UNSTEADY OPEN-CHANNEL FLOW B: RIVER ROUTING

~~RIVER MECHANICS~~ (OPEN-CHANNEL HYDRAULICS) (CE5312)

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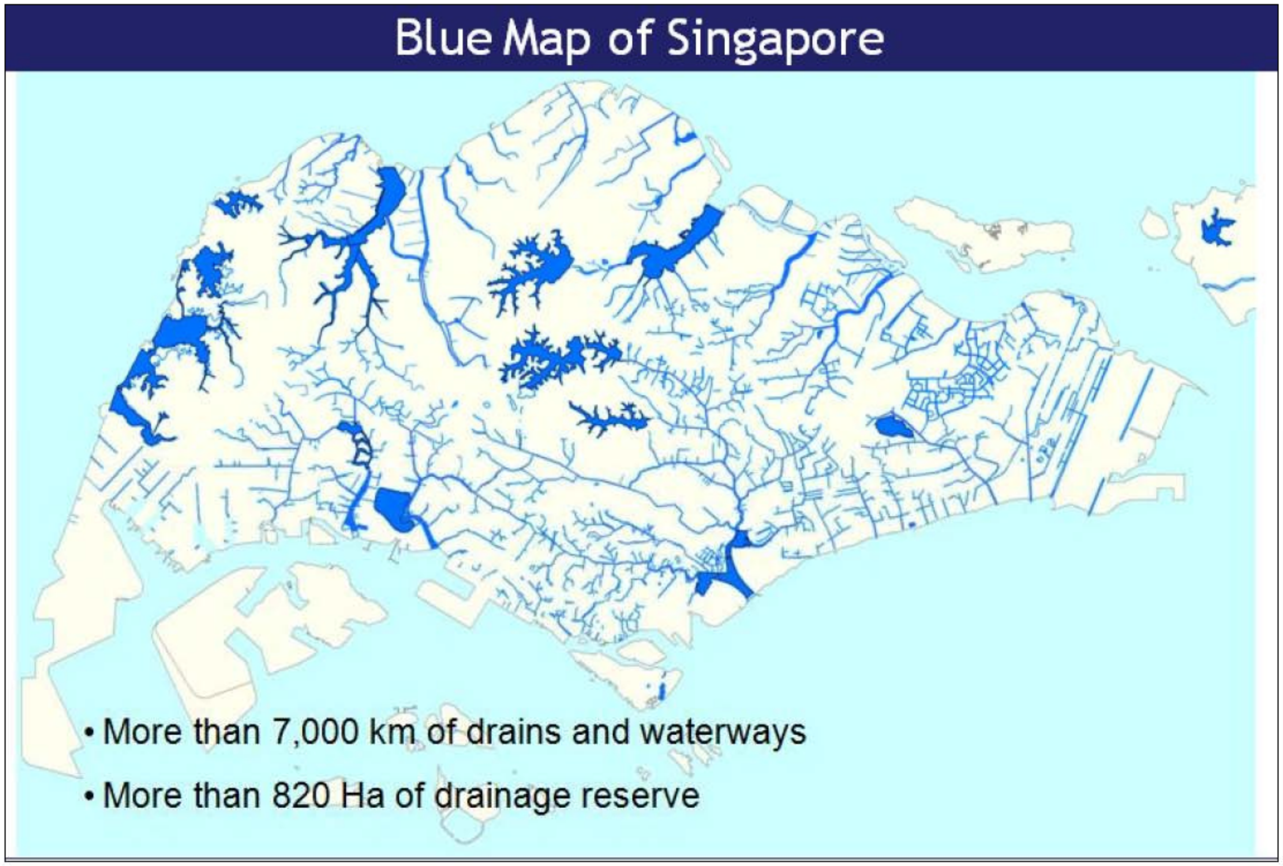
Flood in Singapore

Jun.16.2010 Orchard road
4 inches of rain fell during just two hours

Extreme rainfall produced this flood.

How do we deal with flood?



This image shows the pervasiveness and comprehensiveness of our drainage network. Almost every corner of Singapore is drained.

Over the past few decades, PUB has concentrated its efforts on alleviating the flood prone areas, preventing new flood prone areas from cropping up, and ensuring there is good and sound drainage practice.

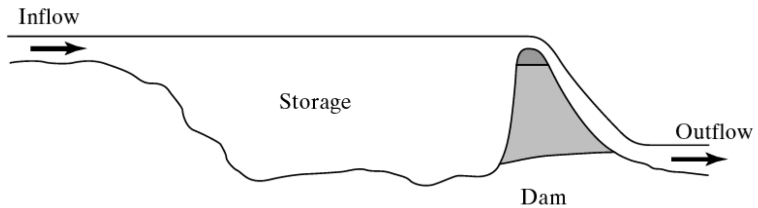
PUB has invested in more than $2.4 billion in drainage development from 1980 to 2010. Another $150 million per year has been budgeted for the next five years for upgrading infrastructure, and the maintenance budget has been doubled to $23 million per year.

Retention and drainage (reservoirs and canals)

River routing

- How does a reservoir/pond reduce the peak of a flood?

Solve for *O*(t), *S*(t)



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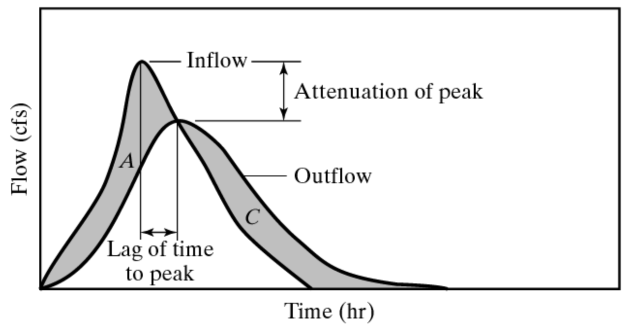
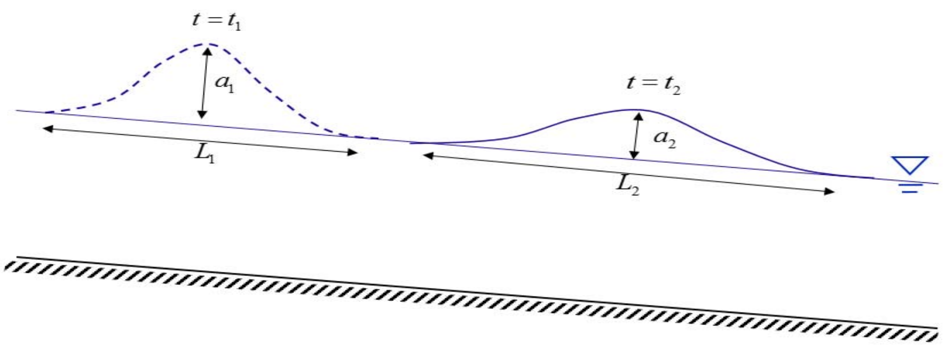


(b)

- How does flood wave travel through a channel?

Solve for *Q*(x,t), *h*(x,t)



Principle for river routing

Continuity equation:

$$\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = 0$$

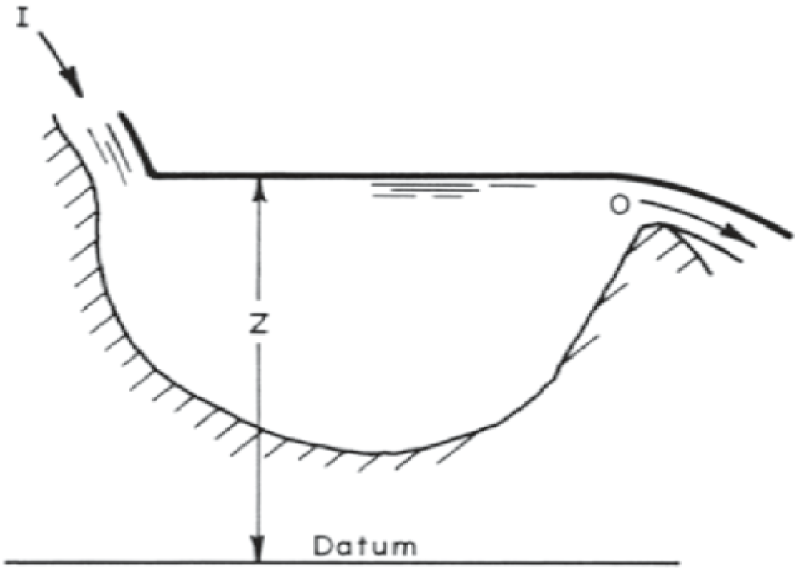
Momentum equation:

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + g\frac{\partial h}{\partial x} = g(S_0 - S_f)$$

Hydrologic routing: just use the continuity equation, but use some empirical as an alternative to momentum equation

Hydraulic routing: simultaneously solve two equations, but some simplifications may be applied to momentum equation.

