Let $\mathbf{D}_j = \mathbf{\Phi}_j^C$ and consider the adjustment matrices

$$\mathbf{A}_j = \mathbf{D}_j' \mathbf{B}_j^{+/2} \mathbf{D}_j,$$

where $\mathbf{B}_j = \mathbf{D}_j \left(\mathbf{I} - \mathbf{H}\right)_j \mathbf{\Phi} \left(\mathbf{I} - \mathbf{H}\right)_j' \mathbf{D}_j'$ and $\mathbf{B}_j^{+/2}$ denotes the symmetric square root of the Moore-Penrose inverse of $\mathbf{B}_j$. Then in order for $\mathbf{V}^R$ to be exactly model-unbiased, we must have that

$$\mathbf{X}_j' \mathbf{W}_j \mathbf{D}_j' \mathbf{B}_j^{+/2} \mathbf{B}_j \mathbf{B}_j^{+/2} \mathbf{D}_j \mathbf{W}_j \mathbf{X}_j = \mathbf{X}_j' \mathbf{W}_j \mathbf{D}_j' \mathbf{D}_j \mathbf{W}_j \mathbf{X}_j$$

or equivalently that

$$\mathbf{X}_j' \mathbf{W}_j \mathbf{D}_j' \mathbf{B}_j^{+} \mathbf{B}_j \mathbf{D}_j \mathbf{W}_j \mathbf{X}_j = \mathbf{X}_j' \mathbf{W}_j \mathbf{D}_j' \mathbf{D}_j \mathbf{W}_j \mathbf{X}_j,$$

where $\mathbf{B}_j^+$ is the Moore-Penrose inverse of $\mathbf{B}_j$.

Now consider the rank-decomposition of $\left(\mathbf{I} - \mathbf{H}\right)_j = \mathbf{C}\mathbf{R}$ for $n_j \times r$ matrix $\mathbf{C}$ with full column-rank and $\mathbf{R}$ is $r \times N$ with full row-rank. Then it can be verified that

$$\mathbf{B}_j^+ = \mathbf{D}_j \mathbf{C} \left(\mathbf{C}' \mathbf{D}_j' \mathbf{D}_j \mathbf{C}\right)^{-1} \left(\mathbf{R} \mathbf{\Phi} \mathbf{R}'\right)^{-1} \left(\mathbf{C}' \mathbf{D}_j' \mathbf{D}_j \mathbf{C}\right)^{-1} \mathbf{C}' \mathbf{D}_j'$$

and therefore that

$$\mathbf{B}_j^+ \mathbf{B}_j = \mathbf{D}_j \mathbf{C} \left(\mathbf{C}' \mathbf{D}_j' \mathbf{D}_j \mathbf{C}\right)^{-1} \mathbf{C}' \mathbf{D}_j'.$$

Thus, the question is to identify conditions on $\mathbf{X}_j$ under which the following equality holds:

$$\mathbf{X}_j' \mathbf{W}_j \mathbf{D}_j' \mathbf{D}_j \mathbf{C} \left(\mathbf{C}' \mathbf{D}_j' \mathbf{D}_j \mathbf{C}\right)^{-1} \mathbf{C}' \mathbf{D}_j' \mathbf{D}_j \mathbf{W}_j \mathbf{X}_j = \mathbf{X}_j' \mathbf{W}_j \mathbf{D}_j' \mathbf{D}_j \mathbf{W}_j \mathbf{X}_j.$$

It seems like to answer this question, we need to find an explicit expression for $\mathbf{C}$ in terms of the components of $\left(\mathbf{I} - \mathbf{H}\right)_j$. Not sure how to do that....