

# ISOLATED POINTS ON MODULAR CURVES OF PRIME-POWER LEVEL

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**ABSTRACT.** An isolated point on an algebraic curve is a closed point not belonging to a collection of points of the same degree parametrized by  $\mathbf{P}^1$  or a positive rank abelian subvariety of the curve's Jacobian. We study the sets of  $j$ -invariants, in extensions of bounded degree, that arise as the  $j$ -invariant of an isolated point on a modular curve. We obtain finiteness results on these sets for families of modular curves with prime-power level. This is related to recent work of Bourdon and Ejder, who classified rational  $j$ -invariants of isolated points on the families  $X_1(n)$  and  $X_0(n)$ , for  $n$  a prime power.

## 1. INTRODUCTION

The closed points of a smooth, projective curve  $C$  over a number field  $k$  are in bijection with  $\text{Gal}_k$ -orbits of points in  $C(\bar{k})$ , where  $\text{Gal}_k := \text{Gal}(\bar{k}/k)$  is the absolute Galois group. The degree of a closed point corresponds to the size of the associated Galois-orbit. A closed point  $x \in C$  of degree  $d$  is said to be **isolated**<sup>1</sup> if it does not belong to a collection of degree  $d$  closed points parametrized by  $\mathbf{P}^1$  or a positive rank abelian subvariety of the curve's Jacobian.

Faltings's Theorem [Fal83] guarantees that if  $C$  has genus strictly greater than 1, then  $C$  has finitely many degree 1 points. Such a curve  $C$  has infinitely many points of arbitrary degree. However, it was shown in [Bou+19] that  $C$  has only finitely many isolated points. Because of this, the study of isolated points on curves generalizes the study of rational points in a natural way.

If  $E$  is an elliptic curve defined over a number field  $k$ , the action of  $\text{Gal}_k$  on the  $n$ -torsion subgroups  $E[n]$ , for positive integers  $n$ , induce representations

$$\begin{aligned}\rho_{E,n} : \text{Gal}_k &\rightarrow \mathbf{GL}_2(\mathbf{Z}/n\mathbf{Z}), \\ \rho_{E,\ell^\infty} : \text{Gal}_k &\rightarrow \mathbf{GL}_2(\mathbf{Z}_\ell) = \varprojlim \mathbf{GL}_2(\mathbf{Z}/\ell^n\mathbf{Z}), \text{ and} \\ \rho_E : \text{Gal}_k &\rightarrow \mathbf{GL}_2(\hat{\mathbf{Z}}) = \varprojlim \mathbf{GL}_2(\mathbf{Z}/n\mathbf{Z}).\end{aligned}$$

If  $H$  is an open subgroup of  $\mathbf{GL}_2(\mathbf{Z}_\ell)$ , the modular curve  $X_H$  is a curve whose non-cuspidal points parametrize elliptic curves with  $\ell$ -adic image of Galois contained in  $H$ . To a closed point  $x \in X_H$ , we associate an algebraic number  $j(x)$  called the  **$j$ -invariant** of  $x$ , which is the image of  $x$  under the natural map to the coarse moduli space of elliptic curves.

For a fixed non-CM elliptic curve  $E$  over a number field  $k$ , Serre showed [Ser72] that the mod- $\ell$  Galois representation  $\rho_{E,\ell}$  is surjective for large primes  $\ell$ . Serre asked [Ser72, Section 4.3] if this result in the case of  $k = \mathbf{Q}$  can be made uniform—that is, if there is prime  $L$  such that  $\rho_{E,\ell}$  is surjective for all non-CM elliptic curves  $E/\mathbf{Q}$  and all primes  $\ell > L$ . This prime  $L$  has since been conjectured by Zywna [Zyw24, Conjecture 1.2] and Sutherland [Sut16, Conjecture 1.1] to be 37. We will refer to this conjecture as **Serre's uniformity problem**. It remains unknown if there exists a prime  $\ell > 37$  and non-CM elliptic curve

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<sup>1</sup>See Section 2.6 for a precise definition.

$E/\mathbf{Q}$  with  $\rho_{E,\ell}(\text{Gal}_k)$  conjugate to  $C_{\text{ns}}^+(\ell)$ , the normalizer of a non-split Cartan subgroup of  $\mathbf{GL}_2(\mathbf{Z}/\ell\mathbf{Z})$ . If such a prime  $\ell$  and elliptic curve  $E$  exists, then  $E$  gives rise to a representative for a degree one point on the modular curve  $X_{C_{\text{ns}}^+(\ell)}$ . This curve has genus greater than 1, so Faltings's Theorem implies there are finitely many degree one points on  $X_{C_{\text{ns}}^+(\ell)}$ , all of which are isolated.

