Steady open-channel flow

1. Introduction

In this module, we focus on the analysis of flows in natural "conduits" such as rivers and streams, i.e. channels in which the fluid flows under the influence of gravity and has a free surface in contact with air as its upper boundary.

For open-channel flow, the area of flow is not prescribed, but is a function of the location of the free surface, i.e. the depth, h, of the flow in the open channel. Water depth is usually a priori unknown, as shown in Figure 1.

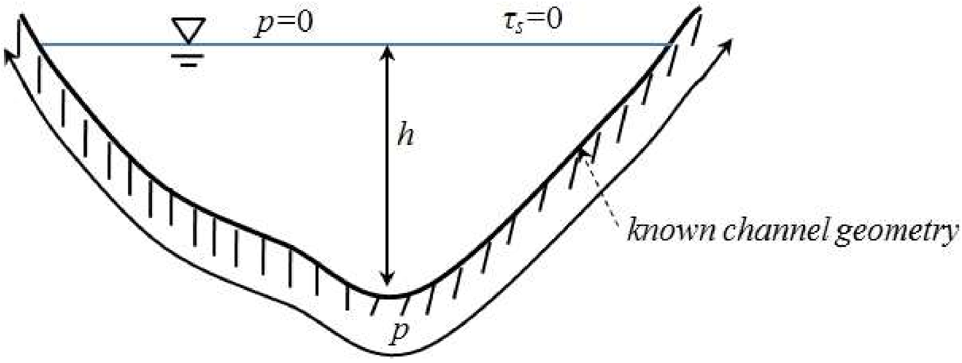


Figure 1 Schematically drawing of the cross section of a river channel

For a known channel geometry we must know h in order to specify the following flow area-related quantities:

- A = flow area = A (h)
- P = wetted perimeter =length of channel surface with fluid-solid contact = $P(h)$
- R_h = hydraulic radius = $A(h)/P(h)$ = R_h (h)

Since $P=P_{atm}$= 0 at the free surface, it is not possible to impose an external pressure gradient on the flow (as we could in closed conduits). A gravity component that acts in the down-slope direction is the primary driving force for open-channel flows.

2. Uniform steady flow

For simplicity we start with discussing the simplest open-channel flow: uniform steady flow. Steady means that the flow does not change with time t, i.e. $\partial/\partial t$=0. A flow is uniform if the cross-sectional properties (mean velocity, water depth, etc) do not change along the direction of flow; that is $\partial/\partial x$=0, where x is the coordinate defining the streamwise direction of flow. A uniform flow is necessarily steady in practical terms and usually occur at distances away from control structures. Usually, a uniform flow requires a long and straight prismatic channel with a constant bottom slope β, i.e. the channel's cross section does not change with x.

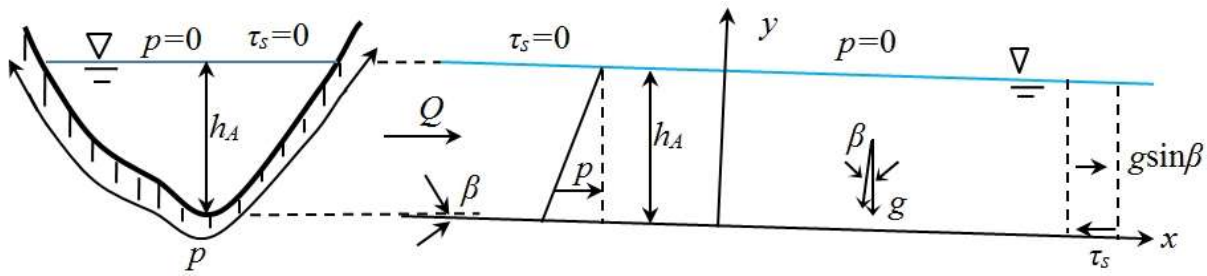


Figure 2 schematic drawing of uniform flow in a prismatic channel

Since the flow is uniform, streamlines are straight lines parallel to bottom, so according to Bernoulli principle the pressure variation perpendicular to streamlines (or bottom), i.e. in y-direction, is hydrostatic.

$$\frac{\partial p}{\partial y} = -\rho g \cos\beta \;\Rightarrow p = \rho g \cos\beta \left(h_A - y\right) \tag{2.1}$$

where h_A is the normal depth of the channel, p=0 at y=h_A=free surface is used. The bottom slope β is generally very small, so cosβ=1 and y is approximately equivalent to the vertical coordinate z.

Consider a small segment of the open-channel flow with a width of δx (dashed lines near the right-side end of Figure 2), the gravity force on this segment in the streamwise direction is:

$$G_x = \rho \delta V g_x = \rho\left(A\delta x\right) g \sin\beta \tag{2.2}$$

The pressure forces on the upstream and downstream sides of the segment completely balance each other, so there is no net pressure force. The boundary shear force is:

$$F_s = \tau_s \left(P\delta x\right) \tag{2.3}$$

where P is the wetted perimeter and τ_s is the average bottom shear stress. The force balance can then be written as:

$$\tau_s \left(P\delta x\right) = \rho g A \sin\beta \delta x \tag{2.4}$$

or:

$$\tau_s = \rho g R_h S_o \tag{2.5}$$

where τ_s =average bottom shear stress along wetted perimeter, $R_h = A/P$= hydraulic radius, S_o =bottom slope ≈β. Eq. (2.5) is the Basic Hydraulic Formula for uniform steady flow. In a prismatic open channel, if you know h, the cross-sectional geometry and the bottom slope β, you can use this basic hydraulic formula to predict the average boundary shear stress τ_b along the wetted perimeter of the channel. However, that is usually not the problem you will have! You want to predict Q, the discharge in the river. You realize that if the bottom shear stress τ_s can be a function of discharge we can use Eq. (2.5) to solve for Q, so naturally the next step is figuring out how you can parameterize τ_b in terms of Q.

2.1. Flow resistance

From dimensional analysis, bottom shear stress can be relate to the mean flow velocity through the Darcy-Weisbach Formula:

$$\tau_s = \frac{1}{8} f \rho V^2 \tag{2.6}$$

where f is the friction factor and V is the mean velocity. Substitute Eq. (2.6) into (2.5) we get:

$$V = \sqrt{\frac{8g}{f}} \sqrt{R_h} \sqrt{S_o} \tag{2.7}$$

Thus, if you know the friction factor f, you could get V and then the discharge is given by:

$$Q = AV$$

From dimensional analysis the friction factor is a function of Reynolds number and relative roughness:

$$f = f\left(\mathrm{Re} = \frac{V(4R_h)}{\nu}, \frac{\varepsilon}{4R_h} \right) \tag{2.8}$$

where ε is channel roughness. The Moody diagram can be used to obtain the friction factor with the Reynolds number and relative roughness defined in Eq. (2.8). Some scenario is just as pipe flow analysis: Guess value of $f = f^{(1)}$, get $V = V^{(1)}$ and then $\mathrm{Re} = \mathrm{Re}^{(1)}$, go to Moody to update $f = f^{(2)}$ etc. If you want the river stage for a given *Q*, things get a bit more involved. For example, guess value of $h = h^{(1)}$ and $f = f^{(1)}$, obtain $V^{(1)}$ and then $A^{(2)} = Q / V^{(1)}$. From $A^{(2)} = A\left(h^{(2)}\right)$ obtain new value of $h^{(2)}$ etc. This is a very long procedure since you have to iterate on both *h* and *f*.

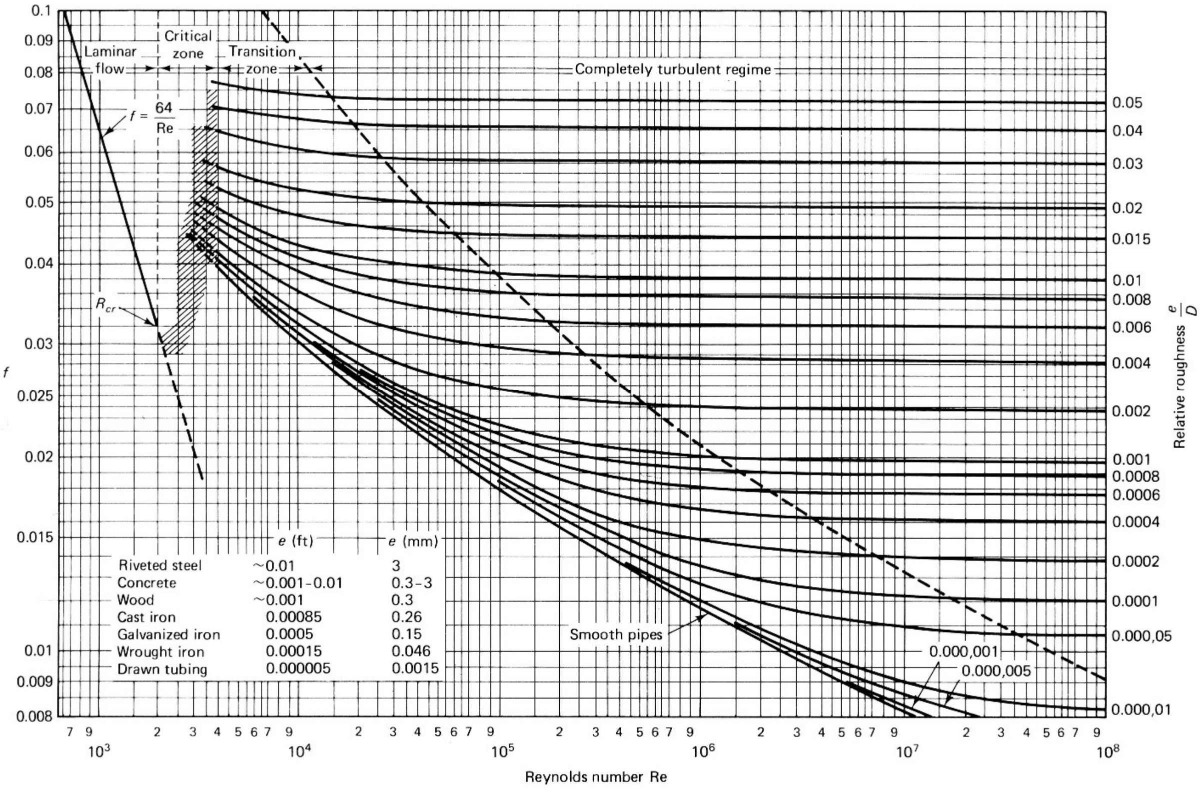


Figure 3 The moody diagram

<u>Chezy Formula</u>

if we simply take *f*=0.02, Eq. (2.7) becomes the Chezy Formula:

$$V = C\sqrt{R_h}\sqrt{S_o} \tag{2.9}$$

where:

$$C = Chezy's\ C = \sqrt{\frac{8g}{f}} \approx 60 \frac{m^{1/2}}{S} \tag{2.10}$$

<u>Manning's Equation:</u>

For fully rough turbulent flow the friction factor is no longer a function of Reynolds number, and it turns out that:

$$f \approx 0.113\left(\varepsilon / R_h\right)^{1/3} \tag{2.11}$$

Substitute Eq. (2.11) into Eq. (2.7), we get:

$$\sqrt{\frac{8g}{f}} = \sqrt{\frac{8g}{0.113\varepsilon^{1/3}}} R_h^{\ 1/6} = \frac{1}{n} R_h^{\ 1/6} \tag{2.12}$$

so:

$$V = \frac{1}{n} R_h^{\ 2/3} \sqrt{S_o} \tag{2.13}$$

where n= Manning's coefficient:

$$n = \sqrt{\frac{8g}{0.113\varepsilon^{1/3}}} = 0.038\varepsilon^{1/6} \tag{2.14}$$

Be cautious, the constant "0.038" can only be used if you are using the SI unit, i.e. ε=channel roughness in [m], then [n]= [$m^{-1/3}s$]. Typical values of Manning's coefficient are shown in Figure 4. We notice that there is a weak dependency of "n" on ε, i.e. n increases by factor of 2 if ε increases by 2^6=64.

One weakness (or rather inconsistency) of Manning's equation is that it assumes a fully-rough turbulent flow, but in applications people usually neglect this assumption, i.e. some people blindly apply Manning' equation for transitional-rough turbulent flow.

Table 4-1. Typical values* of Manning n

Material	n
Metals	
Steel	0.012
Cast iron	0.013
Corrugated metal	0.025
Non-metals	
Lucite	0.009
Glass	0.010
Cement	0.011
Concrete	0.013
Wood	0.012
Clay	0.013
Brickwork	0.013
Gunite	0.019
Masonary	0.025
Rock cuts	0.035
Natural streams	
Clean and straight	0.030
Bottom: gravel, cobbles and boulders	0.040
Bottom: cobbles with large boulders	0.050

Figure 4 Typical value of Manning's coefficient 'n' [$m^{1/3}/s$]

2.2. Uniform flow calculation

In many applications, we can know the geometry of prismatic channel, i.e. $A(h)$, $P(h)$ and $R_h=A/P=R_h(h)$ are known. The channel slope and channel roughness (or Manning's coefficient) can also be considered known as they can be obtained from field survey. Thus, the remaining task is either finding the discharge Q with a given water depth h or finding the water depth h with a given flow discharge Q.

It is quite easy to find Q with a specified h. Using Eq. (2.13), the discharge Q can be obtained as:

$$Q = VA = \frac{1}{n} R_h^{2/3} A\sqrt{S_o} = \frac{1}{n}\left(\frac{A}{P}\right)^{2/3} A\sqrt{S_o} = \frac{1}{n}\frac{A^{5/3}}{P^{2/3}}\sqrt{S_o} \qquad (2.15)$$

Thus, you just need to use the given h to find A and P, and apply Eq. (2.15) to get Q.

Finding h with a specified Q is a bit difficult. This depth is called the <u>normal depth</u> under a specified discharge Q. Eq. (2.15) can be re-arranged as:

$$\frac{A^{5/3}}{P^{2/3}} = K(h) = \frac{nQ}{\sqrt{S_o}} \tag{2.16}$$

The left-hand side is a function of water depth, while the right-hand side is a function of discharge Q, so we can use Eq. (2.16) to plot a discharge-depth diagram for a given prismatic channel, as shown in Figure 5. With this diagram, one can quickly find the corresponding h with a given Q.

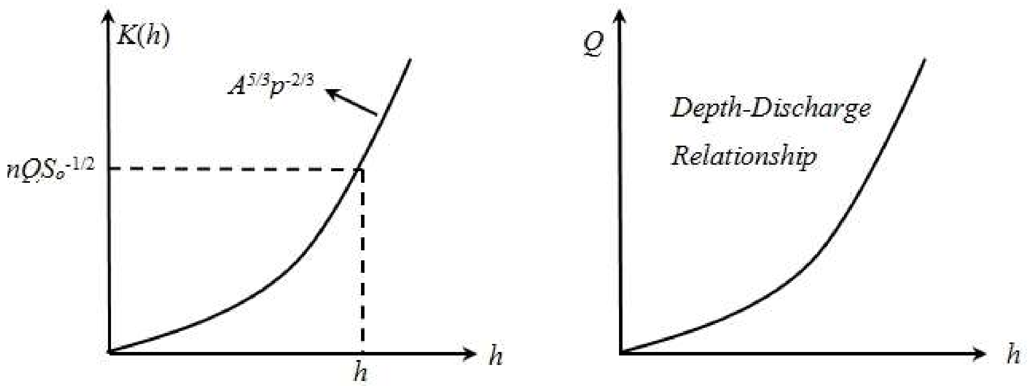


Figure 5 Discharge-depth diagram for uniform flow

For a <u>very wide rectangular channel</u>, the calculation can be much simpler. In such a channel, the hydraulic radius is just the water depth, since:

$$R_h = \frac{A}{P} = \frac{bh}{b+2h} = h(1 - \frac{1}{\frac{b}{2h}+1}) \approx h, \text{ if } b >> h \tag{2.17}$$

Thus, Eq. (2.15) becomes:

$$Q = VA = \frac{1}{n} R_h^{2/3} A \sqrt{S_o} = \frac{1}{n} \frac{h^{2/3}}{\left(1+2h/b\right)^{2/3}} hb\sqrt{S_o} \tag{2.18}$$

or:

$$h = \left(\frac{Qn}{\sqrt{S_o}b}\right)^{3/5} \left(1+2h/b\right)^{2/5} \tag{2.19}$$

We can solve for h by iteration:

$$h^{(n+1)} = \left(\frac{Qn}{\sqrt{S_o}b}\right)^{3/5} \left(1 + 2h^{(n)}/b\right)^{2/5}$$, staring with $h^{(0)}=0$

In many cases we may deal with a compound channel. For example, the channel shown in Figure 6 has different values for bottom roughness (or Manning coefficient) for the flood plains (P_1 and P_3) and the main channel (P_2). The usual practice is separate the cross section according to bottom roughness. We neglect shear stress along fluid-air and fluid-fluid interfaces, so:

$$Q = \sum Q_n = \sum \frac{1}{n_n} A_n \left(\frac{A_n}{P_n}\right)^{2/3} \sqrt{S_o} \qquad (2.20)$$

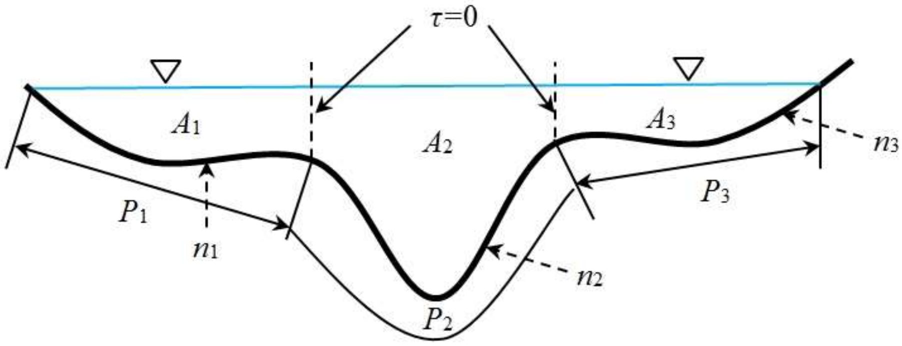


Figure 6 Compound channel

2.3. Energy and Hydraulic Grade Lines

The energy and hydraulic grade lines are useful concepts for describing an open-channel flow, and here we apply them to steady uniform open channel flow.

As shown in Figure 7, we can arbitrarily choose a datum for vertical coordinate z=0. The vertical distance of energy grade line (or total head line) above datum is:

$$z_{EGL} = H = z + \frac{p}{\rho g} + \frac{V^2}{2g} \qquad (2.21)$$

Since flow is steady, uniform and well-behaved $p + \rho gz$=constant perpendicular to streamlines within fluid, so we have:

$$p + \rho gz = p + \rho g\left(z_o + y\cos\beta\right) = constant = \rho g\left(z_o + h\cos\beta\right)$$

since p=0 at the free-surface. Thus, within the fluid:

$$\frac{p}{\rho g}+z=\frac{p+\rho gz}{\rho g}=z_o+h\cos\beta \tag{2.22}$$

The vertical distance of the hydraulic grade line to the datum is:

$$z_{EGL}=H=z_o+h\cos\beta+\frac{V^2}{2g}=z_o+h\cos\beta+\frac{Q^2}{2gA^2} \tag{2.23}$$

Since, in most cases, $\cos\beta\simeq 1.000$:

$$z_{EGL}=H=z_o+h+\frac{Q^2}{2gA^2} \tag{2.24}$$

Differentiating Eq. (2.24) with respect to x, we get:

$$\frac{\partial z_{EGL}}{\partial x}=\frac{\partial z_0}{\partial x} \tag{2.25}$$

or:

$$\frac{\partial H}{\partial x}=-S_0,\ \left(\frac{\partial z_{EGL}}{\partial x}=\frac{\partial H}{\partial x},\frac{\partial z_0}{\partial x}=-S_0\right) \tag{2.26}$$

Eq. (2.25) means that EGL is parallel to bottom, while Eq. (2.26) suggests that the streamwise gradient of the total water head is related to the bottom slope.

