NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF ENGINEERING

EXAMINATION FOR

(Semester I: 2012-2013)

CE5311 - ENVIRONMENTAL MODELLING WITH COMPUTERS

Nov/ Dec 2012 - Time allowed: 3 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **THREE(3)** questions and comprises **TEN(10)** printed pages.
2. Answer ALL **THREE(3)** questions.
3. All questions DO NOT carry equal marks.
4. This is an "**OPEN BOOK**" examination.

<u>Question 1</u> [20 marks]

A mass spring equation is given by:

$$\frac{du}{dt}+\frac{A}{M}u+\frac{K}{M}x=\frac{f\left(t\right)}{M}$$
$$\frac{dx}{dt}-u=0$$
$$u(0)=0,\ x(0)=0$$

Where:
u(t) velocity
x(t) position
f(t) external forcing
M mass
A friction constant
K spring constant

The following constants are given:

M=1 kg, *A*=0 (I) of 0,2 Ns/m (II), *K*=100 N/m en f(t)=sin(πt/2) N

These equations are integrated numerically by the following two methods:

$$\frac{u^{n+1}-u^{n-1}}{2\Delta t}+\frac{A}{M}u^{n}+\frac{K}{M}x^{n}=\frac{f\left(n\Delta t\right)}{M}$$
$$\frac{x^{n+1}-x^{n-1}}{2\Delta t}-u^{n}=0$$

A

$$\frac{u^{n+1}-u^{n}}{\Delta t}+\theta\left(\frac{A}{M}u^{n+1}+\frac{K}{M}x^{n+1}\right)+\left(1-\theta\right)\left(\frac{A}{M}u^{n}+\frac{K}{M}x^{n}\right)=\frac{f\left(\left(n+\theta\right)\Delta t\right)}{M}$$
$$\frac{x^{n+1}-x^{n}}{\Delta t}-\theta u^{n+1}-\left(1-\theta\right)u^{n}=0$$

B

The time step Δt is 0.04 s. For method B, θ has 3 values: θ=0.48, θ=0.5 en θ=1.0. With or without friction this yield 8 combinations given by the table below:

1:Method A	A=0	A=0.2
2:Method B, θ=0.48	A=0	A=0.2
3:Method B, θ=0.5	A=0	A=0.2
4:Method B, θ=1.0	A=0	A=0.2

The Figures Q1(a) – (d) contain the results for x^n, for the cases with or without friction. The figures are in arbitrary order and give results for just one method.

a) How many periods or time scales can you find in the Q1(a) – (d). Give the exact size of these time scales. [4 marks]

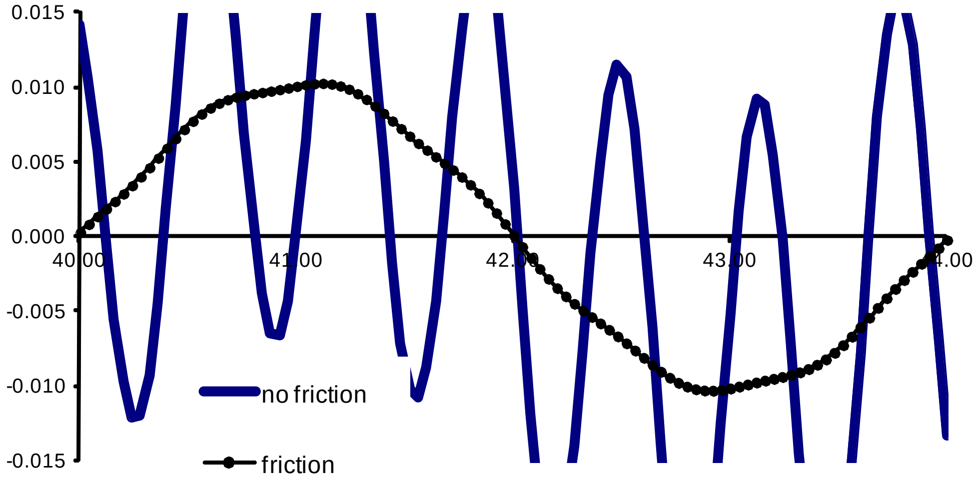
b) Which figure shows a purely numerical time scale? [4 marks]

c) Give for each method the truncation error. [4 marks]

