Proposal for issue of stingray fishing licenses in Malaysian waters (for the period of Oct 2017 to SEP 2027)

LKIM
FISHERIES DEVELOPMENT AUTHORITY OF MALAYSIA

TeamB

Table of Contents

Executive Summary 0

1 Objective 1

2 Background 1

2.1 Overview 1

2.2 Initial conditions and parameters 1

2.2.1 Determination of parameter r 2

2.2.2 Determination of parameter k 4

2.2.3 Determination of parameter m 6

2.2.4 Determination of parameter N 8

3 Numerical method and implementation 10

3.1 Overview 10

3.1.1 Discussion of time-step, h 10

3.2 Euler Explicit method 12

3.2.1 Description 12

3.2.2 Matlab subroutine 12

3.2.3 Results 12

3.3 Trapezoidal method 15

3.3.1 Description 15

3.3.2 Matlab subroutine 15

3.3.3 Results 15

3.4 Modified Trapezoidal method 17

3.4.1 Description 17

3.4.2 Matlab subroutine 17

3.4.3 Results 17

3.5 Runge-Kutta (RK 4) 20

3.5.1 Description 20

3.5.2 Matlab subroutine 20

3.5.3 Results 20

3.6 Ranking of methods 23

4 Stability Analysis 24
4.1 Definition of Stability 24
4.1.1 Stability and Critical Time Step, $h_{critical}$ 24
4.1.2 Stability analysis of non-linear function. 25
4.2 Linearization of population dynamic equation 25
4.3 Range of stability for Euler Explicit 26
4.4 Range of stability for Trap-NR method 28
4.5 Range of stability for Modified Trapezoidal method 30
4.6 Range of stability for RK4 31
4.7 Summary of Stability 33
5 Accuracy Analysis 34
6 Conclusion 36
7 Learning Points 37
8 References 39
Appendix 41
General 41
Euler Explicit method (Subroutine) 41
Trapezoidal method (Subroutine) 42
Modified Trapezoidal method (Subroutine) 43
RK4 method (Subroutine) 43

Executive Summary

This report presents the findings from the analysis conducted by Team B, in response to a project tender request, ID53772017B, by Fisheries Development Authority of Malaysia (FDAM).

The objective of the report is to conduct an analysis on the population dynamics and find the optimum number of fishing licenses to be issued for the period of 1st October 2017 to 30th September 2027. The number of licenses issued must then maintain the population of stingrays above the 1.5 times of minimum population threshold within the 10 years.

This report utilizes the numerical method Runge-Kutta (RK4). It is ranked that this method obtains the best estimate for the solution, considering the stability and accuracy of the method and its ease of implementation. The other methods considered are Euler Explicit, Trapezoidal and Modified Trapezoidal method.

Through extensive research into available data and trends, the parameters chosen to illustrate the population of the stingrays are growth rate, r=3, population capacity, k=38 and minimum threshold, m=13. It is notable that these values are different from what was determined 10 years ago. As such, supporting evidences are carefully made reference to, with certain reservations being made.

The report concludes that the recommended number of fishing licenses to be issued is 10. If the number of licenses were to exceed this amount, drastic consequences for the stingray species may ensue such that any measures taken thereafter to recover the species may be extremely difficult or futile.

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1 Objective

This proposal is drafted as a recommendation to the Fisheries Development Authority of Malaysia (FDAM), a statutory board under the Ministry of Agriculture & Agro-based Industry Malaysia, on the number of stingray fishing licenses to be issued for the period stated between 1^{st} October 2017 to 30^{th} September 2027.

The document will provide detailed discussion and analysis on the determination of the existing parameters affecting the population dynamics; and the selection of a suitable numerical method to be applied so as to determine the number of licenses, N to be issued.

2 Background

2.1 Overview

The population dynamics of stingrays in Malaysian waters can be described by the following equation:

$$\dot{P} = \frac{dP}{dt} = -rP(1-\frac{P}{k})(1-\frac{P}{m}) - 2N$$

where P is population measured in thousand tons

t is the number of years

r is the growth rate of stingrays

k is the population capacity of stingrays

m is the minimum threshold value

N is the number of stingray fishing licenses to be issued

This equation is non-linear, non-homogenous and is a first-order ordinary differential equation (ODE).

2.2 Initial conditions and parameters

The stingray population, P_o as of September 2017, is estimated to be at 30 thousand tons.

The number of stingray fishing licenses to be issued by FDAM is based on the requirement that the population of stingray is maintained at greater than 1.5 times the minimum threshold value ($P \geq 1.5 \times m$) at all times over the next 10 years (2017 to 2027). Using the parameters and conditions in the modelling process, as summarized in Table 1, the minimum population required, P is calculated to be 19.5 thousand tons.

Table 1: Summary of defined parameters and values

Parameter	**Value (Existing expectation)**	**Values (10 years ago)**	**Unit**
Initial population, P_o	30	-	thousand tons
Time, t	0	-	year
Growth rate, r	3	4	thousand tons per year
Population capacity, k	38	55	thousand tons
Minimum threshold, m	13	10	thousand tons

The following sections will elaborate on the parameter values chosen, supplemented with numerical modelling results and evidence from sources. Trapezoidal method, which is unconditionally stable, is utilized to initialize the parameters.

2.2.1 Determination of parameter r

The parameter r refers to the growth rate of the stingray population, which is directly related to the reproduction rate of stingrays. It can be reasoned that there had been a decrease in r value from that adopted 10 years ago (i.e. 4) because of the following reasons, habitat and water temperature:

1) Stingray can only mate when there is adequate habitat and food to support their survival (BioExpedition, 2012). Due to the reclamation projects in Malaysia, such as in Penang, Melaka and Johor, hundreds of hectares of marine and coastal habitats have disappeared (Idris, 2017). The mining of marine sand and aggregates for reclamation has also affected the ecosystem, especially in the benthic zone (Idris, 2017), which provides the main source of food for the stingrays. Thus, this reduction in habitat space and availability of food will cause a decrease in reproduction rate of stingrays and consequently a decrease in their growth rate.

2) There is an optimum water temperature for reproduction for each species of stingray (Spells, n.d.). If the water temperature is higher or lower than optimum, it would negatively affect the reproduction rate. Due to global warming, sea temperature has been on the rise (National Geographic, 2010) and this would cause a decrease in reproduction rate of stingrays.

Though it has been observed that growth rate decreased during the past 10 years, it is difficult to quantify the extent of decline, because of limited available survey data of estimated numbers of stingrays over the past 10 years. The growth rate is projected to decrease gradually and hence *r value of 3* is adopted for this proposal, which is a 25% decrease from the value 10 years ago.

This projection is supported by the numerical analysis of the population, based on different r values from 1 to 4, for the next 10 years. For these simulations, the value of N is fixed at 10 and value of m is fixed at 10, which is the value from 10 years ago. The value of k varies from 25 to 55 because k is postulated to have decreased in the past 10 years (detailed in section 2.2.2). It can be seen from the graphs that as r decreases, the range of k that will not result in the population dying out in 10 years becomes smaller. Thus, it is not reasonable to choose a value of r that is too small. This is because the population will die out easily even with a small decrease in value of k, which is not logical.

For a brief summary of the ranges in Figure 1, Figure 2, Figure 3 and Figure 4, where population stays sustainable, when r=4, k ranges from 27 to 55; when r=3, k ranges from 30 to 55; when r=2, k ranges from 34 to 55; when r=1, k ranges from 45 to 55.

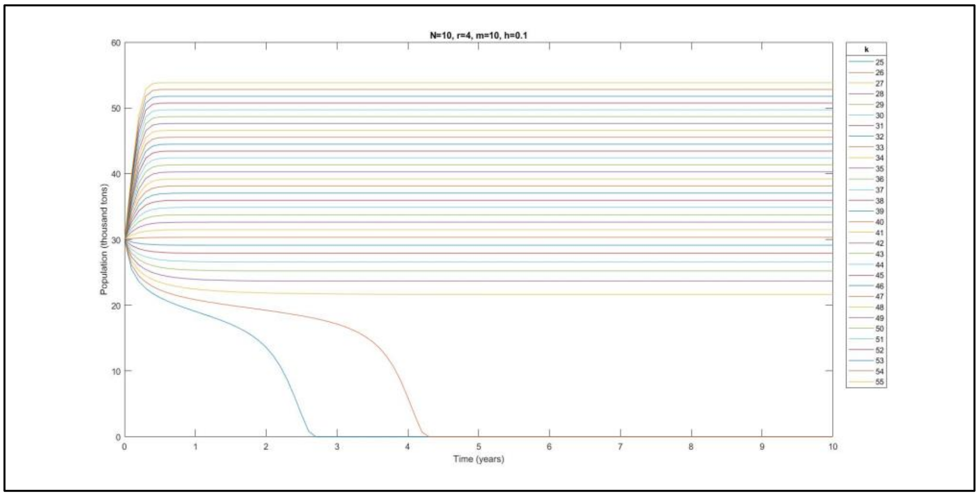

Figure 1: Graph of population when r=4

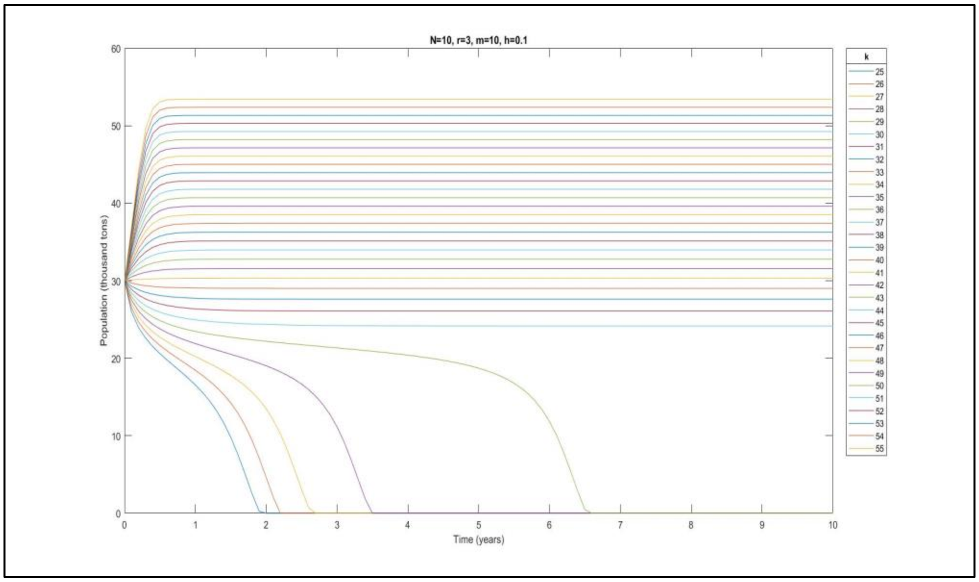

Figure 2: Graph of population when r=3

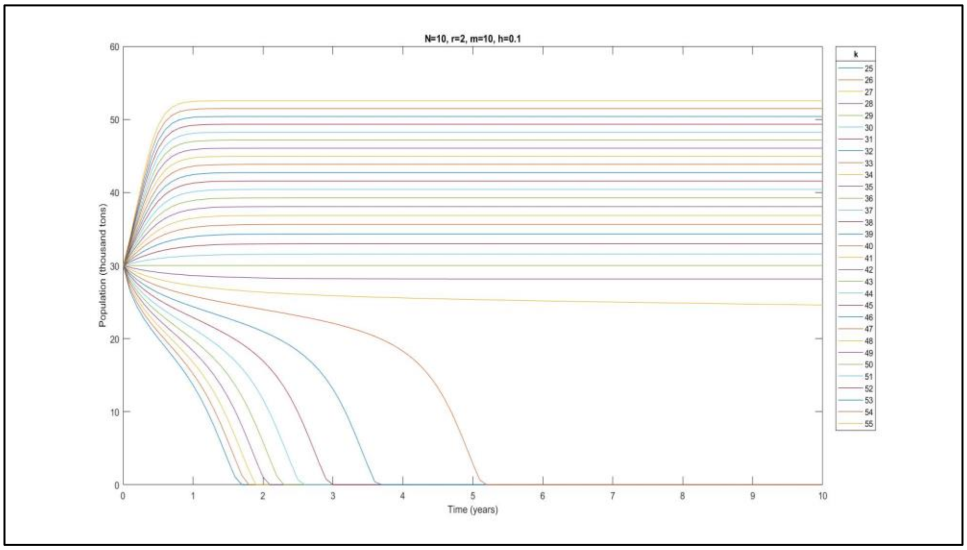

Figure 3: Graph of population when r=2

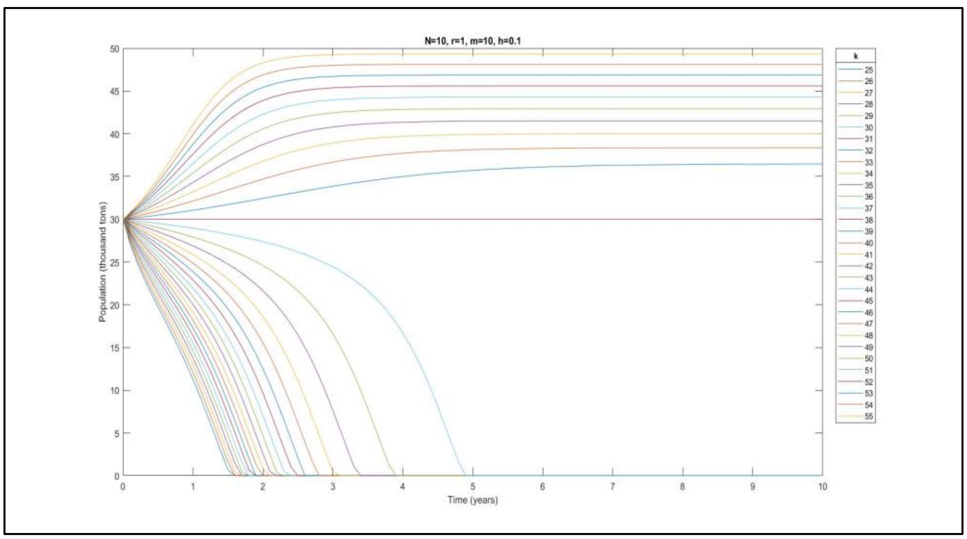

Figure 4: Graph of population when r=1

2.2.2 Determination of parameter k

The parameter k refers to the population capacity of stingrays. The population capacity of a species is defined as the maximum population size of the species that the environment can sustain indefinitely, on the condition that resources essential for the survival of the species are adequately available, e.g. food, habitat. The rationale for choosing a lower k value as compared to that adopted 10 years ago (i.e. 55) are presented in the following points, increased shipping activities and marine pollution.

