ISOLATED POINTS ON MODULAR CURVES OF PRIME-POWER LEVEL

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Abstract. An isolated point on an algebraic curve is a closed point not belonging to a collection of points of the same degree parametrized by $\mathbf{P}^1$ or a positive rank abelian subvariety of the curve's Jacobian. We study the sets of j-invariants, in extensions of bounded degree, that arise as the j-invariant of an isolated point on a modular curve. We obtain finiteness results on these sets for families of modular curves with prime-power level. This is related to recent work of Bourdon and Ejder, who classified rational j-invariants of isolated points on the families $X_1(n)$ and $X_0(n)$, for n a prime power.

1. Introduction

The closed points of a smooth, projective curve C over a number field k are in bijection with Gal_k-orbits of points in $C(\overline{k})$, where $\mathrm{Gal}_k := \mathrm{Gal}(\overline{k}/k)$ is the absolute Galois group. The degree of a closed point corresponds to the size of the associated Galois-orbit. A closed point $x \in C$ of degree d is said to be **isolated**[1] if it does not belong to a collection of degree d closed points parametrized by $\mathbf{P}^1$ or a positive rank abelian subvariety of the curve's Jacobian.

Faltings's Theorem [8] guarantees that if C has genus strictly greater than 1, then C has finitely many degree 1 points. Such a curve C has infinitely many points of arbitrary degree. However, it was shown in [2] that, as a consequence of Faltings's Theorem on rational points on subvarieties of abelian varieties [9], C has only finitely many isolated points. Because of this, the study of isolated points on curves generalizes the study of rational points in a natural way.

In this paper, we study isolated points on modular curves, which parametrize elliptic curves equipped with specific structure. If E is an elliptic curve defined over a number field k, the action of Gal_k on the n-torsion subgroups $E[n]$, for positive integers n, induces representations

$$\begin{aligned} \rho_{E,n} &: \mathrm{Gal}_k \to \mathbf{GL}_2(\mathbf{Z}/n\mathbf{Z}), \\ \rho_{E,\ell^\infty} &: \mathrm{Gal}_k \to \mathbf{GL}_2(\mathbf{Z}_\ell) = \varprojlim \mathbf{GL}_2(\mathbf{Z}/\ell^n\mathbf{Z}), \text{ and} \\ \rho_E &: \mathrm{Gal}_k \to \mathbf{GL}_2(\widehat{\mathbf{Z}}) = \varprojlim \mathbf{GL}_2(\mathbf{Z}/n\mathbf{Z}). \end{aligned}$$

If H is an open subgroup of $\mathbf{GL}_2(\mathbf{Z}_\ell)$, the modular curve X_H is a curve whose non-cuspidal points parametrize elliptic curves with $\rho_{E,\ell^\infty}(\mathrm{Gal}_k)$ contained in H. To a closed point $x \in X_H$, we associate an algebraic number $j(x)$ called the **j-invariant of** x, which is the image of x under the natural map to the coarse moduli space of elliptic curves.

Isolated points on modular curves are related to a number of open questions concerning the arithmetic of elliptic curves. For a fixed non-CM elliptic curve E over a number field k, Serre showed [19] that the mod-ℓ Galois representation $\rho_{E,\ell}$ is surjective for large primes ℓ. Serre asked [19, Section 4.3] if this result in the case of $k = \mathbf{Q}$ can be made uniform—that

[1] See Section 2.6 for a precise definition.

is, if there is prime L such that $\rho_{E,\ell}$ is surjective for all non-CM elliptic curves $E/\mathbf{Q}$ and all primes $\ell > L$. This prime L has since been conjectured by Zywina [24, Conjecture 1.2] and Sutherland [21, Conjecture 1.1] to be 37. We will refer to this conjecture as **Serre's uniformity problem**. It remains unknown if there exists a prime $\ell > 37$ and non-CM elliptic curve $E/\mathbf{Q}$ with $\rho_{E,\ell}(\mathrm{Gal}_k)$ conjugate to $C_{\mathrm{ns}}^+(\ell)$, the normalizer of a non-split Cartan subgroup of $\mathbf{GL}_2(\mathbf{Z}/\ell\mathbf{Z})$. If such a prime ℓ and elliptic curve E exists, then E gives rise to a representative for a degree one point on the modular curve $X_{C_{\mathrm{ns}}^+(\ell)}$. This curve has genus greater than 1, so Faltings's Theorem implies there are finitely many degree one points on $X_{C_{\mathrm{ns}}^+(\ell)}$, all of which are isolated.

The problem of bounding the non-surjective mod-ℓ images of Galois is situated within a broader effort, namely Mazur's "Program B" [14], which asks, given a number field k, for a classification of the images $\rho_E(\mathrm{Gal}_k)$ in $\mathbf{GL}_2(\widehat{\mathbf{Z}})$ for elliptic curves E/k. Work in this paper is related to the ℓ-adic case of the program, which is the problem of classifying images $\rho_{E,\ell^\infty}(\mathrm{Gal}_k)$ in $\mathbf{GL}_2(\mathbf{Z}_\ell)$. Determining the ℓ-adic image of Galois attached to an elliptic curve $E/\mathbf{Q}$ helps us to understand closed and isolated points with j-invariant $j(E)$ on modular curves, and vice-versa.

For each positive integer n, the modular curve $X_1(n)$ is a curve whose non-cuspidal points parametrize elliptic curves with a point of order n. Bourdon, Ejder, Liu, Odumodu, and Viray [2, Corollary 1.7] showed that, assuming an affirmative answer to Serre's uniformity problem, there are finitely many j-invariants in $\mathbf{Q}$ corresponding to isolated points on $X_1(n)$, as n varies over all positive integers.

It is natural to ask for which other families of curves we can obtain similar finiteness results. In this paper, we examine the set of j-invariants in an extension of bounded degree that arise as the j-invariant of an isolated point on a modular curve of prime-power level.

Theorem 1.1. *Fix a prime $\ell \in \mathbf{N}$ and define*

$$\mathcal{I} := \{\, x \in X_H \mid x \text{ is isolated and } H \le \mathbf{GL}_2(\mathbf{Z}_\ell) \text{ an open subgroup} \,\}.$$

For every $d \in \mathbf{N}$, there are finitely many j-invariants of degree d in $j(\mathcal{I})$.

The proof uses group theory techniques and results of Cadoret and Tamagawa [4] and Bourdon, Ejder, Liu, Odumodu, and Viray [2] to obtain a bound m, dependent on ℓ and d, such that for any isolated point on an ℓ-power level modular curve with j-invariant of degree d, there exists another isolated point on a modular curve of ℓ-power level dividing m having the same j-invariant. This method of reducing the level of modular curve on which an isolated point must appear is related to [2, Theorem 1.1], which states that for a non-CM elliptic curve with m-adic Galois representation of level M, for a certain integer m, the natural map $X_1(n) \to X_1(\gcd(n, M))$ sends isolated points to isolated points. We also use results of Terao [23] to describe the degrees of closed points $x \in X_H$ in terms of H and the adelic image of Galois associated to an elliptic curve corresponding to x defined over $\mathbf{Q}(j(x))$.

Remark 1.2. Terao [23, Theorem 1.5] showed that if x is a non-cuspidal, non-CM isolated point on a modular curve X_H, with $H \le \mathbf{GL}_2(\widehat{\mathbf{Z}})$ an open subgroup of level 7 and $j(x) \in \mathbf{Q}$, then $j(x) = 3^3 \cdot 5 \cdot 7^5/2^7$. Theorem 1.1 implies that there are finitely many j-invariants of fixed degree arising from isolated points on modular curves whose level is any power of 7.

Recall the modular curve $X_0(n)$ is a curve whose non-cuspidal points parametrize elliptic curves with a cyclic subgroup of n. In the case of prime-power level curves in the families $X_0(n)$ and $X_1(n)$, Bourdon and Ejder [1] give an unconditional result classifying the rational j-invariants arising from isolated points that may appear, which extends a partial classification given by Ejder [7]. Building on these results, we study the rational j-invariants arising from isolated points on prime-power level curves in the family $X_\Delta(n)$, which contains the families $X_0(n)$ and $X_1(n)$ and the so-called intermediate modular curves living between $X_0(n)$ and $X_1(n)$.

Theorem 1.3. *Let $\ell \in \mathbf{N}$ be a prime, let $n \in \mathbf{N}$ a positive number, and suppose $\Delta \leq (\mathbf{Z}/\ell^n\mathbf{Z})^\times$ is a subgroup containing -1. If $x \in X_\Delta(\ell^n)$ is a non-cuspidal isolated point with non-CM rational j-invariant $j \in \mathbf{Q}$, then j appears as the j-invariant of an isolated point on $X_0(\ell)$, for some $\ell \in \{11, 17, 37\}$.*

The proof uses the work of Bourdon and Ejder [1] and the work of Rouse, Sutherland, and Zureick-Brown [17] and Furio [10] studying ℓ-adic images of Galois attached to non-CM elliptic curves over $\mathbf{Q}$, to determine whether a rational j-invariant $j(E)$ associated to a non-CM elliptic curve $E/\mathbf{Q}$ may appear as the j-invariant of an isolated point on a modular curve $X_\Delta(\ell^n)$.

