Section 1.2

> **Example.**
> The discharge in a channel with bottom width 3 m is $12 \text{ m}^3\text{ s}^{-1}$. If Manning's n is $0.013 \text{ m}^{-1/3}$ s and the streamwise slope is 1 in 200, find the normal depth if:
> (a)   the channel has vertical sides (i.e. rectangular channel);
> (b)   the channel is trapezoidal with side slopes 2H:1V.

$b = 3$ m (base width)
$Q = 12 \text{ m}^3 \text{ s}^{-1}$
$n = 0.013 \text{ m}^{-1/3}$ s
$S = 0.005$

(a)
Discharge:
$$Q = VA$$
where, in normal flow,
$$V = \frac{1}{n} R_h^{2/3} S^{1/2}, \qquad A = bh, \qquad R_h = \frac{bh}{b+2h} = \frac{h}{1+2h/b}$$
Hence,
$$Q = \frac{1}{n} \frac{bh^{5/3}}{(1+2h/b)^{2/3}} S^{1/2}$$
Rearranging as an iterative formula for h:
$$h = \left(\frac{nQ}{b\sqrt{S}} \right)^{3/5} (1+2h/b)^{2/5}$$

Here, with lengths in metres,
$$h = 0.8316(1+2h/3)^{2/5}$$
Iteration (from, e.g., $h = 0.8316$) gives
$$h_n = 1.024 \text{ m}$$

Answer: normal depth = 1.02 m.

(b) Geometry: trapezoidal cross-section with base width b, surface width $b + 2\times(2h)$ and two sloping side lengths $\sqrt{h^2 + (2h)^2} = h\sqrt{5}$.

Area and wetted perimeter:
$$A = \tfrac{1}{2}(b + b + 4h)h = h(b+2h) = hb(1+2h/b)$$
$$P = b + 2h\sqrt{5}$$
Hydraulic radius:
$$R_h \equiv \frac{A}{P} = \frac{h(b+2h)}{b+2h\sqrt{5}} = h\left(\frac{1+2h/b}{1+2\sqrt{5}h/b} \right)$$

Discharge:
$$Q = VA \quad = \frac{1}{n} R_h^{2/3} S^{1/2} A$$

Hence,
$$Q = \frac{1}{n} h^{2/3} \left(\frac{1 + 2h/b}{1 + 2\sqrt{5}h/b} \right)^{2/3} S^{1/2} hb(1 + 2h/b)$$

$$\Rightarrow \quad \frac{nQ}{b\sqrt{S}} = h^{5/3} \frac{(1 + 2h/b)^{5/3}}{(1 + 2\sqrt{5}h/b)^{2/3}}$$

$$\Rightarrow \quad h = \left(\frac{nQ}{b\sqrt{S}} \right)^{3/5} \frac{(1 + 2\sqrt{5}h/b)^{2/5}}{1 + 2h/b}$$

Here, with lengths in metres,
$$h = 0.8316 \frac{(1 + 1.491h)^{2/5}}{1 + 2h/3}$$
Iteration (from, e.g., $h = 0.8316$) gives
$$h_n = 0.7487 \text{ m}$$

Answer: normal depth = 0.749 m.

Section 1.4

Example.
The discharge in a rectangular channel of width 6 m with Manning's $n = 0.012 \text{ m}^{-1/3}$ s is $24 \text{ m}^3 \text{ s}^{-1}$. If the streamwise slope is 1 in 200 find:
(a)   the normal depth;
(b)   the Froude number at the normal depth;
(c)   the critical depth.

State whether the normal flow is subcritical or supercritical.

$b = 6 \text{ m}$
$n = 0.012 \text{ m}^{-1/3}$
$Q = 24 \text{ m}^3 \text{ s}^{-1}$
$S = 0.005$

(a)
Discharge:
$$Q = VA$$
where, in normal flow,
$$V = \frac{1}{n} R_h^{2/3} S^{1/2}, \qquad A = bh, \qquad R_h = \frac{bh}{b + 2h} = \frac{h}{1 + 2h/b}$$
Hence,
$$Q = \frac{1}{n} \frac{bh^{5/3}}{(1 + 2h/b)^{2/3}} S^{1/2}$$
or, rearranging as an iterative formula for h:
$$h = \left(\frac{nQ}{b\sqrt{S}} \right)^{3/5} (1 + 2h/b)^{2/5}$$

Here, with lengths in metres,
$$h = 0.7926(1 + h/3)^{2/5}$$
Iteration (from, e.g., $h = 0.7926$) gives
$$h_n = 0.8783 \text{ m}$$

Answer: normal depth = 0.878 m.

(b) At the normal depth, $h = 0.8783$ m:
$$V = \frac{Q}{A} = \frac{24}{6 \times 0.8783} = 4.554 \text{ m s}^{-1}$$
$$\text{Fr} \equiv \frac{V}{\sqrt{gh}} = \frac{4.554}{\sqrt{9.81 \times 0.8783}} = 1.551$$

Answer: Froude number = 1.55.

(c) The critical depth is that depth (at the given flow rate) for which Fr = 1. It is *not* normal flow, and does not depend on the slope S or the roughness n.

$$\text{Fr} = \frac{V}{\sqrt{gh}}$$

where

$$V = \frac{Q}{A} \qquad \text{(in general)}$$

or

$$V = \frac{Q}{bh} = \frac{q}{h} \qquad \text{(for a rectangular channel; } q \text{ is the flow per unit width)}$$

Hence, for a rectangular channel,

$$\text{Fr}^2 = \frac{(q/h)^2}{gh} = \frac{q^2}{gh^3}$$

For critical flow, Fr = 1 and so

$$h_c = \left(\frac{q^2}{g} \right)^{1/3}$$

Here, the flow per unit width is $q = 24/6 = 4 \text{ m}^2 \text{ s}^{-1}$, so that

$$h_c = \left(\frac{4^2}{9.81} \right)^{1/3} = 1.177 \text{ m}$$

Answer: critical depth = 1.18 m.

The normal depth is supercritical because, when $h = h_n$, then Fr > 1 (part (b)).

Alternatively (and often more conveniently), the normal depth is supercritical because $h_n < h_c$; then speed V is larger, and depth h is smaller in normal flow than critical flow, so that $\text{Fr} \equiv V/\sqrt{gh}$ must be greater than 1.

Section 2.2

Example.
A 3-m wide channel carries a total discharge of 12 m^3 s^{-1}. Calculate:
(a)   the critical depth;
(b)   the minimum specific energy;
(c)   the alternate depths when $E = 4$ m.

$b = 3$ m
$Q = 12$ m^3 s^{-1}

(a)
Discharge per unit width:
$$q = \frac{Q}{b} = \frac{12}{3} = 4 \text{ m}^2 \text{ s}^{-1}$$
Then, for a rectangular channel:
$$h_c = \left(\frac{q^2}{g}\right)^{1/3} = \left(\frac{4^2}{9.81}\right)^{1/3} = 1.177 \text{ m}$$

Answer: critical depth = 1.18 m.

(b) For a rectangular channel,
$$E_c = \frac{3}{2}h_c = \frac{3}{2} \times 1.177 = 1.766 \text{ m}$$

Answer: minimum specific energy = 1.77 m.

(c) As $E > E_c$, there are two possible depths for a given specific energy.
$$E \equiv h + \frac{V^2}{2g}, \quad \text{where} \quad V = \frac{Q}{A} = \frac{q}{h} \quad \text{(for a rectangular channel)}$$
$\Rightarrow \qquad E \equiv h + \frac{q^2}{2gh^2}$

Substituting values in metre-second units:
$$4 \equiv h + \frac{0.8155}{h^2}$$

For the *subcritical* (slow, deep) solution, the first term, associated with potential energy, dominates, so rearrange as:
$$h = 4 - \frac{0.8155}{h^2}$$
Iteration (from, e.g., $h = 4$) gives $h = 3.948$ m.

For the *supercritical* (fast, shallow) solution, the second term, associated with kinetic energy, dominates, so rearrange as:

$$h = \sqrt{\frac{0.8155}{4 - h}}$$

Iteration (from, e.g., $h = 0$) gives $h = 0.4814$ m.

Answer: alternate depths are 3.95 m and 0.481 m.

Section 2.3.1

Example. (Exam 2009 – reworded and extended)
(a)   Define specific energy and explain its relevance in determining critical conditions in a channel flow.

