UNSTEADY OPEN-CHANNEL FLOW A

~~RIVER MECHANICS~~ (OPEN-CHANNEL HYDRAULICS) (CE5312)

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Governing equations

Continuity: $$\frac{\partial h}{\partial t} + h_m(h)\frac{\partial U}{\partial x} + U\frac{\partial h}{\partial x} = 0$$

Momentum: $$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + g\frac{\partial h}{\partial x} = g(S_0 - S_f)$$

Characteristic form of G.E.

Original equation (PDE)

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + g\frac{\partial h}{\partial x} = g(S_0 - S_f)$$

"Sitting on the river bank and observe the entire river"

Compatibility equations (ODE, no spatial variable!)

"sitting on a boat that travels with specific trajectories (the characteristics), and just observe the flow around you"

$$\boxed{\frac{D(U+2c)}{Dt} = g(S_0 - S_f), \text{ along } \frac{dx}{dt} = U + c}$$

$$\boxed{\frac{D(U-2c)}{Dt} = g(S_0 - S_f), \text{ along } \frac{dx}{dt} = U - c}$$

characteristics

$$c = \sqrt{gh_m}$$

Celerity of gravity wave

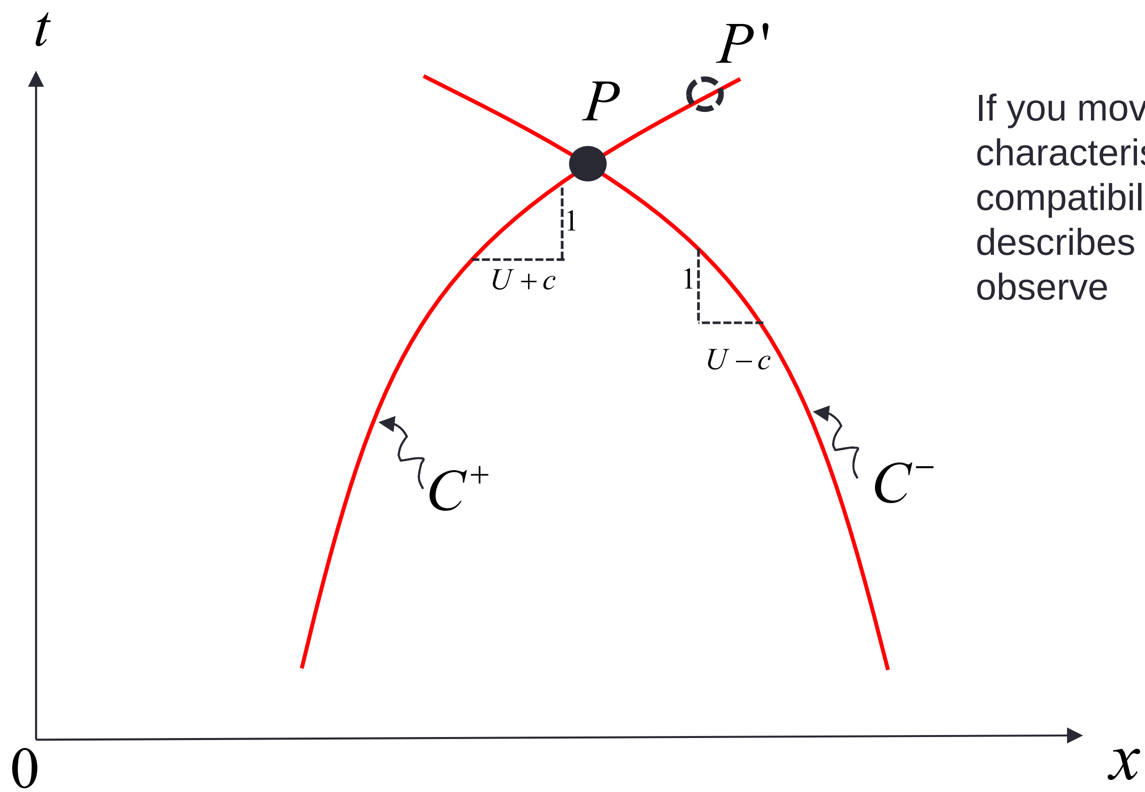
Lines of characteristics

Positive characteristics: C+

$$\frac{dx}{dt} = U + c$$

Negative characteristics: C-

$$\frac{dx}{dt} = U - c$$

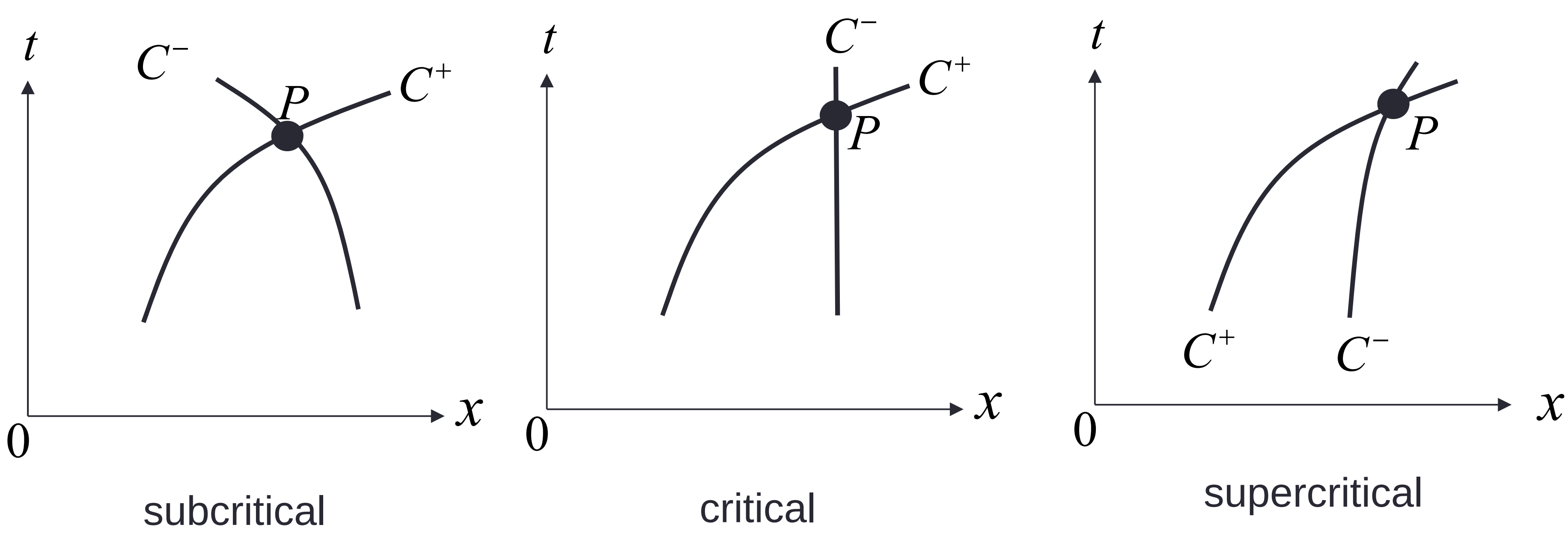


If you move along a characteristic, e.g. P to P', the compatibility equation describes the flow you will observe

Characteristics vs. Froude number

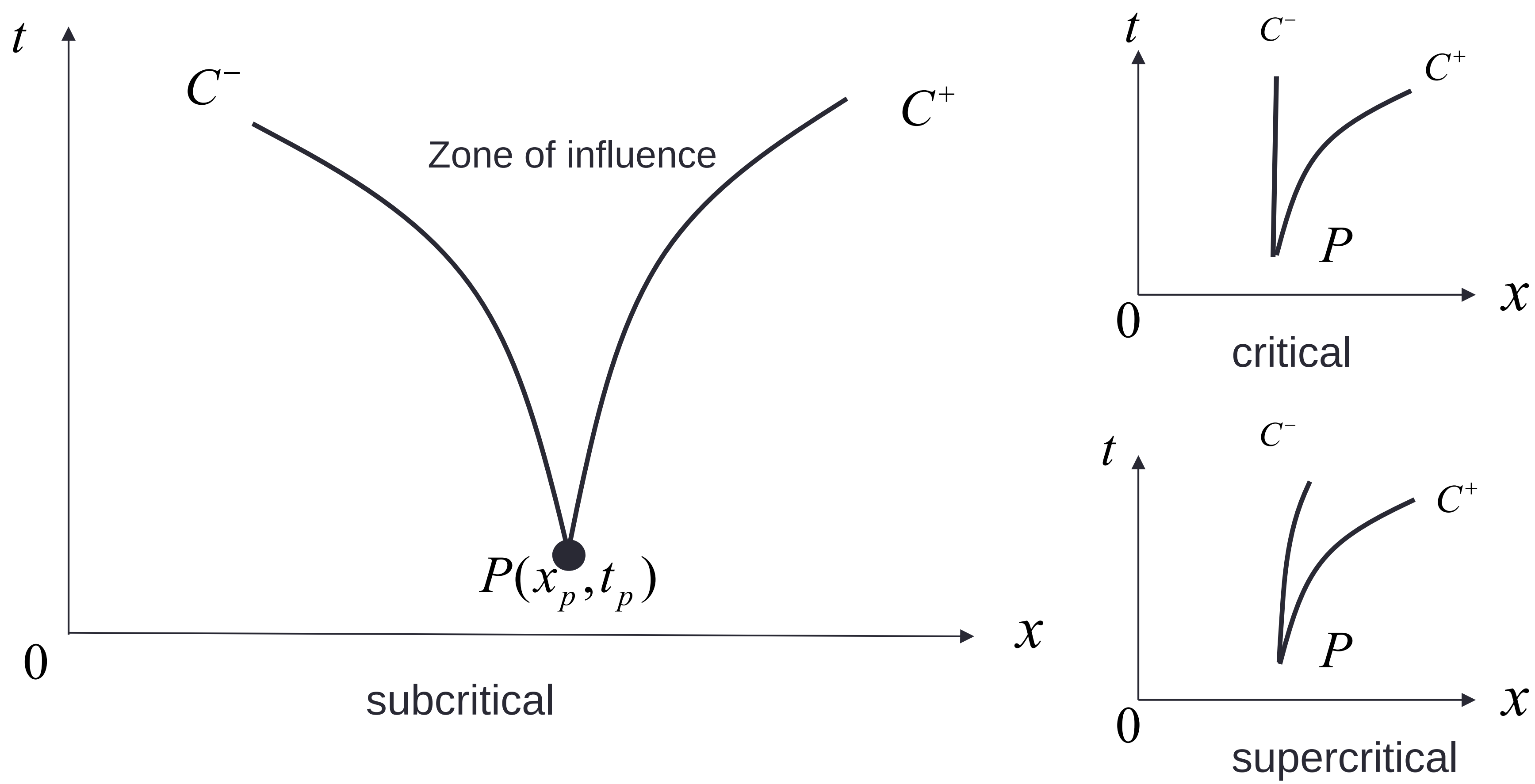
$$Fr = \frac{U}{\sqrt{gh_m}} = \frac{U}{c},$$

$$C^+ : \frac{dx}{dt} = U + c > 0 \qquad C^- : \frac{dx}{dt} = U - c \begin{cases} > 0, \text{ Fr>1, supercritical} \\ = 0, \text{ Fr=1, critical} \\ < 0, \text{ Fr<1, subcritical} \end{cases}$$



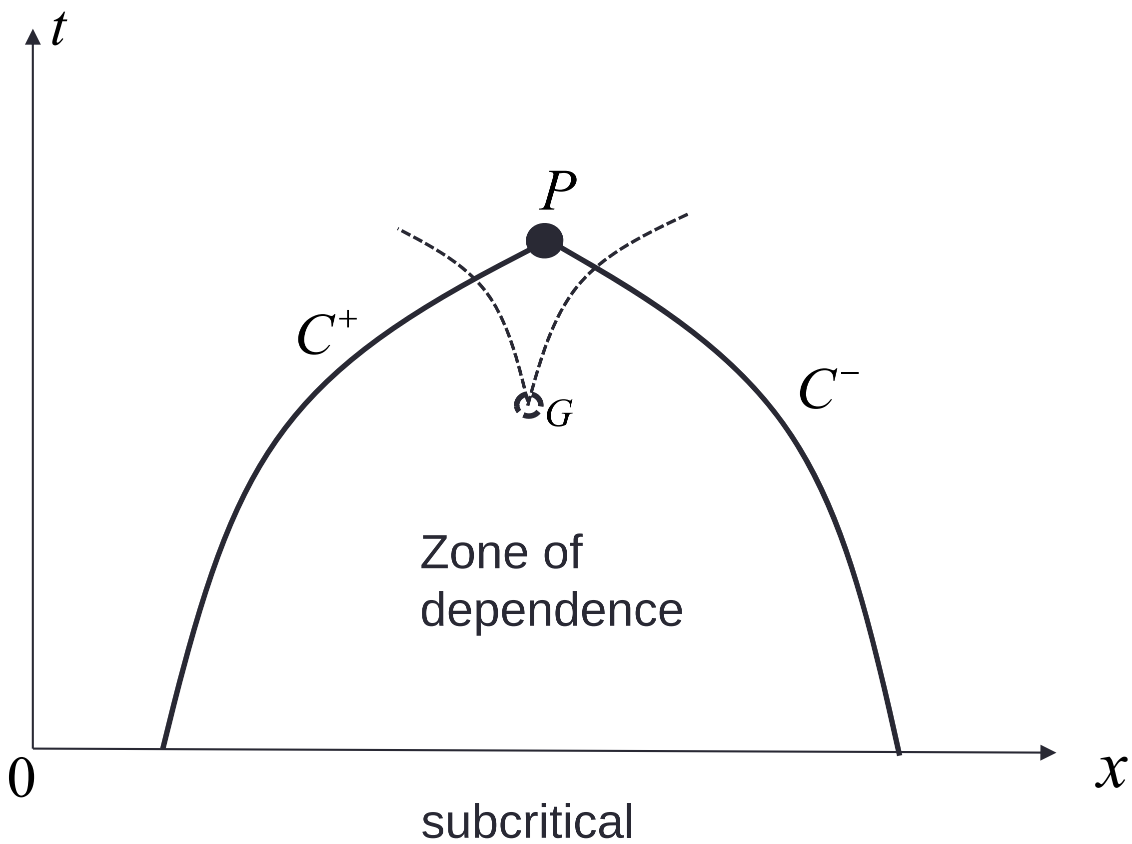
Zone of influence

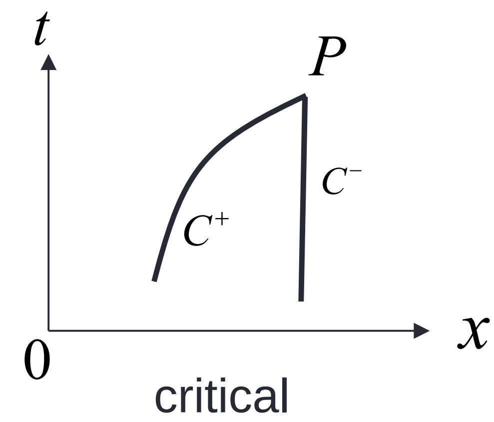
At time level t, any point within the zone of influence of P(x_p,t_p), has been affected by the disturbance initiated at a P.

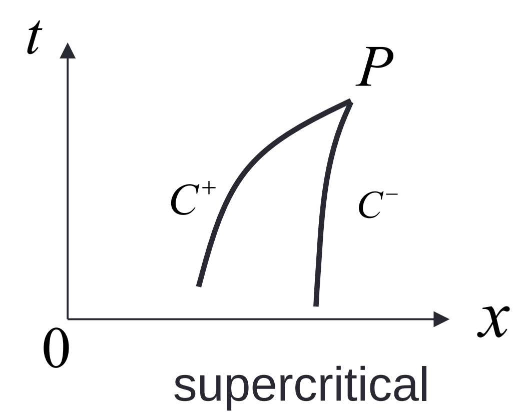


Zone of dependence

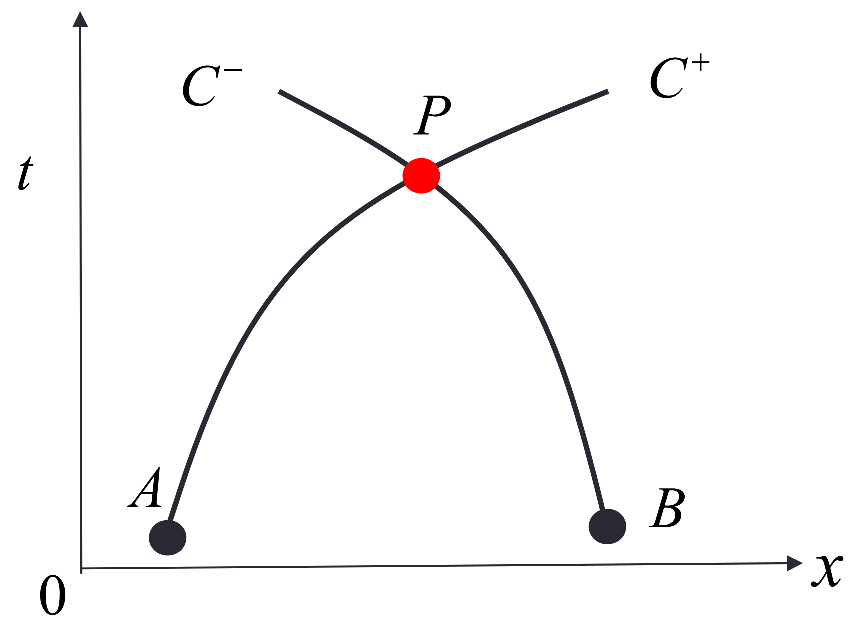
For any points within the zone of dependence of P(x_p,t_p), P is within their zone of influences. In other words, any disturbance initiated within P's zone of dependence has reached x_p before t_p.