Remark 1.4. If we instead consider rational j-invariants arising from isolated points on $X_\Delta(n)$, for n not necessarily a prime power, then the analogue of Theorem 1.3 no longer holds, as noted by Lee [13, Theorem 2]. Specifically, Bourdon, Hashimoto, Keller, Klagsbrun, Lowry-Duda, Morrison, Najman, and Shukla [3, Theorem 2] showed that there is an isolated point on $X_1(28)$ with j-invariant $351/4$, but Lee found that there is no isolated point on a modular curve $X_0(n)$ with j-invariant $351/4$.

1.1. **Related work.** This is also related to work of Menendez [15, Theorem 5.3], who proved the analogue to [2, Theorem 1.1] for the family $X_0(n)$. Lee [13, Algorithm 1] gave an algorithm to test whether a non-CM j-invariant in $\mathbf{Q}$ corresponds to an isolated point on a modular curve $X_0(n)$. Among all elliptic curves in the LMFDB, the j-invariants found in [13, Table 1] are all of the possible j-invariants in $\mathbf{Q}$ corresponding to an isolated point on $X_0(n)$. Terao [23, Theorem 1.6] proved a very similar result, classifying the rational j-invariants arising from isolated points on $X_0(n)$, assuming a conjecture of Zywina [24, Section 14.3]. The j-invariants found by Lee in [13, Table 1] are the same j-invariants appearing in [23, Theorem 1.6].

1.2. **Outline.** In Section 2, we give the relevant background on elliptic curves, modular curves, and isolated points. In Section 3, we prove Theorem 1.1. In Section 4, we give an overview of the family of modular curves $X_\Delta(n)$. In Section 5, we prove Theorem 1.3.

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2. Background

2.1. **Notation and conventions.** Let k be a number field and $\overline{k}$ a fixed algebraic closure of k. We denote Gal_k to be the absolute Galois group $\mathrm{Gal}(\overline{k}/k)$. Let $\mathbf{N}$ denote the set of natural numbers, i.e. the set of positive integers $\{1, 2, 3, \dots\}$. For $n \in \mathbf{N}$, we denote by $\phi(n)$ the number of integers $1 \leq m \leq n$ with $(m, n) = 1$.

By a **nice** curve, we mean a smooth, projective, geometrically integral curve. For a k-scheme X and a field extension K of k, we denote X_K to be the base change $X \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, K$.

2.2. **Elliptic curves.** Let E be an elliptic curve over a number field k. For $n \in \mathbf{N}$, we denote $E[n] = E(\overline{k})[n]$ to be the subgroup of $E(\overline{k})$ consisting of n-torsion points—that is, points with finite order dividing n. The group $E[n]$ is the kernel of the multiplication-by-n map $[n] : E \to E$ and is isomorphic to $(\mathbf{Z}/n\mathbf{Z}) \times (\mathbf{Z}/n\mathbf{Z})$.

We say an elliptic curve E/k has **complex multiplication (CM)** if $\operatorname{End} E_{\overline{k}}$ is not isomorphic to the integers—we will not add the distinction of an elliptic curve having potential CM, as some authors do. We say a j-invariant $j \in \overline{\mathbf{Q}}$ is CM if there exists a CM elliptic curve E with j-invariant equal to j. In this paper, we are interested in showing there are finitely many j-invariants, with a certain property, in number fields of a fixed bounded degree. Our main techniques to do so work only for non-CM j-invariants. However, results from class field theory allow us to restrict to the case of non-CM j-invariants.

Indeed, if E/k has CM, then by [20, Corollary 9.4], there exists an imaginary quadratic field K and an order $\mathcal{O}$ of conductor f in K such that $\operatorname{End}(E_{\overline{k}}) \simeq \mathcal{O}$. By [5, Theorem 11.1], we have $[\mathbf{Q}(j(E)) : \mathbf{Q}] = [K(j(E)) : K] = h(\mathcal{O})$, where $h(\mathcal{O})$ is the class number. There are only finitely many K of a given class number by [12] and so, by [5, Theorem 7.24], there are only finitely many imaginary quadratic orders of a given class number. Each order is associated to only finitely many CM j-invariants by [5, Corollary 10.20]. These results imply the following well-known theorem.

Theorem 2.1. *For every $d \in \mathbf{N}$, there are only finitely many CM j-invariants in number fields of degree d.*

2.3. **Profinite constructions.** For a prime $\ell \in \mathbf{N}$, the ring of **ℓ-adic integers**, denoted $\mathbf{Z}_\ell$, is the inverse limit $\varprojlim \mathbf{Z}/\ell^n\mathbf{Z}$ taken over $n \in \mathbf{N}$ with homomorphisms the evaluation maps $\mathbf{Z}/\ell^j\mathbf{Z} \to \mathbf{Z}/\ell^i\mathbf{Z}$ for $i \leq j$. The **profinite integers**, denoted $\widehat{\mathbf{Z}}$, are the inverse limit $\varprojlim \mathbf{Z}/n\mathbf{Z}$ taken over the directed set consisting of natural numbers ordered by divisibility. There is a ring isomorphism

$$\widehat{\mathbf{Z}} = \varprojlim \mathbf{Z}/n\mathbf{Z} \simeq \prod_{\substack{\ell \in \mathbf{N} \\ \text{prime}}} \varprojlim \mathbf{Z}/\ell^n\mathbf{Z} = \prod_{\ell} \mathbf{Z}_\ell.$$

For an elliptic curve E over a number field k and a prime $\ell \in \mathbf{N}$, we define the **ℓ-adic Tate module of** E to be the inverse limit $T_\ell(E) := \varprojlim E[\ell^n]$ taken over $n \in \mathbf{N}$ with homomorphisms the maps $[\ell^{j-i}] : E[\ell^j] \to E[\ell^i]$ for $i \leq j$. As a $\mathbf{Z}_\ell$-module, $T_\ell(E)$ has the following structure:

$$T_\ell(E) = \varprojlim E[\ell^n] \simeq \varprojlim (\mathbf{Z}/\ell^n\mathbf{Z} \times \mathbf{Z}/\ell^n\mathbf{Z}) = \varprojlim (\mathbf{Z}/\ell^n\mathbf{Z}) \times \varprojlim (\mathbf{Z}/\ell^n\mathbf{Z}) = \mathbf{Z}_\ell \times \mathbf{Z}_\ell.$$

Following a similar construction to $\widehat{\mathbf{Z}}$, we also define the **adelic Tate module of** E to be the inverse limit $T(E) := \varprojlim E[n]$ so that

$$T(E) \simeq \prod_{\ell} T_\ell(E) \simeq \prod_{\ell} \mathbf{Z}_\ell \times \mathbf{Z}_\ell \simeq \widehat{\mathbf{Z}} \times \widehat{\mathbf{Z}}.$$

We will also study the profinite group $\mathbf{GL}_2(\widehat{\mathbf{Z}}) \simeq \varprojlim \mathbf{GL}_2(\mathbf{Z}/n\mathbf{Z})$. For each $n \in \mathbf{N}$, there exists a natural projection map $\pi_n : \mathbf{GL}_2(\widehat{\mathbf{Z}}) \to \mathbf{GL}_2(\mathbf{Z}/n\mathbf{Z})$ induced by the inverse limit. Similarly, by abuse of notation, for a prime number ℓ, we denote π_{ℓ^n} to be the natural projection map $\mathbf{GL}_2(\mathbf{Z}_\ell) \to \mathbf{GL}_2(\mathbf{Z}/\ell^n\mathbf{Z})$. For a subgroup $H \leq \mathbf{GL}_2(\widehat{\mathbf{Z}})$, we denote $H(n)$ to be $\pi_n(H)$. In general, we have $H \leq \pi_n^{-1}(H(n))$. If $H \leq \mathbf{GL}_2(\widehat{\mathbf{Z}})$ is an open subgroup, then there exists a positive integer $n \in \mathbf{N}$ such that $\ker \pi_n \leq H$. We define the **level of** H to be the least such positive integer. If H has level n, then $H = \pi_n^{-1}(H(n))$ and the index of H in $\mathbf{GL}_2(\widehat{\mathbf{Z}})$ (which is finite because H is open) is equal to the index of $H(n)$ in $\mathbf{GL}_2(\mathbf{Z}/n\mathbf{Z})$.

In Section 5, we will consider specific subgroups $H \leq \mathbf{GL}_2(\widehat{\mathbf{Z}})$ with $\det(H) = \widehat{\mathbf{Z}}^\times$. We will refer to these subgroups using the notation of Rouse, Sutherland, and Zureick-Brown [17, Section 2.4], which assigns to each subgroup H a label

`N.i.g.n`

where `N` is the level of H, `i` is the index of H in $\mathbf{GL}_2(\widehat{\mathbf{Z}})$, `g` is the genus of the modular curve X_H, and `n` is an integer which specifies H among other subgroups having the same level, index, and genus.