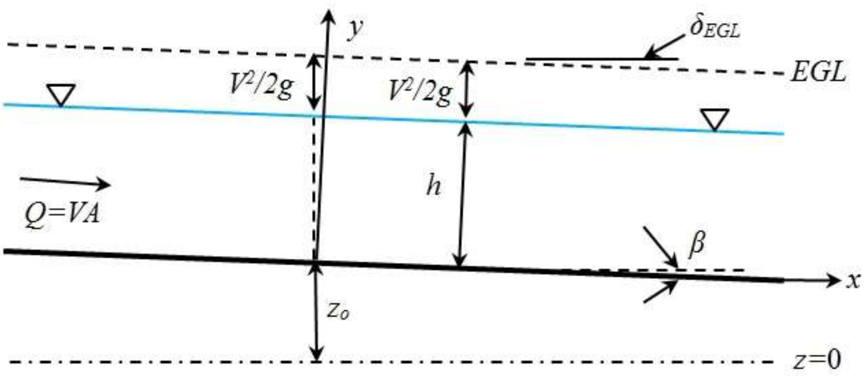


Figure 7 Schematic drawing for Energy grade line and hydraulic grade line for steady uniform flow

In a uniform steady flow, HGL=hydraulic grade line is given by:

$$z_{HGL}=z_o+h\cos\beta\approx z_o+h \tag{2.27}$$

so it is parallel to the bottom and a vertical distance $h\cos\beta$ above it. Since in most cases $\cos\beta \simeq 1$, the hydraulic grade line is "identical" to the free surface. The energy grade line, according to Eq. (2.24) is located a distance of $\frac{V^2}{2g} = \frac{Q^2}{2gA^2}$ =the velocity head above the HGL, i.e. it too is parallel to the bottom.

The basic hydraulic formula, i.e. Eq. (2.5), can be written as:

$$S_f = S_0 \tag{2.28}$$

where

$$S_f = \frac{\tau_b}{\rho g R_h} = \begin{cases} \frac{f}{8g}\frac{P}{A^3}Q^2 \left(Darcy-Weisbach\right) \\ \frac{1}{C^2}\frac{P}{A^3}Q^2 \left(Chezy\text{'}s\ Equation\right) \\ n^2\frac{P^{4/3}}{A^{10/3}}Q^2 \left(Manning\text{'}s\ Equation\right) \end{cases} \tag{2.29}$$

where f is the Darcy-Weisbach friction factor, C is the Chezy coefficient, which is treated as a constant, and n is the Manning's coefficient. Substitute Eq. (2.28) into Eq. (2.26) , we get:

$$\frac{\partial H}{\partial x} = -S_f \tag{2.30}$$

This is the energy equation for steady uniform flow. It suggest that the change of total water head, $\partial H/\partial x$, is due to bottom flow resistance, S_f.

With the above formulae for S_f, Eq. (2.28) reduce to the equations we use to determine uniform steady flow characteristics.

3. Energy principle

3.1. Specific head and critical flow

In open channel flow it turns out to be convenient to refer to the EGL elevation above the channel bottom (located at $z=z_0$ above $z=0$ datum). This elevation difference:

$$H_s = H - z_o = z_{EGL} - z_o = h\cos\beta + \frac{Q^2}{2gA^2} \approx h + \frac{Q^2}{2gA^2} \tag{3.1}$$

is referred to as the specific head, $E=H_s$.

Let's consider a short transition of converging flow, as shown in Figure 8. Suppose we know the discharge and water depth before the transition, and we want to find the water depth after the transition.

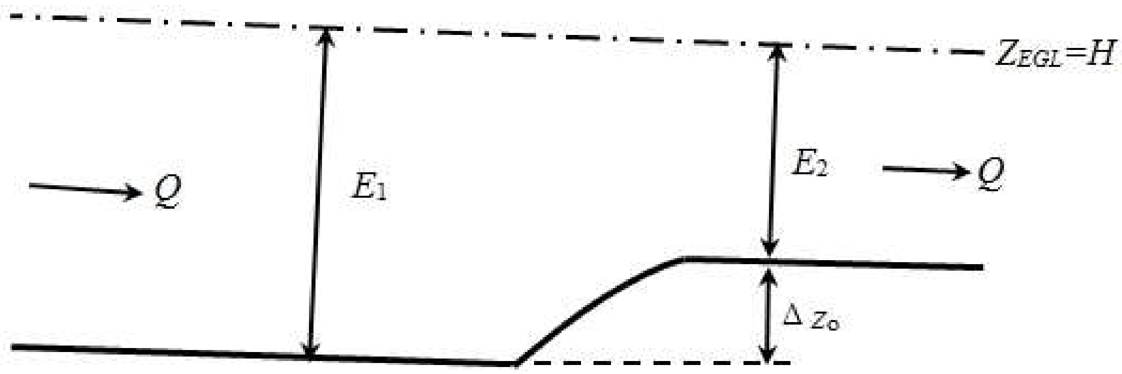


Figure 8 A short transition of converging flow

The specific energy before and after the transition is given by:

$$E_1 = h_1 + \frac{Q^2}{2gA_1^2} \tag{3.2}$$

$$E_2 = h_2 + \frac{Q^2}{2gA_2^2} = E_1 - \Delta z_0 \tag{3.3}$$

From the Bernoulli principle, there will be no head loss (why?), so E_1-Δz_0=E_2, which gives:

$$h_2 + \frac{Q^2}{2gA_2{}^2} = E_1 - \Delta z_0 \tag{3.4}$$

This is an equation for h_2, but it does not have a unique solution, because the left-hand side of Eq. (3.4) goes to infinite as h_2 approaches infinite or zero. Thus, for given E_1-Δz_0, it is possible that we can find two h_2, which satisfies Eq. (3.4), as shown in Figure 9. Mother nature does not like this!

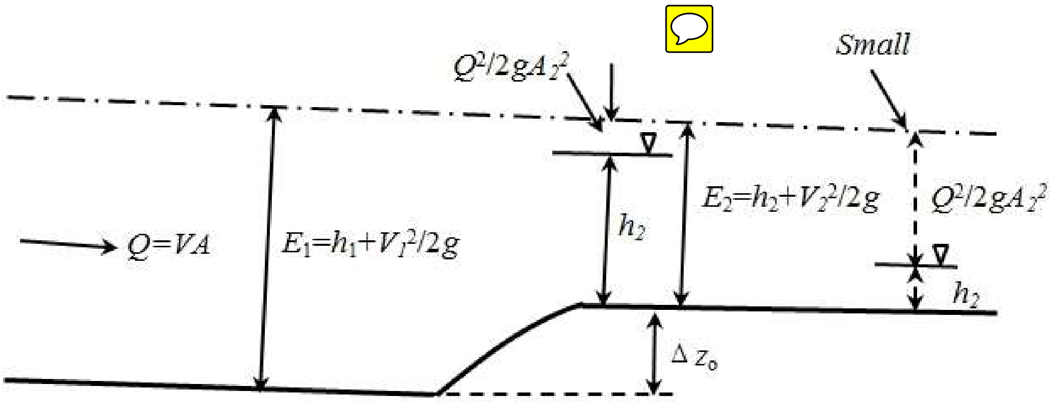


Figure 9 Two downstream water depths with the same specific head

The possibility of multiple depths with the same specific head can be further illustrated use the Specific Head-Depth Diagram. For a specific discharge Q, the specific head is only a function of water depth:

$$E = h + \frac{Q^2}{2gA(h)^2} \tag{3.5}$$

It can be easily shown that:

$$\begin{aligned} &h \to \infty,\ A \to \infty \quad \Rightarrow \quad E \approx h \to \infty \\ &h \to 0,\ A \to 0 \quad \Rightarrow \quad E \approx Q^2/(2gA^2) \to \infty \end{aligned}$$

Thus, if we plot E as a function of h, the curve must looks like the one in Figure 10. E must have a minimum value, which can be found by differentiating E with h:

$$\frac{\partial E}{\partial h} = \frac{\partial}{\partial h}\left(h + \frac{Q^2}{2gA(h)^2} \right) = 0 \tag{3.6}$$

With some algebra, we can show that the minimum of specific head E occurs when:

$$\frac{\partial E}{\partial h} = 1 + \frac{Q^2}{2g}\left(-2A^{-3}\frac{\partial A}{\partial h} \right) = 1 - \frac{Q^2}{gA^3}\frac{\partial A}{\partial h} = 0$$

What is $\partial A/\partial h$? Let's consider the cross section of the river, as shown in Figure 11. A small change of the cross-section area, A, due to a small change of water depth h is given by:

$$\delta A = b_s \delta h \tag{3.7}$$

where b_s is surface width of channel. Thus:

$$\frac{\partial A}{\partial h} = \frac{\delta A}{\delta h} = b_s \tag{3.8}$$

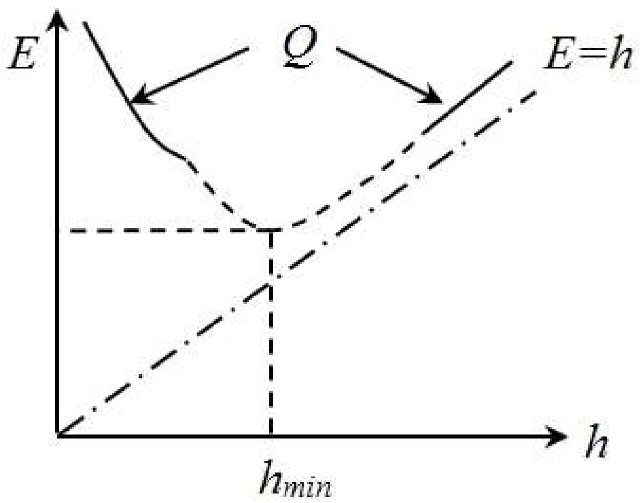


Figure 10 Variation of specific head, *E*, with water depth *h* under a specified discharge *Q*.

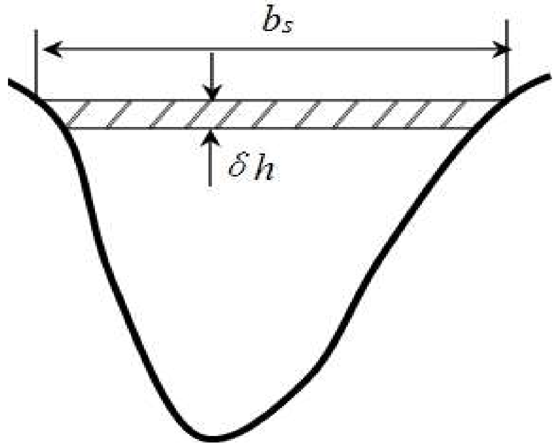


Figure 11 Schematic drawing of the cross section of an open channel

Thus, the specific head reaches minimum when:

$$\frac{Q^2 b_s}{gA^3} = 1$$

Flow condition corresponding to $E=E_{min}$ is referred to as critical flow, and it is defined as

$$\frac{Q^2 b_s}{gA^3} = 1 \text{ with } b_s = b_s(h) \,\&\, A = A(h) \tag{3.9}$$

We can define the mean depth of a channel (see Figure 12) as:

$$h_m = \frac{A}{b_s} \tag{3.10}$$

Then Eq. (3.9) can be written as:

$$\frac{Q^2 b_s}{gA^3} = \frac{(VA)^2 b_s}{gA^2 (h_m b_s)} = \frac{V^2}{gh_m} = 1$$

We here define the Froude number as:

$$F_r = \frac{V}{\sqrt{gh_m}} \tag{3.11}$$

Thus, Eq. (3.9) suggests that the critical flow occurs when:

$$F_r = \frac{V}{\sqrt{gh_m}} = 1 \tag{3.12}$$

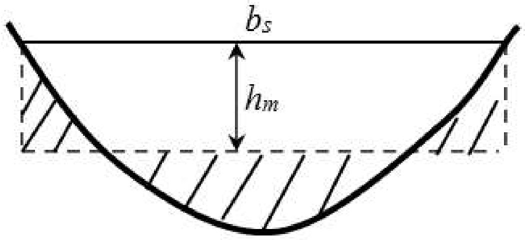


Figure 12 Mean water depth

Let us do more analysis for critical flow. The velocity head for critical flow can be written as:

$$\frac{Q^2}{2gA_c^{\,2}} = \frac{V_c^2}{2g} = \frac{1}{2}\frac{A_c}{b_{sc}} = \frac{1}{2}h_{mc} \tag{3.13}$$

so the specific head is:

$$E_c = h_c + \frac{V_c^2}{2g} = h_c + \frac{1}{2}\frac{A_c}{b_{sc}} = h_c + \frac{1}{2}h_{mc} \tag{3.14}$$

For a rectangular channel, we have:

$b_s = b$ =constant;

$$h_m = \frac{A}{b_s} = \frac{bh}{b_s} = h$$

so the critical flow's specific head for a rectangular channel can be simply written as:

$$E_c = h_c + \frac{Q^2}{2gA_c^2} = h_c + \frac{1}{2}h_{mc} = h_c + \frac{1}{2}h_c = \frac{3}{2}h_c \tag{3.15}$$

Thus, if a channel is rectangular and the specific head is known, the depth corresponding to critical flow is:

$$h_c = \frac{2}{3}E \text{ for rectangular channel} \tag{3.16}$$

Since $Fr = V_c / \sqrt{gh_{mc}} = V_c / \sqrt{gh_c} = 1$ for rectangular channel, the discharge per unit width is given by

$$(Q/b) = q = V_c h_c = \sqrt{g} h_c^{3/2} \tag{3.17}$$

Also, for a rectangular channel, we have:

$$\frac{Q^2}{2gA^2} = \frac{(Q/b)^2}{2gh^2} = velocity\ head = \frac{1}{2}h_{mc} = \frac{1}{2}h_c \tag{3.18}$$

if flow is critical. Thus the critical water depth can be obtained from total discharge with:

$$h_c = \left(\frac{(Q/b)^2}{g} \right)^{1/3} \tag{3.19}$$

For nonrectangular channel, Eq. (3.9) should be used to solve for the critical water depth under a specified discharge. See Chaudhry, section 3-6.

3.2. Specific Head-Depth diagram

With these discussions, we now can introduce the Specific Head-Depth diagram in Figure 13. Using this diagram, we can define two "types" of flow:

Subcritical flow, where most of E is contributed by h with only a small velocity head:

$$h > h_c \Rightarrow F_r^2 = \frac{Q^2}{g\left(A^3 / b_s\right)} < 1 \tag{3.20}$$

Supercritical flow, where most of E is contributed by the velocity head and h is minor:

$$h < h_c \Rightarrow F_r^2 = \frac{Q^2}{g\left(A^3 / b_s\right)} > 1 \tag{3.21}$$

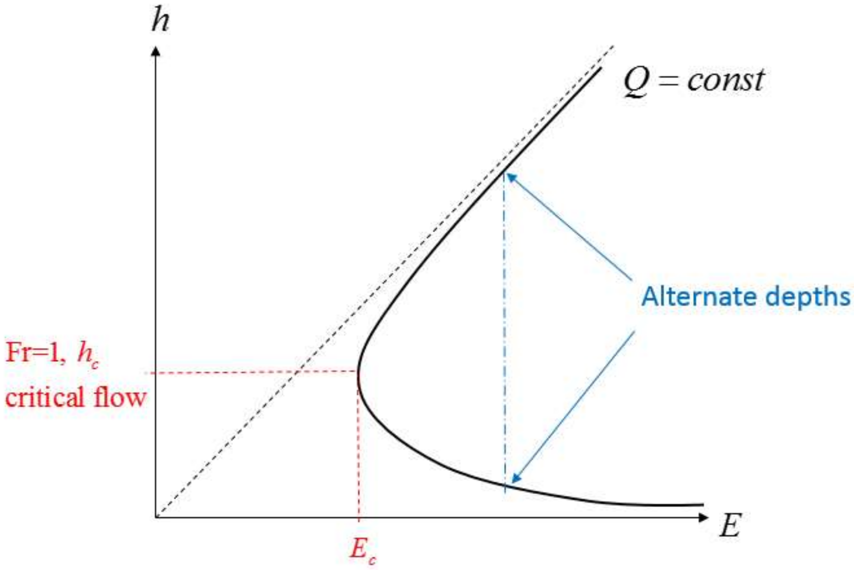


Figure 13 Specific head-depth diagram

For given Q and channel geometry, a specified value of specific head, E, can result in:

(1) If $E>E_c$, two solutions for h. These are known as alternate depths. One corresponds to a subcritical flow, and the other to a supercritical flow, i.e. one $h>h_c$ and the other $h<h_c$.

(2) If $E=E_c$, only one solution corresponding to the critical flow, i.e. $h=h_c$-critical depth.

(3) If $E>E_c$, no solution is physically impossible!

3.3. Steady open channel flow over a short transition

Now, we are properly prepared to return to our transition problem, which "motivated" us to look at E-h diagram:

(1) if E_1 is subcritical: solution for E_2 is subcritical. Free surface slopes downward in flow direction to produce a pressure gradient that accelerates the flow (Q=constant & h decreases, V increases)

(2) if E_1 is supercritical: solution for E_2 is supercritical. Flow is "streaking" uphill against gravity and is slowed down, i.e. V is decreasing. To maintain constant Q=VA, h must increase.

These two scenarios are illustrated in Figure 14.