1) Increased shipping activities in the Malaysian waters over the past 10 years has resulted in disturbance to the marine habitat. The discharge of ballast water,

which is used to maintain the ship's stability, creates a risk of invasive non-native species being introduced into the aquatic environment (Mobilik & Hassan, 2016). Without natural predators available, there is a high possibility of certain invasive species being able to thrive and eventually affect the delicate ecological balance in the sensitive coastal ecosystem. This results in higher competition with native species for food and space, thus disrupting the existing marine food web. In other words, there is a risk of depletion of resources required for the stingray to thrive in the next 10 years. In addition, the increasing shipping activities in Malaysian waters had affected the movement and feeding patterns of the marine species, forcing them to migrate to quieter waters, which may not present the best conditions for survival of the stingray population.

2) Marine pollution has steadily increased over the years. Between 2009 and 2015, a total of 121 cases of oil pollution had been reported, which are either deliberately discharged or spilled as a result of vessel collisions (Alam, 2016). Areas affected by such oil pollution included coastal regions in Terengganu and Pahang. Such incidents caused a great reduction in food and habitat space for the marine species. Moreover, the marine habitat requires time to recover from marine pollution, which on the contrary are happening at a higher frequency as a result of greater volume of coastal and shipping activities.

Taking into account for the possible shrinkages in food resources and habitat space for stingrays in the near future, *a conservative value of k=38* is chosen for the population capacity, k. The value is obtained based on the prediction of shipping activities to increase by a further 30% over the next 10 years with the expansion in Malaysian and Singaporean ports. A directly proportional relationship between marine pollution and negative effect on the marine environment is postulated. To analyze the feasibility of k value as 38, Figure 5 shows the plot of a range of k values. As shown, for k=38, the population can be maintained above 30 thousand tons consistently over the next 10 years even with 10 licenses, which fulfills the minimum population requirement decided by FDAM (section 2.2).

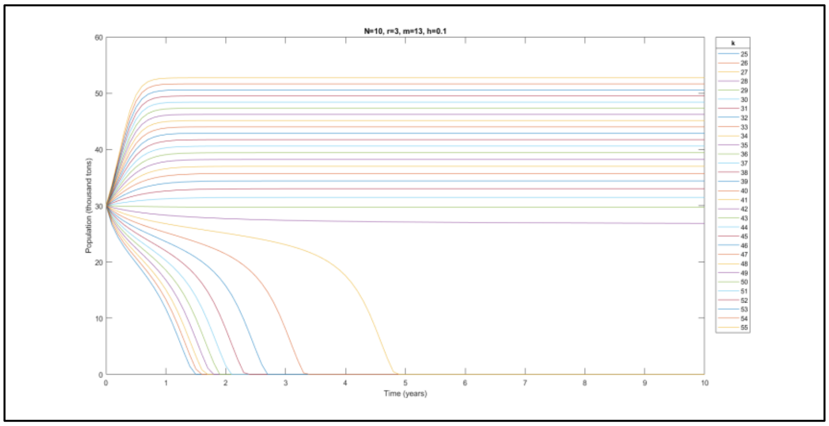

Figure 5: Graph of population for values of k (25 to 55)

2.2.3 Determination of parameter m

The parameter m refers to the threshold population, below which the reproduction rate is less than the death rate. Consequently, the growth rate becomes negative and the population will decrease until it is wiped out. The value of m is predicted to have been required to be increased from 10 years ago to alleviate impacts from the following factors, lower reproduction rate and marine pollution.

1) As mentioned in section 2.2.1, the reproduction rate of stingrays would have decreased in the past 10 years due to destruction of their habitats and source of food, and also due to the warming of seawaters. Assuming the death rate is constant, a lower reproduction rate means that a higher minimum population is needed to maintain the same growth rate.

2) With rapid industrialization and economic development in the region, marine pollution has increased dramatically in Malaysian waters, as described in section 2.2.2. Pollutants discharged to the sea include sewage effluent, industrial discharge, runoff from land-based activities, oil spills from ships and solid waste such as plastic (Idris, 2017). One of the major concerns is emerging pollutants, such as synthetic hormones and pharmaceutical chemicals in aquatic environment, which has been detected in Malaysian waters in the past years (Al-Odaini, Pauzi Zakaria, Ismail Yaziz, & Surif, 2011). These pollutants have been reported to disrupt the endocrine system of marine organisms, possibly leading to lower rates of reproduction in stingray population (Al-Odaini & Zakaria, Management of Pharmaceutical Compounds in Environment: Occurrence, Sources, Impacts and Control, 2006).

Similar to the case of the growth rate r, the value of m should have increased in the past 10 years based on qualitative evidence but it is difficult to quantify the increase. The value of m is postulated to slightly increase in a short time span of 10 years. To understand the population trend in the next 10 years for different values of m, numerical models are run for m value from 10 to 20. The value of r is fixed at 3 and value of k is fixed at 38 as determined in the previous sections.

It is seen from the graph that the population decreases when m increases. In Figure 6, when m increases to 15 and beyond, the population will decrease and die out within 10 years. This is in-line with the postulation that m should not have increased so much from 10 years ago. Looking at Figure 7 with focus on m values from 10 to 14, it can be seen that the effect of m on the stabilized population value is amplified with each increase in m. For example, the stabilized population decreased about 0.5 thousand tons when m increased from 10 to 11, but decreased about 1.3 thousand tons when m increased from 13 to 14. The *m value of 13 is chosen,* representing a reasonable 30% increase from the value 10 years ago, but is not so drastic as to cause a wipe out of the population.

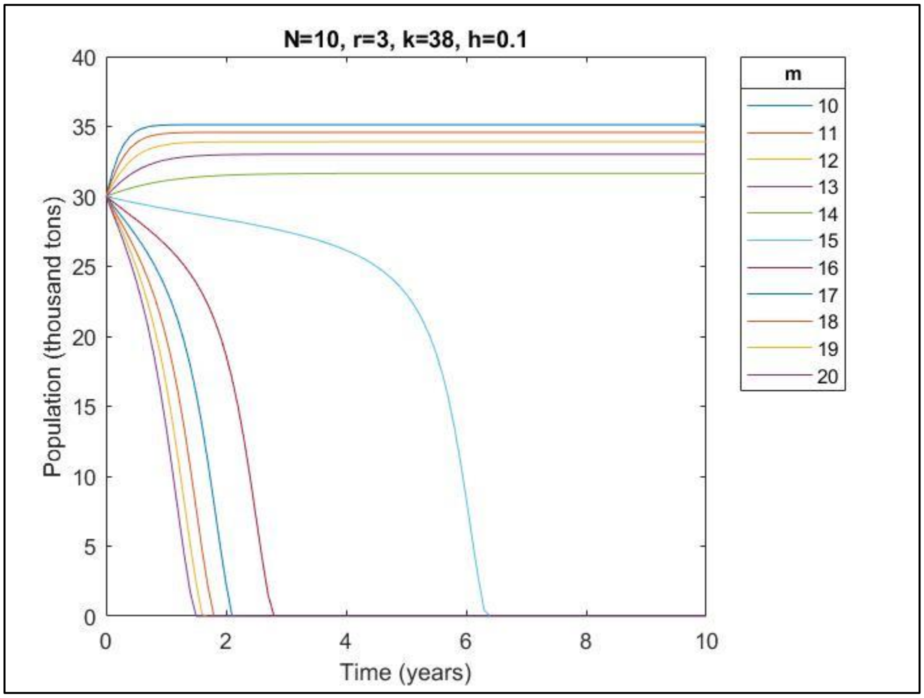

Figure 6: Graph of population for values of m (10 to 20)

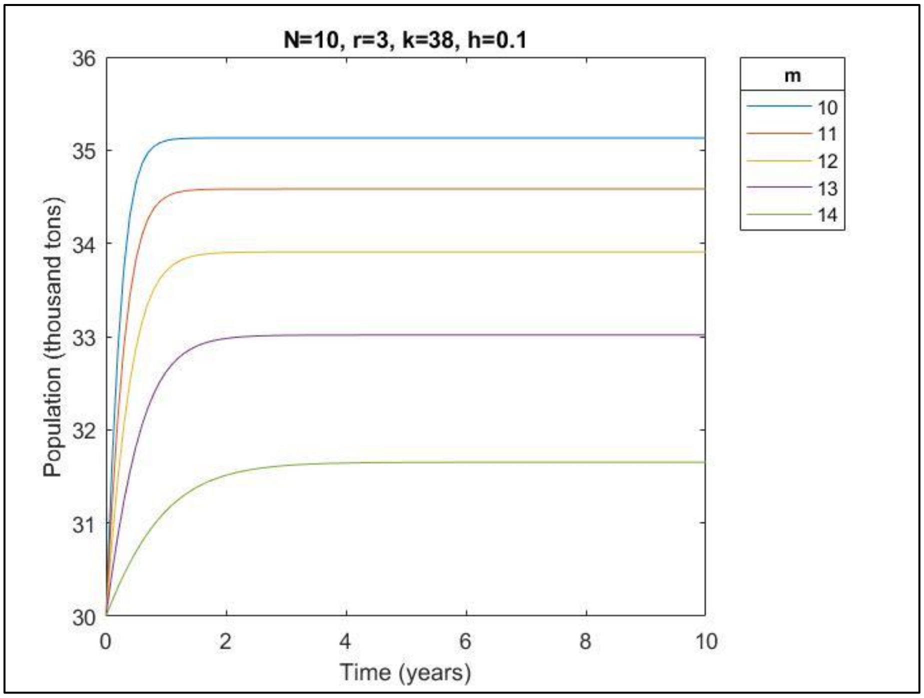

Figure 7: Graph of population for values of m (10 to 14)

2.2.4 Determination of parameter N

The variable N denotes the number of stingray fishing licenses that FDAM issues to entities allowed to fish in Malaysian waters. Each license is entitled to catch up to two thousand tons of stingray per year. Based on the population dynamic model and the initial conditions, the optimal number of licenses to be issued is determined to be 10 licenses for the following reasons, species sustainability and marine biodiversity:

1) A report from World Wide Fund for Nature (WWF) on sustainable seafood products has listed stingrays under the list of seafood to avoid at present as they are considered to be unsustainable, overfished and over-exploited globally (World Wide Fund for Nature (WWF), 2016). The number of licenses issued between 1960 and 2016 has been maintained at more than 10 licenses per year, as illustrated in Figure 8. However, the data recorded over the years along the eastern coast of Malaysia has indicated the occurrence of overfishing in the region (Food and Agriculture Organisation of the United Nations (FAO), 2016).

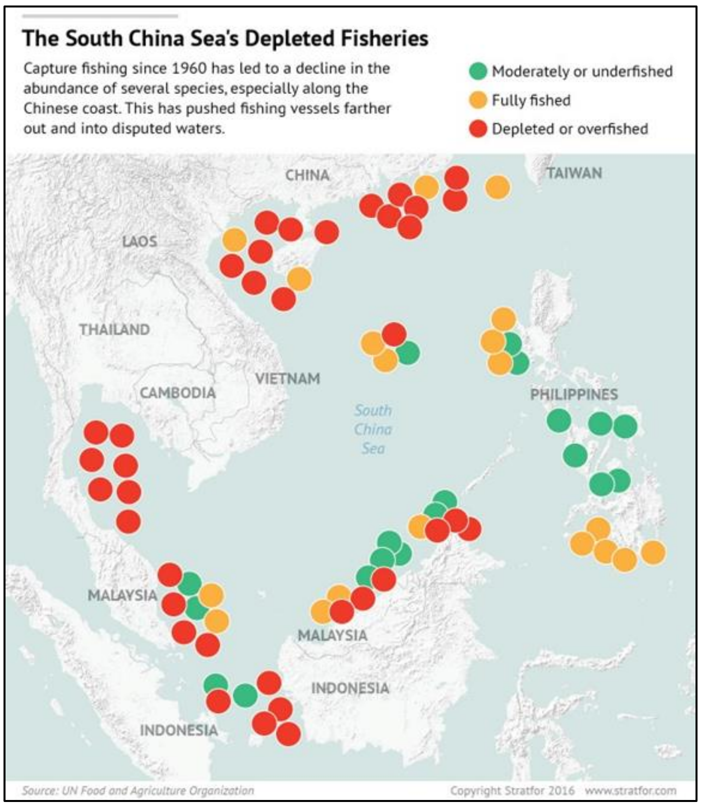

Figure 8: Diagram of fishing situation in the South China Sea

2) Extensive fishing for stingrays has impacted the rich marine biodiversity in the Malaysian waters, which often end up as bycatch in the nets laid out to catch stingrays along the shores (The Straits Times, 2016). One of the most affected species is the Malaysian state Terengganu's turtle, which often ended up trapped in the nets and eventually die from drowning. From 2014 to 2016, an average of 45 deaths per year has been recorded. The actual number could be significantly higher as many deaths have gone unreported. There are concerns from the general public with regard to this bycatch issue. As much as 80% of the people population living in overfished areas are dependent on revenues brought in by tourism, tapping on the draw of pristine dive sites.

