Seminar assignments - Assignment 1-2

Environmental modelling (Technische Universiteit Delft)

SOLUTION TO ASSIGNMENT 1

CIE 4400

QUESTION 1

Convolution: Consider Figure 1. Figure 1c shows a well-mixed system with pollutant concentration c(t) (mg/L) in volume V (L) that varies over time. The pollutant decays by a first order reaction defined by reaction constant k (1/s) and has an outflow of Q(L/s). Figures 1a and b show two types of loading functions. Assume initial concentration before loading to be zero, i.e. c(t) = 0 for t<t1.
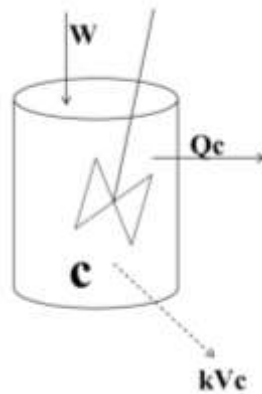
(a) How will the concentration vary over time for a pulse loading as shown in 1a?
(Hint1: a pulse loading is the difference between two step loadings that start at different times. Hint2: response of a system to sum of two loadings can be decomposed into sum of responses to those two loadings)

Load Case 1   Load Case 2
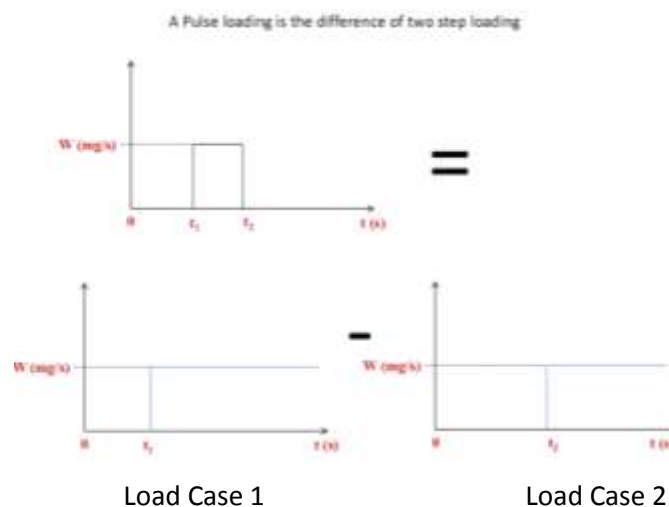
Solution:

Let $C_1(t)$ *and* $C_2(t)$ define the concentration for load case 1 and load case 2 respectively and C(t) define the concentration response for the pulse load.

Since a pulse load is a difference of two step loads (as shown in the above figure), we have the following relationship by principle of superposition

$$C(t) = C_1(t) - C_2(t)$$

Since Case 1 is a step load, we have the following solution

$$C_1(t) = e^{-\lambda t/V}\left[\int_{-\infty}^{t}\frac{e^{\frac{\lambda\tau}{V}}W(\tau)}{V}d\tau + A\right] \quad \text{where } \lambda = Q + kV,$$

There is no load before t_1, thus

$$C_1(t) = 0 = e^{-\lambda t/V}A \quad, \ t \leq t_1, \text{ or}$$

$A = 0$.

$$\Rightarrow C_1(t) = \int_{t_1}^{t}\frac{e^{-\frac{\lambda(t-\tau)}{V}}W}{V}d\tau = \frac{W}{V}\int_{t_1}^{t}e^{-\frac{\lambda(t-\tau)}{V}}d\tau = \frac{W}{V}\frac{V}{\lambda}e^{-\frac{\lambda(t-\tau)}{V}}\Big|_{t_1}^{t} = \frac{W}{\lambda}(e^{-\frac{\lambda(t-t)}{V}} - e^{-\frac{\lambda(t-t_1)}{V}}).$$

Thus,

$$C_1(t) = \frac{W}{\lambda}\left\{1 - \exp\left[-\frac{\lambda}{V}(t - t_1)\right]\right\}, t > t_1$$

