Numerical Methods in Mechanics and Environmental Flows

OCT 13, 2017

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ADMINISTRATIVE MATTERS

To set up appointments for homework and project check IVLE for schedule

- Friday afternoons (E1-08-25)
- Monday or other days if assigned (11 Kent Ridge Road - directions to be given)

For project assignments, teams to be assigned 2 weeks from now.

- Some description

Outline for Environmental Flows

Oct 13
- Introduction
- Delft3D Introduction and Assignment 1: Spin-up?

Oct 20
- Box models and solution methods
- Delft3D Assignment 2 - Boundary conditions

Oct 27
- Solution methods / Transport processes
- Delft3D Assignment 3 – stratification (wind-driven flows)

Nov 3
- Transport processes in flows (1)
- Delft3d Project

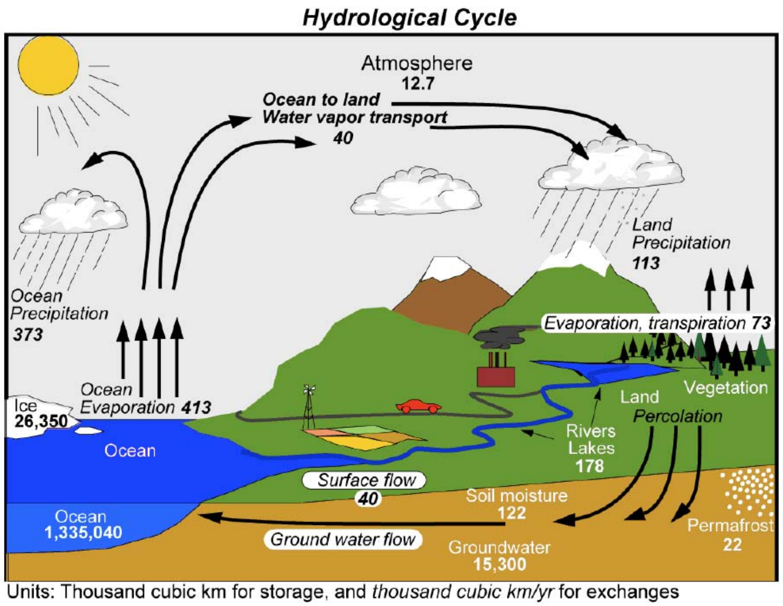
Nov 10
- Transport processes in flows (2)
- Delft3d assignment 4 – model extend (estuarine stratification as an example)

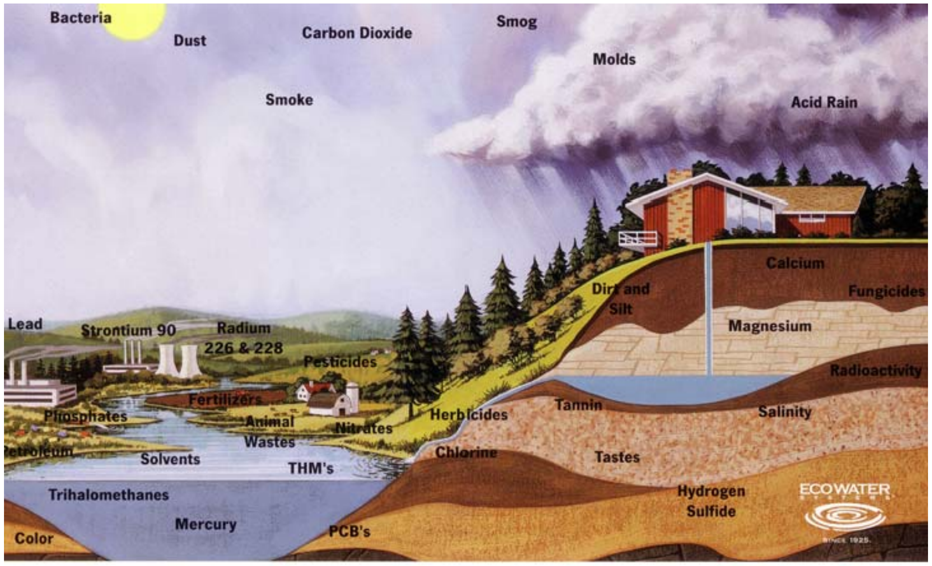
Nov 17
- Presentation of term assignment (5 groups)

Today's Outline

Are there any differences to solving mechanic problems and environmental flow problems?

- What are Environmental flows?
- Box Models as a conceptual framework
- Introduction to numerical models and the use of the mass-spring system equivalent



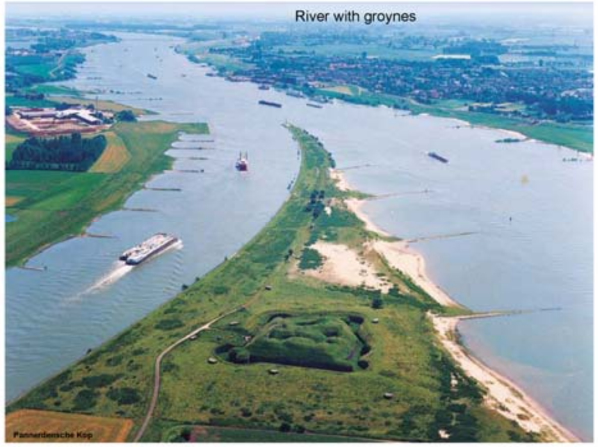


What are environmental flows?

WHAT DO THESE PICTURES TELL YOU?

