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Unsteady open-channel flow

1. Governing equation for one-dimensional open channel flow

For unsteady open-channel flow, we are interested in the variation of water depth, *h*, and cross-sectional velocity, *U*, as a function of streamwise coordinate, *x*, and time *t*. Thus, for the two unknowns, *h* and *U*, we need two governing equations, which can be obtained by considering the two fundamental principles in hydraulics: conservation of volume and momentum.

1.1. Continuity equation:

The governing equation based on the conservation of volume is usually referred to as the continuity equation. As shown in Figure 1, we consider a short reach of an open channel, Δx, as a control volume (CV). According to the conservation of volume, the increase/decrease of the volume of the CV must be equal to a net inflow or outflow. For this short reach, there is no flow across the bottom or the free surface, so the net inflow and outflow is the difference of the upstream and downstream discharge, Q_u and Q_d. Thus, the change of the volume of the CV over a very short period Δt is:

$$\Delta V_{CV} = (Q_u - Q_d)\Delta t \tag{1.1}$$

Since Δx is very short, the volume of the CV can be written as:

$$V_{CV} = A\Delta x$$

where *A* is the cross-section area. The change of V_{CV} over a short time interval Δt is then expressed as:

$$\Delta V_{CV} = \frac{\partial V_{CV}}{\partial t}\Delta t = \frac{\partial A}{\partial t}\Delta x \Delta t \tag{1.2}$$

The downstream discharge is related to the upstream discharge as:

$$Q_d = Q_u + \frac{\partial Q}{\partial x}\Delta x \tag{1.3}$$

Substitute Eq. (1.2) and (1.3) into Eq. (1.1), we get:

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0 \tag{1.4}$$

This is the continuity equation in *A*-*Q* form. We can further convert it into a *U*-*h* form. The *Q*-term can be written as:

$$\frac{\partial Q}{\partial x} = \frac{\partial (AU)}{\partial x} = A\frac{\partial U}{\partial x} + U\frac{\partial A}{\partial x} \tag{1.5}$$

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The derivative of cross-section area can be related to the derivative of water depth as:

$$\frac{\partial A}{\partial x} = b_s \frac{\partial h}{\partial x}, \qquad \frac{dA}{dt} = b_s \frac{\partial h}{\partial t}$$

where b_s is the width of the channel's free surface. Also, A can be related to b_s through:

$$A = h_m(h) \cdot b_s$$

where h_m is the mean water depth (a function of h). Thus:

$$\frac{\partial Q}{\partial x} = b_s h_m \frac{\partial U}{\partial x} + b_s U \frac{\partial h}{\partial x} \tag{1.6}$$

Eq. (1.4) now can be expressed as:

$$b_s \left(\frac{\partial h}{\partial t} + h_m \frac{\partial U}{\partial x} + U \frac{\partial h}{\partial x} \right) = 0 \tag{1.7}$$

Since b_s is non-zero, we get the continuity equation in h-U form:

$$\frac{\partial h}{\partial t} + h_m(h) \frac{\partial U}{\partial x} + U \frac{\partial h}{\partial x} = 0 \tag{1.8}$$

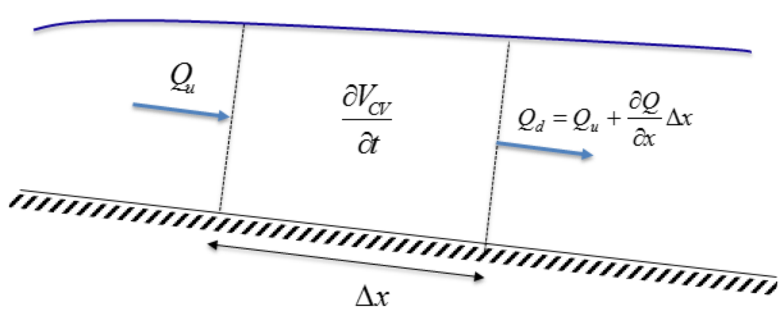

Figure 1 Conservation of volume

1.2. Momentum equation:

The governing equation based on the conservation of momentum is usually called the momentum equation. Here we still consider a short reach of an open channel, i.e. the CV is the flow between two cross sections with a short Δx apart, as shown in Figure 2. The surface of this control volume consists of the two cross sections, the free surface and the channel bottom surface. The flow through the two cross sections are assumed to be well-behaved, i.e. velocity is perpendicular to the surface and is reasonably uniform, so they are flux areas and we can calculate the hydraulic thrusts acting on them. For the channel bottom surface,

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flow must be tangential to it, so it is a streamline area and only surface forces acting on it. The pressure and shear stress acting on the free surface are immaterial, so we do not need to consider the free surface in momentum conservation. The body force acting on the CV is gravity. Thus, the conservation of momentum for the selected CV can be written as:

$$\frac{\partial}{\partial t}\int_{CV}\rho\vec{q}dV = \rho\vec{g}V_{CV} + \overrightarrow{MP}_u + \overrightarrow{MP}_d + \int_{\text{channel bottom}}(-p\vec{n} + \vec{\tau}_s)dA \tag{1.9}$$

where $\vec{q}$ is velocity vector, ρ is water density, MP_u and MP_d are thrusts acting on the upstream and downstream surfaces of the CV, g is gravity acceleration, V_{CV} is the volume of the CV, p is pressure and τ_s is the shear stress acting on the channel bottom. The pressure acting on the bottom of the channel is perpendicular to the flow direction, so the streamwise component of Eq. (1.9) is:

$$\frac{\partial}{\partial t}\int_{CV}\rho q_x dV = \rho g S_0 V_{CV} + MP_u - MP_d + \int_{\text{channel surface}}\tau_s dA \tag{1.10}$$

where τ_s is the mean shear stress acting on the channel surface. Assuming the velocity within the control volume is reasonably uniform, the term on the left-hand side of Eq. (1.10) can be approximately written as:

$$\frac{\partial}{\partial t}\int_{CV}\rho q_x dV = \frac{\partial}{\partial t}(\rho U A \Delta x) = \frac{\partial}{\partial t}(\rho Q \Delta x) = \rho \Delta x \frac{\partial Q}{\partial t} \tag{1.11}$$

The gravity force can be expressed as:

$$\rho g S_0 V_{CV} = \rho g S_0 A \Delta x \tag{1.12}$$

The difference of the two thrusts can be written as:

$$MP_u - MP_d = -\frac{\partial(MP)}{\partial x}\Delta x \tag{1.13}$$

The last term on the right-hand side of Eq. (1.10) can be expressed in terms of the average shear stress acting on the channel bottom:

$$\int_{\text{channel bottom}}\tau_s dA = -\tau_b P \Delta x \tag{1.14}$$

where τ_b is the average shear stress and P is the wetted perimeter. The minus sign is due to the fact that the shear stress is against the flow direction. Recall that:

$$S_f = \frac{\tau_b}{\rho g R_h} = \frac{\tau_b P}{\rho g A}$$

we get:

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$$\int_{\text{channel surface}} \tau_s dA = -\rho g A S_f \Delta x \tag{1.15}$$

Substituting Eq. (1.11), (1.12), (1.13) and (1.15) into Eq. (1.10), we have:

$$\rho \frac{\partial Q}{\partial t} = -\frac{\partial (MP)}{\partial x} + \rho g A (S_0 - S_f) \tag{1.16}$$

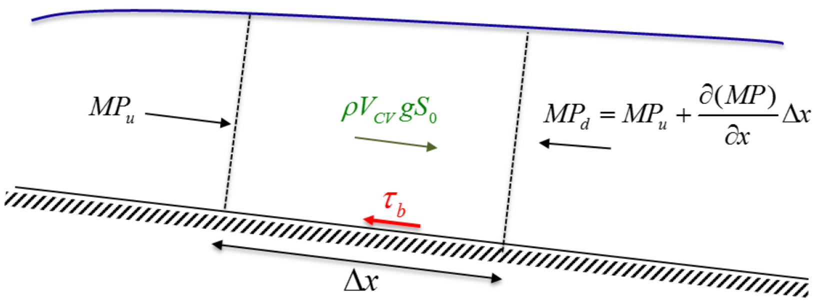

Figure 2 Conservation of momentum

Hydraulic thrust is defined as:

$$MP = \rho A U^2 + \int_0^h b(y) \rho g (h - y) dy \tag{1.17}$$

The last term on the right-hand side is the total hydrostatic pressure force acting on the area. The *x*-derivative of thrust is:

$$\frac{\partial MP}{\partial x} = \rho \frac{\partial (QU)}{\partial x} + \frac{\partial}{\partial x} \left(\int_0^h b(y) \rho g (h - y) dy \right) \tag{1.18}$$

The pressure-related term can be simplified as follows:

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\begin{aligned}
&\frac{\partial}{\partial x}\left(\int_0^h b(y)\rho g(h-y)dy\right) \\
&=\rho g\left[\frac{\partial}{\partial x}\left(h\int_0^h bdy\right)-\frac{\partial}{\partial x}\left(\int_0^h bydy\right)\right] \\
&=\rho g\left[\frac{\partial\left(Ah\right)}{\partial x}-b_s h\frac{\partial h}{\partial x}\right] \\
&=\rho g\left[A\frac{\partial h}{\partial x}+h\frac{\partial A}{\partial x}-b_s h\frac{\partial h}{\partial x}\right] \\
&=\rho g\left[A\frac{\partial h}{\partial x}+hb_s\frac{\partial h}{\partial x}-b_s h\frac{\partial h}{\partial x}\right] \\
&=\rho gA\frac{\partial h}{\partial x}
\end{aligned}
$$

Consequently, the *x*-derivative of the thrust can be written as:

$$\frac{\partial MP}{\partial x}=\rho\frac{\partial\left(QU\right)}{\partial x}+\rho gA\frac{\partial h}{\partial x} \tag{1.19}$$

After substituting Eq. (1.19) into (1.16) and eliminating ρ for all terms, we get:

$$\frac{\partial Q}{\partial t}+\frac{\partial\left(QU\right)}{\partial x}+gA\frac{\partial h}{\partial x}=gA(S_0-S_f) \tag{1.20}$$

Notice that:

$$\frac{\partial Q}{\partial t}=\frac{\partial(UA)}{\partial t}=U\frac{\partial A}{\partial t}+A\frac{\partial U}{\partial t}$$

$$\frac{\partial\left(QU\right)}{\partial x}=U\frac{\partial Q}{\partial x}+Q\frac{\partial U}{\partial x}=U\frac{\partial(UA)}{\partial x}+(UA)\frac{\partial U}{\partial x}=\left(UA\frac{\partial U}{\partial x}+U^2\frac{\partial A}{\partial x}\right)+UA\frac{\partial U}{\partial x}$$

Eq. (1.20) can be written as:

$$U\frac{\partial A}{\partial t}+A\frac{\partial U}{\partial t}+\left(UA\frac{\partial U}{\partial x}+U^2\frac{\partial A}{\partial x}\right)+UA\frac{\partial U}{\partial x}+gA\frac{\partial h}{\partial x}=gA(S_0-S_f) \tag{1.21}$$

Re-arrange the terms:

$$A\frac{\partial U}{\partial t}+AU\frac{\partial U}{\partial x}+U(\frac{\partial A}{\partial t}+U\frac{\partial A}{\partial x}+A\frac{\partial U}{\partial x})+gA\frac{\partial h}{\partial x}=gA(S_0-S_f)$$

According to the continuity equation:

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$$(\frac{\partial A}{\partial t}+U\frac{\partial A}{\partial x}+A\frac{\partial U}{\partial x})=0$$

Thus, the final form of the momentum equation is obtained as (after eliminating *A* for all terms):

$$\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+g\frac{\partial h}{\partial x}=g(S_0-S_f) \tag{1.22}$$

Eq. (1.22) is called the Saint-Venant equation, which is widely used in open-channel flow calculations. As you can see, if we neglect all terms on the left-hand side, we get the equation for uniform flow, i.e.:

$$S_0=S_f$$

If the time-derivative term is neglect, we actually get the equation for gradually-varied steady flow:

$$U\frac{\partial U}{\partial x}+g\frac{\partial h}{\partial x}=\frac{\partial}{\partial x}\left(\frac{U^2}{2g}+h\right)=\frac{\partial H}{\partial x}=g(S_0-S_f)$$

2. Method of characteristics

2.1. Characteristic forms of governing equation

By multiplying Eq. (1.8) by an unknown multiplier, λ, adding it to Eq.(1.22), and rearranging the terms of the resulting equation, we obtain:

$$\left[\frac{\partial U}{\partial t}+(U+\lambda h_m)\frac{\partial U}{\partial x}\right]+\lambda\left[\frac{\partial h}{\partial t}+(U+\frac{g}{\lambda})\frac{\partial h}{\partial x}\right]=g(S_0-S_f) \tag{2.1}$$

The total derivatives are defined as:

$$\begin{aligned}\frac{DU}{Dt}&=\frac{\partial U}{\partial t}+\frac{dx}{dt}\frac{\partial U}{\partial x}\\ \frac{Dh}{Dt}&=\frac{\partial h}{\partial t}+\frac{dx}{dt}\frac{\partial h}{\partial x}\end{aligned} \tag{2.2}$$

A comparison of Eq. (2.1) and (2.2) shows that we can write the terms inside the brackets as total derivatives if the unknown multiplier λ satisfies:

$$U+\lambda h_m=U+\frac{g}{\lambda} \tag{2.3}$$

Thus:

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$$\lambda = \pm\sqrt{\frac{g}{h_m}} = \pm\sqrt{\frac{g}{A/b_s}} \tag{2.4}$$

The celerity of a gravity wave in a prismatic open channel is:

$$c = \sqrt{gh_m} \tag{2.5}$$

Thus, if we define $\lambda=g/c$, we can write Eq. (2.1) as:

$$\frac{DU}{Dt} + \frac{g}{c}\frac{Dh}{Dt} = g(S_0 - S_f) \tag{2.6}$$

For a channel which is not too narrow and surface width varies slowly with water depth, we have (proof?):

$$\frac{dc}{dt} = \frac{d\sqrt{gh_m}}{dt} = \frac{1}{2}\frac{g}{c}\frac{dh_m}{dt} \approx \frac{1}{2}\frac{g}{c}\frac{dh}{dt} \tag{2.7}$$

Thus, Eq. (2.6) can be written as:

$$\frac{D(U+2c)}{Dt} = g(S_0 - S_f) \tag{2.8}$$

It is valid along:

$$\frac{dx}{dt} = U + c \tag{2.9}$$

Similarly, if we define $\lambda=-g/c$, we can write Eq. (2.1) as:

$$\frac{D(U-2c)}{Dt} = g(S_0 - S_f) \tag{2.10}$$

and it is valid along

$$\frac{dx}{dt} = U - c \tag{2.11}$$

Notice that Eq. (2.8) is valid only if Eq. (2.9) is satisfied, and Eq. (2.10) is valid only if (2.11) is satisfied. Eq. (2.8) and (2.10) are called <u>compatibility equations</u>. They are ordinary differential equations, which are easier to solve than Eq. (1.22), but you do not get something for nothing, because they are only valid when (2.9) and (2.11) are satisfied. We can visualize this as follows. If we plot Eq. (2.9) on the x-t plane, we will obtain a curve (or sometimes a straight line) since U and c are not always constant, as shown by AD in Figure 3. For any point (x,t) on this curve, the inverse slope is $U+c$, where U and c are the local velocity and celerity at that point (x,t). This curve is referred to as the <u>positive characteristics</u>. Similarly, if we plot

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Eq. (2.11) on the *x*-*t* plane, we will obtain a curve BD (or sometimes a straight line), of which the inverse slope is *U*-*c*. Such a curve is a negative characteristics. The compatibility equations are only valid along the corresponding characteristics, while the original equation, i.e. Eq.(1.22), is valid everywhere. The difference between compatibility equations and the original equation can also be conceptualized as follows. For an observer, who sits on the river bank and remains stationary, Eq. (1.22) describes the flow he/she will observe. While for an observer, who travels with a speed *U*+*c*, Eq. (2.8) describes the flow he/she will observe. Likewise, Eq. (2.10) describes the flow observed by someone who travels at a speed *U*-*c*.

