NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF ENGINEERING

EXAMINATION FOR

(Semester I: 2011-2012)

CE5311 - ENVIRONMENTAL MODELLING WITH COMPUTERS

Nov/ Dec 2011 - Time allowed: 3 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **THREE(3)** questions and comprises **SEVEN(7)** printed pages.

2. Answer ALL **THREE(3)** questions.

3. All questions carry equal marks.

4. This is an “**OPEN BOOK**” examination.

5. Provide a **clear explanation** with every answer.

Question 1 [4 points per question]

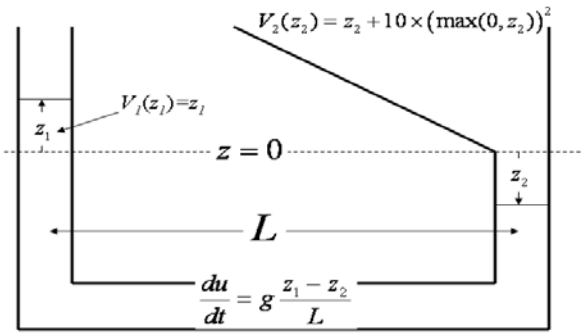

Figure 1

We consider a pipe as given by Figure 1. The pipe has two open ends. One end has the same cross section as the pipe. The size is 1 m^2. The other end contains a small reservoir. The volume of the reservoir, with respect to the reference level z=0, is given by:

$$V_2(z_2) = z_2 + 10 \times \left(\max(0, z_2)\right)^2 \tag{1}$$

At the beginning of the pipe the volume, with respect to the reference level z=0, is simply given by $V_1(z_1) = z_1$. The horizontal part of the pipe is always completely filled. The flow in the pipe is supposed to be described by the following equation:

$$\frac{du(t)}{dt} = g\frac{z_1(t) - z_2(t)}{L} \tag{2}$$

L, the length of the pipe, is 98.1 m.

The initial conditions are given by the following values:

$z_1(0) = 1m, z_2(0) = -1m, u(0) = 0m/s$.

Figure 2 shows the solutions for $z_1(t)$ and $z_2(t)$.

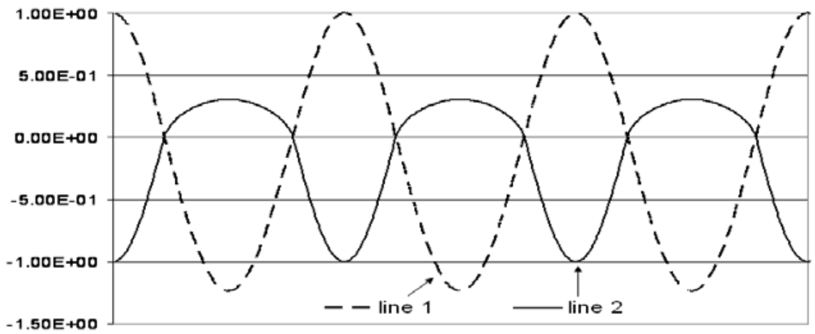

Figure 2

Questions

A) Give the complete set of equations that describes the system of Figure 1.

B) Which line in figure 2 to belongs to $z_1(t)$ **and which one to** $z_2(t)$ **?**

C) Explain why the following relation must hold: $V_1(t)+V_2(t)=0$

D) For different inititial conditions: $z_1(0)=10^{-3}m, z_2(0)=-10^{-3}m, u(0)=0m/s$ **, the solutions for** $z_1(t)$ **and** $z_2(t)$ **are given by figure 3. Explain the difference with Figure 2.**

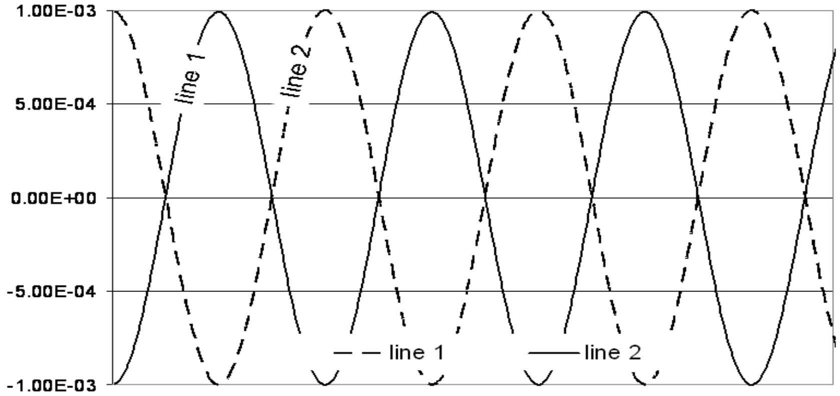

Figure 3.

E: What is the value of the eigen frequency of Figure 3?