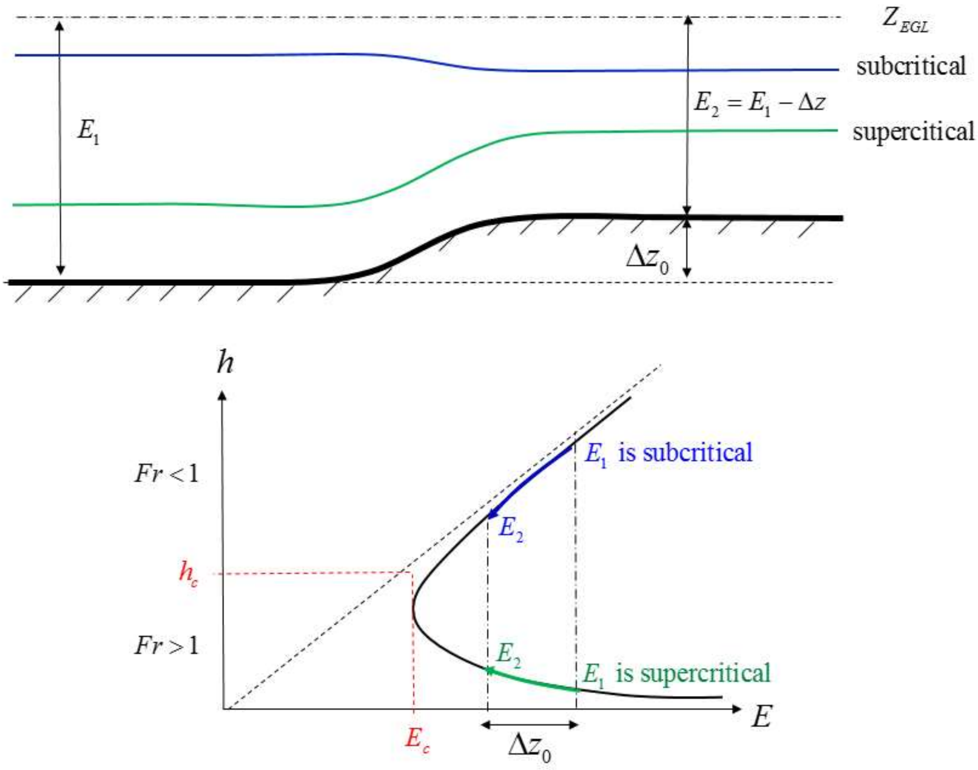


Figure 14 Steady open-channel flow over a short transition

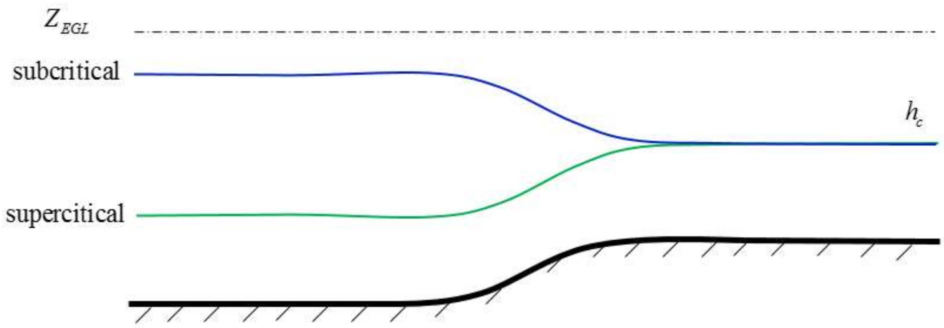


Figure 15 Steady open-channel flow over a short transition with the critical flow occurring downstream

If $E_2 = E_c$ (Figure 15) after transition, flow can just pass the "hump". If $E_2 < E_c$ after transition, there is no physically possible solution for the given value of E_2, which in turn is related to E_1, through $E_2 = E_1 - \Delta z_0$. Only way to get flow over the "hump" is to increase E_1 relative to its specified value, as shown in Figure 16. The channel "chokes", backs up the water upstream until E_1 is just large enough to make flow proceed, i.e.

$$E_{1,final} = E_c + \Delta z_0 \tag{3.22}$$

so that

$$E_{2,final} = E_1 - \Delta z_0 = E_c \tag{3.23}$$

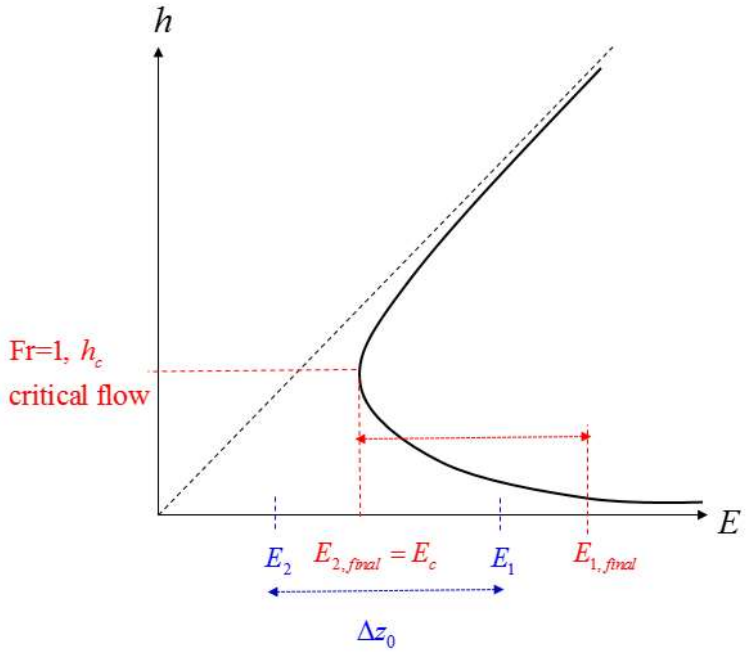


Figure 16 Choke of channel ($E_2<E_c$ initially flow cannot pass, channel chokes and backs up water upstream until $E_{1,final}$ is just enough to produce $E_{2,final} = E_{1,final} - \Delta z_0 = E_c$. Flow can now proceed as critical flow)

We should notice that if the upstream flow is supercritical, velocity is actually decreasing in the direction of flow. Therefore, although the transition may be short, flow is actually not converging but diverging. In such cases, possibility of head loss due to expansion is present, but we assumed ΔH=0 (inconsistency)!

As a golden rule, there is no head loss for a Short Transition of Non-separating Flows. Here "Short" means that head loss due to friction may be neglected. "Short" also means that any change in elevating due to the general slope of the channel may be neglected over the short transition. "Non-separating" means converging (for sure) or so gently expanding that flow separation from boundaries is not expected.

The concept of “short transition” is excellent for subcritical flow transitions from one channel cross-section (e.g. man-made rectangular concrete channel) at (1) through a transition to a natural (e.g. a trapezoidal channel) at (2). That is transition may involve a change in bottom elevation as well as a change in channel cross-section’s geometry. So long as changes are gradual and do not result in severe decreases in velocity in direction of flow (negligible expansion losses) the basic assumption: $\Delta H \simeq 0$ for the transition is reasonably good.

However, for supercritical flow transitions: watch out! The smooth variations in water depth h may potentially be completely unrealistic, and transitions exhibiting hydraulic jump, waves and non-uniform flows downstream may results (consult advanced texts on open channel flow for supercritical transition).

4. Momentum Principle

4.1. Thrust-depth diagram and conjugate depths

When applying the momentum principle, we conceptualize the pressure force and momentum flux through a control surface perpendicular to a well-behaved flow as thrust:

$$MP = (p_{CG} + \rho V^2)A$$

where p_{CG} is the water pressure at the centroid of the control surface and V is the averaged velocity through the surface. (Please consult the supplementary note “Momentum Principle” for basic knowledge on momentum principle first).

As shown in Figure 17, for well-behaved flow we can define a plain control surface perpendicular to the flow, and the thrust acting on it is in the direction parallel to the bottom (the x-direction):

$$MP_x = MP = \left(\rho V^2 + P_{CG}\right)A = \rho\frac{Q^2}{A} + \rho g\left(h - y_{CG}\right)A\cos\beta \qquad (4.1)$$

or:

$$\frac{MP}{\rho g} = \frac{Q^2}{\rho A} + \left(h - y_{CG}\right)A \qquad (4.2)$$

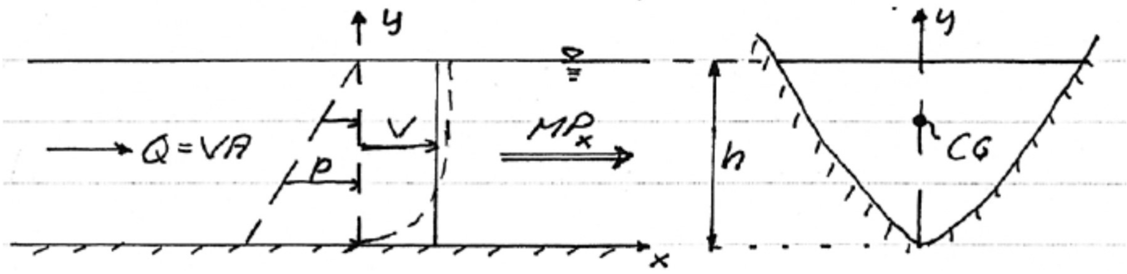


Figure 17 Thrust for well-behaved open channel flow

For a given Q and cross-section, it is obvious that:

$$\frac{MP}{\rho g} \to \infty \text{ as } h \to \infty : \frac{Q^2}{\rho A} \to \infty$$

$$\frac{MP}{\rho g} \to \infty \text{ as } h \to \infty : \left(h - y_{CG}\right) A \to \infty$$

Thus, just as we found for the specific energy, E, the thrust must have a minimum value for given Q and channel cross-section. This minimum is obtained from:

$$\partial\left(\frac{MP}{\rho g}\right) / \partial h = 0 \tag{4.3}$$

With some algebra:

$$\frac{\partial\left(MP / \rho g\right)}{\partial h} = \frac{\partial}{\partial h}\left(\frac{Q^2}{gA} + hA - y_{CG} A\right) = -\frac{Q^2}{gA^2}\frac{\partial A}{\partial h} + A + h\left(\frac{\partial A}{\partial h}\right) - \frac{\partial\left(y_{CG} A\right)}{\partial h}$$

Recall that $y_{CG} A$ is moment of A around y=0 (the channel bottom), we have:

$$\delta\left(y_{CG} A\right) = \left(\delta h \cdot b_s\right) \cdot h$$

so

$$\partial\left(y_{CG} A\right) / \partial h = b_s h = h\frac{\partial A}{\partial h} = b_s h \tag{4.4}$$

Thus,

$$\frac{\partial\left(MP / \rho g\right)}{\partial h} = -\frac{Q^2 b_s}{gA^2} + A = 0 \tag{4.5}$$

or

$$\frac{Q^2 b_{sc}}{gA_c^3} = F_r^2 = 1 \quad (4.6)$$

How about that! Thrust is minimum for a given channel conveying a certain discharge Q when the Froude Number is unity, i.e. when the flow is critical, just as we obtained for the minimum of specific energy, E.

The corresponding minimum value for thrust is given by:

$$\left(\frac{MP}{\rho g}\right)_{\min} = \left[\frac{Q^2}{gA_c^2} + \left(h_c - y_{CG,C}\right)\right]A_c = \left(\frac{A_c}{b_{sc}} + \left(h_c - y_{CG,C}\right)\right)A_c = \left(h_{mc} + h_c - y_{CG,C}\right)A_c \quad (4.7)$$

which, for a rectangular channel is:

$$\left(\frac{MP}{\rho g}\right)_{\min} = \frac{3}{2}h_c^2 \cdot b \quad (4.8)$$

Thus, analogous to the E vs h diagram, we have a thrust-depth diagram (Figure 18):

(1) If $MP > MP_{min}$ There are two solutions for the depth. One corresponds to supercritical flow, the other to subcritical flow. These depths that correspond to the same value of MP (for given channel geometry and discharge Q) are conjugate depths.
(2) If $MP = MP_{min}$ there is only one solution corresponding to critical depth
(3) If $MP < MP_{min}$ there is no solution. The specified combination of MP and Q is physically impossible in the given channel.

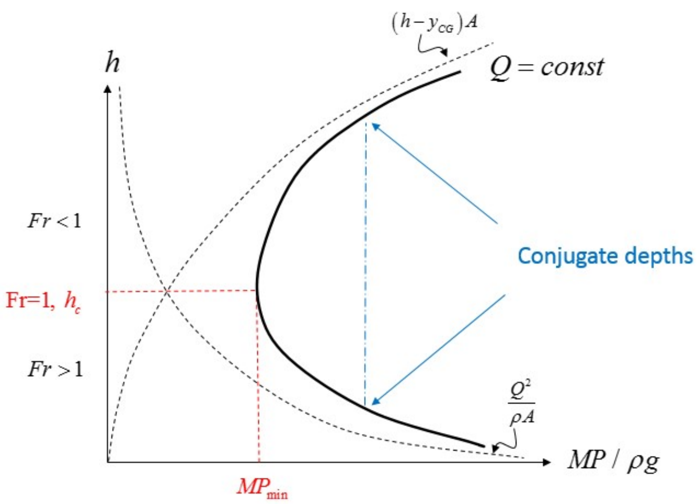


Figure 18 Thrust-Depth diagram

The unassisted (free) hydraulic jump

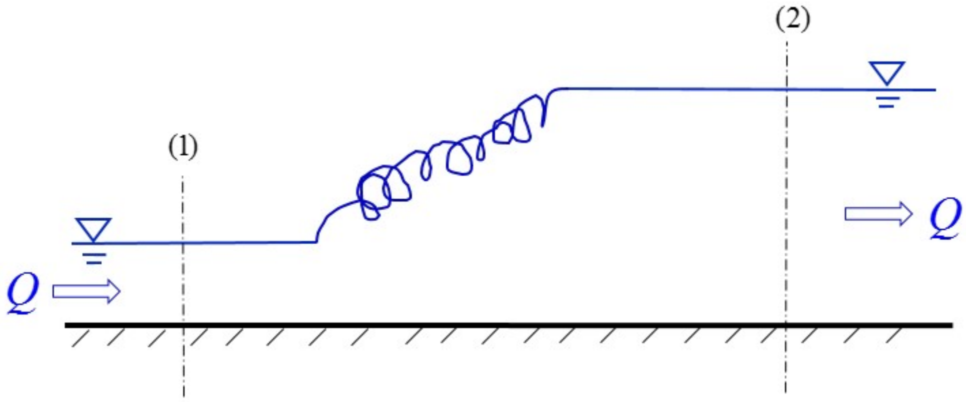


Figure 19 Unassisted hydraulic jump

Here we consider one application of the momentum principle: the unassisted (free) hydraulic jump. The upstream flow quickly transits into the downstream flow through a hydraulic jump, which actively dissipates flow energy. We consider a control volume defined by the flow within the cross-section 1 and 2 in Figure 19. Applying momentum balance for this control volume in the x-direction we get:

$$MP_1\text{+Gravity in } x\text{=}MP_2 + \text{friction force from bottom} + \text{other forces in } x$$

For simplicity, we can make the following assumptions:

1) Channel must be prismatic with a plane bottom, so the "other forces" in x can be neglected. If this were not the case, there would be pressure forces from the sidewalls and the bottom that would not be perpendicular to x, i.e. there would be a contribution of unknown magnitude from "other forces in x".
2) The plane bottom slope β is small, so the gravity component in x can be neglected.
3) The transition from 1) and 2) is "short", so friction force from bottom is small.

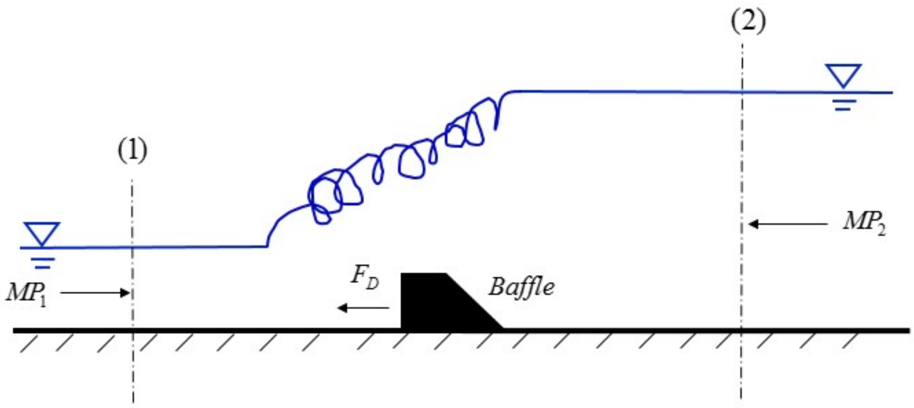
With these assumptions/simplifications, the result is that

$$MP_1 = MP_2$$

This means that h_1 and h_2 are conjugate depths. Thus, for a free (or unassisted) hydraulic jump across the flow changes from supercritical (h_1) to subcritical (h_2) [or vice versa-but this turns out to violate energy conservation].

The Assisted Hydraulic Jump

Keeping the channel prismatic of negligible slope and friction (gravity + friction =0). There may be cases, e.g. in the stilling basin downstream of a dam, when the balance of MP_1=MP_2 cannot be achieved without some "assistance"



An example of this assistance is baffles, i.e. flow obstructions that produce a drag force on the fluid that helps to achieve the necessary force equilibrium for the jump:

$$MP_1 = MP_2 + F_D \tag{4.9}$$

Since F_D would vary depending on the position of the baffles within the jump (large approach velocity = large F_D if close to start of jump, low velocity= small F_D if towards end of jump), the baffles can, within a range, adjust F_D so that the jump takes place across the baffles, i.e. we may control the location of the hydraulic jump, for a range of flow conditions.

4.2. Hydraulic jump in a rectangular channel

In this section, we discuss more about the hydraulic jump. For algebraic simplicity, we consider the unassisted hydraulic jump in a rectangular channel. With channel width being a constant, we treat problem in terms of "per unit width": q=Q/b=discharge per unit width=Vh, as shown in Figure 20.

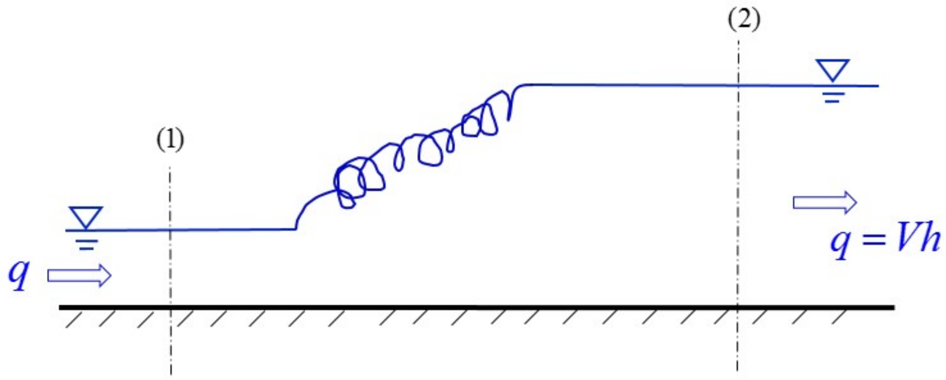


Figure 20 Unassisted hydraulic jump in a rectangular channel

The trusts at sections 1-1 and 2-2 in Figure 20 are:

$$MP_1 = \frac{1}{2}\rho g h_1^2 + \rho V_1^2 h_1 = MP_2 = \frac{1}{2}\rho g h_2^2 + \rho V_2^2 h_2$$

Divide by $\frac{1}{2}\rho g h_1^2$:

$$2\frac{V_1^2 h_1}{g h_1^2} - 2\frac{V_2^2 h_2}{g h_1^2} = \left(h_2 / h_1\right)^2 - 1 \tag{4.10}$$

using $q=Vh$, we can re-write the left-hand side of (4.10) as :

$$2\frac{V_1^2 h_1}{g h_1^2} - 2\frac{V_2^2 h_2}{g h_1^2} = 2\left(\frac{q^2}{g h_1^3} - \frac{q^2}{g h_1^2 h_2}\right) = 2\frac{q^2}{g h_1^3}\left(1 - \frac{h_1}{h_2}\right)$$

and re-write the right-hand side as:

$$\left(\frac{h_2}{h_1}\right)^2 - 1 = \left(\frac{h_2}{h_1} - 1\right)\left(\frac{h_2}{h_1} + 1\right) = \frac{h_2}{h_1}\left(1 - \frac{h_1}{h_2}\right)\left(\frac{h_2}{h_1} + 1\right)$$

Eq. (4.10) then becomes:

$$\frac{q^2}{g h_1^3} = \frac{1}{2}\frac{h_2}{h_1}\left(\frac{h_2}{h_1} + 1\right) \tag{4.11}$$

Notice that:

$$\frac{q^2}{g h_1^3} = \frac{V_1^2}{g h_1} = Fr_1^2$$

The final form of momentum principle is obtained as:

$$Fr_1^2 = \frac{1}{2}\frac{h_2}{h_1}\left(\frac{h_2}{h_1}+1\right) \tag{4.12}$$

This is a quadratic equation in h_2 / h_1, and the solution is:

$$h_2 / h_1 = \frac{1}{2}\left(-1 \pm \sqrt{1+8Fr_1^2}\right) \tag{4.13}$$

Similarly, we can obtain:

$$h_1 / h_2 = \frac{1}{2}\left(-1 \pm \sqrt{1+8Fr_2^2}\right) \tag{4.14}$$

where h_1 = upstream depth, h_2 = downstream depth , h_1 & h_2 are conjugate depths. Eq. (4.13) and Eq. (4.14) are known as the jump conditions. Eq. (4.13) can be used to find h_2 with given upstream conditions, while Eq. (4.14) can be used to obtain h_1 with given downstream conditions.