As we mentioned before, a flow disturbance (depth and/or velocity) propagates in two directions if the flow is subcritical and only in the downstream direction if the flow is supercritical. The absolute velocity at which this disturbance travels is $U\pm c$. If we plot this propagation in the *x*-*t* plane assuming the disturbance is at point *P* at time t=0 and distance $x=x_0$, then its influence will be felt in the shaded region bonded by the positive and negative characteristics originated from P, as shown in Figure 4(a). This region is referred to as the zone of influence. Any point within this region has been affected by the disturbance initiated at point P. Likewise, for any point P(x_p,t_p) in the *x*-*t* plane, the positive and negative characteristics and the *x*-axis determine a zone of dependence for point P, i.e. Figure 4(b). This means that the flow at location $x=x_p$ are influenced by disturbance within the zone of dependence.

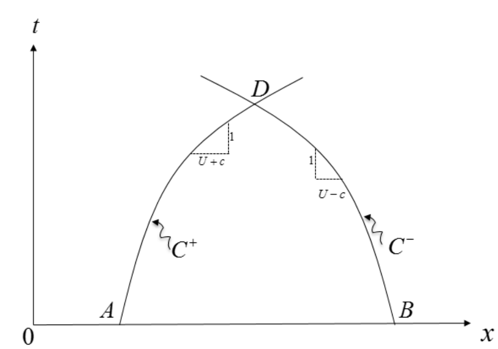

Figure 3 lines of characteristics

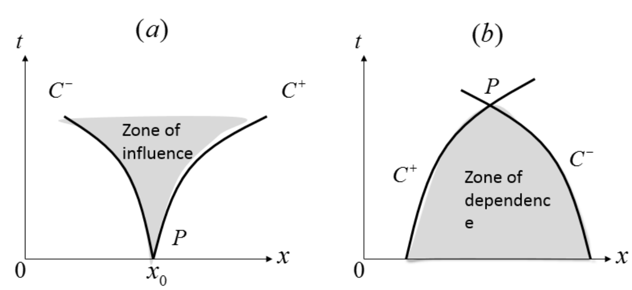

Figure 4 Zone of influence and dependence

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Depending upon the relative magnitude of the flow velocity *U* and the celerity *c*, a disturbance may or may not travel in the upstream direction. The convention used in this module sets *x* positive in the direction of flow, so *U* takes a positive valve. If the flow is subcritical, then the characteristic directions are both positive and negative, but the negative characteristic should be relative steeper since the absolute value of U-c is smaller than $U+c$. For critical flow, $U=c$, so the negative characteristic is perpendicular to the *x*-axis, whereas the positive characteristic has a positive slope. In supercritical flows, both characteristic directions are positive, and the negative characteristic should have steeper slope since $U-c<U+c$. These are shown in Figure 5.

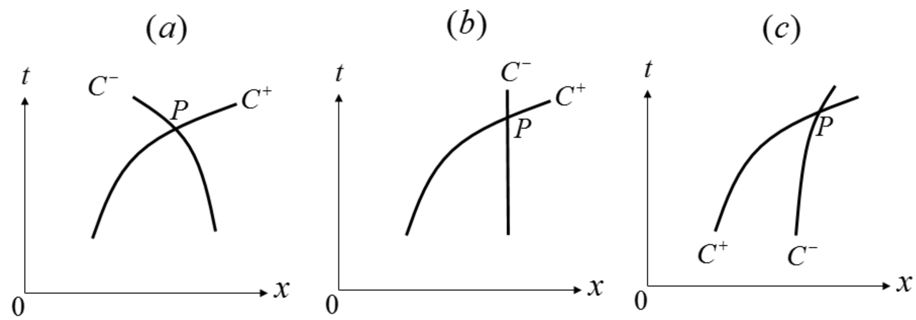

Figure 5 Characteristics for (a) subcritical, (b) critical and (c) supercritical flows

Even though the compatibility equations are ordinary differential equations, the solutions for *U* and *h* are not easy to obtain, because you have to define the characteristics which requires prior knowledge of *U* and *h*. Thus, we often have to rely on numerical techniques to obtain solutions.

2.2. Initial and boundary conditions

For determining the locations and number of the initial and boundary conditions required to solve an unsteady-flow problem, consider a channel reach AB, shown in Figure 6. Also consider that the flow condition at $t=t_1$ in the channel reach, say, at point A1, B1 and P are to be determined. The locations of points A1 and B1 are selected at the upstream and downstream boundaries of the channel reach, respectively, and P is within the channel reach. It should be possible in theory to obtain the zone of dependence for each point, as shown in Figure 6 (a) and (b) for supercritical and subcritical flows, respectively. The zone of dependence for point *P* at time t_1 lies with points 3 and 4. The flow conditions at point P at time t_1 can be determined by solving Eq. (2.8) and (2.10) for *U* and *h*, but the solution requires that the values for *U* and *h* be give at point 3 and 4 at time $t=t_0$. In the other words two conditions in terms of *U* and *h* at some initial time, called initial condition must be specified along the entire channel reach, i.e. one must specify the flow condition along AB at t_0.

For subcritical flow, a portion of the zone of dependence for points A1 and B1 in Figure 6 (a) are outside the channel reach, where the flow conditions are unknown. Therefore, the flow conditions at these points cannot be determined from the flow condition within the channel reach only. For points A1 and B1 only one of the characteristics is within the computation domain, consequently only one equation, either Eq. (2.8) or (2.10) can be utilized. The two unknowns U and h at these points can only be obtained, if one of them or their relationship is specified at both upstream and downstream boundaries. Thus, for subcritical flow one upstream and one downstream boundary conditions must be specified. For example, one can specify the temporal variation of water depth at point A and the temporal variation of discharge at point B.

For supercritical flow, both the characteristics through A_1 are outside the computational domain at t=t_0, while both the characteristics through B_1 are inside the computational domain at t=t_0. Thus, none of Eq. (2.8) and (2.10) can be utilized for A_1, but both of them can be used for B_1. Therefore, for supercritical flow two upstream boundary condition are needed, while no downstream boundary condition is necessary. For instance, one can specify the water depth and discharge at A, and no need to say anything about B.

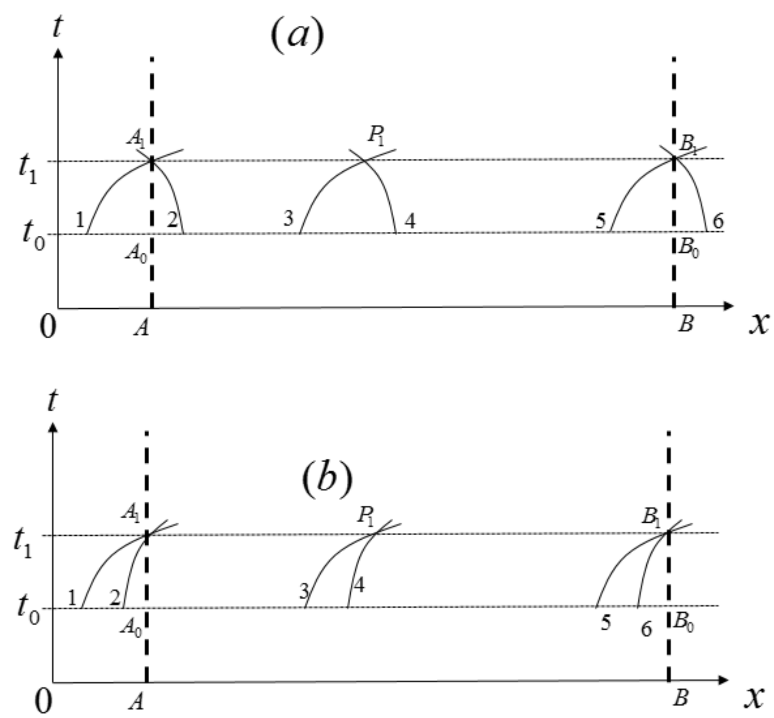

Figure 6 Initial and boundary condition requirement: (a) subcritical flow, (b) supercritical flow

2.3. Numerical solution method

There are many ways to numerically solve the Saint-Venant equation using the method of characteristics, and here we only introduce a very simple one. For this method we choose a regularly spaced grid

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arrangement as shown in Figure 7. Assume that the initial condition is specified at time *t*, i.e. the flow conditions, *U* and *h*, are already obtained for grids at level *t*. We now want to determine the flow condition at an advanced time $t+\Delta t$. This figure shows the advanced time solution point P, with the positive and negative characteristics passing through it, respectively. These intersect with the initial time level line at points L (left) and R (right). Recall the compatibility equations:

$$\frac{D(U+2c)}{Dt} = g(S_0 - S_f), \quad \frac{dx}{dt} = U + c$$
$$\frac{D(U-2c)}{Dt} = g(S_0 - S_f), \quad \frac{dx}{dt} = U - c$$

We will apply these along the positive and negative characteristics. First we must simplify the situation by assuming that the characteristics are straight lines. This is true only if nothing is changing (steady uniform flow) - a situation we are not very interested in. However if the time step is small the assumption usually gives very good solutions. Figure 8 shows a close up of the grid and straight characteristics associated with point P.

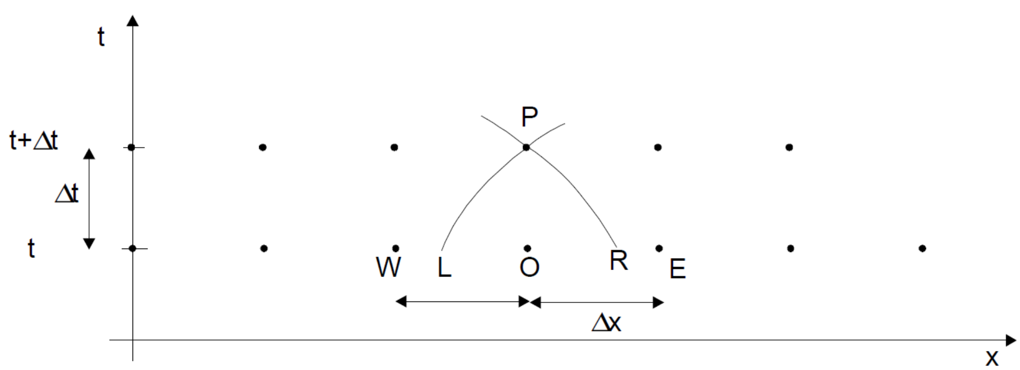

Figure 7 A rectangular grid with characteristics through P.

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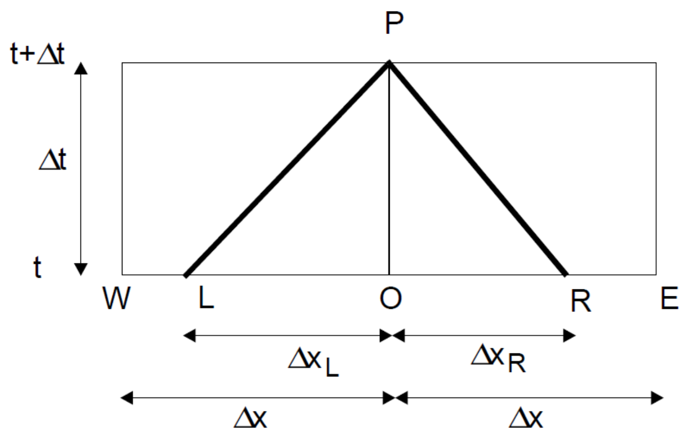

Figure 8 Grid and associated characteristics for point P

Secondly we must construct the characteristics. We know the slope when the positive characteristic crosses the known level, *t*, it is given by:

$$\frac{dx}{dt} = U_L + c_L$$

and the slope when it passes through P is given by:

$$\frac{dx}{dt} = U_P + c_P$$

The problem is that we don't know the values at either L or P! We must make an approximation. The usual approximation to determine the slope of the characteristics is to use the point at the same *x* position as P, but on the known time level i.e. the values at point O. The time step is Δt so the slope of the positive characteristic is:

$$\frac{dx}{dt} = U_o + c_o$$

and the negative characteristic is:

$$\frac{dx}{dt} = U_o - c_o$$

Now we can apply Eq. (2.8) along the positive characteristic, L-P, to give:

$$\left(U_p + 2c_p\right) - \left(U_L + 2c_L\right) = \Delta t g (S_0 - S_f)_L \tag{2.12}$$

Similarly, along the negative characteristic R-P:

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$$\left(U_p - 2c_p\right) - \left(U_R - 2c_R\right) = \Delta t g(S_0 - S_f)_R \tag{2.13}$$

Eq. (2.12) and (2.13) are two equations for two unknowns U_p and c_p. Solving Eq. (2.12) and (2.13) simultaneously gives the condition at point P:

$$U_p = \frac{U_R + U_L}{2} + (c_L - c_R) + \frac{\Delta t g}{2}[(S_0 - S_f)_L + (S_0 - S_f)_R] \tag{2.14}$$

and:

$$c_p = \frac{U_L - U_R}{4} + \frac{c_L + c_R}{2} + \frac{\Delta t g}{4}[(S_0 - S_f)_L - (S_0 - S_f)_R] \tag{2.15}$$

These are the solution at point P.