A long, wide channel has a slope of 1:1000, a Manning's n of 0.015 m$^{-1/3}$ s and a discharge of 5 m^3 s^{-1} per metre width.

(b)   Calculate the normal depth.
(c)   Calculate the critical depth.
(d)   In a region of the channel the bed is raised by a height of 0.5 m over a length sufficient for the flow to be parallel to the bed over this length. Determine the depths upstream, downstream and over the raised bed, ignoring any friction losses. Sketch the flow, including gradually-varied flow upstream and downstream.
(e)   In the same channel, the bed is lowered by 0.5 m from its original level. Again, determine the depths upstream, downstream and over the lowered bed, ignoring any friction losses. Sketch the flow.

(a) *Specific energy* is the head relative to the local bed of the channel and is given by

$$E = h + \frac{V^2}{2g}$$

(Compare the *total head*, where the potential-energy part is the level of the free surface relative to an absolute datum: $H = z_s + \frac{V^2}{2g}$.)

Critical conditions (where Fr = 1) correspond to a minimum of specific energy at the given discharge.

(b)
$S = 0.001$
$n = 0.015$ m$^{-1/3}$ s
$q = 5$ m^2 s^{-1}

Discharge per unit width:

$$q = Vh \quad \text{where} \quad V = \frac{1}{n} R_h^{2/3} S^{1/2} \text{ (Manning)}, \quad R_h = h \text{ ("wide" channel)}$$

$\Rightarrow \quad q = \frac{1}{n} h^{2/3} S^{1/2} h$

$\Rightarrow \quad q = \frac{h^{5/3} \sqrt{S}}{n}$

$\Rightarrow \quad h_n = \left(\frac{nq}{\sqrt{S}}\right)^{3/5} = \left(\frac{0.015 \times 5}{\sqrt{0.001}}\right)^{3/5} = 1.679 \text{ m}$

Answer: normal depth = 1.68 m.

(c) Critical depth:

$$h_c = \left(\frac{q^2}{g}\right)^{1/3} = \left(\frac{5^2}{9.81}\right)^{1/3} = 1.366 \text{ m}$$

Answer: critical depth = 1.37 m.

(d) To determine the type of behaviour over the raised bed, compare the total head under critical conditions (the minimum energy necessary to get over the weir at this flow rate) with that available in the approach flow.

<u>Critical</u>

$$h_c = 1.366 \text{ m}$$
$$E_c = \tfrac{3}{2} h_c = 2.049 \text{ m}$$
$$z_{weir} = 0.5 \text{ m}$$
$$H_c = z_{weir} + E_c = 2.549 \text{ m}$$

<u>Approach Flow</u>
Because the channel is described as "long" it will have sufficient fetch to develop normal flow; hence the approach-flow head is that for the normal depth ($h = 1.679$ m):

$$H_a = E_a = h_n + \frac{V_n^2}{2g}$$

$$= h_n + \frac{q^2}{2gh_n^2}$$

$$= 1.679 + \frac{5^2}{2 \times 9.81 \times 1.679^2}$$

$$= 2.131 \text{ m}$$

At the normal depth the available head (H_a) is less than the minimum required to get over the weir (H_c). Hence the water depth must increase upstream ("backing up"), to raise the head immediately upstream of the weir. Thus:
- critical conditions *do* occur;
- the total head through the device is the critical head ($H = H_c = 2.549$ m).

Over the weir there is a critical flow transition, so the depth here is critical: $h = h_c = 1.366$ m.

Just up- or downstream,

$$H = E = h + \frac{V^2}{2g} \qquad \text{where} \qquad V = \frac{q}{h}$$

$$\Rightarrow \qquad H = h + \frac{q^2}{2gh^2}$$

$$\Rightarrow \qquad 2.549 = h + \frac{1.274}{h^2}$$

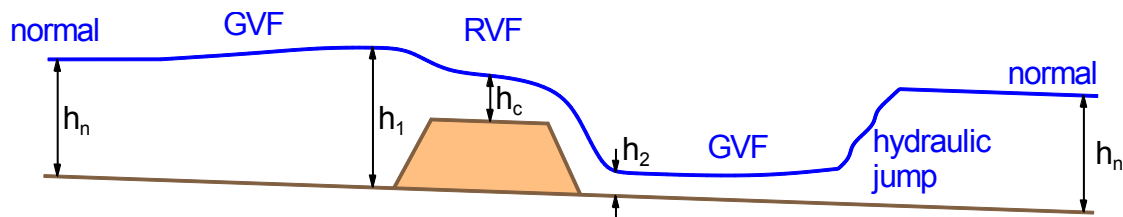Upstream, rearrange for the deep, subcritical, solution:
$$h = 2.549 - \frac{1.274}{h^2}$$
Iteration (from, e.g., $h = 2.549$) gives $h = 2.310$ m.

Downstream, rearrange for the shallow, supercritical solution:
$$h = \sqrt{\frac{1.274}{2.549 - h}}$$
Iteration (from, e.g., $h = 0$) gives $h = 0.8715$ m.

Answer: depths upstream, over, downstream of the weir: 2.31 m, 1.37 m, 0.871 m.

(e) The flow does not require additional energy to pass a depressed section; hence, the total head throughout is that supplied by the approach flow ($H = H_a = 2.131$ m) and the flow remains subcritical. The depths just upstream and downstream of the lowered section are those in the approach flow; i.e. normal depth.

As bed height z_b decreases, specific energy E must increase to maintain the same total head. In the lowered section:
$$H = z_b + E$$
$\Rightarrow \quad 2.131 = -0.5 + E$
$\Rightarrow \quad E = 2.631$ m

Then
$$E = h + \frac{V^2}{2g} \quad \text{where} \quad V = \frac{q}{h}$$
$$\Rightarrow \quad E = h + \frac{q^2}{2gh^2}$$
$$\Rightarrow \quad 2.631 = h + \frac{1.274}{h^2}$$

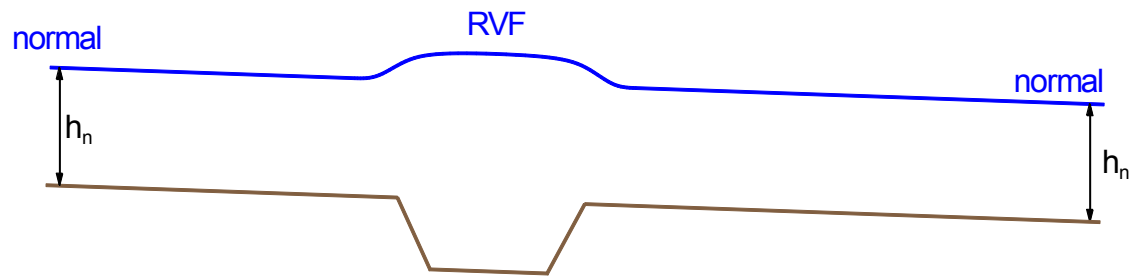As we require the subcritical solution, rearrange as
$$h = 2.631 - \frac{1.274}{h^2}$$
Iteration (from, e.g., $h = 2.631$) gives $h = 2.412$ m.

(Note that this is the depth of the water column. The actual surface level here is

$$z_s = -0.5 + h \quad = 1.912 \text{ m}$$

so the overall water level also rises in this section.)

Answer: depths upstream, within, downstream of the lowered section: 1.68 m, 2.41 m, 1.68 m.

Section 2.3.1

Example.
A long channel of rectangular cross-section with width 3.5 m and streamwise slope 1 in 800 carries a discharge of 15 m^3 s^{-1}. Manning's n may be taken as 0.016 m$^{-1/3}$ s. A broad-crested weir of height 0.7 m is constructed at the centre of the channel. Determine:
(a)   the depth far upstream of the weir;
(b)   the depth just upstream of the weir;
(c)   whether or not a region of supercritical gradually-varied flow exists downstream of
        the weir.