Similarly, since Case 2 is also a step load,

$$C_2(t) = 0 \ , \ t \leq t_2$$

$$C_2(t) = \frac{W}{\lambda}\left\{1 - \exp\left[-\frac{\lambda}{V}(t - t_2)\right]\right\}, t > t_2$$

Finally given that $t_2 > t_1$, we apply principle of superposition to obtain concentration response to a pulse load:

$C(t)=C_1(t)-C_2(t)$

To conclude, the concentration response for a pulse load can be given by:

$$C(t)=\begin{cases} 0 & ,t \leq t_1 \quad (\textbf{no load}) \\ \frac{W}{\lambda}\left\{1 - \exp\left[-\frac{\lambda}{V}(t - t_1)\right]\right\} & ,t_1 < t \leq t_2 \text{ (a step load)} \\ \frac{W}{\lambda}\left\{-\exp\left[-\frac{\lambda}{V}(t - t_1)\right] + \exp\left[-\frac{\lambda}{V}(t - t_2)\right]\right\} & ,t > t_2 \text{ (a pulse load)} \end{cases}$$
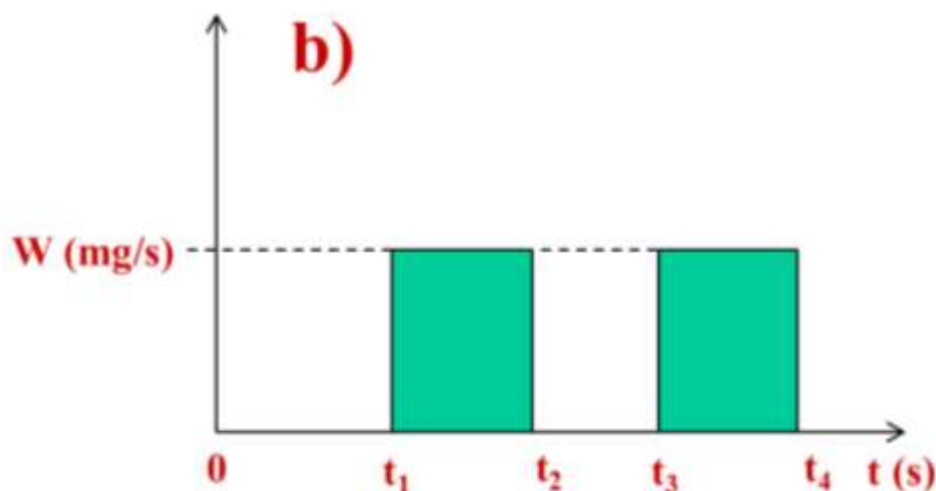
(b) Indicate where is convolution in your derivation

Solution:

Convolution appears in the solution for concentration. We convolute weight loading function $W(\tau)$ with $e^{-\frac{\lambda}{V}(t-\tau)}$, which is $\int_0^t W(\tau)e^{-\frac{\lambda}{V}(t-\tau)}d\tau$ to obtain concentration at any time t (when initial concentration is zero). $\int_0^t W(\tau)e^{-\frac{\lambda}{V}(t-\tau)}d\tau$ is convolution of $W(\tau)$ with $\int_0^t W(\tau)e^{-\frac{\lambda}{V}(t-\tau)}d\tau$.

(c) Repeat a for loading in figure 1b

Solution:

Let $C_1(t)$ be the concentration response to a step load that starts at t1, $C_2(t)$ be the concentration response to a step load that starts at t2, $C_3(t)$ be the concentration response to a step load that starts at t3 and $C_4(t)$ be the concentration response to a

step load that starts at t4 (for t4>t3>t2>t1). Then we can apply principle of superposition to obtain concentration response for time t>t1.