To simplify matters we might have chosen to calculate the friction slope at the midpoint O (as this is at the same *x* position as P) which would give for the positive characteristic:

$$\left(U_p + 2c_p\right) - \left(U_L + 2c_L\right) = \Delta t g(S_0 - S_f)_o \tag{2.16}$$

and for the negative characteristic:

$$\left(U_p - 2c_p\right) - \left(U_L - 2c_L\right) = \Delta t g(S_0 - S_f)_o \tag{2.17}$$

and the solution is:

$$U_p = \frac{U_R + U_L}{2} + (c_L - c_R) + \Delta t g(S_0 - S_f)_o \tag{2.18}$$

and:

$$c_p = \frac{U_L - U_R}{4} + \frac{c_L + c_R}{2} \tag{2.19}$$

To calculate this solution we must obtain the values at the point L and R. They are on the initial time level where we know all the solution, but we only know the solution at the node points (W, O and E).If we know the position of L and R we can interpolate between these know values to get U_L, c_L, U_R, and c_R. A simple linear interpolation procedure is all that is necessary i.e. for U_L:

$$U_L = U_o - \frac{\Delta x_L}{\Delta x}(U_o - U_w) \tag{2.20}$$

and for U_R:

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$$U_R = U_o - \frac{\Delta x_R}{\Delta x}(U_o - U_E) \tag{2.21}$$

The method described above is clearly quite simple to implement - although there are a lot of repetitive calculations involved if we wish to calculate for every point on a channel. A computer makes it particularly straightforward to calculate all the point and advance through time updating all values at every point and at every time level. However one must be extremely careful that the solution remains stable and accurate.

Stability and accuracy are controlled by the values Δt and Δx. We can determine a stability criterion by considering how information is being passed in our solution - or as we have seen by looking at the domain of dependence to see where the information at any point comes from. Consider Figure 9 which shows the characteristics around the point P from our earlier calculation. The positive characteristic through L has slope $1/(U+c)$, and the negative characteristic through R has slope $1/(U-c)$. From our earlier arguments point P can only receive information from points L and R if it lies within the domain of dependence of L and R, i.e. in the region bounded by the two characteristics. Therefore to remain within this domain of dependence this criteria for the forward characteristic must be satisfied:

$$\frac{dx}{dt} > U + c \tag{2.22}$$

This stability condition is often referred to as the Courant or CFL (Courant-Friedrichs-Lewy) condition.
In this case, $dt = \Delta t$ and $dx = \Delta x$, so:

$$\frac{\Delta x}{\Delta t} > (U + c) \tag{2.23}$$

Considering the backward characteristic we get the criteria for stability:

$$\frac{\Delta x}{\Delta t} > |U - c| \tag{2.24}$$

This is unnecessary as it is already taken account of by Eq. (2.22).

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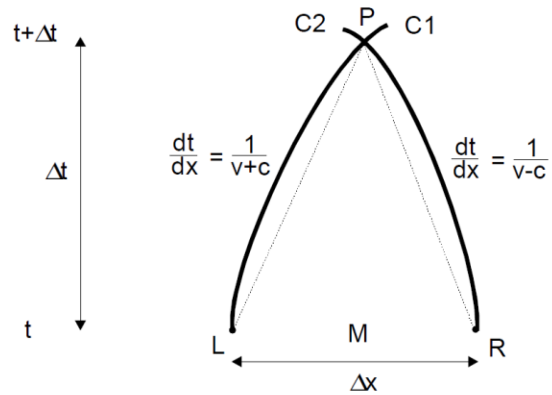

Figure 9 stability criterion

Clearly to apply Eq. (2.22) values of *U* and *c* must be known. But which do we choose? The values at L, R and P are all different (and those at P are not known beforehand). In practice the values at the known level must be used and some factor (≈ 0.9) introduced to take into account that the values at P may be so different to those at L or R as to change the gradient of the characteristics significantly. To actually use the stability condition in a calculation it is usual to choose a fixed grid spacing, Δx, then determine the maximum value of $(U+c)$ for each point on the know time level then calculating a Δt. So for the staggered grid the time step would be:

$$\Delta t = 0.9\frac{\Delta x}{(U+c)_{\max}} \tag{2.25}$$

Choosing a time step like this ensures that the Δx_L and Δx_R are both less than Δx.

3. Applications of method of characteristics: simple wave problems

With some simplifications, the Saint-Venant equation can be analytical solved with the method of characteristics, and the solution, although not really applicable, can illustrated some typical phenomenon for unsteady open-channel flows.

Here we consider flow disturbance propagating in a long horizontal frictionless channel, i.e. S_0=0 and S_f=0, so the compatibility equation for the Saint-Venant equation can be approximated as:

$$\frac{D(U+2c)}{Dt} = 0, \quad \frac{dx}{dt} = U + c \tag{3.1}$$

and:

$$\frac{D(U-2c)}{Dt} = 0, \quad \frac{dx}{dt} = U - c \tag{3.2}$$

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This is called the simple-wave problem. Integrating equations gives:

$$U + 2c = const, \quad \text{along } \frac{dx}{dt} = U + c \tag{3.3}$$

$$U - 2c = const, \quad \text{along } \frac{dx}{dt} = U - c \tag{3.4}$$

where the constants can be obtained from the initial and boundary conditions. This is termed as the simple-wave problem. The requirements for boundary conditions are different for subcritical and supercritical flows, i.e. subcritical flow requires one boundary condition at both the upstream and downstream ends of the channel reach, while supercritical flows requires two boundary conditions at the upstream end. Therefore, the simple wave problem for the two types of flows must be dealt separately.

3.1. Simple surface disturbance problem

3.1.1. Negative surge associated with a falling estuary level (subcritical flow)

Consider a flow in a long rectangular channel that finally enters an estuary, as shown in Figure 10 (a). Initially, we know that the velocity and water depth is uniform everywhere until the entrance, i.e.:

$$U(x,0) = U_0, \; x < 0 \tag{3.5}$$

and:

$$h(x,0) = h_0 \quad (\text{or } c(x,0) = c_0), \; x < 0 \tag{3.6}$$

where x=0 is set at the entrance. Assuming that the water depth at the entrance x=0 gradually decreases from t=0, which serves as a boundary condition for celerity:

$$c(0,t) = \sqrt{gh(t)} = f(t) \tag{3.7}$$

The task is to find how this disturbance propagates upstream into the channel reach.

To apply the method of characteristics, we shall first construct the characteristics. Since we are considering disturbance propagating upstream, the C^- characteristics should be analyzed. As shown in Figure 10b, The C^- characteristic started from the origin, OA, represents the trajectory of the front of the disturbance, where the local velocity and water depth are yet disturbed. Thus, velocity and celerity remain constant along OA, indicating that OA has a constant slope:

$$\frac{dx}{dt} = U_0 - c_0, \; \text{along OA} \tag{3.8}$$

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This slope is negative, because the flow is subcritical and hence $U_0<c_0$. In physical terms the line OA divides the disturbed and undisturbed flows, i.e. the region below OA represents the undisturbed region and the region above OA represents the disturbed region.

We can show that the other C^- characteristics are all straight lines as follows. Consider the C^- characteristic BE in Figure 10(b), which originates from time $t=\tau$ at x=0. We arbitrarily choose two points G and D on OA, and construct two C^+ characteristics from these two points. Although we do not really know these two C^+ characteristics, but we know that they have to intersect with BE, and the intersections are H and P. From G to H:

$$U_G + 2c_G = U_H + 2c_H \tag{3.9}$$

From D to P:

$$U_D + 2c_D = U_P + 2c_P \tag{3.10}$$

Since G and D are on OA, they have the same velocity and celerity:

$$U_G = U_D = U_0, \quad c_G = c_D = c_0 \tag{3.11}$$

Thus:

$$U_P + 2c_P = U_H + 2c_H \tag{3.12}$$

Since P and H are on a C^- characteristic, so:

$$U_P - 2c_P = U_H - 2c_H \tag{3.13}$$

Eq. (3.12) and (3.13) can be satisfied simultaneously only if:

$$U_P = U_H, \; c_P = c_H \tag{3.14}$$

Since P and H are arbitrarily chosen, along BE velocity and celerity remain constant, and its slope is constant, indicating that it is a straight line. The slope of BE can be obtained by considering a C^+ characteristic QB connecting B to OA. Along QB:

$$U_B + 2c_B = U_Q + 2c_Q = U_0 + 2c_0 \tag{3.15}$$

Thus:

$$U_B = U_0 + 2c_0 - 2c_B \tag{3.16}$$

and

$$U_B - c_B = U_0 + 2c_0 - 3c_B \tag{3.17}$$

`

Since B is at the boundary x=0, c_B can be determined from the boundary condition at x=0, so the slope of BE can be obtained from knowledge of initial condition (U_0 and C_0) and boundary condition. Meanwhile, the members of the C^+ family are not straight lines. To demonstrate this, we can consider the slope of GH in Figure 10(b). We have shown that:

$$U_H = U_B = U_0 + 2c_0 - 2c_B, \qquad U_G = U_0$$
$$c_H = c_B, \qquad c_G = c_0$$

Thus:

$$U_G + c_G = U_0 + c_0 \neq U_H + c_H$$

The slope of GH changes from G to H, indicating that GH is not a straight line.

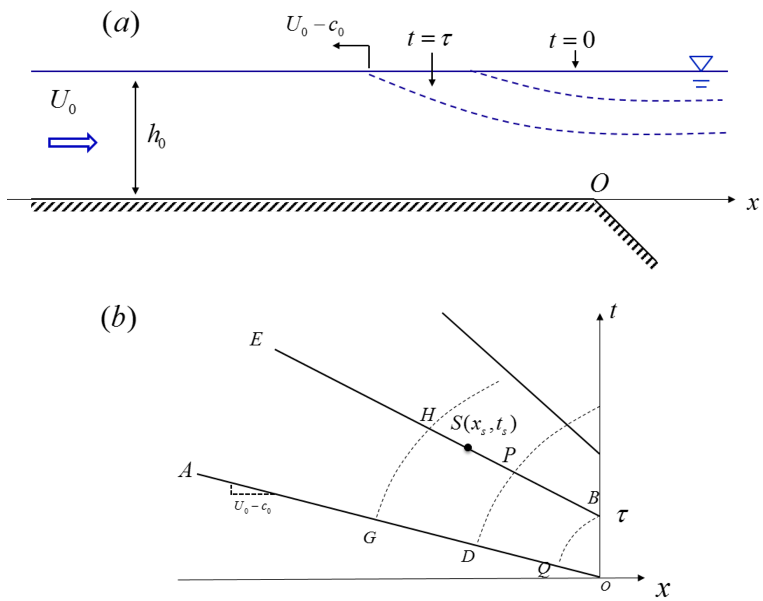

Figure 10 Simple-wave problem: negative disturbance for subcritical flow

Since any point on the t-axis (or x=0) can be connected to OA via a C^+ characteristics, the velocity and celerity at x=0 are related by (consulting Eq. (3.16)):

$$U(0,t) = U_0 + 2c_0 - 2c(0,t) \tag{3.18}$$

Since one boundary condition must be specified at x=0, Eq. (3.18) can be used with the given boundary condition of $c(0,t)$ (Eq. (3.7)) to fully determine $U(0,t)$ at x=0. With $U(0,t)$ and $c(0,t)$ determined the slope of a C^- characteristic started from x=0 at $t=\tau$ is then given by Eq. (3.17):

$$\frac{dx}{dt} = U_0 + 2c_0 - 3c(0,\tau) \tag{3.19}$$

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Thus, all C^- characteristics started from the t-axis can be constructed with the knowledge of initial and boundary conditions. If $c(0,\tau)$ is increasingly smaller than c_0, e.g. due to continuously lowering of water depth at the t-axis, the family of C^- characteristics diverge, as shown for the case in Figure 10(b). For a disturbance which makes the straight-line family of characteristics diverge, it is classified as a <u>negative disturbance</u>. There is also <u>positive disturbance</u> which makes the straight-line family of characteristics converge, e.g. raising the water depth and hence celerity such as those associated with the rising tide in an estuary, and we will discuss this later. For negative disturbance the front of the disturbance propagates fastest, so the disturbance will spread out.

In order to determine the flow conditions at an arbitrary point $S(x_s,t_s)$, we just need to find the C^- characteristic passing through it. For example, BE is the characteristic pass through $S(x_s,t_s)$ in Figure 10b. We just need to determine τ, i.e. the moment BE is started from the t-axis. Notice that the slope of BE can be written as:

$$\frac{dx}{dt}=\frac{x_s}{t_s-\tau} \tag{3.20}$$

Substitute Eq. (3.20) into Eq. (3.19), we get:

$$\frac{x_s}{t_s-\tau}=U_0+2c_0-3c(0,\tau) \tag{3.21}$$

The only unknown is τ, so we just need to solve Eq. (3.21) for τ, and the velocity and celerity at $S(x_s,t_s)$ is:

$$\begin{aligned} &U\left(x_s,t_s\right)=U(0,\tau) \\ &c\left(x_s,t_s\right)=c(0,\tau) \;\Rightarrow\; h\left(x_s,t_s\right)=h(0,\tau) \end{aligned}$$

By fixing the value of t_s but using different x_s, one can obtain the spatial variation of flow condition at $t=t_s$ over the entire channel. Similarly, by fixing the value of x_s but using different t_s, one can obtain the temporal variation of flow condition at $x= x_s$ over the entire channel.

Practice:

Water flows at a uniform depth of 1.52 m and velocity of 0.9 m/s in a wide rectangular channel into a large estuary. The estuary level is initially the same as the river level at the mouth, when it starts of fall at a rate of 0.3 m/hr for the next 3 hours. Neglecting channel friction and assuming that the channel bed is horizontal, determine the time it takes for the level of the river to fall by 0.6 m at a location 1600 m from the estuary. How far upstream will the river level starts to fall at this time?

(Solution to be provided in class).

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3.1.2. Positive surge associated with a rising estuary level (subcritical flow)

In the last subsection, we discussed the propagation of a negative disturbance, which has diverging family of straight characteristics. In this section, we shall investigate the propagation of a positive disturbance, which has a converging family of straight characteristics. This positive disturbance is created by rising water level in the estuary, as shown in Figure 11(a). We still assume the channel is a prismatic rectangular channel, and it is horizontal and frictionless. The C- characteristics of the front of the disturbance should still be a straight line with a slope of U_0-c_0, as indicated by OA in Figure 11(b), using the same argument as we did for the negative disturbance, the other C^- characteristics are all straight lines with a slope:

$$\frac{dx}{dt} = U_0 + 2c_0 - 3c(0,\tau)$$

where τ is the time when the C^- characteristic is initiated at x=0. Since the water depth at x=0 increases with τ, $c(0, \tau)$ increase with τ. Therefore, the slope becomes increasingly negative, meaning that a C^- characteristic initiated later will intersect with all earlier C^- characteristics. On the x-t plane, we shall see that all C^- characteristics converge.