$b = 3.5$ m
$S = 0.00125$
$Q = 15$ m^3 s^{-1}
$n = 0.016$ m$^{-1/3}$ s
$z_{weir} = 0.7$ m

(a) The depth far upstream is normal since the channel is described as "long". For normal flow in a rectangular channel:
$$Q = VA$$
where:
$$V = \frac{1}{n} R_h^{2/3} S^{1/2}, \qquad A = bh, \qquad R_h = \frac{bh}{b+2h} = \frac{h}{1+2h/b}$$
Hence,
$$Q = \frac{1}{n} \frac{bh^{5/3}}{(1+2h/b)^{2/3}} S^{1/2}$$
or, rearranging as an iterative formula for h:
$$h = \left(\frac{nQ}{b\sqrt{S}} \right)^{3/5} (1+2h/b)^{2/5}$$

Here, with lengths in metres,
$$h = 1.488 (1+0.5714h)^{2/5}$$
Iteration (from, e.g., $h = 1.488$) gives
$$h = 2.023 \text{ m}$$

Answer: depth far upstream = 2.02 m.

(b) To establish depths near the weir we need to know the flow behaviour at the weir. Compare the energy in the approach flow with that under critical conditions.

Approach flow
$$h = 2.023 \text{ m} \quad \text{(from part (a))}$$
$$V = \frac{Q}{A} = \frac{Q}{bh} = \frac{15}{3.5 \times 2.023} = 2.118 \text{ m s}^{-1}$$
Specific energy in the approach flow:

$$E_a = h + \frac{V^2}{2g} = 2.023 + \frac{2.118^2}{2 \times 9.81} = 2.252 \text{ m}$$

Referring heads to the undisturbed bed near the weir:
$$H_a = E_a = 2.252 \text{ m}$$

Critical conditions
$$h_c = \left(\frac{q^2}{g}\right)^{1/3} \qquad \text{where} \qquad q = \frac{Q}{b} = \frac{15}{3.5} = 4.286 \text{ m}^2 \text{ s}^{-1}$$

$$\Rightarrow \qquad h_c = \left(\frac{4.286^2}{9.81}\right)^{1/3} = 1.233 \text{ m}$$

$$E_c = \frac{3}{2} h_c = \frac{3}{2} \times 1.233 = 1.850 \text{ m}$$

$$H_c = z_{weir} + E_c = 0.7 + 1.850 = 2.550 \text{ m}$$

Since the head required to flow over the weir ($H_c = 2.550$ m) exceeds that in the approach flow ($H_a = 2.252$ m), the depth just upstream of the weir must increase and the flow back up. The total head at any position in the vicinity of the weir is $H = H_c = 2.550$ m.

Just upstream and downstream of the weir (i.e. at undisturbed bed level):
$$H = E = h + \frac{V^2}{2g}, \qquad \text{with} \qquad V = \frac{Q}{bh}$$

$$\Rightarrow \qquad H = h + \frac{Q^2}{2gb^2 h^2}$$

$$\Rightarrow \qquad 2.550 = h + \frac{0.9362}{h^2} \qquad (*)$$

The depth just upstream is the deep, subcritical solution. Hence, rearrange as
$$h = 2.550 - \frac{0.9362}{h^2}$$

Iteration (from, e.g., $h = 2.550$) gives
$$h = 2.385 \text{ m}$$

Answer: depth just upstream of the weir = 2.39 m.

(c) Since the normal flow is subcritical, the flow must return to it via a hydraulic jump on the downstream side of the weir.

If the flow in the vicinity of the weir is unaffected by the hydraulic jump the flow goes smoothly supercritical on the downstream side, with total head $H = 2.550$ m (equation (*)). Rearranging to get an iterative formula for the supercritical solution:
$$h = \sqrt{\frac{0.9362}{2.550 - h}}$$

Iteration (from, e.g., $h = 0$) gives

$$h = 0.7141 \text{ m}$$

Denote by subscripts A and B respectively the conditions upstream and downstream of the hydraulic jump. On the downstream side conditions may be assumed normal, since the channel is "long" and hence there is sufficient fetch to develop the preferred depth:

$$h_B = 2.023 \text{ m}$$

$$V_B = 2.118 \text{ m s}^{-1} \quad \text{(from part (b))}$$

$$\text{Fr}_B = \frac{V_B}{\sqrt{gh_B}} \quad \frac{2.118}{\sqrt{9.81 \times 2.023}} = 0.4754$$

Hence, from the hydraulic-jump relation for the sequent depths:

$$h_A = \frac{h_B}{2}(-1 + \sqrt{1 + 8\text{Fr}_B^2}) \quad = \frac{2.023}{2}(-1 + \sqrt{1 + 8 \times 0.4754^2}) \quad = 0.6835 \text{ m}$$

Any gradually-varied supercritical flow downstream of the weir would increase in depth until a hydraulic jump occurred (see the lectures on GVF). Since the depth downstream of the weir is already greater than any sequent depth upstream of the hydraulic jump, no such increasing-depth GVF is possible and the hydraulic jump must actually occur at (or just before) the downstream end of the weir.

Section 2.3.1

> **Example.**
> A reservoir has a plan area of 50 000 m^2. The outflow passes over a broad-crested weir of
> width 8 m and discharge coefficient 0.9. Calculate:
> (a)   the discharge when the level in the reservoir is 0.6 m above the top of the weir;
> (b)   the time taken for the level of water in the reservoir to fall by 0.3 m.

(a)
Total head:

$$H = z_s + \frac{V^2}{2g} \quad = z_b + E$$

where levels z can be measured relative to any convenient datum. Relative to the top of the
weir, assuming constant head, still water in the reservoir and critical conditions over the weir:

head upstream = head over weir

$$h_0 = \frac{3}{2} \left(\frac{q^2}{g} \right)^{1/3}$$

Hence, in ideal flow,

$$q = g^{1/2} (2/3)^{3/2} h_0^{3/2}$$

$\Rightarrow \qquad Q_{ideal} = qb \quad = g^{1/2} (2/3)^{3/2} b h_0^{3/2} \quad = 1.705 b h_0^{3/2}$ (in metre-second units)

Representing non-ideal behaviour via a discharge coefficient c_d, and taking $b = 8$ m,

$$Q = c_d Q_{ideal} \quad = 0.9 \times 1.705 \times 8 h_0^{3/2}$$

$\Rightarrow \qquad Q = 12.28 h_0^{3/2}$

When $h_0 = 0.6$ m,

$$Q = 12.28 \times 0.6^{3/2} \quad = 5.707 \text{ m}^3 \text{ s}^{-1}$$

Answer: initial discharge $= 5.71$ m^3 s^{-1}.

(b) Drop the subscript 0 and write the freeboard as h.

Consider the change in volume of a tank, water surface area A_{ws}. When the water level
changes by dh the change in volume is

$$\text{d}(volume) = A_{ws} \text{d}h$$

Hence, by continuity,

$$\frac{\text{d}}{\text{d}t}(volume) = Q_{in} - Q_{out}$$

$\Rightarrow \qquad A_{ws} \frac{\text{d}h}{\text{d}t} = 0 - 12.28 h^{3/2}$

$\Rightarrow \qquad 50000 \frac{\text{d}h}{\text{d}t} = -12.28 h^{3/2}$

Separating variables,

$$50000 \frac{dh}{h^{3/2}} = -12.28 \, dt$$

Apply boundary conditions $h = 0.6$ when $t = 0$ and $h = 0.3$ when $t = T$, and integrate:

$$50000 \int_{0.6}^{0.3} h^{-3/2} \, dh = -12.28 \int_0^T dt$$

$$\Rightarrow \quad 50000 \left[\frac{h^{-1/2}}{(-1/2)} \right]_{0.6}^{0.3} = -12.28 T$$

$$\Rightarrow \quad \frac{50000 \times 2}{12.28} \left(\frac{1}{\sqrt{0.3}} - \frac{1}{\sqrt{0.6}} \right) = T$$

$$\Rightarrow \quad T = 4355 \text{ s}$$

Answer: time = 4360 s (about 73 min).

Section 2.3.2

> **Example** (Exam 2008 – modified, including a change in the value of *n*)
>
> A venturi flume is placed near the middle of a long rectangular channel with Manning's $n = 0.012$ m$^{-1/3}$ s. The channel has a width of 5 m, a discharge of 12.5 m^3 s^{-1} and a slope of 1:2500.
>
> (a)   Determine the critical depth and the normal depth in the main channel.
> (b)   Determine the venturi flume width which will just make the flow critical at the contraction.
> (c)   If the contraction width is 2 m find the depths just upstream, downstream and at the throat of the venturi flume (neglecting friction in this short section).
> (d)   Sketch the surface profile.

