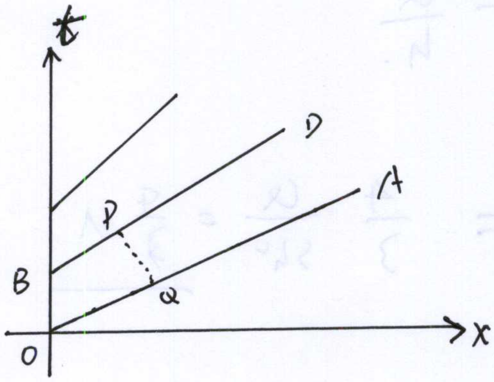
Question 1.

(a)



OA is a C^+ that represent the front of the disturbance:

$$\left.\frac{dx}{dt}\right|_{OA} = u_0 + c_0$$

Any point P on another C^+ can be connected to OA with a C^- PQ:

$$u_P - 2c_P = u_Q - 2c_Q = u_0 - 2c_0 \qquad (1)$$

So along BD: $u - 2c = u_0 - 2c_0 = \text{const}$

Also since BD is a C^+: $u + 2c = \text{const}$

$$\Rightarrow \begin{cases} u = \text{const} \\ c = \text{const} \end{cases}$$

Thus: $\left.\frac{dx}{dt}\right|_{BD} = \text{const}.$

From (1): $u = u_0 - 2c_0 + 2c$

So: $u + c = u_0 - 2c_0 + 3c$

thus, the slope of a C^+ initiated at $t = \tau$ on the t-axis is:

$$dx/dt = u_0 - 2c_0 + 3\sqrt{g h(\tau)}$$

Since $h(\tau)$ decrease with τ, dx/dt decrease with τ, so the C^+ family diverges, until $\tau \geq 1$ hour

(b) The front is represented by OA.

$$dx/dt = u_0 + c_0 = 1.2 + \sqrt{9.8 \cdot 3} = 1.2 + 5.4 = 6.6\ m/s.$$

So the location of the front is given by:

$$X_f\,[m] = 6.6\,[m/s] \cdot t\,[s].$$

The trailing edge is formed at $x=0$ at $t = 1$ hour., so $h = 3 - 0.3 \times 1 = 2.7\ m.$

$$c = \sqrt{gh}$$

$$dx/dt = u_0 - 2c_0 + 3c = 1.2 - 2 \times 5.4 + 3 \times \sqrt{9.8 \times 2.7} = 5.8\ m/s.$$

Thus, the location of the trailing edge is given by:

$$X_t\,[m] = 5.8\,[m/s] \cdot (t\,[s] - 3600\,[s])$$

(c) The c^+ with $h = 2.8m$ is ~~indicated~~ initiated on t-axis

at t_1:
$$h_0 - 0.3 t_1 = 2.8m \Rightarrow t_1 = 2400\ s.$$

The slope of the c^+:
$$dx/dt = u_0 - 2c_0 + 3c = 1.2 - 2 \times 5.4 + 3 \times \sqrt{9.8 \times 2.8} = 6.1\ m/s$$

So:
$$\frac{x}{t - t_1} = 6.1\ m/s$$

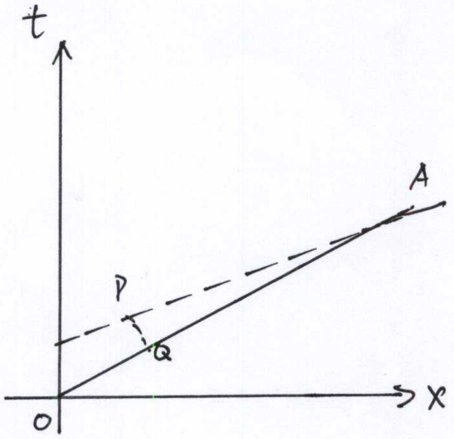
$$\Rightarrow x = 6.1 \times (1.5 \times 3600 - 2400) = \underline{18.2\ km}\ \#$$

Question 2:

$$h_0 = 5m \qquad u_0 = 0\ m/s. \qquad c_0 = \sqrt{gh} = 7 m/s.$$

$$\partial q / \partial \tau = 5 m^2/s \,/\, 500s = \frac{1}{100} m^2/s^2.$$

$$\cancel{\text{So: } \left.\frac{\partial c}{\partial \tau}\right|_{x=0,\,\tau=0} = \frac{g}{2}}$$