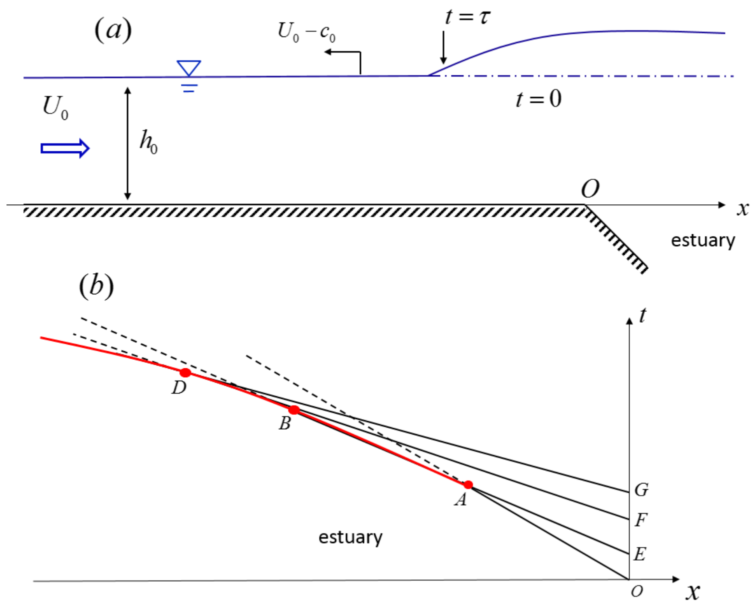

Figure 11 Positive surge propagation

There is a physical significance to the intersection of the C^- characteristics. It means that each incremental rise in the lake level leads to an incremental disturbance traveling at a faster speed than the earlier ones, catching up at a time corresponding to the time represented by the intersections of the C^- lines. At the intersection, there is two possible depths at a given x-location, which in physical terms means that the surface is vertical and a surge is formed, as shown in the lowest subfigure of Figure 12. Beyond the point

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of intersection, the equations of motion no longer hold for, so a surge begins to forms with consequent energy losses. The propagation of surges in a horizontal channel without resistance has been described in the section “moving hydraulic jump” in the notes of “steady open-channel flows”.

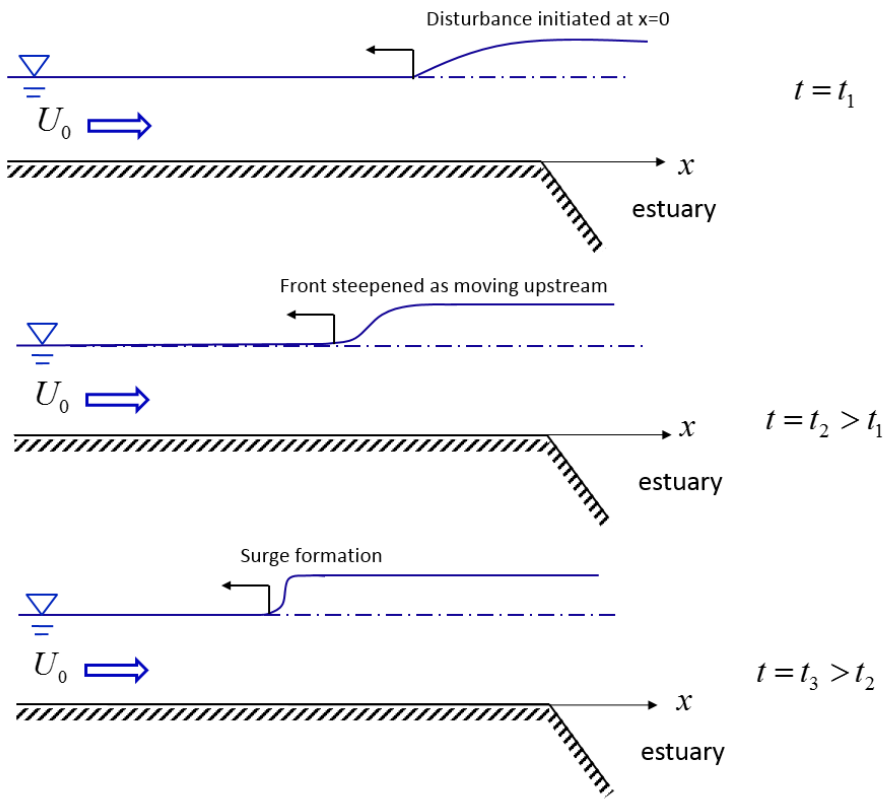

Figure 12 Development and propagation of a positive surge resulting from a gradual rise in the water level in the estuary

The intersections of C^- characteristics give an envelope of intersections, as indicated by the red line in Figure 11b. The envelope can be obtained as follows. As suggested by (3.19) C^- characteristic is described by x, t, and τ (the time of initiation at x=0), some the family of C^- characteristics can be written as:

$$f(x,t,\tau) = x - (t-\tau)[U_0 + 2c_0 - 3\sqrt{gh(0,\tau)}] = 0 \quad (3.22)$$

If two C^- characteristics initiated at the t-axis with a time interval $\Delta\tau$ apart intersects at (x,t), it means that:

$$f(x,t,\tau) = 0,\ f(x,t,\tau+\Delta\tau) = 0$$

Thus:

$$\frac{f(x,t,\tau) - f(x,t,\tau+\Delta\tau)}{\Delta\tau} = 0 \ \Rightarrow\ \frac{\partial f}{\partial\tau} = 0 \quad (3.23)$$

`

That means the intersections, i.e. points on the envelope, must satisfy both Eq. (3.23) and (3.22). Substituted Eq. (3.23) into (3.22):

$$[U_0 + 2c_0 - 3\sqrt{gh(0,\tau)}] + 3(t-\tau)\frac{\partial\sqrt{gh(0,\tau)}}{\partial\tau} = 0 \tag{3.24}$$

Eliminate τ between Eq. (3.24) and (3.22) may not be easy. Instead, the coordinates of the envelope for various values of τ are obtained Eq. (3.22) and (3.24):

$$x = -\frac{[U_0 + 2c_0 - 3\sqrt{gh(\tau)}]^2}{3\dfrac{\partial\sqrt{gh(\tau)}}{\partial\tau}} \tag{3.25}$$

and:

$$t = \tau + \frac{x}{U_0 + 2c_0 - 3\sqrt{gh(0,\tau)}} \tag{3.26}$$

Eq. (3.25) and (3.26) gives the envelope. The first formation of the surge occurs when the leading characteristic (OA in Figure 11(b)) is intersected. Thus, by taking τ=0 in Eq. (3.25), we obtain the location where the surge begins to form:

$$x = -\frac{[U_0 + 2c_0 - 3\sqrt{gh(0,0)}]^2}{3\left.\dfrac{\partial\sqrt{gh(0,\tau)}}{\partial\tau}\right|_{\tau=0}} = -\frac{\left(U_0 - c_0\right)^2}{\dfrac{3}{2}\sqrt{\dfrac{g}{h_0}}\left.\dfrac{\partial h(0,\tau)}{\partial\tau}\right|_{\tau=0}} \tag{3.27}$$

If the boundary condition is specified as an increasing of water depth at x=0:

$$h(0,\tau) = f(\tau) \tag{3.28}$$

then the problem is solved. We can also specified the boundary condition as an increasing of discharge per unit width at x=0:

$$q(0,\tau) = f(\tau) \tag{3.29}$$

The problem is a bit trickier. Notice that:

$$q(0,\tau) = U(0,\tau)\cdot h(0,\tau) = \left.\frac{Uc^2}{g}\right|_{x=0} \tag{3.30}$$

Differentiating Eq. (3.30) with respect to τ:

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$$\frac{\partial q}{\partial \tau} = \frac{1}{g}\left\{2Uc\frac{dc}{d\tau} + c^2\frac{dU}{d\tau}\right\}_{x=0} \tag{3.31}$$

Since:

$$U = U_0 + 2c_0 - 2c$$

$$\frac{\partial q}{\partial \tau} = \frac{1}{g}\left\{2Uc\frac{dc}{d\tau} + c^2\frac{dU}{d\tau}\right\}_{x=0} = \frac{1}{g}\left\{2Uc\frac{dc}{d\tau} - 2c^2\frac{dc}{d\tau}\right\}_{x=0} = \frac{1}{g}\left\{2c(U-c)\frac{dc}{d\tau}\right\}_{x=0} \tag{3.32}$$

Thus:

$$\left.\frac{dc}{d\tau}\right|_{x=0} = g\frac{\partial q}{\partial \tau} / \left[2c(U-c)\right]_{x=0} \tag{3.33}$$

For τ=0:

$$\left.\frac{dc}{d\tau}\right|_{x=0,\tau=0} = \frac{g}{2c_0(U_0-c_0)}\left.\frac{\partial q}{\partial \tau}\right|_{\tau=0} \tag{3.34}$$

This can be used in Eq. (3.27) to obtain the first formation of the surge. We then can use Eq. (3.26) to obtain when the first formation of surge takes place.

3.1.3. Simple wave problem for supercritical flow

Simple-wave problems for supercritical flow is not so simple because you may not have all members of one family of characteristics being straight lines, as shown below.

In supercritical flow, both families of characteristics are positive. Now let us consider a supercritical flow with initially uniform velocity and depth, U_0, and h_0 (or c_0). As shown in Figure 12, we consider the two characteristics originated from the origin, OA (C+) and OF (C-). We assume that OA is a straight line with a slope, U_0+c_0. For subcritical flow, we can demonstrate that all C+ characteristics are straight lines, but for supercritical flow this is no longer valid. Consider two C- characteristics GH and DE and another C+ characteristics BC. GH originates from the t-axis. DE intersects with OA at D. OF intersects with BC at J. Since OA is a straight C+ characteristic, both U+c and $U+2c$ are constant along it, indicating that:

$$U_D = U_0,\;\; c_D = c_0 \tag{3.35}$$

Along BC, which is also a C+ characteristics, $U+2C$ should be a constant:

$$U_H + 2c_H = U_J + 2c_J = U_E + 2c_E \tag{3.36}$$

Because U-$2c$ should be a constant along C- characteristics:

$$U_H - 2c_H = U_G - 2c_G \tag{3.37}$$

$$U_J - 2c_J = U_0 - 2c_0 \tag{3.38}$$

$$U_E - 2c_E = U_D - 2c_D = U_0 - 2c_0 \tag{3.39}$$

Since point G and point O can have different velocity and depth:

$$U_0 - 2c_0 \neq U_G - 2c_G \;\;\Rightarrow\;\; U_H - 2c_H \neq U_J - 2c_J$$

With Eq. (3.36), we conclude:

$$U_H \neq U_J,\;\; c_H \neq c_J$$

However, with Eq. (3.38), (3.39) and (3.36), we can see:

$$U_J = U_E,\;\; c_J = c_E$$

These relations imply that since point J the BC characteristics is a straight line, but before point J BC is a curve. Without BC being a straight line entirely, we cannot do what we did for subcritical flows, e.g. Eq. (3.21), so it is not always possible to obtain simple solution for simple-wave problem for supercritical flows.

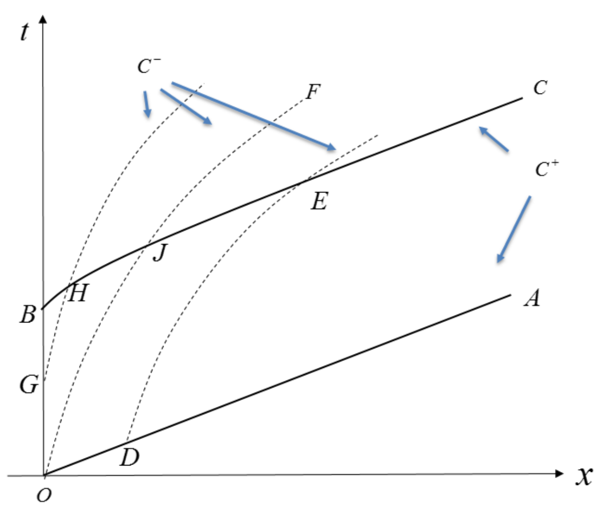

Figure 13 Simple wave problem for supercritical flow

3.2. Dam-break problem

In this section, we consider the flow due to sliding or breaking of a dam. Here we idealize a dam as a vertical plate, and we assume the channel is rectangular. Initially, the dam retains stagnant water to a depth h_0 in

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the channel, and there is no water in the downstream channel, and then the dam moves in the downstream direction, causing a disturbance, as shown in the upper part of Figure 14.

Since t=0, the plate starts to accelerate until it reaches a steady velocity V_p. In the *x*-*t* plane (lower part of Figure 14), the movement of the plate is denoted by OEH. From O to E, the velocity of the plate keeps increasing, so OE is curved. From E onward, the plate has reached the steady velocity V_p, so EH is a straight line. The line OEH works as the boundary condition for this problem, because in the region to the right of OEH, there will be no water. The flow velocity along OEH should just be the plate velocity, because water must closely follows the plate. Since the plate moves downstream and drags some water with it, a negative disturbance is formed and moves upstream. The front of the negative disturbance is the C- characteristic OA. Since the water in the channel is initially stagnant, the flow velocity and water depth at front of the negative disturbance must be zero and h_0, respectively. Therefore, the OA must be straight line, and its slope is:

$$\frac{dx}{dt} = -c_0 = -\sqrt{gh_0}$$

The region to the left of OA, denoted as zone I, is the undisturbed region.