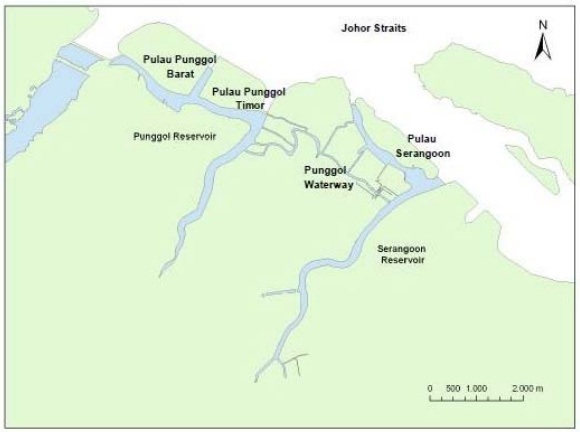
What impacts these flows?

Structures and Outlets

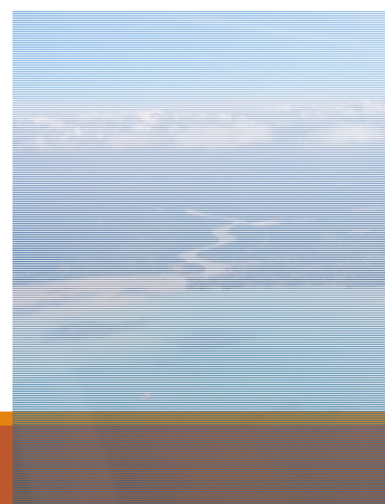


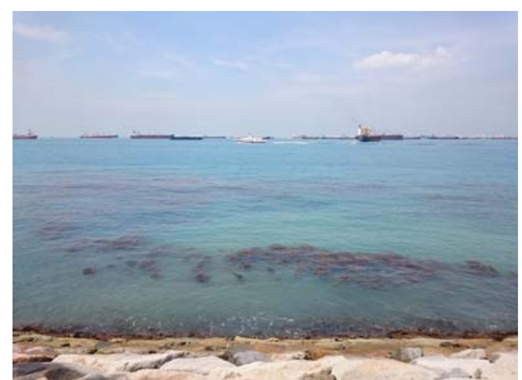
Are there differences between

Inland systems?



Coastal systems?





Some relevant questions…

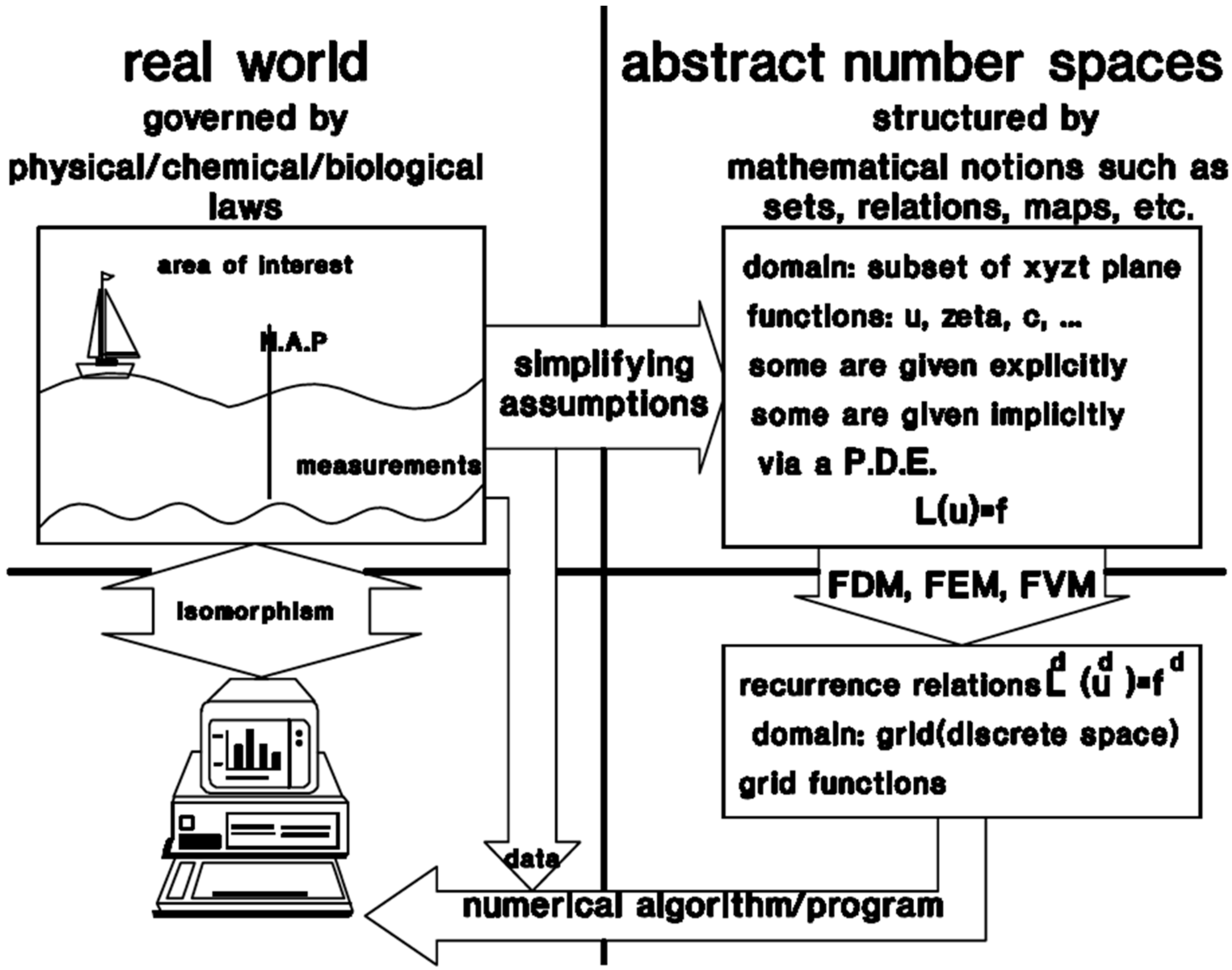
Given that we are only dealing with environmental surface flows and given what you have seen, what do you need to give a reasonably accurate value to a question e.g. what is ***the instantaneous and cumulative flux through the Singapore Straits over a year***

1.