2.4. **Galois representations.** For every $\sigma \in \mathrm{Gal}_k$ and $P \in E$, we denote by P^σ the image of the pair (P, σ) under the natural (right) action $E \times \mathrm{Gal}_k \to E$. For every $P \in E[n]$, we have $[n]P = O$, where O is the base point of E. For every $\sigma \in \mathrm{Gal}_k$,

$$[n](P^\sigma) = ([n]P)^\sigma = O^\sigma = O,$$

so the natural action of Gal_k on $E[n]$ is well-defined. This action defines a representation

$$\rho_{E,n} : \mathrm{Gal}_k \to \mathrm{Aut}(E[n])$$

called the **mod-n Galois representation associated to** E. Let $\alpha : E[n] \to (\mathbf{Z}/n\mathbf{Z}) \times (\mathbf{Z}/n\mathbf{Z})$ be an isomorphism of $(\mathbf{Z}/n\mathbf{Z})$-modules. The **mod-n Galois representation associated to E and α** is the composition of the above representation with the isomorphism $\mathrm{Aut}(E[n]) \simeq \mathbf{GL}_2(\mathbf{Z}/n\mathbf{Z})$ induced by α and is denoted

$$\rho_{E,n,\alpha} : \mathrm{Gal}_k \to \mathrm{Aut}(E[n]) \xrightarrow{\simeq} \mathbf{GL}_2(\mathbf{Z}/n\mathbf{Z}).$$

If $\beta : E[n] \to (\mathbf{Z}/n\mathbf{Z}) \times (\mathbf{Z}/n\mathbf{Z})$ is another isomorphism of groups, then $\rho_{E,n,\alpha}(\mathrm{Gal}_k)$ and $\rho_{E,n,\beta}(\mathrm{Gal}_k)$ are conjugate in $\mathbf{GL}_2(\mathbf{Z}/n\mathbf{Z})$.

Suppose $\ell \in \mathbf{N}$ is a prime and $\alpha : T_\ell(E) \to \mathbf{Z}_\ell \times \mathbf{Z}_\ell$ is an isomorphism. The action of Gal_k on $E[\ell^n]$ for every $n \in \mathbf{N}$ induces an action on $T_\ell(E)$. This defines a representation

$$\rho_{E,\ell^\infty,\alpha} : \mathrm{Gal}_k \to \mathrm{Aut}(T_\ell(E)) \xrightarrow{\simeq} \mathbf{GL}_2(\mathbf{Z}_\ell)$$

called the **ℓ-adic Galois representation associated to E and α**. Similarly, the action of Gal_k on $E[n]$ for every $n \in \mathbf{N}$ extends to an action on $T(E)$. If $\alpha : T(E) \to \widehat{\mathbf{Z}} \times \widehat{\mathbf{Z}}$ is an isomorphism, then this action induces a representation

$$\rho_{E,\alpha} : \mathrm{Gal}_k \to \mathrm{Aut}(T(E)) \xrightarrow{\simeq} \mathbf{GL}_2(\widehat{\mathbf{Z}})$$

called the **adelic Galois representation associated to E and α**. Just as with the mod-n Galois representation, we may not keep track of the isomorphisms α if we only care about the image of Galois up to conjugation.

Serre showed [19, Théorème 3] that for a non-CM elliptic curve E/k, the adelic representation of Galois, $\rho_E : \mathrm{Gal}_k \to \mathbf{GL}_2(\widehat{\mathbf{Z}})$ has open image in $\mathbf{GL}_2(\widehat{\mathbf{Z}})$. Note that since the coset of an open subgroup of a profinite group is open, $\rho_{E,\alpha}(\mathrm{Gal}_k)$ is open regardless of the isomorphism α, as this only changes the image of Galois up to conjugation.

2.5. **Modular curves.** In essence, for H an open subgroup of $\mathbf{GL}_2(\mathbf{Z}/n\mathbf{Z})$ (resp., $\mathbf{GL}_2(\mathbf{Z}_\ell)$; $\mathbf{GL}_2(\widehat{\mathbf{Z}})$), the modular curve X_H is a curve whose non-cuspidal points parametrize elliptic curves having mod-n (resp., ℓ-adic; adelic) image of Galois contained in H. We now give a more precise definition.

Let H be a subgroup of $\mathbf{GL}_2(\mathbf{Z}/n\mathbf{Z})$ and let E be an elliptic curve defined over a number field k. We say two isomorphisms $\alpha, \alpha' : E[n] \to (\mathbf{Z}/n\mathbf{Z})\times(\mathbf{Z}/n\mathbf{Z})$ are H-**equivalent**, denoted $\alpha \sim_H \alpha'$ if there exists $h \in H$ such that $\alpha = h\circ\alpha'$. Let E be an elliptic curve over a number field k. An H-**level structure on** E is an equivalence class $[\alpha]_H$ of isomorphisms $\alpha : E[n] \to (\mathbf{Z}/n\mathbf{Z}) \times (\mathbf{Z}/n\mathbf{Z})$ under H-equivalence.

We define the modular curve Y_H (resp. X_H) to be the coarse moduli space of the stack $\mathcal{M}_H^0$ (resp. $\mathcal{M}_H$), which parametrizes elliptic curves (resp. generalized elliptic curves) with H-level structure. The modular curve Y_H is an affine subscheme of X_H. We call the s of $X_H - Y_H$ **cusps** and the closed points of Y_H **non-cuspidal points**. The curve X_H is a smooth, projective, integral curve over $\mathbf{Q}$ and is geometrically integral if and only if H has full determinant.

If H is an open subgroup of $\mathbf{GL}_2(\widehat{\mathbf{Z}})$ or $\mathbf{GL}_2(\mathbf{Z}_\ell)$ of level n, then we define the modular curve X_H to be the modular curve $X_{H(n)}$. If m is any positive integer divisible by n, then the modular curve $X_H = X_{H(n)}$ is isomorphic to the modular curve $X_{H(m)}$. Even if H does not contain $-I$, we always have that the modular curves X_H and $X_{\pm H}$ are isomorphic as curves. Because of this, we will only consider subgroups H containing $-I$.

We give a precise description of the geometric non-cuspidal points of X_H. The set $Y_H(\overline{k})$ consists of equivalence classes of pairs $(E, [\alpha]_H)$, where E/k is an elliptic curve and $[\alpha]_H$ is an H-level structure on E. We say two pairs $(E, [\alpha]_H)$ and $(E', [\alpha']_H)$ are equivalent if there exists an isomorphism $\phi : E \to E'$ such that the induced isomorphism $\phi : E[n] \to E'[n]$ satisfies $\alpha \sim_H \alpha'\circ\phi$, i.e. $\alpha = h\circ\alpha'\circ\phi$, for some $h \in H$.

The absolute Galois group Gal_k admits a right action on $Y_H(\overline{k})$ as follows: For an automorphism $\sigma \in \mathrm{Gal}_k$ and a point $[(E, [\alpha]_H)] \in Y_H(\overline{k})$, we define

$$[(E, [\alpha]_H)] \cdot \sigma = [(E^\sigma, [\alpha\circ\sigma^{-1}]_H)].$$

This action is well-defined: Indeed, if $(E', [\alpha']_H), [(E, [\alpha]_H)] \in Y_H(\overline{k})$ and $\sigma \in \mathrm{Gal}_k$, then there exists an isomorphism $\phi : E[n] \to E'[n]$ and an element $h \in H$ such that $\alpha = h\circ\alpha'\circ\phi$. This implies

$$\alpha\circ\sigma^{-1} = (h\circ\alpha'\circ\phi)\circ\sigma^{-1} = h\circ(\alpha'\circ\sigma^{-1})\circ(\sigma\circ\phi\circ\sigma^{-1}).$$

The map $\sigma\circ\phi\circ\sigma^{-1} : E^\sigma[n] \to E'^\sigma[n]$ given by $P \mapsto \phi(P^{\sigma^{-1}})^\sigma$ is an isomorphism, so

$$((E', [\alpha']_H) \cdot \sigma) = (E'^\sigma, [\alpha'\circ\sigma^{-1}]_H) \in [(E^\sigma, [\alpha\circ\sigma^{-1}]_H)] = [(E, [\alpha]_H) \cdot \sigma].$$

A **closed point** of X_H is a point $x \in X_H$ such that $\{x\}$ is Zariski closed in X and the **degree** (over $\mathbf{Q}$) of the closed point, denoted $\deg(x)$, is the degree of the residue field of x over $\mathbf{Q}$. By [16, Proposition 2.4.6], there is a one-to-one correspondence between closed points of X_H and Gal_k-orbits of points in $X_H(\overline{k})$ and the degree of a closed point corresponds to the size of the Gal_k-orbit. We will often refer to such a Gal_k-orbit as a closed point.

We say a point in $X_H(\overline{k})$ is k-**rational** if it is fixed by every element of Gal_k. Equivalently, a k-rational point is a closed point of degree 1.

We say a pair $(E, [\alpha]_H)$, consisting of an elliptic curve $E/\mathbf{Q}(j(E))$ and an H-level structure $[\alpha]_H$ on E, is a **minimal representative** for a closed point $x \in X_H$ if x corresponds to the $\mathrm{Gal}_{\mathbf{Q}}$-orbit of $[(E, [\alpha]_H)] \in X_H(\overline{\mathbf{Q}})$. A minimal representative for a non-cuspidal closed point always exists by [23, Lemma 4.9].