Similar to (a),

$$C_1(t) = \frac{W}{\lambda}\left\{1 - \exp\left[-\frac{\lambda}{V}(t - t_1)\right]\right\}, t > t_1$$

$$C_2(t) = \frac{W}{\lambda}\left\{1 - \exp\left[-\frac{\lambda}{V}(t - t_2)\right]\right\}, t > t_2$$

So $C(t) = \frac{W}{\lambda}\left\{1 - exp\left[-\frac{\lambda}{V}(t - t_1)\right]\right\}, t_1 < t \leq t_2$ *(a step load).*

However for $t_2 < t \leq t_3$, we have concentration response to a pulse load (of duration t2-t1 that starts at t=t1) corresponding to the difference between step loads that start at t2 and t1. Using the concentration responses to step loads starting at t1 and t2 (shown above) and the principle of superposition, we have concentration response as C2(t)-C1(t):

$$C(t) = \frac{W}{\lambda}\left\{-exp\left[-\frac{\lambda}{V}(t - t_1)\right] + exp\left[-\frac{\lambda}{V}(t - t_2)\right]\right\}, t_2 < t < t_3 \text{ (a pulse load)}$$

Now for time $t_3 < t \leq t_4$, the system sees a step load starting at t3 in addition to the pulse load (of duration t2-t1 starting at t=t1). Thus the concentration response should now be a sum of concentration response to a pulse load (as calculated above) and a step load. Concentration response for the step load starting at t=t3 is

$$C_3(t) = \frac{W}{\lambda}\left\{1 - exp\left[-\frac{\lambda}{V}(t - t_3)\right]\right\}, \text{ where } \lambda = Q + kV.$$

Thus the concentration response for time $t_3 < t \leq t_4$ is

$$C(t) = C1(t) - C2(t) + C3(t) = \frac{W}{\lambda}\left\{-exp\left[-\frac{\lambda}{V}(t - t_1)\right] + exp\left[-\frac{\lambda}{V}(t - t_2)\right] + 1 - exp\left[-\frac{\lambda}{V}(t - t_3)\right]\right\}, t_3 < t < t_4 \text{ (a pulse load + a step load)}$$

Finally for $t > t_4$, we have concentration response to two pulse loads (of durations t2-t1, t4-t3 and starting at t1,t3 respectively). Applying principle of superposition: we obtain

$$C(t) = C1(t) - C2(t) + C3(t) - C4(t) = \frac{W}{\lambda}\left\{- \exp\left[-\frac{\lambda}{V}(t - t_1)\right] + \exp\left[-\frac{\lambda}{V}(t - t_2)\right] + \right.$$

$$\left. + \exp\left[-\frac{\lambda}{V}(t - t_4)\right] - \exp\left[-\frac{\lambda}{V}(t - t_3)\right]\right\}, t > t_4 \text{ (2 pulse loads)}$$

To conclude,

$$C(t) = \frac{W}{\lambda}\left\{1 - \exp\left[-\frac{\lambda}{V}(t - t_1)\right]\right\}, t_1 < t < t_2 \text{ (a step load)}$$

