\exp(-\lambda_1 t_{10}/V_1) - \exp(-\lambda_2 t_{10}/V_2)) = 0.1c_{\max} \\ &= 0.1c \exp(-\lambda_1 t_{\max}/V_1) - \exp(-\lambda_2 t_{\max}/V_2). \end{aligned}$$

Question: Find out and describe in brief the difference between diffusion, dispersion, conduction, and convection.

Solution: The following definitions were taken from Surface Water Quality Modeling book of Chapra (1997).

Diffusion refers to the movement of mass due to random motion of water or mixing. On microscopic scale molecular diffusion results from the random Brownian motion of water molecules. A similar kind of random motion occurs on a larger scale due to eddies and is called turbulent diffusion. Both have a tendency to minimize concentration gradients by moving mass from regions of high to low concentrations.

Dispersion is a related process but in contrast to random motion of water in time, it is the result of velocity differences in space. For example consider dye introduced into water flowing through a pipe. Molecules near the wall of the pipe would move more slowly than a molecule near the center due to a velocity gradient. The net effect is the mixing of the dye along the pipe axes.

Conduction refers to the transfer of heat by molecular activity from one substance to another or through a substance. Conduction is similar to molecular diffusion of mass, thus diffusion is also used interchangeably with heat conduction.

Convection generally refers to the motions in a fluid that result in mixing of fluid's properties. Free convection refers to vertical atmospheric motions due to the buoyancy of heated or cooled fluid and closely linked with dispersion. In contrast forced convection is due to external forces, for e.g. lateral movement of heat or mass due to wind, and is similar to advection.

Question: Formulate the differential equation for 1 d plug flow model for reaction = $k\Delta Vc$, and $W=0$.

Solution: A general mass balance equation for a differential (constant) volume ΔV can be given as: $\Delta V \partial c / \partial t = J_{in}A_c - J_{out}A_c \pm \text{reactions} + W$.

Here c is the concentration of the pollutant in the differential element, J_{in} is the pollutant mass flux entering the differential element, A_c is the cross sectional area (perpendicular to the flux direction) and J_{out} is the mass flux exiting the differential volume.

With additional specifications that are: reactions = $k\Delta Vc$, and $W = 0$, we have: $\Delta V \frac{\partial c}{\partial t} = J_{in}A_c - J_{out}A_c \pm k\Delta Vc$.

Note that 1D plug flow model is for advection only transport phenomenon, thus: $J_{in} = Uc$, and

$$J_{out} = J_{in} + \frac{\partial J_{in}}{\partial x} \Delta x = Uc + U \frac{\partial c}{\partial x} \Delta x.$$

Substituting the fluxes in conservation of mass equation, we obtain:

$$\Delta V \partial c / \partial t = UcA_c - (Uc + U \frac{\partial c}{\partial x} \Delta x)A_c \pm k\Delta Vc \Rightarrow \Delta V \partial c / \partial t = -U \frac{\partial c}{\partial x} \Delta x A_c \pm k\Delta Vc.$$

Since $\Delta V = \Delta x A_c$, we have $\Delta V \frac{\partial c}{\partial t} = -U \frac{\partial c}{\partial x} \Delta V \pm k\Delta Vc$.

Dividing both the left and right hand side by ΔV , we finally arrive at the Advection equation of 1D plug flow model with given specification, i.e. $\partial c / \partial t = -U \partial c / \partial x \pm kc$

Question: When is a cascade model equivalent to 1D plug flow model? Prove it. [Hint: The following identity holds: $\exp(a) = \lim_{n \rightarrow \infty} (1 + a/n)^n$].

Solution:

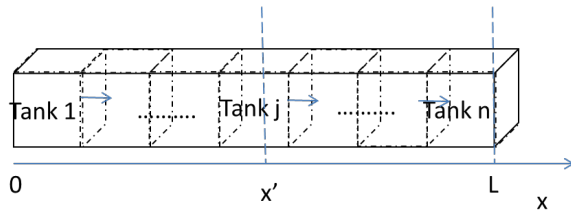


Figure 1.3: A scheme for feed forward cascade model.