Method of characteristics



From A to P:

$$U_P + 2c_P = \left(U_A + 2c_A\right) + \int_{t_A}^{t_P} g(S_0 - S_f)dt$$

From B to P:

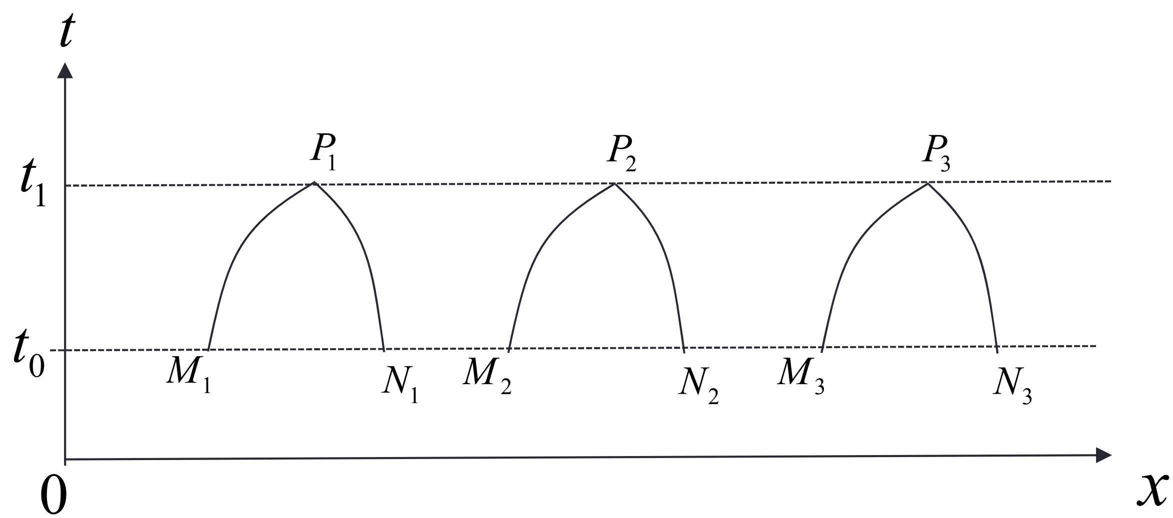
$$U_P - 2c_P = \left(U_B - 2c_B\right) + \int_{t_B}^{t_P} g(S_0 - S_f)dt$$

Two equations for two unknowns!

To give flow conditions (U and c) at point P:

- Construct the two characteristics (C+ and C-) through P
- find reference points with known flow conditions on C+ and C- (A, B)
- Integrate compatibility equations
- Evaluate the source terms, i.e. line integral of $g(S_0\text{-}S_f)$
- Solve for unknowns U_p and c_p

Initial conditions



$$U_{P,i} + 2c_{P,i} = \left(U_{M.i} + 2c_{M,i}\right) + \int_{t_0}^{t_1} g(S_0 - S_f)dt$$

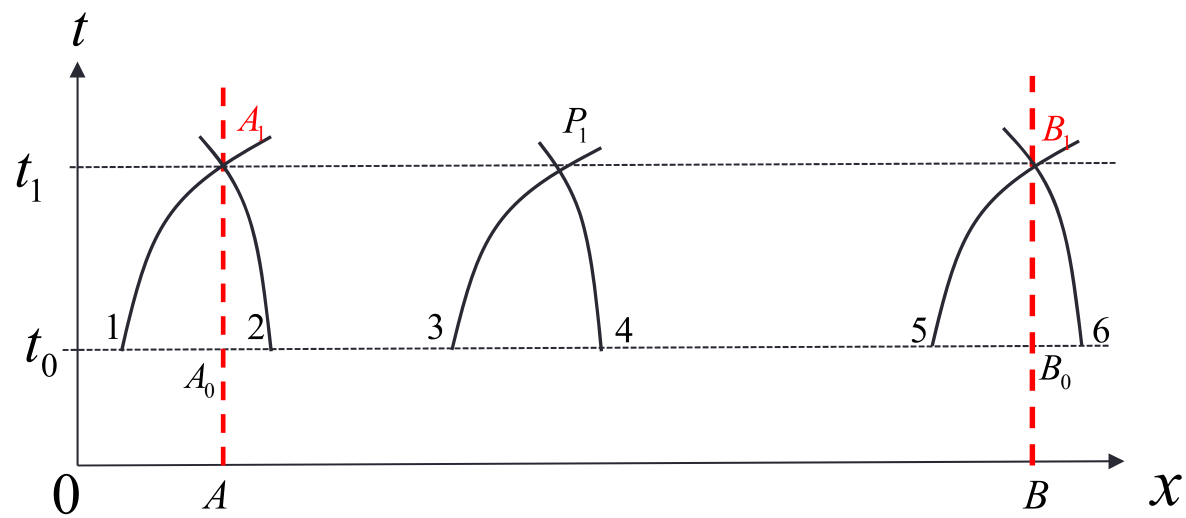
$$U_{P,i} - 2c_{P,i} = \left(U_{N,i} - 2c_{N,i}\right) + \int_{t_0}^{t_1} g(S_0 - S_f)dt$$

You have to tell us the flow conditions at the beginning of calculation, e.g. U and c along the entire channel at $t=t_0$.

This is the same for both subcritical and supercritical flows.

Boundary condition: subcritical flow

Channel reach AB is the computational domain, and we know nothing about the flow outside AB.



$$U_{A1} + 2c_{A1} = \text{flow at point 1?}$$

$$U_{A1} - 2c_{A1} = \left(U_2 - 2c_2\right) + \int_{t_0}^{t_1} g(S_0 - S_f)dt$$

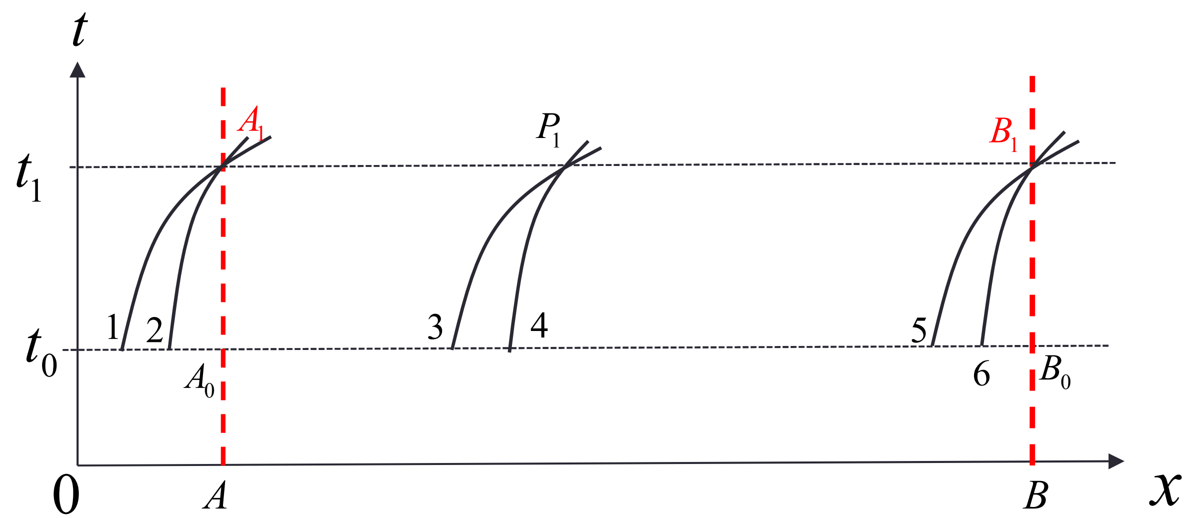
$$U_{B1} + 2c_{B1} = \left(U_5 + 2c_5\right) + \int_{t_0}^{t_1} g(S_0 - S_f)dt$$

$$U_{B1} - 2c_{B1} = \text{flow at point 6?}$$

For subcritical flow one upstream and one downstream boundary conditions must be specified.

Boundary condition: supercritical flow

Channel reach AB is the computational domain, and we know nothing about the flow outside AB.



$$U_{A1} + 2c_{A1} = \text{flow at point 1?}$$

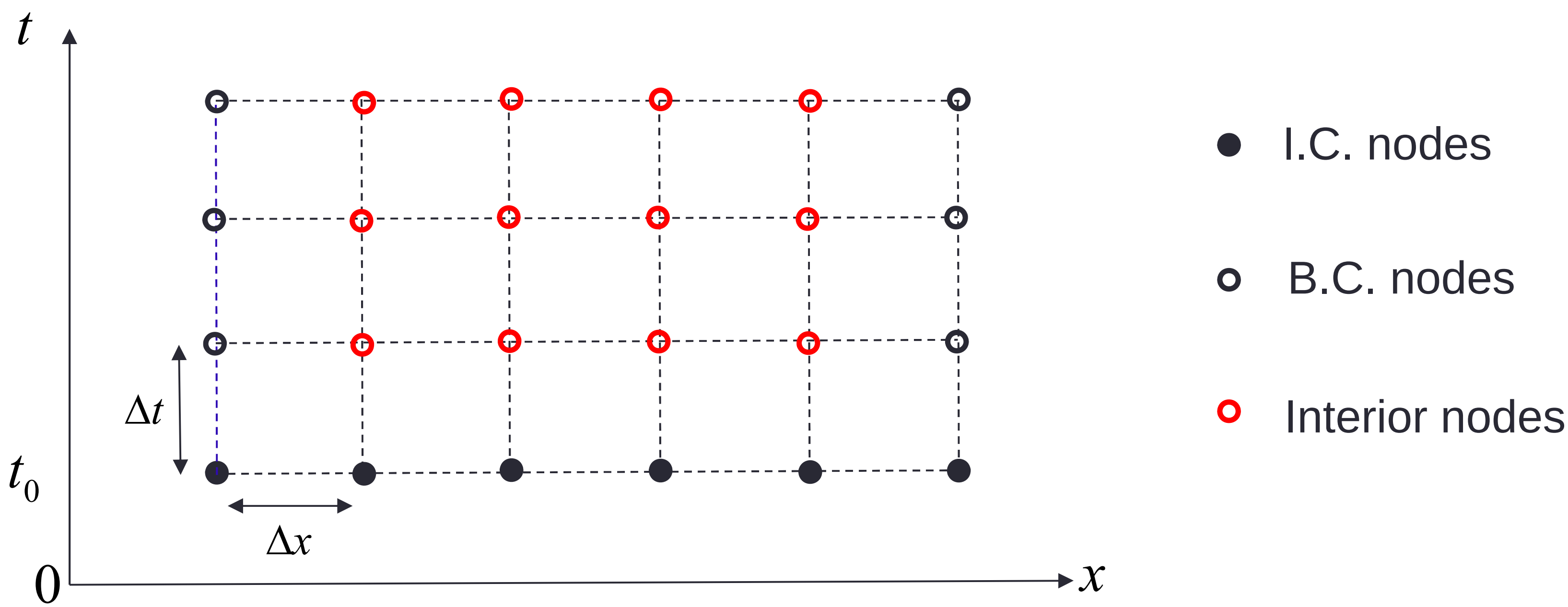
$$U_{A1} - 2c_{A1} = \text{flow at point 2?}$$

$$U_{B1} + 2c_{B1} = \left(U_5 + 2c_5\right) + \int_{t_0}^{t_1} g(S_0 - S_f)dt$$

$$U_{B1} - 2c_{B1} = \left(U_6 - 2c_6\right) + \int_{t_0}^{t_1} g(S_0 - S_f)dt$$

For supercritical flow two upstream B.C. are needed, but no \downstream boundary condition is required.

A numerical solution for subcritical flows

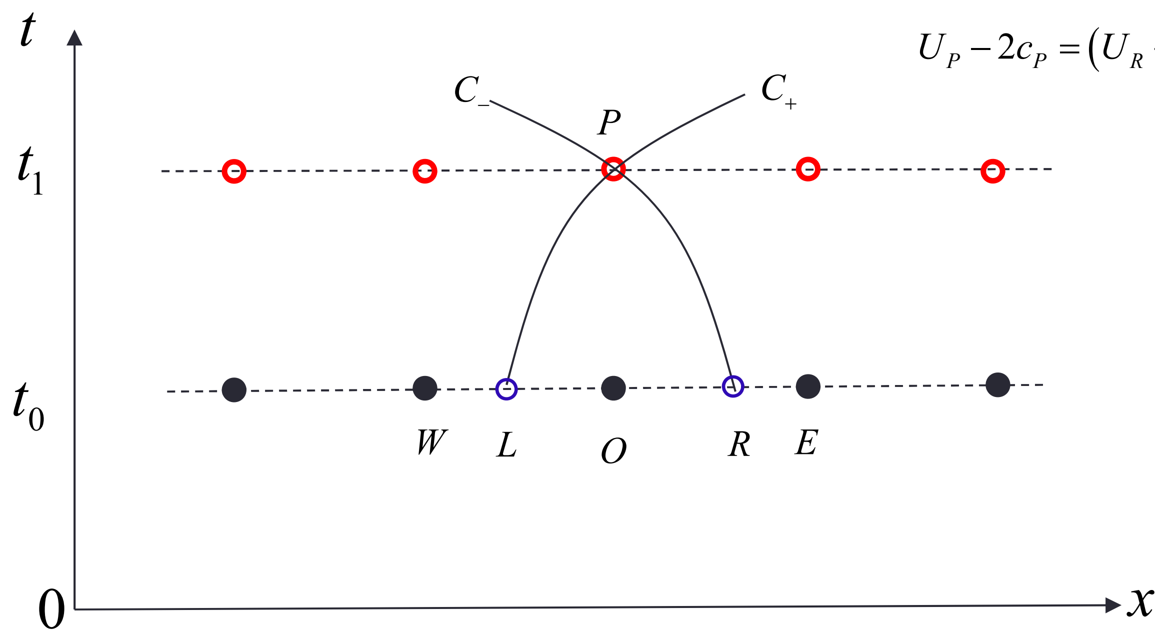


- Discretize x-t plane into rectangular grids
- Specify B.C. and I.C.
- Find solutions for B.C. nodes (if required)
- Find solutions for Interior nodes

Interior nodes: general

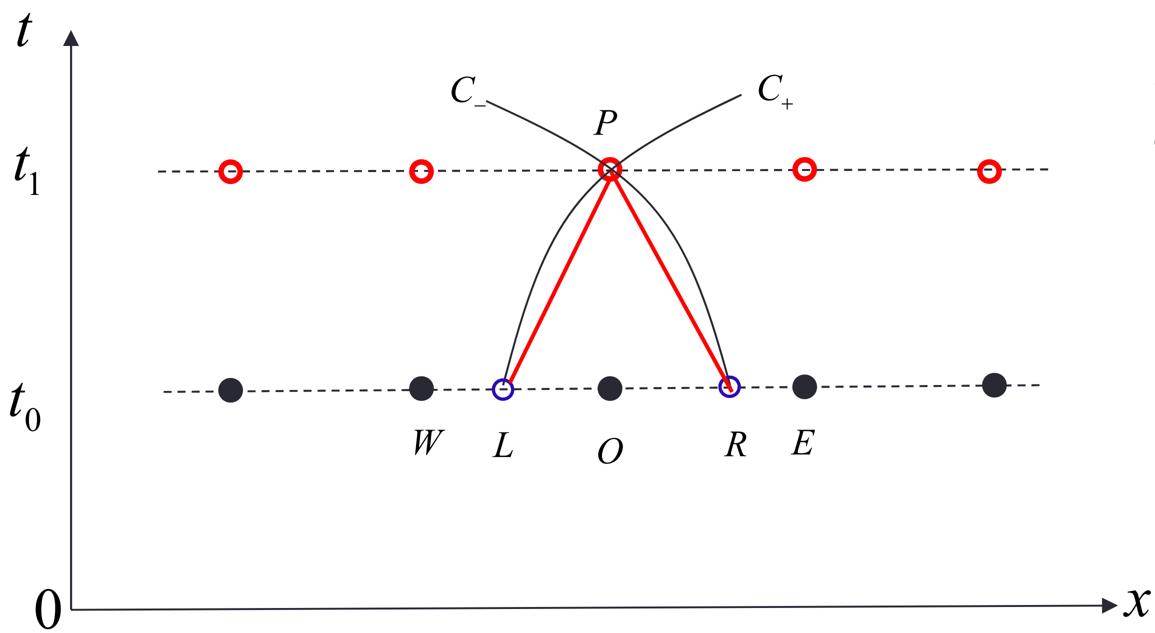
$$U_P + 2c_P = \left(U_L + 2c_L\right) + \int_{t_0}^{t_1} g(S_0 - S_f)dt$$

$$U_P - 2c_P = \left(U_R - 2c_R\right) + \int_{t_0}^{t_1} g(S_0 - S_f)dt$$



- Construct the two characteristics (C+ and C-)
- find reference points at the earlier time level (L, R)
- Evaluate the source terms, i.e. integral of $g(S_0\text{-}S_f)$
- Solve for unknowns

Interior nodes: characteristics

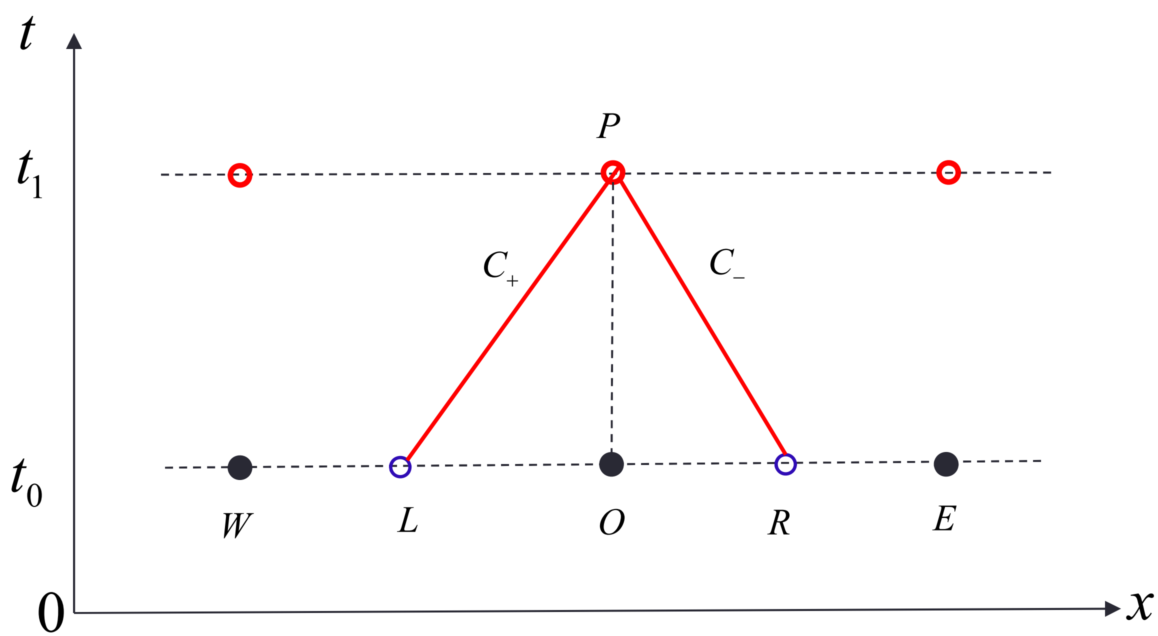


Straight-line approximation!