2.

3.

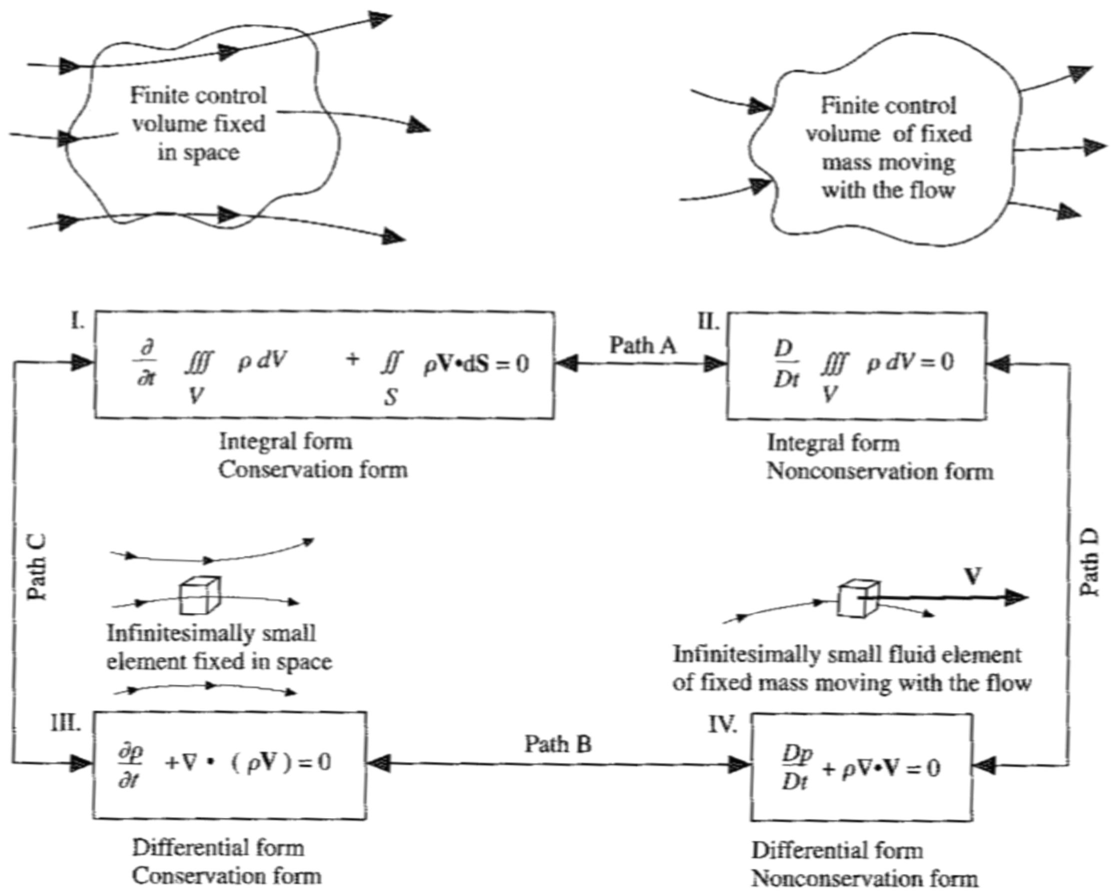
Essentially:



Anything different to mechanics?

LET’S EXPLORE….

For General information Only



A description of surface flows

Common description → conservation laws which are written as equations

What are three basic conservation laws or equations that can describe fluid flow?

A. [VERY BASIC]

B

C

Basic Equations Written mathematically (1)

Differential Form

Equation A

$$\frac{\partial(\rho)}{\partial t} + \frac{\partial(\rho u_i)}{\partial x_i} = 0$$

Equation B

- Starting from: $$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial(\rho u_i u_j)}{\partial x_j} = \sum F$$

- Results in the form: $$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial(\rho u_i u_j)}{\partial x_j} = \frac{\partial \tau_{ij}}{\partial x_j} - \frac{\partial p}{\partial x_i} + b_i$$

Simplifying assumptions for surface flows?

What assumptions can we make for water to simplify our equations?

1.

2.

- Equation A

$$\frac{\partial(\rho)}{\partial t} + \frac{\partial(\rho u_i)}{\partial x_i} = 0 \quad \Rightarrow \quad \frac{\partial u_i}{\partial x_i} = 0$$

- Equation B

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial(\rho u_i u_j)}{\partial x_j} = \frac{\partial \tau_{ij}}{\partial x_j} - \frac{\partial p}{\partial x_i} + b_i \quad \Rightarrow \quad \frac{du_i}{dt} = \frac{\mu}{\rho}\frac{\partial^2 u_i}{\partial x_j^2} - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{1}{\rho} b_i$$

Practical Approximations?