   Without a sustainable model for fishing, the marine biodiversity in the region is projected to plummet over the next 10 years, which affects the region's reputation as a diving hotspot. By limiting the number of licenses, the Malaysian authorities have better control over the fishing situation in the area and safeguard the livelihoods of the people. This helps to promote adoption of more sustainable fishing practices through stricter regulations on the number of licenses issued.

With reference to Figure 9, the model predicts that for the population to maintain more than 19.5 thousand tons over the next 10 years before the population dips to extinction, the largest number of licenses allowed is 12.

As observed from the graph, the population of stingrays maintains steadily at 30 thousand tons for ≤12 licenses issued. However, it should be cautioned that once the population starts to decrease below 30 thousand tons, a steep decline will ensue. Therefore, in order to adequately safeguard the population of stingrays for the next 10 years, the number of fishing *licenses (N) recommended to be issued will be set at 10* to factor for any estimation errors in the model. In accordance with FDAM's efforts to promote sustainable fishing practices, the reduction in licenses issued for the next 10 years will allow the stingray population to stabilize and recover from the previous years of overfishing.

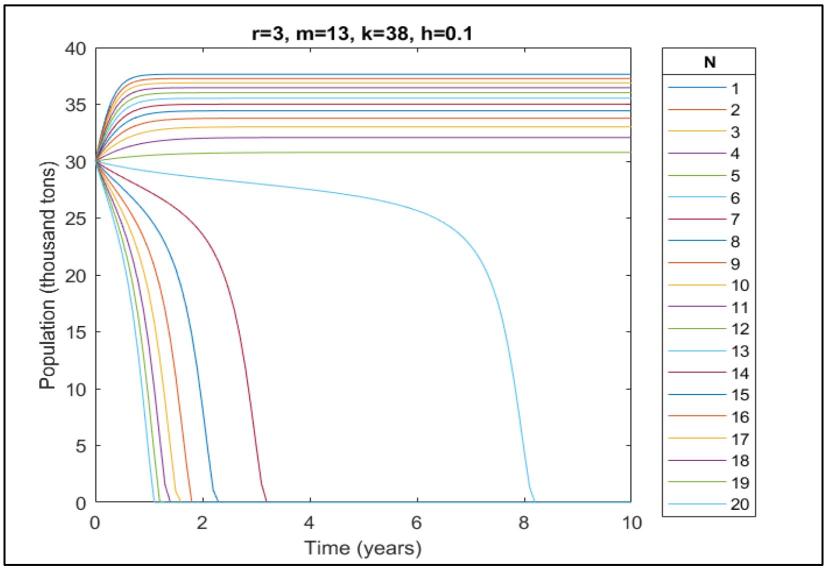

Figure 9: Graph of population for values of N (1 to 20)

3 Numerical method and implementation

3.1 Overview

Table 2 presents the numerical methods considered for solving the nonlinear population dynamics equation.

Table 2: Numerical methods used for modelling

Method	**Type**	**Order of accuracy**
Euler Explicit	Explicit	1^{st} order
Trapezoidal	Implicit	2^{nd} order
Modified Trapezoidal	Explicit	2^{nd} order
Runge-Kutta (RK4)	Explicit	4^{th} order

The numerical methods are compared using stability and accuracy analysis, and *RK4 is selected as the most suitable method* to predict for the number of licenses for the next 10 years. Details of the analysis can be found in the section 3.2 to 3.6.

3.1.1 Discussion of time-step, h

FDAM has requested for the proposal to include data for better monitoring of the true state of fishing in the region. Team B is able to provide modelled data, at a chosen time-step of $\frac{1}{2}$, i.e. h=0.5, which allows FDAM to track and compare with the physically collected data of stingrays caught. The modeled population data is seen in Table 3 below.

Table 3 Population Data from Analysis using Parameters r=3, k=38, m=13, N=10 with RK4 scheme

Population (Model)	Time (year)
30	0
31.810761	0.5
32.602396	1
32.881867	1.5
32.973646	2
33.003096	2.5
33.012476	3
33.015457	3.5
33.016403	4
33.016703	4.5
33.016799	5
33.016829	5.5
33.016839	6
33.016842	6.5
33.016843	7
33.016843	7.5
33.016843	8
33.016843	8.5
33.016843	9
33.016843	9.5
33.016843	10

Team B understands that each license is issued for a duration of 10 years and the fishing limit for each license is set at two thousand tons per year. It is desirable for the population to be depleting at a reasonably gradual rate. In reality, if the physical population drops sharply within a shorter period than expected, the species may be prone to a reduced growth rate, r. Therefore, there is a need for measures to be in place to prevent a sharp drop in the population within a short time period.

The ability to provide model data down to the detail of a half year basis will complement FDAM's existing implementation of physical tracking measures at the various fishery ports. This means that the modelled data can be compared with collated physical data to provide an up to date monitoring.

The following measures are recommended to FDAM:

1) Random physical checks can be conducted to ensure that reported numbers are true. Fishing boats that are found to be under-reporting will be meted out with severe punishments, such as revocation of license for a designated time period.

2) Data figures can be analyzed to ensure that there is no sharp drop in the population size, which can inevitably result in a short period drop in growth rate. If it is found that catching rates are higher than the allowable fishing rate, alert measures can be introduced by FDMA. Fishing activities for rays should then be halted or all fishing boats should be advised to release stingray catches and switch fishing hauls.

3) Subsequent focus can be placed on identifying the relationship between the parameters for the population dynamics and real data. With this, better predictions and adjustments of defined parameters can be made for licensing reviews in the future. In addition, improvements to monitoring of fishing activities can be extended to developing region-specific models, which will help to optimize the maximum amount of stingrays without affecting the population.

3.2 Euler Explicit method

3.2.1 Description

Euler Explicit (EE) is a first order accurate numerical method and works by finding a solution at the next time-step $t_{i+1} = t_i + h$, where h is the increment in time-step, using the known solution at t_i. The method is stable provided the time-step is sufficiently small to satisfy the stability range, i.e. conditionally stable.

$$y_{i+1} = y_i + hf(t_i, y_i)$$

3.2.2 Matlab subroutine

The subroutines for EE method can be found in Appendix.

3.2.3 Results

In Figure 10, the time-step with an increment of 0.1 in the range of 0.1 to 1 was used. In Figure 11, the time-step with an increment of 0.05 in the range of 0.05 to 0.5 was used. It is observed that as h decreases, the fluctuation in the solution decreases and converges towards a smooth curve. A sufficiently smooth curve is only achieved when h is 0.1. This is because EE method is only of first order accuracy and so the error is same order of magnitude as h. The population converges to stabilize at 33 thousand tons in 10 years.

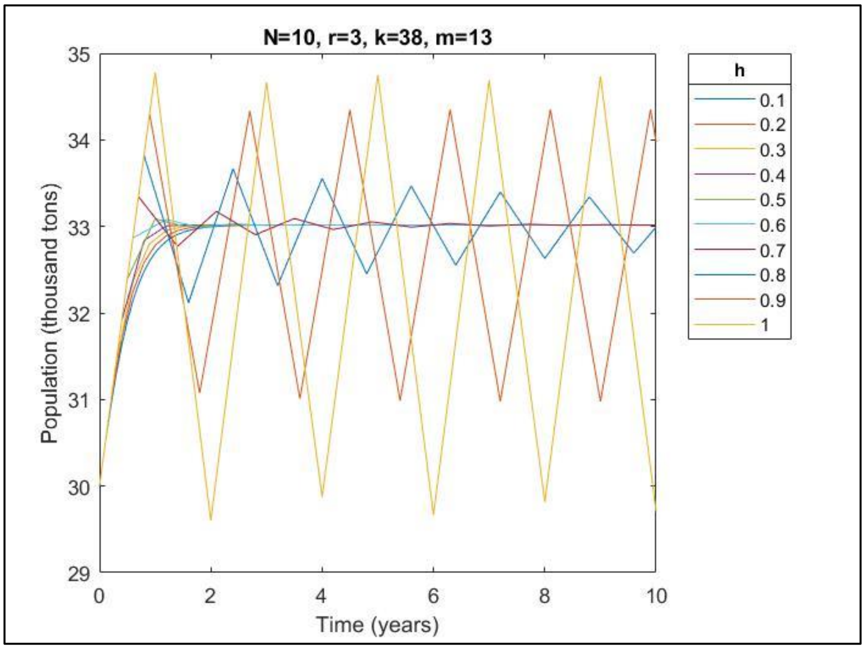

Figure 10: Graph of population for values of h (0.1 to 1) by EE

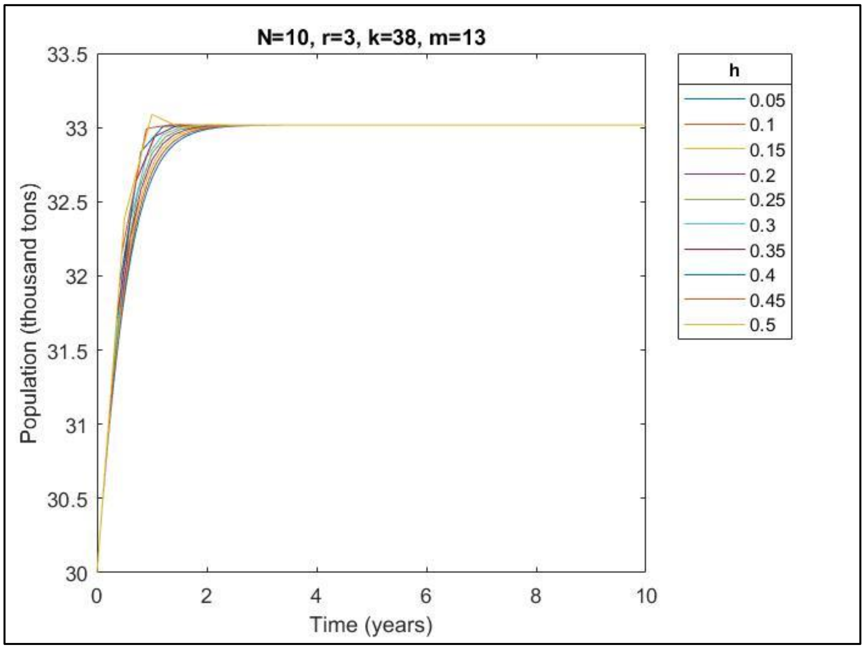

Figure 11: Graph of population for values of h (0.05 to 0.5) by EE

Even though the results fluctuate a lot when time-step is large (i.e. h=0.8, 0.9, 1), it does not mean that the results are not stable. Figure 12 shows the population trend for 100 years and it is seen that the fluctuation does not go unbounded for h=0.9 and 1. For h=0.8, the population does converge to the stable value though only after a very long time of 40-50 years.

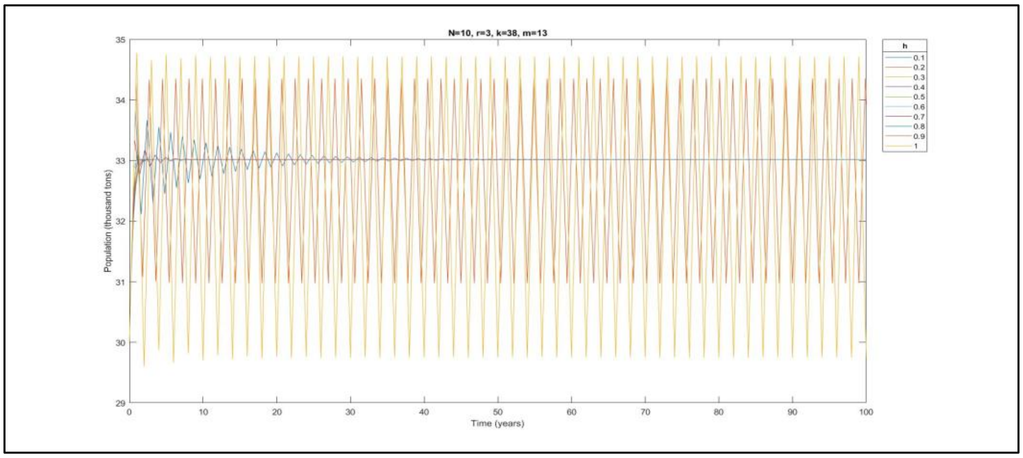

Figure 12: Graph of population for values of h (0.1 to 1) by EE

To find the range of h that the method becomes unstable, h values bigger than 1 are used to run the model. Figure 13 shows that the results are still stable at h=1.2 (though not accurate), but when h is increased to 1.3, the population decreases very fast and die out within 10 years. Thus, for this case, with preliminary attempts to determine the time-step duration, the Euler explicit method is found to be conditionally stable when h is smaller or equal to 1.2.