The study of rational isolated points on modular curves is also related to Mazur's "Program B" [Maz77], which asks, given a number field  $k$ , for a classification of the images  $\rho_E(\text{Gal}_k)$  in  $\mathbf{GL}_2(\widehat{\mathbf{Z}})$ , for elliptic curves  $E/k$ . Work in this paper is related to the  $\ell$ -adic case of the program, which is the problem of classifying images  $\rho_{E,\ell^\infty}(\text{Gal}_k)$  in  $\mathbf{GL}_2(\mathbf{Z}_\ell)$ .

For each positive integer  $n$ , the modular curve  $X_1(n)$  is a curve whose non-cuspidal points parametrize elliptic curves with a point of order  $n$ . Bourdon, Ejder, Liu, Odumodu, and Viray [Bou+19, Corollary 1.7] showed that, assuming an affirmative answer to Serre's uniformity problem, there are finitely many  $j$ -invariants in  $\mathbf{Q}$  corresponding to isolated points on  $X_1(n)$ , as  $n$  varies over all positive integers.

It is natural to ask for which other families of curves we can obtain finiteness results similar to [Bou+19, Corollary 1.7]. In this paper, we examine the set of  $j$ -invariants in an extension of bounded degree that arise as the  $j$ -invariant of an isolated point on a modular curve of prime-power level.

**Theorem 1.1.** *Fix a prime  $\ell \in \mathbf{N}$  and define*

$$\mathcal{I} := \{x \in X_H \mid x \text{ is isolated and } H \leq \mathbf{GL}_2(\mathbf{Z}_\ell) \text{ an open subgroup}\}.$$

*For every  $d \in \mathbf{N}$ , there are finitely many  $j$ -invariants of degree  $d$  in  $j(\mathcal{I})$ .*

The proof uses group theory techniques and results of Cadoret and Tamagawa [CT13] and Bourdon, Ejder, Liu, Odumodu, and Viray [Bou+19] to obtain a bound  $m$ , dependent on  $\ell$  and  $d$ , such that for any isolated point on an  $\ell$ -power level modular curve with  $j$ -invariant of degree  $d$ , there exists another isolated point on a modular curve of  $\ell$ -power level dividing  $m$  having the same  $j$ -invariant. This method of reducing the level of modular curve on which an isolated point must appear is related to [Bou+19, Theorem 1.1], which states that for a non-CM elliptic curve with  $m$ -adic Galois representation of level  $M$ , for a certain integer  $m$ , the natural map  $X_1(n) \rightarrow X_1((n, M))$  sends isolated points to isolated points. We also use results of Terao [Ter24] to describe the degrees of closed points  $x \in X_H$  in terms of  $H$  and the adelic image of Galois associated to an elliptic curve corresponding to  $x$  defined over  $\mathbf{Q}(j(x))$ .

**Remark 1.2.** Terao [Ter24, Theorem 1.5] showed that if  $x$  is a non-cuspidal, non-CM isolated point on a modular curve  $X_H$ , with  $H \leq \mathbf{GL}_2(\widehat{\mathbf{Z}})$  an open subgroup of level 7 and  $j(x) \in \mathbf{Q}$ , then  $j(x) = 3^3 \cdot 5 \cdot 7^5 / 2^7$ . Theorem 1.1 implies that there are finitely many rational  $j$ -invariants arising from isolated points on modular curves whose level is a power of 7.

In the case of prime-power level curves in the family  $X_0(n)$  and  $X_1(n)$ , Bourdon and Ejder [BE25] give an unconditional result classifying the rational  $j$ -invariants arising from isolated points that may appear, which extends a partial classification given by Ejder [Ejd22]. Building on these results, we study the rational  $j$ -invariants arising from isolated points on prime-power level curves in the family  $X_\Delta(n)$ , which contains the families  $X_0(n)$  and  $X_1(n)$  and the so-called intermediate modular curves living between  $X_0(n)$  and  $X_1(n)$ .

**Theorem 1.3.** *Let  $\ell \in \mathbf{N}$  be a prime, let  $n \in \mathbf{N}$  a positive number, and suppose  $\Delta \leq (\mathbf{Z}/\ell^n\mathbf{Z})^\times$  is a subgroup containing  $-1$ . If  $x \in X_\Delta(\ell^n)$  is a non-cuspidal isolated point with non-CM rational*

$j$ -invariant  $j \in \mathbf{Q}$ , then  $j$  appears as the  $j$ -invariant of an isolated point on  $X_0(\ell)$ , for some  $\ell \in \{11, 17, 37\}$ .