Density (Boussinesq approximation)

- Didn't we remove it earlier? Yes but density can also be a function of salinity, temperature or sediment…

Turbulence (Time / Spatial Averaging)

- Averaging results in Equation B becoming:

$$\frac{\partial u_i}{\partial t} + \frac{\partial (u_i u_j)}{\partial x_j} + \frac{\partial \overline{(u_i' u_j')}}{\partial x_j} = \frac{\mu}{\rho_0} \frac{\partial}{\partial x_j} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{1}{\rho_0} \frac{\partial p}{\partial x_i} - f_i - \frac{\rho}{\rho_0} g \delta_{i=3}$$

- Assuming that one can approximate a so-called "eddy-viscosity" results in

$$\frac{\partial u_i}{\partial t} + \frac{\partial (u_i u_j)}{\partial x_j} = \frac{\partial}{\partial x_j} \left(\nu_t \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right) - \frac{1}{\rho_0} \frac{\partial p}{\partial x_i} - f_i - \frac{\rho}{\rho_0} g \delta_{i=3}$$

$$\overline{u_i' u_j'} = -\nu_t \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \frac{2}{3} \delta i_j k$$

A final approximation that is relevant to practical problems

Shallow Water Approximation:

- For this the flow is assumed to satisfy the following criteria:
  - Char. horizontal length >>> char. Vertical length
  - Char. Vertical velocity<< char. Horizontal velocity
- This results in the vertical (z) Equation B reducing to

$$\frac{\partial p}{\partial z} = -\frac{\rho}{\rho_0} g$$

  - Integrating the previous result: $p(x,y,z,t) = g\int_z^{\zeta} \rho \quad dz + p_a$
- The pressure terms in the remaining Equation B directions can then be written as:

$$-\frac{1}{\rho_0}\frac{\partial p}{\partial x_i} = -\frac{\rho g}{\rho_0}\frac{\partial \zeta}{\partial x_i} - \frac{g}{\rho_0}\int_z^{\zeta}\frac{\partial p}{\partial x_i}dz' - \frac{1}{\rho_0}\frac{\partial p_a}{\partial x_i}$$

**NOTE THIS APPROXIMATION IS USED FOR LOTS OF SURFACE FLOW CODES*

Recap and ....

The equations for fluid flows

You have also seen the simplifications and approximations that can be made for some types of environmental surface flows.

Given the above how do we solve the queston posed earlier for environmental flows?

- First ask yourself what type of equations are these?
- How can we solve this?

The issue is this…

Without simplifications the full Navier Stokes equations are 2nd order non-linear equations with 4 independent variables which means that they cannot be directly classified. But

- They do possess the properties inherent in the classification schemes and
- Some simplifications can be made for certain problems which allows us to classify these simplified flows

This leads to the fact that most flow problems cannot be solved analytically and an approximate numerical solution must be obtained.

How do we test and gauge if our solution is correct???

Box Models

A WAY OF LOOKING AT THE PROBLEM

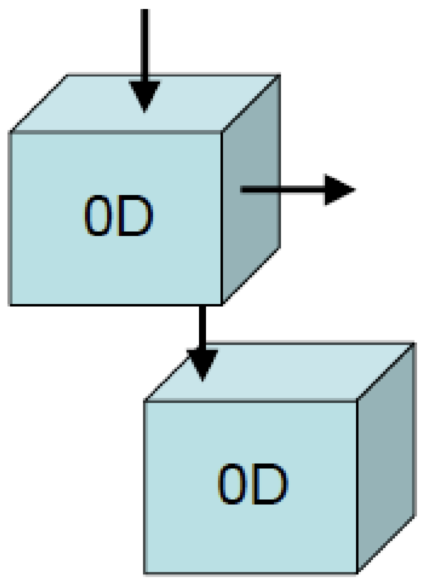
Box Models

Simplified versions of complex systems

Whole system view

Boxes are assumed to be homogeneously mixed

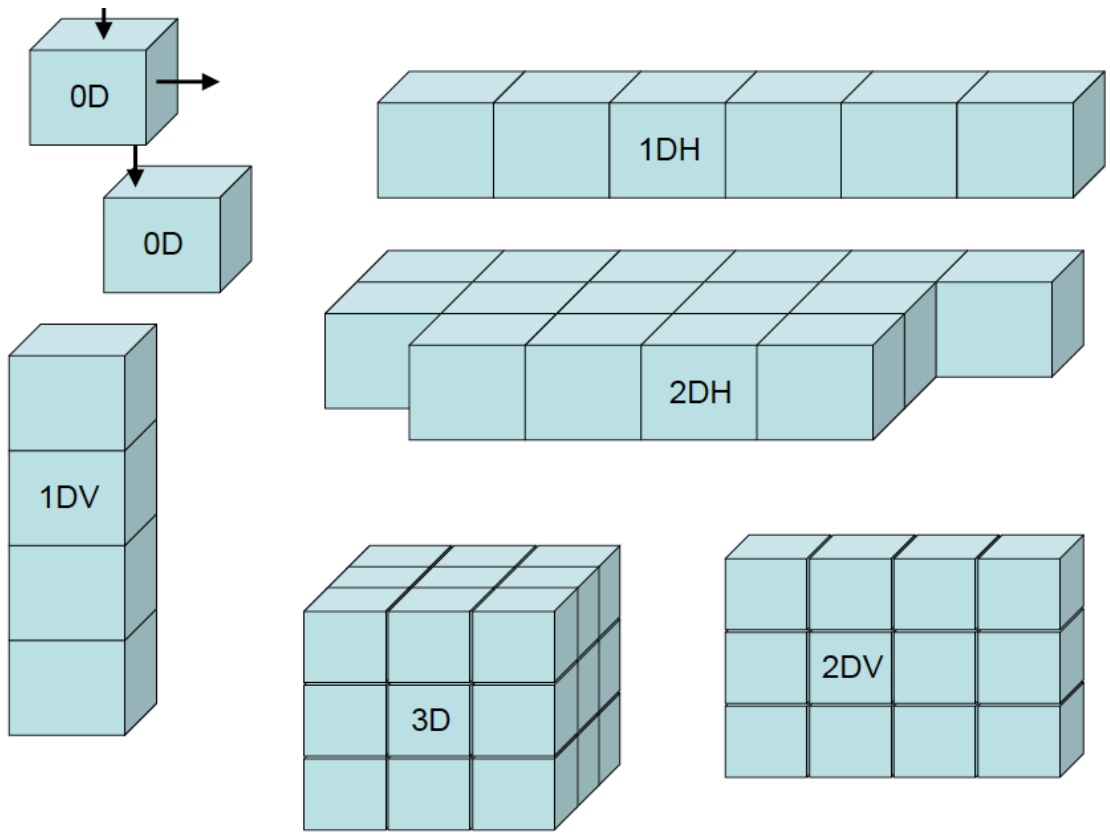
Simplest Box



Extension of Box Models

Remember the assumptions! They still apply.