In the above figure, the tube through which pollutant flows is discretized into n tanks in order to create a cascade model of feed forward reservoirs. The pollutant undergoes first order decay (k) and volumetric flow rate is Q , length of the tube is L and volume V . Given a constant concentration c_0 at $x = 0$ (at the inflow of first reservoir), the concentration in the j th reservoir is given as $c_j = c_0(Q/(Q + V/n))^j$. We note that for large enough n , $j = nx'/L$.

$$\Rightarrow c_j = c_0(Q/(Q + kV/n))^{nx'/L} = c_0(1/(1 + kV/Qn))^{nx'/L} = c_0(1/(1 + kL/Un))^{nx'/L}$$

Since $Q = UA_c$ and $V = A_c L$, this implies $c_j = c_0(1 + kL/Un)^{-nx'/L}$. Finally, we find the limit of c_j when $n \rightarrow \infty$.

$$\lim_{n \rightarrow \infty} c_j = \lim_{n \rightarrow \infty} c_0((1 + kL/Un)^n)^{-x'/L} = c_0(\lim_{n \rightarrow \infty} (1 + kL/Un)^n)^{-x'/L}$$

Let $a = kL/U$, then

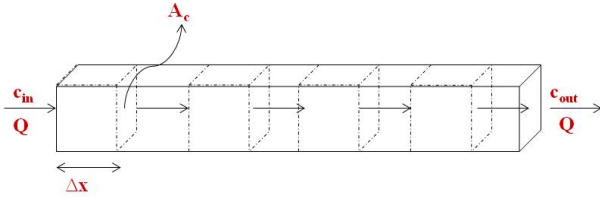
$$(\lim_{n \rightarrow \infty} (1 + kL/Un)^n)^{-x'/L} = (\lim_{n \rightarrow \infty} (1 + a/n)^n)^{-x'/L} = (\exp(a))^{-x'/L}$$

Substituting $a = kL/U$ in the above, we get

$$\lim_{n \rightarrow \infty} c_j = c_0(\exp(kL/U))^{-x'/L} = c_0 \exp(-kx'/U).$$

But $c(x = x') = c_0 \exp(-\frac{k}{U}x')$ for a plug flow model. Hence proved.

Question: Prove that upper and lower limits of mixed flow model (with respect to diffusion coefficient, i.e. what is the limit of the steady state solution for the mixed flow model when the diffusion coefficient tends to infinity and when it tends to zero respectively) is a completely mixed model and a plug flow model respectively. Plot steady state ADE solutions for various values of D ($=1, 100, 500, 1E+5, 1E+8$) (sq m/hr) as well as plug flow equation for $c_0=1\text{mg/L}$, $k= 20/\text{hr}$, $U = 100 \text{ m/hr}$ and $A_c = 10 \text{ sq m}$.

Solution:

a) Upper limit of mixed flow model with a continuous point load at $x = 0$ at steady state is a completely mixed system (a completely mixed system)

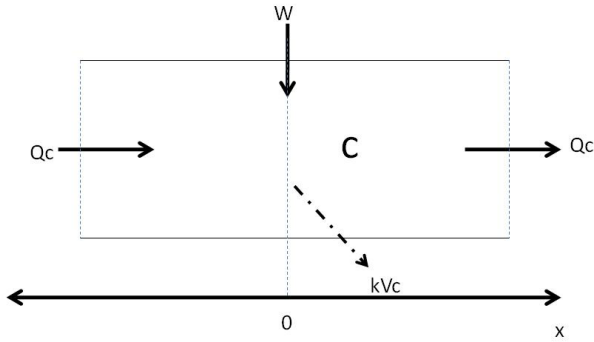
We note that the general solution for ADE at steady state is: $c(x) = F \exp(\lambda_1 x) + G \exp(\lambda_2 x)$ where

$$\lambda_1 = \frac{U}{2D}(1 + \sqrt{1 + 4\eta}), \lambda_2 = \frac{U}{2D}(1 - \sqrt{1 + 4\eta}), \eta = kD/U^2 \quad (1.2)$$

and F and G are integration constants.

In case when $D \rightarrow \infty$ (i.e. diffusion or dispersion coefficient is really large), $\lambda_1, \lambda_2 \rightarrow 0$. Thus, $c(x) = F + G = \text{constant}$.