$$C^+ : \frac{dx}{dt} = U_P + c_P \approx U_o + c_o$$

$$C^- : \frac{dx}{dt} = U_P - c_P \approx U_o - c_o$$

Interior nodes: reference points



$$C^+ : \frac{dx}{dt} = U_o + c_o \Rightarrow \frac{x_o - x_L}{\Delta t} = U_o + c_o$$

$$C^- : \frac{dx}{dt} = U_o - c_o \Rightarrow \frac{x_o - x_R}{\Delta t} = U_o - c_o$$

$$\boxed{\begin{aligned} x_L &= x_o - (U_o + c_o)\Delta t \\ x_R &= x_o - (U_o - c_o)\Delta t \end{aligned}}$$

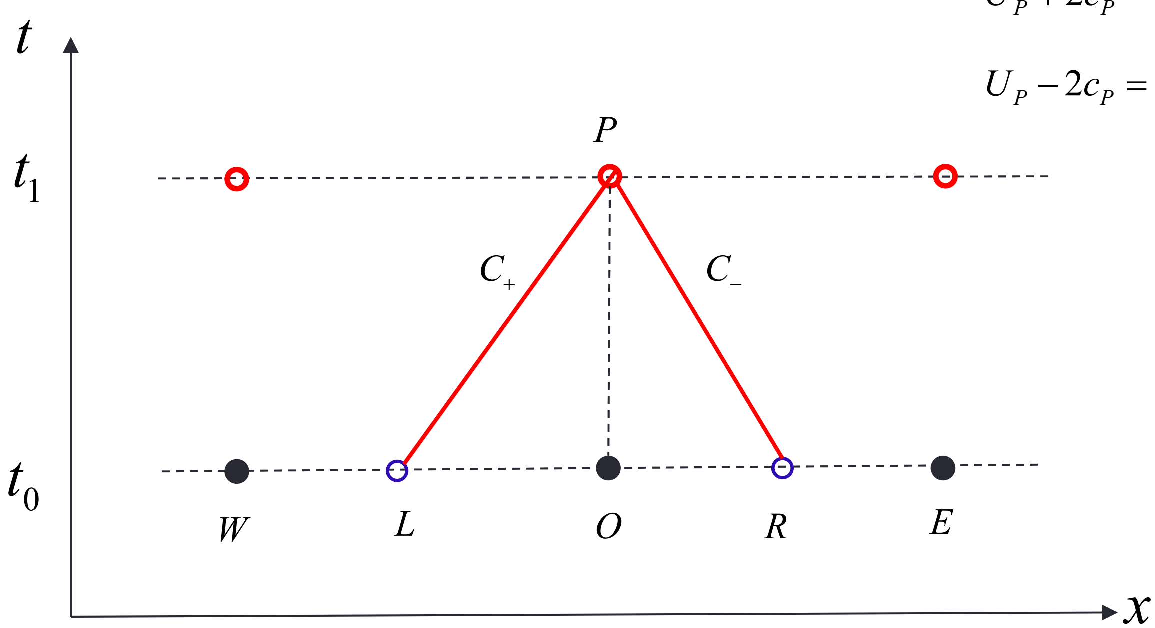
Flow conditions at L and R are given by interpolations:

$$\boxed{\begin{aligned} U_L &= \frac{x_L - x_o}{x_W - x_o}(U_W - U_o) + U_o, \quad c_L = \frac{x_L - x_o}{x_W - x_o}(c_W - c_o) + c_o \\ U_R &= \frac{x_R - x_o}{x_E - x_o}(U_E - U_o) + U_o, \quad c_R = \frac{x_R - x_o}{x_E - x_o}(c_E - c_o) + c_o \end{aligned}}$$

Interior nodes: source term

$$U_P + 2c_P = \left(U_L + 2c_L\right) + \boxed{\int_{t_0}^{t_1} g(S_0 - S_f)dt}$$

$$U_P - 2c_P = \left(U_R - 2c_R\right) + \boxed{\int_{t_0}^{t_1} g(S_0 - S_f)dt}$$



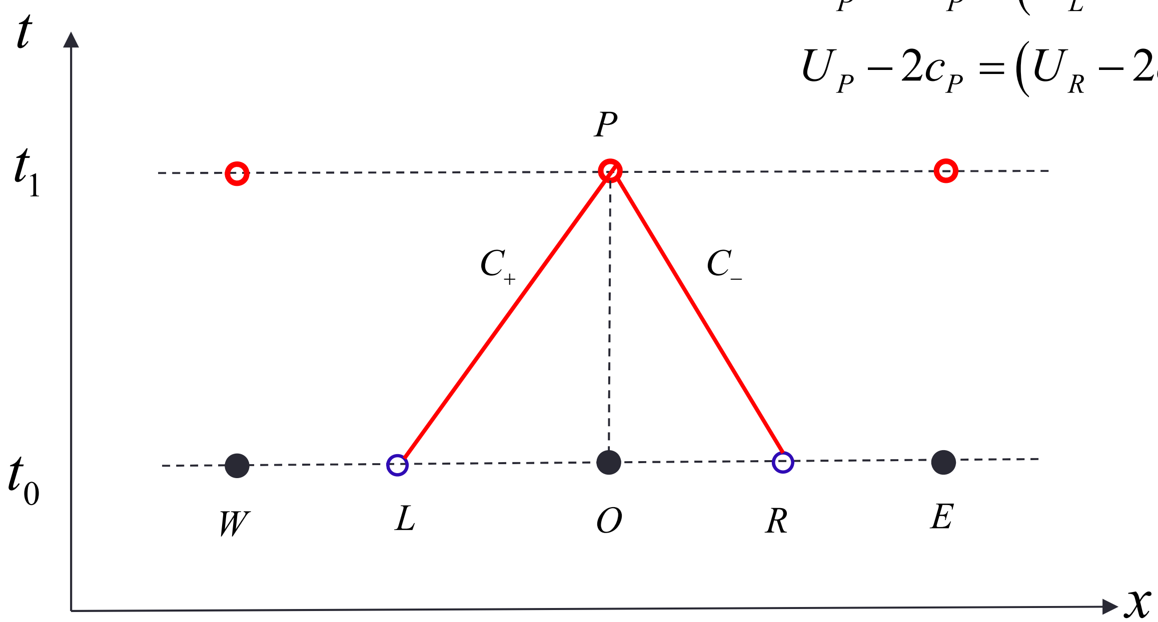
Assume a frictional slope at characteristics, and the value is close to that at point O:

$$C^+ : \int_{t_0}^{t_1} g(S_0 - S_f)dt \approx \Delta t g(S_0 - S_{f,L}) \approx \Delta t g(S_0 - S_{f,O})$$

$$C^+ : \int_{t_0}^{t_1} g(S_0 - S_f)dt \approx \Delta t g(S_0 - S_{f,R}) \approx \Delta t g(S_0 - S_{f,O})$$

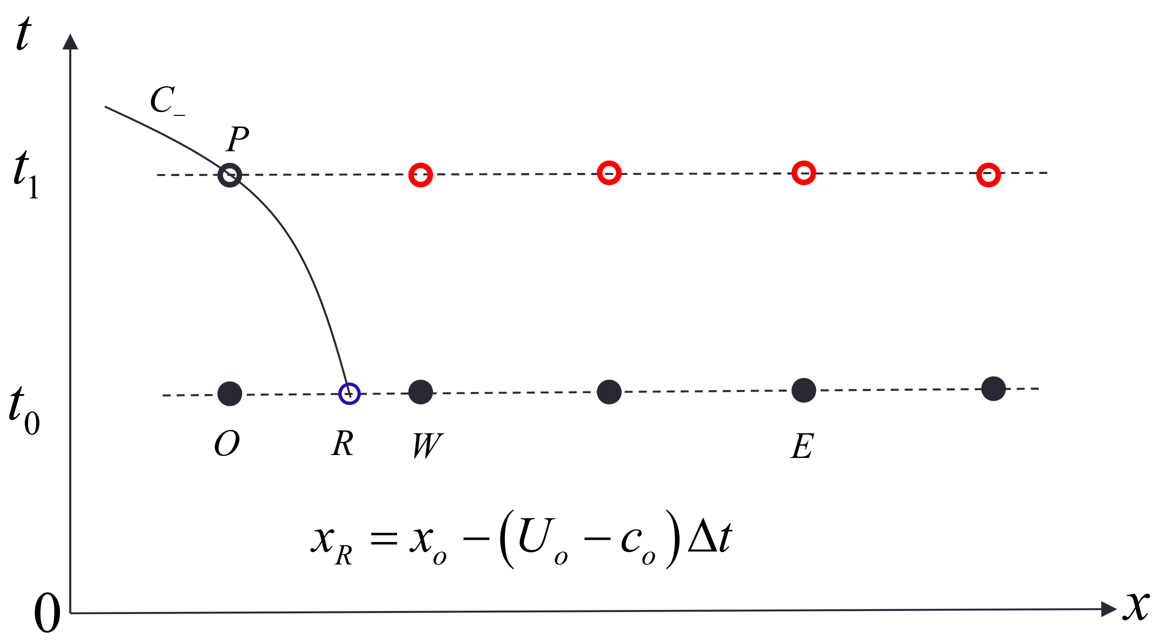
Interior nodes: solution

$$U_P + 2c_P = (U_L + 2c_L) + \Delta t g(S_0 - S_{f,O})$$
$$U_P - 2c_P = (U_R - 2c_R) + \Delta t g(S_0 - S_{f,O})$$



$$U_p = \frac{U_R + U_L}{2} + (c_L - c_R) + \Delta t g(S_0 - S_{f,O})$$
$$c_p = \frac{U_L - U_R}{4} + \frac{c_L + c_R}{2}$$

Boundary nodes:

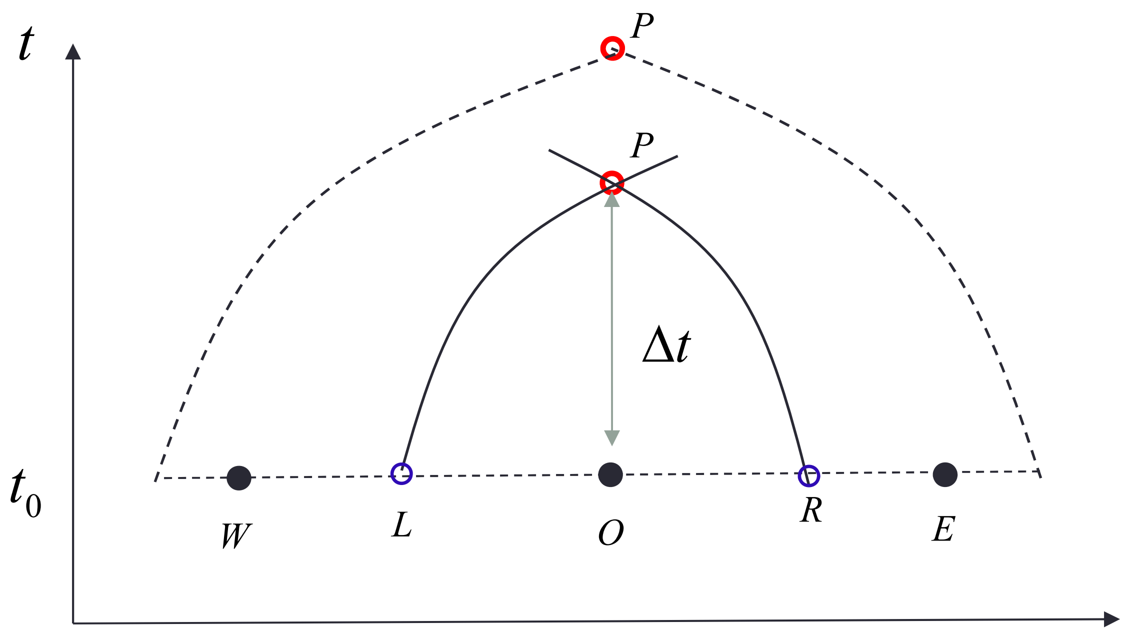


$$U_R = \frac{x_R - x_o}{x_E - x_o}\left(U_E - U_o\right) + U_o, \quad c_R = \frac{x_R - x_o}{x_E - x_o}\left(c_E - c_o\right) + c_o$$

$$\begin{cases} U_P - 2c_P = \left(U_R - 2c_R\right) + \Delta t g(S_0 - S_{f,O}) \\ \text{one B.C.} \end{cases}$$

Solve for U_p and c_p.

Criterion for stability



The method assume flow at P is fully controlled by information within WE, so the zone of dependence of P at level t_0 must be within WE
Otherwise the information outside WE can also effect point P

$$\Delta x > OR = |U_o - c_o|\Delta t$$
$$\Delta x > LO = |U_o + c_o|\Delta t$$

$$\frac{\Delta x}{\Delta t} > \alpha (U + c)_{\max}, \quad (\alpha = 0.9 < 1)$$

$$\text{or: } \Delta t \le \alpha \frac{\Delta x}{(U + c)_{\max}}, \quad (\alpha = 0.9 < 1)$$

Simple wave

Assuming the channel is friction less and the bottom is horizontal:

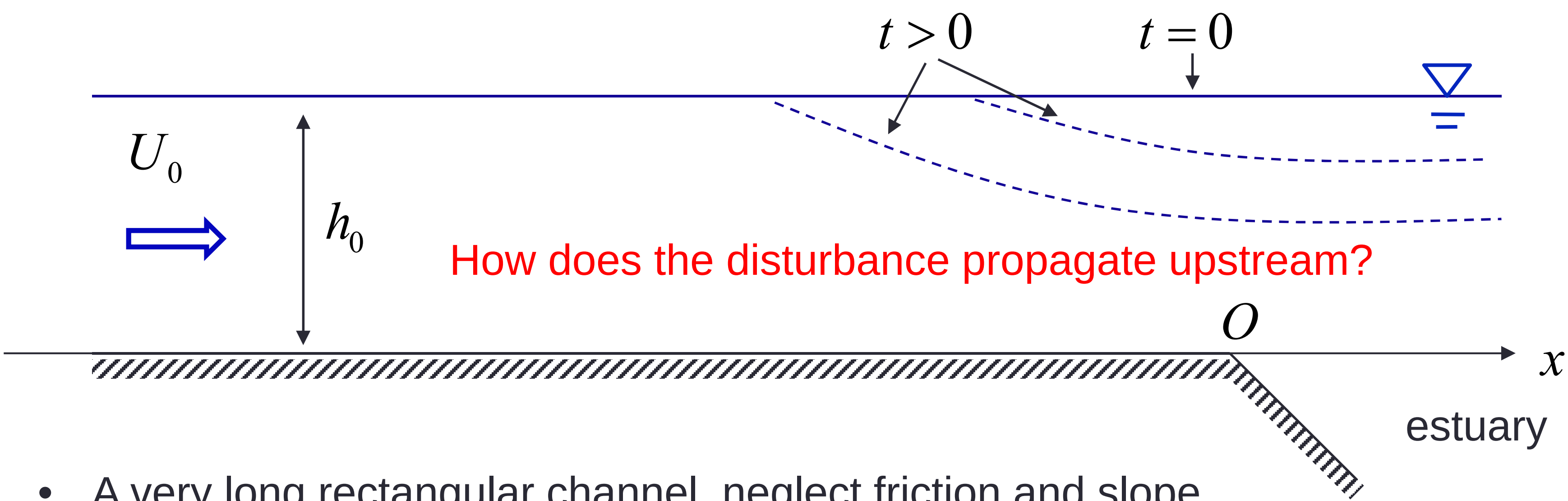
$$\frac{D(U+2c)}{Dt} = g(\cancel{S_0 - S_f}), \text{ along } \frac{dx}{dt} = U + c$$

$$\frac{D(U-2c)}{Dt} = g(\cancel{S_0 - S_f}), \text{ along } \frac{dx}{dt} = U - c$$

$$U + 2c = const, \text{ along } \frac{dx}{dt} = U + c$$

$$U - 2c = const, \text{ along } \frac{dx}{dt} = U - c$$

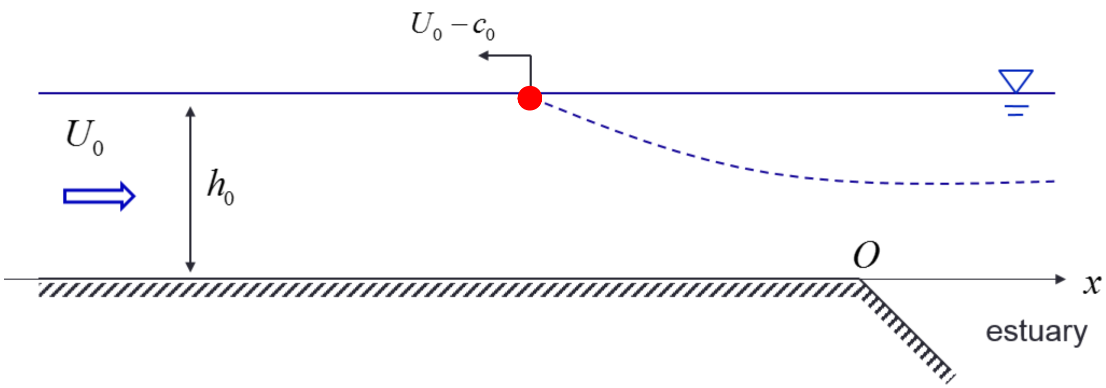
Negative surge due to a falling estuary level



- A very long rectangular channel, neglect friction and slope
- Initially (U_0, h_0) everywhere
- From t=0, the water level in the estuary starts to fall. The B.C. at x=0:

$$c(0,t) = \sqrt{gh(t)} = f(t)$$

Front of the disturbance



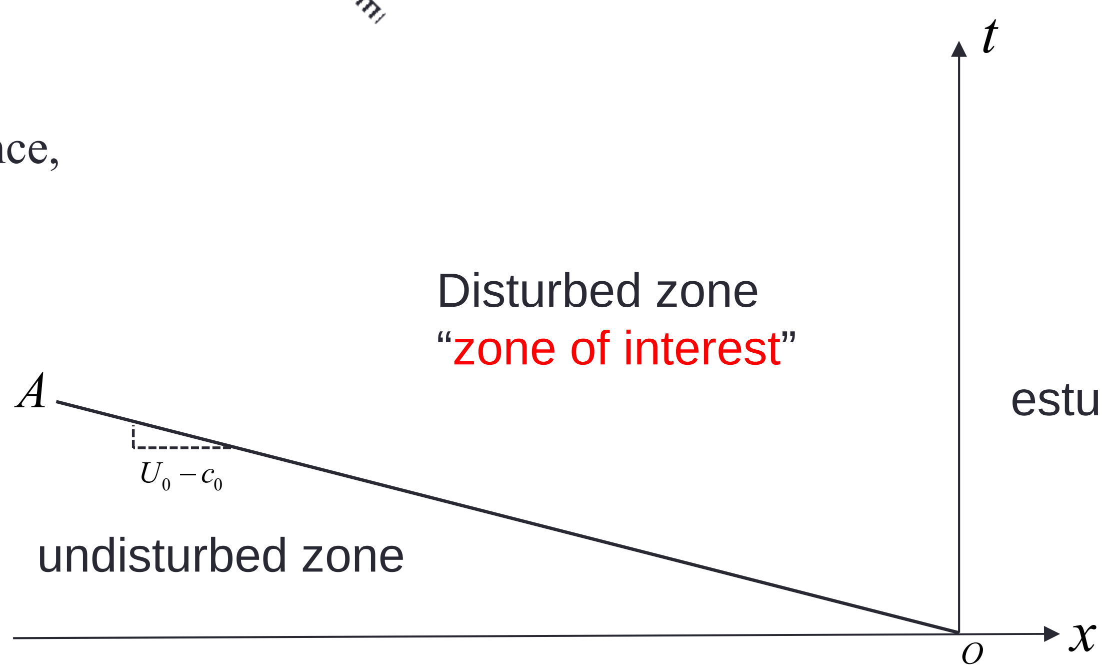
The C^- characteristics initiated at the origin denotes the propagation of the front.