What would you use each system for?



Simple box models for real-world systems

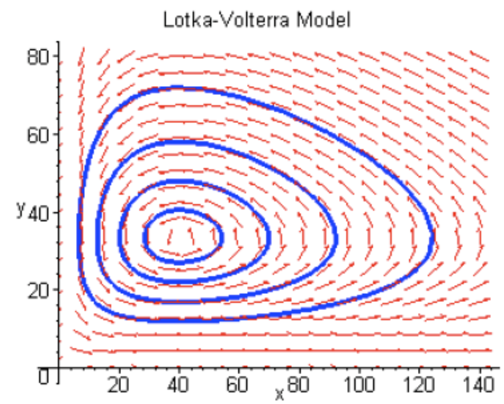
Predator Prey

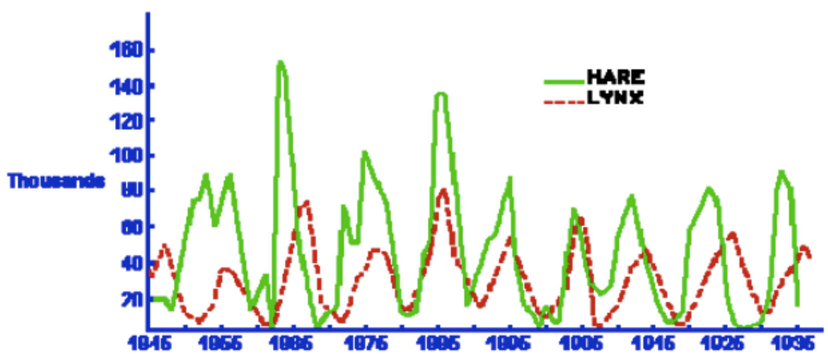
$$\frac{dPRED}{dt} = GROWTH1 * PREY * PRED - DEATH1 * PRED$$

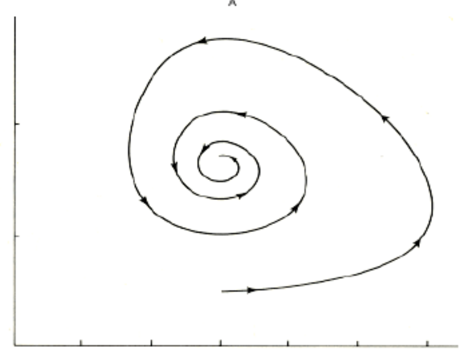
$$\frac{dPREY}{dt} = -CONSUM * PREY * PRED + GROWTH2 * PREY$$

where:

PRED	Predator concentration per unit volume
PREY	Prey concentration per unit volume
DEATH1	Death rate of predators
GROWTH1	Growth rate of predators
CONSUM	Consumption(death) rate of preys
GROWTH	Constant growth rate of preys



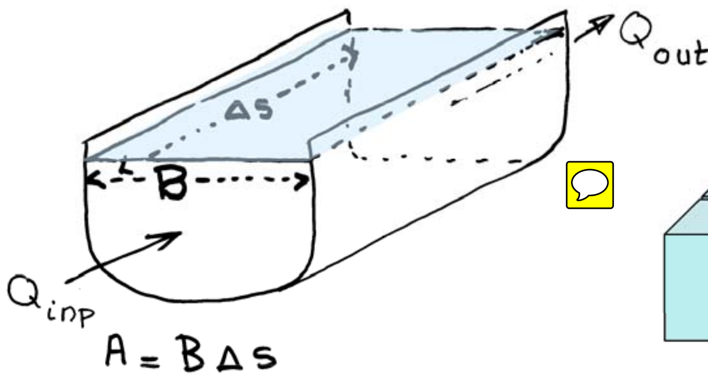


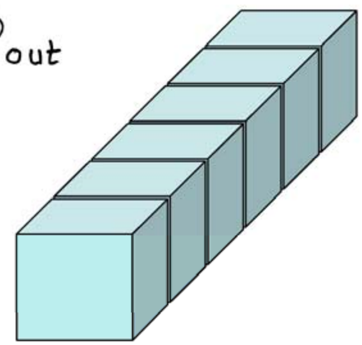


Simple box models (1)

River:

How would you formulate this flow problem?