Theorem 2.2 ([23, Theorem 4.24])**.** *Let $H \leq \mathbf{GL}_2(\widehat{\mathbf{Z}})$ be an open subgroup and let $x \in X_H$ be a non-cuspidal closed point with minimal representative $(E, [\alpha]_H)$ defined over $k := \mathbf{Q}(j(E))$. Let $A_{E,\alpha} \leq \mathbf{GL}_2(\widehat{\mathbf{Z}})$ be the subgroup $\{\alpha\circ\phi\circ\alpha^{-1} \mid \phi \in \mathrm{Aut}(E_{\overline{k}})\}$. Then*

$$\deg(x) = [\mathbf{Q}(j(E)) : \mathbf{Q}][\rho_{E,\alpha}(\mathrm{Gal}_k)A_{E,\alpha} : \rho_{E,\alpha}(\mathrm{Gal}_k)A_{E,\alpha} \cap A_{E,\alpha}H].$$

If $j(x) \neq 0, 1728$—in particular, if E is non-CM—then $\mathrm{Aut}(E_{\overline{k}}) = \{\pm 1\}$, so $A_{E,\alpha} = \{\pm I\}$. Since we are assuming H contains $-I$, we get that $A_{E,\alpha}H = \{\pm I\}H = H$. This yields:

Corollary 2.3. *Let $H \leq \mathbf{GL}_2(\widehat{\mathbf{Z}})$ be an open subgroup containing $-I$ and let $x \in X_H$ be a non-cuspidal closed point with minimal representative $(E, [\alpha]_H)$ defined over $k := \mathbf{Q}(j(E))$ such that $j(E) \neq 0, 1728$. Then*

$$\deg(x) = [\mathbf{Q}(j(E)) : \mathbf{Q}][\rho_{E,\alpha}(\mathrm{Gal}_k) : \rho_{E,\alpha}(\mathrm{Gal}_k) \cap H].$$

If H_1 and H_2 are subgroups of $\mathbf{GL}_2(\mathbf{Z}/n\mathbf{Z})$ containing $-I$ with $H_1 \leq H_2$, then there is a natural inclusion map $X_{H_1} \to X_{H_2}$. Indeed, if $(E, [\alpha]_{H_1})$ is a representative for a non-cuspidal point in $Y_{H_1}(\overline{k})$, then $[\alpha]_{H_1}$ defines an H_2-level structure on E because $\alpha \sim_{H_1} \alpha'$ implies $\alpha \sim_{H_2} \alpha'$. We summarize a result from [6], which describes the degree of this inclusion map.

Theorem 2.4 ([6, Page 66])**.** *Let $H_1, H_2 \leq \mathbf{GL}_2(\widehat{\mathbf{Z}})$ be open subgroups containing -1. Suppose $H_1 \leq H_2$ and denote f to be the natural map $X_{H_1} \to X_{H_2}$ of curves over $\mathbf{Q}$. Then* $\deg(f) = [H_2 : H_1]$.

2.6. **Isolated points.** Let C be a smooth, projective curve over a number field k. Let $\mathbf{Pic}_{C/k}$ be the Picard scheme of C and let $\mathbf{Pic}^0_{C/k}$ be the connected component of the identity. Denote $\mathbf{Div}_{C/k}$ to be the divisor scheme of C. Denote by $\mathbf{A}_{C/k} : \mathbf{Div}_{C/k} \to \mathbf{Pic}_{C/k}$ the Abel map, and by $\mathbf{W}_{C/k}$ its image. We refer the reader to [23, Section 2] for more details.

Let $x \in C$ be a closed point. By taking x to be a sum of Gal_k-conjugates, we can view x as an element of $\mathbf{Div}_{C/k}(k)$.

(1) We say x is **$\mathbf{P}^1$-parametrized** if there exists $x' \in \mathbf{Div}_{C/k}(k)$ with $x' \neq x$ such that $\mathbf{A}_{C/k}(x) = \mathbf{A}_{C/k}(x')$. We say x is **$\mathbf{P}^1$-isolated** if it is not $\mathbf{P}^1$-parametrized.
(2) We say x is **AV-parametrized** if there exists a positive rank abelian subvariety $A \subseteq \mathbf{Pic}^0_C$ such that $\mathbf{A}_{C/k}(x) + A \subseteq \mathbf{W}_{C/k}$. We say x is **AV-isolated** if it is not AV-parametrized.
(3) We say x is **isolated** if it is both $\mathbf{P}^1$-isolated and AV-isolated.
(4) We say x is **sporadic** if there are finitely many points $y \in C$ with $\deg(y) \leq \deg(x)$.

The study of isolated points is motivated by Faltings's theorem [8] which says that if C/k has genus strictly greater than 1, then $C(k)$ is finite and hence every point in $C(k)$ is sporadic. Indeed, it was shown in [2, Theorem 4.2] that on C/k, every sporadic point is isolated and there are only finitely many isolated points. So, if C/k has genus strictly greater than 1, the study of isolated points is a natural extension of the study of rational points.

The following theorem generalizes [7, Lemma 2.3] and will be particularly useful to us in Section 5.

Theorem 2.5 ([23, Theorem 2.17])**.** *Let C be a smooth, projective curve over a number field k and let $K := k(C) \cap \overline{k}$. Let $r = [K : k]$ be the number of geometric components of C and let g be the genus of C_K. If $x \in C$ is a closed point with* $\deg(x) > rg$*, then x is $\mathbf{P}^1$-parametrized.*

For a finite locally free map of curves, we have the following inequality:

Lemma 2.6. *Let C and D be smooth, projective curves over a number field k and let $f : C \to D$ be a finite locally free map. If $x \in C$ is a closed point, then*

$$\deg(x) \leq \deg(f)\deg(f(x)).$$

Proof. Let $x \in C$ be a closed point. Let $f^* : \mathbf{Div}_{D/k} \to \mathbf{Div}_{C/k}$ be the pullback map induced by f and let $f_* : \mathbf{Div}_{C/k} \to \mathbf{Div}_{D/k}$ be the pushforward map induced by f. By [11, Proposition 21.10.4], we have

$$\deg(f^*(f(x))) = \sum_{x' \in f^{-1}(f(x))} e_f(x')\deg(x'),$$

where $e_f(x')$ is the ramification index of f at x'. By [11, Proposition 21.10.18], we have $f_*(f^*(f(x))) = \deg(f)f(x)$, so

$$\deg(f^*(f(x))) = \deg(f_*(f^*(f(x)))) = \deg(f)\deg(f(x)).$$

This implies

$$\deg(x) \leq \sum_{x' \in f^{-1}(f(x))} e_f(x')\deg(x') = \deg(f)\deg(f(x)).$$ ∎

If $f : C \to D$ is a finite map of curves $x \in C$ is an isolated point, it is not true in general that the image $f(x)$ is an isolated point of D. However, if the degree is as large as possible—that is, if $\deg(x) = \deg(f)\deg(f(x))$—then we can in fact conclude that $f(x)$ is isolated. This result was first proved for nice curves in [2, Theorem 4.3] and then generalized by Terao to the setting of smooth, projective curves.

Theorem 2.7 ([23, Theorem 2.15])**.** *Let $f : C \to D$ be a finite locally free map of smooth, projective curves over a number field k. If $x \in C$ and $y \in D$ are closed points such that $y = f(x)$ and $\deg(x) = \deg(y)\deg(f)$, then the following hold:*

(1) If x is $\mathbf{P}^1$-isolated, then y is $\mathbf{P}^1$-isolated.
(2) If x is AV-isolated, then y is AV-isolated.

In particular, if x is isolated, then y is also isolated.

3. Modular curves of prime-power level

In this section we prove Theorem 1.1. Our strategy is to bound the level of modular curves on which a non-CM isolated point with j-invariant of fixed degree must appear.

Proposition 3.1. *Fix positive integers $\ell, m \in \mathbf{N}$ with ℓ prime. Let $H \leq \mathbf{GL}_2(\mathbf{Z}_\ell)$ be an open subgroup containing $-I$ of level ℓ^n with $n \geq m$. Suppose $x \in X_H$ is a non-cuspidal closed point with minimal representative $(E, [\alpha]_H)$ such that E is non-CM and $\rho_{E,\alpha,\ell^\infty}(\mathrm{Gal}_{\mathbf{Q}(j(E))}) \leq \mathbf{GL}_2(\mathbf{Z}_\ell)$ has level dividing ℓ^m. Let $H' := \pi_{\ell^m}^{-1}(H(\ell^m))$ and let f denote the natural map $X_H \to X_{H'}$. Then $\deg(x) = \deg(f)\deg(f(x))$.*

3.1. Preliminary results.

We shall need the following lemma.

Lemma 3.2. *Let H be a subgroup of a group G and let $f : G \to G'$ be a surjective homomorphism of groups.*

(1) If H has finite index in G, then $[G : H] \geq [f(G) : f(H)]$.
(2) If $H' := f^{-1}(f(H))$ and $K := \ker(f)$, then $H' = HK$.

Proof.