$$C(t) = \frac{W}{\lambda}\left\{- \exp\left[-\frac{\lambda}{V}(t - t_1)\right] + \exp\left[-\frac{\lambda}{V}(t - t_2)\right]\right\}, t_2 < t < t_3 \text{ (a pulse load)}$$
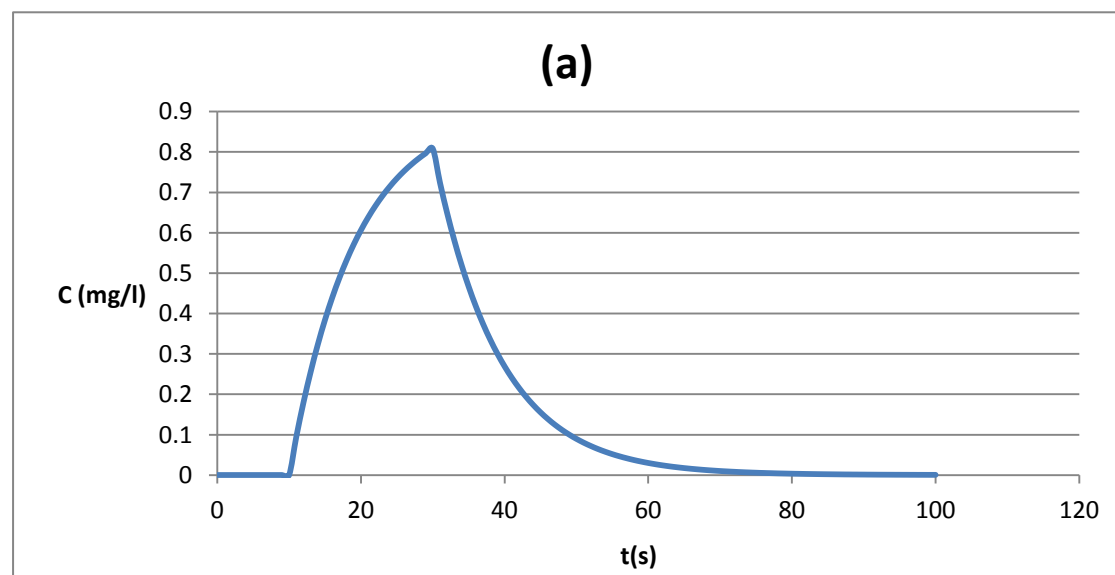
$$C(t) = \frac{W}{\lambda}\left\{- \exp\left[-\frac{\lambda}{V}(t - t_1)\right] + \exp\left[-\frac{\lambda}{V}(t - t_2)\right] + 1 - \exp\left[-\frac{\lambda}{V}(t - t_3)\right]\right\} ,$$
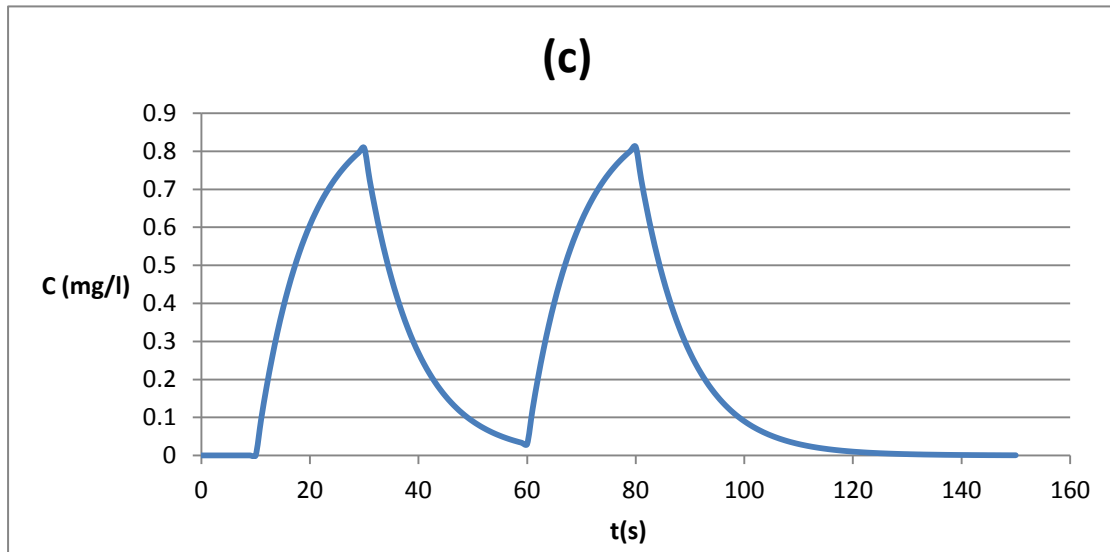$$t_3 < t < t_4 \text{ (a pulse load + a step load)}$$

$$C(t) = \frac{W}{\lambda}\left\{- \exp\left[-\frac{\lambda}{V}(t - t_1)\right] + \exp\left[-\frac{\lambda}{V}(t - t_2)\right] + + \exp\left[-\frac{\lambda}{V}(t - t_4)\right] - \right.$$

$$\left. \exp\left[-\frac{\lambda}{V}(t - t_3)\right]\right\}, t > t_4 \text{ (2 pulse loads)}$$