For any C- characteristic initiated from OEH, we can easily show that it must be straight line. Take BC as an example. Along BC, the definition of C- characteristic requires that:

$$\left(U - 2c\right)_{\text{along BC}} = const$$

An arbitrarily chosen point on BC can be connected to a point on OA via a C+ characteristics. Since velocity and celerity along OA are both constant, so we have:

$$\left(U + 2c\right)_{\text{along OA}} = 2c_0$$

Therefore, along BC:

$$\left(U + 2c\right)_{\text{along BC}} = \left(U + 2c\right)_{\text{along OA}} = 2c_0$$

Since BC is an arbitrarily chosen C- characteristics, we can say that:

$$U + 2c = 2c_0 \tag{3.40}$$

over the entire x-t plane. Both V+2c and V-2c are constant along BC. This can only be satisfied if U and c are both constants, and therefore the slope of BC is constant:

$$\frac{dx}{dt} = U_B - c_B = U_B + 2c_B - 3c_B = 2c_0 - 3c_B \tag{3.41}$$

`

or:

$$\frac{dx}{dt} = U_B - c_B = U_B + \frac{1}{2}U_B - \frac{1}{2}U_B - c_B = \frac{3}{2}U_B - \frac{1}{2}(U_B + 2c_B) = \frac{3}{2}U_B - c_0 \tag{3.42}$$

Since B is an arbitrarily chosen point along OEH, Eq. (3.42) suggests that the slope of a C- characteristic initiated from OEH changes with its flow velocity U_B at its origin on OEH. We know that the flow velocity along OEH is the same as the plate velocity, which increases from 0 at point O to U_p at point E, so the slope of C- characteristics must increases from O to E, and the C- characteristics initiated between O and E must be diverging, e.g. in the lower panel of Figure 14 the slope of BC is between those of OA and EF. The plate velocity becomes constant after point E, so all C- characteristics initiated to the right of point E must be parallel straight lines with a slope:

$$\frac{dx}{dt} = 1.5U_p - c_0$$

We define the zone to the right of EF, where all C- characteristics are parallel straight lines, as zone III. The region between zone I and zone III are defined as zone II, where the C- characteristics are diverging straight lines.

The flow velocity and water depth in zone III can be obtained by simultaneously solving:

$$U - c = 1.5U_p - c_0$$
$$U + 2c = 2c_0$$

which gives:

$$c = c_0 - \frac{1}{2}U_p, \text{ in zone III} \tag{3.43}$$

$$U = U_p \text{ in zone III} \tag{3.44}$$

Thus, within zone III, both velocity and celerity are constant. A constant celerity indicates a constant water depth:

$$h_p = \frac{(c_0 - U_p / 2)^2}{g} \tag{3.45}$$

Thus, once the plate reaches its constant velocity U_p, there is a region of constant water depth, $h=h_p$, developed upstream to the plate, as denoted by the zone III in the upper panel of Figure 14. In zone II, however, the flow conditions are variable, as the slopes of C- characteristics are variable.

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If U_p is not very large, $1.5U_p – c_0$ can be negative, so the C- characteristics in zone III can be negative, which is the case shown in Figure 14. In this case, EF can extend to the upstream of x=0, meaning that zone III, which has a constant water depth and constant flow velocity, will keep extending upstream as time goes on. This makes sense, as the flow in zone III is subcritical ($U<c$).

However, if:

$$1.5U_p - c_0 = 0$$

EF, and all C- characteristics initiated after point E will be vertical lines, so zone III will not be able to extend beyond $x=x_E$. In this case, the flow in zone III is critical, since $U=c$.

If the final velocity of the plate is more than 2/3 of c_0:

$$\frac{dx}{dt} = 1.5U_p - c_0 > 0$$

In this case, EF and all the C- characteristics in zone III will be positive, so zone III will move downstream as time goes on, which agrees with that the flow in zone III is supercritical ($U>c$).

From Eq. (3.43), if the plate final velocity is $U_p=2c_0$, the celerity in zone III will be 0, indicating that the depth in zone III is zero. In such case, all C- characteristics in zone III collapse with EH, and zone III disappear, as shown in the lower panel of Figure 15. For $t_1>t_E$, where t_E is the time when the plate reaches its final velocity, the surface profile is shown in the upper panel of Figure 15. The depth at the leading edge of the flow is zero. The leading edge that is in contact with the plate is moving with a velocity $2c_0$. It can be easily seen that if the plate moves with a final velocity over $2c_0$, it will lose contact with the water behind it. In other words the flow field is not affected by the removal of the plate. In such cases, the plate can be considered “disappeared”, or the dam is broken.

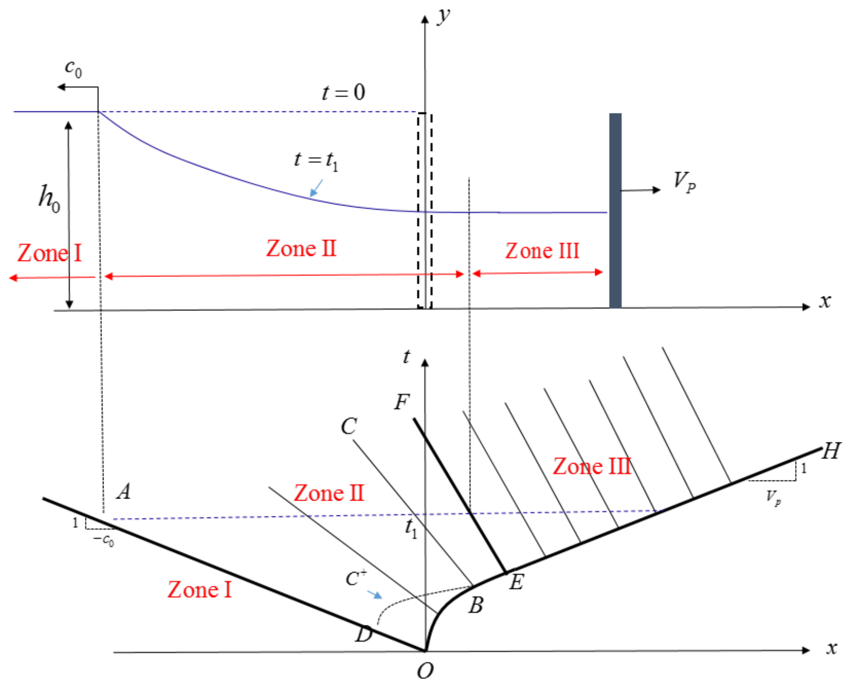

Figure 14 Flow development by movement of a plate

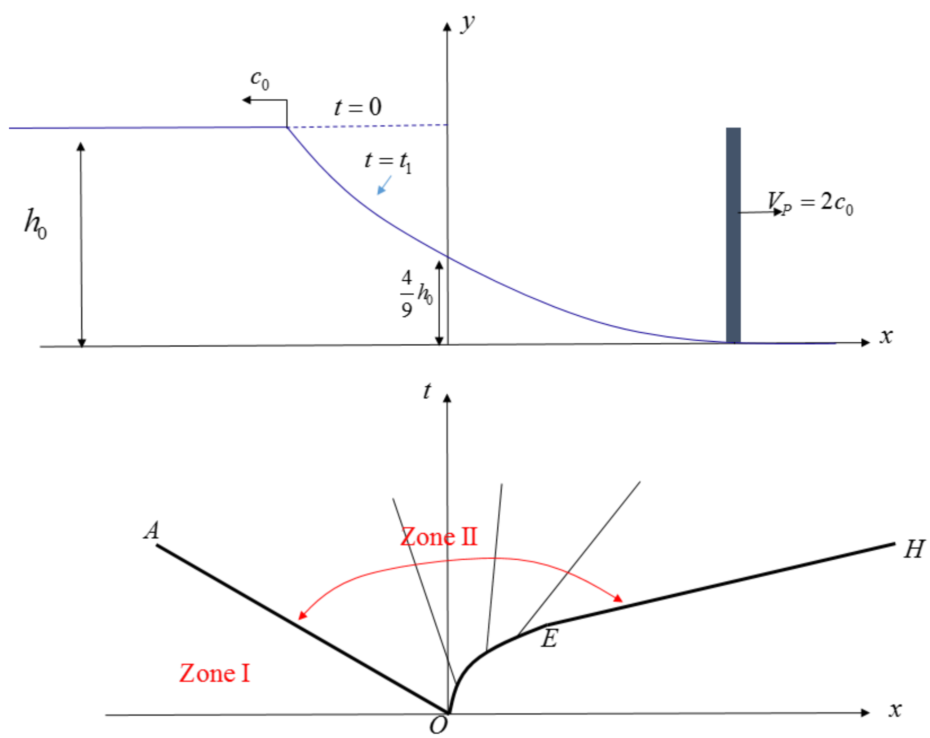

Figure 15 a limiting case wherein U_p=2c_0.

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If the plate velocity can suddenly increase from 0 to $U_p>2c_0$ at t=0, then the problem is equivalent to a sudden removal of the plate, since the water can never catch up with the plate. This is an idealization of a dam-break problem. In such cases, point E coincide with point O, and the line EH originates from the origin, as shown in the lower panel of Figure 16. The flow condition at any point S can be obtained from Eq. (3.41):

$$\frac{dx}{dt}=\frac{x_s}{t_s}=2c_0-3c_S=2\sqrt{gh_0}-3\sqrt{gh_s}$$

Therefore, the surface profile at any instant t can be given by:

$$\frac{x}{t}=2\sqrt{gh_0}-3\sqrt{gh} \tag{3.46}$$

This means the surface profile between the leading and trailing edges of the dam-break wave is a parabola, with the leading edge moving downstream at a speed $2c_0$ and the trailing edge moving upstream at a speed c_0. It should be noted that there must be a C- characteristics coinciding with the t-axis, i.e. its slope is 0. A zero slope means that the flow is just critical! (U-c=0 ==> U=c). Since Eq. (3.40) still hold, we can obtain that:

$$U=c=2c_0/3, \quad h=\frac{4}{9}h_0, \text{ at } x=0$$

The slope of C- characteristics for the first quadrant of the x-t plane is positive (see Figure 16), so the flow is always supercritical in the channel downstream from the dam's initial position (x=0). While in the channel upstream to the dam's initial position, which corresponds to the fourth quadrant of the x-t plane, the C- characteristics are negative, and the flow is subcritical.

The solution for dam-break problem obtained here agrees reasonably well with observations. However, the leading edge of the dam-break wave is observed to be rounded and move at a velocity roughly half of the theoretical value. This is because the bottom resistance effect is pronounced, but is neglected in our analysis.

In many situations, the downstream channel is not dry. For such problems, you can consult advance text books (Subhash C. Jain, Open-Channel Flow, 2001).

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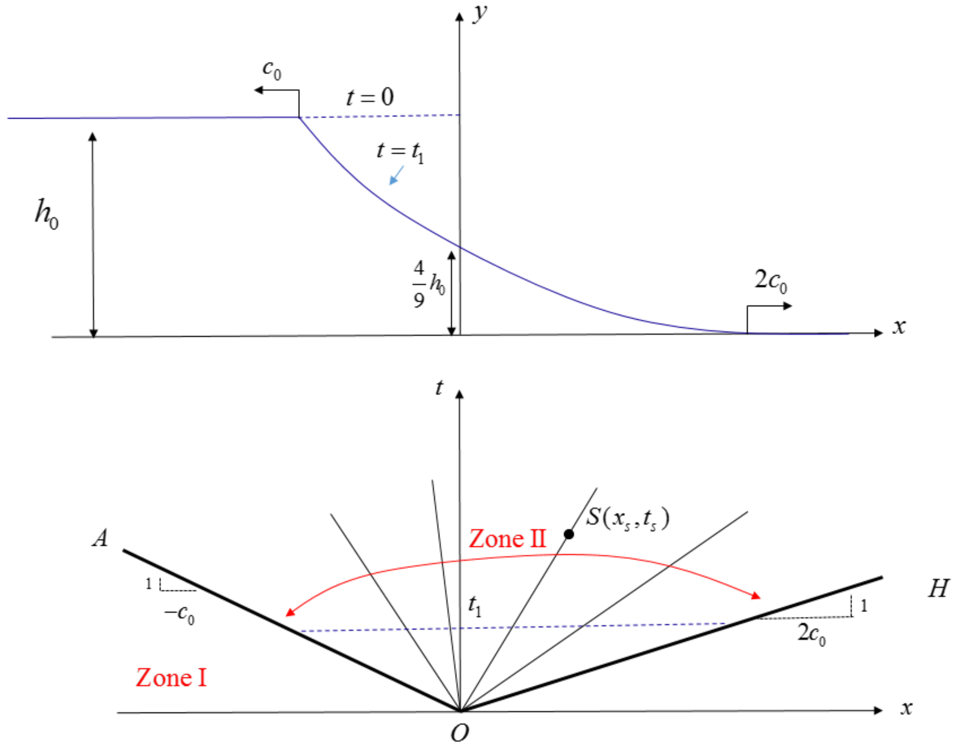

Figure 16 sudden failure of the dam with dry downstream bed

4. Flood routing

In this section, we deal with the problem of flood routing, i.e. the process of tracing by computations the course of a flood wave (how discharge Q and depth h varies with time and space). Imagine a flood wave moving downstream in a channel which is originally having a uniform flow as shown in Figure 17. As the flood wave travels downstream, it is generally observed that its peak gets smaller (attenuation) and the wave spreads longer. The spread arises from the wave front traveling faster than the rear due mainly to the $\partial h / \partial x$ term and also the acceleration terms in the momentum equation, Eq. (1.22). The rise and fall of a flood wave may occur much slower than the changes reflected by some of the terms in the momentum equation which may suggest that some of the acceleration terms have reduced importance and may be neglected. Different flood routing methods are derived based on which terms are neglected in the momentum equation.

In some cases, attenuation is mainly due to storage effects, particularly for the case when the flood passes through a reservoir. In such cases, the momentum equation can be neglected, and we only need to apply the continuity equation, Eq. (1.4), to describe flood routing. There are also methods of flood routing in rivers that are based on the continuity equation along, with the Muskingum method being the most widely used one.