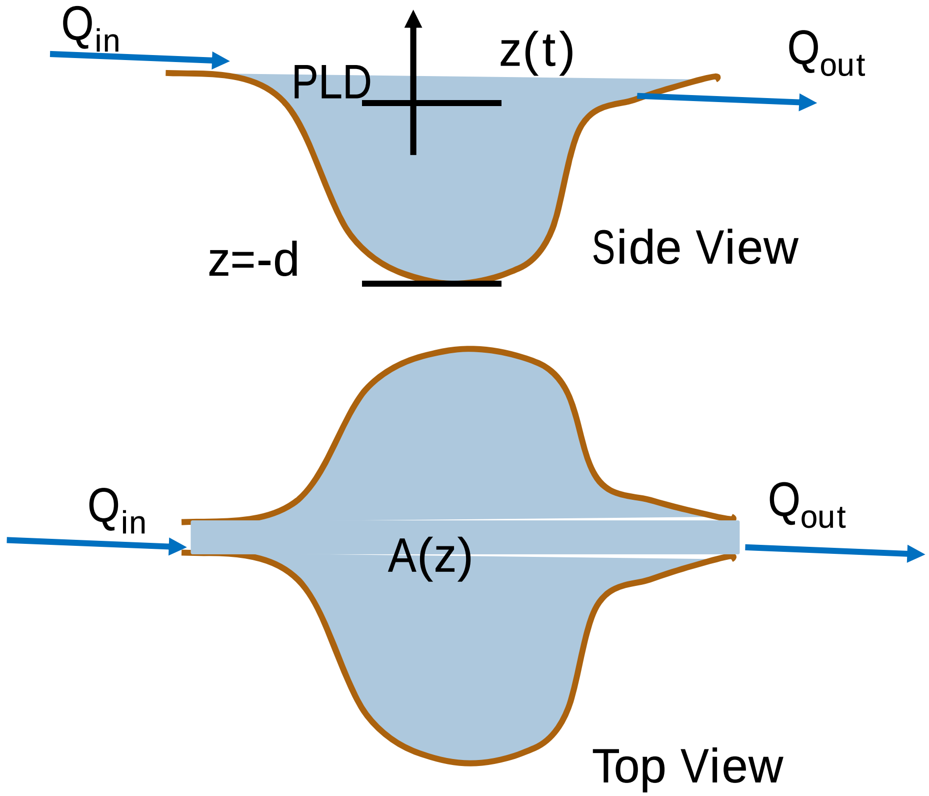




Simple box model (2)

Reservoir / Lake:

What about this?



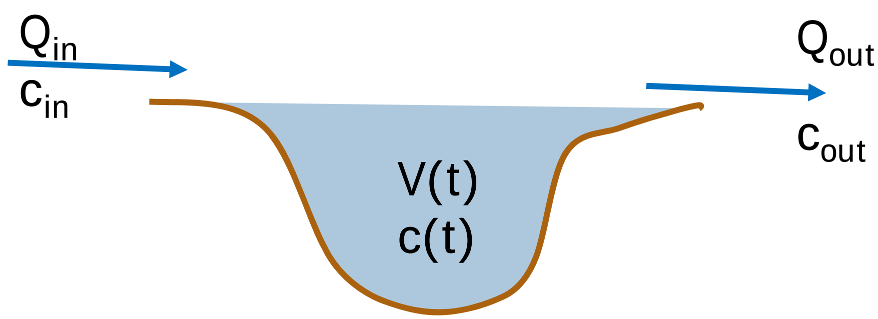
$$h(t) = d + z(t)$$

Solution method?

If you were given the following data, how would you obtain the solution by using a numerical model?

- $A = 1000\ m^2$
- $h(0) = 10\ m$
- $Q_{in} = 9\ m^3/s$
- $Q_{out} = 10\ m^3/s$ when $h > 0$; or 0 when $h \leq 0$

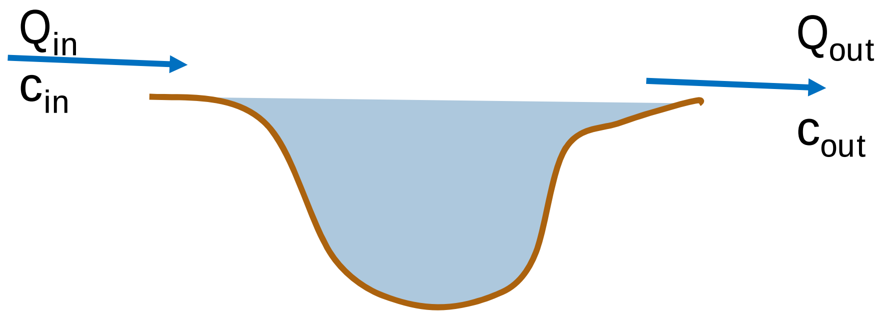
Extending (2) to pollutant transport - 1



What assumptions can or should you make ?

- 1.
- 2.

Extending (2) to pollutant transport - 2



What assumptions can or should you make ?

- 1.
- 2.

Let's stop here …

We know that to solve the problem we have to

1. Define the problem → mathematically

2. Get data to establish a reasonable domain, enable us to provide an initial solution, and if sufficient solve the problem.

3. Then find a method to solve the problem. If not analytically, then numerically.

Next week we will look at some numerical methods to solve the problems that have been laid out and the issues with them

Today's Assignment

A BRIEF INTRODUCTION TO NUMERICAL MODELS

What is a numerical model?

A set of mathematical models that use a numerical procedure to obtain a solution (hopefully unique)

We are looking at surface flows – so what do we solve?

- Equation A

$$\frac{\partial \zeta}{\partial t} + \frac{1}{\sqrt{G_{\xi\xi}}\sqrt{G_{\eta\eta}}} \frac{\partial \left[(d+\zeta) U \sqrt{G_{\eta\eta}}\right]}{\partial \xi} + \frac{1}{\sqrt{G_{\xi\xi}}\sqrt{G_{\eta\eta}}} \frac{\partial \left[(d+\zeta) V \sqrt{G_{\xi\xi}}\right]}{\partial \eta} = Q,$$