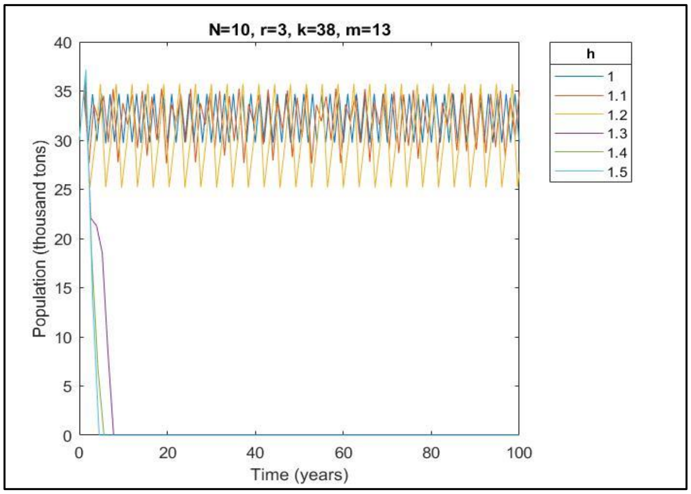

Figure 13: Graph of population for values of h (1 to 1.5) by EE

3.3 Trapezoidal method

3.3.1 Description

The Trapezoidal method is derived from both Euler explicit and implicit method, and it is of second order accuracy. It works by using the average of the gradient at the current time-step and the next time-step, using the formula:

$$y_{i+1} = y_i + 0.5h[f(t_i, y_i) + f(t_{i+1}, y_{i+1})]$$

Since it uses a value from the next time-step, it is an implicit method which requires a form of iteration and so may take a longer computational time. The advantage is that it is unconditionally stable regardless of the time-step chosen.

The iterations for each time-step is done using the Newton Raphson method.

3.3.2 Matlab subroutine

The subroutines for the Trapezoidal-Newton Raphson (Trap-NR) method can be found in Appendix.

3.3.3 Results

Figure 14 and Figure 15 show that the fact that the Trap-NR method is unconditionally stable seems to help in the accuracy of the method as well because the results converge fast even with a relatively big time-step of 1. A time-step of 0.2 is able to produce a sufficiently smooth curve.

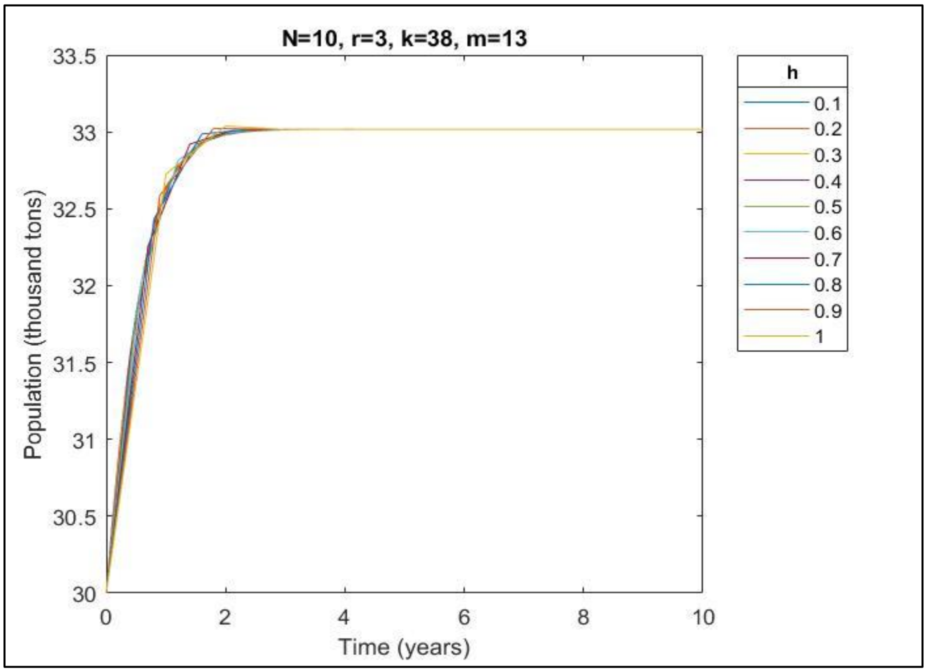

Figure 14: Graph of population for values of h (0.1 to 1) by Trap-NR

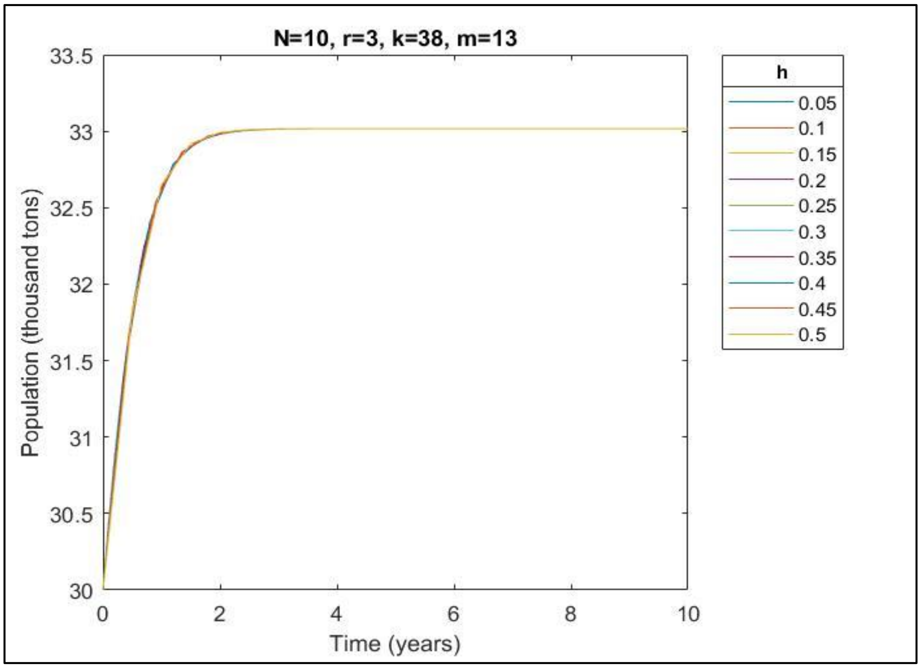

Figure 15: Graph of population for values of h (0.05 to 0.5) by Trap-NR

3.4 Modified Trapezoidal method

3.4.1 Description

The Modified Trapezoidal (Trap-Mod) method is derived from the implicit Trapezoidal method to become an explicit method. Thus, it is conditionally stable. Similar to the Trap-NR method, this method is of second order accuracy. The following steps are taken:

a) Use Euler explicit method to estimate an intermediate value, y_i* using the formula:

$$y_{i+1}* = y_i + hf(t_i, y_i)$$

where $f(t_i, y_i)$ is the gradient at the current time-step t_i

b) Use the estimated value to approximate y_{i+1} using the formula:

$$y_{i+1} = y_i + 0.5h[f(t_i, y_i) + f(t_{i+1}, y_{i+1}*)]$$

3.4.2 Matlab subroutine

The subroutines for the Trap-Mod method can be found in Appendix.

3.4.3 Results

Figure 16 and Figure 17 show the results for the Trap-Modified method for different time-steps h. Though it does not fluctuate drastically when a big time-step is used like the EE method, it can be observed that when a big h is used, the results are not accurate. This is because the magnitude of error is proportional to h^2.

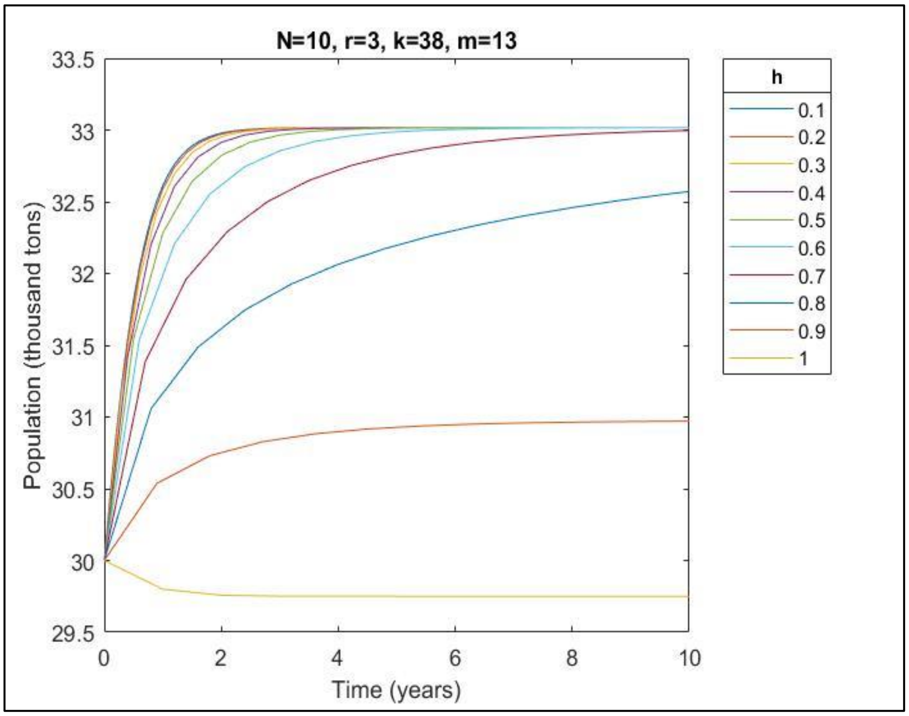

Figure 16: Graph of population for values of h (0.1 to 1) for Trap-mod

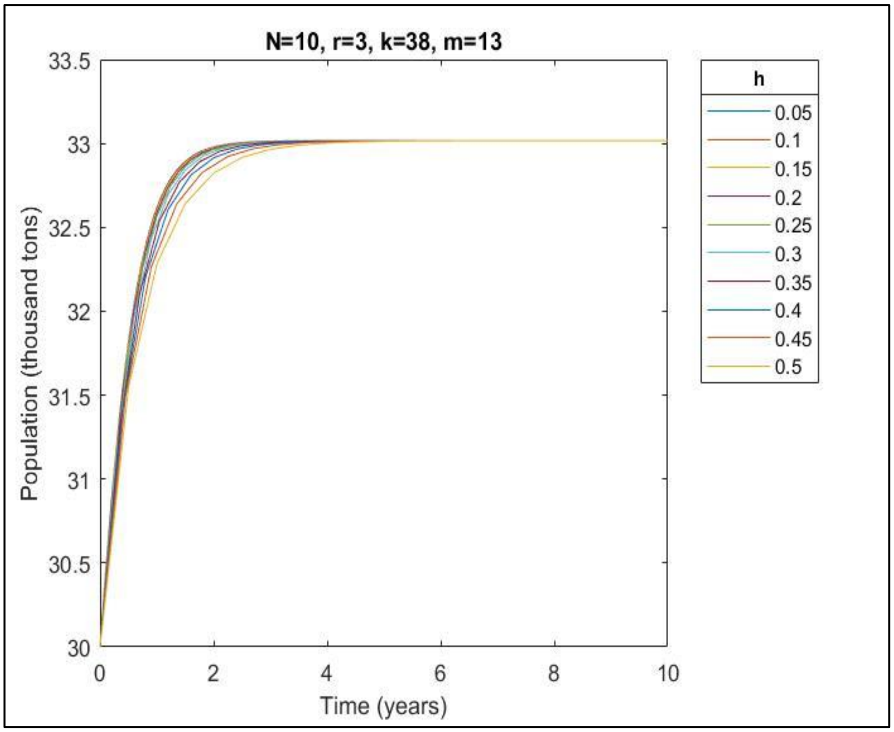

Figure 17: Graph of population for values of h (0.05 to 0.5) for Trap-mod

Similar to the EE method, further analysis is done for a longer period of time to find the limit of h that gives a stable solution. From Figure 18, it is seen that the solution is still stable at h=1.4 but goes unbounded when h=1.5 as shown in Figure 19 . Thus, for this case, with preliminary attempts to determine the time-step duration, the maximum h for the Trap-Mod method to be stable is h=1.4.