(1) Let G/H denote the set of all cosets of H in G. Define a function $\phi : G/H \to f(G)/f(H)$ by $xH \mapsto f(x)f(H)$. First, we show that ϕ is well-defined. Suppose $xH = yH$. Then $y^{-1}x \in H$, so $f(y^{-1}x) \in f(H)$. Then

$$f(H) = f(y^{-1}x)f(H) = f(y)^{-1}f(x)f(H),$$

which implies

$$\phi(xH) = f(x)f(H) = f(y)f(H) = \phi(yH).$$

Thus, ϕ is well-defined. Moreover, ϕ is surjective because f is surjective. Hence, we conclude

$$[G : H] = \#(G/H) \geq \#(f(G)/f(H)) = [f(G) : f(H)].$$

(2) First, we show $HK \subseteq H'$. Let $x \in HK$ be given and suppose $x = hk$ for some $h \in H$ and $k \in K$. Then

$$f(x) = f(hk) = f(h)f(k) = f(h) \in f(H).$$

Thus, $x \in H'$.

Now we show $H' \subseteq HK$. Let $x \in H'$ be given. Then $f(x) \in f(H)$, so there exists $h \in H$ with $f(x) = f(h)$. Notice

$$1 = f(x)f(h)^{-1} = f(xh^{-1}),$$

so $xh^{-1} \in K$. Thus, $x \in KH = HK$. ■

3.2. **Proof of Proposition 3.1.** By Lemma 2.6, we know $\deg(x) \leq \deg(f)\deg(f(x))$, so it suffices to show

$$\frac{\deg(x)}{\deg(f(x))} \geq \deg(f). \tag{1}$$

Let π_{ℓ^∞} denote the projection

$$\mathbf{GL}_2(\widehat{\mathbf{Z}}) \xrightarrow{\simeq} \prod_p \mathbf{GL}_2(\mathbf{Z}_p) \to \mathbf{GL}_2(\mathbf{Z}_\ell).$$

Since H contains $-I$, Theorem 2.4 implies $\deg(f) = [\pi_{\ell^\infty}^{-1}(H') : \pi_{\ell^\infty}^{-1}(H)] = [H' : H]$. Then

$$\begin{aligned}
\frac{\deg(x)}{\deg(f(x))} &= \frac{[\mathbf{Q}(j(E)) : \mathbf{Q}][\operatorname{im}\rho_{E,\alpha} : \operatorname{im}\rho_{E,\alpha} \cap \pi_{\ell^\infty}^{-1}(H)]}{[\mathbf{Q}(j(E)) : \mathbf{Q}][\operatorname{im}\rho_{E,\alpha} : \operatorname{im}\rho_{E,\alpha} \cap \pi_{\ell^\infty}^{-1}(H')]} && \text{(By Corollary 2.3)} \\
&= [\operatorname{im}\rho_{E,\alpha} \cap \pi_{\ell^\infty}^{-1}(H') : \operatorname{im}\rho_{E,\alpha} \cap \pi_{\ell^\infty}^{-1}(H)] \\
&\geq [\pi_{\ell^\infty}(\operatorname{im}\rho_{E,\alpha} \cap \pi_{\ell^\infty}^{-1}(H')) : \pi_{\ell^\infty}(\operatorname{im}\rho_{E,\alpha} \cap \pi_{\ell^\infty}^{-1}(H))] && \text{(By Lemma 1.3.2)} \\
&= [\operatorname{im}\rho_{E,\alpha,\ell^\infty} \cap H' : \operatorname{im}\rho_{E,\alpha,\ell^\infty} \cap H].
\end{aligned}$$

Denote $R := \rho_{E,\alpha,\ell^\infty}(\mathrm{Gal}_{\mathbf{Q}(j(E))})$ and consider the following subset lattice:

$$\begin{tikzcd}
 & & R\cap H \arrow[dll, dash] \arrow[d, dash] \arrow[drr, dash] & & \\
H \arrow[d, dash] \arrow[drr, dash] & & R\cap H' \arrow[dll, dash] \arrow[drr, dash] & & \\
H' \arrow[ddrr, dash] & & & & R \arrow[dll, dash] \arrow[ddll, dash] \\
 & & HR \arrow[d, dash] & & \\
 & & H'R \arrow[d, dash] & & \\
 & & \mathbf{GL}_2(\mathbf{Z}_\ell) & &
\end{tikzcd}$$

The set HR need not be a group. However, HR is a union of left cosets of R, the number of which we denote by $[HR : R]$. By assumption, R and H both have finite index in $\mathbf{GL}_2(\mathbf{Z}_\ell)$, so all subgroups in the above lattice have finite index. The map $H/(R\cap H) \to HR/R$ defined by $h(R\cap H) \mapsto hR$ is a well-defined bijection of sets, so $[H : R\cap H] = [HR : R]$. Similarly, we have $[H' : R\cap H'] = [H'R : R]$.

Notice that Lemma 3.2.2 implies $H' = H(\ker \pi_{\ell^m})$. Moreover, R contains $\ker \pi_{\ell^m}$ because R has level dividing ℓ^m. This implies $H'R = H(\ker \pi_{\ell^m})R = HR$. Then,

$$\begin{aligned}
[R\cap H' : R\cap H] &= [H' : H]\frac{[H : R\cap H]}{[H' : R\cap H']} \\
&= [H' : H]\frac{[HR : R]}{[H'R : R]} \\
&= [H' : H]\frac{[HR : R]}{[HR : R]} \\
&= [H' : H].
\end{aligned}$$

Thus, we have verified inequality (1).

3.3. **Proof of Theorem 1.1.** The proof of Theorem 1.1 relies on a result of Cadoret and Tamagawa [4, Theorem 1.1], which, for a fixed prime ℓ and a fixed positive integer d, gives a uniform bound on the level of $\mathrm{im}\,\rho_{E,\ell^\infty}$, as E varies over all non-CM elliptic curves defined over number fields of degree d.

Theorem 3.3. *Fix a prime $\ell \in \mathbf{N}$ and define*

$$\mathcal{I} := \{\, x \in X_H \,|\, x \text{ is isolated and } H \le \mathbf{GL}_2(\mathbf{Z}_\ell) \text{ an open subgroup} \,\}.$$

For every $d \in \mathbf{N}$, there are finitely many j-invariants of degree d in $j(\mathcal{I})$.

Proof. Let $x \in \mathcal{I}$ be given. Then $x \in X_H$ is isolated for some subgroup $H \le \mathbf{GL}_2(\mathbf{Z}_\ell)$ of level ℓ^n. Since X_H and $X_{\pm H}$ are isomorphic as curves, we can assume H contains $-I$. Define $k := \mathbf{Q}(j(x))$ and $d := [k : \mathbf{Q}]$. Let $(E, [\alpha]_H)$ be a minimal representative for x. By Theorem 2.1, there are only finitely many CM j-invariants in number fields of degree d. As such, it suffices to consider the case where E is non-CM.

By [4, Theorem 1.1], there exists a number $m \in \mathbf{N}$ such that for every degree d number field k' and every non-CM elliptic curve E'/k', the image of the ℓ-adic Galois representation associated to E'/k',

$$\rho_{E',\ell^\infty} : \mathrm{Gal}_{k'} \to \mathrm{Aut}(T_\ell(E)) \simeq \mathbf{GL}_2(\mathbf{Z}_\ell)$$

has level dividing ℓ^m.

For each $n' \leq m$, there exist finitely many subgroups of $\mathbf{GL}_2(\mathbf{Z}/\ell^{n'}\mathbf{Z})$ and thus finitely many modular curves of level $\ell^{n'}$, each of which has finitely many isolated points by [2, Theorem 4.2]. So, we may assume $n > m$.

Let $H' \leq \mathbf{GL}_2(\mathbf{Z}_\ell)$ be the subgroup $\pi_{\ell^m}^{-1}(H(\ell^m))$ and let $f : X_H \to X_{H'}$ denote the natural inclusion map. By Proposition 3.1,

$$\deg(x) = \deg(f)\deg(f(x)).$$

Then Theorem 2.7 implies $f(x)$ is isolated.

Thus, $j(x)$ is the j-invariant of an isolated point on a modular curve of level dividing ℓ^m. There are only finitely many such modular curves and each has finitely many isolated points. Since m was dependent only on d, we conclude $j(\mathcal{I})$ contains finitely many j-invariants of degree d. ■

It is natural to ask if Theorem 3.3 holds as ℓ varies over all primes. This is in fact a stronger claim than an affirmative answer to Serre's uniformity problem. Indeed, suppose the set $\mathcal{J}$ of rational isolated j-invariants on modular curves of prime-power level is finite. In particular, this implies there are only finitely many rational j-invariants corresponding to isolated points on $X_{C_{\mathrm{ns}}^+(\ell)}$, as ℓ ranges over all prime numbers. For each non-CM $j \in \mathcal{J}$, there exists a non-CM elliptic curve $E/\mathbf{Q}$ with $j(E) = j$. Serre's open image theorem [19, Théorème 3] ensures that for sufficiently large primes ℓ, the mod-ℓ representation of Galois associated to E is surjective. So, there are only finitely-many primes ℓ with $\rho_{E,\ell}(\mathrm{Gal}_{\mathbf{Q}})$ conjugate to a subgroup of $C_{\mathrm{ns}}^+(\ell)$. This implies that j arises as an isolated point on only finitely many modular curves $X_{C_{\mathrm{ns}}^+(\ell)}$, each of which has finitely many isolated points.