Head Loss across Hydraulic Jump in a rectangular channel:

The head loss is give by:

$$\Delta H_j = H_1 - H_2 = E_1 - E_2 = h_1 - h_2 + \frac{q^2}{2gh_1^2} - \frac{q^2}{2gh_2^2} \tag{4.15}$$

which can be written as:

$$\Delta H_j = h_1 - h_2 + \frac{q^2}{2gh_1^2}\left(1-\left(\frac{h_1}{h_2}\right)^2\right) = h_1 - h_2 + \frac{h_1}{2}Fr_1^2\left(1-\left(\frac{h_1}{h_2}\right)^2\right)$$

Substitute in Eq. (4.12) for Fr_1, we get:

$$\begin{aligned}\Delta H_j &= h_1 - h_2 + \frac{1}{4}h_1\frac{h_2}{h_1}\left(1+\frac{h_2}{h_1}\right)\left(1-\left(\frac{h_1}{h_2}\right)^2\right)\\ &= -\left(h_2-h_1\right)+\frac{1}{4}\frac{1}{h_1h_2}\left(h_1+h_2\right)\left(h_2+h_1\right)\left(h_2-h_1\right)\\ &= \left(h_2-h_1\right)\left(\frac{1}{4h_1h_2}\left(h_1+h_2\right)^2-1\right)\\ &= \frac{\left(h_2-h_1\right)^3}{4h_1h_2}\end{aligned} \tag{16}$$

This derivation was completely general, but we must require that $H_2 \le H_1$, i.e. $\Delta H_j \ge 0$. This is only $h_2 > h_1$, as illustrated in Figure 21.

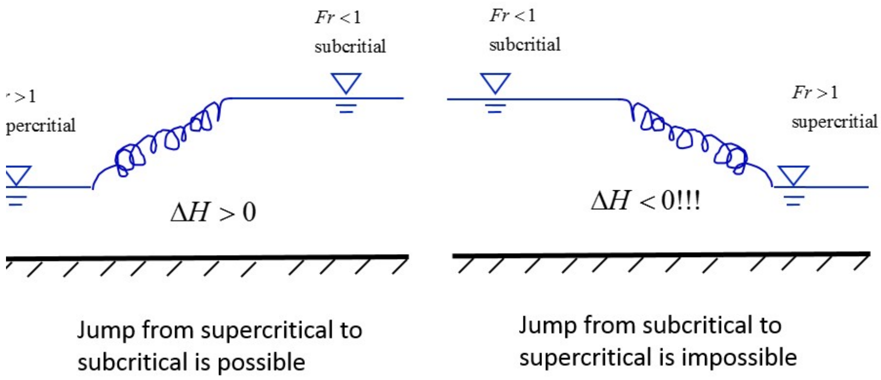


Figure 21 feasibility of a hydraulic jump in rectangular channel

Now let us investigate hydraulic jump by considering the specific head-discharge and thrust-head diagram together.

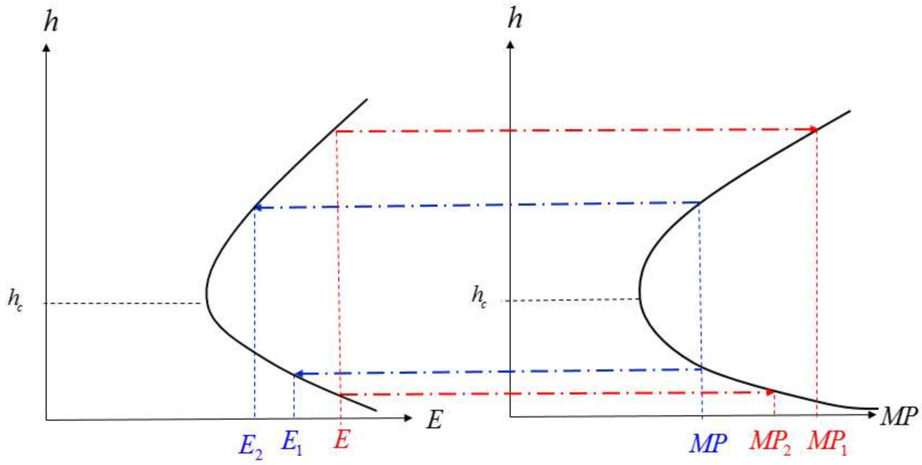


Figure 22 Combined use of E-h & MP-h diagrams

Jump from supercritical to subcritical is possible since the flow is expanding and is associated with a loss in head. $MP_1 = MP_2 \Rightarrow E_1 > E_2$, there is a head loss. It is possible to have a flow going from subcritical to supercritical without violating energy conservation, e.g. flow under a gate, but only if there is an exterior force applied to the fluid, e.g. pressure force from the gate. $E_1 = E_2 \Rightarrow MP_1 > MP_2 \Rightarrow MP_2$, needs help to balance MP_1.

- Supercritical flow: Momentum is important in *MP*, $V^2/2g$ is important in *E*.
- Subcritical flow: Pressure is important in *MP*, *h* is important in *E*.

The moving hydraulic jump (the Bore)

Imagine a steady uniform flow in a rectangular channel, into which a gate is suddenly dropped so that the discharge locally (x=0) is instantaneously (t=0) changed (reduced). This will pile up water upstream of the gate and cause an increase in water depth at the gate. This depth increase will move upstream from the gate at a speed "*C*" with a depth h_1=original depth of flow ahead of it and a depth $h_2>h_1$ behind it. This situation corresponds to a moving hydraulic jump (a.k.a. a bore) and is shown in Figure 23.

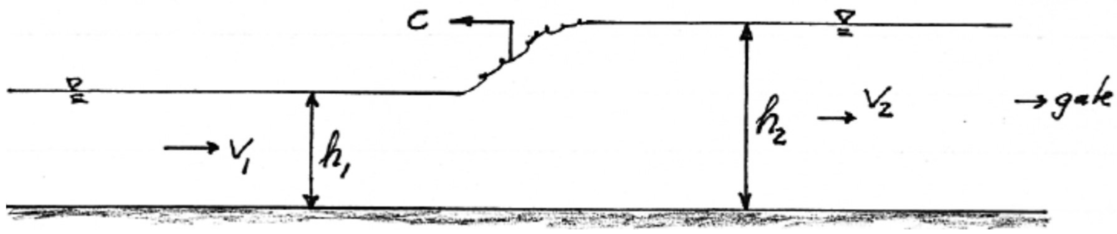


Figure 23 a moving hydraulic jump (bore) in a rectangular channel

This is obviously an unsteady problem! But we may make it steady by adopting a coordinate system that moves along with the bore, i.e. moves at a constant speed "*C*" in the upstream direction. In this moving coordinate system (i) the depth change from h_1 to h_2 is not changing position (we are moving along with it) and (ii) the velocity upstream and downstream of the jump is $C+V_1$, and $C+V_2$, respectively. Thus, in the moving reference frame the moving hydraulic jump appears identical to the stationary hydraulic jump we just analyzed, except for the addition of "*C*" to all velocities, as shown in Figure 24.

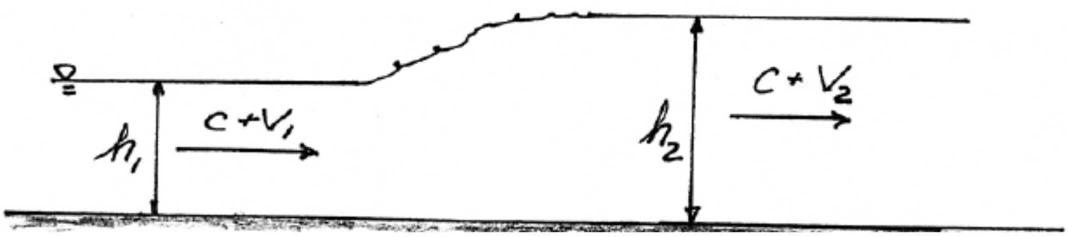


Figure 24 Steady "bore" in moving coordinate system

Conservation of volume gives:

$$(C+V_1)h_1=(C+V_2)h_2 \tag{4.17}$$

Conservation of momentum gives:

$$\left(\rho(C+V_1)^2+\frac{1}{2}\rho g h_1\right)h_1=\left(\rho(C+V_2)^2+\frac{1}{2}\rho g h_2\right)h_2 \tag{4.18}$$

Exactly the same as before when V_1 and V_2 are replaced by (V_1+C) and (V_2+C), respectively. In analogy to the stationary hydraulic jump we obtained:

$$Fr_1^2=\frac{(V_1+C)^2}{gh_1}=\frac{1}{2}\frac{h_2}{h_1}\left(1+\frac{h_2}{h_1}\right) \tag{4.19}$$

From which C= the speed of the bore, may be obtained if V_1, h_1 and h_2 are known. V_1 and h_1 correspond to the original steady uniform flow in the channel and they may be obtained from knowledge of slope S_0, Manning's coefficient "n", channel width b, and discharge Q. If the gate is partially closed the elevation jump h_2 may be estimated by requiring the discharge Q to pass under the gate. In particular, if we assume the jump to be weak, i.e. $h_2-h_1\ll h_1$ so $h_2/h_1\approx 1$ or $h_2/h_1-1\ll 1$, we obtain from the moving jump condition:

$$(V_1+C)^2=\frac{1}{2}\frac{h_2}{h_1}\left(1+\frac{h_2}{h_1}\right)gh_1\approx gh_1 \tag{4.20}$$

or

$$V_1+C=\pm\sqrt{gh_1}\Rightarrow C=-V_1\pm\sqrt{gh_1} \tag{4.21}$$

Thus, an infinitesimally small disturbance on still water (V_1=0) may propagates in the upstream or downstream direction at a speed

$$C_o=-\begin{cases}+\sqrt{gh_1},=\sqrt{gh} & \text{upstream}\\ -\sqrt{gh_1},=-\sqrt{gh} & \text{downstream}\end{cases} \tag{4.22}$$

When the small disturbance moves on running water, its speed may be

$$C=-V_1\begin{cases}+\sqrt{gh_1}\\ -\sqrt{gh_1}\end{cases} \tag{4.23}$$

If $V_1 < \sqrt{gh_1} \Rightarrow$ i.e. $Fr = V_1 / \sqrt{gh} < 1$, we can have:

$$C = \begin{cases} \sqrt{gh_1}\left(1 - Fr\right) > 0 & \text{moving upstream} \\ -\sqrt{gh_1}\left(1 + Fr\right) < 0 & \text{moving downstream} \end{cases} \tag{24}$$

Note C>0 is moving upstreaming, which is defined in Figure 24. Thus, in a subcritical flow disturbances propagate both in the upstream and downstream directions.

If $V_1 > \sqrt{gh_1} \Rightarrow$ i.e. $Fr = V_1 / \sqrt{gh} > 1$, we have:

$$C = \begin{cases} \sqrt{gh_1}\left(1 - Fr\right) < 0 \\ -\sqrt{gh_1}\left(1 + Fr\right) < 0 \end{cases} \quad \text{both move downstream}$$

Thus, in a supercritical flow disturbances can propagate only in the downstream direction. These results demonstrate the significance of the Froude Number:

$$Fr = \frac{V}{\sqrt{gh_m}} = \frac{\text{Fluid velicty}}{\text{Speed of small disturbance}} \tag{4.25}$$

- Fr <1 (subcritical flow): Changes in flow conditions are felt upstream of the location where changes occur. Subcritical flows are controlled by downstream conditions
- Fr >1 (Supercritical flow): Changes in flow conditions are felt only downstream of location where changes occur. Upstream is entirely unaware of what happens downstream. Supercritical flows are controlled by upstream conditions.

5. Gradually varied flow

5.1. Definition of gradually varied flow

Let us review the tools we have developed for free surface flow.

A. In section 2, we discusses the very simple scenario: uniform and steady flow. For such flows, the driving gravity force is balanced by bottom resistance, which can be calculated using various method, e.g. Manning's equation. Therefore, we shall be able to solve for discharge with given water depth, or vice versa. This water depth is termed normal depth.
B. We also developed tools for steady flow over a short transition. For a converging short transition, there is no head loss, and we use the concept of specific energy and alternate depth

to describe the transition. The depth gives the minimum specific energy is called the critical water depth.

C. The concept of short converging transition is good for subcritical-to-supercritical transition. For supercritical-to-subcritical transition, a diverging short transition usually occurs, so there is energy loss during the transition, which requires us to add in the momentum principle. This leads to the discussion of hydraulic jump and the concept of conjugate depth.

Thus, we have the tools to deal with extreme flow conditions that either very rapidly (item B and C) or not at all (item A). We need to fill out the intermediate space, i.e. we need tools to analyze Gradually Varied Flow.

Let us illustrate this "need" by an example of a flow in a prismatic channel in which there is a sudden change in the bottom slope, as shown in Figure 25. In the upstream channel with slope S_{01} we have a normal depth h_{n1}, whereas the normal depth in the downstream channel is h_{n2} corresponding to a slope $S_{02} \neq S_{01}$. The discharge Q is the same in the two channels, so the flow is steady. The change in slope is "sudden", i.e. occurs over a short distance.

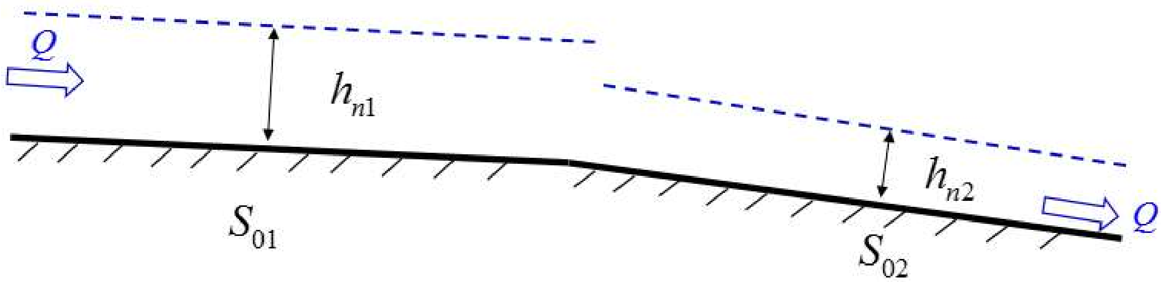


Figure 25 an example of gradually varied flow

Far upstream and downstream of the change in bottom slope the flow must be uniform, since this is the only flow that can exist in an "infinitely long" channel. But if this were the case all the way to the sudden change in slope, then the mismatch in depths ($h_{n1} \neq h_{n2}$) could only be handled by our "rapidly varying flow" tools if h_{n1} and h_{n2} were either alternate or conjugate depths. This cannot be always satisfied, because h_{n1} and h_{n2} are already determined by bottom slopes and discharge. Therefore, somehow the upstream and downstream flow must adjust to meet each other at the transition in slope, i.e. vary gradually in space.

“Gradually” or “slowly” refers to the spatial variation, i.e. the flow is still assumed to be steady. Therefore, with $\partial / \partial t = 0$, and:

$$Q = VA = cons \tag{5.1}$$

However the flow characteristics such as depth, *h*, area, *A*, and average velocity, *V*=*Q/A*, are allowed to vary gradually (or “slowly”) along the channel, i.e. vary slowly in *x*-direction. Locally, the flow behaves as a “well behaved” flow since the “gradual” variation assumes that streamlines are nearly straight and nearly parallel. Thus, we have that ($\cos\beta \simeq 1$):

$$H = \text{total head} = \frac{V^2}{2g} + h + z_0 = \frac{Q^2}{2gA^2} + h + z_0 \tag{5.2}$$

where *A* and *b*, and therefore *H*, are slowly varying functions of *x*.

Restricting the channel to be prismatic so that *A*=*A*(*h*), we then have:

$$\frac{\partial H}{\partial x} = \frac{\partial}{\partial x}\left(\frac{Q^2}{2gA^2} + h\right) + \frac{\partial z_0}{\partial x} = \left(\frac{\partial}{\partial h}\left(\frac{Q^2}{2gA^2}\right) + 1\right)\frac{\partial h}{\partial x} - S_0 \tag{5.3}$$

where $\frac{\partial z_o}{\partial x} = -\sin\beta = -S_o$

In section 3.1 , when determining the minimum value of the specific head, we obtained

$$\frac{\partial}{\partial h}\left(\frac{Q^2}{2gA^2}\right) = -\frac{Q^2}{2gA^3}\frac{\partial A}{\partial h} = -\frac{Q^2 b_s}{gA^3} = -\frac{V^2}{gh_m} = -Fr^2$$

Introducing these expressions:

$$\frac{\partial H}{\partial x} = \left(1 - Fr^2\right)\frac{\partial h}{\partial x} - S_0 = -S_f \tag{5.4}$$

where S_f is the slope of the EGL (which must always be positive since it represents the rate of dissipation of mechanical energy). Rearranging the terms, we obtain an equation for the slope of the free surface:

$$\frac{dh}{dx} = \frac{S_o - S_f}{1 - Fr^2} \tag{5.5}$$

Under the assumptions of gradually varied flow, the only reason for energy dissipation is bottom friction, so S_f can be obtained using Eq. (2.29):

$$S_f = \frac{\tau_b}{\rho g R_h} = \begin{cases} S_f = \dfrac{f}{8g}\dfrac{P}{A^3}Q^2 \left(Darcy - Weisbach\right) \\ S_f = \dfrac{1}{C^2}\dfrac{P}{A^3}Q^2 \left(Chezy\text{'}s\ Equation\right) \\ S_f = n^2 \dfrac{P^{4/3}}{A^{10/3}}Q^2 \left(Manning\text{'}s\ Equation\right) \end{cases}$$

For given Q and bottom resistance information (either f, C or n), S_f is only a function of wetted perimeter P and cross-section area A, which are both functions of water depth h. Thus, Eq. (5.5) can be considered an ordinary differential equation for h, which can be solved for water depth variation along the channel.

5.2. surface profile of gradually varied flow in a prismatic channel

The surface profile $h(x)$ of a gradually varied flow in a prismatic channel often requires numerical solution, but a qualitative estimate of the surface can be easily obtained by consulting Eq. (5.5).

We first recall that for normal flow, $S_0=S_f$, which gives h_n, the normal depth. If $h>h_n$, then $S_f<S_0$, because a larger h gives smaller average velocity V under a given total discharge and therefore small bottom resistance S_f. It is equivalent to say that If $h<h_n$, then $S_f>S_0$, because a smaller h gives larger average velocity V under a given total discharge and therefore larger bottom resistance S_f. Thus, <u>the sign of S_0-S_f is the same as the sign of h-h_n.</u>

Froude number is defined as:

$$Fr = \frac{Q^2 b_s}{gA^3}$$

Fr=1 when the flow is critical, i.e. $h=h_c$. If $h>h_c$, then $Fr^2<1$, because larger h gives larger A^3/b_s and hence smaller Fr^2. On the other hand, If $h<h_c$, then $Fr^2>1$, because smaller h gives smaller A^3/b_s and hence larger Fr^2. Thus, <u>the sign of 1-Fr^2 is the same as the sign of h-h_c.</u>

Thus, we can predict with certainty the nature of the depth variation in the direction of flow by simply considering the signs of the numerator, S_0-S_f, and denominator, 1- Fr^2, in the gradually varied flow equation, and these signs depend, in turn, on the local depth, h, relative to normal depth, h_n, and critical depth, h_c, in the given channel (prismatic) for the given discharge, Q.