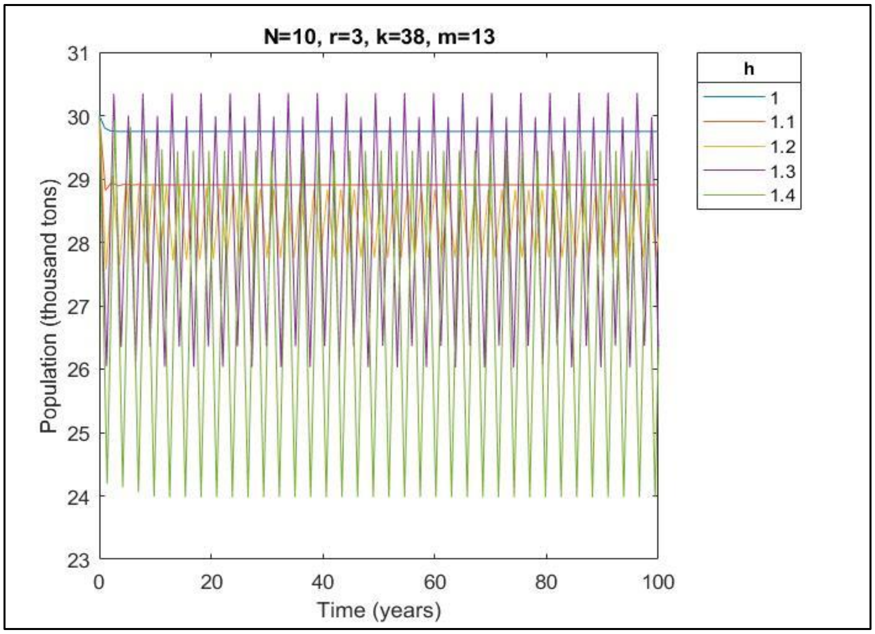

Figure 18: Graph of population of h (1 to 1.4) for Trap-mod

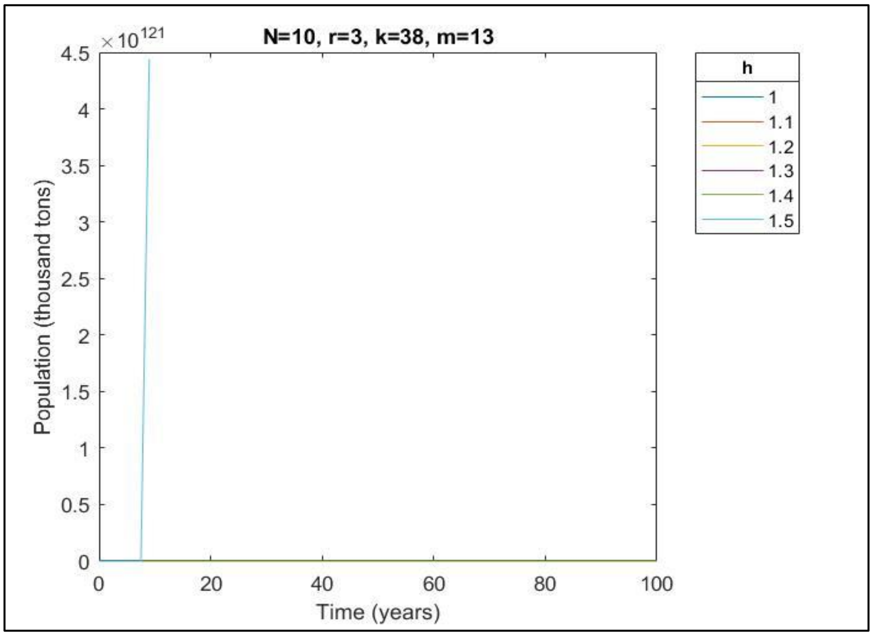

Figure 19: Graph of population of values of h (1 to 1.5) for Trap-mod

3.5 Runge-Kutta (RK 4)

3.5.1 Description

Runge-Kutta fourth-order method (RK4) is one of the most popular predictor-corrector algorithms. RK4 method uses the following equations to solve:

a) Use Euler explicit method at half time-step to estimate an intermediate $y_{i+0.5}*$ value

$$y_{i+0.5}* = y_i + 0.5hf(t_i, y_i)$$

b) Use Euler implicit method at half time-step to estimate an intermediate $y_{i+0.5}**$ value

$$y_{i+0.5}** = y_i + 0.5hf(t_{i+0.5}, y_{i+0.5}*)$$

c) Use midpoint method at a full time-step to estimate an intermediate $y_{i+1}***$ value

$$y_{i+1}*** = y_i + hf(t_{i+0.5}, y_{i+0.5}**)$$

d) Combine the estimated values obtained in parts a, b and c to compute an improved weighted-average slope

$$y_{i+1} = y_i + \frac{1}{6}h[f(t_i, y_i) + 2f(t_{i+0.5}, y_{i+0.5}*) + 2f(t_{i+0.5}, y_{i+0.5}**) + f(t_{i+1}, y_{i+1}***)$$

The solution computed using RK4 method will be more accurate by taking weighted-average, with estimates based on the slope at the midpoint being weighted twice as heavy as those using the slope at the end points.

3.5.2 Matlab subroutine

The subroutines for RK4 method can be found in Appendix.

3.5.3 Results

Figure 20 and Figure 21 show the results using RK4 method with h values from 0.1 to 1 and from 0.05 to 0.5 respectively. It shows that the results can converge to the stable population faster than the EE and Trap-Mod method but slower than the Trap-NR method. This tallies with the expectations because RK4 has higher order of accuracy than the other two explicit methods but is still conditionally stable unlike Trap-NR which is unconditionally stable.

Running the analysis for a longer period of 100 years, it can be seen from Figure 22 and Figure 23 that the population is still stable at h=1.72 but goes unbounded at h =1.73. Thus, for this case, with preliminary attempts to determine the time-step duration, the maximum h for the RK4 method to be stable is h=1.72.

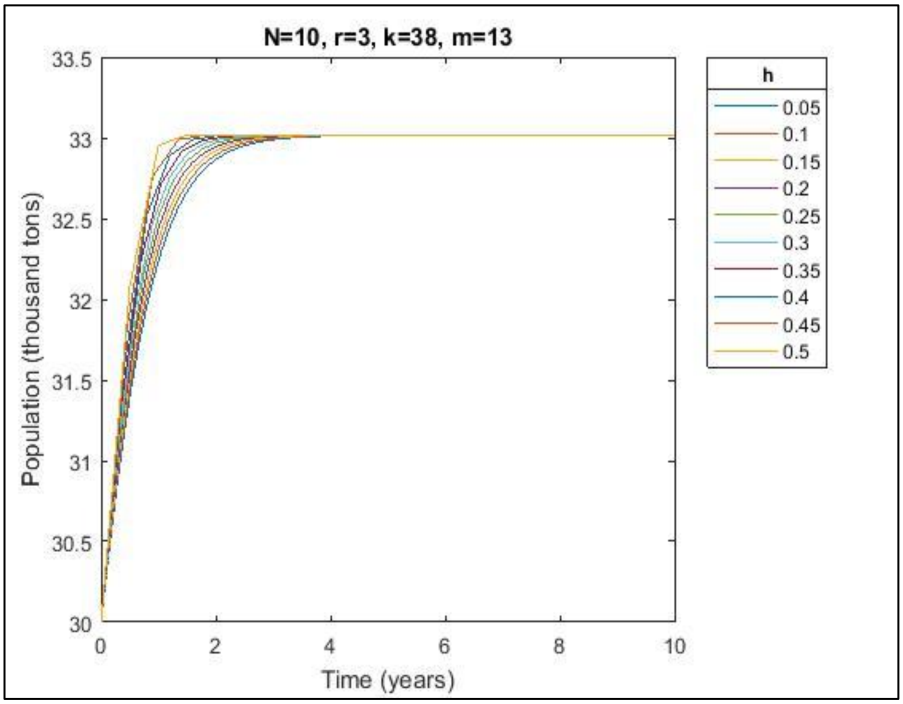

Figure 20: Graph of population of values of h (0.05 to 0.5) for RK4

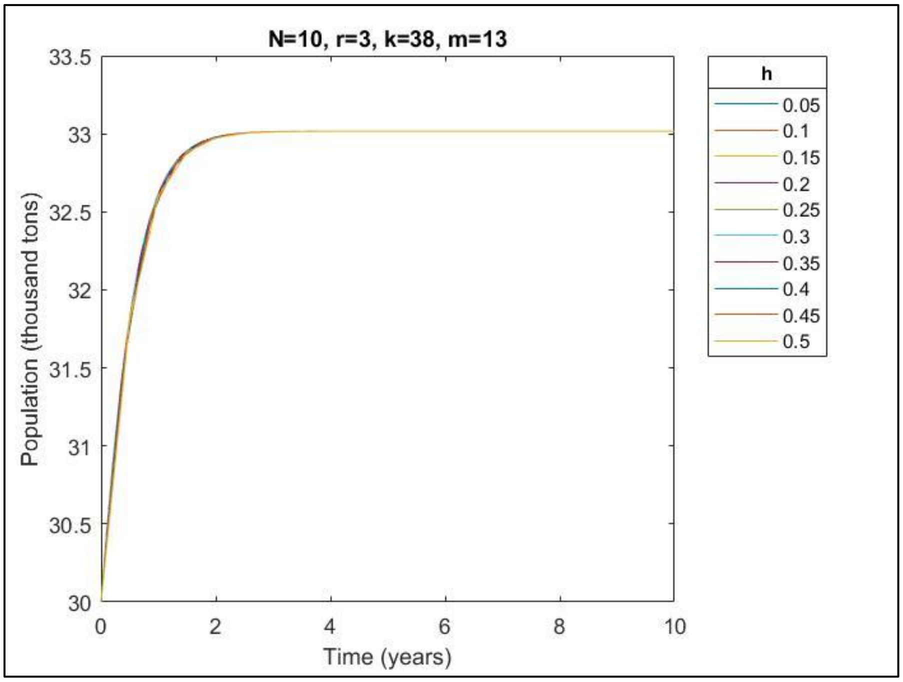

Figure 21: Graph of population for values of h (0.05 to 0.5) for RK4

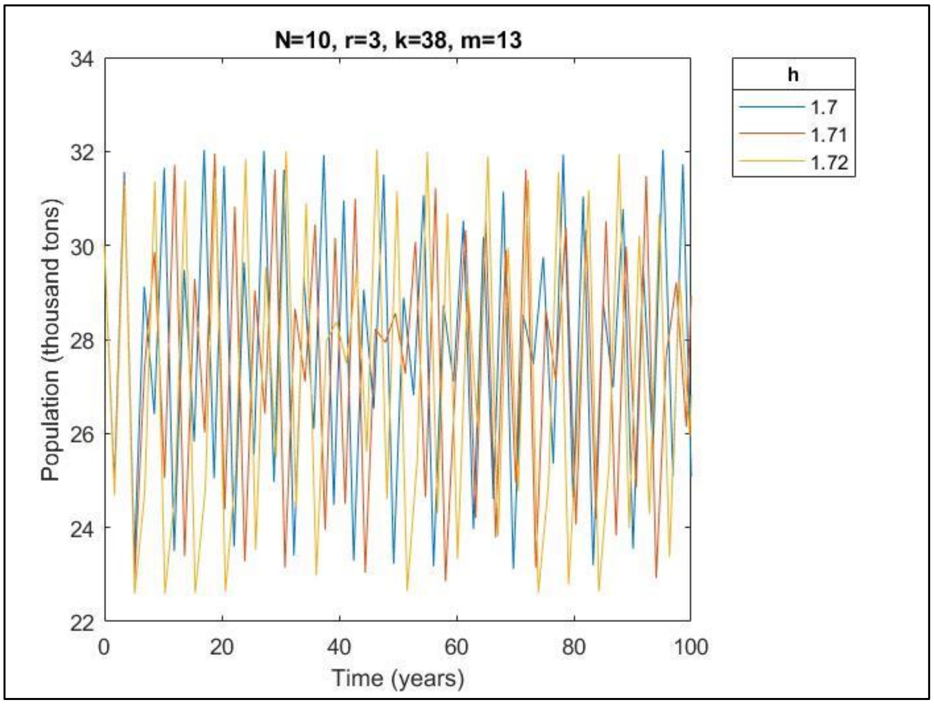

Figure 22: Graph of population for values of h (1.70 to 1.72) for RK4

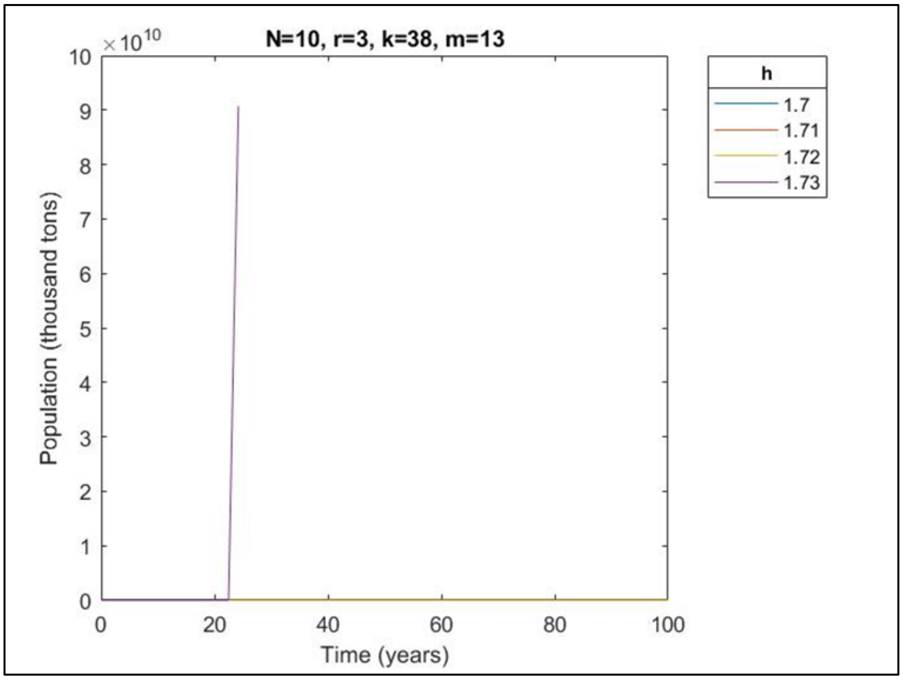

Figure 23: Graph of population for values of h (1.70 to 1.73) for RK4

3.6 Ranking of methods

The various numerical methods as discussed above were used in the determination of the population tendency. In this section, they are ranked accordingly against the three determining factors: 1) computational time, 2) number of time-step and 3) stability and accuracy of method.

The intention is to select the optimal numerical method that provides the required stability and accuracy with the least computational and development time.

The first factor of consideration is computational time. Computational time constitutes the solver time per time step. Total computation time is given by solver time per time step multiply by number of time steps. The number of time steps required is affected by order of accuracy of the numerical method.