At the front of the disturbance,

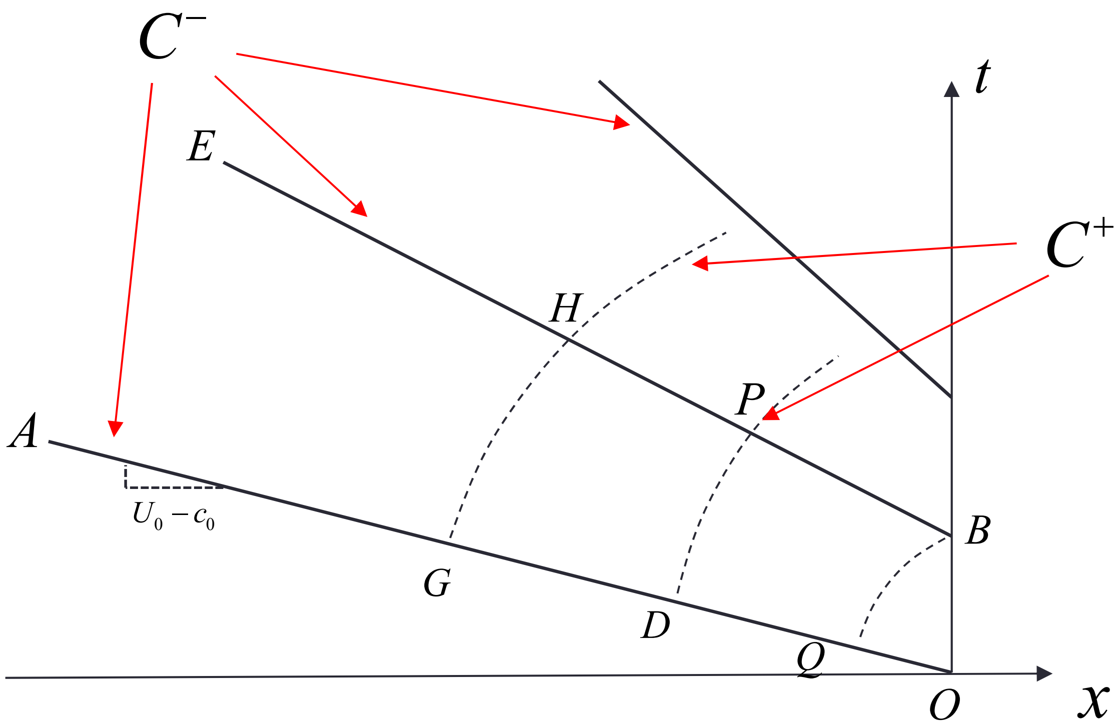
- $U=U_0$
- $h=h_0$, so $c=c_0$

$$\frac{dx}{dt} = U - c = U_0 - c_0$$

OA is a straight line

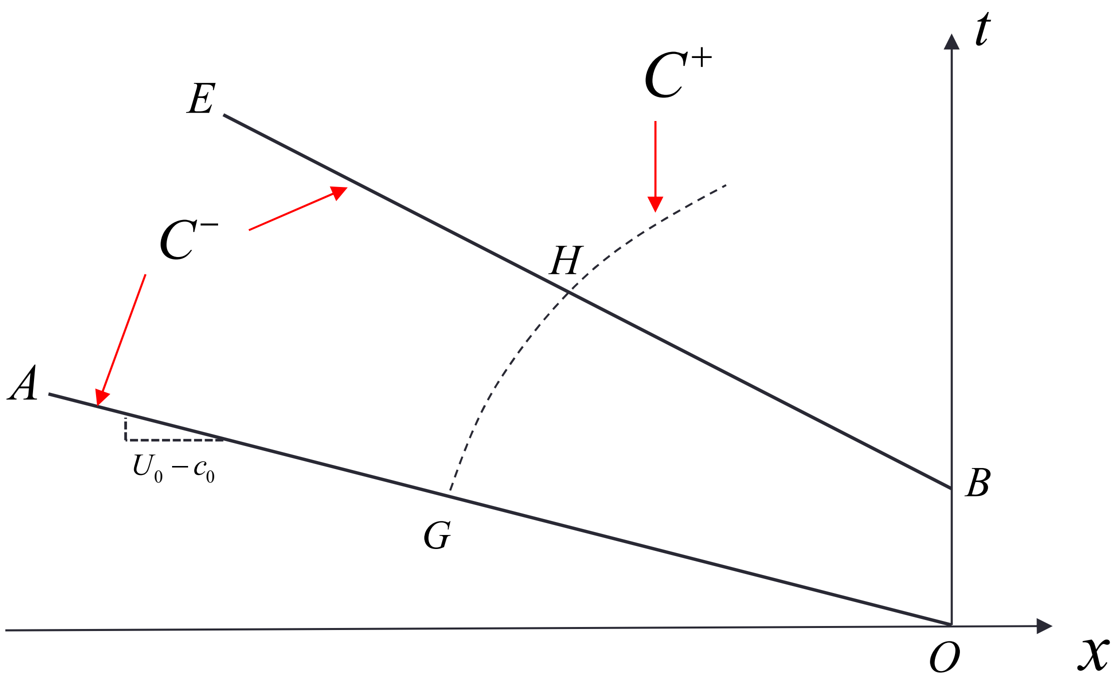


The family of C^- characteristics



All C^- characteristics, e.g. BE, are straight lines.

The family of C^+ characteristics



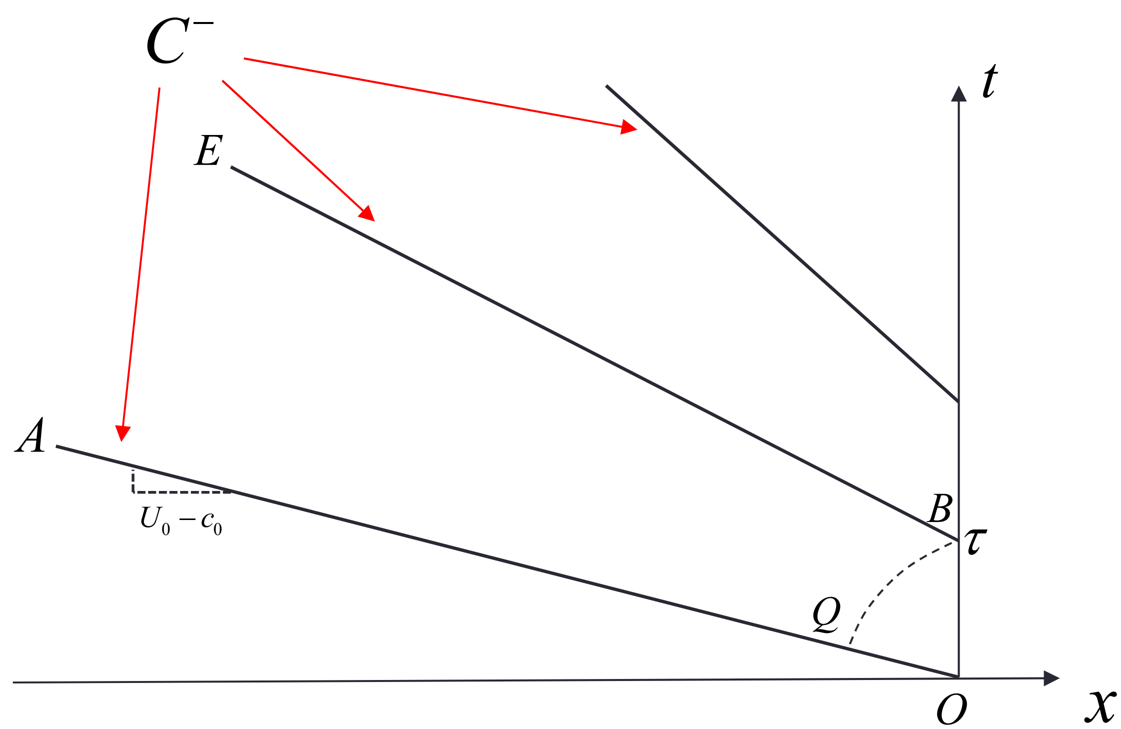
$$\left.\frac{dx}{dt}\right|_G = U_0 + 2c_0, \qquad \left.\frac{dx}{dt}\right|_H = U_0 + 2c_0 - c(\tau),$$

All C^+ characteristics, e.g. GH, are NOT straight lines.

The family of C⁻ characteristics

For a C⁻ characteristic initiated at τ on the t-axis

$$\frac{dx}{dt} = U_0 + 2c_0 - 3c(0,\tau)$$



Negative surge: the C- family of characteristics diverges if $c(0,\tau)$ decreases with τ. The traveling speed of the front is the highest.

Along each C⁻ flow condition remains unchanged:

$$\begin{cases} U = U(0,\tau) = U_0 + 2c_0 - 2c(0,\tau) \\ c = c(0,\tau) \;\Rightarrow\; h = h(0,\tau) \end{cases}$$

Flow condition at a specified *x-t* point

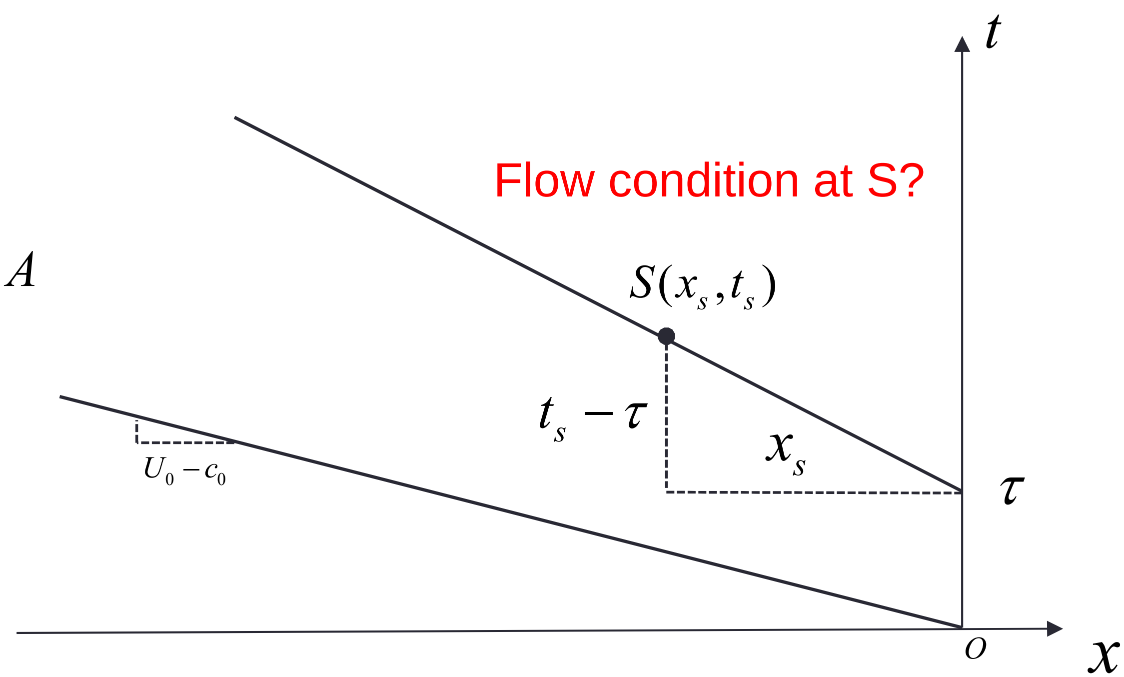
For the C- passing through S:

$$\frac{dx}{dt} = \frac{x_s}{t_s - \tau}$$

$$\frac{dx}{dt} = U_0 + 2c_0 - 3c(0,\tau)$$

$$\boxed{\frac{x_s}{t_s - \tau} = U_0 + 2c_0 - 3\sqrt{gh(0,\tau)}}$$

An equation for τ



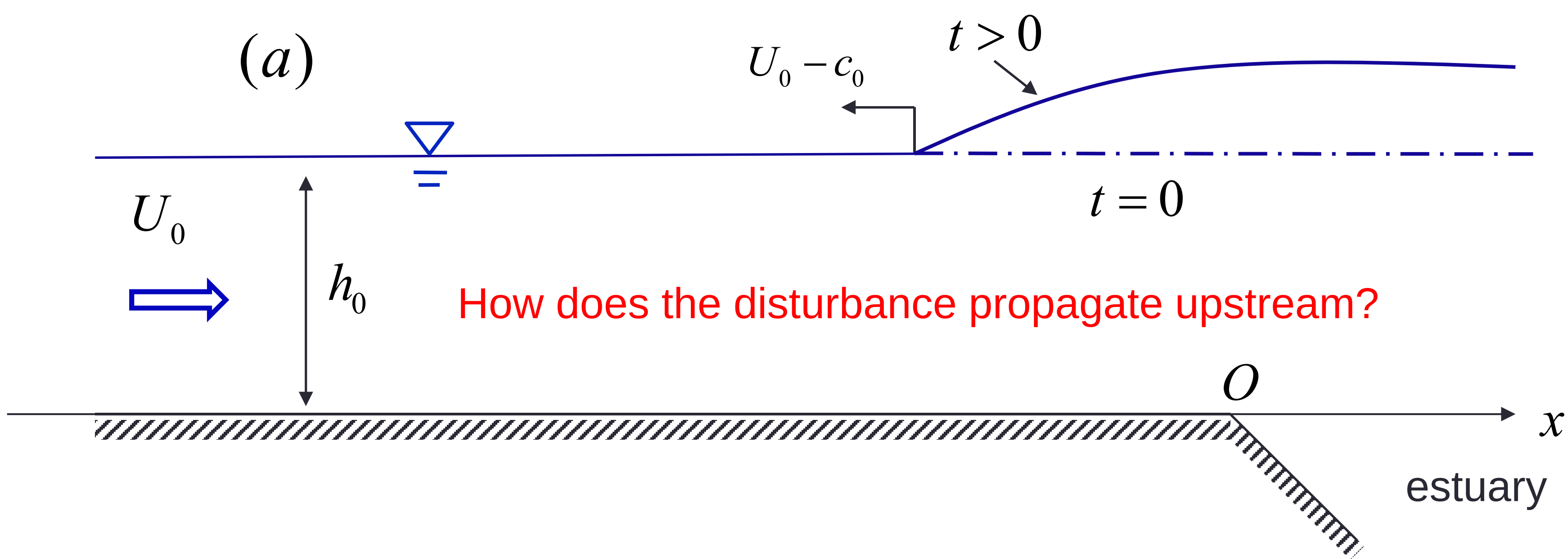
$$U(x_s, t_s) = U(0,\tau)$$

$$c(x_s, t_s) = c(0,\tau) \Rightarrow h(x_s, t_s) = h(0,\tau)$$

Example

Water flows at a uniform depth of 1.52 m and velocity of 0.9 m/s in a wide rectangular channel into a large estuary. The estuary level is initially the same as the river level at the mouth, when it starts of fall at a rate of 0.3 m/hr for the next 3 hours. Neglecting channel friction and assuming that the channel bed is horizontal, determine the time it takes for the level of the river to fall by 0.6 m at a location 1600 m from the estuary. How far upstream will the river level starts to fall at this time?