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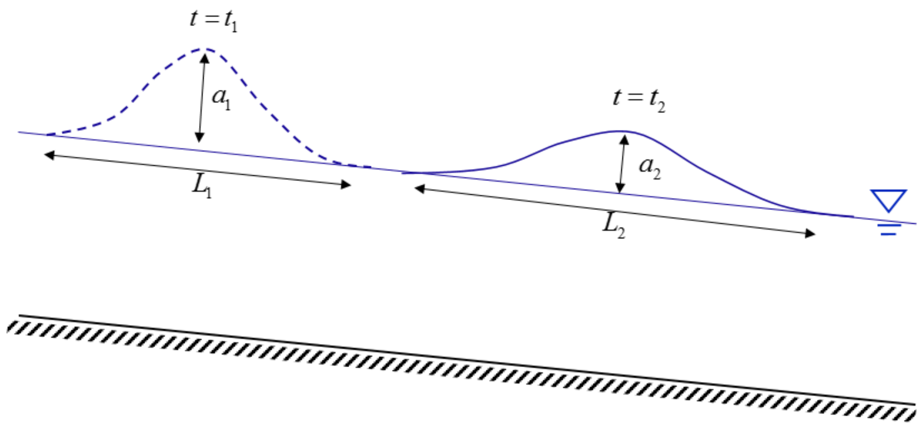

Figure 17 Spreading and attenuation of a flood wave

4.1. Reservoir routing

Here we consider the routing for a reservoir, as shown in Figure 18. The continuity equation obtained before is:

$$\frac{\partial Q}{\partial x}+\frac{\partial A}{\partial t}=0$$

Let us integrate it from $x=x_1$ (upstream end of reservoir) to $x=x_2$ (downstream outlet of a reservoir):

$$Q_1-Q_2+\frac{d}{dt}\int_{x_1}^{x_2}Adx=0 \tag{4.1}$$

We further define the storage over the reservoir as:

$$S=\int_{x_1}^{x_2}Adx \tag{4.2}$$

Note that the discharge at x_1, Q_1, represents the volume of inflow per unit time, while the discharge at x_2, Q_2, represents the volume of outflow per unit time, so Eq. (4.1) can be written as:

$$I-O=\frac{dS}{dt} \tag{4.3}$$

where I denotes inflow discharge and O denotes outflow discharge. This means that for a reservoir the change of water volume in it is the difference between the inflow and the outflow. This equation has two unknowns, S and O, if I (inflow) is given as a boundary condition. Thus, one additional S-O relationship is needed for. If a routing method use an S-O relationship based on the momentum equation, we consider it as a hydraulic routing method. However, if a routing method is based on an S-O relationship that is WITHOUT considering momentum equation, we consider it as a hydrologic routing method. Here we introduce two hydrologic routing method.

Level-pool method

If we consider a short period of time Δt, the average rate of storage change is:

$$\frac{dS}{dt} = \frac{S_2 - S_1}{\Delta t}$$

The inflow and outflow rates can be approximated by the average values before and after Δt:

$$\bar{I} = \frac{I_1 + I_2}{2},\ \bar{Q} = \frac{Q_1 + Q_2}{2}$$

Thus, Eq. (4.3) can be approximated as:

$$\frac{I_1 + I_2}{2} - \frac{O_1 + O_2}{2} = \frac{S_2 - S_1}{\Delta t} \tag{4.4}$$

If the inflow flow condition is known, i.e. I_1 and I_2 is known, and the flow condition at time level 1 is also known, i.e. O_1 and S_1 are given as an initial condition, the unknowns in Eq. (4.4) are outflow rate O_2 and storage S_2 at time level 2. We can re-arrange Eq. (4.4) as:

$$\frac{S_2}{\Delta t} + \frac{O_2}{2} = \frac{S_1}{\Delta t} + \frac{O_1}{2} + \frac{I_1 + I_2}{2} - O_1 \tag{4.5}$$

For simplicity, we can further define:

$$N = \frac{S}{\Delta t} + \frac{O}{2}$$

and Eq. (4.5) finally becomes:

$$N_2 = N_1 + \frac{I_1 + I_2}{2} - O_1 \tag{4.6}$$

To solve Eq. (4.6), an additional relation between N and O is required. Obviously, storage S is only a function of water depth h. If we also assume the outflow O is only a function of h (i.e. if there is a weir controlling the outflow so $Q \sim h^{3/2}$), N is also only a function of depth:

$$N = \frac{S}{\Delta t} + \frac{O}{2} = \frac{S(h)}{\Delta t} + \frac{O(h)}{2} = N(h)$$

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Since O and N are all single-valued functions of h, N can be expressed as a single-valued function of O. Thus, an N-O relation can be established using the S-h and O-h relations from measurements. The S-h relation is given by measured topography information, and the O-h relation is based on previous measurements. Once N-O relation is established, Eq. (4.6) is then used for reservoir routing.

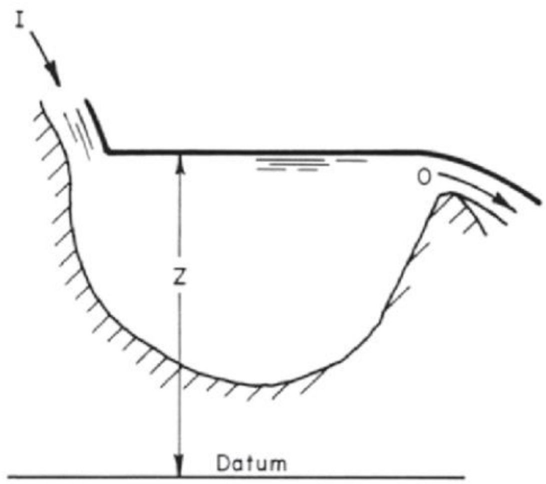

Figure 18 Schematic sketch of reservoir routing

<u>Procedure of the level-pool method is as follows:</u>

Given conditions:

- Inflow hydrograph
- An S-h relationship from topography information
- An O-h relationship from (previous flood record or simple formula)
- Initial flow condition (O,S, I at t=0)

Steps:

- Choose Δt and discretize the time domain
- Develop N-O relationship from S-h and O-h relationships
- Compute N^{i+1} using Eq. (4.6)
- Use the established N-O relationship to obtain O^{i+1}
- Use obtain O and N to get S: $N=S/\Delta t+O/2$

Example: (see lecture PPT)

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The Muskingum Method

The Muskingum method for flood routing was first developed in the 1930s for flood control schemes in the Muskingum River Basin, Ohio. Similar to the level-pool method, this method, which is also based on the continuity equation, requires a relation between storage *S* and in/outflow *I* and *O*. For the level-pool method, *S* is a single value function of *O*. This may not be a suitable assumption for rivers. For example, for non-uniform flow, the storage depends upon the inflow and outflow. The storage in a channel reach may be divided into prism storage, where *S* is proportional to *O*, and wedge storage, where *S* is proportional to the difference between I and O, as shown in Figure 19. The Muskingum method assumes *S* to be linearly related to *I* and *O* as follows:

$$S = KO + KX(I - O) = K[XI + (1 - X)O] \tag{4.7}$$

in which *K* and *X* are the routing parameters. The parameter *K* has a dimension of time, and its value is approximately equal to the travel time of the flood wave through the river reach. The parameter *X* is a dimensionless weighting factor that expresses the relative influence on storage of inflow and outflow. *X* ranges between 0 and 0.5. It accounts for attenuation/dispersion of the flood wave. Substitute Eq. (4.7) into Eq. (4.4) one obtain:

$$\frac{I_1 + I_2}{2} - \frac{O_1 + O_2}{2} = \frac{K[XI_2 + (1 - X)O_2] - K[XI_1 + (1 - X)O_1]}{\Delta t} \tag{4.8}$$

which can be re-arranged to:

$$O_2 = C_0 I_2 + C_1 I_1 + C_2 O_1 \tag{4.9}$$

where:

$$\begin{aligned} C_0 &= \frac{\Delta t - 2KX}{\Delta t + 2K(1 - X)} \\ C_1 &= \frac{\Delta t + 2KX}{\Delta t + 2K(1 - X)} \\ C_2 &= \frac{2K(1 - X) - \Delta t}{\Delta t + 2K(1 - X)} \end{aligned} \tag{4.10}$$

Note that the sum of them should be 1. Since C_0, C_1 and C_2 are constants, Eq. (4.9) can be used to obtain outflow at time level 2 with upstream boundary conditions (I_1 and I_2) and outflow at time level 1 (O_1).

It should be noted that Δt must be larger than $2KX$, otherwise the coefficient of C_0 is negative, which is unreasonable. If *K* and *X* are known, Eq. (4.9) can be used to obtain outflow hydrograph with a given inflow hydrograph. Value of *X* over 0.5 increases the flood peak, which is not in agreement with reality (proof of

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this will be given later). For $K=\Delta t$ and $X=0.5$, we have $C_0=C_2=0$ and $C_1=1$, so $O_2=I_1$, meaning that the flood motion is a pure translation without attenuation or deformation, i.e. what happened at the inflow boundary at time level 1 will be observed at the outflow boundary at time level 2.

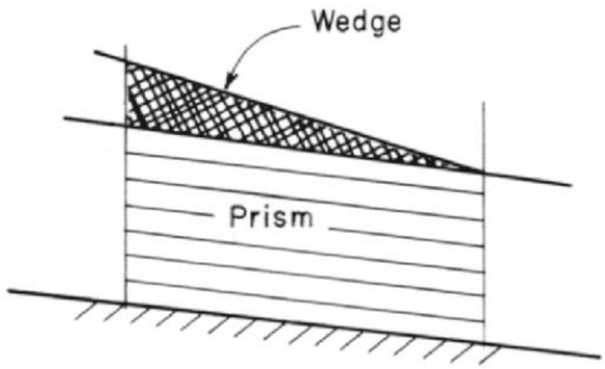

Figure 19 Prism and wedge storage

The parameter K and X for a given river reach are found by trial and error using the past inflow and outflow hydrograph as follows. From Eq. (4.7), K is the ratio of S to the weighted discharge [$XI+X(I-O)$]:

$$K = \frac{S}{XI + (1 - X)O}$$

Channel storage is determined from inflow and outflow hydrographs by solving Eq. (4.4):

$$S_2 = S_1 + \frac{\Delta t}{2}(I_1 + I_2 - O_1 - O_2) \tag{4.11}$$

For various assumed value of X ($0<X<0.5$), the weighted discharge [$XI+X(I-O)$] are calculated. Each weighted discharge is plot against the calculated channel storage. The value that produces the narrowest loop is chosen as X. The value of K is equal to the slope of the straight line through the loop.

Example 1:

For a channel reach the following inflow and outflow hydrographs are obtained from historical data. Using these data to find the routing parameters K and X.

t [day]	0	1	2	3	4	5	6	7	8	9	10	11
inflow [m^3/s]	7.0	19.0	25.0	34.0	30.0	24.0	20.0	15.0	13.0	11.0	8.0	7.0
outflow [m^3/s]	7.0	9.9	20.0	26.9	32.6	28.7	23.3	19.0	14.7	12.6	10.4	7.9

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General procedure for Muskingum routing:

Given:

- inflow hydrograph
- routing parameter K and X (if not you have to obtain them from past flood record)
- initial condition (I, Q, S)

Steps:

- chose a time interval
- Calculate the routing constants, i.e. Eq. (4.10)
- Routing for *O*, i.e. Eq. (4.9)
- Use obtain O and given I to get storage S via Eq. (4.11)

Example 2:

For a channel reach the routing parameters are K=0.8 [day] and X=0.3. Determine the outflow hydrograph with the following inflow hydrograph.

t [day]	0	1	2	3	4	5	6	7	8	9	10	11
inflow [m^3/s]	7	19	25	34	30	24	20	15	13	11	8	7

(solution in lecture PPT).

4.2. Simplification of momentum equation

The momentum equation, Eq. (1.22), can be re-write as:

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + g\frac{\partial h}{\partial x} - gS_0 + gS_f = 0 \tag{4.12}$$

The first two terms on the left-hand side represent the acceleration of flow, so we collectively call them the inertial term. The $g\partial h/\partial x$ terms represent the pressure gradient, so we call it the pressure term. S0 and Sf are naturally called the gravity and friction terms. Through order of magnitude analysis, it is possible to estimate the relative importance of various terms. Henderson (1966) [Henderson FM, (1966) Open Channel Flow, The Macmillan Company, New York, Ltd ISBN 13:9780023535109] has given typical values of each terms in the momentum equation for a natural flood in steep rivers where the rising flood flows were found to increase from 300 m^3/s to 4000 m^3/s and then decrease to 300 m^3/s in 24 hours. These typical values are presented below:

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$$S_0 = 4.9 \cdot 10^{-3}$$
$$\frac{\partial h}{\partial x} = 9.5 \cdot 10^{-5}$$
$$\frac{U}{g}\frac{\partial U}{\partial x} = 2.4 \cdot 10^{-5} \sim 4.7 \cdot 10^{-5}$$
$$\frac{1}{g}\frac{\partial U}{\partial t} = 9.5 \cdot 10^{-5}$$

It can be seen that the pressure and inertial terms are small by comparison with the gravity term and the river discharge can be computed as uniform flow (S_0 balances S_f all the time) where the discharge *Q* is a function only of the depth *h* even if the Manning's '*n*' value changes during the progression of the flood. Such a flood wave has been referred to as a <u>kinematic flood</u> wave on account of the fact that inferences on its characteristics can be derived from the kinematic terms in the momentum equation. This assumption is true for steep channels. For steep slopes, only S_0 is retained on the right-hand side in Eq. (4.12), which is just the formula for uniform flow:

$$S_f = S_0 \tag{4.13}$$

On the other hand, when the channel bed is very flat such that $\partial h/\partial x$ is of the same order as S_0, the Froude number of the flow will be so low that the other slope terms may be neglected, thus, on very flat slopes, we may just retain S_0 and $\partial h/\partial x$:

$$S_f = S_0 - \frac{\partial h}{\partial x}$$

Depending on the number of terms kept in the momentum equation, there are different methods of flood routing in a channel: (i) kinematic-wave, (ii) diffusion-wave, (iii) steady dynamic wave and (iv) dynamic-wave. This is summarized in Figure 20.

$$\underbrace{U_t + \underbrace{UU_x + \underbrace{gh_x + \underbrace{g\left(S_f - S_0\right)}_{kinematic\ \ wave}}_{diffusion\ \ wave}}_{steady\ \ dynamic\ \ wave}}_{dynamic\ \ wave} = 0$$

Figure 20 Momentum equation for different wave models

Non-dimensional form of momentum equation:

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A more rigorous way to investigate the relative magnitude of the terms in momentum equation is to transfer the momentum equation into non-dimensional form. Here we choose the following characteristic values for the involved variables:

$h = h_0 \hat{h}$	h_0: normal depth
$U = U_0 \hat{U}$	U_0: normal flow velocity
$t = T_0 \hat{t}$	T_0: time scale of a flood wave
$x = L_0 \hat{x}$	$L_0 = T_0 * U_0$

Then Eq. (4.12) can be written as:

$$\frac{U_0}{T_0}\frac{\partial \hat{U}}{\partial \hat{t}} + \frac{U_0}{T_0}\hat{U}\frac{\partial \hat{U}}{\partial \hat{x}} + \frac{g h_0}{U_0 T_0}\frac{\partial \hat{h}}{\partial \hat{x}} - gS_0 + gS_f = 0$$

Divide all terms by gS_0:

$$\frac{U_0}{gS_0 T_0}\left(\frac{\partial \hat{U}}{\partial \hat{t}} + \hat{U}\frac{\partial \hat{U}}{\partial \hat{x}}\right) + \frac{h_0}{U_0 T_0 S_0}\frac{\partial \hat{h}}{\partial \hat{x}} - 1 + \frac{S_f}{S_0} = 0 \qquad (4.14)$$

We can neglect the pressure and initial terms if:

$$\Pi_1 = \frac{U_0 T_0 S_0}{h_0} >> 1 \qquad (4.15)$$

and:

$$\Pi_2 = \frac{g S_0 T_0}{U_0} >> 1 \qquad (4.16)$$

For Π_1 to be very large, we need steep slope and long period of flood. The ratio of $\Pi 1/\Pi 2$ is:

$$\frac{\Pi_1}{\Pi_2} = \frac{U_0 T_0 S_0}{h_0} \cdot \frac{U_0}{g S_0 T_0} = \frac{{U_0}^2}{g h_0} = Fr^2$$

Generally speaking we will have subcritical flow when we have flood, so Fr is less than 1. Meaning that $\Pi 1$ is smaller than $\Pi 2$. Therefore if $\Pi 1 \gg 1$ is satisfied, we automatically satisfies $\Pi 2 \gg 1$.