4. Intermediate modular curves

Let n be a positive integer and let Δ be a subgroup of $(\mathbf{Z}/n\mathbf{Z})^\times$. We define a subgroup of $\mathbf{GL}_2(\mathbf{Z}/n\mathbf{Z})$ as follows:

$$B_\Delta(n) := \left\{ \begin{pmatrix} \delta & a \\ 0 & b \end{pmatrix} \,\middle|\, \delta \in \Delta,\ a \in \mathbf{Z}/n\mathbf{Z},\ b \in (\mathbf{Z}/n\mathbf{Z})^\times \right\}.$$

Moreover, we denote $B_1(n)$ (resp. $B_{\pm 1}(n)$; $B_0(n)$) to be $B_\Delta(n)$, with $\Delta = \{1\}$ (resp. $\Delta = \{\pm 1\}$; $\Delta = (\mathbf{Z}/n\mathbf{Z})^\times$).

We will make frequent use of the following formulae.

Proposition 4.1. *Let $n \in \mathbf{N}$ have prime factorization $p_1^{a_1} \cdots p_r^{a_r}$ and let Δ be a subgroup of $(\mathbf{Z}/n\mathbf{Z})^\times$.*

(1) $\#\,\mathbf{GL}_2(\mathbf{Z}/n\mathbf{Z}) = \prod_{i=1}^r p_i^{4a_i-3}(p_i^2-1)(p_i-1)$.
(2) $\#B_\Delta(n) = (\#\Delta)\prod_{i=1}^r p_i^{2a_i-1}(p_i-1)$.
(3) $[\mathbf{GL}_2(\mathbf{Z}/n\mathbf{Z}) : B_\Delta(n)] = \frac{1}{\#\Delta}\prod_{i=1}^r p_i^{2a_i-2}(p_i^2-1)$.

Proof. Formula (1) follows from the short exact sequence

$$1 \to \mathbf{SL}_2(\mathbf{Z}/n\mathbf{Z}) \hookrightarrow \mathbf{GL}_2(\mathbf{Z}/n\mathbf{Z}) \overset{\det}{\twoheadrightarrow} (\mathbf{Z}/n\mathbf{Z})^\times \to 1,$$

formula (2) follows from the definition of $B_\Delta(n)$, and formula (3) follows from Lagrange's theorem. ■

It will occasionally be useful to rewrite Proposition 4.1.3 as

$$[\mathbf{GL}_2(\mathbf{Z}/n\mathbf{Z}) : B_\Delta(n)] = \frac{\phi(n)}{\#\Delta}[\mathbf{GL}_2(\mathbf{Z}/n\mathbf{Z}) : B_0(n)].$$

Let n be a positive integer and let Δ be a subgroup of $(\mathbf{Z}/n\mathbf{Z})^\times$ be given. We define the **modular curve** $X_\Delta(n)$ to be the modular curve $X_{B_\Delta(n)}$. If Δ is such that $\{\pm 1\} \subsetneq \Delta \subsetneq (\mathbf{Z}/n\mathbf{Z})^\times$, we call $X_\Delta(n)$ an **intermediate modular curve**, since it is between $X_1(n)$ and $X_0(n)$. If Δ is a subgroup of $(\mathbf{Z}/n\mathbf{Z})^\times$ and m divides n, then we may write $X_\Delta(m)$ to mean the modular curve $X_{B_\Delta(n)}(m)$, with the understanding that we reduce Δ modulo m.

We briefly describe another construction of $X_\Delta(n)$. Define a congruence subgroup associated to Δ as follows:

$$\Gamma_\Delta(n) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}_2(\mathbf{Z}) \,\middle|\, c \equiv 0 \mod n\,,\ (a \mod n) \in \Delta \right\}.$$

The group $\Gamma_\Delta(n)$ inherits from $\mathbf{SL}_2(\mathbf{Z})$ the action on the upper half plane $\mathbf{H}$ by linear fractional transformations. This action can be extended to $\mathbf{H}^* := \mathbf{H} \cup \mathbf{P}^1(\mathbf{Q})$, the extended upper half plane, and the quotient space $\mathbf{H}^*/\Gamma_\Delta(n)$ is a Riemann surface. There exists a smooth projective curve $X(\Gamma_\Delta(n))/\mathbf{Q}$ and a complex analytic isomorphism $\mathbf{H}^*/\Gamma_\Delta(n) \to X(\Gamma_\Delta(n))(\mathbf{C})$ (c.f. [20, Remark C.13.2]). Moreover, the curves $X(\Gamma_\Delta(n))$ and $X_\Delta(n)$ are isomorphic over $\mathbf{Q}$. Note that $\mathbf{H}^*$ is fixed under the action by $-I \in \mathbf{SL}_2(\mathbf{Z})$, so for any subgroup $\Delta < (\mathbf{Z}/n\mathbf{Z})^\times$, the curves $X_\Delta(n)$ and $X_{\pm\Delta}(n)$ are isomorphic. For this reason, in this paper we only consider subgroups Δ containing -1.

For a point P of order n on an elliptic curve E, we define ΔP to be the set $\{\delta P \mid \delta \in \Delta\}$ and $\mathbf{Q}(\Delta P)$ to be the fixed field of $\left\{\sigma \in \mathrm{Gal}_\mathbf{Q} \,\middle|\, \sigma(\Delta P) = \Delta P\right\}$. Then the non-cuspidal points of $X_\Delta(n)(\overline{\mathbf{Q}})$ parametrize equivalence classes of pairs $(E, \Delta P)$, where E is an elliptic curve and $P \in E$ is a point of order n. Moreover, with this construction, every non-cuspidal k-rational point of $X_\Delta(n)$ is of the form $[(E, \Delta P)]$, with E/k an elliptic curve and $P \in E(k)$ a point of order n such that ΔP is fixed by Gal_k.

Lemma 4.2. *Let $n \in \mathbf{N}$ and $\{\pm 1\} \leq \Delta \leq (\mathbf{Z}/n\mathbf{Z})^\times$ be given. Let $E/\mathbf{Q}(j(E))$ be a non-CM elliptic curve and let $P \in E$ be a point of order n. If $x \in X_\Delta(n)$ is the closed point associated to $(E, \Delta P)$, then*

$$\deg(x) = [\mathbf{Q}(j(E), \Delta P) : \mathbf{Q}].$$

Proof. Let $\mathcal{S} \leq \mathrm{Gal}_\mathbf{Q}$ denote the stabilizer of the action of $\mathrm{Gal}_\mathbf{Q}$ on $[(E, \Delta P)]$. Since E is defined over $\mathbf{Q}(j(E))$, we have $\sigma E = E$ for all $\sigma \in \mathrm{Gal}_{\mathbf{Q}(j(E))}$. Moreover, if $\sigma \in \mathrm{Gal}_\mathbf{Q}$ fixes E, then there exists an isomorphism $\phi : E \to \sigma E$ defined over $\overline{\mathbf{Q}}$. This implies $j(E) = j(\sigma E)$, so $\sigma \in \mathrm{Gal}_{\mathbf{Q}(j(E))}$. We then have

$$\begin{aligned} \mathcal{S} &= \left\{\sigma \in \mathrm{Gal}_\mathbf{Q} \,\middle|\, \sigma[(E, \Delta P)] = [(E, \Delta P)]\right\} \\ &= \left\{\sigma \in \mathrm{Gal}_\mathbf{Q} \,\middle|\, \sigma E = E\right\} \cap \left\{\sigma \in \mathrm{Gal}_\mathbf{Q} \,\middle|\, \sigma(\Delta P) = \Delta P\right\} \\ &= \mathrm{Gal}_{\mathbf{Q}(j(E))} \cap \mathrm{Gal}_{(\mathbf{Q}(\Delta P))}\,. \end{aligned}$$

By the orbit-stabilizer theorem and Galois theory, we conclude

$$[\mathbf{Q}(x) : \mathbf{Q}] = [\mathrm{Gal}_\mathbf{Q} : \mathcal{S}] = [\mathbf{Q}^\mathcal{S} : \mathbf{Q}] = [\mathbf{Q}(j(E))\mathbf{Q}(\Delta P) : \mathbf{Q}] = [\mathbf{Q}(j(E), \Delta P) : \mathbf{Q}]. \qquad \blacksquare$$

5. Intermediate modular curves of prime-power level

5.1. **Rational isolated j-invariants on $X_\Delta(\ell^n)$.** In this section, we consider the rational j-invariants that arise from isolated points on intermediate modular curves of prime-power level. Ejder [7] proved that there are finitely many rational j-invariants arising from isolated points on $X_1(\ell^n)$, ranging over all primes $\ell > 7$, and gave a partial classification of which rational j-invariants may appear. Terao [23, Theorem 1.5] showed that if $H \leq \mathbf{GL}_2(\widehat{\mathbf{Z}})$ has level 7 and $x \in X_H$ is an isolated point with $j(x)$ rational and non-CM, then $j(x) = 3^3 \cdot 5 \cdot 7^5/2^7$ and H is conjugate to one of nine known subgroups. Bourdon and Ejder [1] completed the classification begun by Ejder [7] for $X_1(\ell^n)$ and proved an analogous result for the family of modular curves $X_0(\ell^n)$.

