Consider the matrices $\mathbf{X}_1, ..., \mathbf{X}_m$, where $\mathbf{X}_j$ is an $n_j \times p$ matrix of rank q_j. Let $\mathbf{X} = (\mathbf{X}_1', \mathbf{X}_2', ..., \mathbf{X}_m')'$ and assume that $\mathbf{X}$ has full column rank p. Denote $N = \sum_{j=1}^m n_j$. Let $\mathbf{W}_1, ..., \mathbf{W}_m$ be symmetric matrices of full rank, with $\mathbf{W}_j$ having dimension $n_j \times n_j$. Let $\mathbf{W}$ be the block-diagonal matrix with components $\mathbf{W}_1, ..., \mathbf{W}_m$, i.e., $\mathbf{W} = \bigoplus_{j=1}^m \mathbf{W}_j$. Let $\mathbf{J}_j$ be an $n_j \times N$ matrix consisting of the rows of the identity matrix that correspond to $\mathbf{X}_j$, so that $\mathbf{J}_j \mathbf{X} = \mathbf{X}_j$. Define the $n_j \times N$ matrix

$$\mathbf{K}_j = \mathbf{J}_j \left(\mathbf{I}_N - \mathbf{X} \left(\mathbf{X}' \mathbf{W} \mathbf{X} \right)^{-1} \mathbf{X}' \mathbf{W} \right).$$

Question: Under what conditions is $\mathbf{X}_j$ in the span of $\mathbf{K}_j$, i.e., when does there exist a $N \times p$ matrix $\mathbf{Z}$ such that $\mathbf{K}_j \mathbf{Z} = \mathbf{X}_j$?

My conjecture (or perhaps its more accurate to call it wishful thinking...) is that it is sufficient for $\mathbf{X}_{(j)}$ to have full rank, where $\mathbf{X}_{(j)}$ is the $(N - n_j) \times p$ matrix formed by stacking all of the $\mathbf{X}_k$ where $k \neq j$, i.e., $\mathbf{X}_{(j)} = \left(\mathbf{X}_1', ... \mathbf{X}_{j-1}', \mathbf{X}_{j+1}', ..., \mathbf{X}_m \right)'$.