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4.3. Kinematic-wave routing

The momentum equation for kinematic-wave routing reduced to:

$$S_0 = S_f$$

In the lecture notes for steady uniform open-channel flow, we have introduced the Chezy and Manning's equation:

$$Q = A \cdot C\sqrt{R_h}\sqrt{S_0}$$
$$Q = A \cdot \frac{1}{n} {R_h}^{2/3}\sqrt{S_0}$$

Both equation suggest that Q is proportional to the square root of slope, so we can write:

$$Q = D\sqrt{S_0} \tag{4.17}$$

where D has the dimension of discharge and is only a function of water depth h. Eq. (4.17) requires that flow always corresponds to normal flow, but since Q changes with x and t, the (normal) flow in the channel varies with x and t. Thus:

$$\frac{\partial Q}{\partial t} = \frac{\partial Q}{\partial A}\frac{\partial A}{\partial t} \tag{4.18}$$

The continuity equation is:

$$\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = 0$$

Use Eq. (4.18), we can eliminate the $\partial A/\partial t$ term in the continuity equation:

$$\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = \frac{\partial Q}{\partial x} + \left(\frac{\partial Q}{\partial A}\right)^{-1}\frac{\partial Q}{\partial t} = 0$$

Define:

$$c_k = \frac{\partial Q}{\partial A} \tag{4.19}$$

This c_k is the speed of kinematic wave, and is positive since Q increase with A. This is also known as the Klietz-Seddon law and agrees reasonably with observation.

The continuity equation finally becomes:

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$$\frac{\partial Q}{\partial t} + c_k \frac{\partial Q}{\partial x} = 0 \tag{4.20}$$

Eq. (4.20) can be written as:

$$\frac{\partial Q}{\partial t} + \frac{dx}{dt}\frac{\partial Q}{\partial x} = 0 \tag{4.21}$$

if:

$$\frac{dx}{dt} = c_k$$

This is the definition of characteristics. If c_k is a constant, these characteristics are straight lines with positive slope, i.e. C+ characteristics. There is no C- family of characteristics in this problem, as all disturbance propagates downstream. For an observer who moves at a speed c_k, the flood wave will look stationary. For an observer sitting on the river bank, the flood wave travels downstream **without changing shape**, as shown in Figure 21, which is usually called "translation" of kinematic wave. In reality, a flood wave is usually observed to attenuated, which is due to **diffusion**. A kinematic wave is based on first-order differential equation, so diffusion is **analytically impossible** (but numerically possible, see the Muskingum-Cunge method in next section).

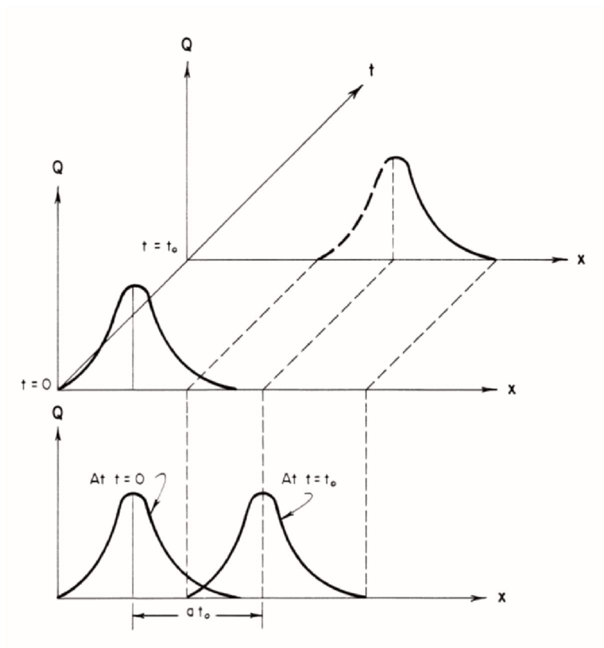

Figure 21 propagation of kinematic wave

If the initial condition is given by:

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$$Q_0 = f_0(x) \tag{4.22}$$

Thus solution at time t is:

$$Q(x,t) = f_0(x - c_k t) \tag{4.23}$$

These conclusion are based on the assumption that c_k is constant. However, in reality c_k may vary with discharge Q. It can be shown from Eq. (4.19) that the ratio γ of the kinematic-wave velocity to the flow velocity is:

$$\gamma = \frac{c_k}{U} = \frac{A}{b_s D}\frac{\partial D}{\partial h} \tag{4.24}$$

where b_s is the width of free surface, A is the cross-section area.

Proof:

$$c_k = \frac{\partial Q}{\partial A} = \sqrt{S_0}\frac{\partial D}{\partial A} = \sqrt{S_0}\frac{\partial D}{\partial h}\frac{\partial h}{\partial A} = \sqrt{S_0}\frac{\partial D}{\partial h}\left(\frac{\partial A}{\partial h}\right)^{-1} = \frac{\sqrt{S_0}}{b_s}\frac{\partial D}{\partial h}$$
$$U = \frac{Q}{A} = \frac{D\sqrt{S_0}}{A}$$

Thus:

$$\gamma = \frac{c_k}{U} = \frac{\dfrac{\sqrt{S_0}}{b_s}\dfrac{\partial D}{\partial h}}{D\sqrt{S_0}\,/\,A} = \frac{A}{b_s D}\frac{\partial D}{\partial h}$$

Assume very wide rectangular channel:

$$Chezy: D = C \cdot bh\sqrt{h}, \quad \frac{\partial D}{\partial h} = \frac{3}{2}Cb\sqrt{h}$$
$$Manning: D = b\frac{1}{n}h^{5/3}, \quad \frac{\partial D}{\partial h} = \frac{5}{3}\frac{1}{n}bh^{2/3}$$

Note $\partial D/\partial h$ is a function of h, so c_k is not really a constant! We can only approximately take it as a constant is water depth changes slightly. Thus:

$$\gamma = \frac{c_k}{U} = \frac{A}{D}\frac{\partial D}{\partial h} = \frac{bh}{b \cdot Cbh\sqrt{h}} \cdot \frac{3Cb}{2}\sqrt{h} = \frac{3}{2}, \text{ for Chezy}$$
$$\gamma = \frac{c_k}{U} = \frac{A}{D}\frac{\partial D}{\partial h} = \frac{bh}{b \cdot b\frac{1}{n}h^{5/3}}\frac{5}{3}\frac{1}{n}bh^{2/3} = \frac{5}{3}, \text{ for Manning}$$

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This suggests that the speed of flood wave is proportional to the flow velocity. For a rising limp of the flood, the peak travels faster than the front, so the **kinematic wave profile becomes steepen**, until it becomes a *kinematic shock*.

Kinematic wave vs. Muskingum method

The kinematic-wave routing and the Muskingum routing both are based on the continuity equation. However, kinematic-wave method gives a flood wave traveling with permanent form, while the Muskingum routing allows wave attenuation. This difference is mainly due to numerical diffusion associated with the finite difference method in deriving the Muskingum method, i.e. Eq. (4.8) (proof is given later). Actually, Eq. (4.20) and Eq. (4.8) are related as follows. Let us discretize Eq. (4.20) for a channel reach of length Δx and time interval Δt, as shown in Figure 22. We apply a weighted finite central difference for the time derivative in Eq. (4.20) and a non-weighted central difference for the spatial derivative:

$$\frac{\partial Q}{\partial t} = \frac{X(I_2 - I_1) + (1-X)(O_2 - O_1)}{\Delta t}$$

$$\frac{\partial Q}{\partial x} = \frac{(O_1 - I_1) + (O_2 - I_2)}{2\Delta x}$$

where I_1 and I_2 are inflow discharges and O_1 and O_2 are out flow discharges.

Eq. (4.20) is discretized to:

$$\frac{X(I_2 - I_1) + (1-X)(O_2 - O_1)}{\Delta t} + c_k \frac{(O_1 - I_1) + (O_2 - I_2)}{2\Delta x} = 0 \tag{4.25}$$

This can be re-written as:

$$\frac{I_1 + I_2}{2} - \frac{O_1 + O_2}{2} = \frac{\left(\Delta x / c_k\right)[XI_2 + (1-X)O_2] - \left(\Delta x / c_k\right)[XI_1 + (1-X)O_1]}{\Delta t}$$

Eq. (4.8) is:

$$\frac{I_1 + I_2}{2} - \frac{O_1 + O_2}{2} = \frac{K[XI_2 + (1-X)O_2] - K[XI_1 + (1-X)O_1]}{\Delta t}$$

Compare these two equations, you can easily see that they are equivalent if:

$$K = \frac{\Delta x}{c_k} \tag{4.26}$$

This means that the routing coefficient K in Muskingum routing is just the time for the flow wave to travel over the channel reach: $T=\Delta x/c_k$.

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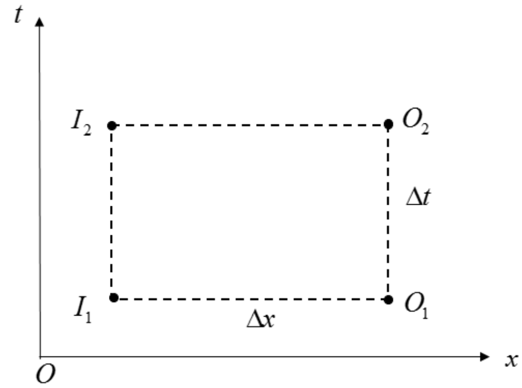

Figure 22 Rectangular mesh for space-time discretization

Kinematic wave vs. Lagrangian wave

The kinematic wave only travels downstream, and its velocity is determined by:

$$c_k = \gamma U$$

We have seen that a small surface wave (recall the lecture on moving hydraulic jump) can travels both in upstream and downstream directions, and the celebrities are:

$$c = U \pm \sqrt{gh}$$

This wave is first proposed by Lagrange, so we call them Lagrangian waves here. How do we understand the difference between kinematic and Lagrangian waves?

The simple answer is that the kinematic "wave" and Lagrangian "wave", although both called waves, are not equivalent concepts. For kinematic wave, the dominate forces are friction and gravity. For this to be true, the length and time scales of the wave must be very long, so that the inertial and pressure terms in Eq. (4.12) can be neglected. Typically, the length scale is hundreds of kilometers and time scale is a few days. Thus, kinematic waves are HUGE. The Lagrangian waves that can travel in two directions are primarily controlled by the balance between inertial and pressure terms in Eq. (4.12). For this to be true, the wave must be short in terms of both length and time scale, e.g. a few hours and a few hundreds of meters (like our simple-wave problems). Therefore, the Lagrangian wave are much smaller.

Both lagrangian and kinematic waves are conceptual waves following certain simplification of the momentum equations. Therefore, a dynamic wave, which is given by solving the entire momentum equation, may have the feature of both. If we are talking about flood wave that with a massive length and time scale, we surely expect that the key behavior of the flood wave is close to that of a kinematic wave. However, if we are talking about a surface ripple wave, then Lagrangian wave should be closer to reality.

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Applicably of kinematic wave:

Following Eq. (4.15), Ponce (1989) gave the following criteria for applying kinematic wave:

$$\frac{T_0 U_0 S_0}{h_0} > 85 \tag{4.27}$$

where T_0 is the time of rise of the inflow hydrograph, U_0 and h_0 are velocity and depth for normal flow and S_0 is slope.

4.4. Diffusion-wave routing

The momentum equation for diffusion wave is (see Figure 20):

$$S_f = S_0 - \frac{\partial h}{\partial x} \tag{4.28}$$

Recall Eq. (2.29) in the note "Steady open-channel flow":

$$S_f = \begin{cases} \dfrac{f}{8g}\dfrac{P}{A^3}Q^2 \left(Darcy - Weisbach\right) \\ \dfrac{1}{C^2}\dfrac{P}{A^3}Q^2 \left(Chezy\text{'}s\ Equation\right) \\ n^2 \dfrac{P^{4/3}}{A^{10/3}}Q^2 \left(Manning\text{'}s\ Equation\right) \end{cases}$$

We can see that:

$$S_f = Q^2 / D^2 \tag{4.29}$$

where

$$D^2(h) = \begin{cases} \left[\dfrac{f}{8g}\dfrac{P(h)}{A(h)^3}\right]^{-1} \left(Darcy - Weisbach\right) \\ \left[\dfrac{1}{C^2}\dfrac{P(h)}{A(h)^3}\right]^{-1} \left(Chezy\text{'}s\ Equation\right) \\ \left[n^2 \dfrac{P(h)^{4/3}}{A(h)^{10/3}}\right]^{-1} \left(Manning\text{'}s\ Equation\right) \end{cases}$$

Thus, Eq. (4.28) becomes:

$$\frac{Q^2}{D^2} = S_0 - \frac{\partial h}{\partial x} \tag{4.30}$$

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Differentiation of the continuity equation with respect to x and Eq. (4.30) with respect to t and use $dA=b_s dh$:

$$\frac{1}{b_s}\frac{\partial^2 Q}{\partial x^2}+\frac{\partial^2 h}{\partial x \partial t}=0$$
$$\frac{2Q}{D^2}\frac{\partial Q}{\partial x}-\frac{2Q^3}{D^3}\frac{dD}{dA}\frac{\partial A}{\partial t}=-\frac{\partial^2 h}{\partial x \partial t}$$

Eliminate the $\partial^2 h/\partial x \partial t$ term between the above two equations, and substitute for $\partial A/\partial t=-\partial Q/\partial x$, we get:

$$\frac{\partial Q}{\partial t}+c_f\frac{\partial Q}{\partial x}=\mu\frac{\partial^2 Q}{\partial x^2} \tag{4.31}$$

where:

$$c_f=\frac{Q}{D}\frac{dD}{dA} \tag{4.32}$$

$$\mu=\frac{D^2}{2b_s Q}\approx\frac{Q}{2b_s S_0} \tag{4.33}$$

Eq. (4.31) is a classical convention-diffusion equation with known analytical solution, if c_f and μ are constants. The second-x-derivative term in Eq. (4.31) accounts for the diffusion of a flood wave. The ratio of diffusion-wave celerity to flow velocity is:

$$\gamma=\frac{c_f}{U}=\frac{Q}{D}\frac{dD}{dA}/(Q/A)=\frac{A}{D}\frac{dD}{dA}=\frac{A}{D}\frac{dD}{dh}\left(\frac{\partial h}{\partial A}\right)=\frac{A}{D}\frac{dD}{dh}b_s^{-1}=\frac{A}{b_s D}\frac{dD}{dh} \tag{4.34}$$

Although this is identical to Eq. (4.24), the diffusion-wave celerity is not necessarily equal to the kinematic-wave celerity, because the flow velocity U in this two routing method are based on different momentum equations. However, the difference is negligible in most cases, so $c_f=c_k$. The diffusivity μ apparently increases with Q, but decreases with S_0 and b_s, meaning that diffusion will be more severe if the slope is close to horizontal, the river is very narrow, or the discharge is very large.