Reservoir routing: level-pool method



$$I - O = \frac{dS}{dt}$$

Given I, find $S(t)$, $O(t)$

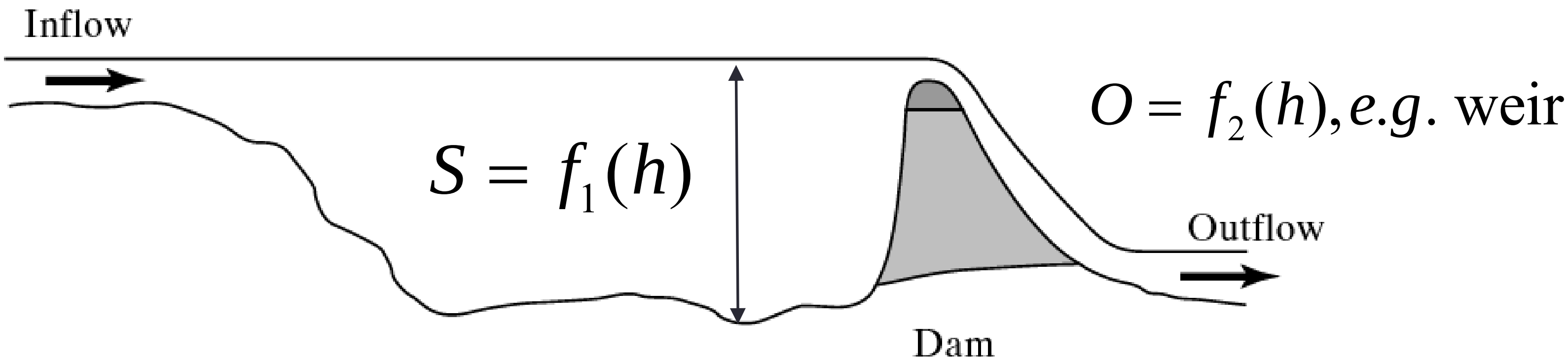
$$\frac{I_1 + I_2}{2} - \frac{O_1 + O_2}{2} = \frac{S_2 - S_1}{\Delta t}$$

$$\boxed{N_2 = N_1 + \frac{I_1 + I_2}{2} - O_1}$$

$$N = \frac{S}{\Delta t} + \frac{O}{2}$$

Need O=f(N)!

N-O relationship



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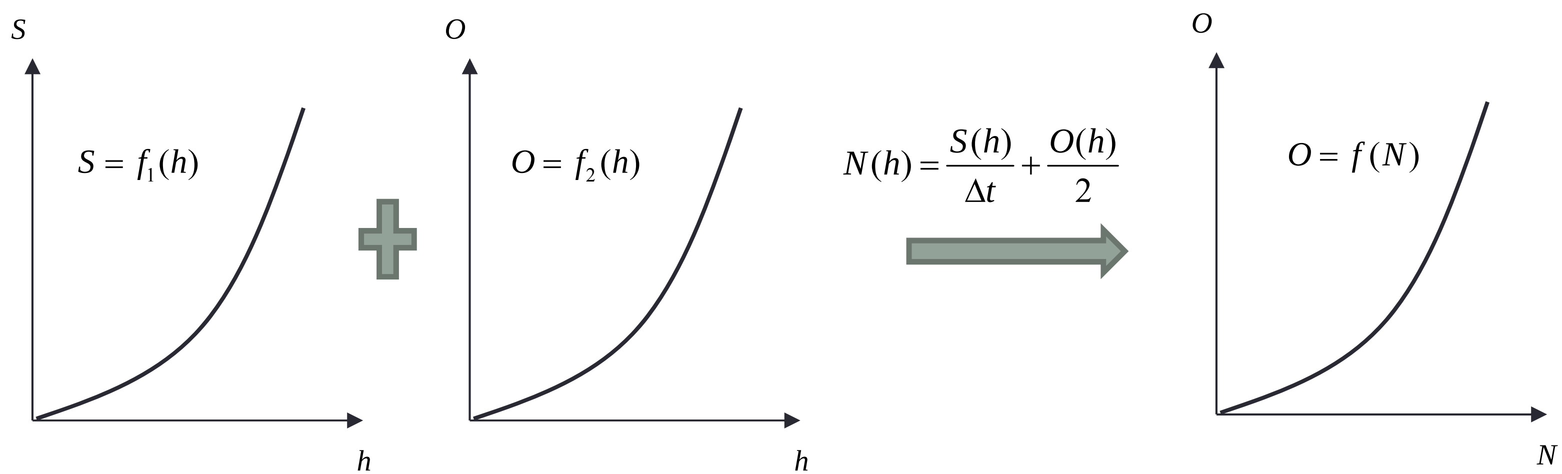


O-N can be obtained from known S-h and O-h relationships

General procedure for level-pool method

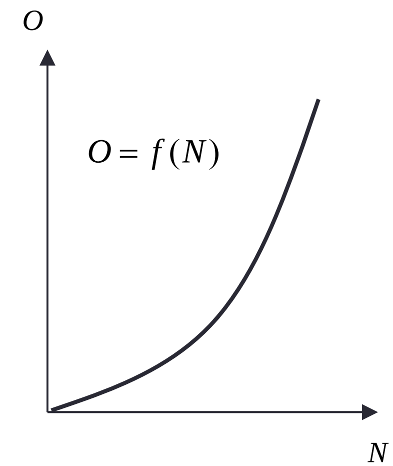
Given conditions:

- Inflow hydrograph
- An S-h relationship from topography information
- An O-h relationship (previous flood record or simple formula)
- Initial flow condition (O_1,N_1, I_1 at t=0)

Steps:

- Choose Δt and discretize the time domain
- Develop N-O relationship from S-h and O-h relationships
- Compute N^{i+1}
- Use the established N-O relationship to obtain O^{i+1}
- Use obtain O and N to get S: $N=S/\Delta t+O/2$

$$N = \frac{S}{\Delta t} + \frac{O}{2}$$



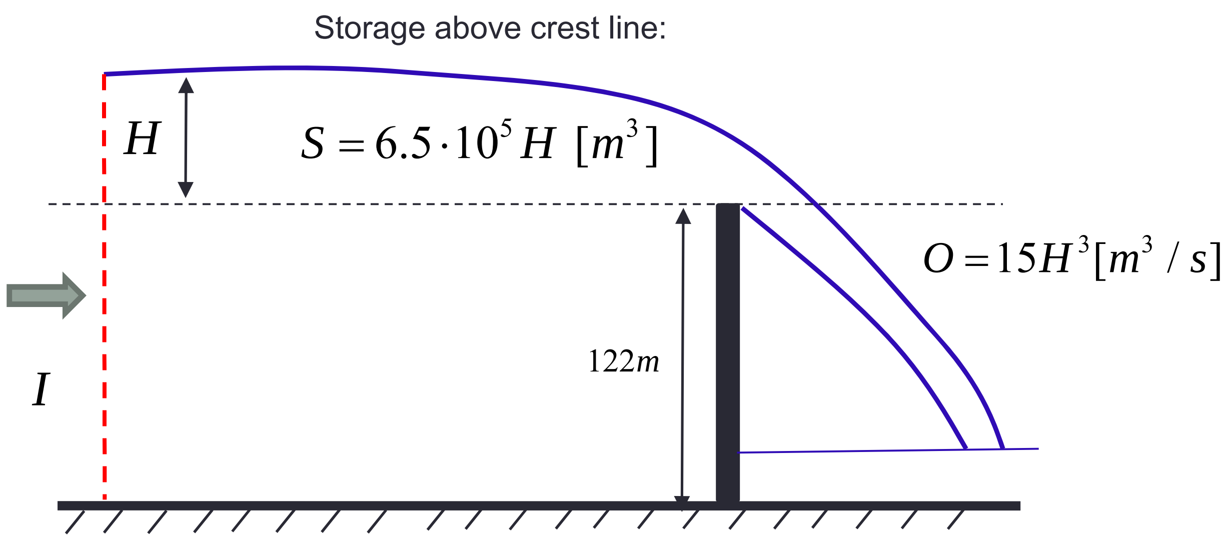
$$N_{i+1} = N_i + \frac{I_i + I_{i+1}}{2} - O_i$$

$$O_{i+1} = f(N_{i+1})$$

Example: level-pool method

Route the following inflow hydrograph through a reservoir created by a weir.