(d) Plot how concentration varies over time for loading in figure 1 a and b.
Solution:

The below plots concentration responses that we derived above for one and two pulse loads respectively.

(c)

QUESTION2:

Q2.a) **Non-steady Feed forward systems:** Consider a feedforward system composed of two completely mixed systems with their characteristics identified by subscripts 1 and 2 in Figure 2a below.
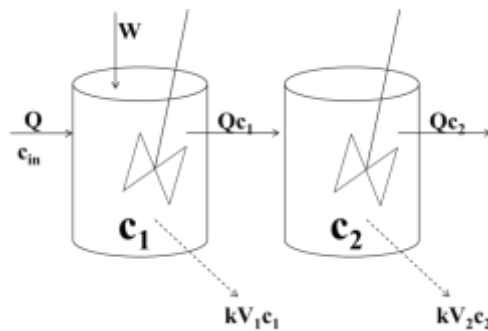

Figure 2a: A feedforward system of two completely mixed systems.

Solution:

We have the following conservation of mass for lake 1:

$$V_1 \frac{dc}{dt} = Q c_{in} - k V_1 c_1 - Q c_1.$$

At steady state we have:

$$\frac{dc}{dt} = 0$$

$$\Rightarrow 0 = Qc_{in} - kV_1c_1 - Qc_1, \text{ or}$$

$$c_1(Q + kV_1) = Qc_{in}.$$

Thus, the steady state concentration of lake1 is $c_1 = \frac{Qc_{in}}{\lambda_1}$ where $\lambda_1 = Q + kV_1$.

Similarly for lake 2 we have the following steady state condition:
$$0 = Qc_1 - kV_2c_2 - Qc_2$$

$$\Rightarrow c_2(Q + kV_2) = Qc_1.$$

From the expression of steady state concentration, c_1 in lake1, we have

$$c_2(Q + kV_2) = \frac{Q^2c_{in}}{\lambda_1}.$$

Thus the steady state concentration in lake is:

$$c_2 = \frac{Q^2c_{in}}{\lambda_1\lambda_2},$$

where $\lambda_2 = Q + kV_2$.

Q2.b) Consider figure 2b where each of the two lakes are conceptualized by two completely mixed systems as in figure 2a. Let $W = m\delta(0)$ be impulse load of mass m [M] at time t = 0 and let $c_1(0) = c_2(0) = 0$ (Initial concentrations are 0).
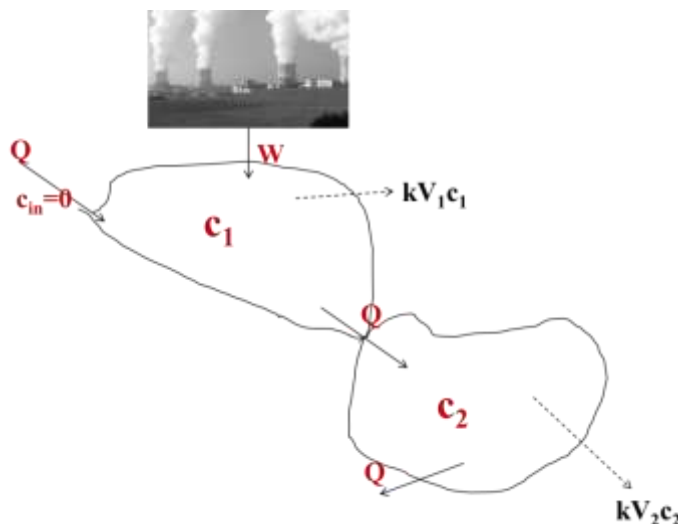

Figure 2b: A toy example with two connected completely mixed lakes for impulse radioactive loading and system's recovery

Solution:

b.1. Can a steady state model for such a case make physical sense?