This already proves that the system is completely mixed as the concentration is constant (and doesn't vary with x). We now need to estimate what is c . In order to do so, consider the following figure which shows the volume under study.



Here Q is the volumetric flow rate, W is the loading rate at $x = 0$, c is the constant concentration, k is the first order kinetic reaction's constant, L is the length of the element and V is its volume. Solving for mass balance for this element,

$$Qc + W = Qc + kVc \Rightarrow c = W/kV.$$

Note that c goes to 0 as the volume that is implicitly defined for the ADE steady state solution is infinite. Recall that the steady state ADE solution with continuous point loading W at $x = 0$ is

$$\begin{aligned} c(x) &= W/(Q\sqrt{1 + 4\eta}) \exp\left(\frac{U}{2D}(1 - \sqrt{1 + 4\eta})x\right) \quad \text{for } x > 0, \\ c(x) &= W/(Q\sqrt{1 + 4\eta}) \exp\left(\frac{U}{2D}(1 + \sqrt{1 + 4\eta})x\right) \quad \text{for } x \leq 0 \end{aligned}$$

where $\eta = kD/U^2$.

Note that we imposed the boundary condition that $c(x) = 0$ at $x = \pm\infty$ for the above steady state solution for ADE. Thus completely mixed limit of the above steady state ADE solution is 0.

An alternate proof can be obtained by directly taking the limit of the steady state ADE solution, which is zero. This is because as $D \rightarrow \infty$, $\exp\left(\frac{U}{2D}(1 \pm \sqrt{1 + 4\eta})x\right) \rightarrow 1$ since $(1 \pm \sqrt{1 + 4\eta})$ in the numerator

approaches to $\pm\infty$ slower than D in the denominator since square-root of a number larger than 1 is always smaller than the number itself. Thus as $D \rightarrow \infty$,

$$\exp\left(\frac{U}{2D}(1 \pm \sqrt{1+4\eta})x\right) \rightarrow 1$$

while $W/(Q\sqrt{1+4\eta}) \rightarrow 0$. Thus $c(x) \rightarrow 0$.

b) Lower limit of mixed flow with a continuous point load at $x=0$ at steady state is a plug flow (an advective system)

This is the case when dispersion coefficient D goes to zero. As can be seen, this is the case when advection dominates. Thus we only choose the concentration equation for $x > 0$, which is the direction of volumetric flow rate,

$$c(x) = W/(Q\sqrt{1+4\eta}) \exp\left(\frac{U}{2D}(1 - \sqrt{1+4\eta})x\right) \quad \text{for } x > 0, \quad \text{where } \eta = kD/U^2 \quad (1.3)$$

Since the above solution for ADE at steady state (advection dispersion equation) includes both advection and dispersion effects, we cannot remove the effect of dispersion from the solution by substituting $D = 0$. We need to find the limit of the above equation with respect to D , i.e.

$$\lim_{D \rightarrow 0} c(x) \quad (1.4)$$

First we obtain the Taylor series expansion upto first order of $\sqrt{1+4\eta}$ around $D = 0$ for some D' close to 0:

$$\sqrt{1+4kD'/U^2} \approx \sqrt{1+4kD/U^2} \Big|_{D=0} + 1/2 \frac{4kD'/U^2}{\sqrt{1+4kD/U^2}} \Big|_{D=0} \quad (1.5)$$

$$= 1 + 2kD'/U^2 \quad (1.6)$$

Substituting the above in the equation for $c(x)$ for dispersion coefficient D' , we obtain:

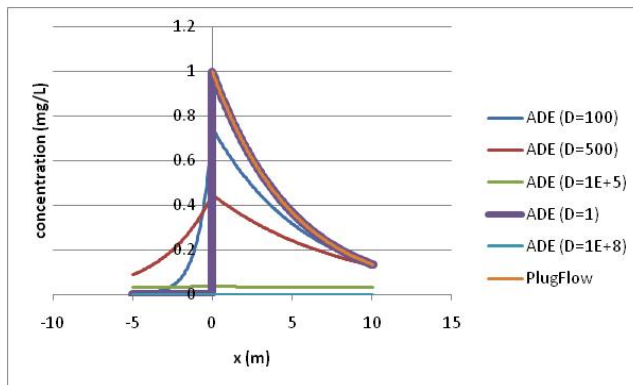
$$c(x) \approx W/(Q\sqrt{1+4kD'/U^2}) \exp\left(\frac{U}{2D'}(1 - 1 - 2kD'/U^2)x\right) \quad (1.7)$$

$$= W/(Q\sqrt{1+4kD'/U^2}) \exp(-kx/U). \quad (1.8)$$

Finally, from the above we can obtain:

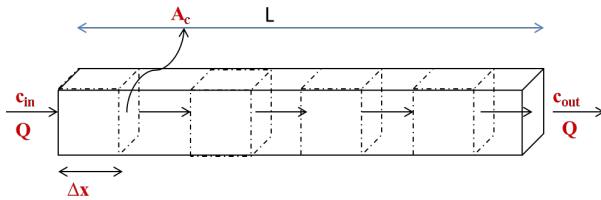
$$\lim_{D' \rightarrow 0} c(x) = \lim_{D' \rightarrow 0} \frac{W}{Q\sqrt{1+4kD'/U^2}} \exp(-kx/U) = \frac{W}{Q} \exp(-kx/U) \quad (1.9)$$

The following attached figure shows ADE solution for various values of D (sq m./hr) as well as plug flow equation for $c_0=1\text{mg/L}$, $k=20/\text{hr}$, $U=100\text{ m/hr}$ and $A_c=10\text{ sq m}$. Note that for $D=1\text{ sq m/hr}$, ADE steady state solution is the same as the solution of plug flow model for continuous point loading at $x=0$.

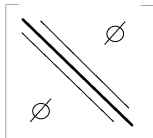


Question: Formulate the system of equations in matrix form for the tank below with $A_c = 10 \text{ m}^2$, $L = 100 \text{ m}$, $U = 100 \text{ m/hr}$, $k = 20 \text{ hr}^{-1}$, $c_0 = 1 \text{ mg/L}$, $D = 500 \text{ m}^2/\text{hr}$ and $n = 50$ using control volume approach.

Solution:



Because D and Q are constants the matrix which has to be constructed turns out to be a simple A matrix (of slide 7) of the following form (See Slide 18 of lecture 2 on incompletely mixed systems):



Where the entries above the diagonal is formed by $-Q \cdot D (= -1500)$ terms, the diagonal by $Q + D + D + k \cdot A \cdot \Delta x (= 2020)$ terms and the entries below the diagonal by $-D (= -500)$ terms. Here \emptyset imply 0 entries.

The concentration in each of the n control volumes, represented by vector \vec{c} , obey the equality $A\vec{c} = W$ (where W is n -dimensional vector containing loading rates into each of the n control volumes) or $\vec{c} = A^{-1}W$. Hence the vector \vec{c} can be calculated using the inverse of A and multiplying it with the load vector W . In the application below, only the first element of W is non-zero, the rest $n - 1$ entries are 0, i.e. $W = [W, 0, \dots, 0]^T$. The following MATLAB code provides the details.

```
clear all
c0 = 1; k = 20; u = 100; A = 10; Q = A*u; D = 500; delX = 1;
n = 50;
V2 = ones(1,n-1)*(-Q-D); V1 = ones(1,n)*(Q+D+D+k*A*delX); V3 = ones(1,n-1)*(-D);
M = diag(V1,0) + diag(V2,-1) + diag(V3,1); W = [10;zeros(n-1,1)];
c = inv(M)*W;
plot(c)
```

Question: How taxation affects the dynamics (co-evolution) of quality and growth? Let $E_0 = 0.2$, $k_0 = 0.01$ (low initial environment quality and a poor society). All other parameters (Slide 44) remain the same except tax rate τ , i.e. the society is environmentally conscious and forms habit for good environment quality over time. Consider $\tau = 0, 0.05$ and 0.10 . What do you think is the optimal tax policy for the society so parameterized? Discuss your results.

