An introduction into finite difference modeling of groundwater flow for CIE4420 Principles of Geohydrology

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One-dimensional steady flow in homogeneous aquifers

Consider one-dimensional steady flow in a homogeneous aquifer with thickness H. The aquifer is divided into cells of length Δx. For any cell, the water balance may be written as

$$\text{In} - \text{Out} = 0 \tag{1}$$

Consider an arbitrary cell flanked by a cell on its western side and a cell on its eastern side (Fig. 1). There is inflow from the west and east and there is a sink with an outflow Q

$$Q_W + Q_E - Q = 0 \tag{2}$$

The flow from the west and east may be expressed in terms of the heads at the centers of the cells using Darcy's law

$$Q_W = kH\Delta y(h_W - h)/\Delta x \qquad Q_E = kH\Delta y(h_E - h)/\Delta x \tag{3}$$

Equations are derived here for square cells so that $\Delta x = \Delta y$. Substitution of (3) for Q_W and Q_E into (2) and rearrangement of terms gives

$$h_W - 2h + h_E = \frac{Q}{kH} \tag{4}$$

or

$$h = \frac{h_W + h_E}{2} - \frac{Q}{2kH} \tag{5}$$

Hence, the head is equal to the average of its two neighboring cells minus a sink term divided by $2kH$.

When the right side of the cell is impermeable, $Q_E = 0$ and (2) simplifies to

$$Q_W - Q = 0 \tag{6}$$

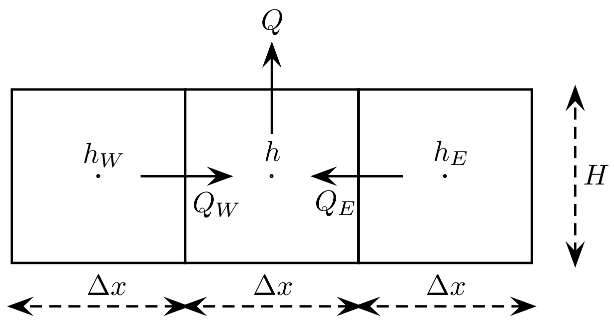

Figure 1: Water balance for a cell in steady one-dimensional flow; cross-sectional view.

so that

$$h_W - h = \frac{Q}{kH} \tag{7}$$

and

$$h = h_W - \frac{Q}{kH} \tag{8}$$

In this case h is equal to the head in the cell next to it minus the sink term divided by kH.

One-dimensional steady flow in heterogeneous aquifers

The hydraulic conductivity varies spatially in heterogeneous aquifers and each cell may be assigned a different k value. The three cells in Fig. 1 are assigned k values equal to k_W, k and k_E from left to right. The water balance (2) still holds, but the equations for the flow from the west and east need to be modified as follows

$$Q_W = C_W(h_W - h) \qquad Q_E = C_E(h_E - h) \tag{9}$$

where C_W and C_E are the conductance values which may be obtained from the equation for flow in a system with two hydraulic conductivity values in series (Fitts, 2002, Eq. 3.23).

$$C_W = \frac{2kk_W}{k + k_W}H \qquad C_E = \frac{2kk_E}{k + k_E}H \tag{10}$$

where it is used again that $\Delta x = \Delta y$. Substitution of (9) for Q_W and Q_E into (2) gives

$$C_W h_W - (C_W + C_E)h + C_E h_E = Q \tag{11}$$

or

$$h = \frac{C_W}{C_W + C_E}h_W + \frac{C_E}{C_W + C_E}h_E - \frac{Q}{C_W + C_E} \tag{12}$$

System of equations for one-dimensional, steady flow in homogeneous aquifer

Consider a finite difference model for one-dimensional steady flow consisting of n uniform cells with $\Delta x = \Delta y$. The head in the first and last cell are given as h_1^* and h_n^*, respectively. The system of equations may be written as

$$
\begin{array}{cccccccl}
h_1 & & & & & & = & h_1^* \\
h_1 & -2h_2 & +h_3 & & & & = & Q_2/(kH) \\
& h_2 & -2h_3 & +h_4 & & & = & Q_3/(kH) \\
& & \ddots & \ddots & \ddots & & = & \vdots \\
& & & h_{n-2} & -2h_{n-1} & +h_n & = & Q_{n-1}/(kH) \\
& & & & & h_n & = & h_n^*
\end{array}
$$

of in matrix form

$$
\begin{pmatrix}
1 & & & & & \\
1 & -2 & 1 & & & \\
 & 1 & -2 & 1 & & \\
 & & \ddots & \ddots & \ddots & \\
 & & & 1 & -2 & 1 \\
 & & & & & 1
\end{pmatrix}
\begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ \vdots \\ h_{n-1} \\ h_n \end{pmatrix}
=
\begin{pmatrix} h_1^* \\ Q_2/(kH) \\ Q_3/(kH) \\ \vdots \\ Q_{n-1}/(kH) \\ h_n^* \end{pmatrix}
$$