$n = 0.012$ m$^{-1/3}$ s
$b = 5$ m (main channel)
$Q = 12.5$ m^3 s^{-1}
$S = 4 \times 10^{-4}$

(a)
In the main channel,
$$q = \frac{Q}{b} = \frac{12.5}{5} = 2.5 \text{ m}^2 \text{ s}^{-1}$$

Critical Depth
$$h_c = \left(\frac{q^2}{g}\right)^{1/3} = \left(\frac{2.5^2}{9.81}\right)^{1/3} = 0.8605 \text{ m}$$

Normal Depth
$$Q = VA$$
where:
$$V = \frac{1}{n} R_h^{2/3} S^{1/2}, \qquad A = bh, \qquad R_h = \frac{bh}{b+2h} = \frac{h}{1+2h/b}$$
Hence,
$$Q = \frac{1}{n} \frac{bh^{5/3}}{(1+2h/b)^{2/3}} S^{1/2}$$
or, rearranging as an iterative formula for *h*:
$$h = \left(\frac{nQ}{b\sqrt{S}}\right)^{3/5} (1+2h/b)^{2/5}$$

Here, with lengths in metres,
$$h = 1.275(1+0.4h)^{2/5}$$
Iteration (from, e.g., $h = 1.275$) gives
$$h_n = 1.546 \text{ m}$$

Answer: critical depth $= 0.860$ m;   normal depth $= 1.55$ m.

(b) The flow will just go critical if the head in the throat (H_c) is exactly equal to that in the approach flow (H_a). Measure heads relative to the bed of the channel in the vicinity of the venturi.

Critical Head

$$H_c = E_c \quad = \frac{3}{2} h_c \quad = \frac{3}{2} \left(\frac{q_m^2}{g} \right)^{1/3} \quad \text{where} \quad q_m = \frac{Q}{b_m}$$

(Note that the critical depth is different at the throat to that in the main channel, due to the narrower width.)

$$\Rightarrow \quad H_c = \frac{3}{2} \left(\frac{Q^2}{g b_m^2} \right)^{1/3}$$

Approach Flow
The approach flow is normal, since the channel is "long". Hence,

$$h_a = 1.546 \text{ m}$$

$$V_a = \frac{Q}{b h_a} \quad = \frac{12.5}{5 \times 1.546} \quad = 1.617 \text{ m s}^{-1}$$

$$H_a = E_a \quad = h_a + \frac{V_a^2}{2g} \quad = 1.546 + \frac{1.617^2}{2 \times 9.81} \quad = 1.679 \text{ m}$$

For the flow just to go critical at the throat,

$$H_c = H_a$$

$$\Rightarrow \quad \frac{3}{2} \left(\frac{Q^2}{g b_m^2} \right)^{1/3} = 1.679$$

$$\Rightarrow \quad b_m = \frac{12.5}{\left(\frac{2}{3} \times 1.679 \right)^{3/2} \sqrt{9.81}} \quad = 3.370 \text{ m}$$

Answer: throat width = 3.37 m.

(c) If the throat width is reduced further, then the flow will back up and undergo a critical transition at the throat.

At the throat,

$$q_m = \frac{Q}{b_m} \quad = \frac{12.5}{2} \quad = 6.25 \text{ m}^2 \text{ s}^{-1}$$

$$h_c = \frac{3}{2} \left(\frac{q_m^2}{g} \right)^{1/3} \quad = \left(\frac{6.25^2}{9.81} \right)^{1/3} \quad = 1.585 \text{ m}$$

$$E_c = \tfrac{3}{2} h_c \quad = \tfrac{3}{2} \times 1.585 \quad = 2.378 \text{ m}$$

$$H_c = z_b + E_c \quad = 0 + 2.378 \quad = 2.378 \text{ m}$$

The head throughout the venturi will be the critical head ($H = H_c = 2.378$ m).

Anywhere in the flume,
$$H = z_s + \frac{V^2}{2g} \qquad \text{where} \qquad z_s = h, \qquad V = \frac{Q}{bh}$$
$$\Rightarrow \qquad H = h + \frac{Q^2}{2gb^2h^2}$$

At the throat the depth will be the critical depth there; i.e. $h = h_c = 1.585$ m.

Just upstream and downstream, $b = 5$ m; hence,
$$2.378 = h + \frac{0.3186}{h^2}$$

Upstream
Rearrange for the deep, subcritical solution:
$$h = 2.378 - \frac{0.3186}{h^2}$$
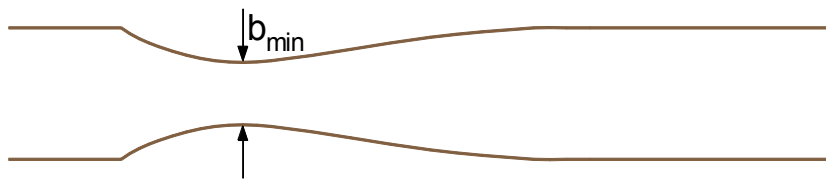Iteration (from, e.g., $h = 2.378$) gives $h = 2.319$ m.

Downstream
Rearrange for the shallow, supercritical solution:
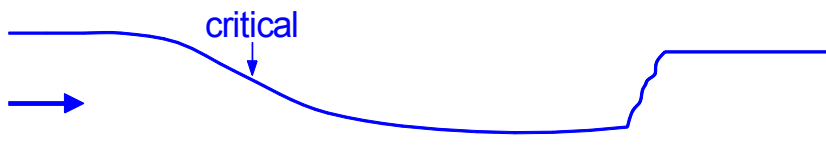$$h = \sqrt{\frac{0.3186}{2.378 - h}}$$
Iteration (from, e.g., $h = 0$) gives $h = 0.4015$ m.

Answer: depths upstream, in the throat, downstream = 2.32 m, 1.59 m, 0.401 m.

(d)
PLAN VIEW

WATER PROFILE

Section 2.3.3

Example.
The water depth upstream of a sluice gate is 0.8 m and the depth just downstream (at the vena contracta) is 0.2 m. Calculate:
(a) the discharge per unit width;
(b) the Froude numbers upstream and downstream.

$h_1 = 0.8$ m
$h_2 = 0.2$ m

(a) Assuming total head the same on either side of the gate:
$$z_{s1} + \frac{V_1^2}{2g} = z_{s2} + \frac{V_2^2}{2g}$$

Substituting $z_s = h$ and $V = \frac{q}{h}$:
$$h_1 + \frac{q^2}{2gh_1^2} = h_2 + \frac{q^2}{2gh_2^2}$$

From the given data, in metre-second units:
$$0.8 + 0.0796q^2 = 0.2 + 1.2742q^2$$
$$\Rightarrow \qquad 0.6 = 1.1946q^2$$
$$\Rightarrow \qquad q = 0.7087 \text{ m}^2 \text{ s}^{-1}$$

Answer: discharge per unit width $= 0.709$ m^2 s^{-1}.

(b) Use, on each side of the gate,
$$V = \frac{q}{h}$$
$$\text{Fr} = \frac{V}{\sqrt{gh}}$$

to get
$$V_1 = 0.8859 \text{ m s}^{-1}$$
$$V_2 = 3.544 \text{ m s}^{-1}$$

and then
$$\text{Fr}_1 = 0.3162$$
$$\text{Fr}_2 = 2.530$$

Answer: Froude numbers upstream, downstream $= 0.316, 2.53$.

Section 2.3.3

Example.
A sluice gate controls the flow in a channel of width 2 m. If the discharge is 0.5 m³ s⁻¹ and the upstream water depth is 1.5 m, calculate the downstream depth and velocity.

$b = 2$ m
$Q = 0.5$ m³ s⁻¹
$h_1 = 1.5$ m

Use upstream conditions to get total head. Then, assuming no losses, find the supercritical flow with the same head.