t [hour]	I [m^3/s]
0	10
2	100
4	180
6	210
8	160
10	90
12	60
14	46
16	30
18	16
20	10



Choose Δt=2 hour for simplicity

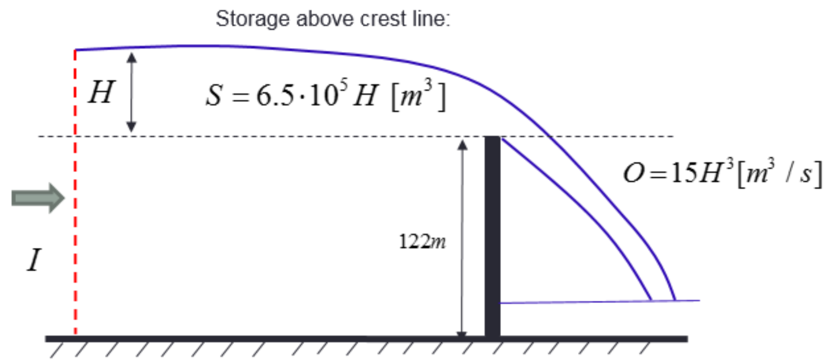
N-O relationship

$$N = \frac{S}{\Delta t} + \frac{O}{2}$$

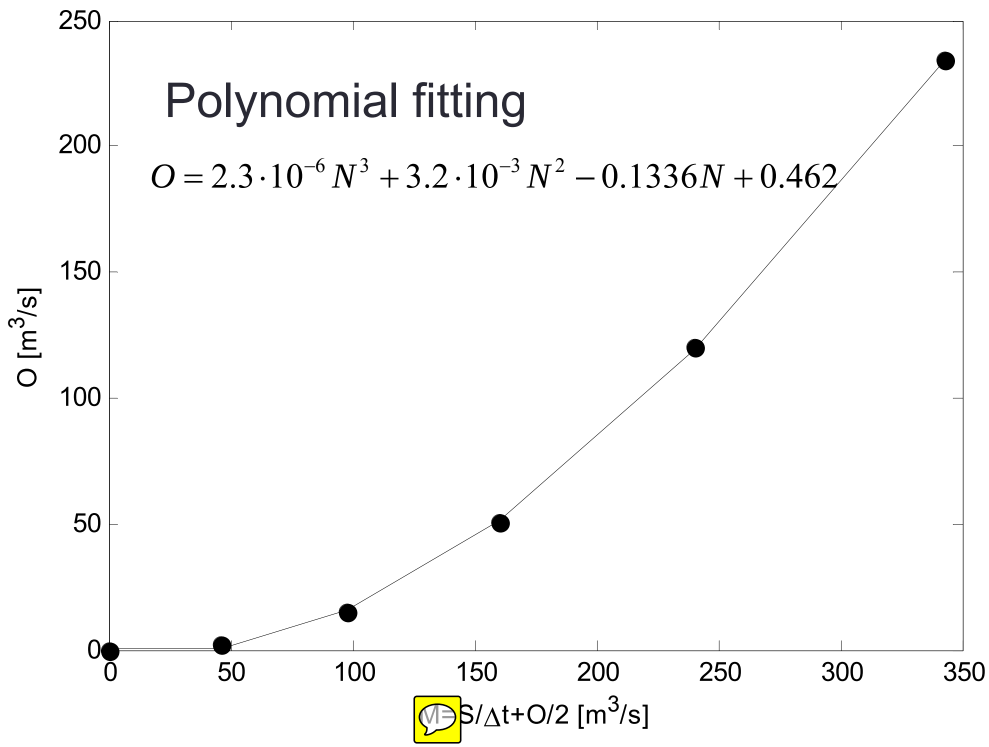
$$S = 6.5 \cdot 10^5 H \ [m^3]$$

$$O = 15H^3 [m^3 / s]$$

$$\Delta t = 2 \ hour$$



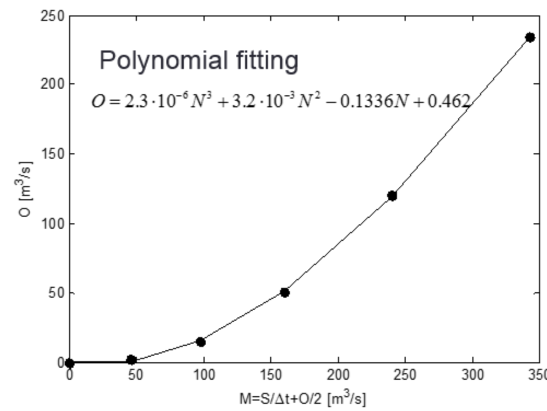
Delta_t [s]	7200		
H [m]	S [m^3]	O [m^3/s]	N [M^3/s]
0	0	0	0
0.5	325000	1.875	46.1
1	650000	15	97.8
1.5	975000	50.625	160.7
2	1300000	120	240.6
2.5	1625000	234.375	342.9



* In this example, both S and O are simple functions of H, so S-H and O-H are easy to establish. For real cases, you need measurements to get S-H and O-H

Prediction of O

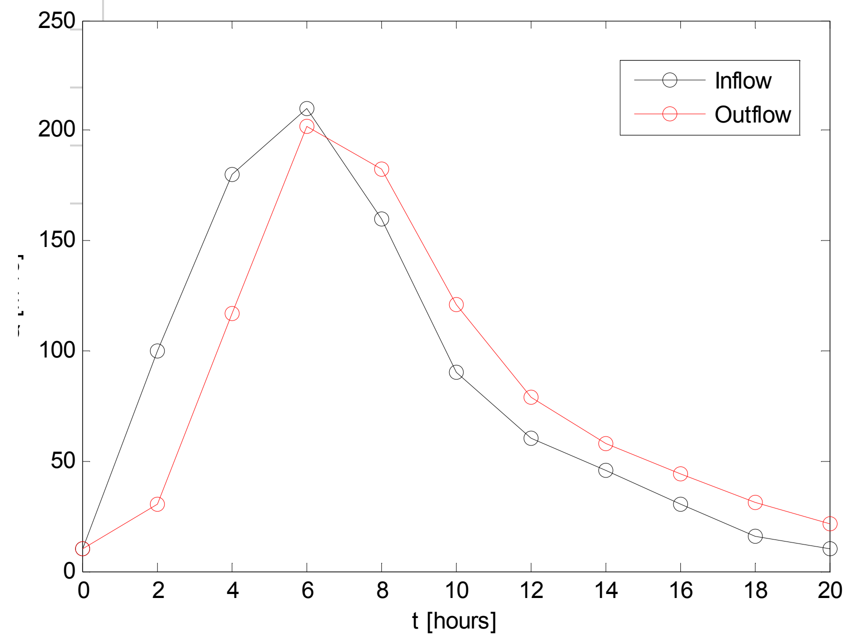
t [hour]	I [m^3/s]	(I1+I2/2) [m^3/s]	N [m^3/s]	O [m^3/s]
0	10		83.5	10.0
2	100	55		
4	180	140		
6	210	195		
8	160	185		
10	90	125		
12	60	75		
14	46	53		
16	30	38		
18	16	23		
20	10	13		



$$N_i = N_{i-1} + \frac{I_i + I_{i-1}}{2} - O_{i-1}$$

$$O_i = f(N_i)$$

Flood peak is reduced due to reservoir storage.