No, because the system would be in a transient state due to "pulse" (or instantaneous) type of loading. We have therefore implicitly assumed a loading that is continuous rather than a pulse in our solution to question 2.a.

b.2. Solve for the non-steady state solution of the system.

(Hint: solution for $\frac{\partial y}{\partial x} + \mu y = we^{-\theta x}$ is $y = \frac{w}{(\mu - \theta)}\left(e^{-\theta x} - e^{-\mu x}\right) + ce^{-\mu x}$, where is a constant of integration).

Mass balance for lake 1 yields:

$$V_1 \frac{\partial c_1}{\partial t} = W - kV_1 c_1 - Qc_1.$$

We simplify the above by substituting $\lambda_1 = Q + kV_1$ in the above to obtain:

$$V_1 \frac{\partial c_1}{\partial t} + c_1 \lambda_1 = W, \text{ or}$$

$$\frac{\partial c_1}{\partial t} + c_1 \frac{\lambda_1}{V_1} = \frac{W}{V_1}.$$

Recall from Week1 lecture slides that solution to the above ordinary differential equation is:

$$c(t) = \int_{-\infty}^{t} \frac{W}{V_1} e^{(-\frac{\lambda_1}{V_1}(t-\tau))} \partial\tau + Ae^{-(\frac{\lambda_1 t}{V_1})}, \text{ where A is a constant of integration.}$$

Since for t=0, c=0 and $W = m\delta(0)$, we have

$$c_1(t) = \frac{m}{V_1} \cdot e^{-\frac{\lambda t}{V_1}}$$

Similarly we solve for the mass balance equation in lake 2

$$V_2 \frac{\partial c_2}{\partial t} = Qc_1 - kV_2 c_2 - Qc_2.$$

With $\lambda_2 = Q + kV_2$, we have

$$V_2 \frac{\partial c_2}{\partial t} + c_2 \lambda_2 = Qc_1.$$

Substituting transient solution for lake 1 in the above:

$$\frac{\partial c_2}{\partial t} + c_2 \frac{\lambda_2}{V_2} = \frac{Qm}{V_1 V_2} \cdot e^{-\frac{\lambda_1 t}{V_1}}.$$

We note that the mass balance equation for lake 2 of form $\frac{\partial y}{\partial x} + \mu y = w e^{-\theta x}$ with

$$y = c_2, \mu = \frac{\lambda_2}{V_2}, W = \frac{Qm}{V_1 V_2}, \theta = \frac{\lambda_1}{V_1} \; and \; x = t.$$

Thus (from Hint), we have transient concentration solution for lake 2,

$$c_2(t) = \left(\frac{\left(\frac{Qm}{V_1 V_2} \right)}{\left(\frac{\lambda_2}{V_2} \right) - \left(\frac{\lambda_1}{V_1} \right)} \right) \cdot \left(e^{\left(-\frac{\lambda_1}{V_1} \cdot t \right)} - e^{\left(-\frac{\lambda_2}{V_2} \cdot t \right)} \right) + A \cdot e^{-\left(\frac{\lambda_2}{V_2} \cdot t \right)}, \text{ where A is a constant of}$$

integration.

Finally we note that $A = 0$, as initial condition for lake 2 is $c_2(0) = 0$. Thus,

$$c_2(t) = \left(\frac{\left(\frac{Qm}{V_1 V_2} \right)}{\left(\frac{\lambda_2}{V_2} \right) - \left(\frac{\lambda_1}{V_1} \right)} \right) \cdot \left(e^{\left(-\frac{\lambda_1}{V_1} \cdot t \right)} - e^{\left(-\frac{\lambda_2}{V_2} \cdot t \right)} \right).$$

b.3. Discuss the conditions under which the concentration in the second reservoir will never rise.