Total head (either side):

$$H = z_s + \frac{V^2}{2g} \quad \text{where} \quad z_s = h \quad \text{and} \quad V = \frac{Q}{bh}$$

$$\Rightarrow \quad H = h + \frac{Q^2}{2gb^2h^2} \quad = h + \frac{3.186 \times 10^{-3}}{h^2}$$

The upstream depth $h_1 = 1.5$ m gives

$$H = 1.5 + \frac{3.186 \times 10^{-3}}{1.5^2} = 1.501 \text{ m} \quad \text{(dominated by } h_1\text{)}$$

Hence,

$$1.501 = h + \frac{3.186 \times 10^{-3}}{h^2}$$

Rearrange for the shallow, supercritical solution:

$$h = \sqrt{\frac{3.186 \times 10^{-3}}{1.501 - h}}$$

Iteration (from, e.g., $h = 0$) gives

$$h_2 = 0.04681 \text{ m}$$

$$V_2 = \frac{Q}{bh_2} = \frac{0.5}{2 \times 0.04681} = 5.341 \text{ m s}^{-1}$$

Answer: downstream depth = 0.0468 m;   velocity = 5.34 m s⁻¹.

Section 2.4

Example. (Exam 2007)
Water flows down a wide channel with a steep slope of 1:50 on to a mild slope of 1:1000. The flow rate is $0.645 \text{ m}^3 \text{ s}^{-1}$ per metre width and the Manning's n on both slopes is $0.015 \text{ m}^{-1/3}$ s. Blocks are placed just downstream of the change in slope to induce a hydraulic jump with a sequent depth equal to the normal depth of the smaller slope.

(a)   Making the wide channel assumption, determine the force per metre width which the blocks must impart.

(b)   If the blocks are not present, explain how the hydraulic jump will change position. Sketch the surface profile (but do not calculate the position of the jump).

$S_1 = 0.02$; $S_2 = 0.001$
$q = 0.645 \text{ m}^2 \text{ s}^{-1}$
$n = 0.015 \text{ m}^{-1/3}$ s

(a) Find depths (assumed normal) and velocities; then use the momentum principle to deduce the force on the blocks.

Normal flow:

$$ q = Vh \quad \text{where} \quad V = \frac{1}{n} R_h^{2/3} S^{1/2} \text{ (Manning)}, \quad R_h = h \text{ ("wide" channel)} $$

$$ \Rightarrow \quad q = \frac{1}{n} h^{2/3} S^{1/2} h $$

$$ \Rightarrow \quad q = \frac{h^{5/3} \sqrt{S}}{n} $$

$$ \Rightarrow \quad h = \left(\frac{nq}{\sqrt{S}} \right)^{3/5} $$

For the two slopes this gives
$h_1 = 0.2000$ m
$h_2 = 0.4913$ m

(Note that the critical depth is

$$ h_c = \left(\frac{q^2}{g} \right)^{1/3} = \left(\frac{0.645^2}{9.81} \right)^{1/3} = 0.3487 \text{ m} $$

This confirms that the normal flows are supercritical on the first slope and subcritical on the second; i.e. the first slope is mild and the second one steep.)

The corresponding velocities are deduced from $V = \dfrac{q}{h}$, whence:

$V_1 = 3.225 \text{ m s}^{-1}$
$V_2 = 1.313 \text{ m s}^{-1}$

Let f be the *magnitude* of the force per unit width exerted by the fluid on the blocks and, by reaction, the blocks on the fluid. The force on the fluid clearly acts in the upstream direction. On each side of the blocks the hydrostatic pressure force is given by

$$\textit{average pressure} \times \textit{area}$$

or

$$\tfrac{1}{2}\rho g h \times h \quad \text{(per unit width)}$$

Hence, from the steady-state momentum principle:

force = rate of change of momentum (mass flux × change in velocity)

$$-f + \tfrac{1}{2}\rho g h_1^2 - \tfrac{1}{2}\rho g h_2^2 = \rho q (V_2 - V_1)$$

$$\Rightarrow \quad -f - 987.7 = -1233.2$$

$$\Rightarrow \quad f = 245.5 \text{ N}$$

Answer: force (per metre width) = 246 N.

(b) If blocks are not present a hydraulic jump will still occur (as the flow must get from supercritical to subcritical); however, it will move further downstream. To see this, observe the following.

The depth at the change of slope is supercritical. On a mild slope, any supercritical flow must increase in depth; (you will prove this in the lectures on GVF). To show that the hydraulic jump takes place downstream (as opposed to immediately) we must therefore show that the sequent depth just upstream of the hydraulic jump (call it h_A) exceeds the depth at the change of slope (0.2000 m), so that there is room for GVF of increasing depth.
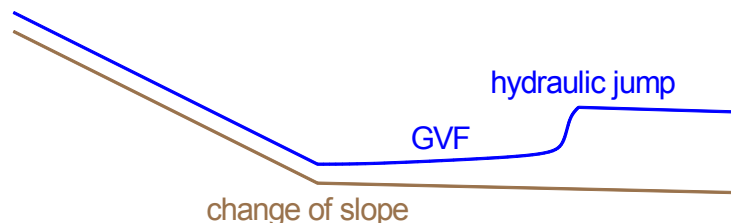
Downstream of the jump (normal flow for the milder slope):

$$h_B = 0.4913 \text{ m}$$

$$V_B = \frac{q}{h_B} = \frac{0.645}{0.4913} = 1.313 \text{ m s}^{-1}$$

$$\text{Fr}_B = \frac{V_B}{\sqrt{g h_B}} = \frac{1.313}{\sqrt{9.81 \times 0.4913}} = 0.5981$$

The sequent depth upstream of the jump is then, by formula,

$$h_A = \frac{h_B}{2}(-1 + \sqrt{1 + 8\text{Fr}_B^2}) = \frac{0.4913}{2}(-1 + \sqrt{1 + 8 \times 0.5981^2}) = 0.2371 \text{ m}$$
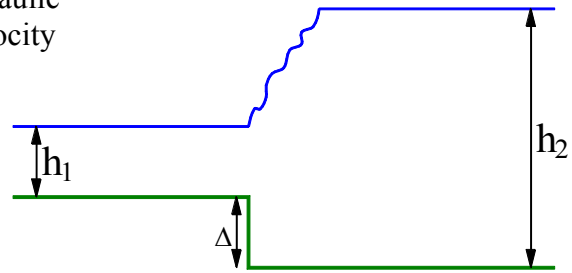
This exceeds the depth at the change of slope (0.2000 m), so the hydraulic jump takes place further downstream.

Example.
A downward step of height 0.5 m causes a hydraulic jump in a wide channel when the depth and velocity of the flow upstream are 0.5 m and 10 m s^{-1}, respectively.

(a)   Find the downstream depth.

(b)   Find the head lost in the jump.

(a) The downstream depth can be deduced from the momentum principle if the reaction force from the step is known. The approximation is that this is the same as would occur if it were in equilibrium with a hydrostatic pressure distribution here.

Flow rate per unit width:
$$q = V_1 h_1 \quad = 10 \times 0.5 \quad = 5 \text{ m}^2 \text{ s}^{-1}$$

Steady-state momentum principle
        force = rate of change of momentum (mass flux × change in velocity)
Per unit width:
$$\tfrac{1}{2}\rho g (h_1 + \Delta)^2 - \tfrac{1}{2}\rho g h_2^2 = \rho q (V_2 - V_1)$$

Since $V_2 = \dfrac{q}{h_2} = \dfrac{5}{h_2}$ this gives, in metre-second units,

$$4905(1 - h_2^2) = 5000(\frac{5}{h_2} - 10)$$

$\Rightarrow \qquad 1 - h_2^2 = 1.019(\dfrac{5}{h_2} - 10)$

$\Rightarrow \qquad 11.19 = h_2^2 + \dfrac{5.095}{h_2}$

Rearrange for the deep, subcritical solution:
$$h_2 = \sqrt{11.19 - \frac{5.095}{h_2}}$$
Iterating (from, e.g., $h_2 = \sqrt{11.19}$) gives $h_2 = 3.089$ m.

Answer: downstream depth = 3.09 m.

(b) Head either side is given by:
$$H = z_s + \frac{V^2}{2g}$$
The datum is not important as it is only the difference in head that is required. For convenience, measure z relative to the bed of the expanded part. Then,

---

$z_{s1} = 1$ m (note: water surface level, not depth), $V_1 = 10$ m s^{-1} $\Rightarrow$ $H_1 = 6.097$ m

$z_{s2} = 3.089$ m, $V_2 = \dfrac{q}{h_2} = 1.619$ m s^{-1} $\Rightarrow$ $H_2 = 3.223$ m

Hence,

head lost $= 6.097 - 3.223 = 2.874$ m

Answer: head lost $= 2.87$ m.