The assumption here is that mathematical operations such as addition, subtraction, multiplication, division, inversion and exponent of real numbers require roughly the same amount of computational time. This is a reasonable assumption given modern computers' capabilities. The algorithms consist of similar amount of mathematical operations per function, thus, computational time can be estimated using the number of functions required by the algorithm.

Euler Explicit method ranks first, as the algorithm only entails one line of code. Modified Trapezoidal method ranks second, as the algorithm entails two lines of code. RK4 ranks third, as the algorithm entails 5 lines of code. Trap NR ranks fourth, as the algorithm requires iteration using Newton-Raphson method, which is affected by the tolerance level set for the iterative process.

The second factor of consideration in the ranking is development time. Development time, or the ease of implementation, is the time taken to write a code based on the complexity of the numerical method. Implementing the algorithm based on a numerical scheme may be straight forward if the method is explicit, or may be tricky if the method is implicit. Euler Explicit ranks first due to its simplicity, followed by Modified Trapezoidal method and then RK4. Trap NR ranks fourth as it requires implementing iteration using the Newton Raphson method, hence takes up the highest development time.

The third factor of consideration in the ranking is the stability and accuracy of the numerical method. The methods of stability and accuracy analysis will be further discussed in the following sections. Stability is ranked based on the largest allowable time step which the solution remained bounded. Accuracy is ranked based on the largest time step which the solution attains 32.618 at $t = 1$. As part of the ranking, RK4 is the most accurate of the four methods, while Trap NR is the most stable.

Weightage are given to the different factors to reflect their relative importance in solving a model to the required accuracy in the least amount of time. Development time is given the least weightage because there are resources and sample codes readily available on the internet to help any programmers implement the numerical method. Computational time, which is the solver time per time step, is given the second lowest weightage as modern computers are more than capable to handle the explicit algorithms quickly. Thus, the number of time step required to have an accurate solution becomes the limiting factor for the total amount of computation time required to solve a model. Accuracy is given the highest weightage because it has the largest effect on the number of time step required. Stability is given the second highest weightage because it represents the upper bound of the allowable time step.

Overall, RK4 represents the optimal numerical method as it possesses both high degree of accuracy and stability. Trap NR and Trap Mod are close in comparison, with Trap NR being a better method as it has high degree of stability and accuracy as well. The only severe disadvantage of Trap NR is that it requires iteration and loses out on Computational Time and Development Time. Lastly, Euler Explicit is the least favorable method as it is least stable and accurate, even though it is the easiest to implement and fastest to solve per time step. The ranking is summarized in Table 4

Table 4: Summary of method ranking

Factors		Weightage	EE	Trap-Mod	Trap-NR	RK4
CT	solver time per time step	0.2	1	2	4	3
DT	complexity	0.15	1	2	4	3
Stability	stability region	0.3	4	3	1	2
Accuracy	order of accuracy	0.35	4	3	2	1
Weightage Summation		1	2.95	2.65	2.4	2
Ranking			4	3	2	1

4 Stability Analysis

4.1 Definition of Stability

4.1.1 Stability and Critical Time Step, $h_{critical}$

The problems that has bounded solution are considered. A numerical scheme is considered to be stable if its solution does not grow unbounded with time. However, a stable solution may not be accurate as the solution may:

1) Oscillate as it converges to the asymptote
2) Converge to a different asymptote
3) Take a different convergence path
4) Be out of phase with the solution (for solutions that has oscillation)

The motivation to perform stability analysis for various numerical methods is to find out the critical time step, $h_{critical}$. The critical time step is the largest allowable time step without the solution diverging. The objective is to choose the largest time-step to satisfy the modelling purpose in the fastest time-span and at the same time, achieve the required accuracy and stability for the estimated solution.

4.1.2 Stability analysis of non-linear function.

Stability analysis of non-linear functions are performed using the following steps:

1) Linearize the non-linear function
2) Use the linearized function as a test function for explicit method
3) Obtain stability expression for explicit methods
4) Find critical time step, $h_{critical}$
5) Verify with actual solution

4.2 Linearization of population dynamic equation

The population dynamics equation is expanded to give the following:

$$\dot{P} = \frac{dP}{dt} = -rP(1-\frac{P}{k})(1-\frac{P}{m}) - 2N$$

$$= -rP(1-\frac{P}{m}-\frac{P}{k}+\frac{P^2}{mk}) - 2N$$

$$= -rP(1-\frac{(m+k)P}{mk}+\frac{P^2}{mk}) - 2N$$

$$= -\frac{r}{mk}P^3 + r(\frac{m+k}{mk})P^2 - rP - 2N$$

As the dynamic model is a cubic function, there are three roots representing three equilibrium points, of which two are stable (maximum and minimum) and one is unstable. The non-linear dynamic model is linearized about the maximum (stable) root, P_r .

Following Taylor's expansion series:

$$L \approx f(P_r) + f'(P_r)(P - P_r)$$

$$L = f(P_r) + f'(P_r)P - f'(P_r)P_r$$

$$L = f'(P_r)P + f(P_r) - f'(P_r)P_r$$

$$L = f'(P_r)P + constant$$

$$L = (\frac{-3rP_r^{\ 2}}{mk} + \frac{2r(m+k)P_r}{mk} - r)P + constant$$

$$L = \left(\frac{-3rP_r^{\ 2} + 2r(m+k)P_r - mkr}{mk}\right)P + constant$$

Linearized form:

$$L = \left(\frac{-3rP_r^{\ 2} + 2r(m+k)P_r - mkr}{mk}\right)P + constant$$

In the form $L = \lambda P$. Hence, this linearised form can be substituted as the test function for stability analysis.

4.3 Range of stability for Euler Explicit

$$P_{i+1} = P_i + hf(P_i)$$

Substituting $f(P_i) = \lambda P$

$$P_{i+1} = P_i + h\lambda P_i$$

$$P_{i+1} = (1 + h\lambda)P_i$$

For stability, $|1 + h\lambda| \leq 1$

Since $\lambda = \left(\frac{-3rP_r^{\ 2} + 2r(m+k)P_r - mkr}{mk}\right)$

The stability expression is given by:

$$-1 \leq 1 + h\left(\frac{-3rP_r^{\ 2} + 2r(m+k)P_r - mkr}{mk}\right) \leq 1$$

$$-2 \leq h\left(\frac{-3rP_r^{\ 2} + 2r(m+k)P_r - mkr}{mk}\right) \leq 0$$

$$h \geq \frac{-2mk}{-3rP_r^{\ 2} + 2r(m+k)P_r - mkr}$$

As the h value obtained from section 4.3 is performed using a linearized function, there is a need to verify it against the actual output of the numerical scheme. The Euler Explicit program is ran using the calculated h value and compared against outputs using other h values in the vicinity of the calculated version. Table 5 and Table 6 provide the information on the parameters and values used in the comparison process.

Table 5: Summary of parameters and their acronyms for Euler Explicit

Acronym	Parameter
r	Growth Rate
m	Minimum threshold
k	Population capacity
N	Fishing licenses
P	Population
λ	Linearized coefficient of P
$C._{hEE}$	Calculated critical time-step, h for Euler Explicit
$E._{hEE}$	Empirical critical time-step, h for Euler Explicit
δ	Difference between $\lvert(E._{hEE} - C._{hEE})\rvert / C._{hEE}$

Table 6: Summary of trial values used for each parameter

Parameter	Trial 1	Trial 2	Trial 3	Trial 4	Trial 5	Trial 6
r	3	3	3	2	2	2
k	55	38	65	55	40	65
m	10	13	8	10	10	8
N	10	10	10	10	10	10
λ	-11.82	-2.41	-20.01	-7.28	-3.22	-12.89

Table 7 Calculated and empirical $h_{critical}$ for various trial cases of r, m, k, N for Euler Explicit

Parameter	Trial 1	Trial 2	Trial 3	Trial 4	Trial 5	Trial 6
$C._{hEE}$	0.169	1.204	0.0998	0.275	0.621	0.155
$E._{hEE}$	0.28	1.28	0.17	0.44	0.97	0.24
δ	65.68%	6.31%	70.34%	60.00%	56.20%	54.84%

Table 7 shows that the calculated $h_{critical}$ is always lower than the empirical $h_{critical}$. The difference ranges from 55% to 70%, with an outlier of 6.3%. This shows that the calculated $h_{critical}$ value is actually a conservative estimate of the empirical $h_{critical}$, and thus can be confidently used as a guide for the maximum time step when applied with EE method.

For some values of r, m, k, N, such as in trial 1, the calculated $h_{critical}$ gives a good solution in which the solution starts to converge to an asymptote. However, this is not always the case as seen in trial 2, whereby the h value which the solution converges is h = 0.8, while the calculated $h_{critical}$ is 1.20, as shown in Figure 24.

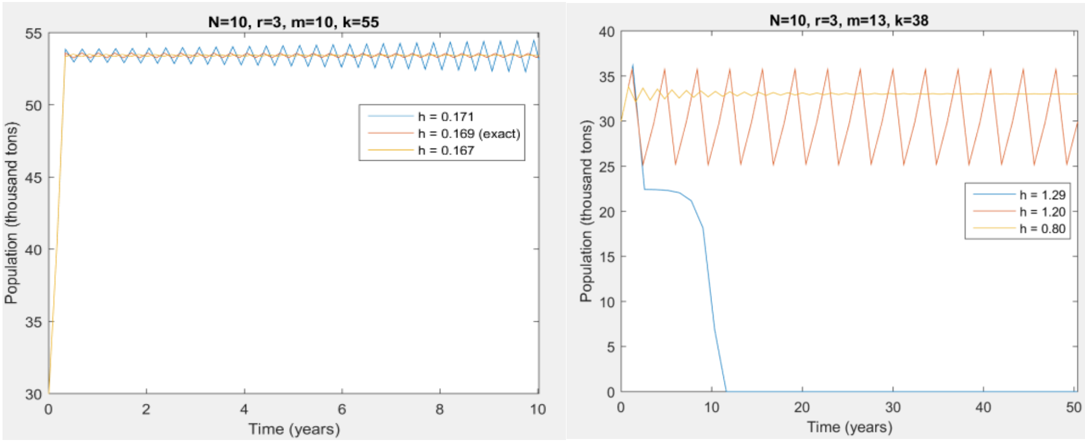

Figure 24: Solutions at various h values for Trial 1 (left) and Trial 2 (right)

4.4 Range of stability for Trap-NR method

As described in section 3.3, Trap-NR in this report, is an implicit method and it is unconditionally stable for any value of h. Thus, the selection of h value is based purely on accuracy. The stability analysis is as follows:

$$P_{i+1} = P_i + 0.5h\{f(P_i) + f(P_{i+1})\}$$

Substituting $f(P_i) = \lambda P_i$ and $f(P_{i+1}) = \lambda P_{i+1}$

$$P_{i+1} = P_i + 0.5h\{\lambda P_i + \lambda P_{i+1}\}$$

$$P_{i+1} = P_i + 0.5h\lambda P_i + 0.5h\lambda P_{i+1}$$

$$(1 - 0.5h\lambda)P_{i+1} = (1 + 0.5h\lambda)P_i$$

$$P_{i+1} = \frac{(1 + 0.5h\lambda)}{(1 - 0.5h\lambda)} P_i$$

For stability, $\left|\frac{(1+0.5h\lambda)}{(1-0.5h\lambda)}\right| \leq 1$

$$-1 \leq \frac{(1 + 0.5h\lambda)}{(1 - 0.5h\lambda)} \leq 1$$

$$h\lambda < 0$$

Since h > 0, and λ < 0 for the solution to be bounded, hλ is always less than 0 and the Trap-NR method is unconditionally stable.

Since h is always positive, and λ must be negative for the solution to be bounded, hλ is always negative and the Trap-NR method is unconditionally stable as shown in Figure 25. Figure 26 illustrates stability is achieved even at h = 3.