- Equation B

$$\begin{aligned}\frac{\partial u}{\partial t} &+ \frac{u}{\sqrt{G_{\xi\xi}}}\frac{\partial u}{\partial \xi} + \frac{v}{\sqrt{G_{\eta\eta}}}\frac{\partial u}{\partial \eta} + \frac{\omega}{d+\zeta}\frac{\partial u}{\partial \sigma} - \frac{v^2}{\sqrt{G_{\xi\xi}}\sqrt{G_{\eta\eta}}}\frac{\partial \sqrt{G_{\eta\eta}}}{\partial \xi} + \\ &+ \frac{uv}{\sqrt{G_{\xi\xi}}\sqrt{G_{\eta\eta}}}\frac{\partial \sqrt{G_{\xi\xi}}}{\partial \eta} - fv = -\frac{1}{\rho_0 \sqrt{G_{\xi\xi}}}P_\xi + F_\xi + \\ &+ \frac{1}{(d+\zeta)^2}\frac{\partial}{\partial \sigma}\left(\nu_V \frac{\partial u}{\partial \sigma}\right) + M_\xi,\end{aligned}$$

$$\begin{aligned}\frac{\partial v}{\partial t} &+ \frac{u}{\sqrt{G_{\xi\xi}}}\frac{\partial v}{\partial \xi} + \frac{v}{\sqrt{G_{\eta\eta}}}\frac{\partial v}{\partial \eta} + \frac{\omega}{d+\zeta}\frac{\partial v}{\partial \sigma} + \frac{uv}{\sqrt{G_{\xi\xi}}\sqrt{G_{\eta\eta}}}\frac{\partial \sqrt{G_{\eta\eta}}}{\partial \xi} + \\ &- \frac{u^2}{\sqrt{G_{\xi\xi}}\sqrt{G_{\eta\eta}}}\frac{\partial \sqrt{G_{\xi\xi}}}{\partial \eta} + fu = -\frac{1}{\rho_0 \sqrt{G_{\eta\eta}}}P_\eta + F_\eta + \\ &+ \frac{1}{(d+\zeta)^2}\frac{\partial}{\partial \sigma}\left(\nu_V \frac{\partial v}{\partial \sigma}\right) + M_\eta.\end{aligned}$$

$$\begin{aligned}\frac{\partial \zeta}{\partial t} &+ \frac{1}{\sqrt{G_{\xi\xi}}\sqrt{G_{\eta\eta}}}\frac{\partial \left[(d+\zeta) u \sqrt{G_{\eta\eta}}\right]}{\partial \xi} + \frac{1}{\sqrt{G_{\xi\xi}}\sqrt{G_{\eta\eta}}}\frac{\partial \left[(d+\zeta) v \sqrt{G_{\xi\xi}}\right]}{\partial \eta} + \\ &+ \frac{\partial \omega}{\partial \sigma} = H\left(q_{in} - q_{out}\right).\end{aligned}$$

Why numerical methods?

Our system is a coupled system of equations

Almost impossible to obtain an exact solution except for idealized problems

Therefore one has to use numerical methods (also known as numerical analysis)

Going back to the unique solution issue in our context

A unique solution to mathematical equations requires a set of

- I…
- B…
- For …

$$\frac{\partial \zeta}{\partial t} + \frac{1}{\sqrt{G_{\xi\xi}}\sqrt{G_{\eta\eta}}} \frac{\partial \left[(d+\zeta) U \sqrt{G_{\eta\eta}} \right]}{\partial \xi} + \frac{1}{\sqrt{G_{\xi\xi}}\sqrt{G_{\eta\eta}}} \frac{\partial \left[(d+\zeta) V \sqrt{G_{\xi\xi}} \right]}{\partial \eta} = Q,$$

$$\frac{\partial \zeta}{\partial t} + \frac{1}{\sqrt{G_{\xi\xi}}\sqrt{G_{\eta\eta}}} \frac{\partial \left[(d+\zeta) u \sqrt{G_{\eta\eta}} \right]}{\partial \xi} + \frac{1}{\sqrt{G_{\xi\xi}}\sqrt{G_{\eta\eta}}} \frac{\partial \left[(d+\zeta) v \sqrt{G_{\xi\xi}} \right]}{\partial \eta} + \\ + \frac{\partial \omega}{\partial \sigma} = H \left(q_{in} - q_{out} \right).$$

$$\frac{\partial u}{\partial t} + \frac{u}{\sqrt{G_{\xi\xi}}} \frac{\partial u}{\partial \xi} + \frac{v}{\sqrt{G_{\eta\eta}}} \frac{\partial u}{\partial \eta} + \frac{\omega}{d+\zeta} \frac{\partial u}{\partial \sigma} - \frac{v^2}{\sqrt{G_{\xi\xi}}\sqrt{G_{\eta\eta}}} \frac{\partial \sqrt{G_{\eta\eta}}}{\partial \xi} + \\ + \frac{uv}{\sqrt{G_{\xi\xi}}\sqrt{G_{\eta\eta}}} \frac{\partial \sqrt{G_{\xi\xi}}}{\partial \eta} - fv = -\frac{1}{\rho_0 \sqrt{G_{\xi\xi}}} P_\xi + F_\xi + \\ + \frac{1}{(d+\zeta)^2} \frac{\partial}{\partial \sigma} \left(\nu_V \frac{\partial u}{\partial \sigma} \right) + M_\xi,$$

$$\frac{\partial v}{\partial t} + \frac{u}{\sqrt{G_{\xi\xi}}} \frac{\partial v}{\partial \xi} + \frac{v}{\sqrt{G_{\eta\eta}}} \frac{\partial v}{\partial \eta} + \frac{\omega}{d+\zeta} \frac{\partial v}{\partial \sigma} + \frac{uv}{\sqrt{G_{\xi\xi}}\sqrt{G_{\eta\eta}}} \frac{\partial \sqrt{G_{\eta\eta}}}{\partial \xi} + \\ - \frac{u^2}{\sqrt{G_{\xi\xi}}\sqrt{G_{\eta\eta}}} \frac{\partial \sqrt{G_{\xi\xi}}}{\partial \eta} + fu = -\frac{1}{\rho_0 \sqrt{G_{\eta\eta}}} P_\eta + F_\eta + \\ + \frac{1}{(d+\zeta)^2} \frac{\partial}{\partial \sigma} \left(\nu_V \frac{\partial v}{\partial \sigma} \right) + M_\eta.$$

