Consider the matrices $\mathbf{X}_1, ..., \mathbf{X}_m$, where $\mathbf{X}_j$ is an $n_j \times p$ matrix of rank q_j. Let $\mathbf{X} = (\mathbf{X}'_1, \mathbf{X}'_2, ..., \mathbf{X}'_m)'$ and assume that $\mathbf{X}$ has full column rank p. Denote $N = \sum_{j=1}^m n_j$. Let $\mathbf{W}_1, ..., \mathbf{W}_m$ be symmetric matrices of full rank, with $\mathbf{W}_j$ having dimension $n_j \times n_j$. Let $\mathbf{W}$ be the block-diagonal matrix with components $\mathbf{W}_1, ..., \mathbf{W}_m$, i.e., $\mathbf{W} = \bigoplus_{j=1}^m \mathbf{W}_j$. Let $\mathbf{M} = (\mathbf{X}'\mathbf{W}\mathbf{X})^{-1}$. Let $\mathbf{J}_j$ be an $n_j \times N$ matrix consisting of the rows of the identity matrix that correspond to $\mathbf{X}_j$, so that $\mathbf{J}_j\mathbf{X} = \mathbf{X}_j$.

Lemma. *Let* $\mathbf{L} = (\mathbf{X}'\mathbf{W}\mathbf{X} - \mathbf{X}_j\mathbf{W}_j\mathbf{X}_j)$ *and assume that* $\mathbf{L}$ *has full rank* p. *Then*

$$\mathbf{X}_j \subset \mathit{span}\left[\mathbf{J}_j\left(\mathbf{I}_N - \mathbf{X}\mathbf{M}\mathbf{X}'\mathbf{W}\right)\right].$$

Proof. Let $\mathbf{Z}_j = \mathbf{0}$ be a $n_j \times p$ matrix of zeros. Define the $n_k \times p$ matrices $\mathbf{Z}_k = -\mathbf{X}_k\mathbf{L}^{-1}\mathbf{X}'\mathbf{W}\mathbf{X}$, for $k \neq j$. Set $\mathbf{Z} = (\mathbf{Z}'_1, ..., \mathbf{Z}'_m)'$. Then

$$\begin{aligned}
\mathbf{J}_j\left(\mathbf{I}_N - \mathbf{X}\mathbf{M}\mathbf{X}'\mathbf{W}\right)\mathbf{Z} &= \mathbf{Z}_j - \mathbf{X}_j\mathbf{M}\sum_{k=1}^m \mathbf{X}'_k\mathbf{W}_k\mathbf{Z}_k \\
&= \mathbf{X}_j\mathbf{M}\left(\sum_{k\neq j} \mathbf{X}'_k\mathbf{W}_k\mathbf{X}_k\right)\mathbf{L}^{-1}\mathbf{X}'\mathbf{W}\mathbf{X} \\
&= \mathbf{X}_j\mathbf{M}\mathbf{L}\mathbf{L}^{-1}\mathbf{M}^{-1} \\
&= \mathbf{X}_j.
\end{aligned}$$

Thus, there exists an $N \times p$ matrix $\mathbf{Z}$ such that $\mathbf{J}_j\left(\mathbf{I}_N - \mathbf{X}\mathbf{M}\mathbf{X}'\mathbf{W}\right)\mathbf{Z} = \mathbf{X}_j$.

□