The slope of c^+ issued at $t=\tau$ on the t-axis:

$$\frac{dx}{dt} = u + c$$

Since any c^+ can be related to OA, we can get:

$$u - 2c = u_0 - 2c_0$$

Thus: $u + c = u_0 - 2c_0 + 3c$

and: $$\frac{dx}{dt} = u_0 - 2c_0 + 3\sqrt{gh(\tau)}$$

or: $$\frac{x}{t-\tau} = u_0 - 2c_0 + 3\sqrt{gh(\tau)}$$

or $$f(x,t,\tau) = x - (t-\tau)\left[u_0 - 2c_0 + 3\sqrt{gh(\tau)}\right] = 0$$

The envelope of intersections is given by:

$$\partial f / \partial \tau = 0$$

$$\Rightarrow: \ (u_0 - 2c_0 + 3c) - 3(t-\tau)\frac{\partial c}{\partial \tau} = 0$$

So the intersections can be fully determined from:

$$\begin{cases} \dfrac{x}{(t-\tau)} = u_0 - 2c_0 + 3c \\[2ex] 3(t-\tau)\dfrac{\partial c}{\partial \tau} = u_0 - 2c_0 + 3c \end{cases}$$

$$\Rightarrow \quad x = (u_0 - 2c_0 + 3c)^2 \Big/ 3 \cdot \frac{\partial c}{\partial \tau}. \qquad (1)$$

Since the question does not directly give $\partial c/\partial \tau$, we need to express $\partial c/\partial \tau$ as $\partial q/\partial \tau$:

$$\partial q/\partial \tau = \frac{\partial}{\partial \tau}(u \cdot h) = \frac{\partial}{\partial \tau}(u \cdot c^2/g)$$

$$= \frac{1}{g}\left(2uc\frac{\partial c}{\partial \tau} + c^2 \frac{du}{d\tau}\right)$$

Since $u = u_0 - 2c_0 + 2c \Rightarrow \partial u/\partial \tau = 2\,\partial c/\partial \tau$.

$$\partial q/\partial \tau = \frac{1}{g}\left(2c(u+c)\frac{\partial c}{\partial \tau}\right)$$

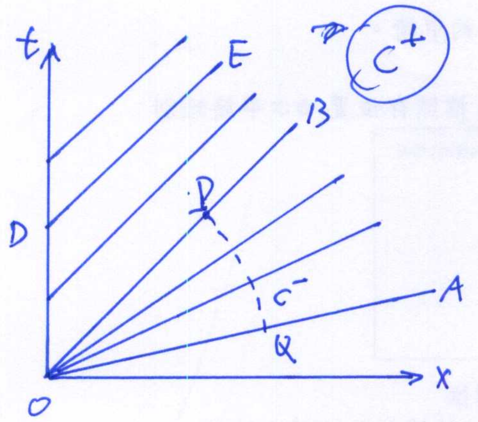
So: $$\partial c/\partial \tau \Big|_{\tau=0} = g\,\partial q/\partial \tau \Big/ \left[2c_0(u+c_0)\right]$$

$$= 9.8 \times \frac{1}{100} \cdot \frac{1}{2 \cdot 7 \cdot 7} = \frac{1}{1000}\,[m/s^2]$$

Plug this into (1)

$$x\Big|_{\tau=0} = \frac{(0 - 2c_0 + 3c_0)^2}{3 \cdot \partial c/\partial \tau\big|_{\tau=0}} = \frac{7^2}{3 \cdot \frac{1}{1000}} = 16.3 \text{ km} \; \#$$

Question 3



(a): Since the water flow is cut off at the gate at $t=0$, the local velocity suddenly changes from u_0 to 0. This sudden change indicates that there are multiple C^+ initiated at the origin.

The front: OA:

$$\left.\frac{dx}{dt}\right|_{OA} = u_0 + c_0.$$

Any other C^+ can be proved to be straight line, their slope is

$$dx/dt = u + c$$

Since any point on a C^+ other than OA can be connected with a point on OA ~~with~~ a C^-, e.g. P on OB can be connected to Q on OA.