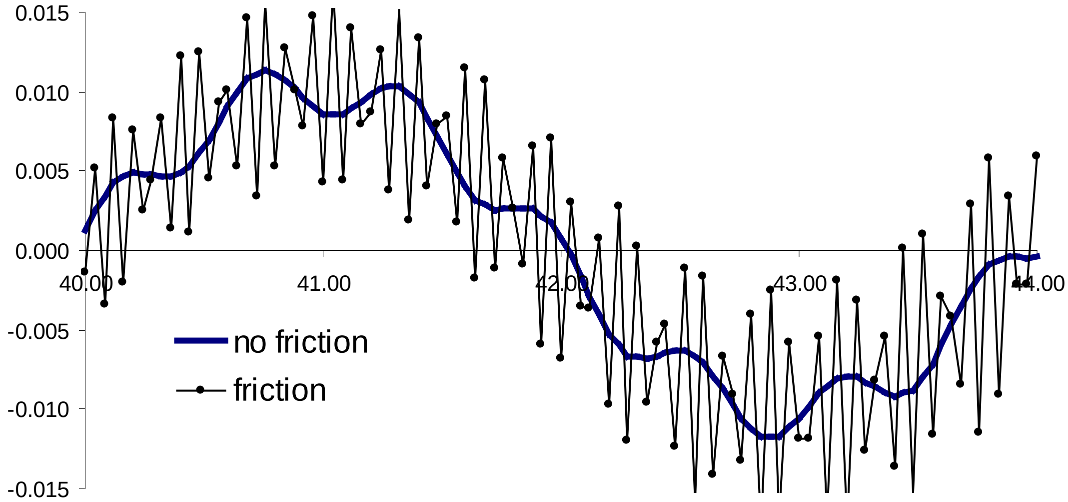
d) Mention the unstable combinations in the table. Give a clear explanation. [4 marks]

e) Which figure number belongs to which row number in the table? [4 marks]

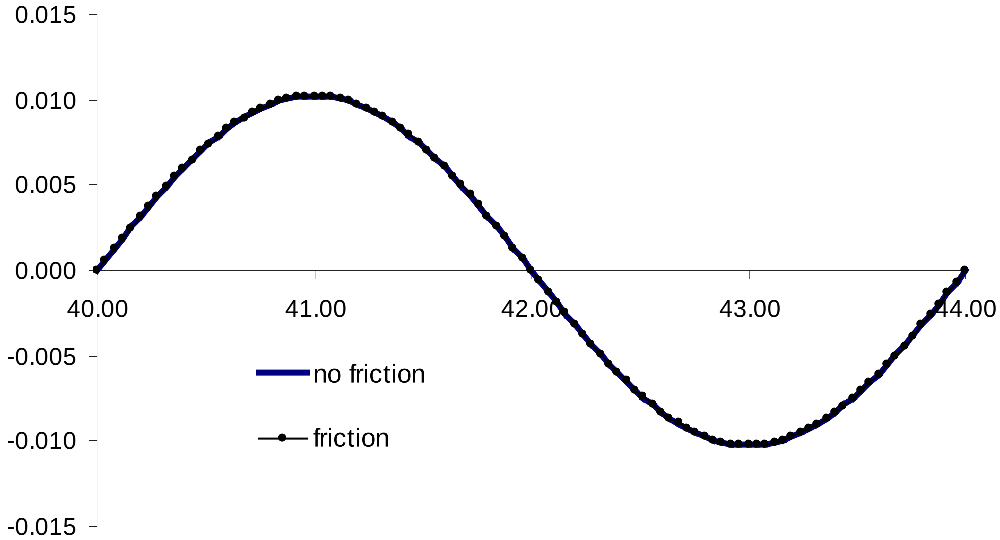
Explain all your answers clearly!