Muskingum Method

Wedge storage in reach

$$S_{\text{Prism}} = KQ$$

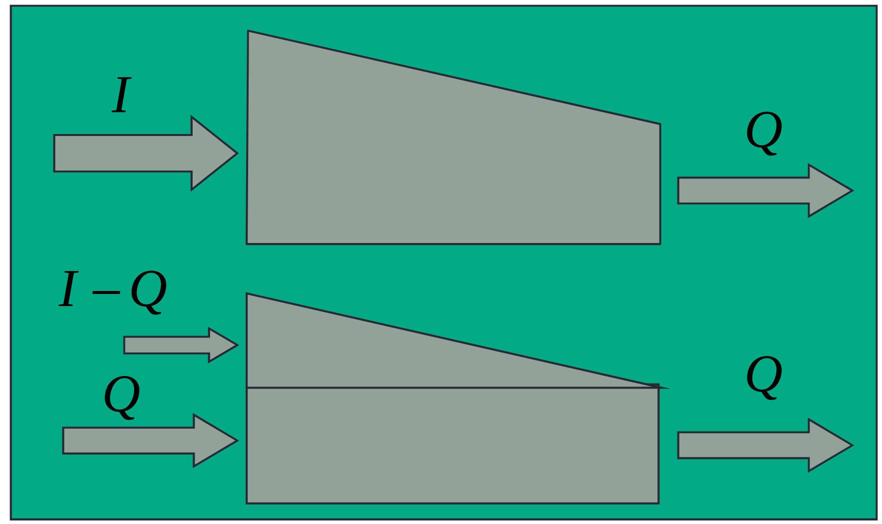
$$S_{\text{Wedge}} = KX(I - Q)$$

- K = travel time of peak through the reach
- X = weight on inflow versus outflow (0 ≤ X ≤ 0.5)
- X over 0.5 increases flood peak (unrealistic)
- X = 0 ➔ level-pool, storage depends on outflow, no wedge
- X = 0.0 - 0.3 ➔ Natural stream

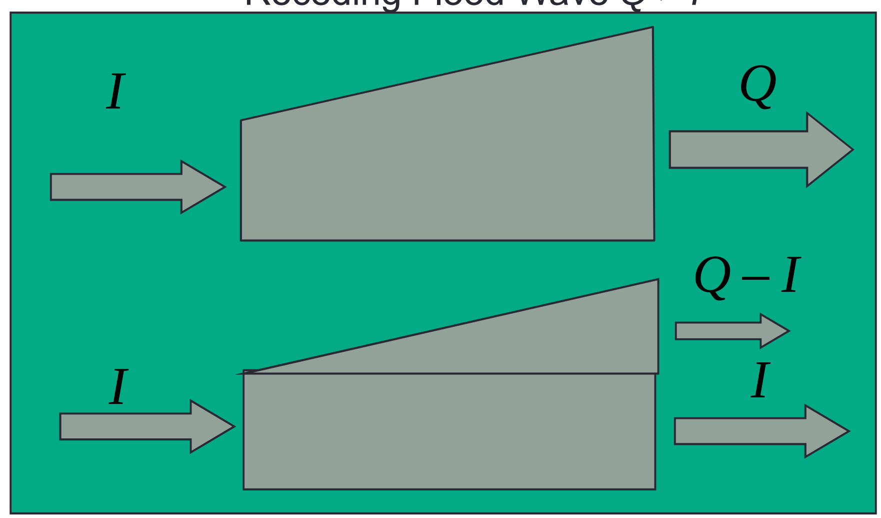
$$S = KQ + KX(I - Q)$$

$$\boxed{S = K[XI + (1 - X)Q]}$$

Advancing Flood Wave *I* > *Q*



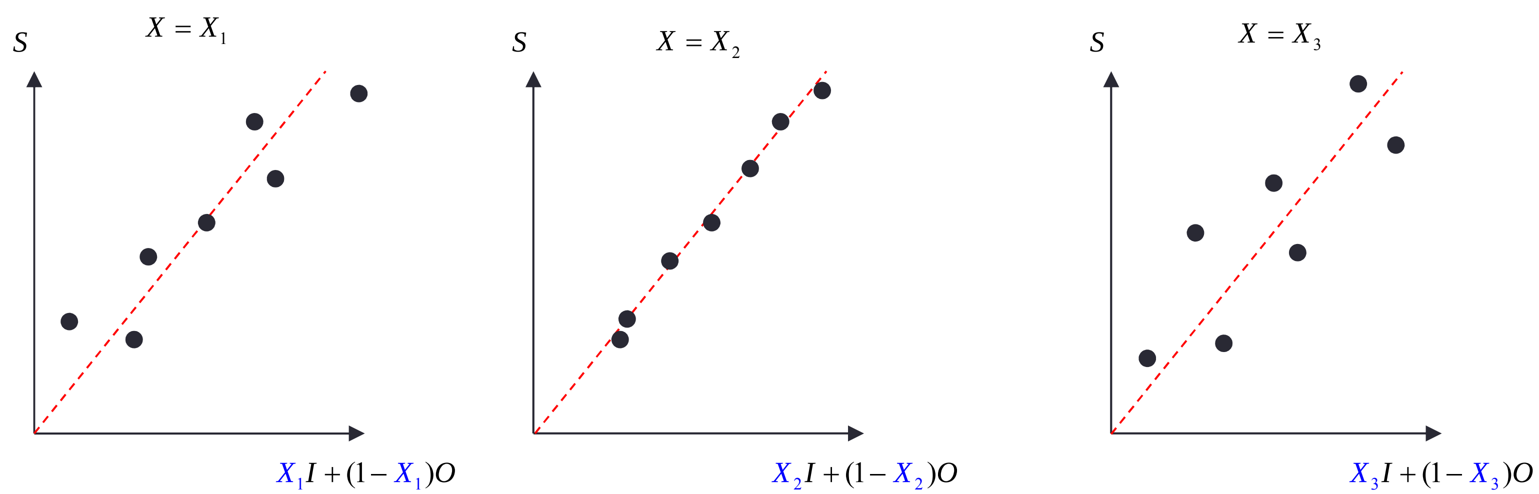
Receding Flood Wave *Q* > *I*



Obtaining routing parameter use past flood record

$$S = K[XI + (1 - X)Q] \quad \Longrightarrow \quad \boxed{K = \frac{S}{XI + (1 - X)O}}$$

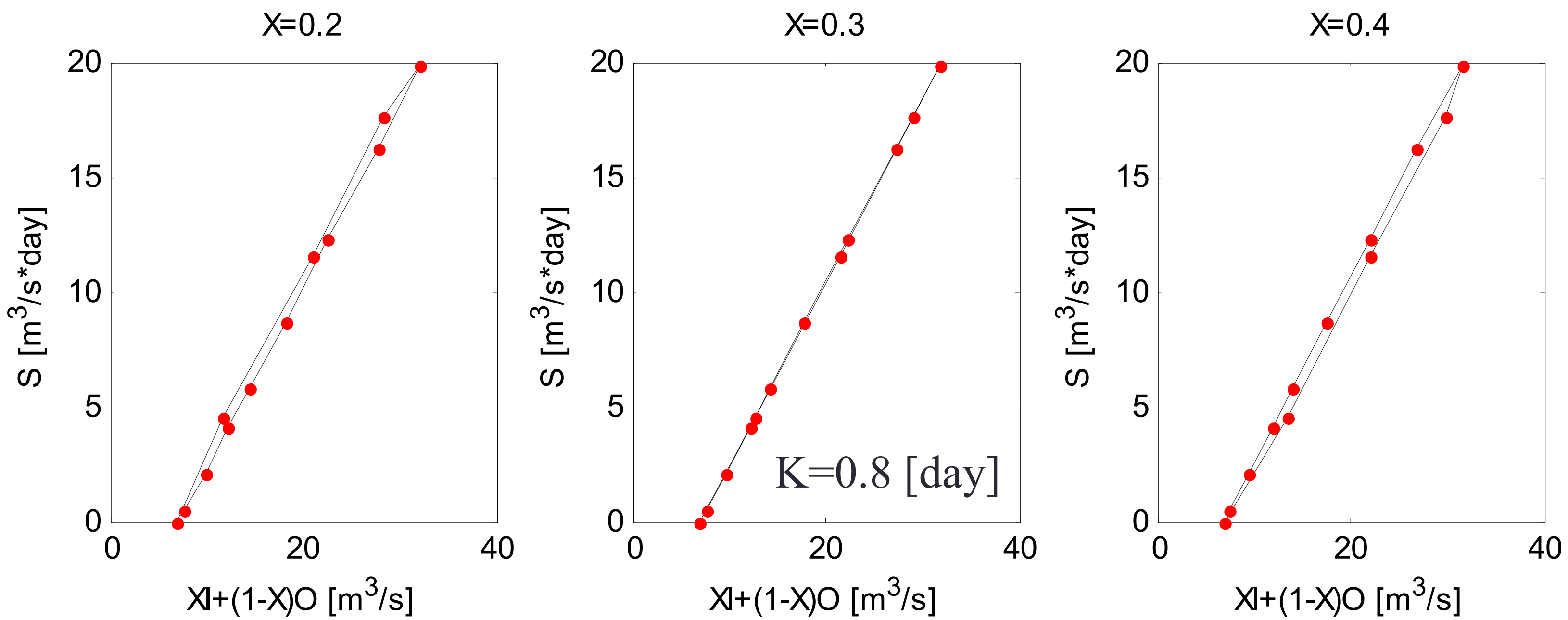
- Use past flood record (S, I and O).
- Try different value of X. With $X = X_2$, the straight-line fit works best.
- X_2 is the value for X, and the corresponding slope of the fit is K.