So today

We are going to look at the impact of

I …

- And

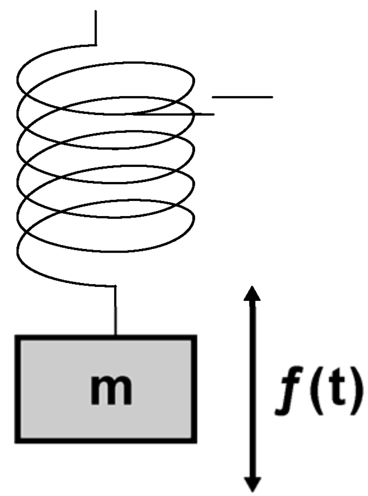
B …

On your solution

An equivalent system to the assignment?

Is this an equivalent system for our assignment today?

I propose this system is.



$$\frac{dx}{dt} - v = 0$$

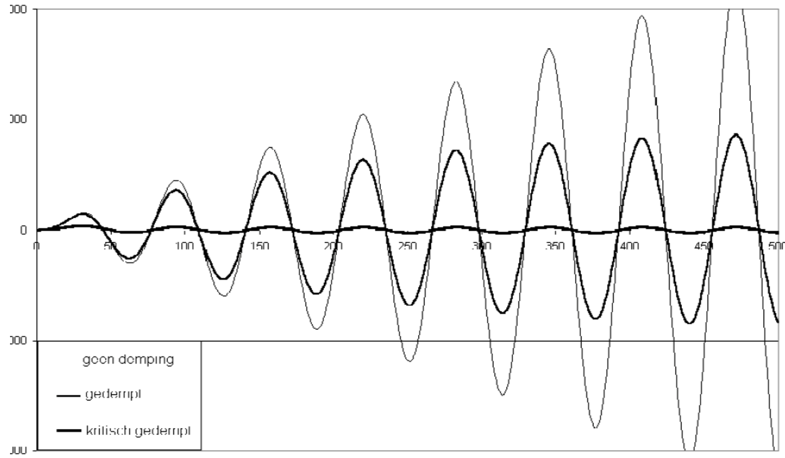
$$\frac{dv}{dt} + \frac{A}{m}v + \frac{K}{m}x = \frac{f_0}{m}\cos(\omega t + \varphi_0)$$

Which means that one can assume that you may see solutions of this form

$$\begin{bmatrix} \dfrac{dv}{dt} \\ \\ \dfrac{dx}{dt} \end{bmatrix} + \begin{bmatrix} \dfrac{A}{m} & \dfrac{K}{m} \\ \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} v \\ \\ x \end{bmatrix} = 0$$

$$\begin{bmatrix} x_H(t) \\ v_H(t) \end{bmatrix} = \alpha_1 \vec{y}_1 e^{-\lambda_1 t} + \alpha_2 \vec{y}_2 e^{-\lambda_2 t}$$

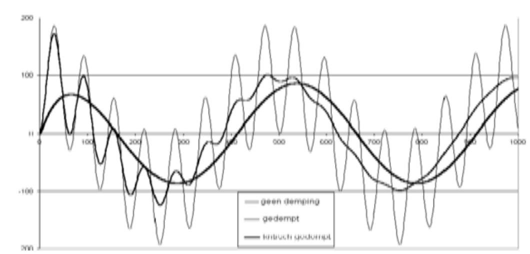
$$\lambda_{1,2} = \frac{A}{2m} \pm \sqrt{\frac{A^2}{4m^2} - \frac{K}{m}} \qquad \left((i)\, A = 0,\ (ii)\ 0 < A < 4mK, (iii)\, A \geq 4mK\right)$$



resonance

$$x_I(t) = \frac{}{\sqrt{m^2(\omega_0^2-\omega^2)^2+A^2\omega^2}}\cos(\omega t - \varphi)$$

$$\cos\varphi = \frac{m\left(\omega_0^2-\omega^2\right)}{\sqrt{m^2\left(\omega_0^2-\omega^2\right)^2+A^2\omega^2}} \qquad \omega_0 = \sqrt{\frac{K}{m}}$$



Exercise 1
Modeling of Free and Forced Behaviour

THE APPROACH THROUGH DELFT3D

We will go through these steps:

How to start up Delft3D

- Follow up on Chapter 5 to 7 on your own after class

Comments on Good Practise (take whatever notes you want)

CREATING YOUR DOMAIN (GRID Workflow)

- Note that for the weekly assignments we will stay with a rectangular grid

PREPARING AND SOLVING YOUR PROBLEM (FLOW Workflow)

- Create Input File to Run Delft3D for Ex. 1
  - Boundaries
  - Water levels / Bathymetry Depths
  - Monitoring (Time History)
  - Outputs to store and plot
- Discussion of Text Editors and Excel [Maybe next week]

CHECKING YOUR SOLUTION (Run-through of QUICKPLOT)

All of that to answer these questions (CE5377):

To assess the impact of boundary conditions, initial conditions and physical and numerical parameters by modeling the natural (Eigen) and forced behaviour of a harbour (Fig. 1-1).

Assess the influence of bottom friction (roughness and depth)

Try to minimize and maximise the spin-up time by varying the initial conditions (water levels and velocities) and boundary type (water levels and velocities)

When (conditions-wise) do the waves behave (roughly) linearly?