All that is needed is the relative magnitude of h_n and h_c, and this is related to the slope of the channel. So,

"If normal flow is $\begin{Bmatrix} \text{subcritical} \\ \text{supercritical} \end{Bmatrix}$ then $Fr_n \begin{Bmatrix} < \\ > \end{Bmatrix} 1$ and the slope, S_o is referred to as $\begin{Bmatrix} \text{mild} \\ \text{steep} \end{Bmatrix}$"

or:

$$\begin{cases} h_n > h_c, \text{ mild slope} \\ h_n < h_c \text{ steep slope} \end{cases}$$

5.2.1. Mild slope:

For mild slope, we always have $h_n>h_c$. Thus, depending the water depth, there can be three scenarios as shown in Figure 26:

- M1 profile: $h>h_n>h_c$
- M2 profile: $h_n>h>h_c$
- M3 profile: $h_n>h_c>h$

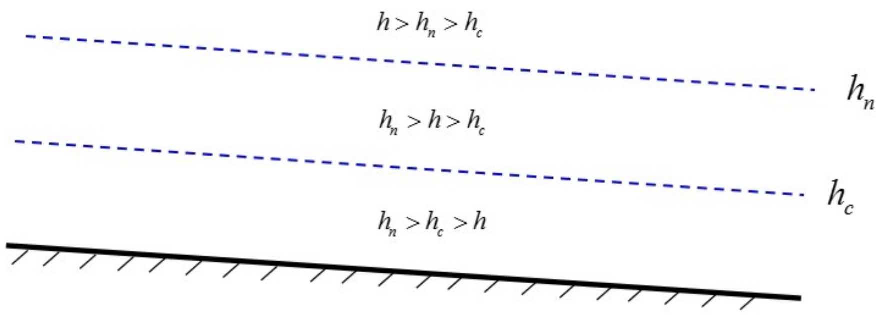


Figure 26 Classification of mild-slope surface profile

(1) M1 Profile : $h>h_n>h_c$

$$\frac{\mathrm{d}h}{dx} = \frac{S_o - S_f}{1 - Fr^2} = \frac{+}{+} > 0$$

M1 profiles must have a depth that increases in the direction of flow-or conversely- a depth that decrease in the upstream direction.

To further determine the shape of M1, let us consider its limiting behavior (*h* approaches infinite and normal depth). As the *h* approaches infinite, we have:

$$h \to \infty \Rightarrow S_f \text{ and } Fr^2 \to 0 \Rightarrow \frac{dh}{dx} \to S_o = \sin\beta$$

This means that the free surface becomes horizontal as $h \to \infty$. Since $h \to \infty$ gives $V \to 0$ conditions approach hydrostatics: Hence a horizontal free surface, as shown in Figure 27.

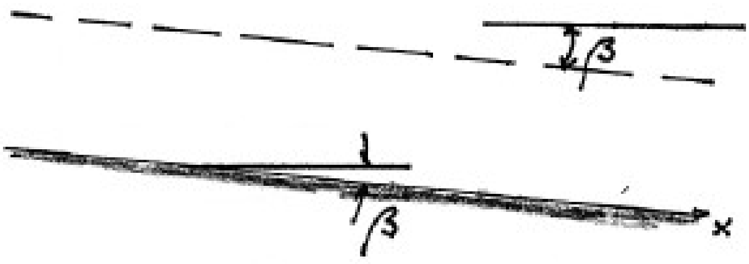


Figure 27 Limiting behavior of M1 profile as water depth increases to infinite

When the water depth *h* approaches the normal depth h_n:

$$h \to h_n \Rightarrow S_f \to S_o, Fr^2 \to Fr_n^{\ 2} < 1 \Rightarrow \frac{dh}{dx} \to 0 \tag{5.6}$$

Depth approaches normal depth which is an obvious result- but how does it approach "normally"

To answer this, we consider a simple channel of the "very wide, rectangular" variety. For this case we have:

$$\frac{dh}{dx} = \frac{S_o}{1 - Fr_n^2}\left(1 - \frac{S_f}{S_o}\right) = \frac{S_o}{1 - Fr_n^2}\left(1 - \left(\frac{h_n}{h}\right)^3\right) \tag{5.7}$$

where S_f is expressed by the Chezy formula. Now, since we are looking for the behavior as $h \to h_n$ from above, we take:

$$h = h_n + \delta_x(x) \text{ with } \delta_x(x) << h \tag{5.8}$$

For this the equation may be written:

$$\frac{d\delta_n}{dx} = \frac{S_o}{1-Fr_n^2}\left(1-\left(\frac{h_n}{h_n+\delta_n}\right)^3\right) = \frac{S_o}{1-Fr_n^2}\left(1-\left(1+\frac{\delta_n}{h_n}\right)^{-3}\right) \qquad (5.9)$$

Finally,

$$\left(1+\frac{\delta_n}{h_n}\right)^{-3} \approx 1-3\frac{\delta_n}{h_n}$$

and the behavior of δ_n, i.e. the manner in which h_n is approached is given by:

$$\frac{d\delta_n}{dx} - \frac{3}{h_n}\frac{S_o}{1-Fr_n^2}\delta_n = \frac{d\delta_n}{dx} - \alpha_n\delta_n = 0$$

The solution to this equation is an exponential variation

$$\delta = \delta_o e^{\alpha_n x}$$

where δ_o =value of δ at "*x*=0". Thus, $\delta \to 0$, i.e. $h \to h_n$, as $x \to -\infty$. So, normal depth is approached "asymptotically" [gets close, but "never" get there] as we go upstream (*x* positive in downstream direction)

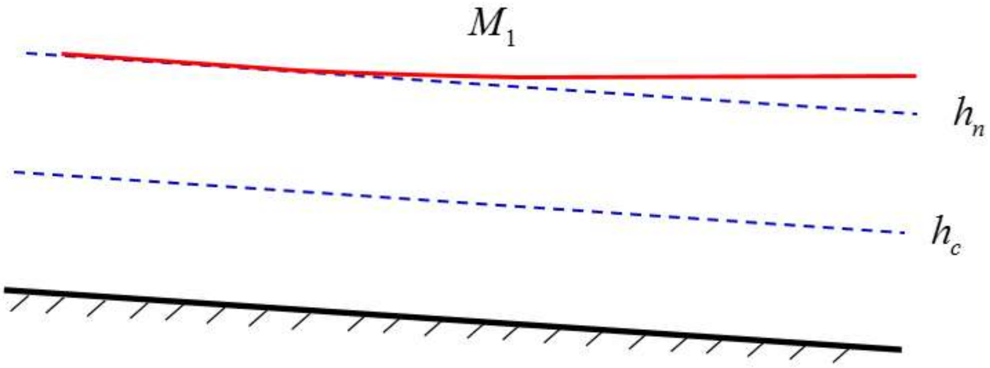


Figure 28 Limiting behavior of M1 profile as the normal depth is approach

The M1-profile is often referred to as a "backwater profile" or "backwater curve" since it is encountered when a flow obstruction such as an underflow gate or a dam "backs up" the water into the mildly sloping channel. Notice, it is a downstream control or condition that imposes a requirement of a depth $h>h_n$ and this control is propagated and felt upstream of control. This is ,of course, related to the fact that the flow along the entire backwater profile has $h> h_n > h_c$ and therefore a subcritical flow which is controlled by downstream conditions.

(2) M2 Profile : $h_n > h > h_c$

$$\frac{dh}{dx} = \frac{S_o - S_f}{1 - Fr^2} = \frac{-}{+} < 0$$

M2-profiles must have a depth that decrease in the downstream (increases in the upstream) direction. For this reason, M2-profiles are often referred to as "draw-down profiles" or "draw-down curves".

As $h \to h_n$, the surface profile behaves in the same manner as obtained for the M1-profile in this limit, i.e. h approaches h_n asymptotically as we go upstream ($h > h_c$ flow is subcritical and has downstream control).

As $h \to h_c$

$$h \to h_c \Rightarrow S_o - S_f < 0, Fr \to 1 \Rightarrow 1 - Fr^2 \to 0 \Rightarrow \frac{dh}{dx} \to -\infty \qquad (5.10)$$

So, critical flow is approached very abruptly! In fact, $\frac{dh}{dx} \to -\infty$ as $h \to h_c$ suggest that the free surface is perpendicular to the bottom $h = h_c$. Even if the bottom slope is small, this means that the free surface is "leaning" forward, which is an absurd result.

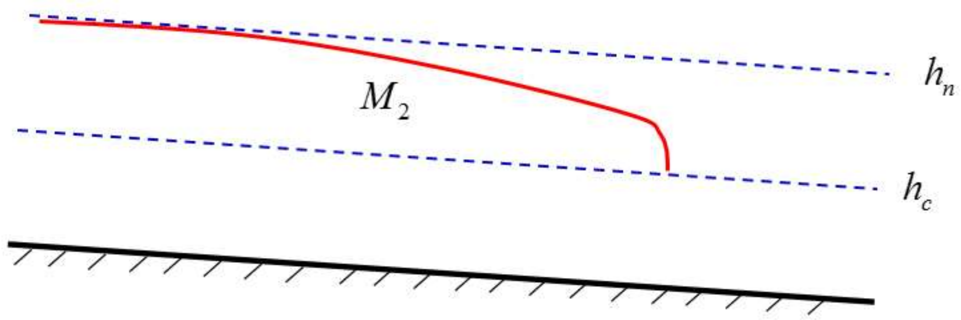


Figure 29 Limiting behavior of M2 profile as the critical depth is approached

First, we recall that the equation governing gradually varied flow profiles was derived under the assumption of well-behaved flow everywhere. This implies that the pressure distribution is hydrostatic and that streamlines are straight and parallel (nearly). Near critical flow both assumptions do not hold, i.e. assumptions are violated and solution is invalid! How serious are the consequences of this violation? To answer this question, we again simplify our analysis by assuming a wide rectangular channel and Chezy's formula for flow resistance and examine the behavior of h as h_c is approached by taking

$$h = h_c + \delta_c(x) \text{ ; with } \delta_c \ll h_c$$

With these simplifications we have

$$\frac{dh}{dx} = \frac{d\delta_c}{dx} = \frac{S_o - S_f}{1 - Fr^2} = \frac{S_o\left(1 - S_f / S_o\right)}{1 - \dfrac{Q^2 / b^2}{gh^3}} = \frac{S_o\left(1 - \left(h_n / h\right)^3\right)}{1 - \left(h_c / h\right)^3} \tag{5.11}$$

In the numerator $h = h_c$ whereas $h = h_c + \delta_c$ in the denominator. Thus, we obtain

$$\frac{d\delta_c}{dx} = -\frac{S_o\left(h_n^3 - h_c^3\right)}{h_c^3\left(1 - \left(1 + \delta_c / h_c\right)^{-3}\right)} \approx -\frac{S_o\left(h_n^3 - h_c^3\right)}{3h_c^3\delta_c} \tag{5.12}$$

or

$$2\delta_c \frac{d\delta_c}{dx} = \frac{d\left(\delta_c\right)^2}{dx} = -\frac{2}{3}\frac{S_o\left(h_n^3 - h_c^3\right)}{h_c^2} = \alpha_c \tag{5.13}$$

Therefore,

$$\delta_c = \left(-\alpha_c x\right)^{1/2} = \sqrt{\alpha_c}\sqrt{-x} \tag{5.14}$$

Valid for conditions upstream of critical flow conditions, i.e. for $x<0$. By differentiation by x we obtain the free surface slope relative to the bottom

$$-\frac{d\delta_c}{dx} = \frac{1}{2}\sqrt{\alpha_c}\frac{1}{\sqrt{-x}} > 0 \tag{5.15}$$

Requiring this to be small, we obtain

$$\frac{d\delta_c}{dx} < S_0 \Rightarrow \frac{1}{2}\sqrt{\alpha_c / S_o} < \sqrt{-x} \tag{5.16}$$

or:

$$-x > \frac{1}{4}\frac{\alpha_c}{S_o} = \frac{1}{6}\left[\left(\frac{h_n}{h_c}\right)^3 - 1\right]h_c = \frac{1}{6}\frac{1 - Fr_n^2}{Fr_n}h_c \tag{5.17}$$

Hence, even for a very mild slope, corresponding to, say, $Fr_n = 0.01$, we only have to go a distance of $\approx 16h_c$ upstream of the location of critical flow to render our assumption of nearly well behaved flow valid. In the context of gradually varied flow profiles this distance of $\approx 16h_c$ is "nothing" and we need not be concerned by our violation of our assumptions. However, we should never forget that our solution is more conceptual than real as we get close to h_c.

(3) M3-Profile: $h_n > h_c > h$

$$\frac{dh}{dx} = \frac{S_o - S_f}{1 - Fr^2} = \frac{-}{-} > 0$$

For an imposed depth $h < h_c < h_n$ the depth must increase in the direction of flow. As the depth approaches the critical depth h_c, we have dh/dx approaches infinite, since $1 - Fr^2 \to 0$ and the behavior is analogous to the one we analyzed as $h \to h_c$ for M2-profiles.

When h approaches zero:

$$h \to 0 \Rightarrow S_f \to \infty \;\&\; Fr^2 \to \infty \Rightarrow \frac{dh}{dx} \to \frac{S_f}{Fr^2} \tag{18}$$

Again, a wide rectangular channel and Chezy's resistance gives

$$\frac{dh}{dx} \to \frac{(Q/b)^2 h^{-3} C^{-2}}{(Q/b)^2 h^{-3} g^{-1}} = \frac{g}{C^2} = \frac{f}{8} \tag{19}$$

i.e. for very small values of the imposed depth [a depth of zero is obviously non-sense, since finite Q would give infinite velocity!] the depth increases nearly linearly in the downstream direction at a rate governed by the channel's frictional characteristics (e.g. the bottom roughness).

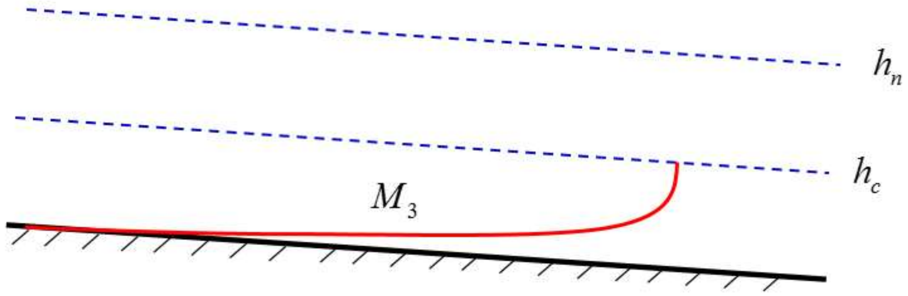


Figure 30 Limiting behavior of M3 profile

The M3-profile does not to my knowledge, have a more descriptive "name" like M1=backwater, M2=draw-down. It is generally encountered when a flow enters a channel often passing under an underflow gate.

5.2.2. Steep slope:

For steep slope, we always have $h_n<h_c$. Thus, depending on the water depth, there can be three scenarios as shown in Figure 31:

- S1 profile: $h>h_c>h_n$
- S2 profile: $h_c>h>h_n$
- S3 profile: $h_c>h_n>h$

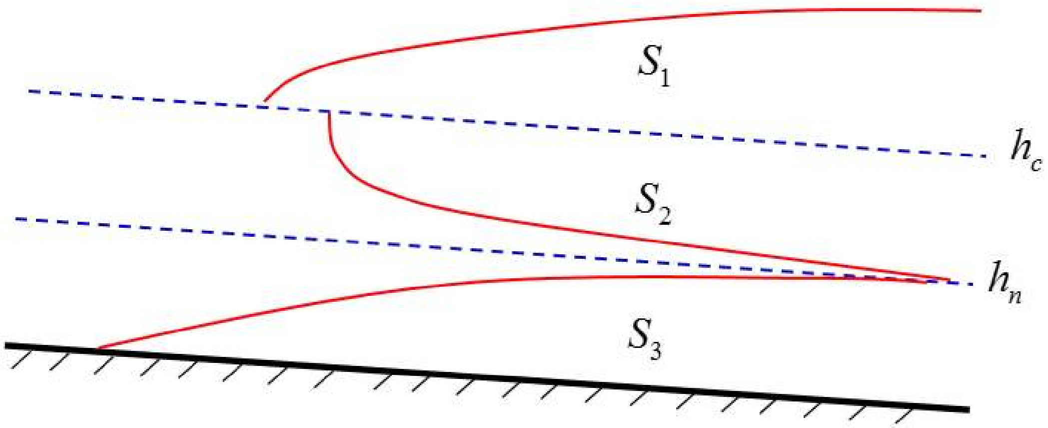


Figure 31 Classification of surface profile for steep-slope gradually varied flow

(1) S1-profile $h>h_c>h_n$

$$\frac{dh}{dx}=\frac{S_o-S_f}{1-Fr^2}=\frac{+}{+}>0$$

Depth increases in downstream (decreases in upstream) direction. Encountered when a flow obstruction (dam or underflow gate) back up the water into a steep channel. Notice that even though channel is steep, the flow is everywhere subcritical along the S1 profile. That's why an S1-profile has a downstream control. The limiting behavior is similar to mild slope profiles, i.e. $h \to \infty \Rightarrow dh/dx \to S_o$ free surface approaches horizontal; $h \to h_c \Rightarrow dh/dx \to +\infty$ i.e. unreliable detail as $h \to h_c$.

(2) S2 profile: $h_c>h>h_n$

$$\frac{dh}{dx}=\frac{S_o-S_f}{1-Fr^2}=\frac{+}{-}<0$$

The S2-profile has a depth that decreases in the direction of flow. It starts out, for $h \to h_c$, with $dh/dx \to +\infty$ and ends for $h \to h_n$, by approaching normal depth asymptotically. Notice that since $h<h_c$, the flow is supercritical and the depth is decreasing not so much because it is "drawn down" but more because it obtained out too high and wants to "get down" to its normal flow value. This situation may occur when a flow exits from under an underflow gate and enters a very steep channel.

(3) S3 profile: $h_c>h_n>h$

$$\frac{dh}{dx}=\frac{S_o-S_f}{1-Fr^2}=\frac{-}{-}>0$$

Depth increases in the direction of flow. Behavior as $h \to 0$ same as M3-profile, and h_n is approached asymptotically in the downstream direction. Again, the flow under a gate may result is a S3-profile downstream of the outflow.