From the expression provided above for the second lake, its concentration will always rise (and then fall) for positive loading in the first lake. Thus the condition under which the concentration in the second reservoir never rises is that loading in the first lake is 0.

b.4. what is the time scale of the recovery for the second lake.

We define time scale of recovery of the second lake as the time it takes for the lake to reach 10% (or another percentage, for eg use of 50% is equivalent to half life estimation of pollutant in the lake) of its maximum concentration (after attaining its maximum). Let time to achieve maximum concentration be given by t_{max} , which can be obtained by solving the following:

$$\frac{\partial c_2}{\partial t} = 0.$$

Solving for the above maximum concentration condition yields,

$$c_{max} = \hat{c} \left[\left(\frac{\lambda_2/V_2}{\lambda_1/V_1}\right)^{-\frac{\lambda_1/V_1}{\left(\frac{\lambda_2}{V_2}-\frac{\lambda_1}{V_1}\right)}} - \left(\frac{\lambda_2/V_2}{\lambda_1/V_1}\right)^{-\frac{\lambda_2/V_2}{\left(\frac{\lambda_2}{V_2}-\frac{\lambda_1}{V_1}\right)}} \right], \text{ where}$$

$$\hat{c} = \left(\frac{\left(\frac{Qm}{V_1 V_2}\right)}{\left(\frac{\lambda_2}{V_2}\right)-\left(\frac{\lambda_1}{V_1}\right)} \right), \text{ and}$$

$$t_{max} = \left. ln\left(\frac{\lambda_2/V_2}{\lambda_1/V_1}\right) \middle/ \left(\frac{\lambda_2}{V_2} - \frac{\lambda_1}{V_1}\right) \right. .$$

Let t_{10} be the time when concentration in the second lake reaches 10% of c_{max}, defined as c_{10}. Then t_{10} can be estimated by solving the following equation:

$$c_{10} = \hat{c}\left(e^{\left(-\frac{\lambda_1}{V_1}.t_{10}\right)} - e^{\left(-\frac{\lambda_2}{V_2}.t_{10}\right)} \right) = 0.1 c_{max} = 0.1\hat{c}\left(e^{\left(-\frac{\lambda_1}{V_1}.t_{max}\right)} - e^{\left(-\frac{\lambda_2}{V_2}.t_{max}\right)} \right), \text{ or}$$

$$e^{\left(-\frac{\lambda_1}{V_1}.t_{10}\right)} - e^{\left(-\frac{\lambda_2}{V_2}.t_{10}\right)} = e^{\left(-\frac{\lambda_1}{V_1}.t_{max}+ln0.1\right)} - e^{\left(-\frac{\lambda_2}{V_2}.t_{max}+ln0.1\right)}.$$

1.   Consider an advection only polluted water system. This means that there is no dispersion but advection due to presence of flow velocity U [LT^{-1}] (=Q/A$_c$) in the positive x [L] direction. The pollutant loading remains the same, which is a continuous loading W_d [ML^{-3}T^{-1}] distributed uniformly over length a [L] from the origin and the pollutant decays with a first order kinetic rate of k [T^{-1}]. The concentration at the origin is c$_0$ [ML^{-3}]. 1) Formulate the mass balance equation for the system. 2) Solve its steady state concentration profile. 3) Plot the steady state concentration profile for the following characteristics: c$_0$ = 16 mg/L, Q = 12 x 10^6 m^{3}d^{-1}, A$_c$ (cross sectional area) = 2000 m^2, k = 0.8 d^{-1}, S$_d$ = 15 gm^{-3}d^{-1} and a = 8km.
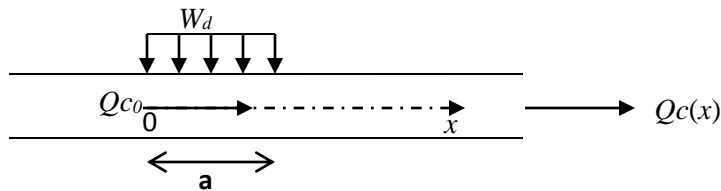
Solution:

Conservation of mass equation for 0 ≤ x ≤ a is given by:

$$\frac{\partial c}{\partial x} + \frac{k}{U}c = \frac{W_d}{U}.$$

Since loading only starts at x = a and this is an advective system, concentration is 0 for x<0. We therefore only solve for x≥0.

We first solve for 0 ≤ x ≤ a, followed by solution for x ≥a.

For 0 ≤ x ≤ a:

Complementary solution:-

$$\frac{\partial c}{\partial x} + \frac{k}{U}c = 0$$

Let

$$f = \frac{\partial}{\partial x}$$

$$\therefore f + \frac{k}{U} = 0$$

$$f = -\frac{k}{U}$$

$$\Rightarrow c_c = c_1 e^{-\frac{k}{U}x}$$

Particular Solution:-

$$y = c_1 y_1(x)$$

Where,

$$y_1(x) = e^{-\frac{k}{U}x}$$

Wronskian,

$$\overline{W} = |y_1|$$

$$\overline{W} = \left| e^{-\frac{k}{U}x} \right|$$

$$\overline{W} = e^{-\frac{k}{U}x}$$

$$\Rightarrow D_1 = 1 \qquad \text{(For 1 x 1 matrix)}$$

And let $g(x') = \frac{W_d}{U}$

$$\therefore c_p = y_1 \int_{-\infty}^{x} \frac{g(x')}{\overline{W}} D_1 dx'$$

$$c_p = e^{-\frac{k}{U}x} \int_{0}^{x} \frac{W_d}{U} e^{-\frac{k}{U}x'} dx'$$

$$c_p = \frac{W_d}{k}$$

Therefore, general solution is

$$c = c_c + c_p$$

$$c = c_1 e^{-\frac{k}{U}x} + \frac{W_d}{k}$$

When x=0 , c=c$_0$. Thus for $0 \leq x \leq a$:

$$c_0 = c_1 + \frac{W_d}{k}$$

$$c_1 = c_0 - \frac{W_d}{k}$$

$$\therefore \underline{\underline{c = \left(c_0 - \frac{W_d}{k}\right)e^{-\frac{k}{U}x} + \frac{W_d}{k}}} \,.$$
(1)

For x ≥ a:

Note that there is no loading from this point on. Thus, the particular solution is given by the following.

Then the general solution (here only complementary solution is needed) can be given by:

$$c = c_1 e^{-\frac{k}{U}x} \,.$$

By continuity at x = a and using (1) for concentration at x= a from the left hand side,

$$c_1 e^{-\frac{k}{U}a} = \left(c_0 - \frac{W_d}{k}\right)e^{-\frac{k}{U}a} + \frac{W_d}{k}$$

$$\Rightarrow c_1 = \left(c_0 - \frac{W_d}{k}\right) + \frac{W_d}{k}e^{\frac{k}{U}a} \,.$$

Thus, we obtain the solution as:

$$c = \left(c_0 - \frac{W_d}{k}\right)e^{-\frac{k}{U}x} + \frac{W_d}{k}e^{-\frac{k}{U}(x-a)} \,.$$

The following is the plot for the above concentration profiles with given parameters. Note that concentration should be 0 for x<0 for the reasons stated previously.

Steady State Solution for Advection Only Flow

Should be at 0 because loading starts at x = 0 and this is advection system only

Y-axis: **Concentration, c (mg/L)** — 0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20

X-axis: **Distance, x (m)** — -5000, 0, 5000, 10000, 15000, 20000, 25000, 30000