Positive surge with a rising estuary level



- A very long rectangular channel, neglect friction and slope
- Initially (U_0, h_0) everywhere
- From t=0, the water level in the estuary starts to rise. The B.C. at x=0:

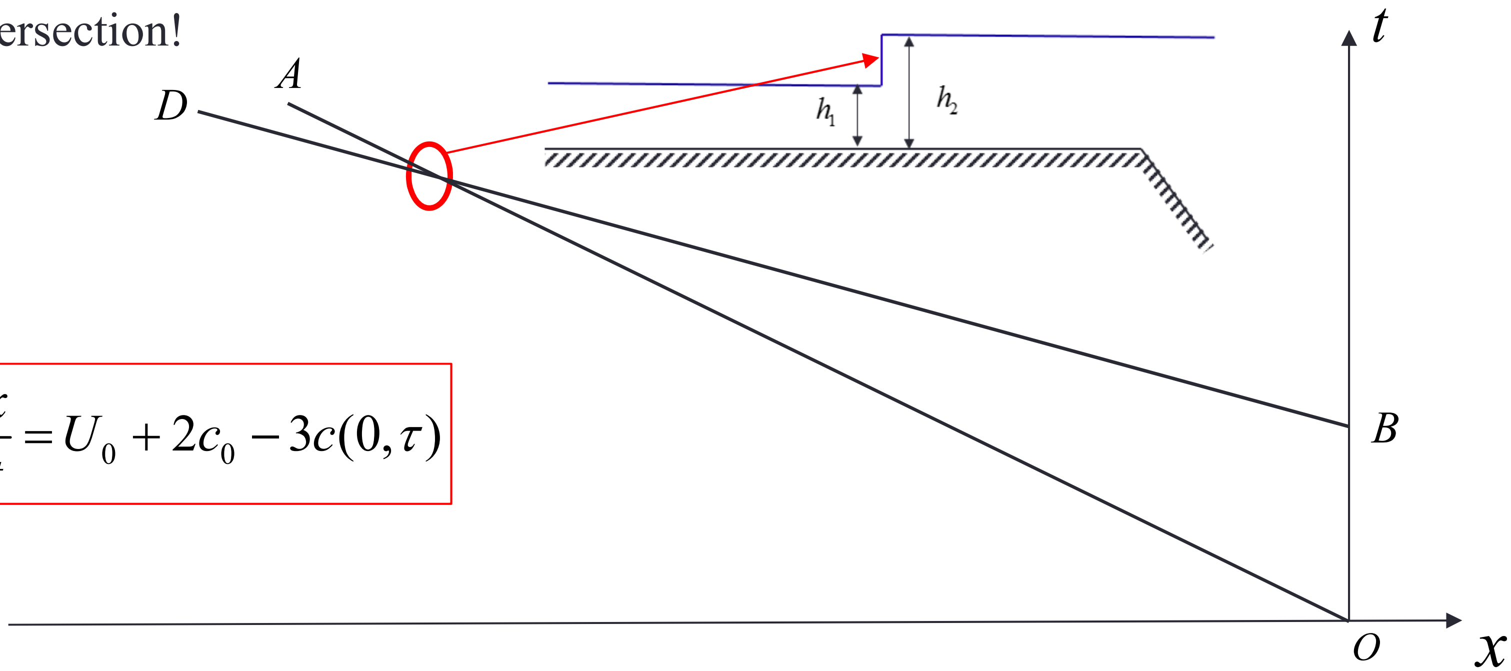
$$c(0,t) = \sqrt{gh(t)} = f(t)$$

The C- characteristics

Similar to negative surge:

- OA denotes the front
- All C- are straight lines
- All C+ are curves

Solution is invalid beyond the intersection!

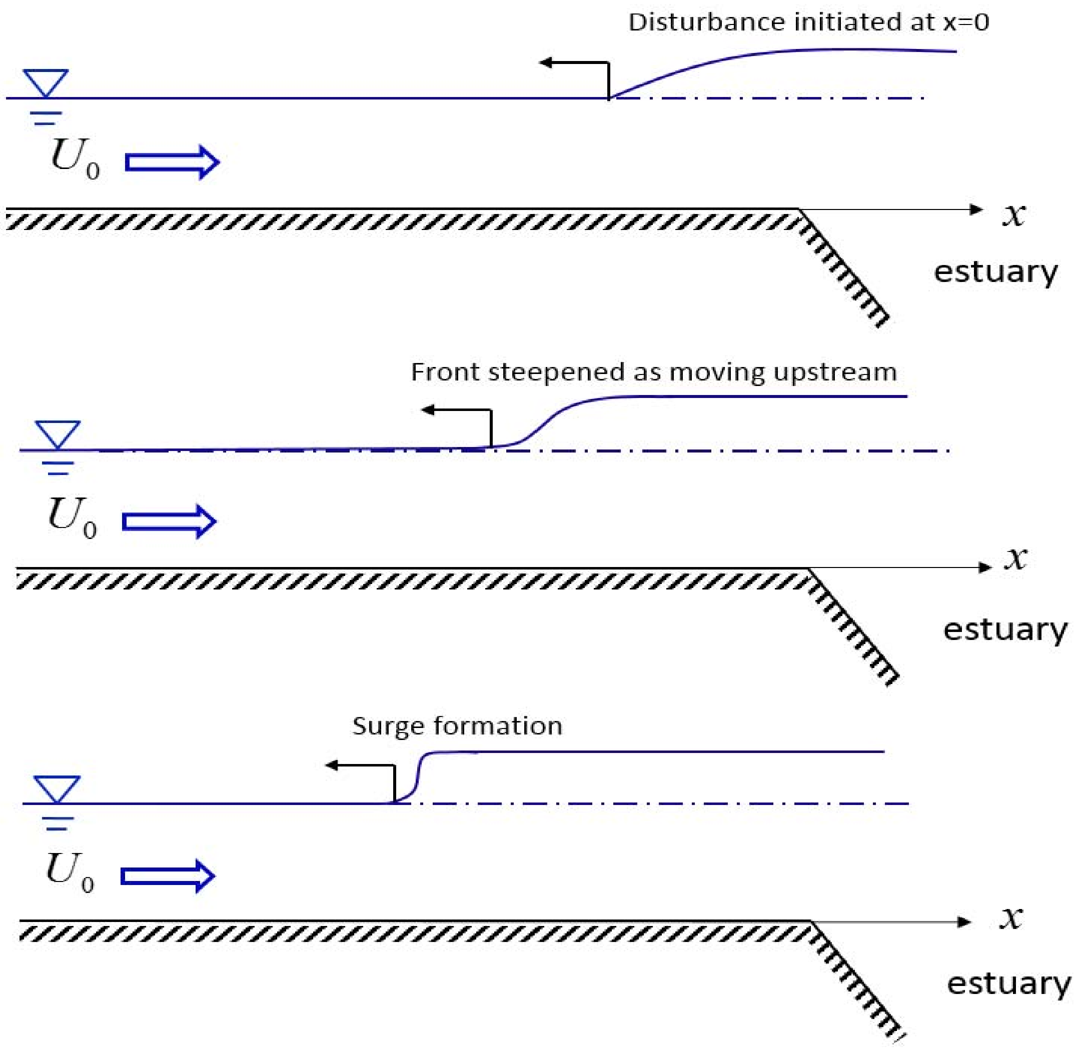


$$\frac{dx}{dt} = U_0 + 2c_0 - 3c(0,\tau)$$

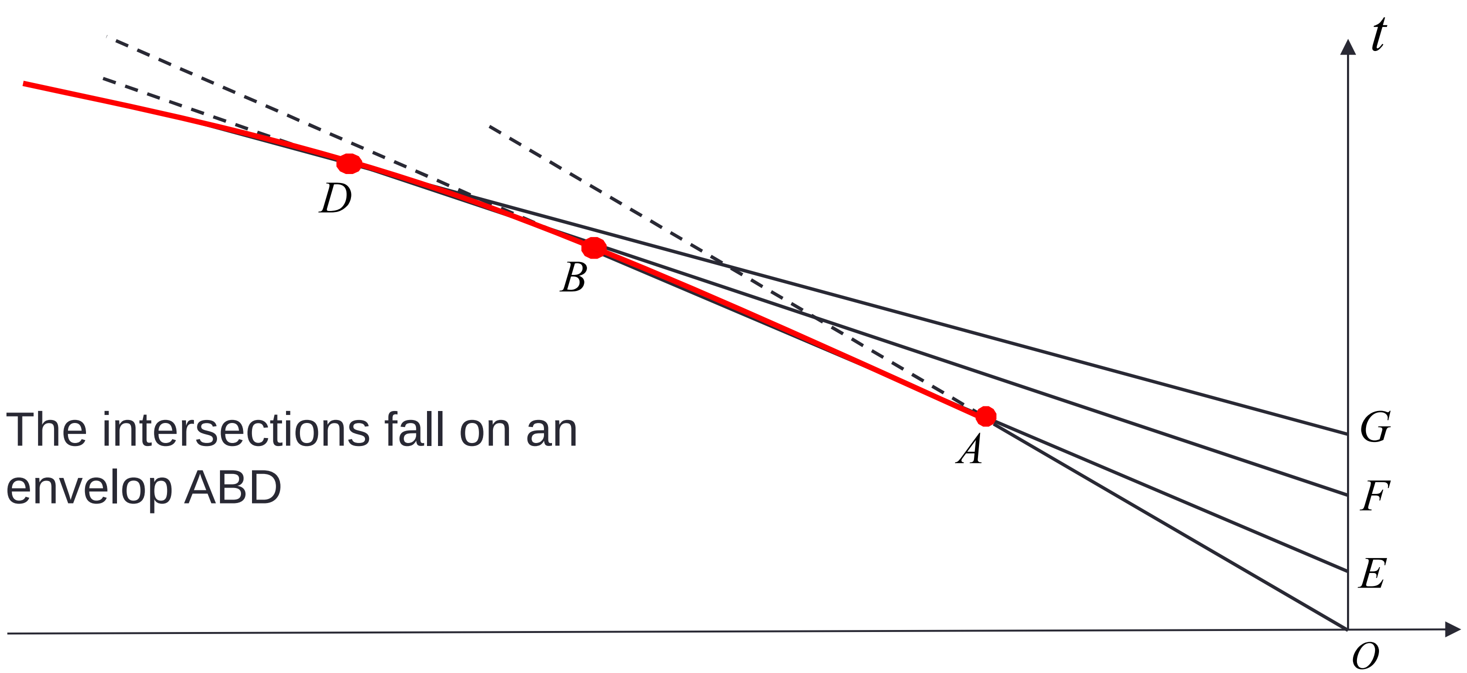
The slope of C- decreases with τ, so a C^- characteristic initiated later will intersect with all earlier C^- characteristics.

What happened at the intersections? Two possible depths!

Formation of surge



Envelop of intersections

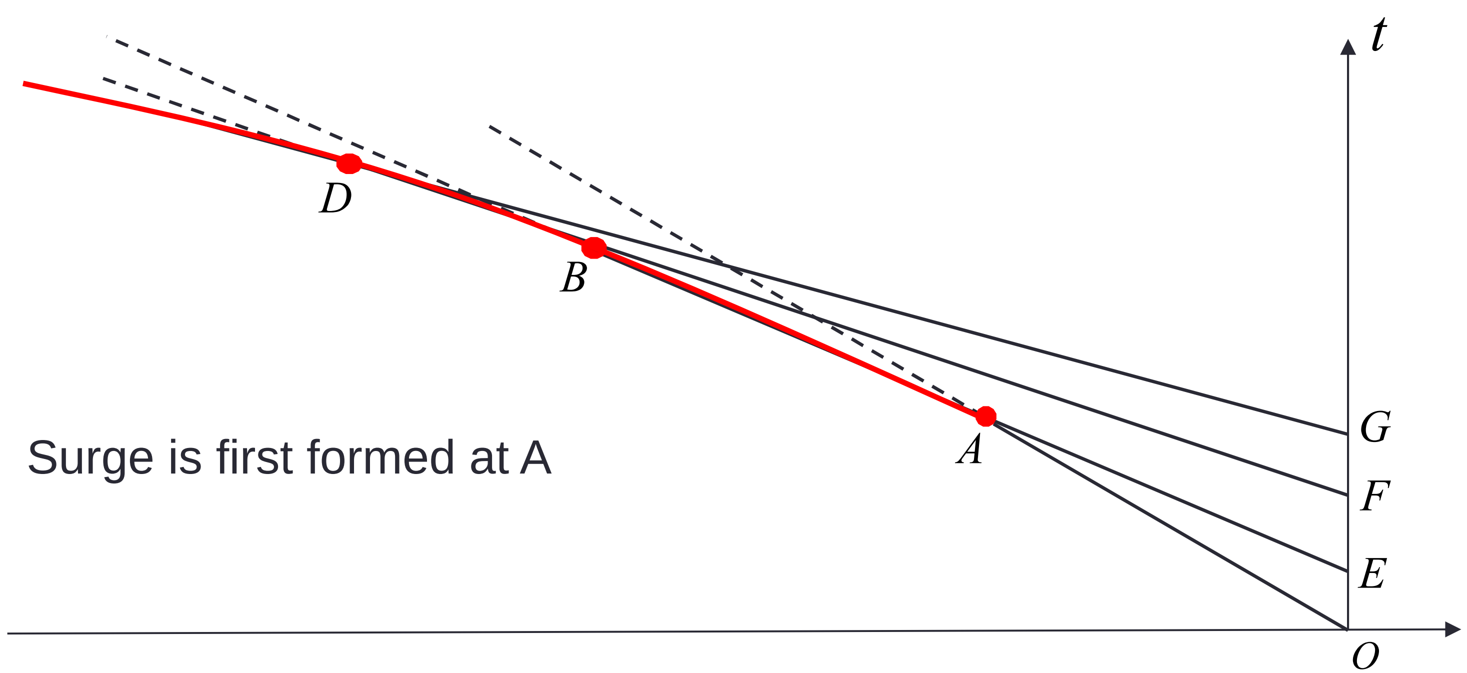


The intersections fall on an envelop ABD

$$x = \frac{[U_0 + 2c_0 - 3\sqrt{gh(\tau)}]^2}{3\dfrac{\partial\sqrt{gh(\tau)}}{\partial\tau}}$$

$$t = \tau + \frac{x}{U_0 + 2c_0 - 3\sqrt{gh(0,\tau)}}$$

Incipient of surge



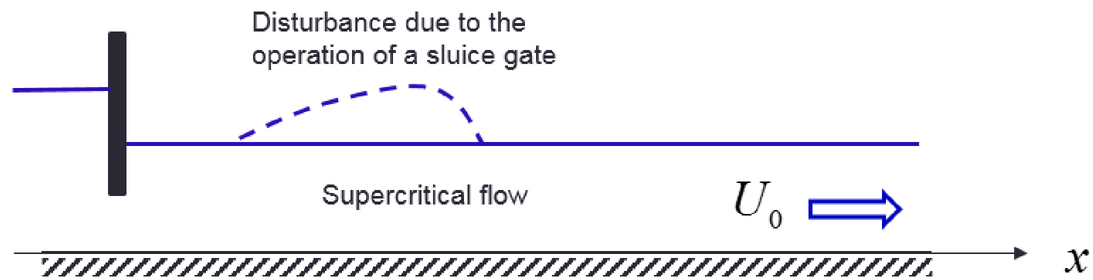
Surge is first formed at A

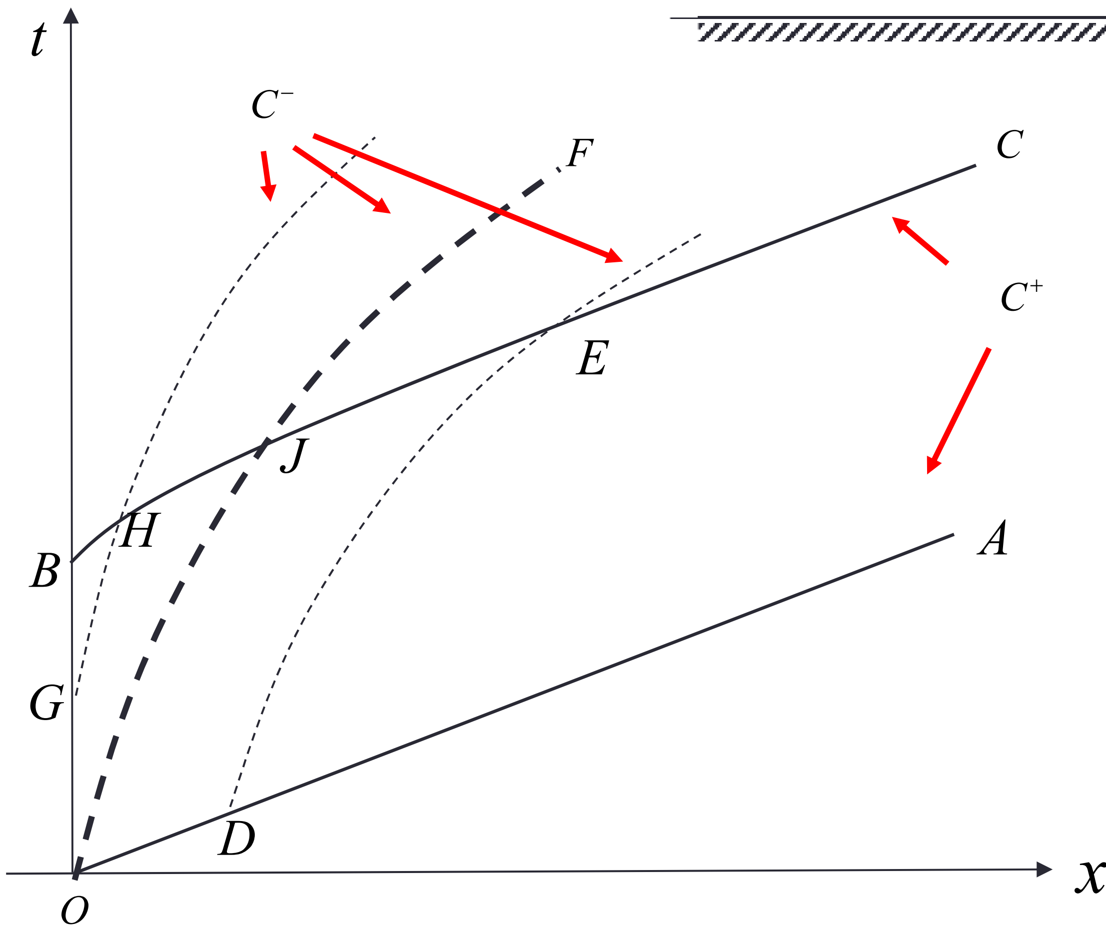
$$x = \frac{(U_0 - c_0)^2}{3\left.\dfrac{\partial c(0,\tau)}{\partial \tau}\right|_{\tau=0}} = \frac{(U_0 - c_0)^2}{\dfrac{3}{2}\sqrt{\dfrac{g}{h_0}}\left.\dfrac{\partial h(0,\tau)}{\partial \tau}\right|_{\tau=0}}$$

$$t = \frac{x}{U_0 - \sqrt{gh_0}}$$

Q(0,t) as B.C.?