Figures Q1(a)

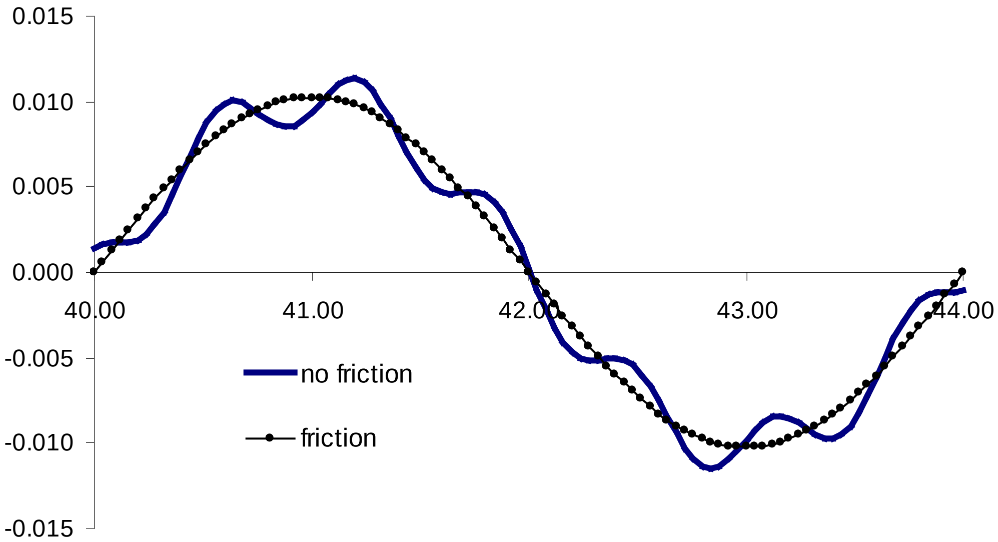


Figures Q1(b)



Figures Q1(c)

.../5


0.015
0.010
0.005
0.000
-0.005
-0.010
-0.015
40.00
41.00
42.00
43.00
44.00
no friction
friction

Figures Q1(d)

<u>Question 2</u> [44 marks]

Consider the following simplified transport equation:

$$\frac{\partial c}{\partial t}+U\frac{\partial c}{\partial x}=0$$

The transport speed U is assumed to have a constant positive value of 1 ms^{-1} in the positive x-direction. For the numerical calculations a domain of total length L= 10 m is selected. The spatial domain starts at x=0 and ends at x=L. A grid size of 4 cm is chosen.

a) The following numerical approximations are tested. State which approximations are explicit and which are implicit. Sketch the stencil of each method.

Method 1 $$\frac{c_j^{n+1}-c_j^n}{\Delta t}+U\frac{c_j^n-c_{j-1}^n}{\Delta x}=0$$

Method 2 $$\frac{c_j^{n+1}-c_j^n}{\Delta t}+U\left[\frac{c_j^n-c_{j-1}^n}{\Delta x}+\left(1-U\frac{\Delta t}{\Delta x}\right)\left(\frac{c_{j+1}^n-2c_j^n+c_{j-1}^n}{2\Delta x}\right)\right]=0$$

Method 3 $$\frac{c_j^{n+1}-c_j^n}{\Delta t}+U\left[\frac{c_j^n-c_{j-1}^n}{\Delta x}+\left(1-U\frac{\Delta t}{\Delta x}\right)\left(\frac{c_j^n-2c_{j-1}^n+c_{j-2}^n}{2\Delta x}\right)\right]=0$$

Method 4

$$\frac{c_j^{n+1}-c_j^n}{\Delta t}+U\frac{\left[c_j^n+\frac{1-\sigma}{2}f\left(\Delta c_j^n,\Delta c_{j-1}^n\right)\right]-\left[c_{j-1}^n+\frac{1-\sigma}{2}f\left(\Delta c_{j-1}^n,\Delta c_{j-2}^n\right)\right]}{\Delta x}=0$$

where $\sigma=U\frac{\Delta t}{\Delta x}$, which is the Courant number, f denotes a slope limiter given by:

$$f(\Delta_1,\Delta_2)=\min\bmod(\Delta_1,\Delta_2)=\begin{cases}0 & if\ \Delta_1\cdot\Delta_2\le 0\\ \frac{\Delta_1}{|\Delta_1|}\cdot\min(|\Delta_1|,|\Delta_2|) & if\ \Delta_1\cdot\Delta_2>0\end{cases}$$ and:

$$\Delta c_j=c_{j+1}-c_j$$

[4 marks]

b) What is the essential difference between absolute stability and zero stability? What is the absolute stability condition of method 1? [4 marks]

c) Determine the stability of method 1 with the modified equation approach. [4 marks]

d) Is any of the methods 1 to 4 unconditionally stable? [4 marks]

e) For which of the methods 1,2,3, as given above, is the stability condition based on the CFL condition: σ<2. [4 marks]

f) Derive the order of consistency of method 2. [4 marks]

g) Determine the amount of numerical diffusion for method 1 with the modified equation approach. Under which conditions is the method positive? Explain your answer.
[4 marks]

h) The methods are applied to solve an initial boundary value problem using the above numerical specifications. The initial condition is sketched in each figure. The boundary conditions are assumed to be cyclic: $c(0,t)=c(L,t)$ $(0 \le t \le L/U)$. Then 2 different calculations are carried out for each method using 2 values for σ, $\sigma=0.1$ and $\sigma=0.9$. This yields 8 numerical results, numbered A1,A2,B1,…,D2. Each of the figures A to D contain just 1 method with 2 Courant numbers. Which result contains the largest amount of numerical dissipation? [4 marks]

i) Which results show clearly a leading phase error? [4 marks]

j) For which results has the minmod limiter been applied? [4 marks]

k) Give for all the results A1 to D2, the method that is applied and the Courant number that is used (i.e.: $\sigma=0.1$ or $\sigma=0.9$), and clearly explain your reasoning. [4 marks]

N.B. σ is the Courant number.

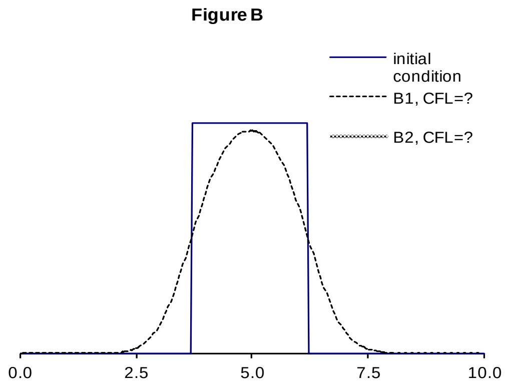


Figure C

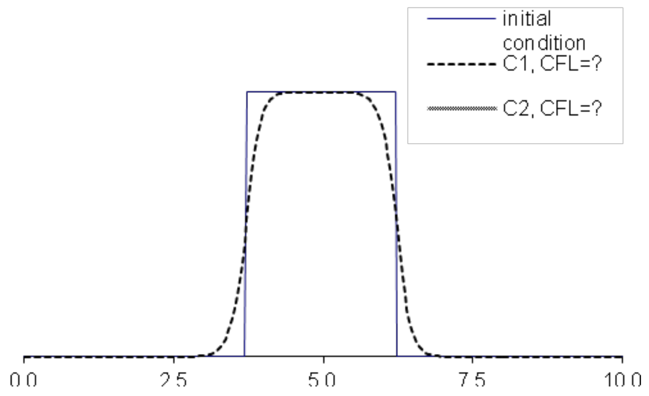
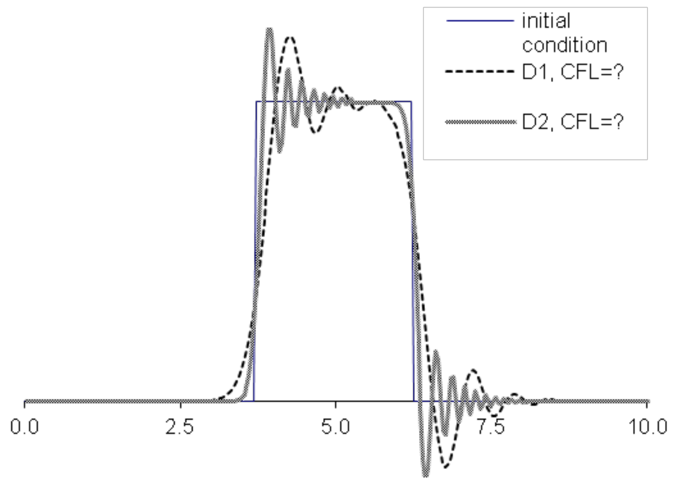


