Notation:

- Let $\mathbf{X}_j$ be an $n_j \times p$ matrix of rank q. Let $\mathbf{X} = (\mathbf{X}_1', \mathbf{X}_2', ..., \mathbf{X}_m')'$ and assume that $\mathbf{X}$ has full rank p.
- Let $\mathbf{W}_j$ and $\mathbf{\Phi}_j$ be $n_j \times n_j$ matrices of full rank; denote $\mathbf{W} = \bigoplus_{j=1}^m \mathbf{W}_j$ and $\mathbf{\Phi} = \bigoplus_{j=1}^m \mathbf{\Phi}_j$.
- Let $\mathbf{M} = (\mathbf{X}'\mathbf{W}\mathbf{X})^{-1}$ and $\mathbf{H} = \mathbf{X}\mathbf{M}\mathbf{X}'\mathbf{W}$.
- Let $(\mathbf{I} - \mathbf{H})_j$ denote the rows of $\mathbf{I} - \mathbf{H}$ corresponding to cluster j.
- Let $\mathbf{D}_j = \mathbf{\Phi}_j^C$, where $\mathbf{\Phi}_j^C$ is the upper-triangular Cholesky decomposition of $\mathbf{\Phi}_j$.

Consider the adjustment matrices

$$\mathbf{A}_j = \mathbf{D}_j'\mathbf{B}_j^{+/2}\mathbf{D}_j,$$

where $\mathbf{B}_j = \mathbf{D}_j (\mathbf{I} - \mathbf{H})_j \mathbf{\Phi} (\mathbf{I} - \mathbf{H})_j' \mathbf{D}_j'$ and $\mathbf{B}_j^{+/2}$ denotes the symmetric square root of the Moore-Penrose inverse of $\mathbf{B}_j$. Then in order for $\mathbf{V}^R$ to be exactly model-unbiased, we must have that

$$\mathbf{X}_j'\mathbf{W}_j\mathbf{D}_j'\mathbf{B}_j^+\mathbf{B}_j\mathbf{D}_j\mathbf{W}_j\mathbf{X}_j = \mathbf{X}_j'\mathbf{W}_j\mathbf{D}_j'\mathbf{D}_j\mathbf{W}_j\mathbf{X}_j, \tag{1}$$

where $\mathbf{B}_j^+$ is the Moore-Penrose inverse of $\mathbf{B}_j$.

Now consider the rank-decomposition of $(\mathbf{I} - \mathbf{H})_j = \mathbf{C}\mathbf{R}$ for $n_j \times r$ matrix $\mathbf{C}$ with full column-rank and $\mathbf{R}$ is $r \times N$ with full row-rank. Then it can be verified that

$$\mathbf{B}_j^+ = \mathbf{D}_j\mathbf{C}\left(\mathbf{C}'\mathbf{D}_j'\mathbf{D}_j\mathbf{C}\right)^{-1}\left(\mathbf{R}\mathbf{\Phi}\mathbf{R}'\right)^{-1}\left(\mathbf{C}'\mathbf{D}_j'\mathbf{D}_j\mathbf{C}\right)^{-1}\mathbf{C}'\mathbf{D}_j'$$

and therefore that

$$\mathbf{B}_j^+\mathbf{B}_j = \mathbf{D}_j\mathbf{C}\left(\mathbf{C}'\mathbf{D}_j'\mathbf{D}_j\mathbf{C}\right)^{-1}\mathbf{C}'\mathbf{D}_j'.$$

Thus, the question is to identify conditions on $\mathbf{X}_j$ under which the following equality holds:

$$\mathbf{X}_j'\mathbf{W}_j\mathbf{D}_j'\mathbf{D}_j\mathbf{C}\left(\mathbf{C}'\mathbf{D}_j'\mathbf{D}_j\mathbf{C}\right)^{-1}\mathbf{C}'\mathbf{D}_j'\mathbf{D}_j\mathbf{W}_j\mathbf{X}_j = \mathbf{X}_j'\mathbf{W}_j\mathbf{D}_j'\mathbf{D}_j\mathbf{W}_j\mathbf{X}_j.$$

Equivalently, under what conditions are the columns of $\mathbf{D}_j\mathbf{W}_j\mathbf{X}_j$ in the column space of $\mathbf{D}(\mathbf{I} - \mathbf{H})_j$?

One approach to answering this question would be to find an explicit expression for $\mathbf{C}$ in terms of the components of $(\mathbf{I} - \mathbf{H})_j$. Not yet sure how to do that.... Also, I would speculate that a necessary condition may be that each column of $\mathbf{X}$ must be identified in more than one cluster.