Simple wave problem for supercritical flow



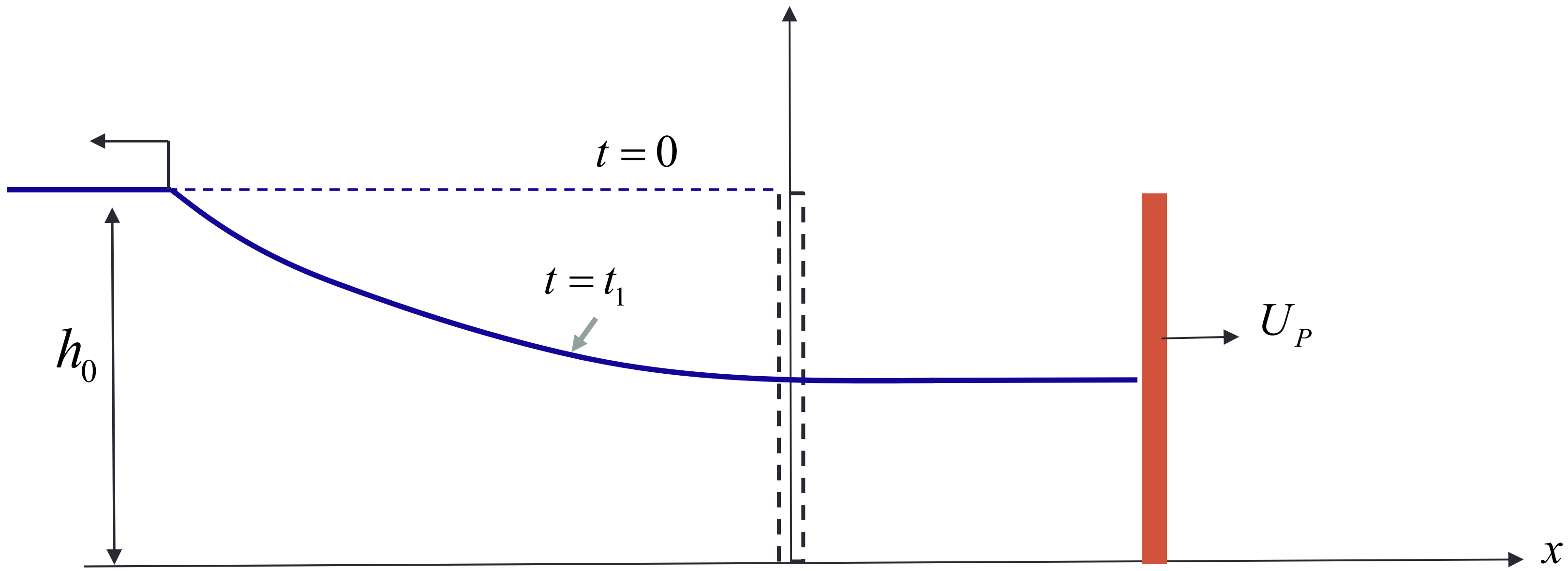


Different requirement on B.Cs.

No family of characteristic are all straight lines!

No simple solution.

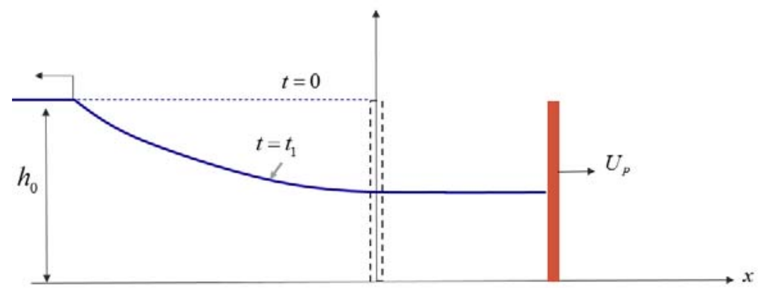
Dam-break problem



- Initially stagnant water in the upstream, no water in the downstream
- Assume rectangular channel, neglect friction and slope
- Plate starts to move downstream until a constant speed is reached
- Find the surface profile
- Find the average velocity

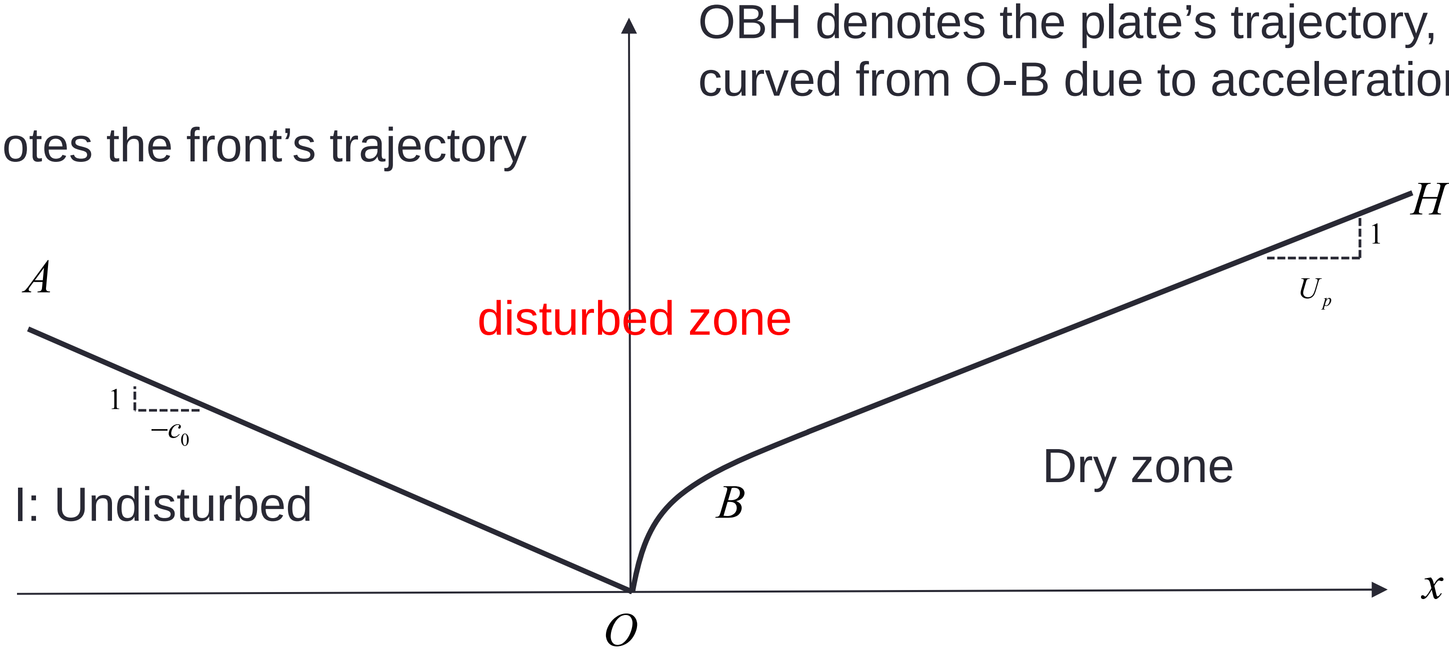
The disturbed zone on x-t plane

The region between the plate and the front of upstream-moving disturbance is the disturbed zone



OBH denotes the plate's trajectory, curved from O-B due to acceleration

OA denotes the front's trajectory



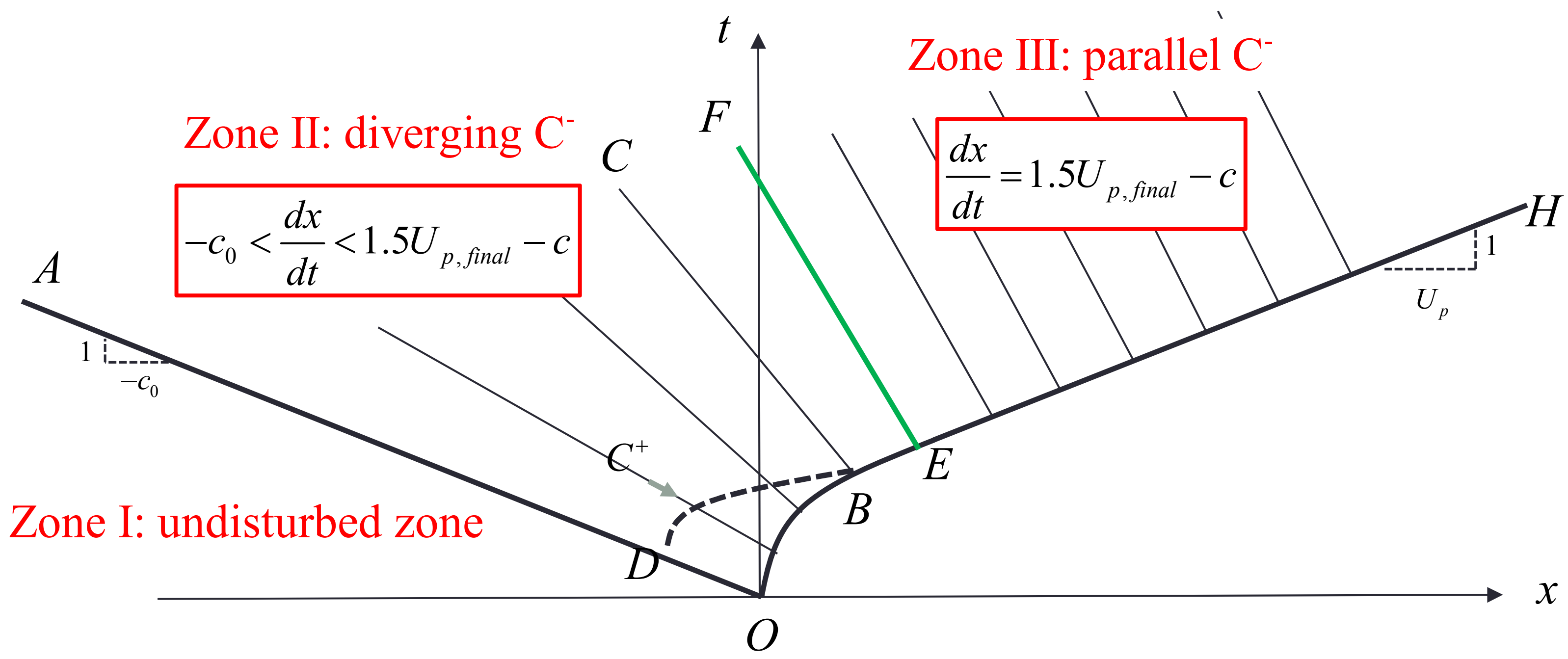
Family of C- characteristics

All C- characteristics are straight lines:

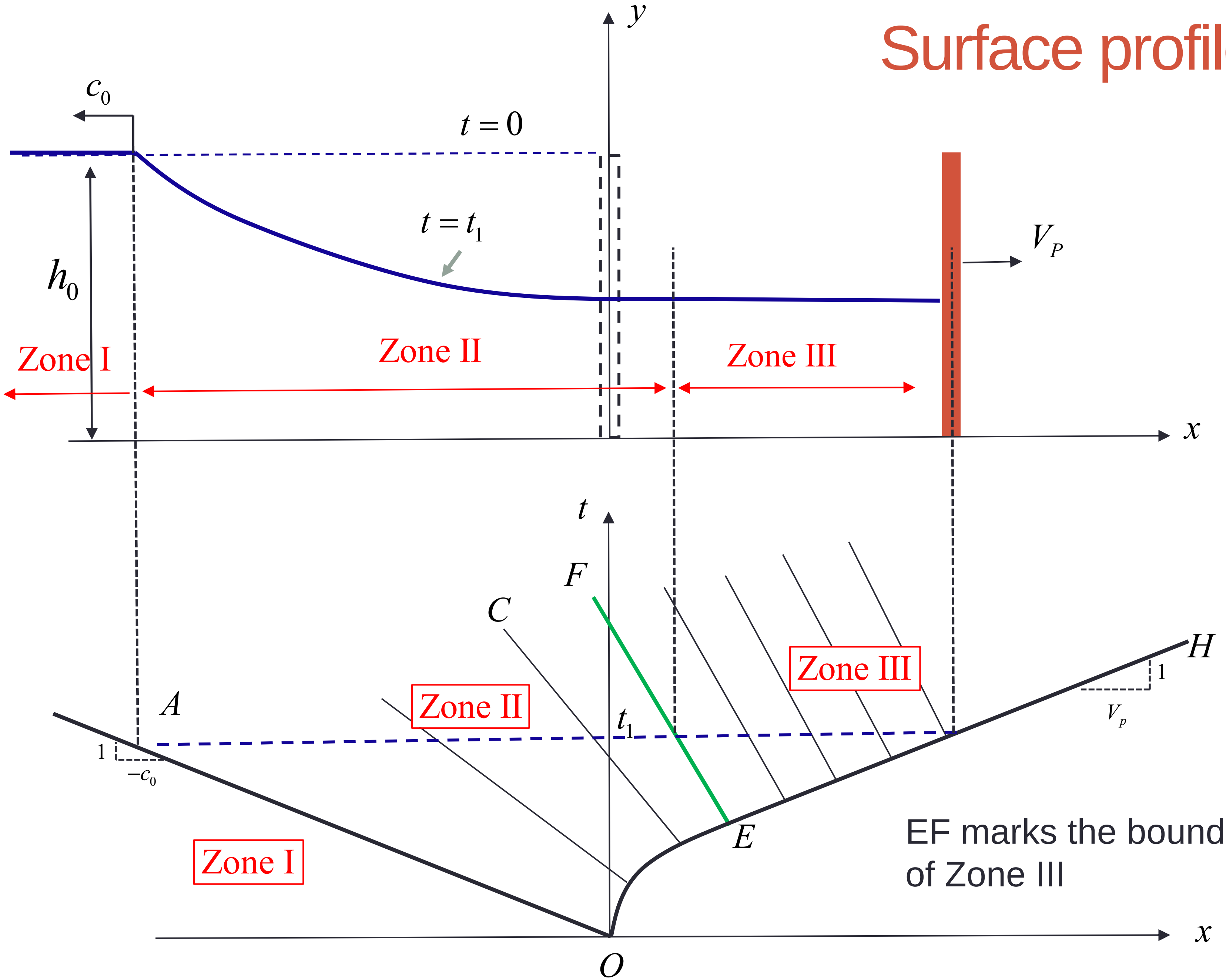
$$U = U_p(\tau)$$
$$c = c_0 - 0.5U_p(\tau)$$