Example: Muskingum method's routing parameters

t [day]	0	1	2	3	4	5	6	7	8	9	10	11
inflow [m^3/s]	7.0	19.0	25.0	34.0	30.0	24.0	20.0	15.0	13.0	11.0	8.0	7.0
outflow [m^3/s]	7.0	9.9	20.0	26.9	32.6	28.7	23.3	19.0	14.7	12.6	10.4	7.9
Storage	0.0	4.5	11.6	17.6	19.9	16.2	12.2	8.6	5.8	4.1	2.1	0.5

$$S_{i+1} = S_i + \frac{\Delta t}{2}(I_i + I_{i+1} - O_i - O_{i+1})$$



Some algebra

$$I - O = \frac{dS}{dt} \Rightarrow \frac{I_1 + I_2}{2} - \frac{O_1 + O_2}{2} = \frac{S_2 - S_1}{\Delta t}$$

$$S = KQ + KX(I - Q) \Downarrow$$

$$\frac{I_1 + I_2}{2} - \frac{O_1 + O_2}{2} = \frac{K[XI_2 + (1 - X)O_2] - K[XI_1 + (1 - X)O_1]}{\Delta t}$$

$$\Downarrow$$

Basic equation for Muskingum routing

$$\boxed{O_2 = C_0 I_2 + C_1 I_1 + C_2 O_1}$$

$$\boxed{\begin{aligned} C_0 &= \frac{\Delta t - 2KX}{\Delta t + 2K(1 - X)} \\ C_1 &= \frac{\Delta t + 2KX}{\Delta t + 2K(1 - X)} \\ C_2 &= \frac{2K(1 - X) - \Delta t}{\Delta t + 2K(1 - X)} \end{aligned}}$$

- $2KX < \Delta t$ to ensure C_0>1
- $C_0+C_1+C_2 = 1$
- X=0.5 and K= Δt: $O_{i+1}=I_i$

General procedure for Muskingum routing

Given:

- Inflow hydrograph: I(t)
- routing parameter K and X
- initial condition (I, Q, S)

$$C_0 = \frac{\Delta t - 2KX}{\Delta t + 2K(1-X)}$$

$$C_1 = \frac{\Delta t + 2KX}{\Delta t + 2K(1-X)}$$

$$C_2 = \frac{2K(1-X) - \Delta t}{\Delta t + 2K(1-X)}$$

Steps:

- chose a time interval Δt
- Calculate the routing constants
- Routing for *O*

$$O_2 = C_0 I_2 + C_1 I_1 + C_2 O_1$$

- Use obtain O and given I to get storage S

$$S_{i+1} = S_i + \frac{\Delta t}{2}(I_i + I_{i+1} - O_i - O_{i+1})$$

Example: Muskingum routing

For a channel reach the routing parameters are K=0.8 [day] and X=0.3. Determine the outflow hydrograph with the following inflow hydrograph.

t [day]	0	1	2	3	4	5	6	7	8	9	10	11
inflow [m^3/s]	7	19	25	34	30	24	20	15	13	11	8	7

Choose Δt=1 day for simplicity

$$C_0 = \frac{\Delta t - 2KX}{\Delta t + 2K(1-X)} = \frac{1-2\cdot 0.8\cdot 0.3}{1+2\cdot 0.8\cdot(1-0.3)} = 0.245$$

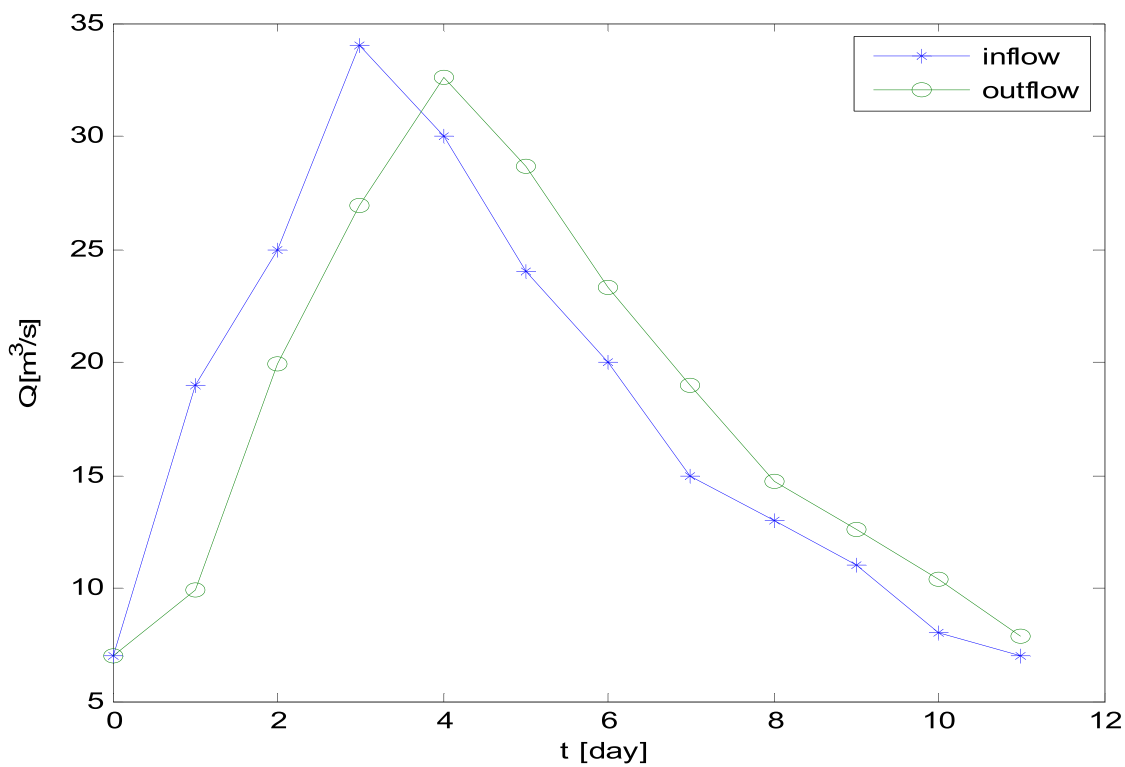
$$C_1 = \frac{\Delta t + 2KX}{\Delta t + 2K(1-X)} = \frac{1+2\cdot 0.8\cdot 0.3}{1+2\cdot 0.8\cdot(1-0.3)} = 0.698$$

$$C_2 = \frac{2K(1-X) - \Delta t}{\Delta t + 2K(1-X)} = \frac{2\cdot 0.8(1-0.3)-1}{1+2\cdot 0.8\cdot(1-0.3)} = 0.057$$

Example: Muskingum routing (cont.')

$$O_{i+1} = C_0 I_{i+1} + C_1 I_i + C_2 O_i$$

C0	0.245	
C1	0.698	
C2	0.057	
t [day]	I [m^3/s]	O [m^3/s]
0	7	7
1	19	9.9
2	25	20.0
3	34	26.9
4	30	32.6
5	24	28.7
6	20	23.3
7	15	19.0
8	13	14.7
9	11	12.6
10	8	10.4
11	7	7.9



Flood peak is attenuated.