Theorem 5.1 ([1, Theorems 1 and 2.]). *Let ℓ be a prime and let $j \in \mathbf{Q}$ be a rational number.*

1. *There exists an isolated point $x \in X_1(\ell^n)$ with $j(x) = j$ for some $n \in \mathbf{N}$ if and only if j is a CM j-invariant, $-7 \cdot 11^3$, or $-7 \cdot 137^3 \cdot 2083^3$. Moreover, the non-CM j-invariants occur if and only if $\ell = 37$.*
2. *There exists an isolated point $x \in X_0(\ell^n)$ with $j(x) = j$ for some $n \in \mathbf{N}$ if and only if j is a CM j-invariant, $-11 \cdot 131^3$, -11^2, $-17^2 \cdot 101^3/2$, $-17 \cdot 373^3/2^{17}$, $-7 \cdot 11^3$, or $-7 \cdot 137^3 \cdot 2083^3$. Moreover, the non-CM j-invariants in this list correspond to isolated rational points on $X_0(\ell)$.*

Our goal for this section is to consider the above theorem in the setting of intermediate modular curves of prime-power level. Notice that in the above theorem, the non-CM rational j-invariants arising from isolated points on $X_1(\ell^n)$ all appear as j-invariants arising from isolated points on $X_0(\ell)$. We will see that any rational j-invariant corresponding to an isolated point on a modular curve $X_\Delta(\ell^n)$ must appear in the list of rational isolated j-invariants arising from $X_0(\ell)$.

5.2. **Possible ℓ-adic images of Galois.** We shall need a classification by Rouse, Sutherland, and Zureick-Brown [17] of the possible ℓ-adic images of Galois associated to non-CM elliptic curves over $\mathbf{Q}$. The classification builds on work of Sutherland and Zywina [22] and of Rouse and Zureick-Brown [18].

Theorem 5.2 ([17, Theorem 1.1.6]). *Let $E/\mathbf{Q}$ be a non-CM elliptic curve, let ℓ be a prime, and denote $R := \rho_{E,\ell^\infty}(\mathrm{Gal}_{\mathbf{Q}})$. Then exactly one of the following is true:*

- (a) *The modular curve X_R is isomorphic to $\mathbf{P}^1$ or a rank one elliptic curve.*
- (b) *The modular curve X_R has an exceptional rational point for known R.*
- (c) *R is conjugate to a subgroup of $C_{ns}^+(3^3), C_{ns}^+(5^2), C_{ns}^+(7^2), C_{ns}^+(11^2)$ or $C_{ns}^+(\ell)$ for $\ell \geq 19$.*
- (d) *R is conjugate to a subgroup of* `49.147.9.1` *or* `49.196.9.1`.

For a non-CM j-invariant attached to an elliptic curve $E/\mathbf{Q}$, Theorem 5.2 gives us four cases to consider. If R is as in case (a), the possible subgroups that can arise as R are all known. Specifically, [22, Corollary 1.6] says that for $\ell = 2, 3, 5, 7, 11, 13$, there are $1201, 47, 23, 15, 2, 11$ subgroups of $\mathbf{GL}_2(\mathbf{Z}_\ell)$ that can arise as R, respectively. And for $\ell > 13$, the only possible subgroup is $R = \mathbf{GL}_2(\mathbf{Z}_\ell)$. If R is as in case (b), there are 23 possible known images for R, which have levels $2^4, 2^3, 5^2, 7, 11, 13, 17, 37$. It is conjectured [17, Conjecture 1.1.5] that these known subgroups are all of the exceptional groups of prime-power level. If R is as in case (d) and is conjugate to a subgroup of `49.196.1`, then Bourdon and Ejder [1, Proposition 1] showed that R must be conjugate to `49.196.9.1`.

Proposition 5.3 ([7, Proposition 3.1], [1, Proposition 4]). *Let $E/\mathbf{Q}$ be a non-CM elliptic curve. Suppose $\rho_{E,\ell}(\mathrm{Gal}_{\mathbf{Q}})$ is conjugate to the normalizer of a nonsplit Cartan subgroup for some prime $\ell > 13$. For $P \in E(\overline{\mathbf{Q}})$ of order ℓ^n, let $x = [(E,P)] \in X_1(\ell^n)$ be the associated closed point on the modular curve. Then*

$$\deg(x) = \frac{1}{2}(\ell^2 - 1)\ell^{2n-2} = \deg(X_1(\ell^n) \to X(1)).$$

As noted in [1], there was in an error in a result used to prove [7, Proposition 3.1], so the proof is given in [1, Proposition 4]. From the above proposition, we immediately obtain the following corollary.

Corollary 5.4. *Let $E/\mathbf{Q}$ be a non-CM elliptic curve. Suppose $\rho_{E,\ell}(\mathrm{Gal}_{\mathbf{Q}})$ is conjugate to the normalizer of a nonsplit Cartan subgroup for some prime $\ell > 13$. For $P \in E(\overline{\mathbf{Q}})$ of order ℓ^n and $\{\pm 1\} \leq \Delta \leq (\mathbf{Z}/\ell^n\mathbf{Z})^\times$, let $x = [(E, \Delta P)] \in X_\Delta(\ell^n)$ be the associated closed point on the modular curve. Then*

$$\deg(x) = \frac{1}{\#\Delta}(\ell^2 - 1)\ell^{2n-2} = \deg(X_\Delta(\ell^n) \to X(1)).$$

We will make use of the following theorem.

Theorem 5.5 ([23, Theorem 1.3]). *Let $n \geq 1$, let $H \leq \mathbf{GL}_2(\mathbf{Z}/n\mathbf{Z})$ be a subgroup, and let x be a non-cuspidal isolated point on X_H with $j(x) \neq \{0, 1728\}$. Let $E/\mathbf{Q}$ be an elliptic curve such that $j(E) = j(x)$, and let $G_n := \rho_{E,n}(\mathrm{Gal}_{\mathbf{Q}(j(x))})$. Then the modular curve X_{G_n} contains an isolated point with j-invariant equal to $j(x)$.*

If the modular curve X_{G_n} in the above theorem has genus 0, then Theorem 2.5 implies that there are no isolated points on X_{G_n} and hence x cannot be isolated. This will allow us to rule out certain cases in the following proposition.

Proposition 5.6. *Let $\ell \in \mathbf{N}$ be prime and $n \in \mathbf{N}$ a positive integer. Let $E/\mathbf{Q}$ be a non-CM elliptic curve and suppose $\rho_{E,\ell^\infty}(\mathrm{Gal}_{\mathbf{Q}})$ is conjugate to a known image, as in cases (a) or (b) of Theorem 5.2. If $\ell \neq 11, 17, 37$, then there are no isolated points above $j(E)$ on a modular curve $X_\Delta(\ell^n)$.*

Proof. First, suppose $R := \rho_{E,\ell^\infty}(\mathrm{Gal}_{\mathbf{Q}})$ is equal to $\mathbf{GL}_2(\mathbf{Z}_\ell)$, for some prime ℓ. Then for every $n \in \mathbf{N}$ and $\Delta \leq (\mathbf{Z}/\ell^n\mathbf{Z})^\times$, the fiber of $X_\Delta(\ell^n) \to X(1)$ over $j(E)$ is a single closed point of maximal degree. So, if there is an isolated point on $X_\Delta(\ell^n)$ above $j(E)$, then Theorem 2.7 implies $j(E) \in X(1)$ is isolated. But $X(1)$ has genus 0, so by Theorem 2.5 this is not possible.

We now assume R is conjugate to a subgroup not equal to $\mathbf{GL}_2(\mathbf{Z}_\ell)$ that appears in [22, Tables 1-4] or in [17, Table 1]; these are cases (a) and (b) of Theorem 5.2. By Theorem 5.1, there are no isolated points on $X_0(\ell^n)$ or $X_1(\ell^n)$ associated to E because $\ell \neq 11, 17, 37$, so it suffices to consider intermediate modular curves $X_\Delta(\ell^n)$. So, let n be a positive integer and $\{\pm 1\} \subsetneq \Delta \subsetneq (\mathbf{Z}/\ell^n\mathbf{Z})^\times$ a subgroup. Suppose $x \in X_\Delta(\ell^n)$ is an isolated point with $j(x) = j(E)$ and say $x = [(E, \Delta P)]$. By Theorem 5.2, and because we are assuming $\ell \neq 11, 17, 37$, we know R has level ℓ^m for some

$$\ell^m \in \mathcal{L} := \{1, 2, 4, 8, 16, 32, 3, 9, 27, 5, 25, 7, 13\}.$$

If $n \geq m$, then by Proposition 3.1 and Theorem 2.7, the image of x under the natural map $X_\Delta(\ell^n) \to X_\Delta(\ell^m)$ is isolated. So, it suffices to consider the case when $\ell^n \in \mathcal{L}$. Note there are no proper subgroups $\{\pm 1\} \subsetneq \Delta \subsetneq (\mathbf{Z}/\ell^n\mathbf{Z})^\times$ for

$$\ell^n \in \{1, 2, 4, 8, 3, 9, 5, 7\}.$$

Moreover, every intermediate modular curve of level 13 and 16 has genus zero, and hence has no isolated points by Theorem 2.5.