Section 3.6.2

Example. (Exam 2008 – modified)

A long, wide channel has a slope of 1:2747 with a Manning's n of 0.015 $m^{-1/3}$ s. It carries a discharge of 2.5 m^3 s^{-1} per metre width, and there is a free overfall at the downstream end. An undershot sluice is placed a certain distance upstream of the free overfall which determines the nature of the flow between sluice and overfall. The depth just downstream of the sluice is 0.5 m.

(a)   Determine the critical depth and normal depth.

(b)   Sketch, with explanation, the two possible gradually-varied flows between sluice and overfall.

(c)   Calculate the particular distance between sluice and overfall which determines the boundary between these two flows. Use one step in the gradually-varied-flow equation.

$S_0 = 1/2747 = 3.640 \times 10^{-4}$
$n = 0.015$ $m^{-1/3}$ s
$q = 2.5$ m^2 s^{-1}

(a)
Critical Depth

$$h_c = \left(\frac{q^2}{g}\right)^{1/3} = \left(\frac{2.5^2}{9.81}\right)^{1/3} = 0.8605 \text{ m}$$

Normal Depth

$$q = Vh \quad \text{where} \quad V = \frac{1}{n}R_h^{2/3}S^{1/2} \text{ (Manning)}, \quad R_h = h \text{ ("wide" channel)}$$

$$\Rightarrow \quad q = \frac{1}{n}h^{2/3}S^{1/2}h$$

$$\Rightarrow \quad q = \frac{h^{5/3}\sqrt{S}}{n}$$

$$\Rightarrow \quad h = \left(\frac{nq}{\sqrt{S}}\right)^{3/5} = \left(\frac{0.015 \times 2.5}{\sqrt{1/2747}}\right)^{3/5} = 1.500 \text{ m}$$

Answer: critical depth = 0.860 m;   normal depth = 1.50 m.

(b) The depth just downstream of the sluice is supercritical (0.5 m < h_c). However, the preferred depth is subcritical ($h_n > h_c$). Hence, if the channel is long enough then there will be a downstream hydraulic jump, with the flow depth then decreasing to pass through critical again near the overfall.

If the channel is too short, however, the region of supercritical flow from the sluice will extend to the overfall.
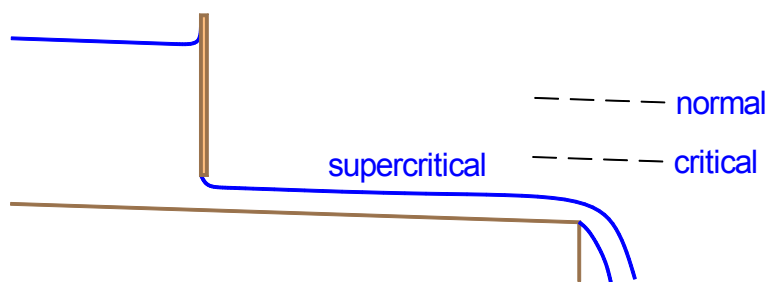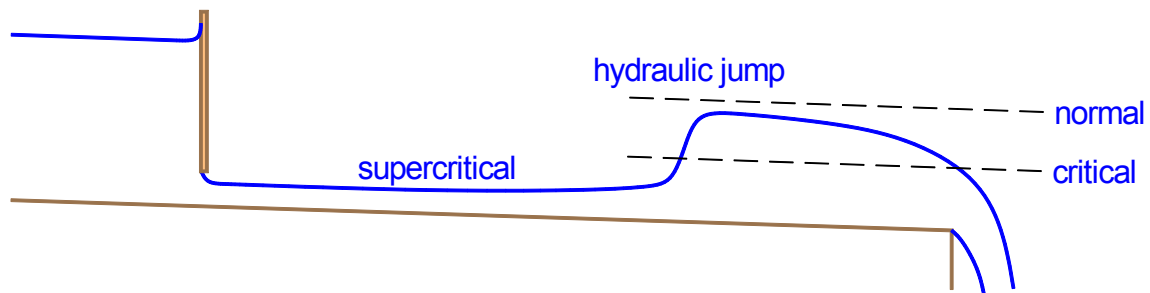
(c) As the channel shortens, the depth change across the hydraulic jump diminishes. The boundary between the two possible flow behaviours occurs when the supercritical GVF just reaches critical depth at the overfall (i.e. the limiting depth change across the hydraulic jump is zero).

As the flow is supercritical, integrate the GVF equation forward from the downstream side of the sluice gate (where $h = 0.5$ m) to the overfall (where $h = h_c = 0.8605$ m). Use 1 step.

GVF equation:
$$\frac{dh}{dx} = \frac{S_0 - S_f}{1 - \text{Fr}^2}$$
For the direct-step method invert the GVF equation:
$$\frac{dx}{dh} = \frac{1 - \text{Fr}^2}{S_0 - S_f} \qquad \text{and} \qquad \Delta x \approx \left(\frac{dx}{dh}\right)\Delta h$$

For the working, write the derivative as a function of h; (all lengths in metres).
$$\text{Fr} = \frac{V}{\sqrt{gh}} = \frac{q}{\sqrt{gh^3}} \qquad \Rightarrow \qquad \text{Fr}^2 = \frac{q^2}{gh^3} = \frac{0.6371}{h^3}$$

$$S_f = \left(\frac{nq}{h^{5/3}}\right)^2 = \frac{1.406 \times 10^{-3}}{h^{10/3}}$$

$$\Delta h = 0.8605 - 0.5 = 0.3605$$

Working formulae:

$$\Delta x = \left(\frac{dx}{dh}\right)_{mid} \Delta h$$

where

$$\frac{dx}{dh} = \frac{1 - \dfrac{0.6371}{h^3}}{3.640 \times 10^{-4} - \dfrac{1.406 \times 10^{-3}}{h^{10/3}}} , \qquad \Delta h = 0.3605$$

i	h_i	x_i	h_{mid}	$\left(\dfrac{dx}{dh}\right)_{mid}$	Δx
0	0.5	0			
			0.6803	217.1	78.26
1	0.8605	**78.26**			

Answer: length to overfall = 78 m.

Example.
A sluice gate discharges water at 9 m³ s⁻¹ into a 6 m wide rectangular channel laid on a slope of 0.0004 with $n = 0.015$ m⁻¹ᐟ³ s. The depth at the vena contracta is 0.15 m.

(a)   Find the normal and critical depths.

(b)   Compute the position of the hydraulic jump, assuming normal depth downstream. Use one step in the GVF equation.

$Q = 9$ m³ s⁻¹
$b = 6$ m
$S_0 = 4 \times 10^{-4}$
$n = 0.015$ m⁻¹ᐟ³ s

(a)

Normal Depth

$$Q = VA \qquad \text{where} \qquad V = \frac{1}{n} R_h^{2/3} S^{1/2}$$

and

$$A = bh$$
$$R_h = \frac{bh}{b + 2h} = \frac{h}{1 + 2h/b} \qquad \text{(rectangular channel)}$$

Hence,

$$Q = \frac{1}{n}\left(\frac{h}{1 + 2h/b}\right)^{2/3} S^{1/2} bh$$

$$\Rightarrow \qquad \frac{nQ}{b\sqrt{S}} = \frac{h^{5/3}}{(1 + 2h/b)^{2/3}} \tag{*}$$

For the given slope, rearrange as an iterative formula for h:

$$h = \left(\frac{nQ}{b\sqrt{S_0}}\right)^{3/5} (1 + 2h/b)^{2/5}$$

or, with numerical values:
$$h = 1.073(1 + h/3)^{2/5}$$
Iteration (from, e.g., $h = 1.073$) gives
$$h_n = 1.231 \text{ m}$$

Critical Depth

$$h_c = \left(\frac{q^2}{g}\right)^{1/3}$$

Here,

$$q = \frac{Q}{b} = \frac{9}{6} = 1.5 \text{ m}^2 \text{ s}^{-1}$$

Hence,

$$h_c = \left(\frac{q^2}{g}\right)^{1/3} = \left(\frac{1.5^2}{9.81}\right)^{1/3} = 0.6121 \text{ m}$$

Answer: normal depth = 1.23 m; critical depth = 0.612 m.