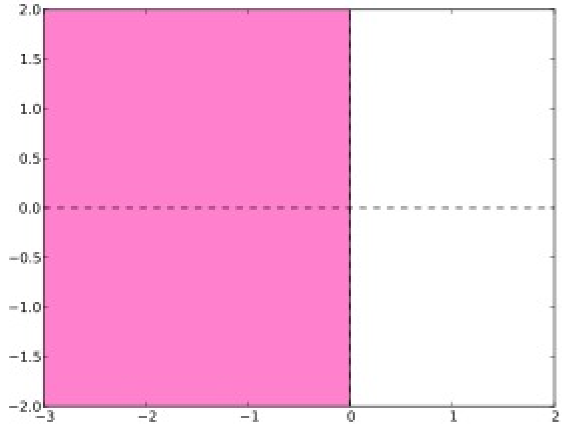

Figure 25: Stability plot (Complex Plane) of Trap-NR method

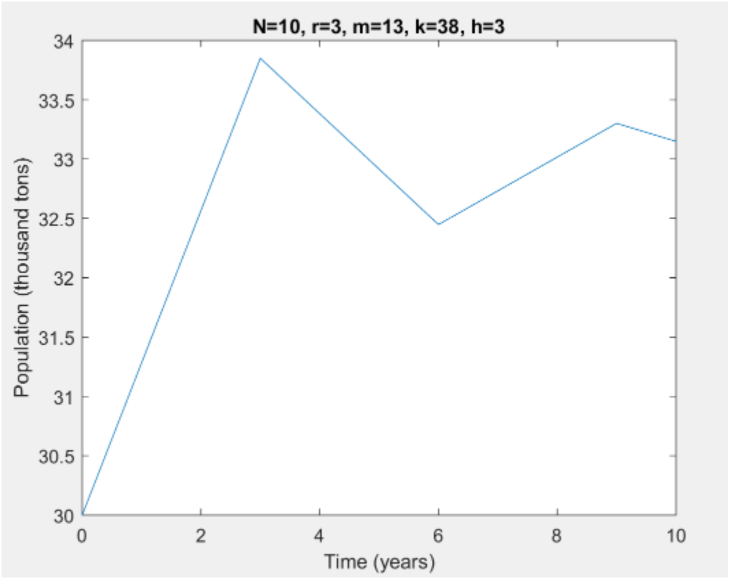

Figure 26: Trap-NR method showing stability even at h = 3

4.5 Range of stability for Modified Trapezoidal method

$$P_{i+1} = P_i + 0.5h\{f(P_i) + f(P^*{}_{i+1})\}$$

$$P^*{}_{i+1} = P_i + h\,f(P_i)$$

$$P^*{}_{i+1} = P_i + h\lambda P_i$$

$$P^*{}_{i+1} = (1 + h\lambda)P_i$$

Substitute $P^*{}_{i+1} = (1 + h\lambda)P_i$

$$P_{i+1} = P_i + 0.5h\{\lambda P_i + \lambda(1 + h\lambda)P_i\}$$

$$P_{i+1} = P_i + 0.5h\lambda P_i + 0.5h\lambda(1 + h\lambda)P_i$$

$$P_{i+1} = (1 + h\lambda + 0.5h^2\lambda^2)P_i$$

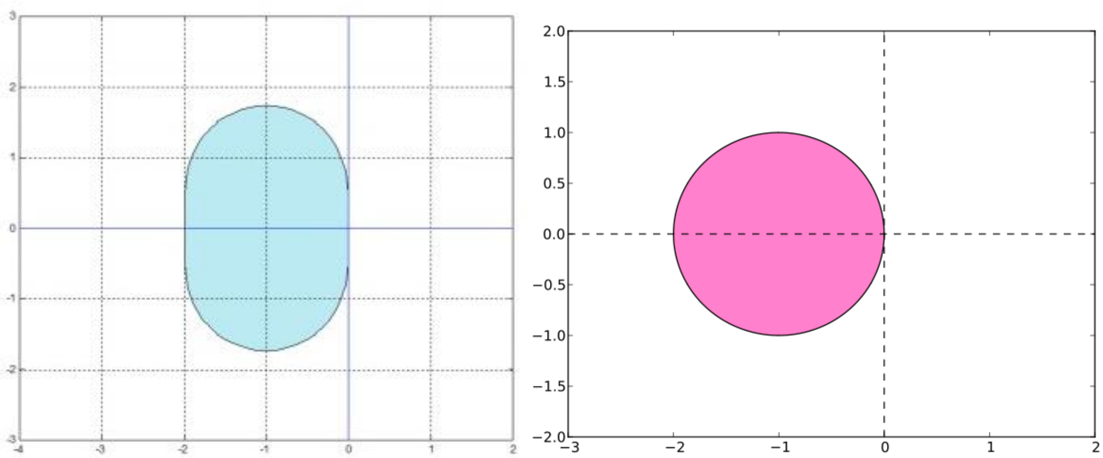

Figure 27: Stability plot (Complex Plane) of Modified Trapezoid Method (left) and Euler Explicit (right)

Interestingly, even though the stability expression for the Trap-Mod method is different from that of EE, the result $h_{critical}$ value is exactly the same ($C._{hEE}$ in Table 7 and $C._{hMT}$ in Table 9), given by $\frac{2}{\lambda}$ if λ is a real number. This is because the stability plot of Trap-Mod Method crosses the real axis at -2, and $h_{critical}$ scales the value of λ along the real axis (and imaginary axis if λ is complex). The difference between EE and Trap-Mod method is that the latter has a larger stability area in the complex region compared to former as shown in Figure 27.

Table 6 and Table 8 provide the information on the parameters and values used in the comparison process. Table 9 shows that the calculated $h_{critical}$ is always lower than the empirical $h_{critical}$. The difference ranges from 25% to 30%, with two outliers of 154.5% and 352%. This shows that the calculated $h_{critical}$ value is actually a conservative estimate of the empirical $h_{critical}$, and thus can be used confidently as a guide for the maximum time step when applied with Trap-Mod method.

Table 8: Summary of parameters and their acronyms for Trap-Mod method

Acronym	Parameter
r	Growth Rate
m	Minimum threshold
k	Population capacity
N	Fishing licenses
P	Population
λ	Linearized coefficient of P
$C._{hMT}$	Calculated critical time-step, h for Trap-Mod method
$E._{hMT}$	Empirical critical time-step, h for Trap-Mod method
δ	Difference between $\|(E._{hMT} - C._{hMT})\|/ C._{hMT}$

Table 9: Calculated and empirical $h_{critical}$ for various trial cases of r, m, k, N for Trap-Mod method

Parameter	Trial 1	Trial 2	Trial 3	Trial 4	Trial 5	Trial 6
$C._{hMT}$	0.169	1.204	0.0998	0.275	0.621	0.155
$E._{hMT}$	0.31	1.46	0.18	0.99	1.11	0.99
δ	83.43%	21.26%	80.36%	260.00%	78.74%	538.71%

4.6 Range of stability for RK4

$$\lambda_1 = h\lambda P$$

$$\lambda_2 = h\lambda P(1 + \frac{h\lambda}{2})$$

$$\lambda_3 = h\lambda P[1 + \frac{1}{2}h\lambda + \frac{1}{4}h^2\lambda^2]$$

$$\lambda_4 = h\lambda P[1 + h\lambda + \frac{1}{2}h^2\lambda^2 + \frac{1}{4}h^3\lambda^3]$$

$$P_{n+1} = P_n[1 + h\lambda + \frac{1}{2}h^2\lambda^2 + \frac{1}{6}h^3\lambda^3 + \frac{1}{24}h^4\lambda^4]$$

$$\left|1 + h\lambda + \frac{1}{2}h^2\lambda^2 + \frac{1}{6}h^3\lambda^3 + \frac{1}{24}h^4\lambda^4\right| < 1$$

Stability plot of RK4 crosses the real axis at -2.81 in Figure 28. Since $h_{critical}$ serves as a scaling factor along the real axis (and imaginary axis if λ is complex), $h_{critical}$ can be determined by $\frac{2.81}{\lambda}$.

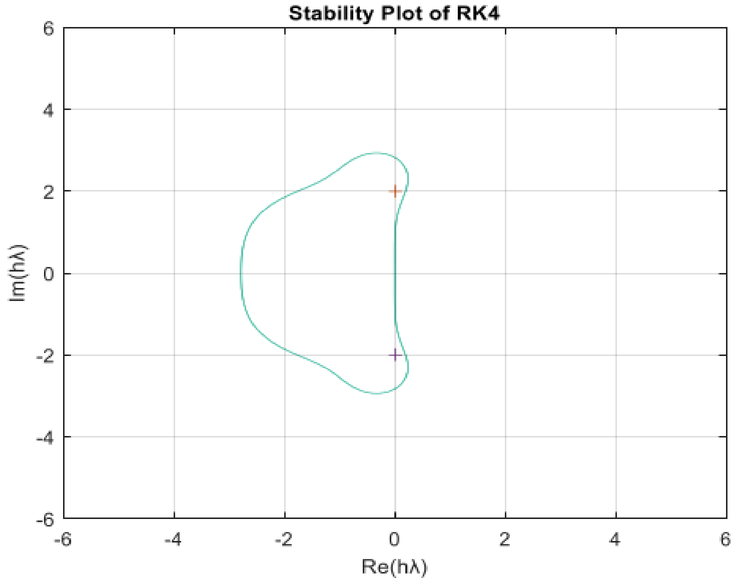

Figure 28: Stability plot (Complex Plane) of RK4

Table 6 and Table 10 provide the information on the parameters and values used in the comparison process. Table 11 shows that the calculated $h_{critical}$ is <u>always lower</u> than the empirical $h_{critical}$. The difference ranges from 48% to 51%, which is the smallest difference range as compared to the other three methods. This shows that the calculated $h_{critical}$ value is actually a <u>conservative estimate</u> of the empirical $h_{critical}$, and thus can be used confidently as a guide for the maximum time step when applied with RK4 method.

Table 10: Summary of parameters and their acronyms for RK4

Acronym	Parameter
r	Growth Rate
m	Minimum threshold
k	Population capacity
N	Fishing licenses
P	Population
λ	Linearized coefficient of P
$C._{hRK4}$	Calculated critical time-step, h for RK4 method
$E._{hRK4}$	Empirical critical time-step, h for RK4 method
δ	Difference between $\|(E._{hRK4} - C._{hRK4})\|/ C._{hRK4}$

Table 11: Calculated and empirical $h_{critical}$ for various trial cases of r, m, k, N for RK4

Parameter	**Trial 1**	**Trial 2**	**Trial 3**	**Trial 4**	**Trial 5**	**Trial 6**
$C._{hRK4}$	0.238	1.166	0.140	0.386	0.873	0.218
$E._{h\ RK4}$	0.36	1.72	0.21	0.58	1.30	0.33
δ	51.43%	47.52%	49.54%	50.26%	48.97%	51.38%

4.7 Summary of Stability

Table 12 shows a summary of the calculated, empirical $h_{critical}$ and convergent h for the four numerical methods discussed previously. Convergent h is defined as the time-step, h, for which the solution:

- Tends towards an accurate (truncation error tends towards zero) asymptote
- Does not exhibit any oscillation
- Does not exhibit any overshoot

Convergent h provides an indication of the time-step which the solution starts to become accurate. Table 12 illustrates a few important points:

1. The calculated $h_{critical}$ value is always smaller than the empirical $h_{critical}$ value, hence, it is considered as a conservative estimate of the empirical $h_{critical}$ and can be trusted as a method to choose a h value that will ensure stability.
2. Even though the calculated $h_{critical}$ values for all methods may be quite similar for the given set of r, m, k and N values, the empirical $h_{critical}$ value shows that it increases with the order of the method's accuracy as well.
3. Stability does not necessarily mean accuracy, and the convergent h values shows that it depends on the order of accuracy of the method. We see a pattern of increasing convergent h from EE to RK4. EE is a first order method, and its convergent h is 38% of the calculated $h_{critical}$. Trap-Mod method is a second order method, and its convergent h is 51% of the calculated $h_{critical}$. RK4 is a fourth order method, and its convergent h is 82% of the calculated $h_{critical}$. While there is no formula to calculate the convergent h exactly, the order of method serves as a guide for an initial estimate of its value.

Table 12: Summary table of calculated, empirical $h_{critical}$ and convergent h for 4 numerical methods based on r = 3, m = 38, k = 13, N = 10

	EE	Trap-Mod	Trap-NR	RK4
Calculated $h_{critical}$	1.204	1.204	-	1.166
Empirical $h_{critical}$	1.28	1.46	-	1.72
Convergent h	0.46	0.62	0.88	0.96

Figure 29 gives a visual interpretation of convergent h of 0.96 for RK4.

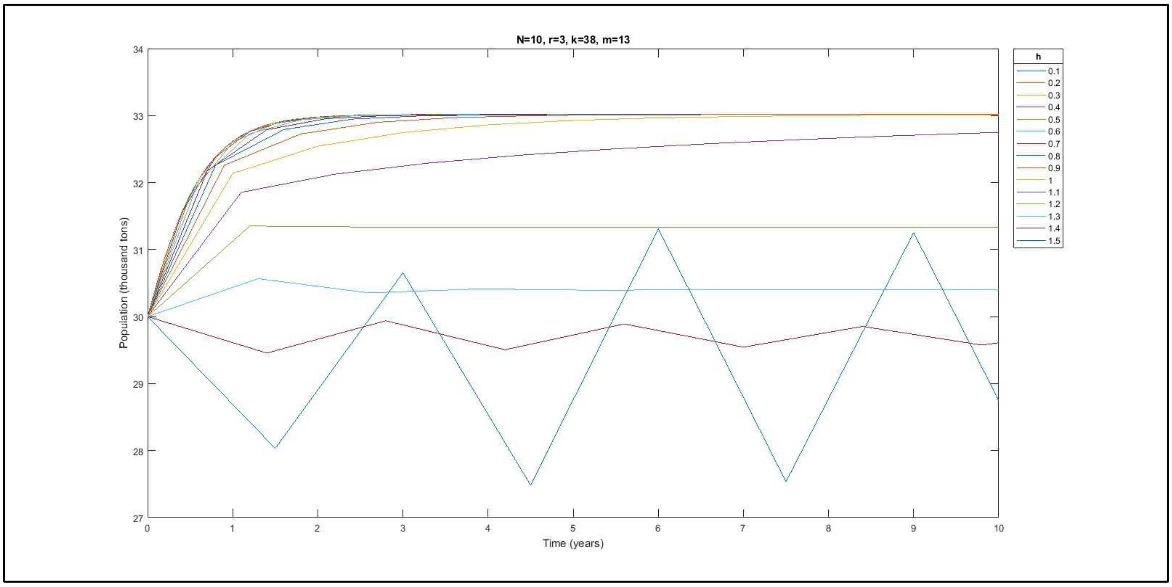

Figure 29: Graph of population of values of h (0.1 to 1.5) for RK4

5 Accuracy Analysis

Accuracy reflects the ability of the numerical scheme to tend towards an exact value with a given time-step duration. To review the accuracy between the different methods, a set of varying time-step durations were implemented for a set of r, k, m, N values for each of the four numerical methods.