Momentum equation

$$\underbrace{\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x}}_{\text{inertial}} + \underset{\text{pressure}}{g\frac{\partial h}{\partial x}} = \underset{\text{gravity}}{gS_0} - \underset{\text{friction}}{gS_f}$$

Typical values:

$$S_0 = 4.9 \cdot 10^{-3}$$

$$\frac{\partial h}{\partial x} = 9.5 \cdot 10^{-5}$$

$$\frac{U}{g}\frac{\partial U}{\partial x} = 2.4 \cdot 10^{-5} \sim 4.7 \cdot 10^{-5}$$

$$\frac{1}{g}\frac{\partial U}{\partial t} = 9.5 \cdot 10^{-5}$$

Very possible that we can neglect inertial and pressure terms

Another way to look at this

Normalize momentum equation

$$\boxed{\frac{U_0}{gS_0T_0}\left(\frac{\partial \hat{U}}{\partial \hat{t}} + \hat{U}\frac{\partial U}{\partial \hat{x}}\right) + \frac{h_0}{U_0T_0S_0}\frac{\partial \hat{h}}{\partial x} - 1 + \frac{S_f}{S_0} = 0}$$

$h = h_0\hat{h}$	h_0: normal depth
$U = U_0\hat{U}$	U_0: normal flow velocity
$t = T_0\hat{t}$	T_0: time scale of a flood wave
$x = L_0\hat{x}$	$L_0 = T_0 * U_0$

We can neglect the pressure and initial terms if:

$$\Pi_1 = \frac{U_0T_0S_0}{h_0} >> 1 \qquad \Pi_2 = \frac{gS_0T_0}{U_0} >> 1$$

$S_0 = 0.001$

$h_0 = 10\ m$

$U_0 = 1$ m/s

T=1 day=86400 s

$\Pi_1 = 17.2 >> 1$

$\Pi_2 = 432 >> 1$

Simplification of momentum equation

$$\underbrace{U_t + \underbrace{UU_x + \underbrace{gh_x + \underbrace{g\left(S_f - S_0\right)}_{\textit{kinematic wave}}}_{\textit{diffusion wave}}}_{\textit{steady dynamic wave}}}_{\textit{dynamic wave}} = 0$$

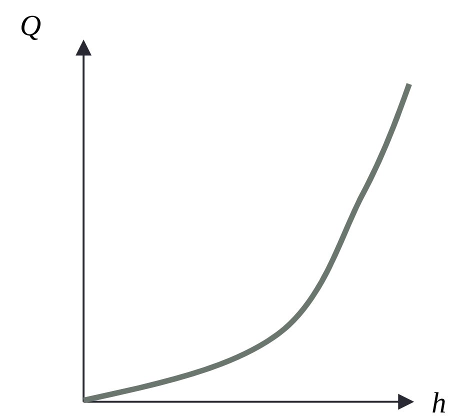
- Kinematic wave: gravity and friction
- Diffusion wave: gravity, friction and pressure
- Dynamic wave: gravity, friction, pressure and inertial

Kinematic-wave routing: Governing equation

$$S_0 = S_f \quad \Rightarrow \quad Q = \begin{cases} A \cdot C\sqrt{R_h}\sqrt{S_0} & \text{Chezy} \\ A \cdot \dfrac{1}{n} R_h^{2/3}\sqrt{S_0} & \text{Manning} \end{cases}$$

In general:

$$Q = D(h)\sqrt{S_0}$$



Ant any point and any time, the flow is "quasi-steady", so the stage-discharge relationship of steady uniform flow can be applied.

The unsteadiness comes from the continuity equation

$$\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = 0$$

Kinematic-wave equation

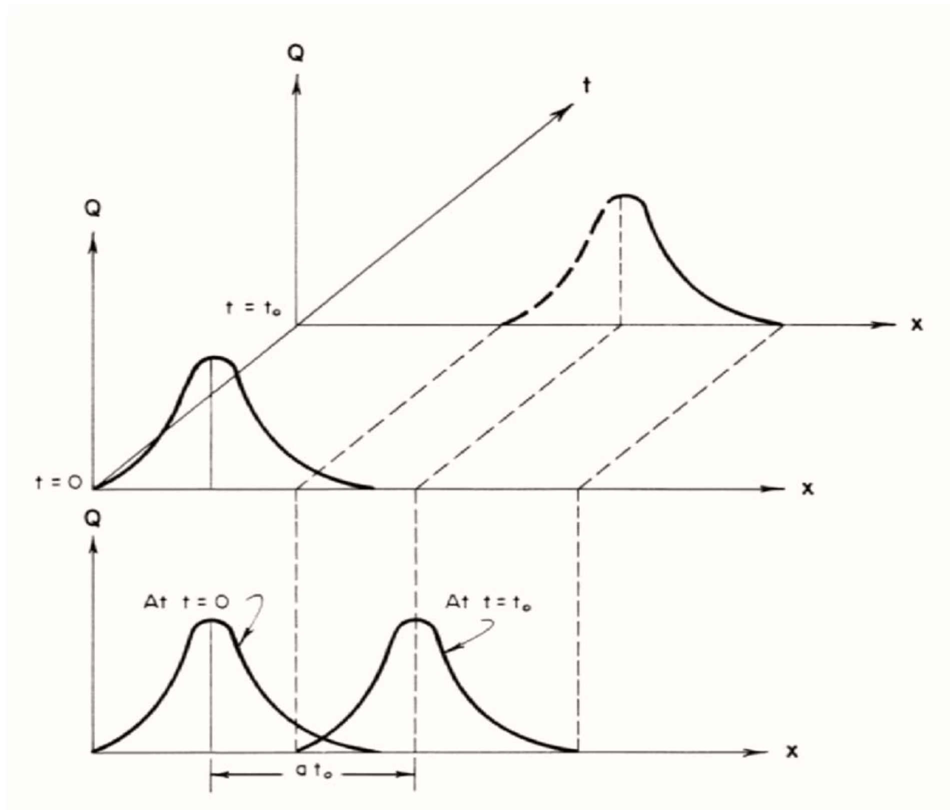
$$\boxed{\frac{\partial Q}{\partial t} + c_k \frac{\partial Q}{\partial x} = 0}$$

$$c_k = \frac{\partial Q}{\partial A} > 0$$

Klietz-Seddon law

$$\frac{DQ}{Dt} = 0, \text{ along } \frac{dx}{dt} = c_k$$

Kinematic wave moves downstream without changing shape, if c_k is constant (pure translation)

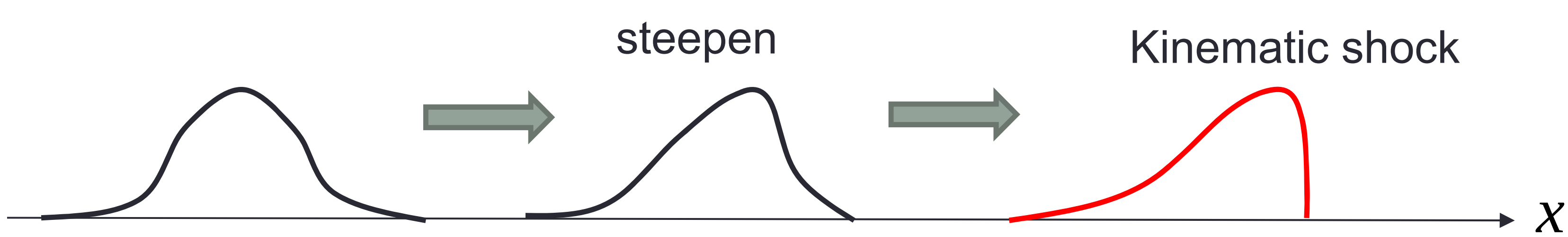


Celerity of kinematic wave

For a wide channel:

$$c_k = \gamma U \begin{cases} \gamma = 3/2 \text{ Chezy} \\ \gamma = 5/3 \text{ Manning} \end{cases}$$

Celerity increases with velocity:



As kinematic waves moves downstream, the peak moves faster than the front and tail, so the wave becomes steepen and eventually becomes a kinematic shock
No spreading and attenuation!

