User Guide for `ffr-LFDFT`

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Contents

1 Introduction

Welcome to `ffr-LFDFT` documentation.
`ffr-LFDFT` is a poor man's program (or collection of subroutines, as of now) to carry out electronic structure calculations based on density functional theory and Lagrange basis set.

2 Kohn-Sham equations

3 Lagrange basis function

3.1 Periodic Lagrange function

For a given interval $[0, L]$, with $L > 0$, the grid points x_i appropriate for periodic Lagrange function are given by:

$$x_i = \frac{L}{2}\frac{2i-1}{N} \tag{1}$$

with $i = 1, \ldots, N$. Number of points N should be an odd number.
The periodic cardinal functions $L_i^{\text{per}}(x)$, defined at grid point i are given by:

$$L_i^{\text{per}}(x) = \frac{1}{\sqrt{NL}} \sum_{n=1}^{N} \cos\left(\frac{\pi}{L}(2n - N - 1)(x - x_i)\right). \tag{2}$$

The expansion of periodic function in terms of Lagrange functions:

$$f(x) = \sum_{i=1}^{N} c_i L_i^{\text{per}}(x) \tag{3}$$

with expansion coefficients $c_i = \sqrt{L/N} f(x_i)$. When doing variational calculation, the cofficients c_i are the variational parameters. The actual function values $f(x_i)$ at grid points x_i is obtained by $f(x_i) = \sqrt{N/L} c_i$. The prefactor is sometimes abbreviated by $h = L/N$ and is also referred to as scaling factor.

Consider periodic potential in one dimension:

$$V(x+L) = V(x). \tag{4}$$

Floquet-Bloch theorem states that the wave function solution for periodic potentials can be written in the form:

$$\psi_k(x) = e^{\imath kx}\phi_k(x) \tag{5}$$

where function $\phi_k(x)$ and its first derivative $\phi'_k(x)$ have the same periodicity as $V(x)$ and k is a constant called the crystal momentum. Substituting this expression to Schrodinger equation we obtain:

$$\left[-\frac{\hbar^2}{2m}\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 2\imath k\frac{\mathrm{d}}{\mathrm{d}x} - k^2\right) + V(x)\right]\phi_k(x) = E\phi_k(k). \tag{6}$$

An alternative way of enforcing periodicity of the wave function is to require that:

$$\psi_k(x+L) = e^{\imath kL}\psi_k(x). \tag{7}$$

This condition follows from:

$$\begin{aligned}\psi_k(x+L) &= e^{\imath k(x+L)}\phi_k(x+L)\\ &= e^{\imath k(x+L)}\phi_k(x)\\ &= e^{\imath kL}e^{\imath kx}\phi_k(x)\\ &= e^{\imath kL}\psi_k(x)\end{aligned}$$

Using periodic cardinal the Schrodinger equation for periodic potential can be written as:

$$\sum_{j=1}^{N}\left[-\frac{\hbar^2}{2m}\left(D^{(2)}_{jl} + 2\imath kD^{(1)}_{jl} - k^2\delta_{jl}\right) + V(j)\delta_{jl}\right]\phi(j) = E\phi(l) \tag{8}$$

with $l = 1, \ldots, N$. $D^{(1)}_{jl}$ are matrix elements of the first derivatives:

$$D^{(1)}_{jl} = \begin{cases} 0 & j = l \\ -\dfrac{2\pi}{L}(-1)^{j-l}\left(2\sin\dfrac{\pi(j-l)}{N}\right)^{-1} & j \neq l \end{cases} \tag{9}$$

and $D^{(2)}_{jl}$ are matrix elements of the second derivatives, $N' = (N-1)/2$:

$$D^{(2)}_{jl} = \begin{cases} -\left(\dfrac{2\pi}{L}\right)^2\dfrac{N'(N'+1)}{3} & j = l \\ -\left(\dfrac{2\pi}{L}\right)^2(-1)^{j-l}\dfrac{\cos\left(\pi(j-l)/N\right)}{2\sin^2\left[\pi(j-l)/N\right]} & j \neq l \end{cases} \tag{10}$$

Note that, $D^{(1)}_{jl}$ is not symmetric, but $D^{(1)}_{jl} = -D^{(1)}_{lj}$. Meanwhile, the second derivative matrix $D^{(2)}_{jl}$ is symetric, i.e. $D^{(2)}_{jl} = D^{(2)}_{lj}$. With the above expressions, first and second derivative of periodic cardinals can be expressed as

$$\frac{\mathrm{d}}{\mathrm{d}x}L^{\mathrm{per}}_i(x) = \sum_{j=1}^{N} D^{(1)}_{ji}L^{\mathrm{per}}_j(x) \tag{11}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}L^{\mathrm{per}}_i(x) = \sum_{j=1}^{N} D^{(2)}_{ji}L^{\mathrm{per}}_j(x) \tag{12}$$

The previous approach also can be extended to periodic potential in 3D:

$$V(\mathbf{r}) = V(x,y,z) = V\left(x+L_x, y+L_y, z+L_z\right)$$

Using periodic LF, Schrodinger equation can be casted into the following form:

$$\left[-\frac{\hbar^2}{2m}\left(\nabla^2 + 2\imath\mathbf{k}\cdot\nabla - \mathbf{k}^2\right) + V(\mathbf{r})\right]\phi_{\mathbf{k}}(\mathbf{r}) = E\ \phi_{\mathbf{k}}(\mathbf{r}) \tag{13}$$

3.2 Cluster Lagrange function

For a given interval $[A, B]$, with $B > A$, the grid points x_i appropriate for cluster Lagrange function are given by:

$$x_i = A + \frac{B-A}{N+1} i$$

where $i = 1, \ldots, N$. Number of points N can be either odd or even number.
The cluster Lagrange functions $L_i^{\mathrm{clu}}(x)$, defined at grid point i are given by:

$$L_i^{\mathrm{clu}}(x) = \frac{2}{\sqrt{(N+1)(B-A)}} \sum_{n=1}^{N} \sin\left(k_n(x_i - A)\right) \sin\left(k_n(x - A)\right). \tag{14}$$

where $k_n = \pi n/(B-A)$. The expansion of a function $f(x)$ in terms of cluster Lagrange functions:

$$f(x) = \sum_{i=1}^{N} c_i L_i^{\mathrm{clu}}(x) \tag{15}$$

with expansion coefficients $c_i = \sqrt{(B-A)/(N+1)} f(x_i)$. When doing variational calculation, the cofficients c_i are the variational parameters. The actual function values $f(x_i)$ at grid points x_i is obtained by $f(x_i) = \sqrt{(N+1)/(B-A)} c_i$.
Matrix elements $D_{jl}^{(2)}$ of the second derivatives for cluster Lagrange functions are

$$D_{jl}^{(2)} = \begin{cases} -\frac{1}{2}\left(\frac{\pi}{B-A}\right)^2 \frac{2(N+1)^2+1}{3} - \frac{1}{\sin^2\left[\pi j/(N+1)\right]} & j = l \\ -\frac{1}{2}\left(\frac{\pi}{B-A}\right)^2 (-1)^{j-l} \left[\frac{1}{\sin^2\left[\frac{\pi(j-l)}{2(N+1)}\right]} - \frac{1}{\sin^2\left[\frac{\pi(j+l)}{2(N+1)}\right]}\right] & j \neq l \end{cases} \tag{16}$$

For free or cluster boundary condition, we don't need $D_{jl}^{(1)}$.

4 Implementation

Description of unit cell and atomic structures

```
PROGRAM testprog
  IMPLICIT NONE
END PROGRAM
```