Answer: Following is the state space co-evolutionary trajectory of environmental growth and environmental quality for $\tau = 0$:

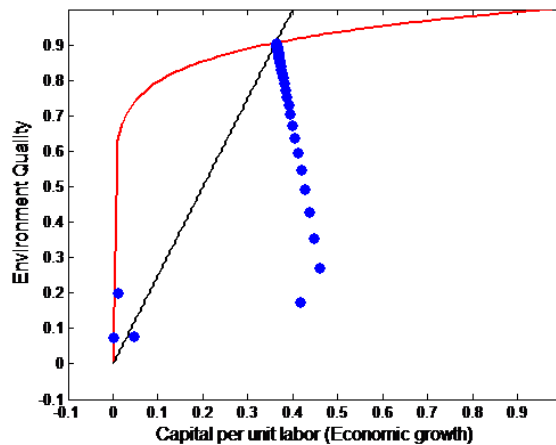


Figure 1.4: Coevolution of quality and growth (path of blue dots) under consumer tax rate of 0% for environment improvement. The black and red lines are steady state paths for capital and environment quality. This society has moderate preference for environment quality and high degree of habit formation for quality. 'X' represents the initial condition.

And the following is the state space co-evolution when $\tau = 0.1$:

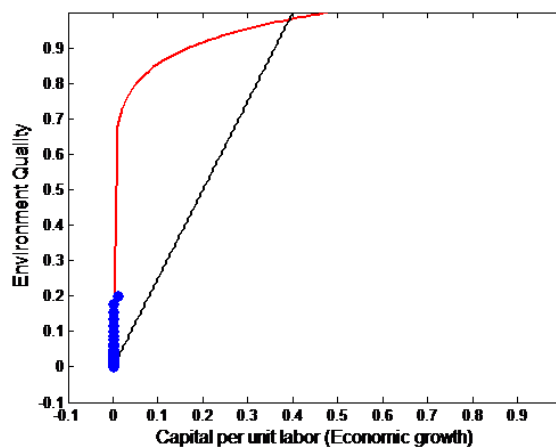


Figure 1.5: Coevolution of quality and growth (path of blue dots) under consumer tax rate of 10% for environment improvement. The black and red lines are steady state paths for capital and environment quality. This society has moderate preference for environment quality and high degree of habit formation for quality. 'X' represents the initial condition.

While counter-intuitive, the above two plots therefore suggest that increasing the tax rate on consumption to pay for environmental clean-up leads to the collapse of the parameterized system. This may make sense for the society that has high own-willingness to take care of its environment. The society as parameterized has high degree of habit formation of good environment quality ($\phi = 0.9$) and prefers not just high consumption but also better environment quality (as indicated by $\eta = 0.5$). The society therefore may on its own be able to invest in improving environment quality.

Taxes have “wedge effect” that suppresses growth and hence reduces an individual’s capacity to save and invest in environment quality. It also suppresses future income and thus future tax revenue for improving quality. Thus higher taxes have an overall negative effect on the coevolution of growth and quality leading to an eventual decline of the society. The optimal tax policy for such a society is no tax!

In order to demonstrate that zero tax policy is not the optimal policy for all societies, consider a society with low preference for environmental quality ($\eta = 0.1$) and no degree of habit formation ($\phi = 0.0$). The co-evolutionary plot below demonstrates the eventual decline under no tax policy. The decline is quite rapid.

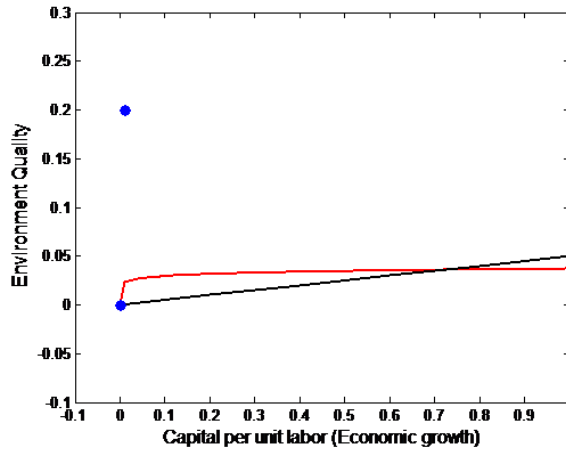


Figure 1.6: Coevolution of quality and growth under no tax policy for environment improvement (indicated by blue dots). The black and red lines are steady state paths for capital and environment quality. This society has low preference for environment quality and no habit formation for quality. The decline to zero is rapid compared to figure 1.5. 'X' represents the initial condition.