So: along any C^+: $\quad u - 2c = u_0 - 2c_0$

Thus: $\quad c = c_0 + \frac{u - u_0}{2}$

$$dx/dt = u + c = 3u/2 + \left(c_0 - \frac{u_0}{2}\right)$$

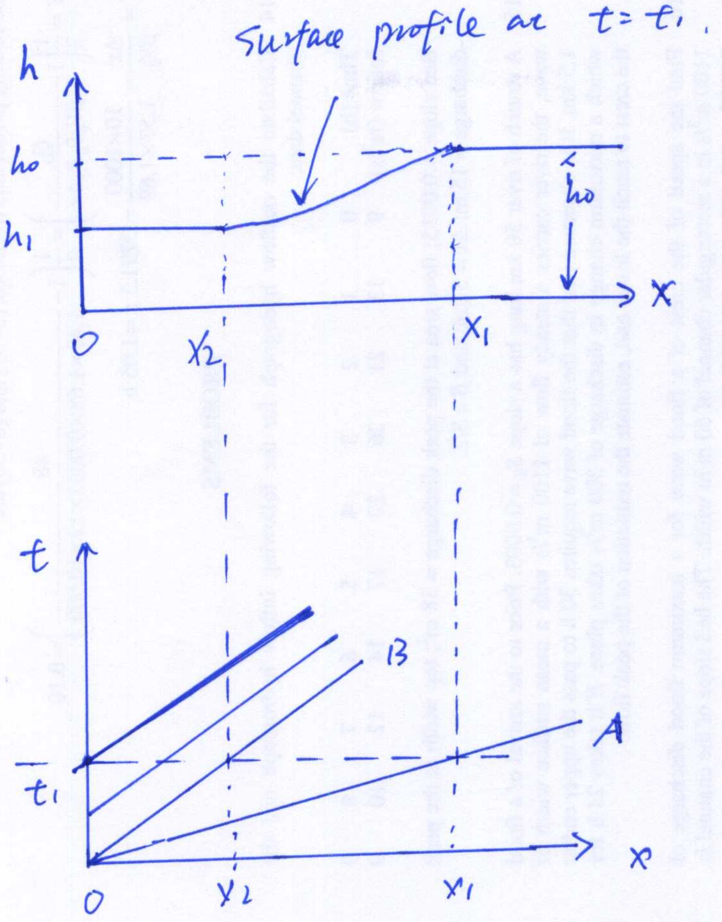
Since u changes from u_0 to 0 at $t=0$, the C^+ issued at the origin has a minimum slope with $u=0$:

$$\left.dx/dt\right|_{OB} = c_0 - \frac{u_0}{2}$$

For any C^+ ~~initial~~ initiated at $t=\tau>0$: e.g DE:

$$\left.dx/dt\right|_{DE} = c_0 - \frac{u_0}{2}$$

So the C^+ initiated after $t=0$ are all parallel straight lines, with the slope of OB.



At $t = t_1$ ~~the region to the downstream of the intersection of~~ the location of the front of the disturbance is:

$$x_1 = (u_0 + c_0)\, t_1 .$$

The trailing edge of the non-uniform region is

$$x_2 = \left(c_0 - \frac{u_0}{2}\right) t_1$$

~~For~~ For

$x > x_1 \quad h = h_0$

$x < x_2 \quad h = h_1 = \left(c_0 - \frac{u_0}{2}\right)^2 / g \; = 1.6\ \text{m/s}$

$x_2 \le x \le x_1 \quad h_1 < h < h_0 .$

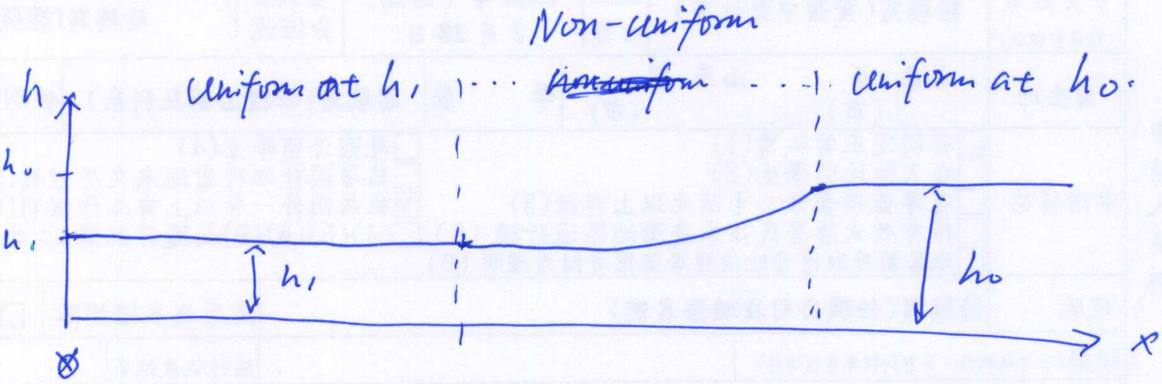
This means that the region to the left of OB has a uniform celerity:

$$c = c_0 - u_0/2$$

Which indicates a uniform depth

$$h_1 = (c_0 - u_0/2)^2/g = 1.6 \text{ m/s}.$$

Thus, the surface profile should be:



The leading edge of the non-uniform region is moving at:

$$dx/dt = u_0 + c_0.$$

The trailing edge moves at:

$$dx/dt = c_0 - u_0/2.$$

So the non-uniform region elongates as it moves downstream.

(b): The uniform region behind the gate has a water depth.

$$h_1 = (c_0 - u_0/2)^2 / g$$

$$h_1 = 0 \Rightarrow c_0 = u_0/2$$

This is the condition for the depth downstream of the gate go to zero.