The Muskingum-Cunge Method

Cunge (1969) showed that the equation for Muskingum method, Eq. (4.9), is just a finite difference representation of the diffusion-wave equation, Eq. (4.31), when:

$$\mu=c_f\Delta x(1/2-X) \tag{4.35}$$

and:

$$c_f=\frac{\Delta x}{K} \tag{4.36}$$

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Thus, instead of using past flood data, Eq. (4.35) and (4.36) can be used to determine the routing parameters. From Eq. (4.33) to (4.35):

$$X = \frac{1}{2}\left(1 - \frac{D^2}{\Delta x c_f b_s Q}\right) \tag{4.37}$$

and from Eq. (4.36):

$$K = \frac{\Delta x}{c_f} \tag{4.38}$$

where $c_f = c_k$ can be given by the flow velocity and the ratio $\gamma = c_k/U$ (the example for very wide rectangular channel is given before). In most practical application the routing parameters are assumed to be constant at some reference discharge Q_0 that is taken as either the average or the peak discharge. Eq. (4.37) and (4.38) at the reference discharge can be written as:

$$X = \frac{1}{2}\left(1 - \frac{Q_0}{\gamma U_0 S_0 b_{s0} \Delta x}\right) \tag{4.39}$$

$$K = \frac{\Delta x}{\gamma U_0} \tag{4.40}$$

<u>Why Muskingum-Cunge method is equivalent to a diffusion-wave method?</u>

We start with the governing equation for kinematic wave:

$$\frac{\partial Q}{\partial t} + c_k \frac{\partial Q}{\partial x} = 0 \tag{4.41}$$

We discrete the time derivative as follows:

$$\frac{\partial Q}{\partial t} = \frac{X(Q_i^{k+1} - Q_i^k) + (1 - X)\left(Q_{i+1}^{k+1} - Q_{i+1}^k\right)}{\Delta t} \tag{4.42}$$

and the x-derivative:

$$c_k \frac{\partial Q}{\partial x} = c_k \frac{Y(Q_{i+1}^k - Q_i^k) + (1 - Y)\left(Q_{i+1}^{k+1} - Q_i^{k+1}\right)}{\Delta x} \tag{4.43}$$

Define:

$$c_N = \frac{c_k \Delta t}{\Delta x} \tag{4.44}$$

`

Eq, (4.43) becomes:

$$c_k \frac{\partial Q}{\partial x} = c_N \frac{Y(Q_{i+1}^k - Q_i^k) + (1-Y)\left(Q_{i+1}^{k+1} - Q_i^{k+1}\right)}{\Delta t} \tag{4.45}$$

Combine Eq. (4.42) and (4.45):

$$\frac{X(Q_i^{k+1} - Q_i^k) + (1-X)\left(Q_{i+1}^{k+1} - Q_{i+1}^k\right)}{\Delta t} + c_N \frac{Y(Q_{i+1}^k - Q_i^k) + (1-Y)\left(Q_{i+1}^{k+1} - Q_i^{k+1}\right)}{\Delta t} = 0 \tag{4.46}$$

Re-arrange terms, we get:

$$Q_{i+1}^{k+1} = C_0 Q_i^{k+1} + C_1 Q_i^k + C_2 Q_{i+1}^k \tag{4.47}$$

where:

$$\begin{aligned} C_0 &= \frac{-X + c_N(1-Y)}{1 - X + c_N(1-Y)} \\ C_1 &= \frac{X + c_N Y}{1 - X + c_N(1-Y)} \\ C_2 &= \frac{(1-X) - c_N Y}{1 - X + c_N(1-Y)} \end{aligned} \tag{4.48}$$

Eq. (4.47) is equivalent to Eq. (4.9):

$$Q_{i+1}^{k+1} \sim O_2, \quad Q_i^{k+1} \sim I_2, \quad Q_i^k \sim I_{1,} \quad Q_{i+1}^k \sim O_1$$

Compare Eq. (4.48) with Eq. (4.10):

$$\begin{aligned} C_0 &= \frac{\Delta t - 2KX}{\Delta t + 2K(1-X)} \\ C_1 &= \frac{\Delta t + 2KX}{\Delta t + 2K(1-X)} \\ C_2 &= \frac{2K(1-X) - \Delta t}{\Delta t + 2K(1-X)} \end{aligned}$$

We can see that these two sets of constants are equivalent if:

$$Y = 0.5$$

and:

$$K = \frac{\Delta x}{c_K}$$

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Thus, the Muskingum method is just a finite-difference method for kinematic-wave routing, and we have proved why $K=\Delta x/c_k$. Now, the remaining question is to prove that if using Cunge's parameterization for X, the numerical diffusion will be equal to the actual diffusion.

The numerical diffusion comes from truncation error. Here we first express Q_{i+1}^{k+1}, Q_i^{k+1} and Q_{i+1}^{k} in terms of the Taylor expansion evaluated at Q_i^k to the second order, e.g.:

$$f(x+\Delta x, y+\Delta y) = f(x,y) + \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\Delta x^2 + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}\Delta y^2 + \frac{\partial^2 f}{\partial x \partial y}\Delta x \Delta y + O(\Delta x^3, \Delta y^3)$$

Q_{i+1}^k can be written as:

$$Q_{i+1}^k = Q_i^k + \frac{\partial Q}{\partial x}\Delta x + \frac{1}{2}\frac{\partial^2 Q}{\partial x^2}\Delta x^2 \tag{4.49}$$

Q_i^{k+1} can be written as:

$$Q_i^{k+1} = Q_i^k + \frac{\partial Q}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 Q}{\partial t^2}\Delta t^2 \tag{4.50}$$

Q_{i+1}^{k+1} can be written as:

$$Q_{i+1}^{k+1} = Q_i^k + \frac{\partial Q}{\partial t}\Delta t + \frac{\partial Q}{\partial x}\Delta x + \frac{1}{2}\frac{\partial^2 Q}{\partial x^2}\Delta x^2 + \frac{1}{2}\frac{\partial^2 Q}{\partial t^2}\Delta t^2 + \frac{\partial^2 Q}{\partial x \partial t}\Delta t \Delta x \tag{4.51}$$

Substitute Eq. (4.49), (4.50) and (4.50) into Eq. (4.42):

$$\begin{aligned}
&\frac{X(Q_i^{k+1} - Q_i^k) + (1-X)\left(Q_{i+1}^{k+1} - Q_{i+1}^k\right)}{\Delta t} \\
&= X\left[\frac{\partial Q}{\partial t} + \frac{1}{2}\frac{\partial^2 Q}{\partial t^2}\Delta t\right] + (1-X)\left[\frac{\partial Q}{\partial t} + \frac{1}{2}\frac{\partial^2 Q}{\partial t^2}\Delta t + \frac{\partial^2 Q}{\partial t \partial x}\Delta x\right] \\
&= \frac{\partial Q}{\partial t} + \frac{1}{2}\frac{\partial^2 Q}{\partial t^2}\Delta t + (1-X)\frac{\partial^2 Q}{\partial t \partial x}\Delta x
\end{aligned} \tag{4.52}$$

we want to convert the $\partial^2 Q/\partial^2 t$ and $\partial^2 Q/\partial t \partial x$ terms to $\partial^2 Q/\partial^2 x$. Take t-derivative of Eq. (4.41):

$$\frac{\partial^2 Q}{\partial t^2} + c_k \frac{\partial^2 Q}{\partial x \partial t} = 0$$

Thus:

$$\frac{\partial^2 Q}{\partial t^2} = -c_k \frac{\partial^2 Q}{\partial x \partial t} = -c_k \frac{\partial}{\partial x}\left(\frac{\partial Q}{\partial t}\right) = -c_k \frac{\partial}{\partial x}\left(-c_k \frac{\partial Q}{\partial x}\right) = {c_k}^2 \frac{\partial Q}{\partial x} \tag{4.53}$$

`

Take x-derivative of Eq. (4.41):

$$\frac{\partial^2 Q}{\partial x \partial t} + c_k \frac{\partial^2 Q}{\partial x^2} = 0$$

so:

$$\frac{\partial^2 Q}{\partial x \partial t} = -c_k \frac{\partial^2 Q}{\partial x^2} \tag{4.54}$$

We can further express Δt as:

$$\Delta t = c_N \Delta x / c_k \tag{4.55}$$

Thus, Eq. (4.52) becomes:

$$\begin{aligned} &\frac{X(Q_i^{k+1} - Q_i^k) + (1-X)\left(Q_{i+1}^{k+1} - Q_{i+1}^k\right)}{\Delta t} \\ &= \frac{\partial Q}{\partial t} + \frac{1}{2}\frac{\partial^2 Q}{\partial t^2}\Delta t + (1-X)\frac{\partial^2 Q}{\partial t \partial x}\Delta x \\ &= \frac{\partial Q}{\partial t} + \frac{1}{2} c_N c_k \frac{\partial^2 Q}{\partial x^2}\Delta x - c_k (1-X)\frac{\partial^2 Q}{\partial x^2}\Delta x \end{aligned} \tag{4.56}$$

Substitute Eq. (4.49), (4.50) and (4.50) into Eq. (4.43):

$$\begin{aligned} &c_k \frac{Y(Q_{i+1}^k - Q_i^k) + (1-Y)\left(Q_{i+1}^{k+1} - Q_i^{k+1}\right)}{\Delta x} \\ &= c_k Y\left[\frac{\partial Q}{\partial x} + \frac{1}{2}\frac{\partial^2 Q}{\partial x^2}\Delta x\right] + c_k (1-Y)\left[\frac{\partial Q}{\partial x} + \frac{1}{2}\frac{\partial^2 Q}{\partial x^2}\Delta x - \frac{\partial^2 Q}{\partial x \partial t}\Delta t\right] \\ &= c_k \left[\frac{\partial Q}{\partial x} + \frac{1}{2}\frac{\partial^2 Q}{\partial x^2}\Delta x\right] - c_k c_N (1-Y)\frac{\partial^2 Q}{\partial x^2}\Delta x \end{aligned} \tag{4.57}$$

Add Eq. (4.57) to (4.56):

$$\begin{aligned} &\frac{X(Q_i^{k+1} - Q_i^k) + (1-X)\left(Q_{i+1}^{k+1} - Q_{i+1}^k\right)}{\Delta t} + c_k \frac{Y(Q_{i+1}^k - Q_i^k) + (1-Y)\left(Q_{i+1}^{k+1} - Q_i^{k+1}\right)}{\Delta x} \\ &= \frac{\partial Q}{\partial t} + \frac{1}{2} c_N c_k \frac{\partial^2 Q}{\partial x^2}\Delta x - c_k (1-X)\frac{\partial^2 Q}{\partial x^2}\Delta x + c_k\left[\frac{\partial Q}{\partial x} + \frac{1}{2}\frac{\partial^2 Q}{\partial x^2}\Delta x\right] - c_k c_N (1-Y)\frac{\partial^2 Q}{\partial x^2}\Delta x \\ &= \left(\frac{\partial Q}{\partial t} + c_k \frac{\partial Q}{\partial x}\right) + c_k \frac{\partial^2 Q}{\partial x^2}\Delta x\left[\frac{1}{2}c_N - (1-X) + \frac{1}{2} - c_N (1-Y)\right] \\ &= \left(\frac{\partial Q}{\partial t} + c_k \frac{\partial Q}{\partial x}\right) + c_k \frac{\partial^2 Q}{\partial x^2}\Delta x\left[\left(X - \frac{1}{2}\right) + c_N\left(Y - \frac{1}{2}\right)\right] \end{aligned} \tag{4.58}$$

`

or:

$$\frac{X(Q_i^{k+1}-Q_i^k)+(1-X)\left(Q_{i+1}^{k+1}-Q_{i+1}^k\right)}{\Delta t}+c_k\frac{Y(Q_{i+1}^k-Q_i^k)+(1-Y)\left(Q_{i+1}^{k+1}-Q_i^{k+1}\right)}{\Delta x}$$
$$=\frac{\partial Q}{\partial t}+c_k\frac{\partial Q}{\partial x}+c_k\frac{\partial^2 Q}{\partial x^2}\Delta x\left[\left(X-\frac{1}{2}\right)+c_N\left(Y-\frac{1}{2}\right)\right] \tag{4.59}$$

Substitute Eq. (4.59) to (4.46), we see that the discretized kinematic-wave equation is equivalent to:

$$\frac{\partial Q}{\partial t}+c_k\frac{\partial Q}{\partial x}=-c_k\Delta x\left[\left(X-\frac{1}{2}\right)+c_N\left(Y-\frac{1}{2}\right)\right]\frac{\partial^2 Q}{\partial x^2} \tag{4.60}$$

For Muskingum method Y=1/2:

$$\frac{\partial Q}{\partial t}+c_k\frac{\partial Q}{\partial x}=c_k\Delta x\left[\left(\frac{1}{2}-X\right)\right]\frac{\partial^2 Q}{\partial x^2} \tag{4.61}$$

The right-hand side is a truncation error that is proportional to $\partial^2 Q/\partial^2 x$, so it can be equivalent to the diffusion term in Eq. (4.31), if:

$$\mu=c_k\left(\frac{1}{2}-X\right)\Delta x \tag{4.62}$$

If X>1/2, we have μ<0, this will leads to an increasingly peaky flood, which is against observation, so in Muskingum method, we require that X<1/2.