One-dimensional transient flow in homogeneous aquifers

Consider one-dimensional transient flow in a homogeneous aquifer with thickness H, hydraulic conductivity k, and storage coefficient S. The aquifer is divided into cells of length Δx. For any cell, the water balance may be written for a time period Δt as

$$\text{In} - \text{Out} = \text{Increase in storage} \tag{13}$$

Consider an arbitrary cell flanked by a cell on its western side and a cell on its eastern side (Fig. 1). There is inflow from the west and east and there is a sink with an outflow Q. The water balance for period Δt is

$$(Q_W + Q_E - Q)\Delta t = S[h(t + \Delta t) - h(t)]\Delta x \Delta y \tag{14}$$

The flow from the west and east may be expressed in terms of the heads using (3), where square cells are considered again ($\Delta x = \Delta y$), but it has to be decided whether the heads used in (3) are the heads at time t or at time Δt. In case the heads are at time t are used, the inflow is

$$Q_W = kH[h_W(t) - h(t)] \qquad Q_E = kH(h_E(t) - h(t)) \tag{15}$$

so that the head at time $(t + \Delta t)$ can be computed explicitly from the heads at time t

$$h(t + \Delta t) = h(t) + \alpha \left[h_W(t) - 2h(t) + h_E(t) - \frac{Q}{kH} \right] \tag{16}$$

where

$$\alpha = \frac{kH\Delta t}{S(\Delta x)^2} \tag{17}$$

This explicit scheme only provides numerically stable results when the time step is chosen small enough, or more generally, when the factor α is small enough. For the case that $Q = 0$, a numerically stable (but not necessarily accurate) solution is obtained when $\alpha < 0.5$, which means the time step should be chosen as

$$\Delta t < \frac{1}{2}\frac{S(\Delta x)^2}{kH} \tag{18}$$

Alternatively, the flow from west and east may be computed from the heads at the end of the time step

$$Q_W = kH[h_W(t+\Delta t) - h(t+\Delta t)] \qquad Q_E = kH(h_E(t+\Delta t) - h(t+\Delta t)) \tag{19}$$

which results in

$$h(t+\Delta t) = \frac{h(t) + \alpha\left[h_W(t+\Delta t) + h_E(t+\Delta t) - \frac{Q}{kH}\right]}{1+2\alpha} \tag{20}$$

This is called an implicit scheme, as the head at time $t + \Delta t$ depends on all the other heads at time $t + \Delta t$. An implicit scheme is always stable, independent of the time step, but accurate solutions are only obtained when the time step is small enough.

An even better solution is to compute the inflow from the average between the heads at time t and $t + \Delta t$, known as the Crank-Nicholson scheme.

Two-dimensional steady flow in homogeneous aquifers

For two-dimensional flow, the water balance equation (1) still holds, of course. A cell is now flanked by four cells, on its eastern, northern, western, and southern sides (Fig. 2). The water balance for the cell in the middle is

$$Q_E + Q_N + Q_W + Q_S - Q = 0 \tag{21}$$

Expressions for flow from the four surrounding cells are obtained in the same fashion as in (3), again for the case that $\Delta x = \Delta y$

$$Q_E = kH(h_E - h) \qquad Q_N = kH(h_N - h) \qquad Q_W = kH(h_W - h) \qquad Q_S = kH(h_S - h) \tag{22}$$

Substitution of these four discharges for the equivalent terms in (21) and rearrangement of terms gives

$$h_E + h_N + h_W + h_S - 4h = \frac{Q}{kH} \tag{23}$$

or

$$h = \frac{h_E + h_N + h_W + h_S}{4} - \frac{Q}{4kH} \tag{24}$$

Hence, the head in a cell is equal to the average of the heads around it minus a sink term divided by $4kH$.

If we have an impermeable boundary on a side, an equation for the head in a cell may be derived in a similar fashion. Consider, for example the case of Fig. 3. The north face of the cell is impermeable. The water balance now becomes

$$Q_E + Q_W + Q_S - Q = 0 \tag{25}$$

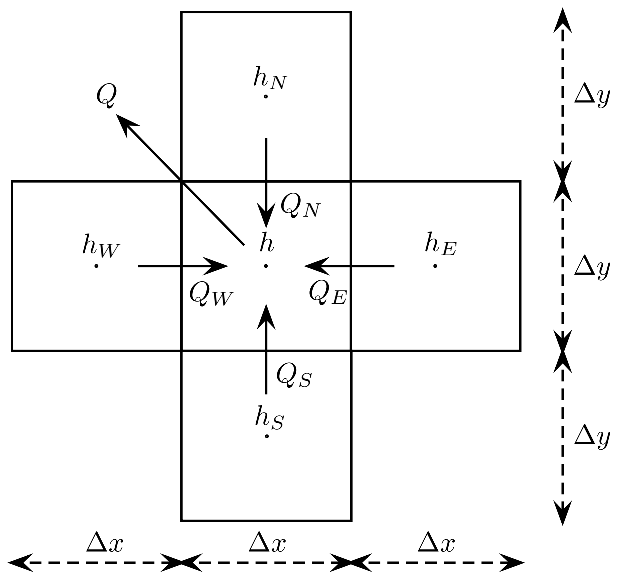

Figure 2: Water balance for a cell in steady two-dimensional flow; plan view.