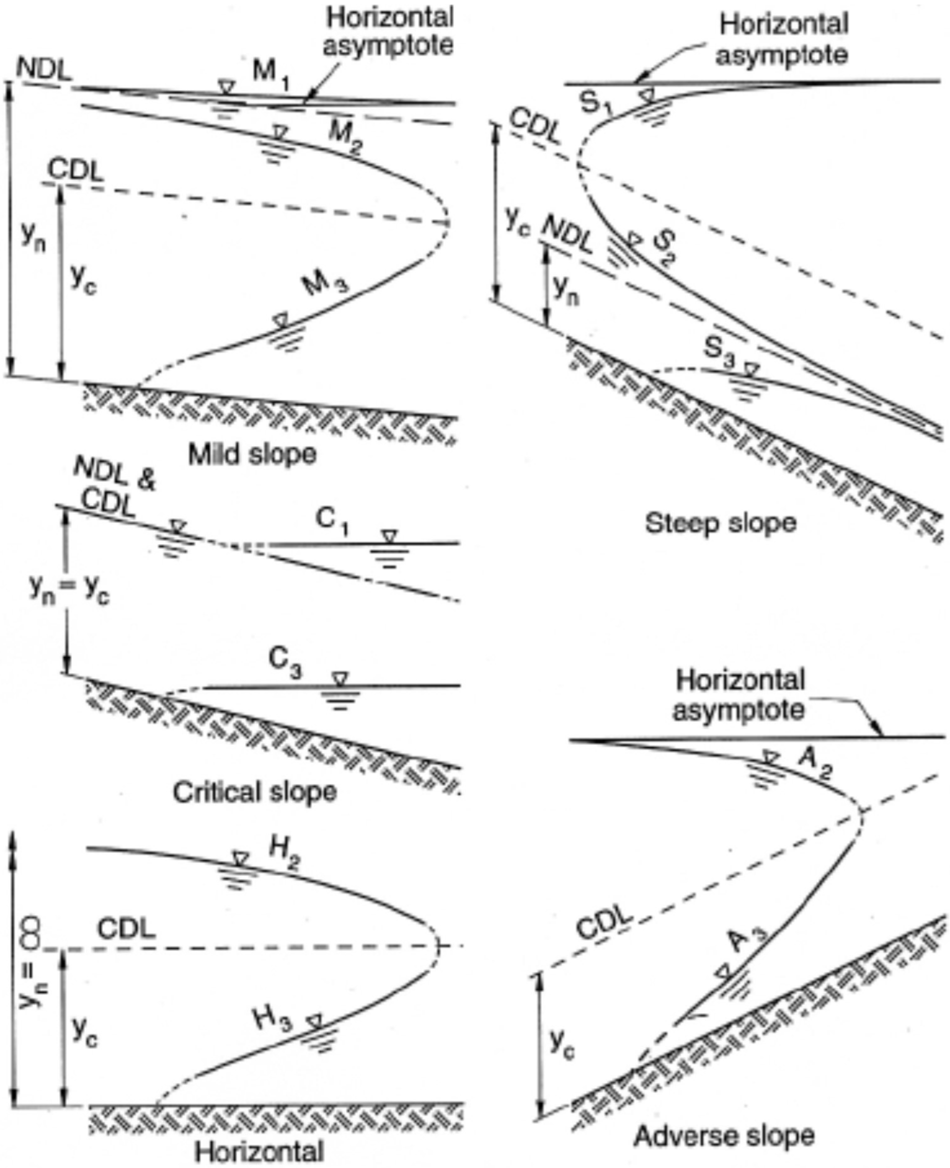


Figure 32 summary of surface profile for gradually varied flow in a prismatic channel (NDL=normal depth line (y_n=h_n); CDL=critical depth line (y_c=h_c))

5.3. Estimate of Channel length affected by gradually varied flow

Making certain simplifying assumption regarding channel cross-section variables solutions for gradually varied flow profiles in prismatic channels may be obtained from tabulated values of the gradually varied flow function. Alternatively, the governing differentiated equation may be solved numerically (without the restriction of the channel being prismatic).

Here we just present a very approximate procedure that can be used to obtain a rough estimate of the distance required for a gradually varied flow to achieve a given change of depth.

First, of course, we have known:

- Q= discharge,
- S_o=channel slope,
- $n = 0.038\varepsilon^{1/6}$ -Manning's n
- $A(h)$ = channel cross-section

With this information available, we can compute

- h_n=normal depth, from $S_o = S_{fn} = \dfrac{n^2 Q^2}{A_n^{10/3} / P_n^{4/3}}$
- h_c=critical depth, from $Fr_c^2 = \dfrac{Q^2 b_{sc}}{A_c^3} = 1$

This will tell us if channel has a mild or a sleep slope and give us an idea about the type of gradually varied flow profile to expect (M-types or S-types). Now, we assume that two depths are prescribed, say:

$$h = h_1 \quad @x = x_1;$$
$$h = h_2 \quad @x = x_2$$

where x_2 is unknown, i.e. we seek the distance from x_1, where the depth reaches h_2.

We have

$$\frac{h_2 - h_1}{x_2 - x_1} \approx \left(\overline{\frac{dh}{dx}}\right)_{1-2} \text{=average surface slope} \tag{5.20}$$

Now, we may obtain an estimate of the surface slope between h_1 and h_2 from the gradually varied flow equation

$$\left(\overline{\frac{dh}{dx}}\right)_{1-2} \approx \frac{S_o - \overline{S_f}}{1 - \overline{Fr^2}} \tag{5.21}$$

where

$$\overline{S_f} = \left(S_{f1} + S_{f2}\right)/2 = n^2Q^2\left(\frac{1}{A_1^{10/3}/P_1^{4/3}} + \frac{1}{A_2^{10/3}/P_2^{4/3}}\right)/2 \qquad (5.22)$$

and

$$\overline{Fr^2} = \frac{Q^2}{g}\left(\frac{b_{s1}}{A_1^3} + \frac{b_{s2}}{A_2^3}\right)/2 = \left(Fr_1^2 + Fr_2^2\right)/2 \qquad (5.23)$$

An alternative would be to take the average depth $\bar{h} = \left(h_1 + h_2\right)/2$ and compute from this the corresponding $S_f = \bar{S}_f$ and $Fr^2 = \overline{Fr^2}$.

Thus, we have a rough estimate for:

$$x_2 = x_1 + \frac{h_2 - h_1}{\left(\overline{dh/dx}\right)_{1-2}} = x_1 + \frac{\left(1 - \overline{Fr^2}\right)}{\left(S_o - \bar{S}_f\right)}\left(h_2 - h_1\right) \qquad (5.24)$$

If $h_2 - h_1$ is very large, and therefore the above approximation of a single step is very rough, one may subdivide $h_2 - h_1$ into several steps and sum the individual step length to get the desired result [this would amount to a numerical solution procedure]. Although very rough, the estimate of $x_2 - x_1$ from a single step at least will tell you if the distance required to reach h_2 is measure in 10's or 100's of meters or in km.

Before performing the actual calculations check the type of profile expected to connect h_1 and h_2 in particular, and lookout for hydraulic jumps along the way.

5.4. Determination of surface profile

In the previous section we introduced surface profiles for gradually varied flows, which are essentially qualitative solutions of a differential equation. The final solution, of course, requires boundary conditions, which can be called control points, i.e. locations with known water depths.

5.4.1. Entrance condition for mild slopes

In the absence of a downstream control a subcritical flow entering a mildly sloping channel must do so at normal depth, as shown in Figure 33. We call this Normal Depth Control.

Proof: if the flow entered the channel at a depth $h > h_n$ it would have to continue along an M1-profile-this would be impossible unless some downstream control were present. If the flow entered the channel at a depth $h<h_n$ it would have to follow a M2-Profile, which would bring the flow to critical depth and this flow could not continue down the channel since only normal flow can exist in a uniformly sloping channel for an "infinite distance".

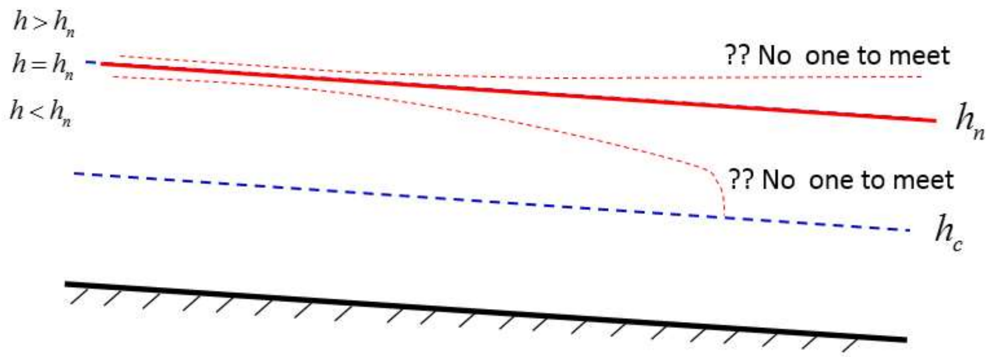


Figure 33_Enhance condition for mild slopes

5.4.2. Occurrence of critical flow

If the slope of a channel changes from mild to steep, the flow must pass through critical flow at the location of the change in slope, as shown in Figure 34. We call this Critical Depth Control.

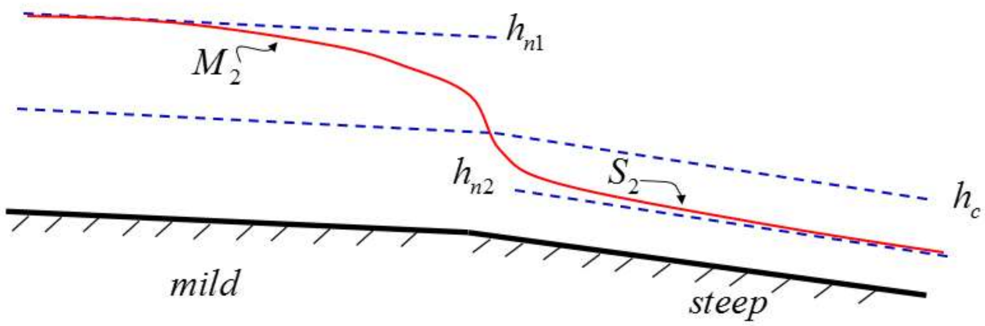


Figure 34 Occurrence of critical flow

Flow in mildly sloping section will be drawn down to h_c at the change in slope. Upon entering the steep channel it starts at h_c and proceeds toward h_n through a S2-Profile. If the M2-Profile did not hit h_c at entrance to the steep channel, the flow would follow an S1-Profile which would be impossible in the absence at a downstream control.

Computation would start at the change in slope where $h=h_c$; proceed upstream in the subcritical flow ($h>h_c$) and downstream in the supercritical ($h<h_c$) channels!

5.4.3. Entrance from a lake

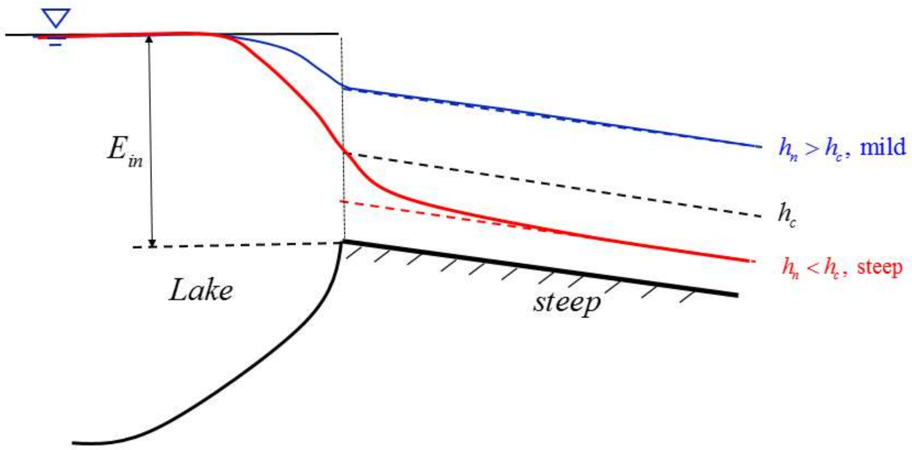


Figure 35 Discharge determination from a lake

If channel slope is steep, this is a special case of transition from sub to supercritical flow (Example 2). So flow must pass through critical at entrance and follow an S2-profile until reaching normal depth, i.e. the critical depth control. The discharge from the lake to the channel is obtained from solving the following two equations:

$$E_{in} = h_c + \frac{Q^2}{2gA_c^2} \quad \& \quad {Fr_{in}}^2 = \frac{Q^2 b_{sc}}{gA_c^3} = 1$$

With this Q it remains to be shown that normal flow in the channel is supercritical, i.e. $h_n<h_c$, since this was assumed to be the case initially. If $h_n<h_c$ then channel is indeed "steep" for the given Q and this is the discharge from the lake to the river.

If channel slope is mild, suggesting by a low value $S_o<\sim10^{-3}$, or obtained after first assuming it to be steep and finding this to be in error, then the outflow from lake to channel is a special case of mild-slope channel entrance, i.e. inflow from lake must be at normal depth right from the start. For this case we have the following two equations:

$$E_{in} = h_n + \frac{Q^2}{2gA_n^2} \quad \& \quad Q = \frac{1}{n}\frac{A_n^{5/3}}{P_n^{2/3}}\sqrt{S_o}$$

which may be solved for h_n=normal depth and Q=discharge from lake to channel.

5.4.4. The two lake problem

For simplicity, here we limit our channel to be wide and rectangular, so formulae become simple. The nice thing about wide and rectangular channel is that the hydraulic radius is simply the depth:

$$R_h = \frac{A}{P} = \frac{bh}{b+2h} = \frac{h}{1+2h/b} \simeq h$$

. The following are some results, which we have obtained before.

A. Normal depth for a wide rectangular channel:

$$q = \frac{Q}{b} = \frac{1}{n} h_n^{5/3} \sqrt{S_o} \Rightarrow h_n = \left(\frac{nq}{\sqrt{S_o}} \right)^{3/5}$$

B. Critical depth for a wide rectangular channel:

$$Fr = \frac{V}{\sqrt{gh}} = \frac{q}{\sqrt{g} h^{3/2}} = 1 \quad \Rightarrow \quad h_c = \left(\frac{q^2}{g} \right)^{1/3}$$

C. Specific energy for critical flow for a wide rectangular channel:

$$E_c = h_c + \frac{V_c^2}{gh_c} \frac{h_c}{2} = h_c + 1 \cdot \frac{h_c}{2} = \frac{3}{2} h_c$$

D. Hydraulic jump condition for a wide rectangular channel (upstream flow must be supercritical):

$$h_{conj} = \frac{h}{2}\left(-1 + \sqrt{1+8Fr^2} \right)$$

The two-lake problem is shown in Figure 36. We assume that the upper lake level, H_u, is fixed, but the lower lake level, H_L, is variable. Channel is wide rectangular, prismatic, of constant slope S_o, and known Manning's coefficient 'n'. Channel is assumed long, so that normal depth can be reached.

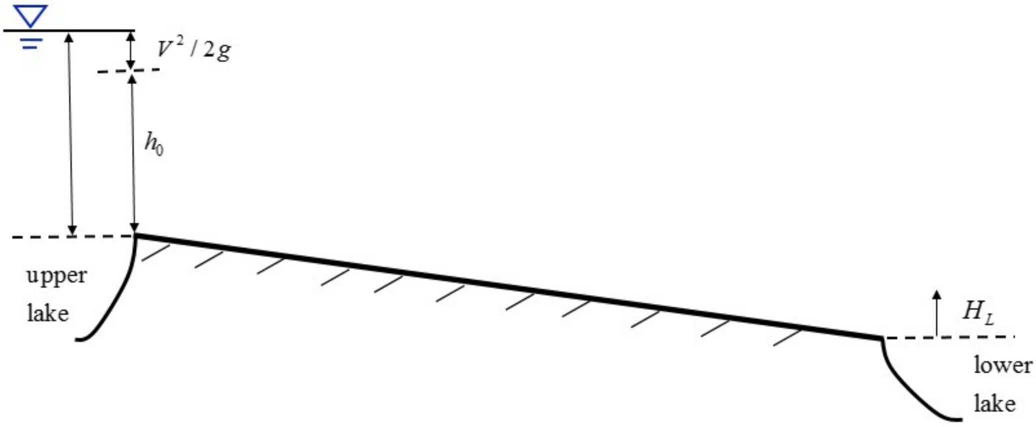


Figure 36 two-lake problem

(1) Discharge from upper lake into channel

Flow from lake to channel is a short transition of a converging flow without energy loss:

$$H_U = h_0 + \frac{V^2}{2g} \tag{5.25}$$

where h_0 and V are depth and velocity at the entrance, respectively. If slope is "steep" flow goes from subcritical (in lake) to supercritical (in "steep" long channel). Flow must pass critical at transition, i.e.

$$h_0 = h_c = \frac{2}{3}H_U \tag{5.26}$$

or:

$$q = q_c = h_c V_c = \frac{2}{3}H_U\sqrt{g\frac{2}{3}H_U} = \left(\frac{2}{3}\right)^{3/2}\sqrt{g}H_U^{3/2} \tag{5.27}$$

But is the slope really steep? We don't know until we have found the normal depth. However, with q=q_c (assuming "steep" is correct) we have:

$$h_n = \left(\frac{nq_c}{\sqrt{S_o}}\right)^{3/5} \begin{cases} < h_c & \text{slope is steep} \\ > h_c & \text{slope is mild} \end{cases} \tag{5.28}$$

If slope is steep, $h_n < h_c$, $Fr_n > 1$, then flow passes critical depth at entrance to channel and proceeds along an S2-cruve until h_n is reached (channel assumed long enough for this to happen) with $q = q_c$.

If slope is not steep, it is mild, $h_n > h_c$, $Fr_n < 1$, and does not pass through critical depth at entrance to channel. In the absence (assumed from the moment) of any downstream control, flow must hit normal flow at the entrance, i.e. $h_o = h_n$, and we have

$$H_U = h_n + \frac{V_n^2}{2g} = h_n + \frac{q_n^2}{2gh_n^2} \tag{5.29}$$

but since flow is normal we also have:

$$q_n = \frac{1}{n} h_n^{5/3} \sqrt{S_o} \Rightarrow \frac{q_n^2}{2gh_n^2} = \frac{S_o}{n^2} \frac{h_n^{10/3}}{2gh_n^2} = \frac{S_o}{2gn^2} h_n^{4/3} \tag{5.30}$$

Substitution now gives:

$$H_U = h_n + \frac{S_o}{2gn^2} h_n^{4/3} = h_n \left(1 + \frac{S_o}{2gn^2} h_n^{1/3}\right) \tag{5.31}$$

or:

$$h_n = H_U / \left(1 + \frac{S_o}{2gn^2} h_n^{1/3}\right) \tag{5.32}$$

from which h_n is readily obtained by iteration. With h_n know we have:

$$q = q_n = V_n h_n = \frac{1}{n} h_n^{5/3} \sqrt{S_o} \tag{5.33}$$

Just to make sure, you may want to check if slope now is mild by evaluating and showing:

$$Fr_n = \frac{q_n^2}{gh_n^3} < 1$$

But this is really not necessary since we know that q is maximum for a given specific energy, H_u, when flow is critical, $q = q_c$. For any other condition than $q = q_c$ we therefore know that $q = q_n < q_c$, but $h_n > h_c$ since inflow is subcritical.