Question: Coevolution under different tax regimes for moderately rich society with high initial environment. Let us consider a society that is environmentally conscious and has a strong tendency for habit formation for environment quality. Let $E_0 = 1, k_0 = 0.5$. Discuss and interpret the results in the light of previous answer.

Solution: This is yet another example of a different society that may have similar preference for environment quality ($\eta = 0.5$) and high degree of habit formation for environment quality ($\phi = 0.9$) but are endowed different in terms of initial capital and environment quality. The inherent dynamics of such a society leads to an eventual decline under zero tax policy as Figure 1.7 demonstrates.

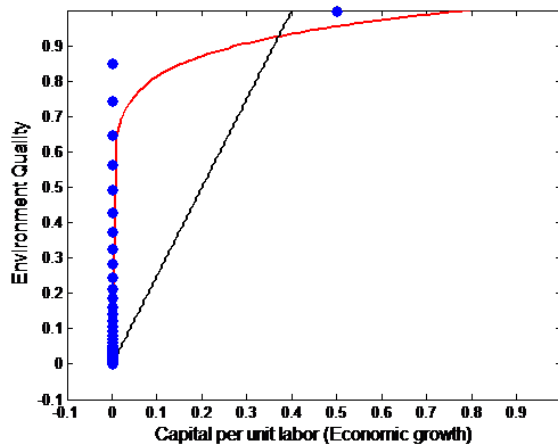


Figure 1.7: Same as Figure 1.4 except $E_0 = 1.0$ and $k_0 = 0.5$. 'X' represents the initial condition.

However for this society high tax rates work. A tax rate of 0.10 leads the society to the better long run equilibrium of environment quality and economic growth. This is demonstrated by Figure 1.8.

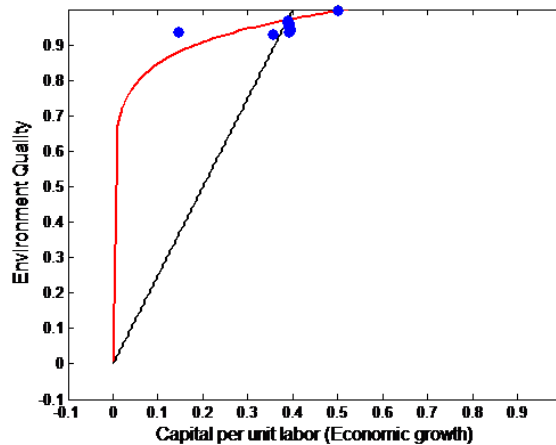


Figure 1.8: Same as Figure 1.5 except $E_0 = 1.0$ and $k_0 = 0.5$. 'X' represents the initial condition.

Question: How about societies with no habit formation for environment quality? Let $E_0 = 0.2$, $k_0 = 0.01$. Discuss and interpret the results in the light of previous answers.

Solution: The society in this case is able to escape eventual decline even under really high tax rate for environment quality improvement. Figure 1.6 displays the case for zero tax while Figure 1.9 below demonstrates the case for a tax rate of 0.50. It therefore appears that high taxation is a solution to avoid an eventual societal collapse.

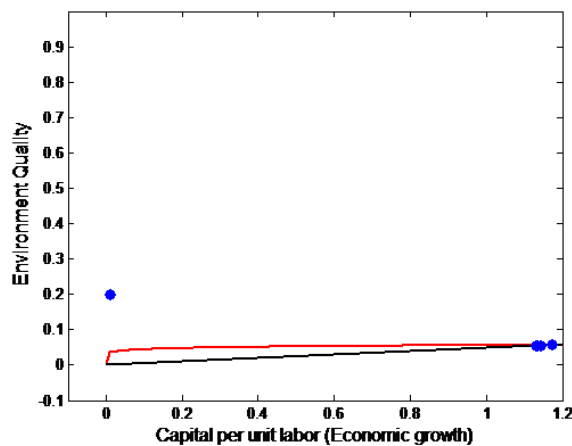


Figure 1.9: Same as figure 1.6 except that tax rate of 0.50 is applied. 'X' represents the initial condition.

Bibliography