(b) At the hydraulic jump the *downstream* depth is the normal depth; i.e.

$$h_2 = 1.231 \text{ m}$$

$$V_2 = \frac{Q}{bh_2} = \frac{9}{6 \times 1.231} = 1.219 \text{ m s}^{-1}$$

$$\text{Fr}_2 = \frac{V}{\sqrt{gh_2}} = \frac{1.219}{\sqrt{9.81 \times 1.231}} = 0.3508$$

Just *upstream* of the hydraulic jump then

$$h_1 = \frac{h_2}{2}(-1 + \sqrt{1 + 8\text{Fr}_2^2}) = \frac{1.231}{2}(-1 + \sqrt{1 + 8 \times 0.3508^2}) = 0.2516 \text{ m}$$

We must therefore do a GVF calculation from just downstream of the sluice gate (where $h = 0.15$ m) to just upstream of the hydraulic jump (where $h = 0.2516$ m).

GVF equation:

$$\frac{dh}{dx} = \frac{S_0 - S_f}{1 - \text{Fr}^2}$$

For the direct-step method invert the GVF equation:

$$\frac{dx}{dh} = \frac{1 - \text{Fr}^2}{S_0 - S_f} \qquad \text{and} \qquad \Delta x \approx \left(\frac{dx}{dh}\right)\Delta h$$

For the working, write the derivative as a function of h; (all lengths in metres).

$$\text{Fr} = \frac{V}{\sqrt{gh}} = \frac{q}{\sqrt{gh^3}} \qquad \Rightarrow \qquad \text{Fr}^2 = \frac{q^2}{gh^3} = \frac{0.2294}{h^3}$$

$$S_f = \left(\frac{nQ}{bh^{5/3}}\right)^2 (1 + 2h/b)^{4/3} = 5.0625 \times 10^{-4} \frac{(1 + h/3)^{4/3}}{h^{10/3}}$$

$$\Delta h = 0.2516 - 0.15 = 0.1016$$

Working formulae:

$$\Delta x = \left(\frac{dx}{dh}\right)_{mid} \Delta h$$

where

$$\frac{dx}{dh} = \frac{1 - \dfrac{0.2294}{h^3}}{[4 - 5.0625 \times \dfrac{(1+h/3)^{4/3}}{h^{10/3}}] \times 10^{-4}}, \qquad \Delta h = 0.1016$$

i	h_i	x_i	h_{mid}	$(dx/dh)_{mid}$	Δx
0	0.15	0			
			0.2008	235.6	23.9
1	0.2516	**23.9**			

Answer: distance from sluice gate to hydraulic jump = 24 m.

Using the specific energy as dependent variable and $\Delta x = \dfrac{\Delta E}{(S_0 - S_f)_{av}}$ leads to a distance of 17.8 m. An accurate answer using a very small step size and many steps is 23.7 m.

Example. (Exam 2014)
An undershot sluice is used to control the flow of water in a long wide channel of slope 0.003 and Manning's roughness coefficient 0.012 $m^{-1/3}$ s. The flow rate in the channel is 2 $m^3 s^{-1}$ per metre width.

(a)   Calculate the normal depth and critical depth in the channel and show that the channel is hydrodynamically "steep" at this flow rate.

(b)   The depth of flow just downstream of the sluice is 0.4 m. Assuming no head losses at the sluice calculate the depth just upstream of the sluice.

(c)   Sketch the depth profile along the channel, indicating clearly any flow transitions brought about by the sluice and indicating where water depth is increasing or decreasing.

(d)   Use 2 steps in the gradually-varied flow equation to determine how far upstream of the sluice a hydraulic jump will occur.

$S_0 = 0.003$
$n = 0.012$ $m^{-1/3}$ s
$q = 2$ $m^2 s^{-1}$

(a)
Normal depth

$$q = Vh \quad = \frac{1}{n} R_h^{2/3} S^{1/2} h \qquad \text{where} \qquad R_h = h \ \text{(wide channel)}$$

$$\Rightarrow \qquad \frac{nq}{\sqrt{S}} = h^{5/3} \qquad\qquad (*)$$

$$\Rightarrow \qquad h_n = \left(\frac{nq}{\sqrt{S_0}} \right)^{3/5} \quad = \left(\frac{0.012 \times 2}{\sqrt{0.003}} \right)^{3/5} \quad = 0.6095 \text{ m}$$

Critical Depth

$$h_c = \left(\frac{q^2}{g} \right)^{1/3} \quad = \left(\frac{2^2}{9.81} \right)^{1/3} \quad = 0.7415 \text{ m}$$

The normal depth is smaller than the critical depth. Hence the normal flow is supercritical; i.e. the channel is steep at this discharge.

Answer: $h_n = 0.610$ m; $h_c = 0.742$ m.

(b) Assuming the same total head on either side of the gate:

$$z_{s1} + \frac{V_1^2}{2g} = z_{s2} + \frac{V_2^2}{2g}$$

$$\Rightarrow \quad h_1 + \frac{q^2}{2gh_1^2} = h_2 + \frac{q^2}{2gh_2^2}$$

$$\Rightarrow \quad h_1 + \frac{2^2}{2gh_1^2} = 0.4 + \frac{2^2}{2g \times 0.4^2}$$

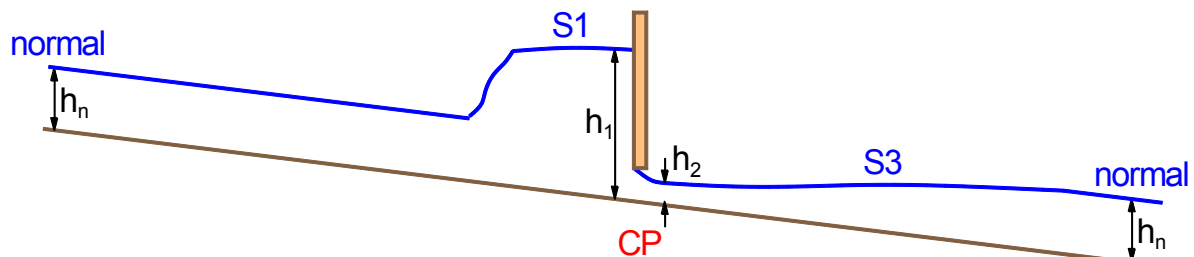$$\Rightarrow \quad h_1 + \frac{0.2039}{h_1^2} = 1.674$$

We require the subcritical solution, so rearrange for iteration as

$$h_1 = 1.674 - \frac{0.2039}{h_1^2}$$

Iteration (e.g. from 1.674) gives $h_1 = 1.594$ m

Answer: $h_1 = 1.59$ m.

(c) The depth increases in each of the GVF regions (S1 and S3) shown below.

(d) Upstream of the sluice there is GVF between the hydraulic jump and the sluice. This starts at the subcritical sequent depth of the hydraulic jump (to be found below) and ends at the depth h_1 found in part (b).

For the hydraulic jump, on the upstream side (normal flow):

$h_A = 0.6095$ m

$$V_A = \frac{q}{h_A} = \frac{2}{0.6095} = 3.281 \text{ m s}^{-1}$$

$$\text{Fr}_A = \frac{V}{\sqrt{gh_A}} = \frac{3.281}{\sqrt{9.81 \times 0.6095}} = 1.342$$

On the downstream side of the hydraulic jump:

$$h_B = \frac{h_A}{2}(-1 + \sqrt{1 + 8\text{Fr}_A^2}) = \frac{0.6095}{2}(-1 + \sqrt{1 + 8 \times 1.342^2}) = 0.8915 \text{ m}$$

Do a GVF calculation (subcritical, so physically it should start at the fixed downstream control and work upstream, although mathematically it can be done the other way) from just upstream of the sluice gate (where $h = 1.594$ m) to just downstream of the hydraulic jump (where $h = 0.892$ m). Using two steps the depth
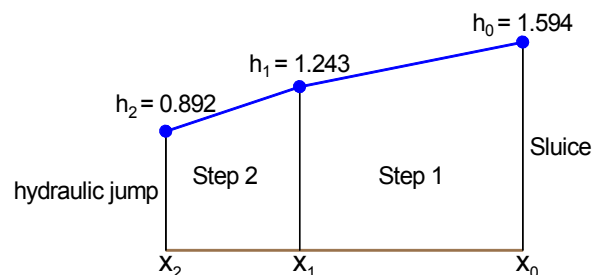
increment per step is

$$\Delta h = \frac{0.892 - 1.594}{2} \quad = -0.351 \text{ m}$$

GVF equation:

$$\frac{dh}{dx} = \frac{S_0 - S_f}{1 - Fr^2}$$

For the direct-step method rewrite the GVF equation "the other way up":

$$\frac{dx}{dh} = \frac{1 - Fr^2}{S_0 - S_f} \qquad \text{and} \qquad \Delta x \approx \left(\frac{dx}{dh}\right)\Delta h$$