Finally, we have for $q = q_n$ the corresponding critical depth for the mild slope case:

$$h_{c,mild} = \left(\frac{q_n^2}{g} \right)^{1/3} \quad (5.34)$$

(2) Downstream to the lower lake

Steep slope profiles

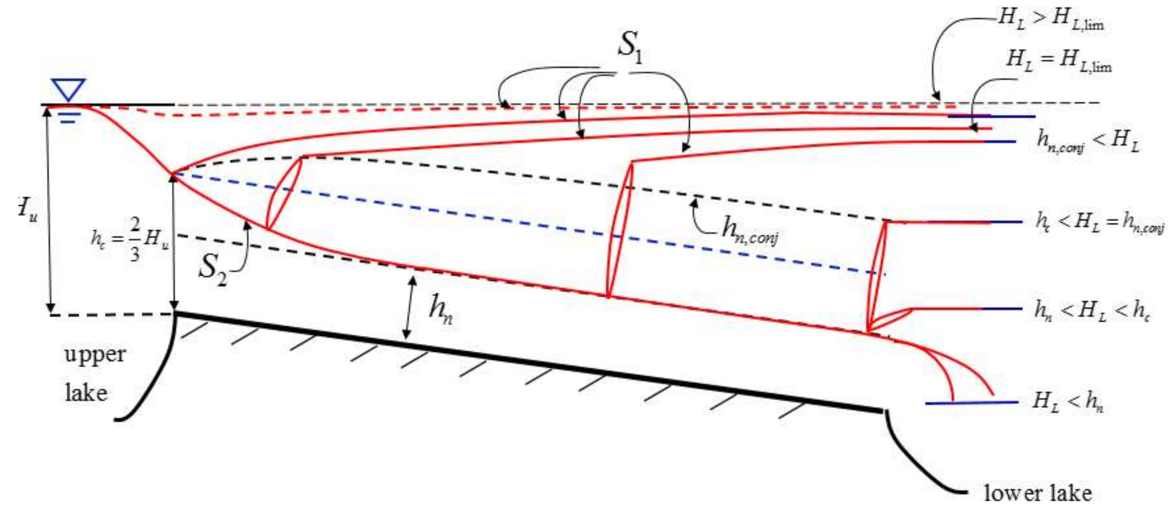


Figure 37 flow enters the lower lake; steep slope

If H_L=lower lake level is less than h_n, the flow is supercritical all the way. It reaches h_n through an S2-profile and since $h_n<h_c$ or $Fr_n>1$ the flow cannot respond to anything happening downstream, i.e. it remains at h_n until it discharges in a jet-like fashion into the lower lake.

The only way the flow in the channel is affected by the downstream (lower) lake level is if the channel flow somehow becomes subcritical (downstream control). Since normal flow is reached and is supercritical, the only way to get ta a subcritical flow in the channel is through a hydraulic jump. To have a hydraulic jump the depth downstream of the jump must be conjugate to the upstream depth. The conjugate depth to the depth along the S2-profile ($h>h_n$ but $h<h_c$) is shown in Figure 37. At any point along the channel the flow can jump from its upstream supercritical value to its conjugate (subcritical) depth. The first time this becomes a possibility is when the lower lake level, H_L, is equal to $h_{n,conj}$. Thus:

$H_L=h_{n,conj}$ Flow jumps from h_n to $h_{n,conj}$ just before it exists channel into lower lake

For $h_n<H_L<h_{n,conj}$, a partial jump starts in the channel and extends into the lower lake. Now, for $H_L>h_{n,conj}$ the lower lake level is so high that the flow is backed up into the steep channel, i.e. profile starts at $H_L=h_L$ at the outflow and follows a S1-profile up into the channel. Since the depth is $h>h_{n,conj}>h_c$ the flow is subcritical (and that is why it can feel what is going on downstream!). When the S1-profiles depth has decreased until it reaches a value of $h_{n,conj}$ a jump lakes place. As H_L increases this jump moves further upstream in the channel, until the S1-profile just manages to reach h_c at the entrance from the upper lake. This value of $H_L=H_{L,lim}$ signals a change in outflow conditions from the upper lake and therefore q will no longer be q_c since the flow no longer can get down to $2H_U/3$ at the channel entrance.

For $H_{n,conj}<H_L<H_{L,lim}$, flow enters channel passing through critical depth, follows an S2-profile until it makes a jump to its conjugate depth and proceeds along an S1-profile until reaching the lower lake. As H_L increases, the jump forms closer to the upper lake and from $H_L=H_{L,lim}$ there is no longer a hydraulic jump in the channel since the S1-profile starts at $h=h_c$ at the entrance.

Note, when $H_L=H_{L,lim}$ the flow is subcritical along the entire length of the channel, except right at the entrance where it is right at critical. Thus, any further increase in H_L will make the flow subcritical everywhere including right at the entrance to the channel.

For $H_L>H_{L,lim}$ $q \neq q_c$, so $q < q_c$, but it is also not normal flow as in the case of a mild sloping channel. What do we do? We notice that

$$H_U = h_0 + \frac{q^2}{2gh_0^2} \tag{5.35}$$

still holds with h_o=depth at start of channel and we know that $h_0>2H_U/3$. Thus we can pick a value of h_0 such that

$$\frac{2}{3}H_U < h_0 < H_U \tag{36}$$

and for this h_0 the corresponding value of q is obtained. With this value of q and starting h from h_0 the surface profile in the channel (S1-profile) can be calculated and the value of h obtained at the outflow of the channel into the lower lake must be the level in the lake, H_L. Obviously, if $h_0=H_u$ there will be no flow into the channel, i.e. q=0, and the lower lake will be at a level equal to that of the upper lake.

Mild slope profiles

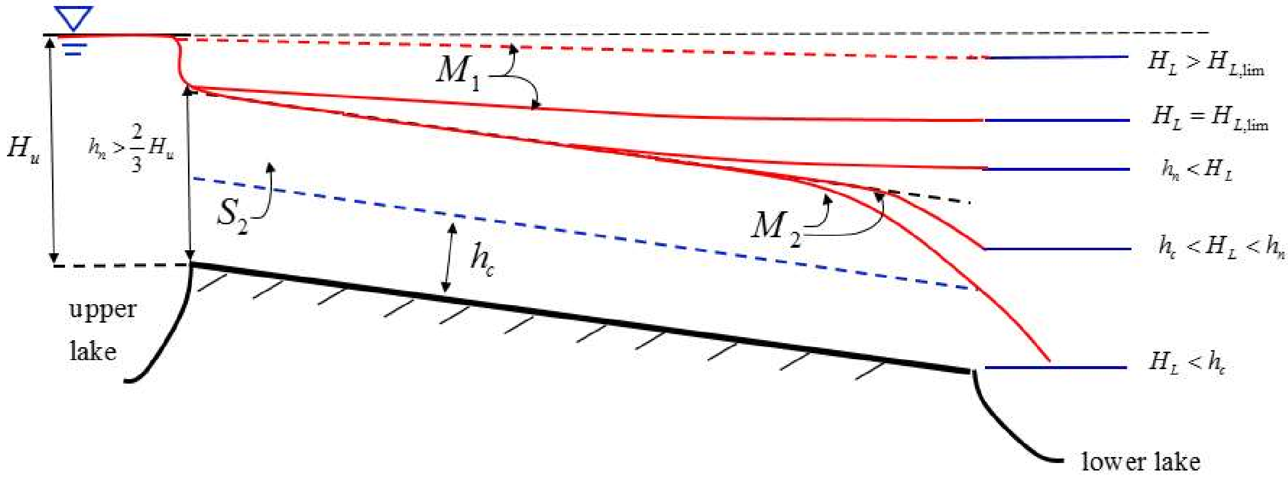


Figure 38 flow into the lower lake: mild slope

Discharge and normal depth are given by Eq. (5.33), and h_n is hit immediately at start of channel and is maintained until influenced by conditions imposed by the lower lake (flow is subcritical $h_n > h_c$ so there is downstream control!).

- $H_L < h_c$: flow is drawn down in channel from h_n to critical depth at the outflow from the channel to the lower lake (free outflow over a brink). Follows an M2-profile since channel is assumed long enough to reach h_n.
- $h_c < H_L < h_n$: flow is drawn down through an M2-profile to meet level in lower lake.
- $h_n < H_L < H_{L,lim}$: flow is backed into the channel. Meets H_L at lower lake and h_n upstream. But when $H_L = H_{L,lim}$ the lower lake level is so high that h_n is not reached until right at the entrance from the upper lake. $q = q_n$ still holds.
- $H_L > H_{L,lim}$: h_0=depth at entrance>h_n. Discharge changes! Solution is as for steep channel when $H_L > H_{L,lim}$, i.e.

$$H_U = h_o + \frac{q^2}{2gh_o^2} \quad (h_n < h_o < H_U) \tag{5.37}$$

Pick h_o, get q. With this q and starting from h_o surface profile is computed and depth at outflow to lower lake meets (and defines) $H_L > H_{L,lim}$. When $h_o = H_U$, $q=0$ and there is no flow between the lakes-free surface is horizontal everywhere.

5.4.5. General Comments for determining surface profile:

Rule 1: subcritical flow, $Fr<1$ or $h>h_c$, is always controlled from downstream location

Rule 2: supercritical flow, $Fr>1$ or $h < h_c$, is always having upstream control

Rule 3: in the absence of any control the only possible flow is normal, $h=h_n$.

Rule 4: gradually varied flow must follow surface profiles given by M1-3 or S1-3

Rule 5: transition from a supercritical to a subcritical flow possible only through a hydraulic jump.

6. Artificial channel control

6.1. Sluice gate

In this example, we consider a free outflow through a sluice gate. As shown in Figure 39, the upstream flow goes through the gate with depth h_g, and keeps converging until a vena contracta with water depth:

$$h_v = C_v h_g \tag{6.1}$$

is reached. The flow at the vena contracta can be considered well-behaved, as streamlines are straight and parallel there.

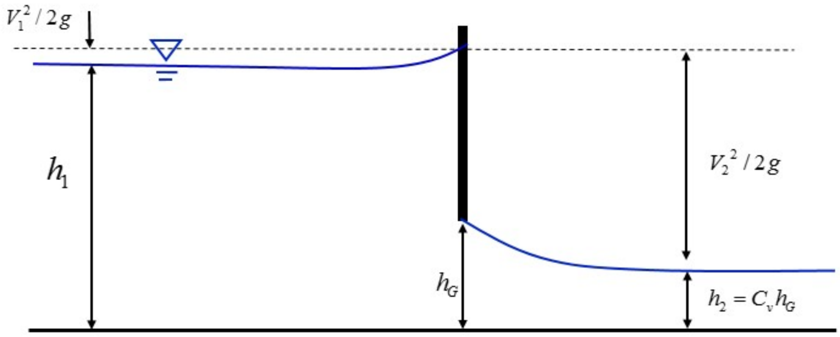


Figure 39 Discharge and flow characteristics of underflow gates

Since the flow is converging, there is no head loss, so the upstream water depth, h_1, and the downstream water depth at the vena contracta, h_2, should be alternate depths:

$$E_1 = E_2 \ , \ V_1 h_1 = V_2 h_2 = Q$$

$$h_1 + \frac{(Q/b)^2}{2gh_1^2} = h_2 + \frac{(Q/b)^2}{2gh_2^2} \tag{6.2}$$

As we know for alternate depths, the larger one corresponds to subcritical flow condition, while the smaller one corresponds to supercritical flow condition, so in this case:

- Upstream of gate $h = h_1$ is $> h_c$: subcritical flow
- Downstream of gate $h = h_2$ is $< h_c$: supercritical flow

Since h_2 is given by (6.1) (under certain condition which we will discuss later), Eq. (6.2) actually can be used to obtain h_1 with a given Q or obtain Q with a given h_1. Thus, we pretty much knows all we need for the short transition, but what happens to upstream and downstream water profile?

(1) Upstream flow characteristics

Far upstream from the gate, the flow will asymptotically approach the normal flow depth. If upstream channel's slope is mild, then h_1 upstream of gate is reached through an M1-profile (backwater curve). If upstream channel's slope is steep, normal flow is supercritical but the flow near the gate is subcritical. Thus, a hydraulic jump is needed to make the transition from supercritical to subcritical. The uniform supercritical (normal) flow proceeds until a location where the normal depth, $h_n<h_c$, is conjugate to the backwater curve-the S1-profile followed by

the flow upstream of the gate starting from h_1 and decreasing in the upstream direction. The jump condition is given by:

$$\left(\rho\frac{Q^2}{A_n^2}+P_{CG,n}\right)A_n=\left(\rho\frac{Q^2}{A_{\mathrm{conj}}^2}+P_{CG,\mathrm{conj}}\right)A_{\mathrm{conj}} \qquad (6.3)$$

where the left-hand side is based on the normal flow and the right-hand side is the conjugate we are looking for. An S1-profile goes upstream from $h=h_1$ at the gate to meet $h=h_{n,conj}$ at the downstream end of the jump. Figure 40 shows the sketch of the surface profile.

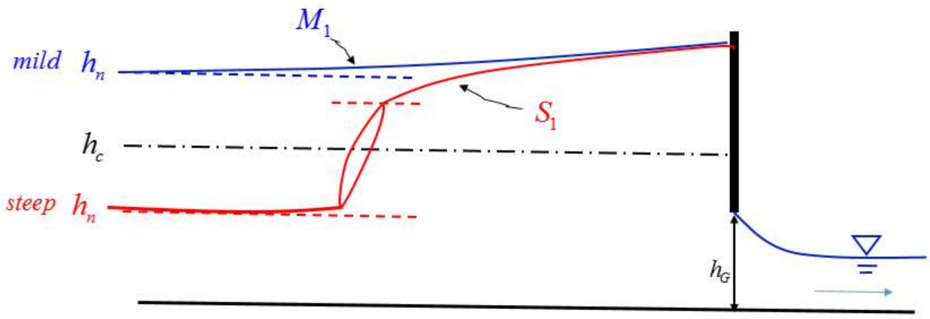


Figure 40 Upstream flow characteristics

a) Downstream flow characteristics

If the channel slope downstream of the gate is steep then the flow will proceed from $h=h_2$ at the exit from under the gate (it is guaranteed to be supercritical!) following on S2 or S3-profile until h_n is reached.

If the channel slope downstream of the gate is mild, then a jump forms bringing the flow from supercritical to subcritical, where the subcritical depth is $h_n>h_c$ normal depth in the channel [must be so since there is no downstream control]. The jump condition is again:

$$\left(\rho\frac{Q^2}{A_{\mathrm{conj}}^2}+P_{CG,\mathrm{conj}}\right)A_{\mathrm{conj}}=\left(\rho\frac{Q^2}{A_n^2}+P_{CG,n}\right)A_n$$

The flow from vena contract after the gate, $h=h_2$, follows an M3-profile until it reaches $h_{n,conj}$ where a hydraulic jump to h_n takes place. The smaller the downstream slope, the large the normal depth for the given Q (corresponding to a free outflow under the gate). But the larger h_n is the smaller $h_{n,conj}$, which is the depth the M3-curve form vena contract must reach in order for the jump to form. Thus, as h_n increases and $h_{n,conj}$ decreases (it will still be located on the same M3-curse since Q is the same) the location of the downstream jump moves towards the gate. The limiting case is reached when $h_{n,conj} = h_{vc}$= depth of vena contract. Any further increase in h_n would make $h_{n,conj} < h_{vc}$ and there is no way for the flow exiting from under the gate to get down to this depth. Result is that the outflow form under the gate no longer is free, i.e. into the atmosphere. The outflow becomes "drowned" and the discharge under the gate changes, or if the discharge is to remain constant, the depth upstream of the gate, h_1, must increase.

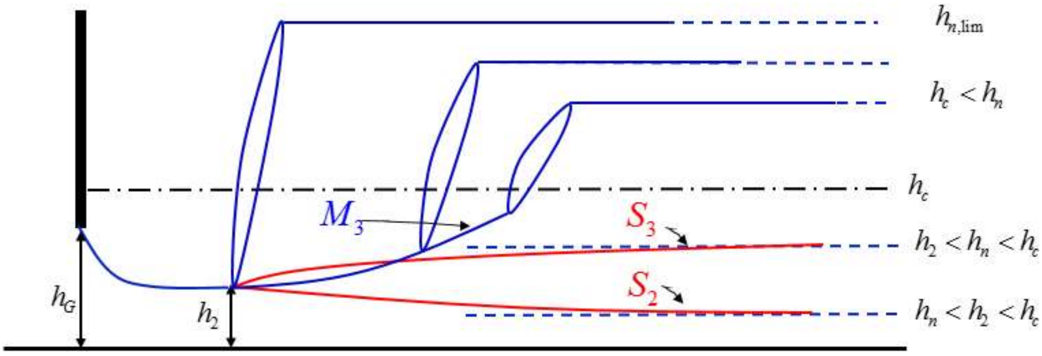


Figure 41 downstream flow characteristics

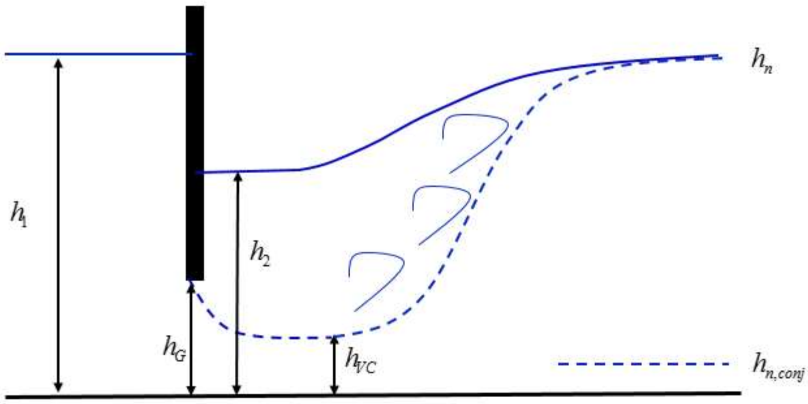


Figure 42 Drowned outflow

For drowned outflow shown in Figure 42, we simply assume that the outflow with still goes through a conceptual vena contracta, but above the vena contracta there is stagnate water, so the water depth at the conceptual vena contracta is h_2. From continuity:

$$V_{vc}h_{vc}=\frac{Q}{b}\Rightarrow V_{vc}=\frac{Q}{b}/(C_v h_g) \tag{6.4}$$

Apply momentum equation from the conceptual vena contracta to the downstream normal depth:

$$\rho V_{vc}^2 h_{vc}+\frac{1}{2}\rho g h_2^2=\rho V_n^2 h_n+\frac{1}{2}\rho g h_n^2 \;\Rightarrow\; h_2=\sqrt{\frac{2V_n^2 h_n+g h_n^2-2V_{vc}^2 h_{vc}}{g}} \tag{6.5}$$

From h_1 to the conceptual vena contracta there is no head loss, so:

$$h_1+\frac{V_1^2}{2g}=h_2+\frac{V_{vc}^2}{2g} \;\Rightarrow\; h_1=h_2+\frac{V_{vc}^2}{2g}-\frac{V_1^2}{2g} \tag{6.6}$$

6.2. Weir

A weir may be defined as any regular obstructions built across a channel over which the flow takes place. There are many types of weirs for various engineering purposes, here we only consider two simple cases: sharp-crested weir and broad-crested weir. For simplicity, we assume the weirs are very wide, so the problem is 2-dimensional.

6.2.1. Sharp-crested weir

A sharp-crested weir consists of a vertical plate mounted perpendicularly to the flow direction, as shown in Figure 43. The approaching flow's free-surface level is with a distance H above the crest of the weir (denoted by C), and the approaching velocity is V_0. The depth of the weir is W. As the flow pass through the crest of the weir, the water surface curved rapidly, and a jet is formed. The total discharge per unit width, q, apparently is zero when the head over the weir, H, is zero, and increases with H. To obtain the q-H relationship, we make the following assumptions:

(a) The velocity distribution of the approaching flow is uniform, i.e. $V=V_0$,
(b) The streamlines are horizontal over the crest so that the pressure within the jet is zero through the depth of the jet, i.e. from the free surface (point A), to the crest (point C).
(c) The pressure under the jet is atmospheric.
(d) The effect of viscosity is negligible.