$$\frac{dx}{dt} = 1.5U_p(\tau) - c_0$$

τ is when the C^- characteristic initiated on OEH



Surface profile



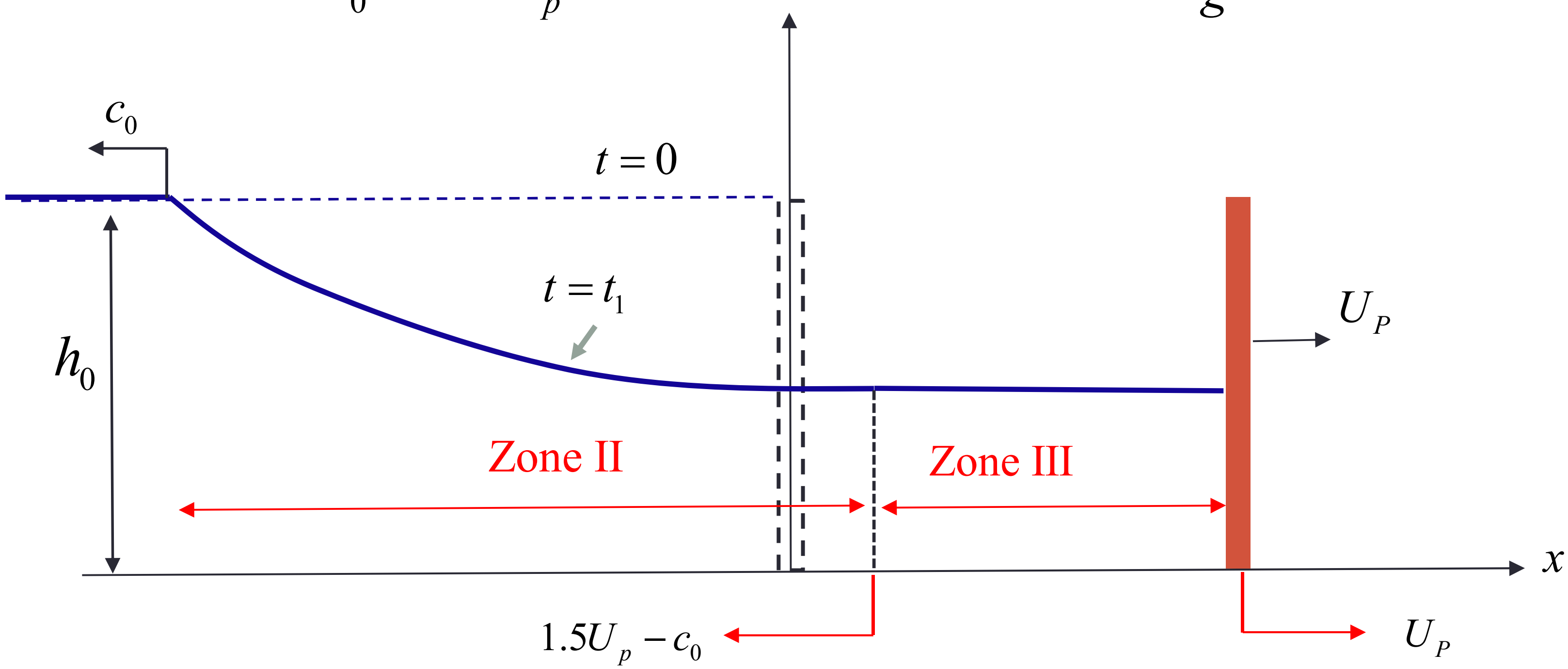
EF marks the boundary of Zone III

Zone III: constant water depth and velocity

$$U = U_p$$

$$c = c_0 - 0.5U_p$$

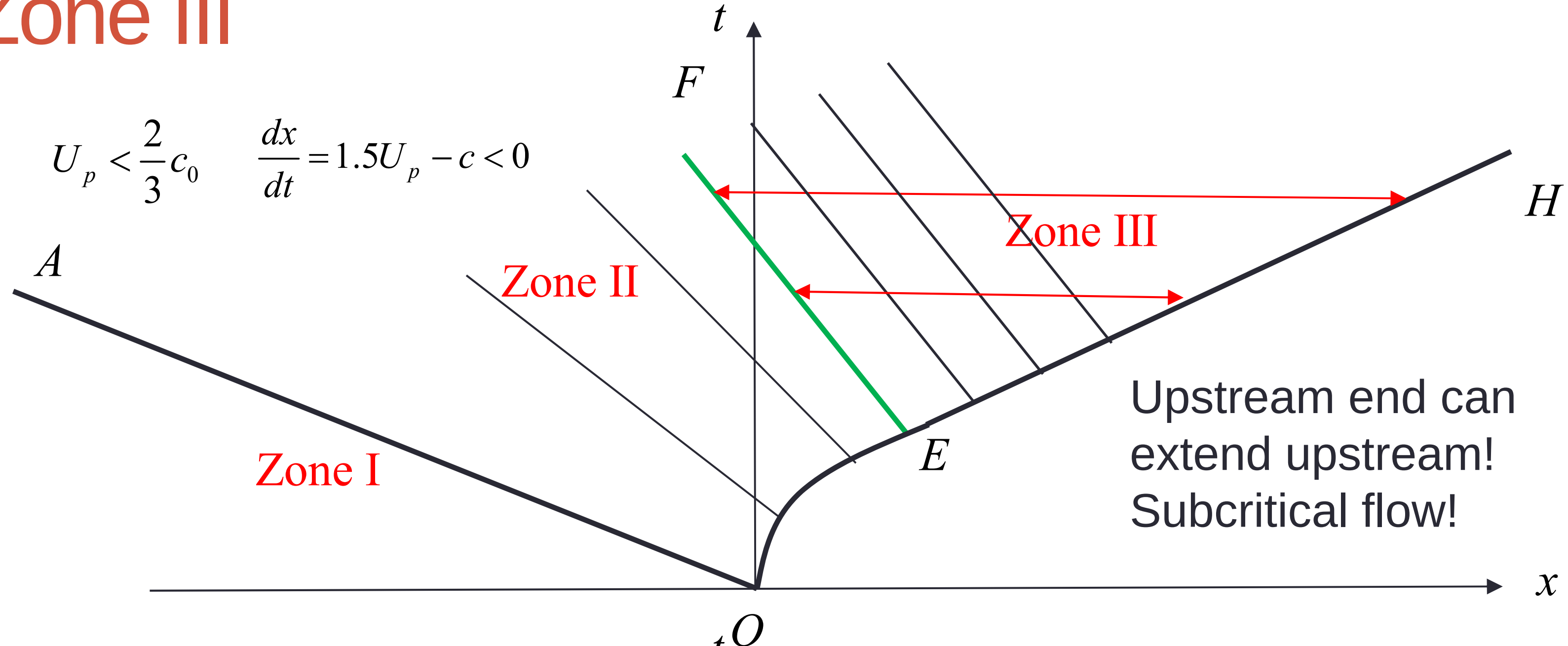
$$h = \frac{(c_0 - U_p/2)^2}{g}$$



- Upstream end of zone III moves at a speed $1.5U_p$-c_0
- Downstream end of zone III moves with the plate

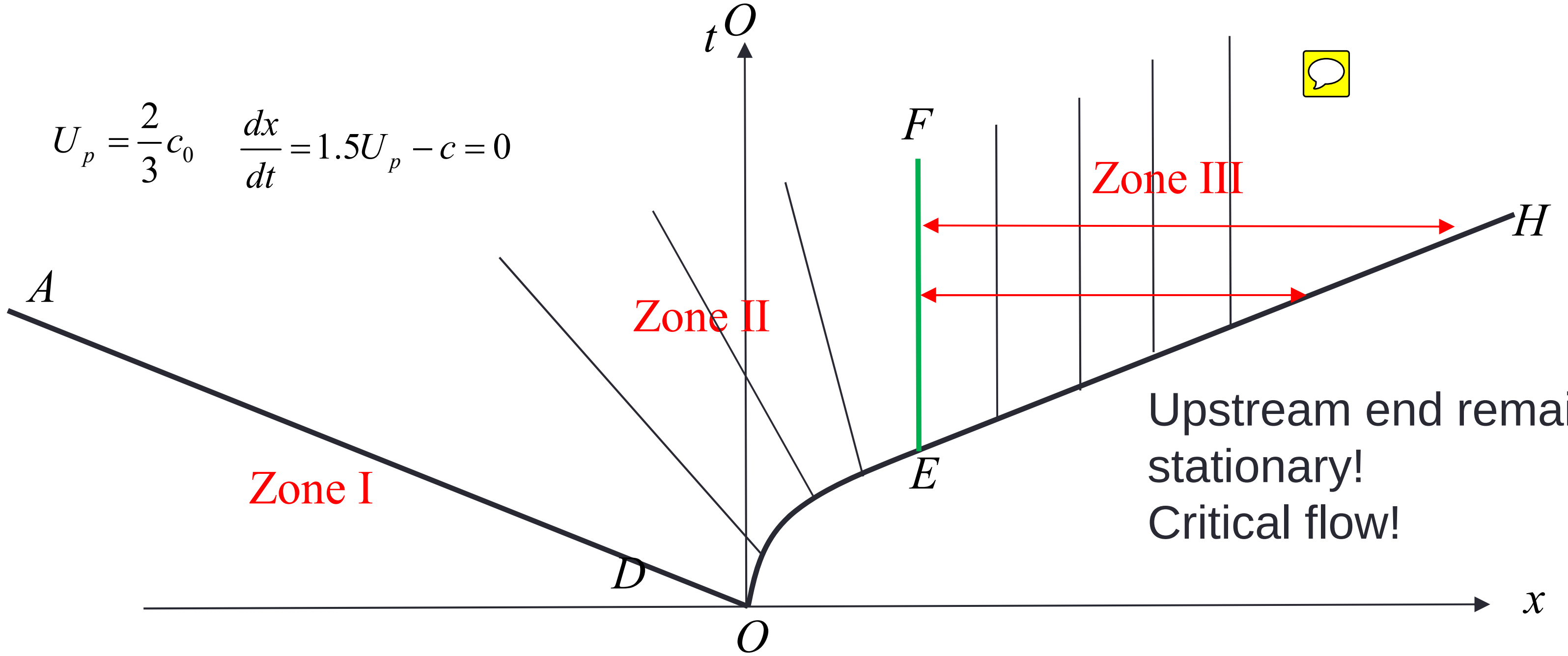
Zone III

$$U_p < \frac{2}{3}c_0 \qquad \frac{dx}{dt} = 1.5U_p - c < 0$$



Upstream end can extend upstream!
Subcritical flow!

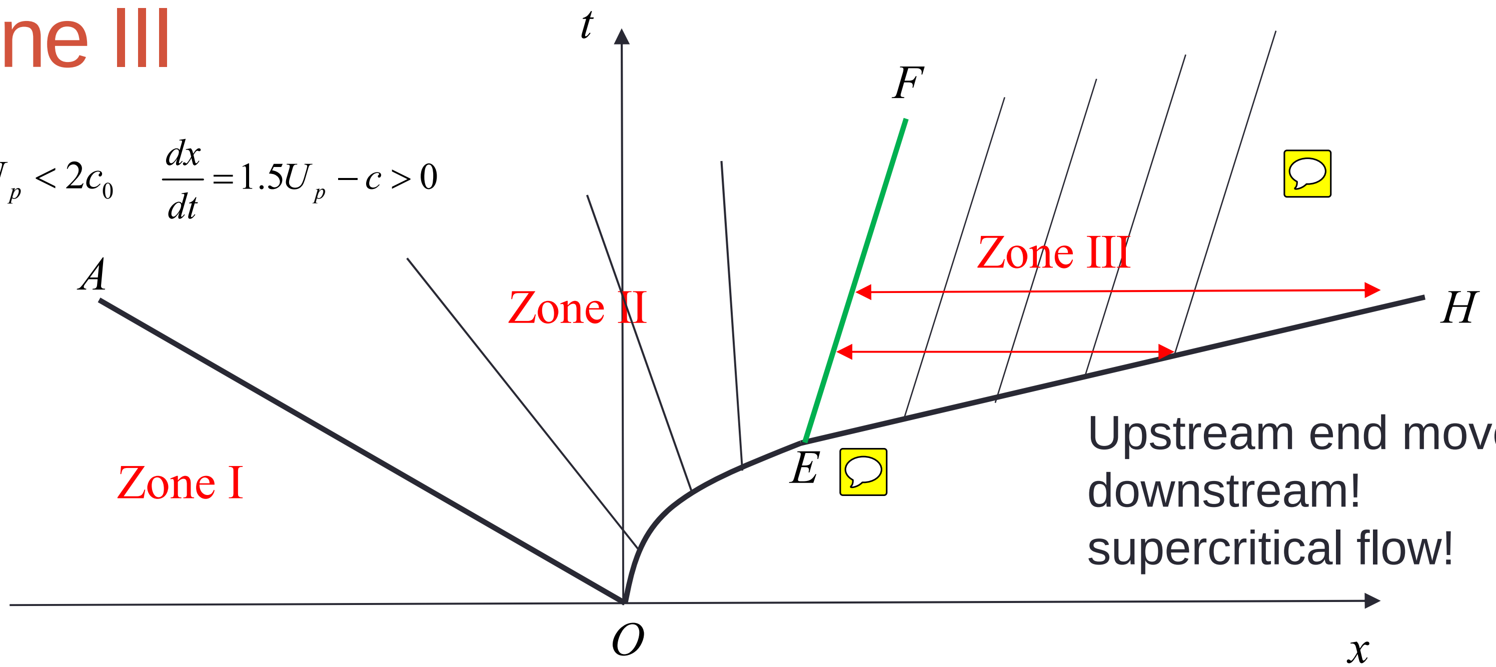
$$U_p = \frac{2}{3}c_0 \qquad \frac{dx}{dt} = 1.5U_p - c = 0$$



Upstream end remain stationary!
Critical flow!

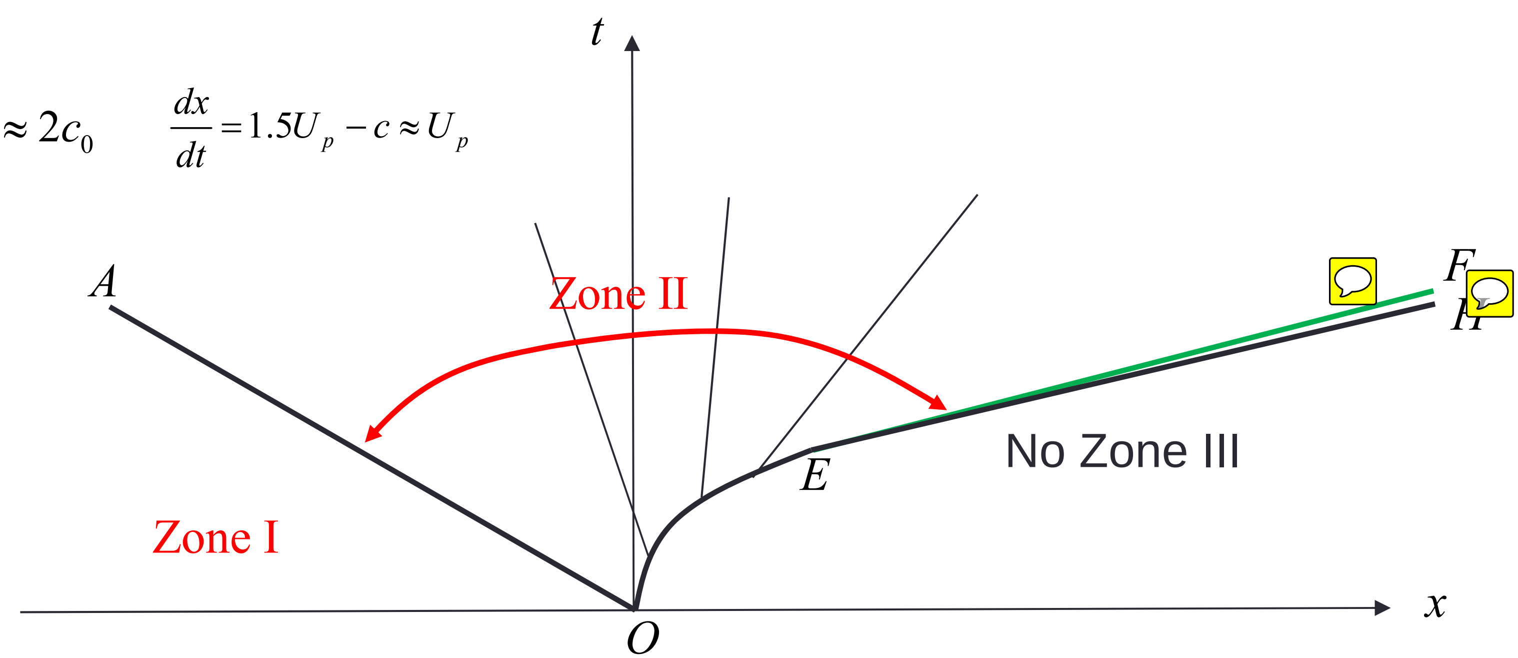
Zone III

$$\frac{2}{3}c_0 < U_p < 2c_0 \qquad \frac{dx}{dt} = 1.5U_p - c > 0$$



Upstream end moves downstream!
supercritical flow!

$$U_p \approx 2c_0 \qquad \frac{dx}{dt} = 1.5U_p - c \approx U_p$$



No Zone III

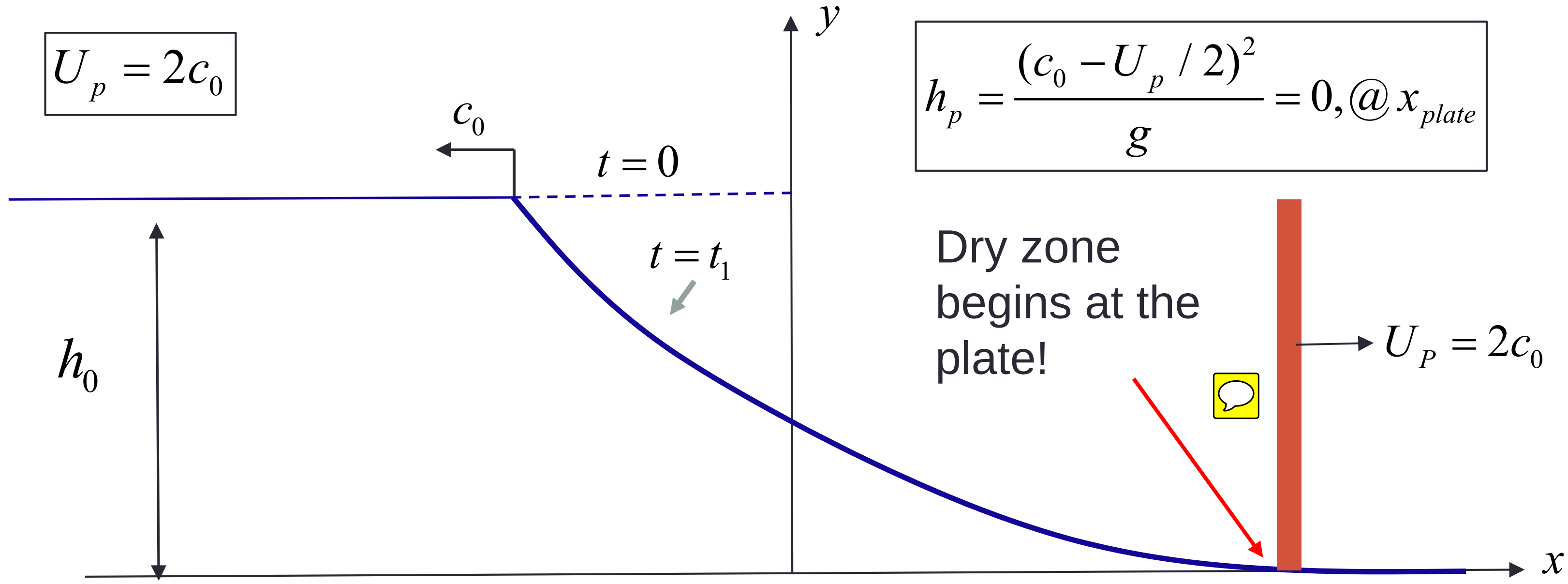
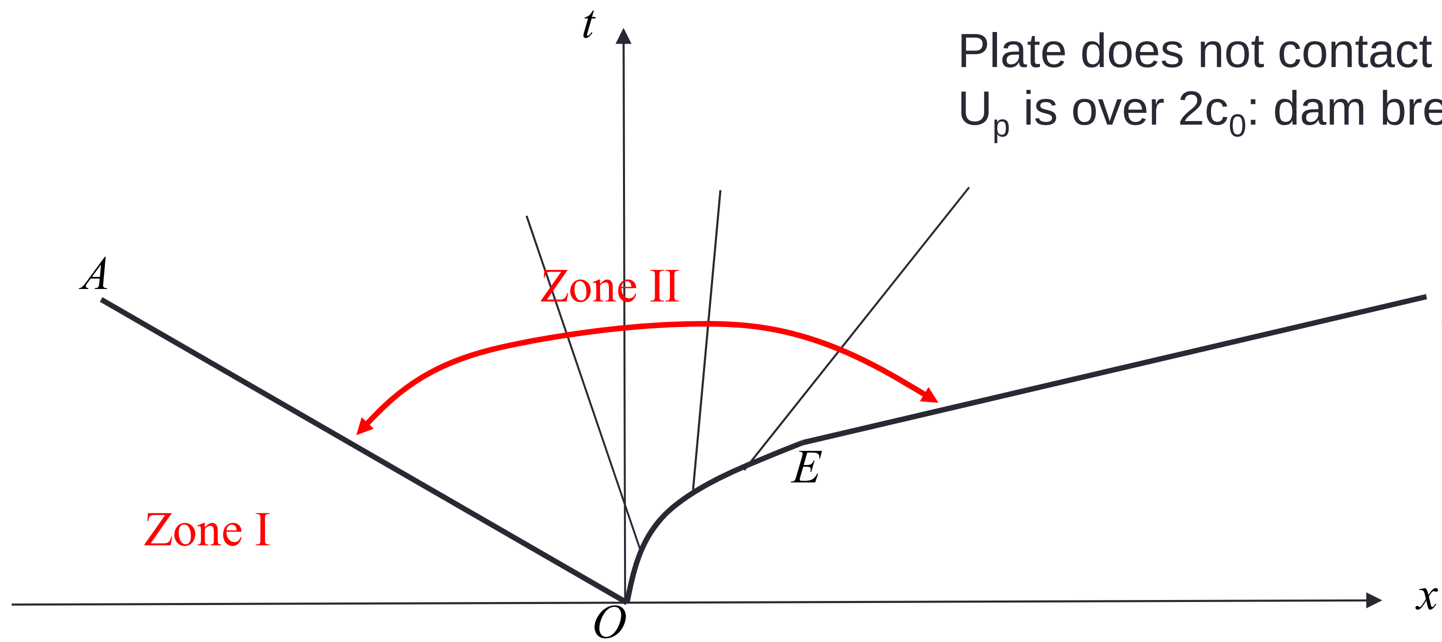