In addition, due to the non-linear and non-homogenous nature of the population model, no analytical solution can be obtained by the separation of variables. Therefore, an exact value is obtained at the t=1-year step using a small time-step duration with the RK4 numerical method as it has the highest order of accuracy. The population tends to 32,618 tons after a year as shown in Table 13 and this estimated value is used for comparison of accuracy analysis between numerical methods.

From the Figure 30 shown below, it can be seen that comparing using time step duration, h= 1 year, the accuracy can be attained best by the implicit Trap-NR method, followed by RK4, EE, and lastly Trap-Mod. This deviates slightly from the understanding that the accuracy is highest in the following arrangement RK4, followed by Trap-Mod and Trap-NR and lastly EE, as the order of accuracies are to the order of four, two and one respectively. The reason for this is possibly because of the region of stability that is required for the explicit methods. This means that the explicit method may not have been in the stable region when the time-step duration is set at h=1 year, which is coherent to what was discussed in the earlier sections.

In this report, a time-step duration of 0.1 have been generally used for trending analysis as it allows the explicit methods to attain stability even at the first time-step. It is thus recognized that 0.1 is able to provide a stable analysis for all the explicit numerical schemes. Hence, when the accuracy observation is made at the time-step duration of h=0.1 year, it is safe to extract the relationship between the various methods. The accuracy is shown to be attained best by RK4, followed by Trap-NR, Trap-Mod and lastly EE. At time step durations h smaller than 0.1, it can be seen that this relationship is coherent.

Table 13: Population (after 1 year) vs Time-step for the four numerical methods

EE				
-LOG(h)	h	Point	P	err
0.0000	1	2	34.7773	-0.066199644
0.3010	0.5	3	33.0884	-0.014421485
0.6990	0.2	6	32.7803	-0.00497578
1.0000	0.1	11	32.6963	-0.002400515
1.3010	0.05	21	32.6565	-0.00118033
1.6990	0.02	51	32.6332	-0.000466
2.0000	0.01	101	32.6256	-0.000233

Trap_NR				
-LOG(h)	h	Point	P	err
0.0000	1	2	32.7263	-0.003320253
0.3010	0.5	3	32.6482	-0.000925869
0.6990	0.2	6	32.6227	-0.000144092
1.0000	0.1	11	32.6192	-3.67895E-05
1.3010	0.05	21	32.6183	-9.19738E-06
1.6990	0.02	51	32.6181	-3.06579E-06
2.0000	0.01	101	32.6180	0

Trap_mod				
-LOG(h)	h	Point	P	err
0.0000	1	2	29.7997	0.086403213
0.3010	0.5	3	32.2811	0.010328653
0.6990	0.2	6	32.5876	0.000932001
1.0000	0.1	11	32.6116	0.000196211
1.3010	0.05	21	32.6165	4.59869E-05
1.6990	0.02	51	32.6178	6.13158E-06
2.0000	0.01	101	32.6180	0

RK4				
-LOG(h)	h	Point	P	err
0.0000	1	2	32.1380	0.014715801
0.3010	0.5	3	32.6024	0.000478264
0.6990	0.2	6	32.6178	6.13158E-06
1.0000	0.1	11	32.6180	0
1.3010	0.05	21	32.6180	0
1.6990	0.02	51	32.6180	0
2.0000	0.01	101	32.6180	0

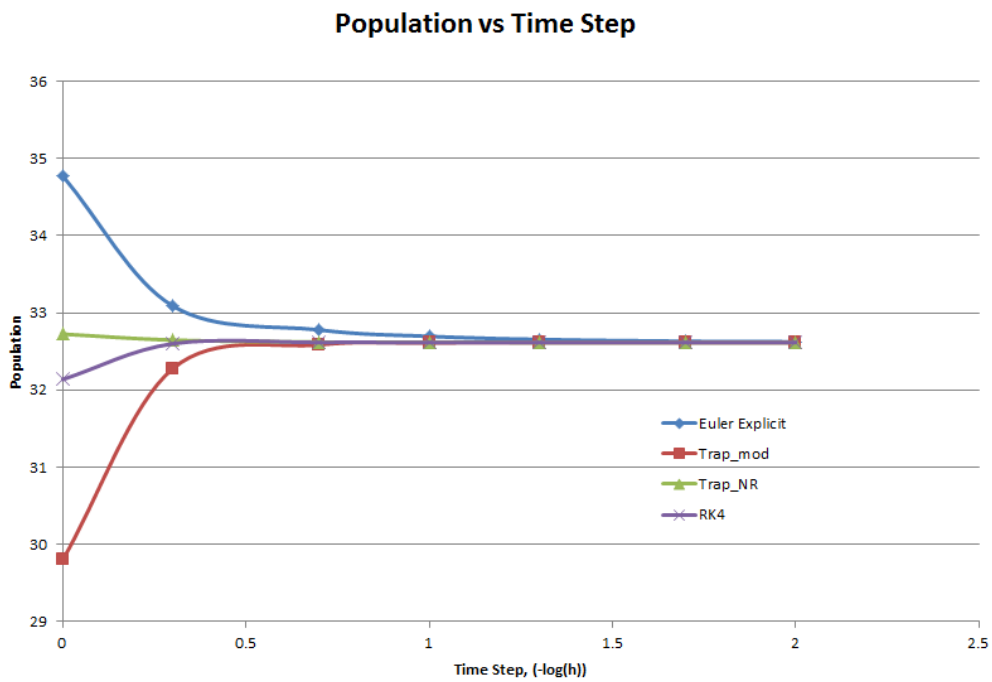

Figure 30: Population (after 1 year) vs Time-step for the four numerical methods

6 Conclusion

To achieve a population of $1.5 \times m$, the proposed number of fishing licenses to be issued by FDAM for the next 10 years is capped at 10. This value is estimated through the application of initialized parameters and conditions via RK4 method. The RK4 numerical scheme yields the most stable and accurate solution, as compared to the others evaluated.

It is recommended for FDAM to engage all interested parties in discussion in the proposed process of monitoring stingray population via random physical checks, with comparison to the modelled data results. This would effectively lead to possible improvements to the predictive accuracy of future population dynamic models as well as to support FDAM's strong interest in the implementation of sustainable fishing practices.

7 Learning Points

During the course of the analysis, Team B has been able to gather numerous learning lessons with regards to the solving of such initial value problems for real world dynamic models.

For the purpose of this proposal, Team B assumes the population dynamics equation postulated 10 years ago offers the ideal estimation of the stingray population in Malaysian waters. This equation is a simplification of the system in which complex relationships and interactions exist between the stingrays, human actions and environment factors. It is difficult to justify if this equation is still applicable in the present situation because of lack of detailed quantifiable data in this area of study. In reality, it is noteworthy that the population dynamics probably would be constantly changing. Even if there is existing data on the stingray population for an adequately long period of time, it would require extensive in-depth research to know if any changes in the population are affected by changes in the parameters or by additional undefined factors.

Due to limited data available, our postulations of the quantitative changes in parameters were based on logic, preliminary modelling while keeping other parameters constant at some assumed values, or assumptions of proportional effect of some environment factors on the parameter, for example a 30% increase in shipping will cause a 30% drop in population capacity k. In reality the parameters might be interdependent and so cannot be analyzed separately or assumed constant while the other parameters vary.

Practically, the usage of numerical modeling goes far beyond the basic use of solving for a solution. It can be used as part of a predictor model for trends like in this case, to complement policy making decisions. Yet, as mentioned, it is important that the user of the data and conclusions of the model must exercise caution and active assessment. Through the years of the usage of the concluded model population, they have to ensure that the conditions assumed in the model stay true or is within the bounds of the assumption. Take for example, should the parameters be observed to drop beyond those set in the analysis, there is a need for the user to reassess the model and re-establish a new set of data. To prevent dire consequences, contingencies should have already been put in place as part of a proper risk assessment. It is also necessary that the implementer understand the data that is produced from the model, such that whilst the dynamic model is manipulated, physical relationships and laws are still obeyed and remotely followed.

Technically, non-linear models are challenging to perform stability analysis because the eigenvalue of the system coefficient depends on the solution itself. This means that the key stability determinant, $h\lambda$, of the numerical scheme is always changing depending on the value of the solution. The most common method is to linearize the model to provide some preliminary insights to the critical time step for stability. However, it may not be obvious which point the model should be linearized about. In addition, the critical time step derived from the linearized model may not necessarily hold true for the entire range of solution. Therefore, it is important to provide some margin from the calculated critical time step for stability when dealing with non-linear model that has been linearized.

Another challenge is stiff equations. If the model remains the same, but the nature of the organism is different e.g. bacteria whereby the growth rate can be high, large coefficients of the equations can make it stiff, thus creating another set of challenge for explicit methods as extremely small time steps must be used to ensure that the solution does not "blow up".

As such, the above presents the different perspectives that Team B has considered during the course of the analysis and also the learning points, technically and practically in the implementation of numerical schemes for real life problems.

8 References

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Appendix

This section contains the Matlab subroutines for numerical methods.

General

```
%Define population dynamic equation
function [ Pdot ] = Pdot( P, N, r, k, m )

Pdot = -r*P*(1-P/k)*(1-P/m)-2*N;

end
```

Euler Explicit method (Subroutine)

```
%Euler Explicit method
clear all

% define parameters and initial values
P0 = 30; %initial population
N = 10; %number of licenses
r = 3; %growth rate
k = 38; %population capacity
m = 13; %threshold population
h = 0.46; %time-step size
n = 10; %total number of years
steps = ceil(n/h);
n = steps*h;

T = 0:h:n;
P = zeros(1+steps,1);
P(1) = P0;

% Euler Explicit Forward Method
for i = 1:steps
    P(i+1) = P(i) + h*Pdot(P(i),N,r,k,m);

    if P(i+1)<0
        P(i+1) = 0;
    end
end

plot(T,P);
hold on;
title(['N=',num2str(N),', r=',num2str(r),', m=',num2str(m),',
k=',num2str(k),', h=',num2str(h)]);
xlabel('Time (years)');
ylabel('Population (thousand tons)');
xlim([0 n]);
```

Trapezoidal method (Subroutine)

```
%Trapezoidal method
close all
clear all

tol=0.01;
r=3;
m=13;
k=38;
N=10;
h=1.26;
n=10; %total number of years
steps = ceil(n/h);
n = steps*h;

T=0:h:n;
P=zeros(steps+1,1);
P(1)=30;

for i=1:steps
    Yt1=P(i)+h*Pdot(P(i),N,r,k,m); % assume an initial value
    eps=1;
    while eps>tol
        %form a function
       Fx=Yt1-P(i)-h/2*(Pdot(Yt1,N,r,k,m)+Pdot(P(i),N,r,k,m));
        %1st-order derivative depends on f
       dFx=1-h/2*(-r+2*r*Yt1/m+2*r*Yt1/k-3*r*Yt1^2/k/m);
        % define a new x
       Yt2=Yt1-Fx/dFx;
       eps=abs(Yt2-Yt1); % decide when to abort
       Yt1=Yt2;
    end
    P(i+1)=P(i)+h*1/2*(Pdot(P(i),N,r,k,m)+Pdot(Yt1,N,r,k,m));

    if P(i+1)<0
        P(i+1)=0;
    end
end

plot(T,P);
title(['N=',num2str(N),', r=',num2str(r),', m=',num2str(m),',
k=',num2str(k),', h=',num2str(h)]);
xlabel('Time (years)');
ylabel('Population (thousand tons)');
xlim([0 10]);
```

Modified Trapezoidal method (Subroutine)

```
%Modified Trapezoidal method
close all
clear all

r=3;
m=13;
k=38;
N=10;
h=0.97;
n=10; %total number of years
steps = ceil(n/h);
n = steps*h;

T=0:h:n;
P=zeros(steps+1,1);
P(1)=30;

for i=1:steps
    Yt=P(i)+h*Pdot(P(i),N,r,k,m);
    P(i+1)=P(i)+h*0.5*(Pdot(P(i),N,r,k,m)+Pdot(Yt,N,r,k,m));

    if P(i+1)<0
        P(i+1)=0;
    end
end

plot(T,P);
title(['N=',num2str(N),', r=',num2str(r),', m=',num2str(m),',
k=',num2str(k),', h=',num2str(h)]);
xlabel('Time (years)');
ylabel('Population (thousand tons)');
xlim([0 n]);
```

RK4 method (Subroutine)

```
%RK4
close all
clear all

P0=30;
r=3;
m=13;
k=38;
N=10;
%h=0.1;
%n=10; %total number of years


for h = 0.22:0.01:0.25 %CHANGE THE RANGE OF H HERE
    n = 10;
    steps = ceil(n/h);
```

```
    n = steps*h;
    T = 0:h:n;
    P = zeros(1+steps,1);
    P(1) = P0;

    for i=1:steps
    P1=h*Pdot(P(i),N,r,k,m);
    P2=h*(Pdot(P(i)+(P1/2),N,r,k,m));
    P3=h*(Pdot(P(i)+(P2/2),N,r,k,m));
    P4=h*(Pdot(P(i)+(P3),N,r,k,m));

    P(i+1)=P(i)+(P1+2*P2+2*P3+P4)/6;
        if P(i+1)<0
            P(i+1)=0;
        end

    end
plot(T,P,'DisplayName',num2str(h));
hold on
end

title(['N=',num2str(N),', r=',num2str(r),', k=',num2str(k),',
m=',num2str(m)]);
xlabel('Time (years)');
ylabel('Population (thousand tons)');
xlim([0 10]);
lgd = legend('show','Location','bestoutside');
title(lgd,'h')
```