For the working, write the derivative as a function of h; (all lengths in metres).

$$Fr = \frac{V}{\sqrt{gh}} = \frac{q}{\sqrt{gh^3}} \qquad \Rightarrow \qquad Fr^2 = \frac{q^2}{gh^3} = \frac{0.4077}{h^3}$$

$$S_f = \left(\frac{nq}{h^{5/3}}\right)^2 = \frac{5.76 \times 10^{-4}}{h^{10/3}}$$

Working formulae:

$$\Delta x = \left(\frac{dx}{dh}\right)_{mid} \Delta h$$

where

$$\frac{dx}{dh} = \frac{1 - \dfrac{0.4077}{h^3}}{[30 - \dfrac{5.76}{h^{10/3}}] \times 10^{-4}}, \qquad \Delta h = -0.351$$

| i	h_i	x_i	h_{mid}	$(dx/dh)_{mid}$	Δx
0	1.594	**0**			
			1.419	303.9	−106.7
1	1.243	**−106.7**			
			1.068	262.2	−92.0
2	0.892	**−198.7**			

Answer: upstream distance from sluice to hydraulic jump = 199 m.

Section 4.2

Example. (From White, 2006)
A pencil point piercing the surface of a wide rectangular channel flow creates a wedgelike 25° half-angle wave, as in the figure right. If the channel has a Manning's n of 0.014 m$^{-1/3}$ s and the depth is 350 mm, determine: (a) the Froude number; (b) the critical depth; and (c) the critical slope.
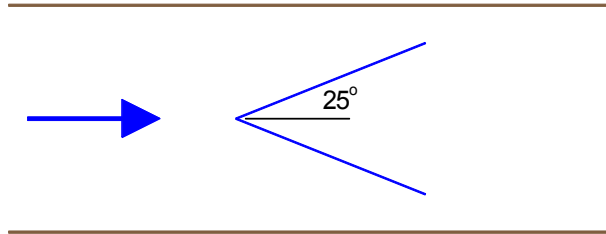
$\alpha = 25°$
$n = 0.014$ m$^{-1/3}$ s
$h = 0.35$ m

(a)

$$\sin \alpha = \frac{1}{\text{Fr}}$$

$\Rightarrow \quad \text{Fr} = \frac{1}{\sin \alpha} = \frac{1}{\sin 25°} = 2.366$

Answer: Froude number = 2.37.

(b)

$$\text{Fr} \equiv \frac{V}{\sqrt{gh}}$$

$\Rightarrow \quad V = \text{Fr}\sqrt{gh} = 2.366 \times \sqrt{9.81 \times 0.35} = 4.384$ m s^{-1}

$\Rightarrow \quad q = Vh = 4.384 \times 0.35 = 1.534$ m^2 s^{-1}

$\Rightarrow \quad h_c = \left(\frac{q^2}{g} \right)^{1/3} = 0.6213$ m

Answer: critical depth = 0.621 m.

(c) The critical slope is that at which the normal depth is equal to the critical depth.

Normal flow:

$$q = Vh \quad \text{where} \quad V = \frac{1}{n} R_h^{2/3} S^{1/2} \text{ (Manning)}, \quad R_h = h \text{ ("wide" channel)}$$

$$\Rightarrow \quad q = \frac{1}{n} h^{2/3} S^{1/2} h$$

Rearranging, and setting a depth h equal to the critical depth ($h_c = 0.6213$ m),

$$\Rightarrow \quad S = \frac{(nq)^2}{h^{10/3}} = \frac{(0.014 \times 1.534)^2}{0.6213^{10/3}} = 2.254 \times 10^{-3}$$

Answer: critical slope $= 2.25 \times 10^{-3}$.

Sediment Transport; Section 2.1
Note: removed from the present curriculum because of the reduction in teaching hours.

Example. (Exam 2007 – part)

An undershot sluice is placed in a channel with a horizontal bed covered by gravel with a median diameter of 5 cm and density 2650 kg m^{-3}. The flow rate is 4 m^3 s^{-1} per metre width and initially the depth below the sluice is 0.5 m. Assuming a critical Shields parameter τ^*_{crit} of 0.06 and friction coefficient c_f of 0.01:

(a)   find the depth just upstream of the sluice and show that the bed there is stationary;

(b)   show that the bed below the sluice will erode and determine the depth of scour.

$d = 0.05$ m
$\rho_s = 2650$ kg m^{-3} ($s = 2.65$)
$q = 4$ m^2 s^{-1}
$\tau^*_{crit} = 0.06$
$c_f = 0.01$

(a) The bed is mobile if and only if the bed shear stress exceeds the critical shear stress for incipient motion. (Alternatively, one could find the velocity at which incipent motion occurs.)

First find the critical shear stress for incipient motion. The local shear stress can be found by finding velocities using open-channel flow theory and using the friction coefficient.

The critical shear stress for incipient motion is
$$\tau_{crit} = \tau^*_{crit}(\rho_s - \rho)gd \quad = 0.06 \times (2650 - 1000) \times 9.81 \times 0.05 \quad = 48.56 \text{ N m}^{-2}$$

In the accelerated flow just below the sluice gate,
$$h = 0.5 \text{ m}$$
$$V = \frac{q}{h} \quad = \frac{4}{0.5} \quad = 8 \text{ m s}^{-1}$$
The total head relative to the undisturbed bed is thus (since $z_s = h$ initially):
$$H = z_s + \frac{V^2}{2g} \quad = 0.5 + \frac{8^2}{2 \times 9.81} \quad = 3.762 \text{ m}$$

Upstream of the sluice one must have the same total head:
$$H = z_s + \frac{V^2}{2g} \quad = h + \frac{q^2}{2gh^2}$$
but now we look for the subcritical (deep) solution:
$$h = H - \frac{q^2}{2gh^2}$$
Substituting numerical values, in metre-second units:
$$h = 3.762 - \frac{0.8155}{h^2}$$

Iterating (from, e.g., $h = 3.762$) gives
$$h = 3.703 \text{ m}$$

The velocity upstream of the sluice is
$$V = \frac{q}{h} = \frac{4}{3.703} = 1.080 \text{ m s}^{-1}$$
and the bed shear stress upstream of the sluice is
$$\tau_b = c_f(\tfrac{1}{2}\rho V^2) = 0.01 \times \tfrac{1}{2} \times 1000 \times 1.080^2 = 5.832 \text{ N m}^{-2}$$

Since $\tau_b < \tau_{crit}$ the bed here is stationary.

Answer: depth upstream of sluice = 3.70 m; as demonstrated, the bed here is stationary.

(b) Initially, beneath the sluice,
$$V = 8 \text{ m s}^{-1}$$
$$\tau_b = c_f(\tfrac{1}{2}\rho V^2) = 0.01 \times \tfrac{1}{2} \times 1000 \times 8^2 = 320 \text{ N m}^{-2}$$
Since $\tau_b > \tau_{crit}$ the bed here is initially mobile.

The bed will continue to erode, the flow depth h increasing and velocity V decreasing until
$$\tau_b = \tau_{crit}$$
At this point,
$$c_f(\tfrac{1}{2}\rho V^2) = \tau_{crit}$$
whence
$$V = \sqrt{\frac{2}{c_f} \frac{\tau_{crit}}{\rho}} = \sqrt{\frac{2}{0.01} \times \frac{48.56}{1000}} = 3.116 \text{ m s}^{-1}$$
$$h = \frac{q}{V} = \frac{4}{3.116} = 1.284 \text{ m}$$

This is the depth of flow. Since the downstream water level is set (at 0.5 m above the undisturbed bed) by the gate the depth of the scour hole is
$$1.284 - 0.5 = 0.784 \text{ m}$$

Answer: depth of scour hole = 0.784 m.