AS2

According to question(a):

$$\dot{Y}-3Y=\mathrm{e}^{t}, Y(0)=0$$

$$Y=-0.5\mathrm{e}^{t}+0.5e^{3t}$$

We're using Euler's method to solve IVP 1[st]-order problem. The basic principal is to generate Taylor Series Expansion at different point. It is divided by forward method and backward method.

1. Euler Explicit Method

1) Principal

By using Taylor's equation $y_{i+1}=y_i+hf(t_i,y_i)$, We firstly define a time series called T(i) and compare the results between numerical ones and exact ones in interval [0,2].Also this interval is divided by n=10, which means T(i) equals $0+i\times 0.2$(time step is 0.2).Y(i) is our numerical result while Q(i) is exact result and error(i) is absolute difference between two results.

2) Subroutines

```
function value=explicit(f,a,b,n)
Y=zeros(1,n+1);
T=zeros(1,n+1);
Q=zeros(1,n+1);
error=zeros(1,n+1);
h=(b-a)/n;
Y(1)=0;
Q(1)=Y(1);
T=a:h:b;
error(1)=0;
for i=1:n
    Y(i+1)=Y(i)+h*feval(f,T(i),Y(i));
    Q(i+1)=(-0.5)*exp(T(i+1))+0.5*exp(3*T(i+1));
    error(i+1)=Q(i+1)-Y(i+1);
end
value=[T' Y' Q' error'];
end
```

3) Outcome

Calling this function we get these answers and we put them into a form in convenience of visualization.

i	T	Y	Q	error
1	0.2	0.2000	0.3004	0.1004
2	0.4	0.5643	0.9141	0.3499
3	0.6	1.2012	2.1138	0.9126

4	0.8	2.2864	4.3988	2.1125
5	1.0	4.1033	8.6836	4.5803
6	1.2	7.1089	16.6391	9.5301
7	1.4	12.0383	31.3156	19.2773
8	1.6	20.0723	58.2787	38.2064
9	1.8	33.1063	107.6784	74.5721
10	2.0	54.1801	198.0199	143.8398

4) Stability Analysis

In the application of test problem, the solution is bounded if $|1+hk|\le 1$. In question (a), the k equals to 3. As we can see, $-2/3\le h\le 0$ when it is bounded. While h cannot be smaller than 0, so according to question (a) , the function is instability and error goes up as time series increase.

2. Trapezoidal Method

1) Principal

Also by Taylor's equation, $y_{i+1}=y_i+\frac{h}{2}[f(t_{i+1},y_{i+1})+f(t_i,y_i)]$ .But unknown variable is y_{i+1} which exists in both sides of the equation, and thus cannot be computed explicitly. We can use two ways to address it.

a. Modified Trapezoidal Method

By modified method, we assume a initial value y^*,here I use 1st value of explicit method to iterate $y^*=y_i+hf(t_i,y_i)$ .Then get a new value by equation $y_{i+1}=y_i+\frac{h}{2}[f(t_{i+1},y^*)+f(t_i,y_i)]$.

b. Newton-Raphson

By NR method, we assume a initial value as Iteration method. Next, form a function of unknown variable $Fx=y_{i+1}-y_i-\frac{h}{2}[f(t_{i+1},y_{i+1})+f(t_i,y_i)]$. Find its 1st-order derivative $dFx=1+\frac{3}{2}h$ (according to question (a)) and update it $y_{i+1}=y_i-Fx/dFx$.

2) Subroutines

a. Modified Trapezoidal Method

```
function v=Trapezoidal_modified(f,a,b,n)
h=(b-a)/n;
T=a:h:b;
Y=zeros(1,n+1);
Y(1)=0;
Q(1)=Y(1);
err=zeros(1,n+1);
```

```
for i=1:n
    Yt=Y(i)+h*feval(f,T(i),Y(i));
    Y(i+1)=Y(i)+h*1/2*(feval(f,T(i+1),Yt)+feval(f,T(i),Y(i)));
    Q(i+1)=(-0.5)*exp(T(i+1))+0.5*exp(3*T(i+1));
    err(i+1)=Q(i+1)-Y(i+1);
end
v=[T' Y' Q' err'];
end
```