So, it remains to consider the case when $\ell^n \in \{32, 27, 25\}$. There are 132 possible choices for R to consider, and these can be found in the file `elladicgens.txt` associated to [1]. Let f denote the natural map $X_1(\ell^n) \to X_\Delta(\ell^n)$ and let $x' \in X_1(\ell^n)$ be a closed point in the fiber of f over x. We use `Magma` to find that all but 28 of the 132 images have the property that every closed point in the fiber of f over x has degree strictly greater than $(\#\Delta/2)\,\mathrm{genus}(X_\Delta(\ell^n))$. The data in Table 1 gives the exact bound on $\deg(x')$ for each level ℓ^n. If $\deg(x') > (\#\Delta/2)\,\mathrm{genus}(X_\Delta(\ell^n))$, then Lemma 2.6 implies

$$\deg(x) \geq \frac{2\deg(x')}{\#\Delta} > \frac{2\,\mathrm{genus}(X_1(\ell^n))}{\#\Delta} > \mathrm{genus}(X_\Delta(\ell^n)),$$

which contradicts that x is isolated by Theorem 2.5. So, we may assume that R is one of the remaining 28 images. A `Magma` computation shows that the modular curve $X_R(\ell)$ has genus zero, so Theorem 2.5 implies there are no isolated points on $X_R(\ell)$. But this contradicts Theorem 5.5, so we conclude that x cannot be isolated. See the website of the author for the `Magma` code used. ∎

ℓ^n	$\#\Delta$	$\mathrm{genus}(X_\Delta(\ell^n))$	$\frac{\#\Delta}{2}\,\mathrm{genus}(X_\Delta(\ell^n))$
25	4	4	8
25	10	0	0
27	6	1	3
32	4	5	10
32	8	1	4

TABLE 1. Data for intermediate modular curves $X_\Delta(\ell^n)$.

We now consider the situation when $R := \rho_{E,\ell^\infty}(\mathrm{Gal}_{\mathbf{Q}})$ is as in case (c) of Theorem 5.2. That is, we have $R \leq C_{\mathrm{ns}}^+(3^3), C_{\mathrm{ns}}^+(5^2), C_{\mathrm{ns}}^+(7^2), C_{\mathrm{ns}}^+(11^2)$ or $C_{\mathrm{ns}}^+(\ell)$ for $\ell \geq 19$. For this, a key ingredient is work of Furio [10, Theorem 1.9], which characterizes the possibilities for R in this case.

Proposition 5.7. *Let ℓ be an odd prime and $n \in \mathbf{N}$. Let $E/\mathbf{Q}$ be a non-CM elliptic curve and Δ a subgroup $\{\pm 1\} < \Delta < (\mathbf{Z}/\ell^n\mathbf{Z})^\times$. If $\rho_{E,\ell}(\mathrm{Gal}_{\mathbf{Q}})$ is conjugate to a subgroup of the normalizer of a non-split Cartan subgroup, then there is no isolated point $x \in X_\Delta(\ell^n)$ with $j(x) = j(E)$.*

Proof. Let $x \in X_\Delta(\ell^n)$ be such that $j(x) = j(E)$ and say $x = [(E, \Delta P)]$. By Proposition 5.6, we may assume $R := \rho_{E,\ell^\infty}(\mathrm{Gal}_{\mathbf{Q}})$ is not a known image. By [10, Theorem 1.9], either R has level ℓ^d and is conjugate to $C_{\mathrm{ns}}^+(\ell^d)$ or R has level ℓ^2 and

$$R(\ell^2) \simeq C_{\mathrm{ns}}^+(\ell) \ltimes \left\{ I + \ell \begin{pmatrix} a & \varepsilon b \\ -b & c \end{pmatrix} \right\},$$

with the semidirect product defined by the conjugation action.

Suppose R has level ℓ^d and is conjugate to $C_{\mathrm{ns}}^+(\ell^d)$. The proof of [1, Theorem 6], which proves the claim for the case of $X_\Delta(\ell^d) = X_1(\ell^d)$, shows that $[(E, P)] \in X_1(\ell^n)$ has degree $\ell^{2n-2}(\ell^2 - 1)/2$, so $\deg(x) = \ell^{2n-1}(\ell^2 - 1)/(\#\Delta)$. But then Theorem 2.7 implies that the image of x under the natural map $X_\Delta(\ell^n) \to X(1)$ is isolated, which is absurd, since $X(1)$ has genus 0.

Now suppose R has level ℓ^2 and is isomorphic to the semidirect product above. The proof of [1, Theorem 6] shows that $[(E,P)] \in X_1(\ell^2)$ has degree at least $\ell(\ell^2-1)/2$, so we have $\deg(x) \geq \ell(\ell^2-1)/\#\Delta$. If $\ell > 13$, then Corollary 5.4 and Theorem 2.7 imply there are no isolated points on $X_\Delta(\ell^n)$.

If $3 \leq \ell \leq 11$, then there are no intermediate modular curves of level ℓ because $\phi(\ell)/2$ is prime. There is one intermediate modular curve of level 13, but this curve has genus 0 and hence has no isolated points by Theorem 2.5. Hence, we may assume $3 \leq \ell \leq 13$ and $n \geq 2$. Proposition 3.1 and Theorem 2.7 imply that if x is isolated, then the image of x under the natural map $f : X_\Delta(\ell^n) \to X_\Delta(\ell^2)$ is isolated. Note that there are no intermediate modular curves of level 9 because $\phi(9)/2 = 3$ is prime, so we may assume $\ell \geq 5$. The data in Table 2 combined with Theorem 2.5 shows that $f(x)$, and hence x, is not isolated. ∎

ℓ	$\#\Delta$	genus$(X_\Delta(\ell^2))$	$\ell(\ell^2-1)/\#\Delta$
5	4	4	30
5	10	0	12
7	6	19	56
7	14	3	24
11	10	106	132
11	22	26	60
13	4	516	546
13	6	340	364
13	12	164	182
13	26	50	84
13	52	24	42
13	78	16	28

TABLE 2. Data for $X_\Delta(\ell^2)$ with $\ell \leq 13$.

By the previous proposition and [1, Proposition 1], it remains to consider the case when $\rho_{E,\ell^\infty}(\mathrm{Gal}_{\mathbf{Q}})$ is conjugate to `49.196.9.1`.

Proposition 5.8. *Let $E/\mathbf{Q}$ be a non-CM elliptic curve and suppose $\rho_{E,7^\infty}(\mathrm{Gal}_{\mathbf{Q}})$ is conjugate to* `49.196.9.1`. *Then there are no isolated points above $j(E)$ on a modular curve $X_\Delta(7^n)$.*

Proof. Let n be a positive integer and let $E/\mathbf{Q}$ be a non-CM elliptic curve with $\rho_{E,7^\infty}(\mathrm{Gal}_{\mathbf{Q}})$ conjugate to `49.196.9.1`. Suppose $x \in X_\Delta(7^n)$ is an isolated point with $j(x) = j(E)$. If $n = 1$, then $X_\Delta(7^n)$ is not an intermediate modular curve because $\phi(7)/2 = 3$ is prime, so Theorem 5.1 implies x cannot be isolated. If $n \geq 2$, then by Proposition 3.1 and Theorem 2.7 imply that the image of x under the natural map to $X_\Delta(49)$ is isolated. So, it suffices to show that $[(E, \Delta P)] \in X_\Delta(49)$ is not isolated.

A `Magma` computation shows that if $x = [(E, \langle P \rangle)] \in X_0(49)$ is a closed point with $j(x) = j(E)$, then $\deg(x) = 56$. The genus of each intermediate modular curve $X_\Delta(49)$ is strictly less than 56, so $[(E, \Delta P)] \in X_\Delta(49)$ is not isolated by Theorem 2.5. ∎

5.3. **Proof of Theorem 1.3.** We restate Theorem 1.3 below for convenience.

Theorem 5.9. *Let $\ell \in \mathbf{N}$ be a prime, let $n \in \mathbf{N}$ a positive number, and suppose $\Delta \leq (\mathbf{Z}/\ell^n\mathbf{Z})^\times$ is a subgroup containing -1. If $x \in X_\Delta(\ell^n)$ is a non-cuspidal isolated point with non-CM rational*

j-invariant $j \in \mathbf{Q}$, then j appears as the j-invariant of an isolated point on $X_0(\ell)$, for some $\ell \in \{11, 17, 37\}$.

Proof. Let $E/\mathbf{Q}$ be a non-CM elliptic curve and let $\ell \in \mathbf{N}$ be prime. By Theorem 5.2, there are four cases for the image $R := \rho_{E,\ell^\infty}(\mathrm{Gal}_{\mathbf{Q}}) \leq \mathbf{GL}_2(\mathbf{Z}_\ell)$. If R is in case (a) or (b) and $\ell \neq 11, 17, 37$, then Proposition 5.6 implies that $j(E)$ is not the j-invariant of an isolated point on a modular curve $X_\Delta(\ell^n)$. Similarly, if R is in case (c) or case (d), then Proposition 5.7 and Proposition 5.8 imply that $j(E)$ does not appear as the j-invariant of an isolated point on a modular curve $X_\Delta(\ell^n)$.

It remains to consider the case when $\ell = 11, 17, 37$ and R is a known image. Because $\phi(11)/2$ is prime, there are no intermediate modular curves of level 11. The j-invariants of the known images of levels 17 and 37 all arise from rational points on $X_0(17)$ or $X_0(37)$, as desired. ■

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