For the following questions we first consider a set of numerical recipes for the approximation of the equations of question A.

Recipe 1:

$$u_{1,n+1}=u_{1,n}+\Delta t g\frac{z_{1,n}-z_{2,n}}{L}$$
$$z_{1,n+1}=z_{1,n}-\Delta t u_{1,n}$$
$$z_{2,n+1}=z_{2,n}+\Delta t\frac{u_{1,n}}{1+20\times\max(0,z_{2,n})}$$

Recipe 2:

$$u_{1,n+1}=u_{1,n}+\Delta t g\frac{z_{1,n}-z_{2,n}}{L}$$
$$z_{1,n+1}=z_{1,n}-\Delta t u_{1,n}$$
$$V_2\left(z_{2,n+1}\right)=V_2\left(z_{2,n}\right)+\Delta t u_{1,n}, \text{ (to be solved iteratively)}$$

Recipe 3:

$$u_{1,n+1/2} = u_{1,n-1/2} + \Delta t g \frac{z_{1,n} - z_{2,n}}{L}$$
$$z_{1,n+1} = z_{1,n} - \Delta t u_{1,n+1/2}$$
$$z_{2,n+1} = z_{2,n} + \Delta t \frac{u_{1,n+1/2}}{1 + 20 \times \max(0, z_{2,n})}$$

Recipe 4:

$$u_{1,n+1/2} = u_{1,n-1/2} + \Delta t g \frac{z_{1,n} - z_{2,n}}{L}$$
$$z_{1,n+1} = z_{1,n} - \Delta t u_{1,n+1/2}$$
$V_2\left(z_{2,n+1}\right) = V_2\left(z_{2,n}\right) + \Delta t u_{1,n+1/2}$, (to be solved iteratively)

F) Which recipes are consistent with the equations of question A and what is the order of the local truncation error of the consistent approximations?

G) Which recipes are unstable?

H) Which recipes are mass conservative?

I) Figure 4 shows two lines for the quantity $V_{1,n} + V_{2,n}, n = 1, \ldots$. Explain which line belongs to which recipe.

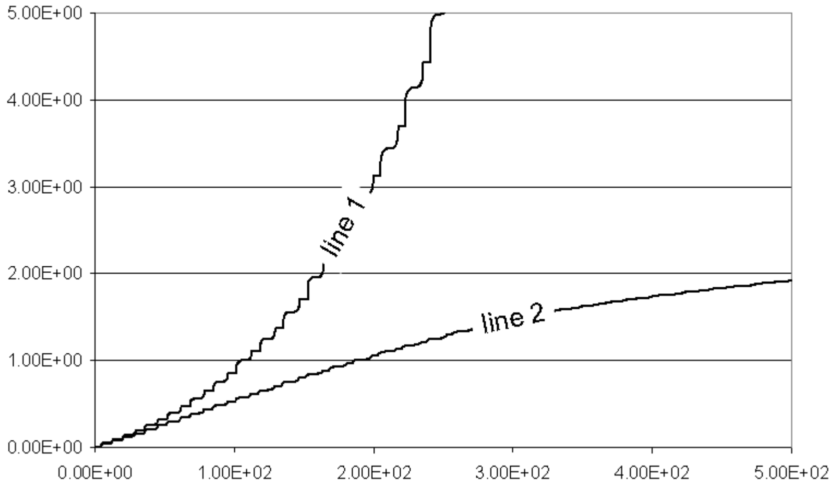

Figure 4

For the following questions we have added friction, by means of a linear friction coefficient: β_1 =0.1 1/s or β_2 =0.895 1/s, to equation 2.

J) Give equation 2 including the linear friction term β.

K) Figure 5 gives the solution for z_1 combined with β_1 =0.1 1/s or β_2 =0.895 1/s. Explain which line belongs to which β.

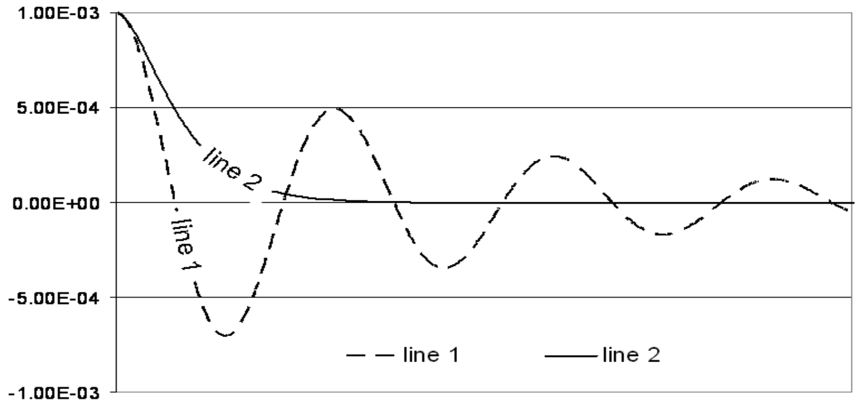

Figure 5

L) Explain why line 2 does not show any oscillation. Above which value for β oscillations will not occur?