b. NR

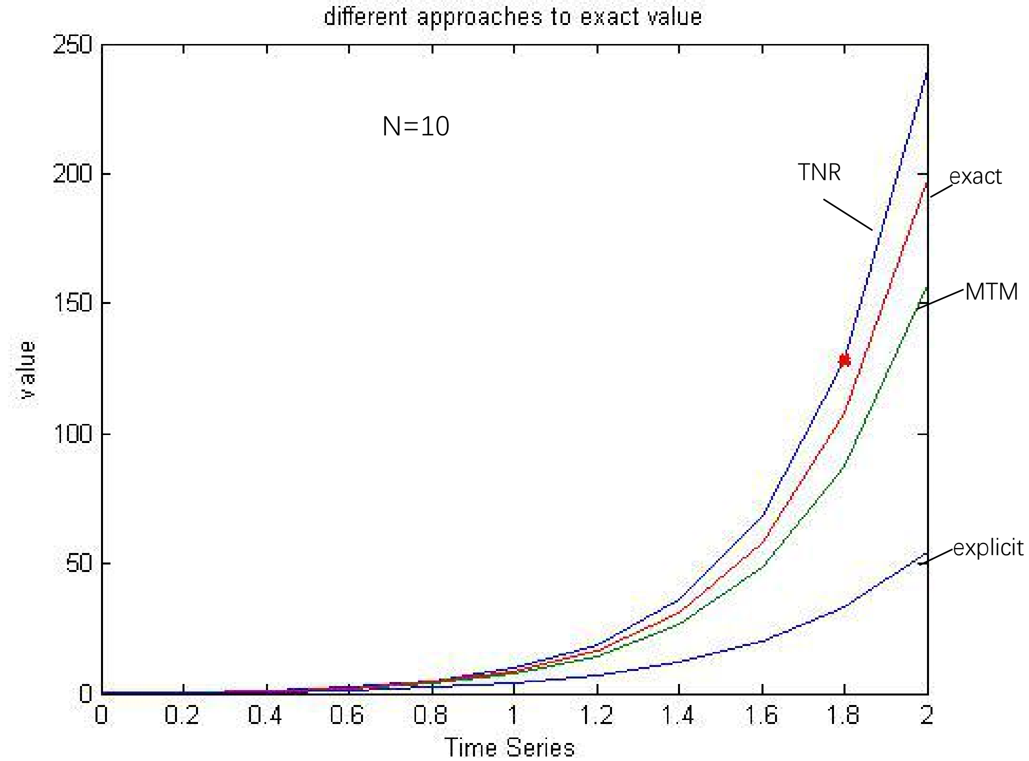
```
function outcome=Trapezoidal_Newton(f,a,b,n,tol)
h=(b-a)/n;
T=a:h:b;
Y=zeros(1,n+1);
Q=zeros(1,n+1);
Q(1)=Y(1);
err=zeros(1,n+1);
for i=1:n
    Yt1=Y(i)+h*feval(f,T(i),Y(i)); % assume a initial value
    eps=1;
    while eps>tol
        %form a function
       Fx=Yt1-Y(i)-h/2*(feval(f,T(i+1),Yt1)+feval(f,T(i),Y(i)));
        %1st-order derivative depends on f
       dFx=1-h*3/2;
        % define a new x
       Yt2=Yt1-Fx/dFx;
       eps=abs(Yt2-Yt1); % decide when to abort
       Yt1=Yt2;
    end
    Y(i+1)=Y(i)+h*1/2*(feval(f,T(i),Y(i))+feval(f,T(i+1),Yt1));
    Q(i+1)=(-0.5)*exp(T(i+1))+0.5*exp(3*T(i+1));
    err(i+1)=abs(Q(i+1)-Y(i+1));
end
outcome=[T' Y' Q' err'];
End
```

3) Outcome

i	T	Y_1(MTM)	Y_2(TNR)	Q	Err_1(MTM)	Err_2(TNR)
1	0.2	0.2821	0.3173	0.3004	0.0182	0.0170
2	0.4	0.8468	0.9770	0.9141	0.0673	0.0628
3	0.6	1.9282	2.2878	2.1138	0.1855	0.1740
4	0.8	3.9464	4.8269	4.3988	0.4525	0.4281
5	1.0	7.6524	9.6706	8.6836	2.2508	0.9870

6	1.2	14.3883	18.8223	16.6391	2.2508	2.1832
7	1.4	26.5478	36.0093	31.3156	4.7677	4.6937
8	1.6	48.3993	68.1613	58.2787	9.8794	9.8826
9	1.8	87.5482	128.1571	107.6784	20.1302	20.4787
10	2	157.5426	239.9259	198.0199	40.4773	41.9060

Here is the image of 3 methods and exact value, we can explicitly see the difference of these methods.



4) Stability Analysis

For Trapezoidal method

$$y_{n+1} = y_n + h/2(Ky_n + Ky_{n+1})$$

$$y_{n+1} = (1+\frac{Kh}{2})/(1-\frac{Kh}{2})y_n$$

$$y_{n+1} = \left[(1+Kh/2)/(1-Kh/2)\right]^{n+1} y_0$$

Because every y_0 has a error inside, we define initial error as e_0 and get this:

$$y_{n+1} = \left[(1+Kh/2)/(1-Kh/2)\right]^{n+1}(y_0^{*} - e_0)$$

$$y_{n+1} = y_n - [(1+Kh/2)/(1-Kh/2)]^{n+1} e_0$$

From the equation above, we can draw a conclusion that if $\left|(1+Kh/2)/(1-Kh/2)\right| \le 1$, the error will die out as time approaches infinite. In question (a), when k equals 3 it turns out h should be in the interval $\left[0, \frac{2\sqrt{2}}{3}\right]$.

To test this, we make h equals to 1 , and with the same function in [0,10]. It is clear that error accumulates with T.