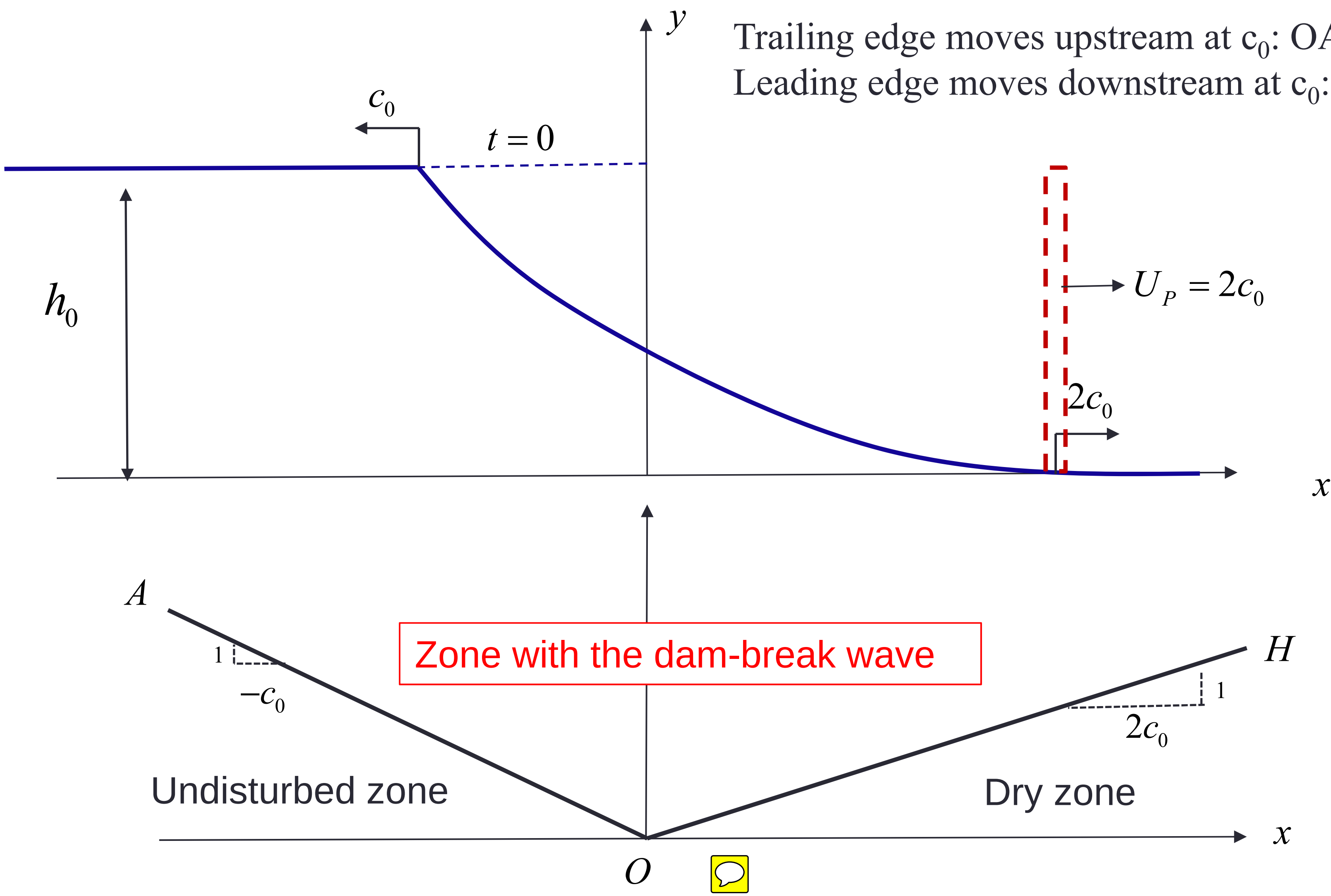
Plate does not contact water is U_p is over $2c_0$: dam break!



Dam-break wave

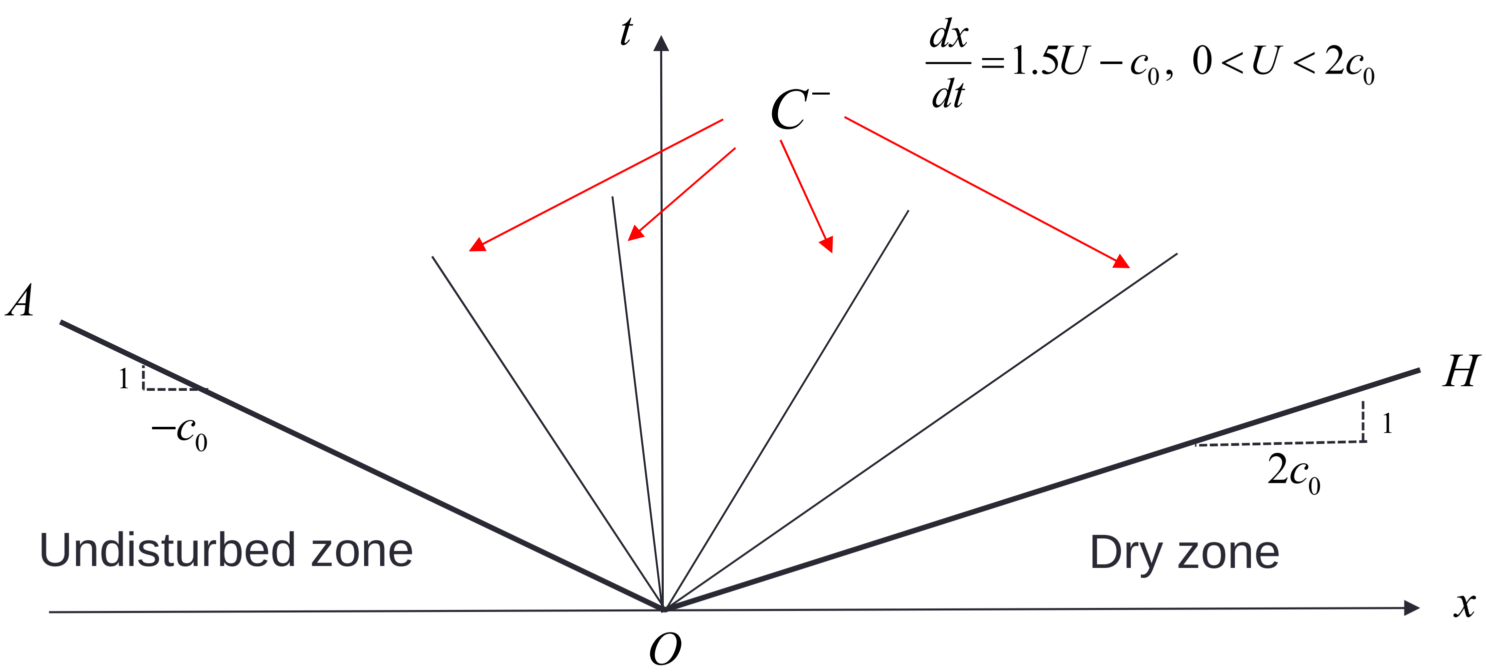
Dam break is equivalent to the plate immediately moving at $U_p = 2c_0$

Trailing edge moves upstream at c_0: OA
Leading edge moves downstream at c_0: OH

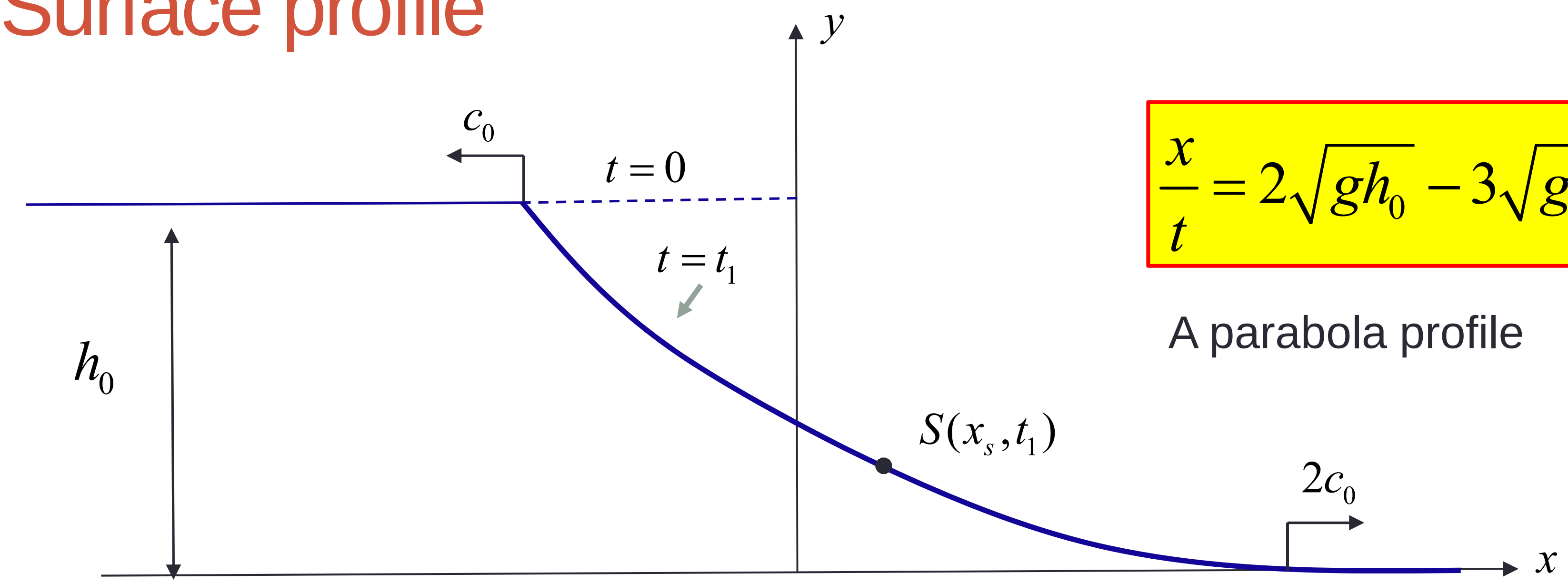


C- family of characteristics

- Many C- is initiated at the origin, since U at the origin can take any value from 0 to $2c_0$.

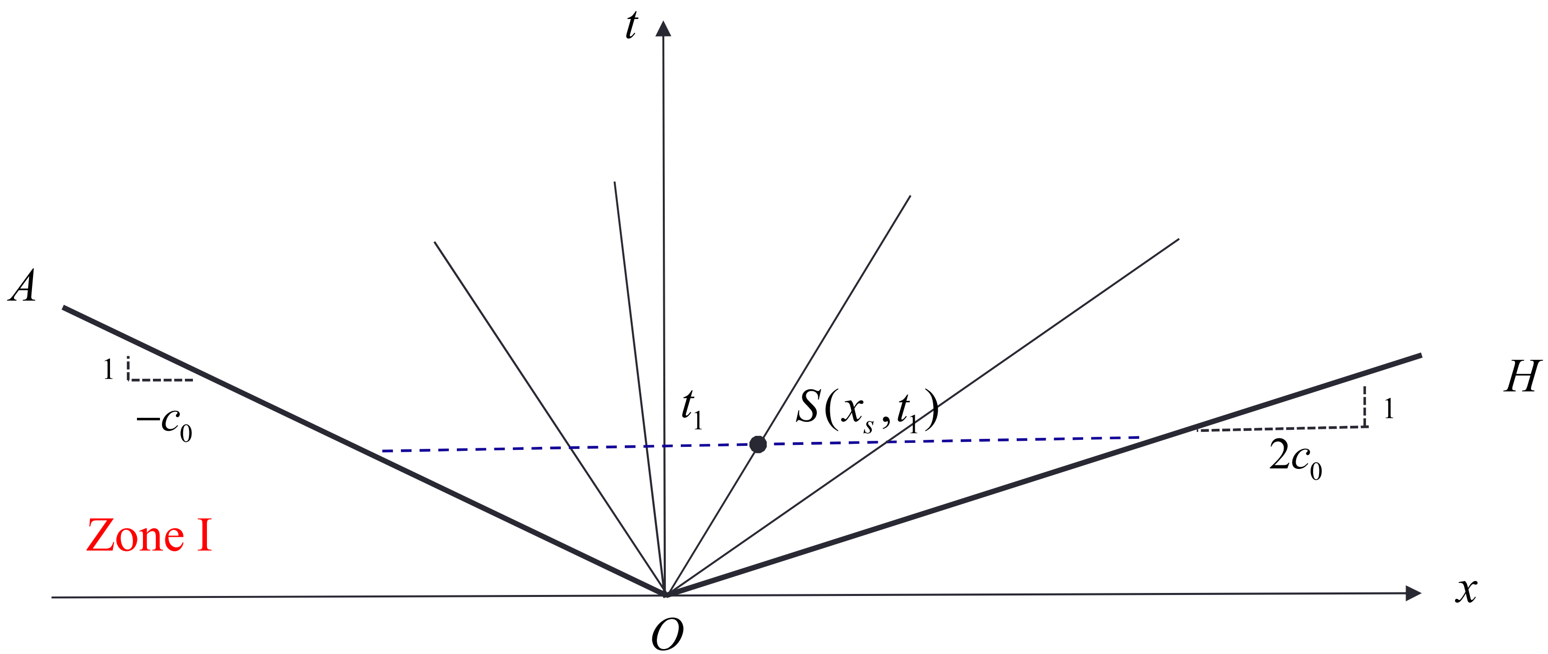


Surface profile

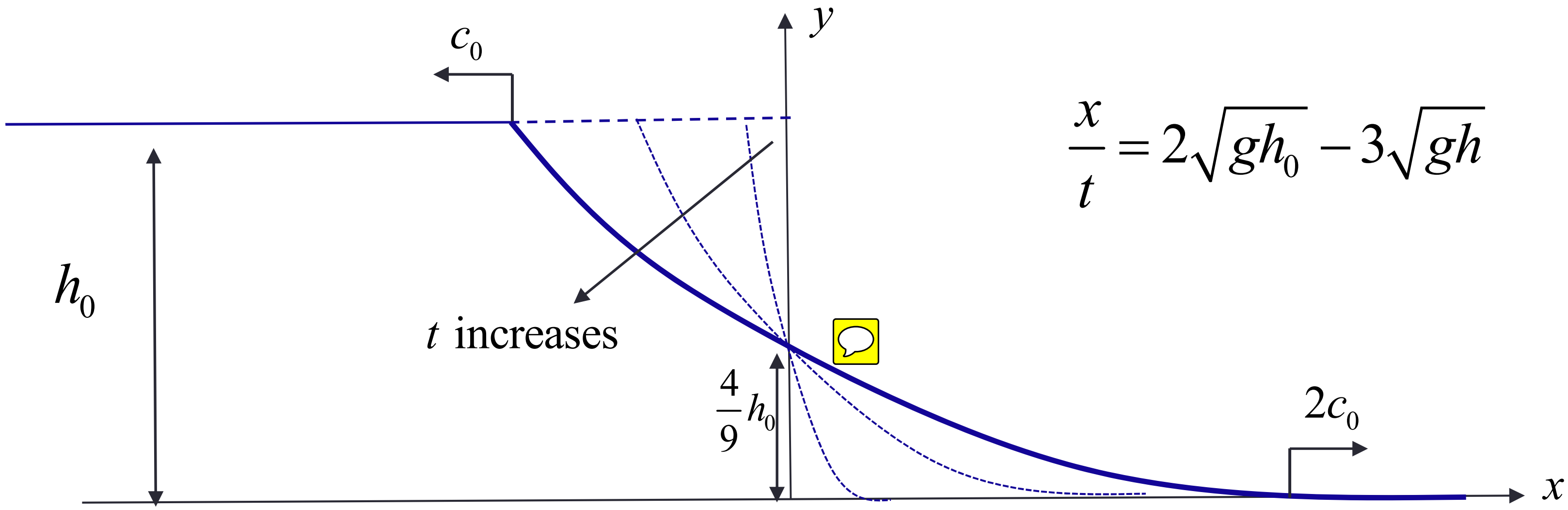


$$\boxed{\frac{x}{t} = 2\sqrt{gh_0} - 3\sqrt{gh}}$$

A parabola profile



Flow at the dam's location (x=0)



$$\frac{x}{t} = 2\sqrt{gh_0} - 3\sqrt{gh}$$

$$\frac{0}{t} = 2\sqrt{gh_0} - 3\sqrt{gh} \Rightarrow h(0,t) = \frac{4}{9}h_0$$

$$\left.\begin{aligned} c(0,t) &= \frac{2}{3}c_0 \\ U(0,t) &= 2c_0 - 2c(0,t) = \frac{2}{3}c_0 \end{aligned}\right\} \quad Fr(0,t) = 1: \text{ critical flow}$$

Conclusions for dam-break wave

- Trailing edge moves upstream at c_0
- Leading edge moves downstream at $2c_0$
- Surface profile is a parabola
- At dam's location the flow is always critical with constant water depth and mean velocity.

These conclusions are not 100% valid:

- Idealization of immediate removal of dam
- No friction (leading edge)
- No water in the downstream