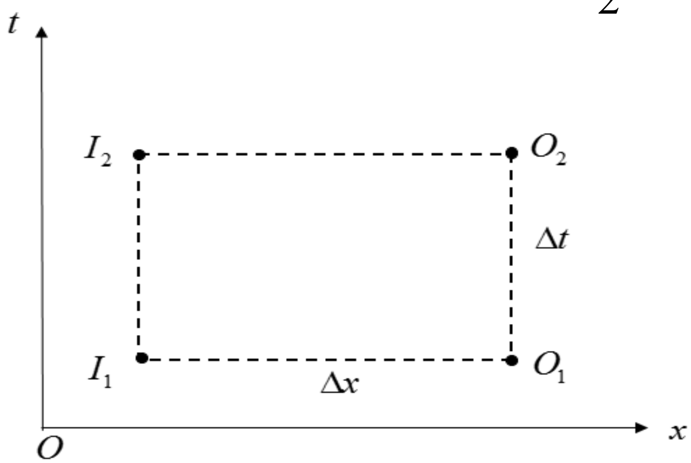
Kinematic wave vs. Muskingum

$$\frac{\partial Q}{\partial t} + c_k \frac{\partial Q}{\partial x} = 0 \Rightarrow \frac{X(I_2 - I_1) + (1-X)(O_2 - O_1)}{\Delta t} + c_k \frac{(O_1 - I_1) + (O_2 - I_2)}{2\Delta x} = 0$$

$$\Downarrow$$

$$\frac{I_1 + I_2}{2} - \frac{O_1 + O_2}{2} = \frac{(\Delta x / c_k)[XI_2 + (1-X)O_2] - (\Delta x / c_k)[XI_1 + (1-X)O_1]}{\Delta t}$$

Muskingum: $$\frac{I_1 + I_2}{2} - \frac{O_1 + O_2}{2} = \frac{K[XI_2 + (1-X)O_2] - K[XI_1 + (1-X)O_1]}{\Delta t}$$

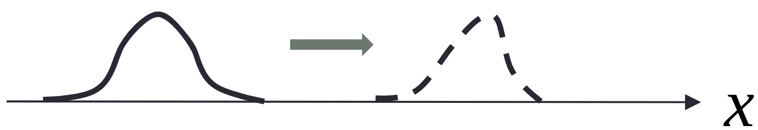


$$\boxed{K = \frac{\Delta x}{c_k}}$$

- Muskingum is just a finite difference representation of kinematic-wave equation
- K is the time for kinematic wave to pass the channel reach Δx

Kinematic wave vs. Lagrangian wave

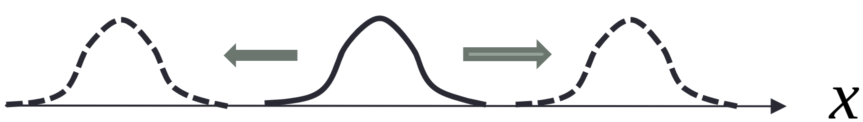
Kinematic wave



$$c_k = \gamma U$$

Only moves downstream

Lagrangian wave



$$c_k = U \pm \sqrt{gh}$$

Can move in both upstream and downstream direction

- Kinematic wave is a HUGE wave that gravity and friction are important
- Lagrangian wave is a SMALL wave that inertial and pressure are important
- They are idealized “waves” based on different simplification of momentum equation.
- A real dynamic wave should have their features.

Kinematic-wave routing: applicability

Recall:

$$\frac{U_0}{gS_0T_0}\left(\frac{\partial \hat{U}}{\partial \hat{t}}+\hat{U}\,\frac{\partial \hat{U}}{\partial \hat{x}}\right)+\frac{h_0}{U_0T_0S_0}\,\frac{\partial \hat{h}}{\partial x}-1+\frac{S_f}{S_0}=0$$

Ponce (1989) gave the following criteria for applying kinematic wave:

$$\frac{T_0U_0S_0}{h_0}>85$$

Diffusion-wave routing: governing equation

$$S_f = S_0 - \frac{\partial h}{\partial x}$$

$$S_f = \frac{Q^2}{D}$$

$$\boxed{\frac{\partial Q}{\partial t} + c_f \frac{\partial Q}{\partial x} = \mu \frac{\partial^2 Q}{\partial x^2}}$$

$$D(h) = \begin{cases} \left[\frac{f}{8g}\frac{P(h)}{A(h)^3}\right]^{-1} (Darcy - Weisbach) \\ \left[\frac{1}{C^2}\frac{P(h)}{A(h)^3}\right]^{-1} (Chezy\text{'}s\ Equation) \\ \left[n^2 \frac{P(h)^{4/3}}{A(h)^{10/3}}\right]^{-1} (Manning?s\ Equation) \end{cases}$$

$$\boxed{c_f = \frac{Q}{D}\frac{dD}{dA}}$$

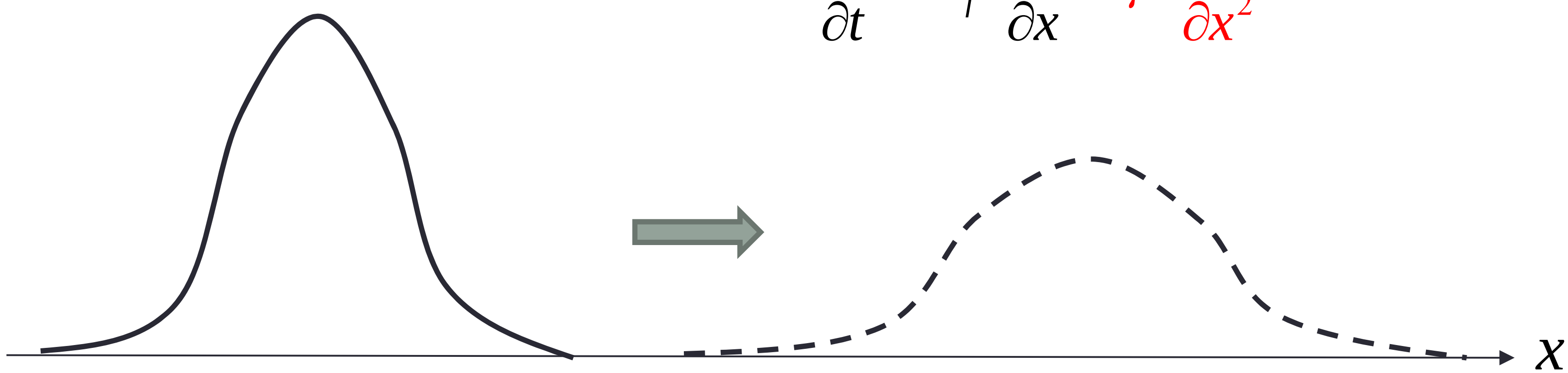
Celerity of diffusion wave

$$\boxed{\mu = \frac{D^2}{2b_s Q}}$$

Diffusion parameter

Behavior of diffusion wave

$$\frac{\partial Q}{\partial t} + c_f \frac{\partial Q}{\partial x} = \mu \frac{\partial^2 Q}{\partial x^2}$$



As a diffusion wave moves downstream, its peak is attenuated, and the wave becomes longer (spread out)

Diffusion-wave routing: celerity

$$\frac{c_f}{U} = \frac{Q}{D}\frac{dD}{dA} / (\frac{Q}{A}) = \frac{A}{D}\frac{dD}{dA} = \frac{A}{D}\frac{dD}{dh}\left(\frac{\partial h}{\partial A}\right) = \frac{A}{D}\frac{dD}{dh} b_s^{-1} = \frac{A}{b_s D}\frac{dD}{dh}$$

$$\frac{c_k}{U} = \frac{A}{b_s D}\frac{dD}{dh}$$

$$c_f \approx c_k$$

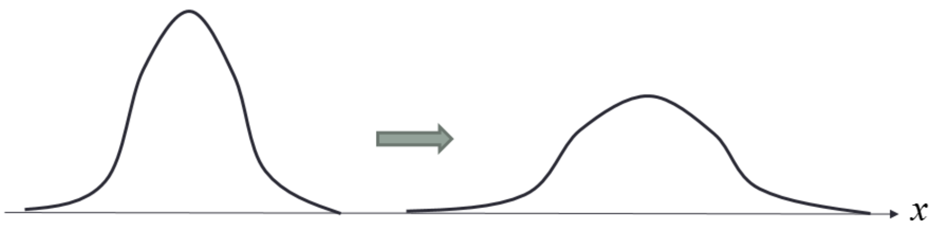
Diffusion-wave celerity is not necessarily equal to the kinematic-wave celerity, because the flow velocity U in this two routing method are based on different momentum equations. However, the difference is negligible in most cases, so $c_f = c_k$.