With assumptions (b), (c) and (d), the total water head along AC is constant, so for an arbitrarily chosen point B, we have:

$$H_B = \frac{V_B^2}{2g} + Z_B = \frac{V_0^2}{2g} + H \tag{6.7}$$

Here we put the z=0 datum at the crest C. Thus, the flow velocity at B is:

$$V_B = \sqrt{2g(\frac{V_0^2}{2g} + H - Z_B)} \tag{6.8}$$

Notice that the distance from the Energy grade line to point B is:

$$h_B = \frac{V_0^2}{2g} + H - Z_B$$

so we can write Eq. (6.8) as:

$$V_B = \sqrt{2gh_B}$$

Therefore, the velocity at any point between A and C is:

$$V = \sqrt{2gh} \tag{6.9}$$

where h is the distance to the Energy grade line. The total discharge per unit width can be obtained by integrating velocity over AC:

$$q = \int_{V_0^2/2g}^{V_0^2/2g+H} \sqrt{2gh}\, dh = \frac{2}{3}\sqrt{2g}\left[\left(H + \frac{V_0^2}{2g}\right)^{3/2} - \left(\frac{V_0^2}{2g}\right)^{3/2}\right] \tag{6.10}$$

Usually the approaching flow has a low velocity, so:

$$\frac{V_0^2}{2g} << H$$

So we can neglect the $V_0^2/2g$ terms in Eq. (6.10):

$$q \approx \frac{2}{3}\sqrt{2g}H^{3/2}$$

We also notice that Eq. (6.10) carries some inaccuracy due to the simplifications. All these inaccuracy can be represented by a discharge coefficient C_d:

$$q \approx \frac{2}{3}C_d\sqrt{2g}H^{3/2} \tag{6.11}$$

C_d is primarily controlled by the ratio H/W, but also can be determined by other factors such as surface tension (when H is very small). It is usually note too far away from 1, and can be obtained through lab experiments.

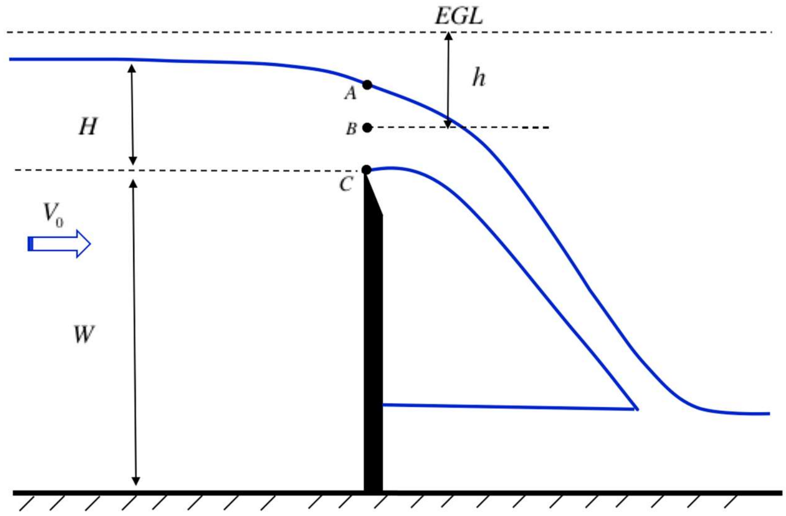


Figure 43 Flow over a sharp-crest weir

6.2.2. Broad-crested weir

It was shown before that if the height of a hump in a channel floor exceeds a critical value, and the hump is long enough for parallel flow to occur over it, the flow over the hump is critical. Such a "hump" is called a broad-crested weir. The channel can be considered rectangular, and the task here is to find the discharge per unit width, q, as a function of the water head over the crest, H.

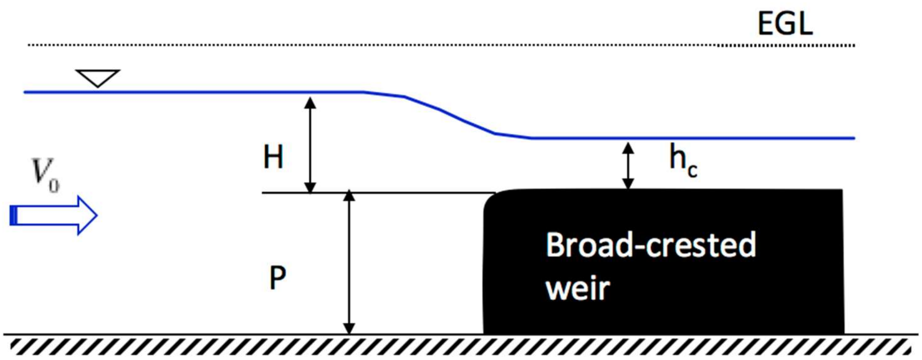


Figure 44 Broad-crested weir

Since the flow is converging from upstream to the point of critical flow, total energy is conserved:

$$H + \frac{V_0^2}{2g} = h_c + \frac{V_c^2}{2g} \tag{6.12}$$

For a rectangle channel the velocity water head is ½ of the critical water depth:

$$\frac{V_c^2}{2g} = \frac{1}{2} h_c \tag{6.13}$$

, so Eq. (6.12) can be further written as:

$$H + \frac{V_0^2}{2g} = \frac{3}{2} h_c$$

The velocity head of the approaching flow can usually be neglected, i.e.:

$$H >> \frac{V_0^2}{2g}$$

Thus, we finally have:

$$h_c = \frac{2}{3} H \tag{6.14}$$

Substitute Eq. (6.14) into Eq. (6.13), we obtain that:

$$V_c = \sqrt{\frac{2}{3} gH} \tag{6.15}$$

The discharge per unit width is:

$$q = V_c h_c = \sqrt{\frac{8}{27} g} \cdot H^{3/2} \tag{6.16}$$

Since Eq. (6.16) is valid only if the length of the weir, L, is not too long for neglecting friction, and also not too short for allowing parallel flow to be developed over the crest. Hence, we must have:

$$0.1 < \frac{H}{L} < 0.35 \tag{6.17}$$

This means that the length of the weir must be within 3-10 times the water head over the crest. For a longer weir, i.e. $H/L<0.1$, the frictional effect becomes significant, while for shorter weir, i.e. $H/L>0.35$, the discharge should asymptotically approach that of the sharp-crested weir.

The actual weir discharge is different from, Eq. (6.16), due to the simplifications, which can be accounted for by introducing a discharge coefficient.

$$q = C_d \sqrt{\frac{8}{27} g} \cdot H^{3/2} \tag{6.18}$$

The value of C_d primarily depends on the value of H/L and also H/W.

6.3. Channel with a constriction

Critical flow can be produced by a hump on the channel bottom, e.g. broad-crested weir. Another way to produce critical flow is producing a constriction on the channel. Figure 45(a) shows a top view of such a channel. The flow is from section (1) to section (3) through a constriction, which gives the narrowest channel width at section (2). The channel bottom remains horizontal. Since the channel width varies over the transition, the channel is not prismatic, so the depth-specific-energy diagram changes from section to section. The problem is to determine flow condition in section (2) for known flow condition in section 1.

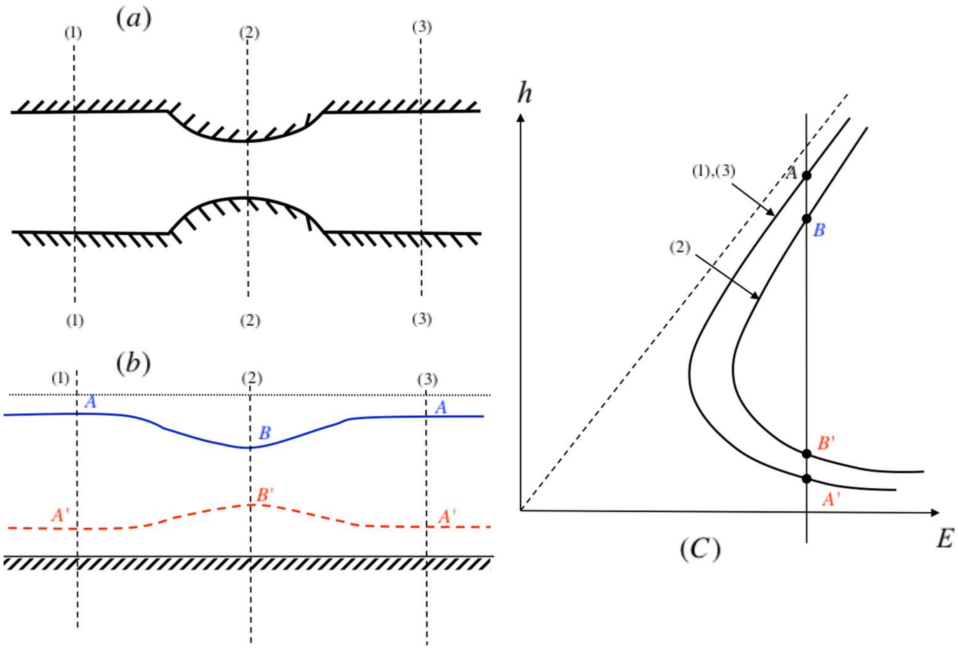


Figure 45 Channel with a constriction: flow energy is sufficient to pass through.

We first construct the h-E diagram for section (1) and (2). We should notice that the total discharge is the same for these two sections. Thus, for the same water depth, the flow velocity is higher at section (2) due to smaller cross-section area, and therefore the specific energy at section (2) is higher. This means the h-E curve for section (2) is to the right of that for section (3), as shown in Figure 45 (c). It is obvious that the specific energy for critical flow is higher for section (2). In another word, the specific energy for critical flow increases as the channel width decreases.

The basic equation to be solved is energy equation. Since flow is converging from section (1) to (2), no head loss is expected. Since the channel bottom is horizontal, specific energy is also conserved from (1) to (2):

$$E_1 = E_2 \tag{6.19}$$

With the known condition at section (1), which corresponds to point A (if subcritical flow) or A' (if supercritical flow), we produce a vertical line on Figure 45c which may or may not intersects with the h-E curve for section (2).

Case I: Two intersections

As shown in Figure 45 (c), the incoming flow has enough specific energy so that the vertical curve through A or A' gives two intersections B and B' with the h-E curve for section (2).

If the incoming flow is subcritical, i.e. point A, the flow at section (2) corresponds to point B, which is also subcritical. Since B is under A, the water depth decreases from A to B (blue solid line in Figure 45(b)). It is impossible for the flow to reach B' at section (2), because this requires the flow point to move along BB' (to conserve specific energy), where the channel width is less than b_2. However, b_2 is already the narrowest width.

If the incoming flow is supercritical, i.e. point A', the flow at section (2) corresponds to point B', which is also supercritical. Since B' is above A', the flow depth increases from A' to B' (red dashed line in Figure 45(b)) Following similar arguments for subcritical-flow scenario, it is impossible for the flow to become subcritical, i.e. reach B.

Downstream from section (2), the flow recovers to its upstream condition, i.e. flow point A for subcritical flow and flow point A' for supercritical flow, as shown in Figure 45(b).

Case II: one intersection

When the channel width at section (2) is sufficiently narrow, or the incoming flow energy is sufficiently low, there will be only one solution to Eq. (6.19). In the h-E diagram, the vertical line through flow point A or A' only has one intersection with the h-E curve for section (2), as shown in Figure 46(c). This intersection has the minimum specific energy, and hence corresponds to critical flow condition at section (2).

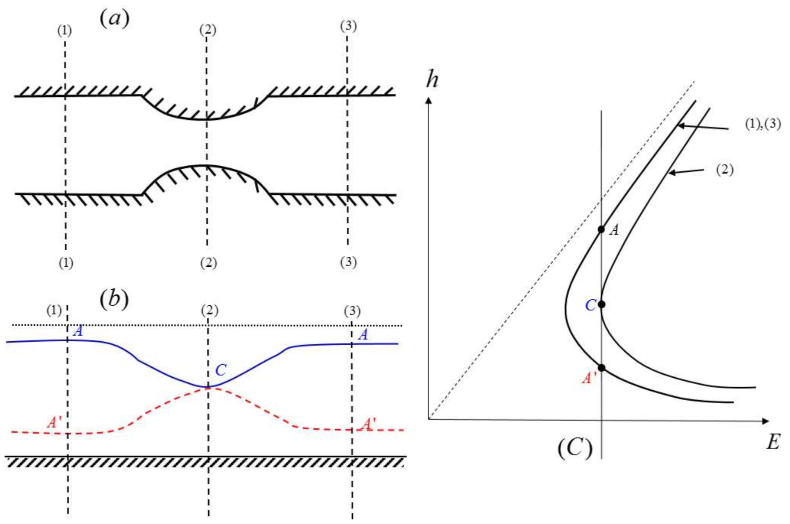


Figure 46 Flow through a channel constriction: flow energy is just enough for passing through

Assuming the channel is rectangular, the width that creates critical flow in section (2) can be determined as follows:

$$E_1 = E_c = \frac{3}{2}h_c = \frac{3}{2}\sqrt[3]{(Q/b_c)^2/g} \tag{6.20}$$

Thus, the critical channel width is:

$$b_c = \left(\frac{3}{2}\right)^{3/2}\frac{Q}{\sqrt{gE_1^3}} \tag{6.21}$$

The surface profile is shown in Figure 46(b). If the incoming flow at section (1) is subcritical, the water depth will drop until it reaches the critical flow depth at section (2), i.e. from A to C. If the incoming flow at section (1) is supercritical the water depth will increase to the critical flow depth at section (2), i.e. from A' to C. Downstream from section (2), theoretically the flow can arbitrarily reach A' or A, but the flow condition represented by A' is more stable, so the flow will be subcritical (A') at downstream of the transition, regardless of the incoming flow condition.

Case III: no intersection

If the channel is too narrow or the incoming flow energy is too low, it is possible that there is no solution to Eq. (6.19). In such case, we have to separately discuss subcritical and supercritical incoming flows.

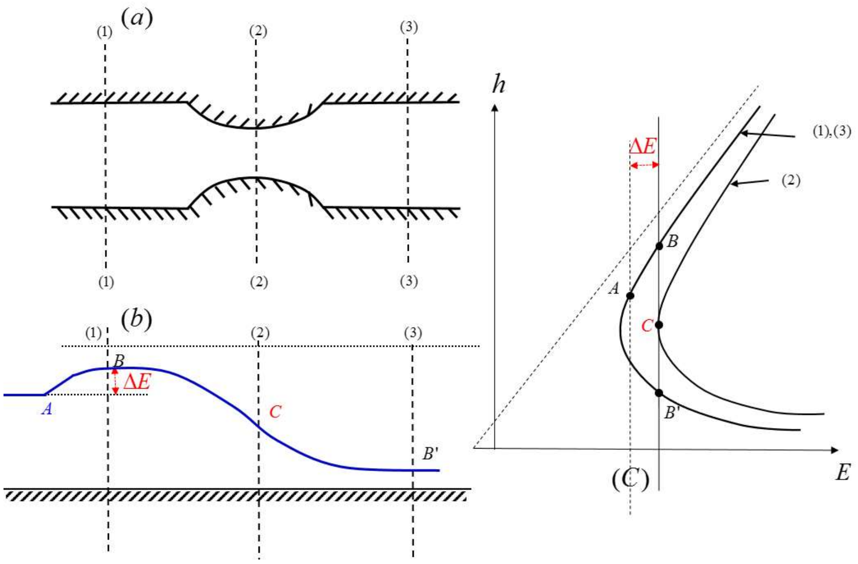


Figure 47 Flow through a channel constriction: flow is subcritical and its energy cannot let it pass directly

As shown in Figure 47(c), for a subcritical incoming flow with low energy (point A), the vertical line through point A on the h-E diagram represents solving Eq. (6.19). Apparently, this vertically gives no intersection with the h-E curve for section (2), and hence no solution is possible. This means that the flow cannot directly pass the constriction. In such a situation, the channel is "choked" to increase the upstream specific energy to a value, which just allows the critical flow to occur at section (2). In the h-E diagram, the upstream flow depth increases from point A to point B, which satisfices:

$$E_B = E_{C2} = \frac{3}{2}\sqrt[3]{(Q/b_2)^2/g}$$

The change in specific energy is:

$$\Delta E = E_B - E_1 = \frac{3}{2}\sqrt[3]{(Q/b_2)^2/g} - E_1$$

The A-B transition corresponds to a M1 profile in Figure 47(b). From point B downstream, the water depth decreases until reaching the critical water depth at section (2). Downstream of section (2), the flow becomes supercritical, i.e. reaching point B' in the h-E diagram, so the water depth decreases from section (2) to (3). Downstream of section (3), a hydraulic jump will occur to bring the flow back to subcritical or a given water level.

If the incoming flow is supercritical, as shown in Figure 48, the flow specific energy has to increases from that of point A to that of point B or B'. You may think the transition will be from A to B'. This "short-cut" is impossible, as the flow depth must decreases from A to B', and no such surface profile exists. Therefore, we must go through a long journey to reach point B, which corresponds to subcritical flow. The flow first goes through a hydraulic jump to its conjugate subcritical flow, which is denoted by point D in the h-E diagram. From point D downstream, it's the same situation as for subcritical flow.

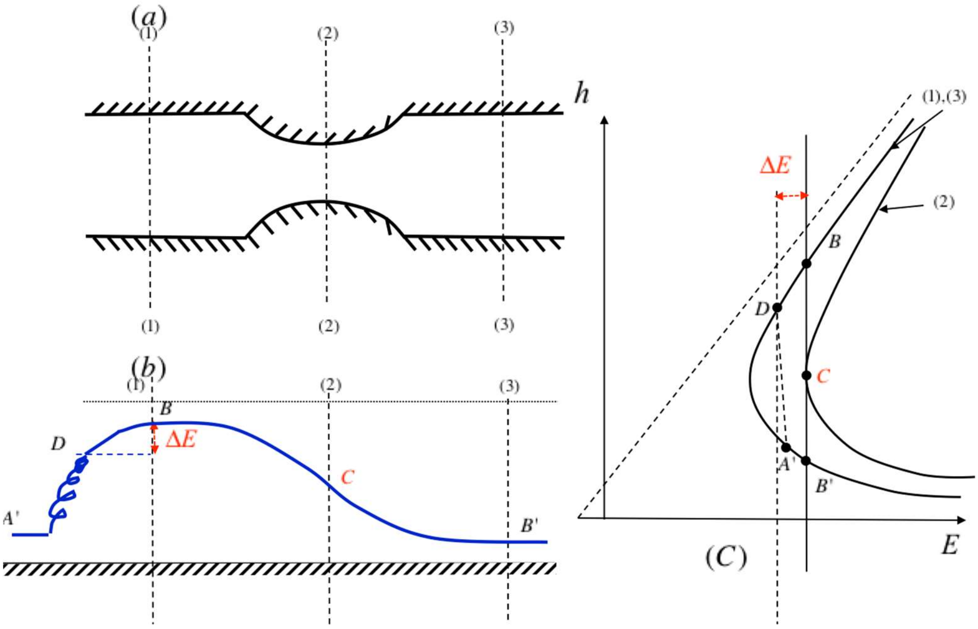


Figure 48 Flow through a channel constriction: flow is supercritical and its energy cannot let it pass directly

The most important implication is that one can create a critical-flow control by building a sufficiently narrow channel constrain. With the upstream specific energy and the width at the narrows point of the channel, the flow discharge can be directly obtained as:

$$Q = b_2\sqrt{g}\left(\frac{2}{3}E\right)^{3/2} \tag{6.22}$$

This is the operation principle for Venturi flume.