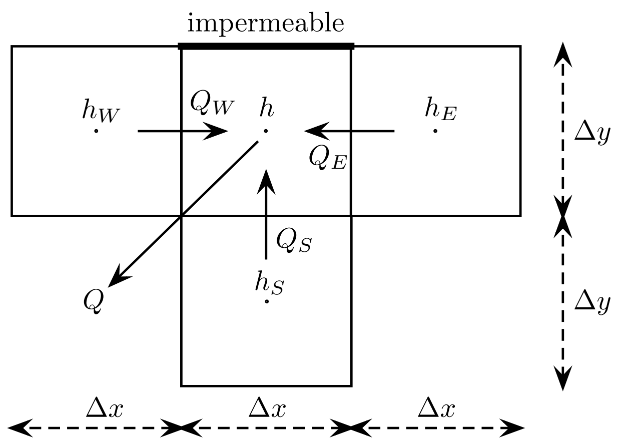

Figure 3: Water balance for a cell that is impermeable along its north face in steady two-dimensional flow; plan view.

and substitution of the equations for the discharges from the surrounding cells gives

$$h_E + h_W + h_S - 3h = \frac{Q}{kH} \tag{26}$$

which may be rewritten as

$$h = \frac{h_E + h_W + h_S}{3} - \frac{Q}{3kH} \tag{27}$$

Again, the head is equal to the average of the surrounding cells (in this case only 3 cells) minus a source term divided by $3kH$.

Two-dimensional steady flow in heterogenous aquifers

The hydraulic conductivity varies spatially and each cell may be assigned a different k value. The inflows in the water balance (21) are written as (compare (9))

$$Q_E = C_E(h_E - h) \qquad Q_N = C_N(h_N - h) \qquad Q_W = C_W(h_W - h) \qquad Q_S = C_S(h_S - h) \tag{28}$$

where the conductances C_E, C_N, C_W, and C_S may be written as

$$C_E = \frac{2kk_E}{k + k_E}H \qquad C_N = \frac{2kk_N}{k + k_N}H \qquad C_W = \frac{2kk_W}{k + k_W}H \qquad C_S = \frac{2kk_S}{k + k_S}H \tag{29}$$

where it is used again that $\Delta x = \Delta y$. Substitution of the inflows in the water balance equation gives

$$C_E h_E + C_N h_N + C_W h_W + C_S h_S - (C_E + C_N + C_W + C_S)h = \frac{Q}{kH} \tag{30}$$

or

$$h = \frac{C_E h_E + C_N h_N + C_W h_W + C_S h_S - Q/(kH)}{C_E + C_N + C_W + C_S} \tag{31}$$

When the northern boundary of the cell is impermeable, $C_N = 0$, and likewise for the other sides.

Stream function

The definition of the stream function is that (1) the stream function is constant along a stream line, and (2) the difference between the stream function values at two points is equal to the amount of water that flows between the two points. The stream function value by itself has no physical meaning. Furthermore, the stream function increases to the right when you look in the direction of flow. A finite difference model may be used to compute the flow between cells as shown in Fig. 4. The stream function may be computed at the corners of cells by simply summing the intercell discharges as shown in the figure.

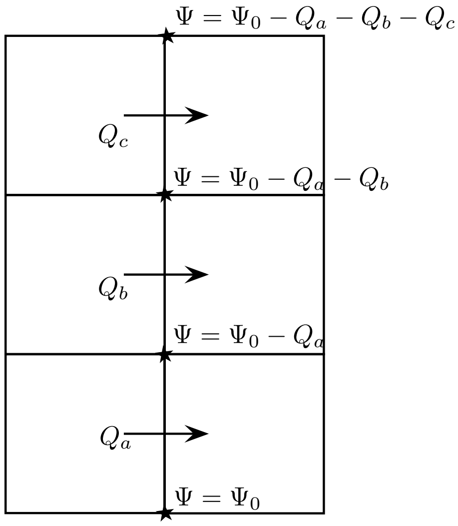

Figure 4: Stream function in cell corners may be computed by summing the intercell discharges. The streamfunction value is set arbitrarily in one point (here it is set to Ψ_0).