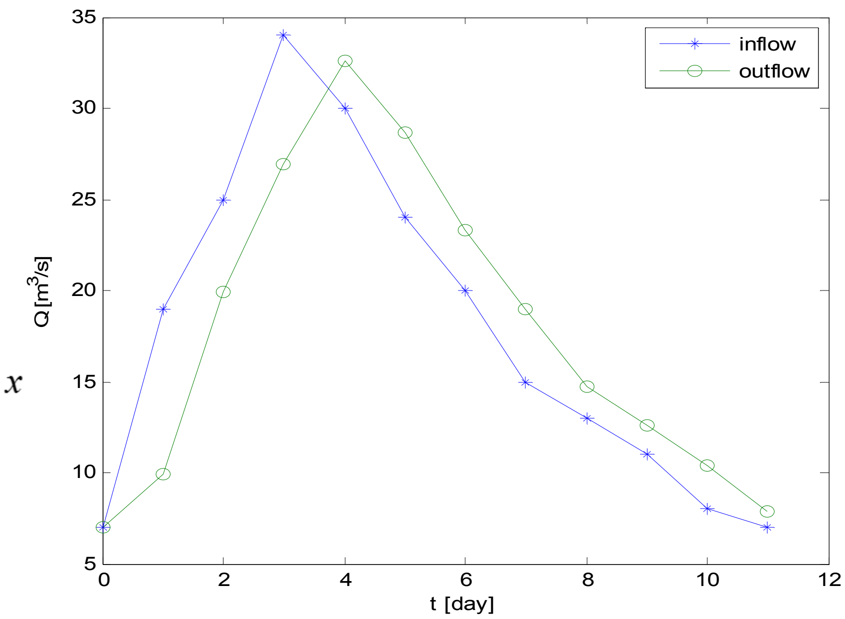
Diffusion-wave routing: the diffusivity

$$\boxed{\mu = \frac{D^2}{2b_s Q} \approx \frac{Q}{2b_s S_0}}$$

The diffusivity μ apparently increases with Q, but decreases with S_0 and b_s, meaning that diffusion will be more severe if the slope is close to horizontal, or the river is very narrow, or the discharge is very large.

We have see the diffusion feature before!



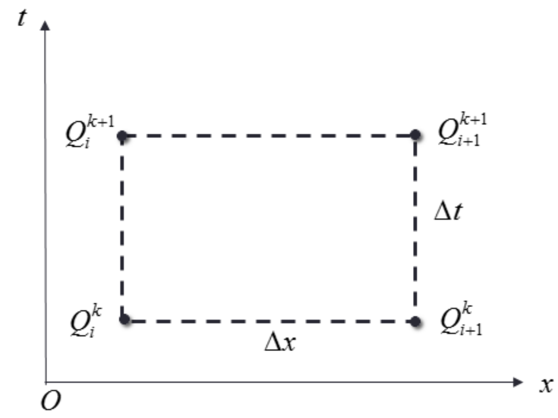


We see the diffusion feature with Muskingum method!

But Muskingum method is proved to be a finite difference representation of kinematic wave, which should not have any diffusion.

Why?

Numerical diffusion



$$\frac{\partial Q}{\partial t} + c_k \frac{\partial Q}{\partial x} = 0$$

Weighted finite differencing

$$\frac{\partial Q}{\partial t} + c_k \frac{\partial Q}{\partial x} = \frac{X(Q_i^{k+1} - Q_i^k) + (1-X)\left(Q_{i+1}^{k+1} - Q_{i+1}^k\right)}{\Delta t} + c_k \frac{0.5(Q_{i+1}^k - Q_i^k) + 0.5\left(Q_{i+1}^{k+1} - Q_i^{k+1}\right)}{\Delta x} + error$$

Truncation error:
(Taylor expansion)

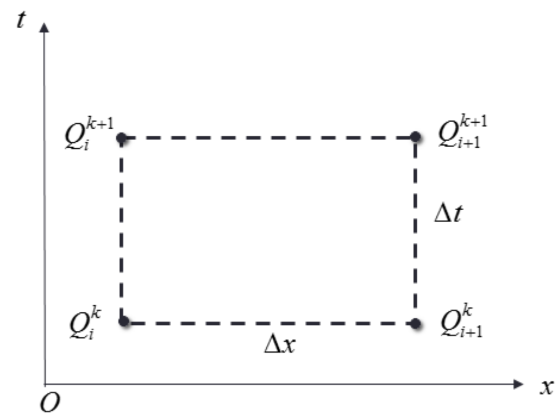
$$\boxed{error = c_k \Delta x \left[\left(\frac{1}{2} - X\right)\right] \frac{\partial^2 Q}{\partial x^2}}$$

$$\frac{X(Q_i^{k+1} - Q_i^k) + (1-X)\left(Q_{i+1}^{k+1} - Q_{i+1}^k\right)}{\Delta t} + c_k \frac{0.5(Q_{i+1}^k - Q_i^k) + 0.5\left(Q_{i+1}^{k+1} - Q_i^{k+1}\right)}{\Delta x} = 0$$

The discretized equation is equivalent to the following equation

$$\frac{\partial Q}{\partial t} + c_k \frac{\partial Q}{\partial x} - c_k \Delta x \left[\left(\frac{1}{2} - X\right)\right] \frac{\partial^2 Q}{\partial x^2} = 0$$

Numerical diffusion (cont.')



$$\frac{\partial Q}{\partial t} + c_k \frac{\partial Q}{\partial x} - c_k \Delta x \left[\left(\frac{1}{2} - X \right) \right] \frac{\partial^2 Q}{\partial x^2} = 0$$

$$\Downarrow$$

$$\frac{\partial Q}{\partial t} + c_k \frac{\partial Q}{\partial x} = c_k \Delta x \left[\left(\frac{1}{2} - X \right) \right] \frac{\partial^2 Q}{\partial x^2}$$

Compare: $\Leftrightarrow$

$$\frac{\partial Q}{\partial t} + c_k \frac{\partial Q}{\partial x} = \mu \frac{\partial^2 Q}{\partial x^2}$$

$$\boxed{\mu = c_k \left(\frac{1}{2} - X \right) \Delta x}$$

By choosing X properly, the numerical diffusion is equivalent to the actual diffusion.

X cannot be over ½, otherwise μ<0, meaning the flood will increase in amplitude

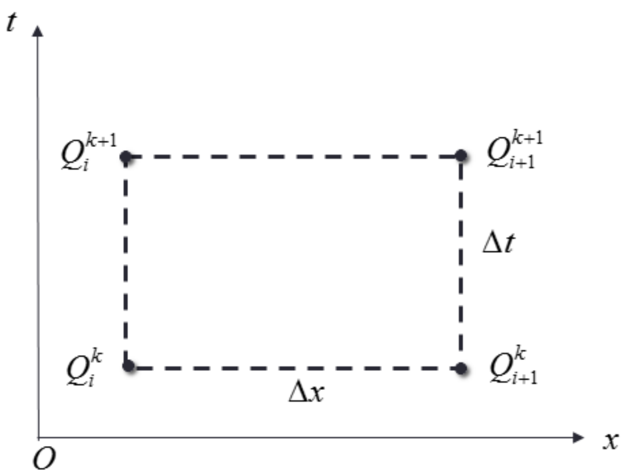
Muskingum-Cunge routing

A Muskingum method is equivalent to kinematic wave method if:

$$\boxed{K = \frac{\Delta x}{c_k}}$$

A Muskingum method can represent the diffusion wave if:

$$\mu = c_k\left(\frac{1}{2} - X\right)\Delta x \Rightarrow \boxed{X = \frac{1}{2}\left(1 - \frac{D^2}{\Delta x c_k b_s Q}\right)}$$



Muskingum method's routing parameter K and X can now be analytically determined (Muskingum-Cunge method)

When there is nothing you can do…