Figure D



Question 3 [36 marks]

For 3D hydrostatic flow models two different grid systems are applied: (1) z-planes and (2) σ-planes. Each system has its own pros and cons.

a) Give the mathematical formulation of the sigma transformation. [4 marks]

b) Which term in the momentum equation is especially more complex due to this transformation? [4 marks]

c) Which term in the momentum equation is not always transformed completely, yielding a major difference with z-plane models? What is the reason for this incomplete transformation? [4 marks]

For flow in a uniform river we consider two vertical grids in Figure A and Figure B. The flow is supposed to be normal. This implies equilibrium between bottom shear stress and gravitational forces. The bottom slope is constant. The water depth is also constant.

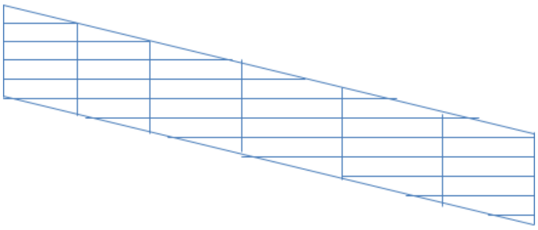


Figure A, river with z-planes

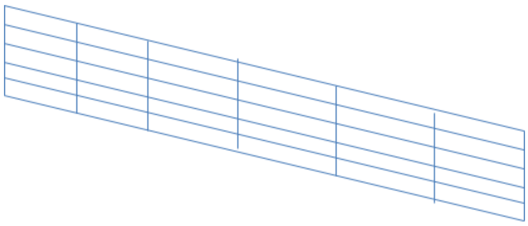


Figure B, river with σ-planes

d) What is the common mathematical formulation for the vertical distribution of uniform river flow? [4 marks]

e) Which of the two grids is more accurate for the representation of uniform river flow? [4 marks]

f) If cooling water is released at the free surface, which of the two grids will generate more artificial vertical mixing? [4 marks]

Next we consider a lake with temperature stratification. Again we use two grids, given by Figure C and D.

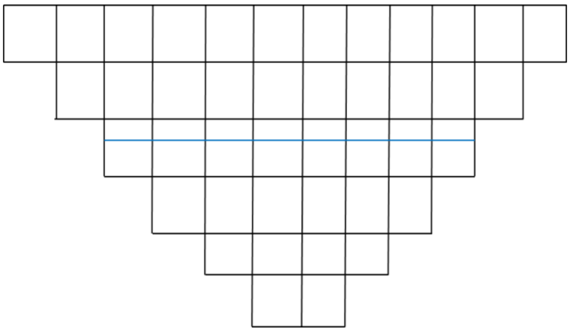


Figure C, lake with z-planes

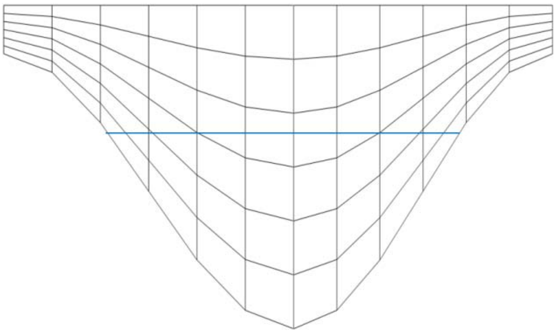


Figure D, lake with σ-planes

As initial condition zero flow is assumed, strictly horizontal water levels and a lower temperature below the thick line than above.

g) What will happen with the flow in the different grids after the simulations starts? [4 marks]

h) What is probably the final result of the spin-up process of figure D? [4 marks]

i) Is there a way to reduce this effect? [4 marks]

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