Question 2 [4 points per question]

The shallow water equations are given by:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+g\frac{\partial \zeta}{\partial x}+c_f\frac{|u|u}{h}=0$$
$$\frac{\partial \zeta}{\partial t}+\frac{\partial}{\partial x}(uh)=0$$

The simplified form is given by:

$$\frac{\partial u}{\partial t}+g\frac{\partial \zeta}{\partial x}+\alpha\frac{u}{D}=0$$
$$\frac{\partial \zeta}{\partial t}+D\frac{\partial u}{\partial x}=0$$

Questions:

A) Give a staggered spatial grid scheme for these equations with the θ method for time integration. Draw the stencil of your scheme.

B) Consider the following situation. A 1D river model must be setup for the region of the river mouth.

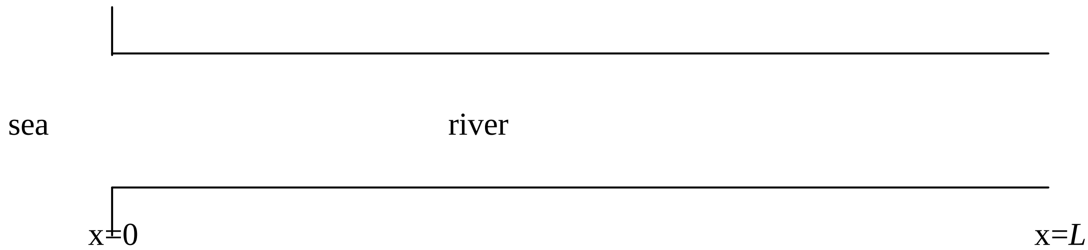

Figure 6

Explain what boundary conditions are needed in this situation. How can you chose them and why?

C) Numerical models take some time to spin up. In what way does the spin up behavior of your model depend on the choice of boundary conditions?

D) For what value of theta does the model spin up fastest?

E) What is the time scale of the spin up time if the simplified equations are used with the following boundary conditions: $\zeta(0,t)=A\cos\left(\frac{2\pi}{T}t\right), u(L,t)=0$ **. What is the frequency of the spin up oscillations?**

F) What is the time scale of the spin up time if the simplified equations are used with the following boundary conditions: $\zeta(0,t)=A\cos\left(\frac{2\pi}{T}t\right), u(L,t)-\zeta(L,t)\sqrt{\frac{g}{D}}=0.$ **What is the frequency of the spin up oscillations? Explain the difference with question E.**

G) In general, what value of θ will yield the fastest convergence to the exact solution?

Question 3 [6 points per question]

Consider 2 scalar transport equations, with diffusion only, given by:

$$\frac{\partial c}{\partial t} = \frac{\partial}{\partial x}\left(K_x \frac{\partial c}{\partial x} \right) + \frac{\partial}{\partial z}\left(K_z \frac{\partial c}{\partial z} \right) \tag{3}$$

$$\frac{\partial c}{\partial t} = \frac{\partial}{\partial x'}\left(K_x \frac{\partial c}{\partial x'} \right) + \frac{1}{H}\frac{\partial}{\partial \sigma}\left(K_z \frac{1}{H}\frac{\partial c}{\partial \sigma} \right) \tag{4}$$

Equation (3) is given in z coordinates and equation (4) in σ coordinates.

Questions:

A) Which value is for practical applications in general larger, K_x or K_z?

B) What are the differences between z- and σ coordinates?

C) Are the equations (3) and (4) completely equivalent, except from the fact that they are denoted in different coordinates, or what are the differences?

D) Consider an initial situation as in figure 7 for a concentration without density differences. The basin is sloshing, while the pressure is hydrostatic, friction is neglected and $K_z = 0$. For which equation (3) or (4) will the concentration be fully mixed in the vertical after a sufficiently long period of time and for which equation the stratification of the concentration c will remain?

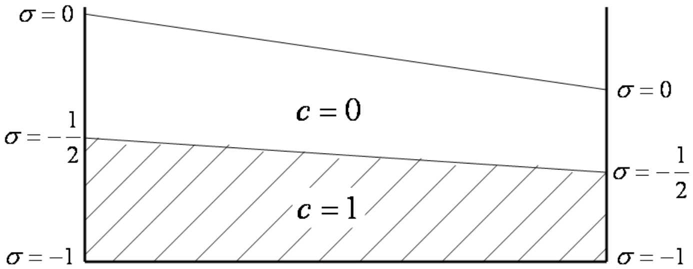

Figure 7

- END OF PAPER -