The proof uses the work of Bourdon and Ejder [BE25] and the work of Rouse, Sutherland, and Zureick-Brown [RSZ22] and Furio [Fur25] studying  $\ell$ -adic images of Galois attached to non-CM elliptic curves over  $\mathbf{Q}$ , to determine whether a rational  $j$ -invariant  $j(E)$  associated to a non-CM elliptic curve  $E/\mathbf{Q}$  may appear as the  $j$ -invariant of an isolated point on a modular curve  $X_\Delta(\ell^n)$ .

*Remark 1.4.* If we instead consider rational  $j$ -invariants arising from isolated points on  $X_\Delta(n)$ , for  $n$  not necessarily a prime power, then the analogue of Theorem 1.3 no longer holds, as noted by Lee [Lee25, Theorem 2]. Specifically, Bourdon, Hashimoto, Keller, Klagsbrun, Lowry-Duda, Morrison, Najman, and Shukla [Bou+25, Theorem 2] showed that there is an isolated point on  $X_1(28)$  with  $j$ -invariant  $351/4$ , but Lee found that there is no isolated point on a modular curve  $X_0(n)$  with  $j$ -invariant  $351/4$ .

**1.1. Related work.** This is also related to work of Menendez [Men22, Theorem 5.3], who proved the analogue to [Bou+19, Theorem 1.1] for the family  $X_0(n)$ . Lee [Lee25, Algorithm 1] gave an algorithm to test whether a non-CM  $j$ -invariant in  $\mathbf{Q}$  corresponds to an isolated point on a modular curve  $X_0(n)$ . If we assume an affirmative answer to Serre's uniformity problem, then the  $j$ -invariants found in [Lee25, Table 1] are all of the possible  $j$ -invariants in  $\mathbf{Q}$ . Terao [Ter24, Theorem 1.6] proved a very similar result, classifying the rational  $j$ -invariants arising from isolated points on  $X_0(n)$ , assuming a conjecture of Zywna [Zyw24, Section 14.3]. The  $j$ -invariants found by Lee in [Lee25, Table 1] are the same  $j$ -invariants appearing in [Ter24, Theorem 1.6].

**1.2. Outline.** In Section 2, we give the relevant background on elliptic curves, profinite groups, modular curves, and isolated points. In Section 3, we prove Theorem 1.1. In Section 4, we give an overview of the family of modular curves  $X_\Delta(n)$ . In Section 5, we prove Theorem 1.3.

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## 2. BACKGROUND

**2.1. Notation and conventions.** Let  $k$  be a number field and  $\bar{k}$  a fixed algebraic closure of  $k$ . We denote  $\text{Gal}_k$  to be the absolute Galois group  $\text{Gal}(\bar{k}/k)$ . Let  $\mathbf{N}$  denote the set of natural numbers, i.e. the set of positive integers  $\{1, 2, 3, \dots\}$ . For  $n \in \mathbf{N}$ , we denote by  $\phi(n)$  the number of integers  $1 \leq m \leq n$  with  $(m, n) = 1$ .

By a **nice** curve, we mean a smooth, projective, geometrically integral curve. For a  $k$ -scheme  $X$  and a field extension  $K$  of  $k$ , we denote  $X_K$  to be the base change  $X \times_{\text{Spec } k} \text{Spec } K$ .

**2.2. Elliptic curves.** Let  $E$  be an elliptic curve over a number field  $k$ . For  $n \in \mathbf{N}$ , we denote  $E[n] = E(\bar{k})[n]$  to be the subgroup of  $E(\bar{k})$  consisting of  $n$ -torsion points—that is, points with finite order dividing  $n$ . The group  $E[n]$  is the kernel of the multiplication-by- $n$  map  $[n] : E \rightarrow E$  and is isomorphic to  $(\mathbf{Z}/n\mathbf{Z}) \times (\mathbf{Z}/n\mathbf{Z})$ .

We say an elliptic curve  $E/k$  has **complex multiplication (CM)** if  $\text{End } E_{\bar{k}}$  is not isomorphic to the integers—we will not add the distinction of an elliptic curve having potential CM, as

some authors do. We say a  $j$ -invariant  $j \in \overline{\mathbf{Q}}$  is CM if there exists a CM elliptic curve  $E$  with  $j$ -invariant equal to  $j$ . In this paper, we are interested in showing there are finitely many  $j$ -invariants, with a certain property, in number fields of a fixed bounded degree. Our main techniques to do so work only for non-CM  $j$ -invariants. However, results from class field theory allow us to restrict to the case of non-CM  $j$ -invariants.

Indeed, if  $E/k$  has CM, then by [Sil09, Corollary 9.4], there exists an imaginary quadratic field  $K$  and an order  $\mathcal{O}$  in  $K$  such that  $\text{End}(E_{\bar{k}}) \simeq \mathcal{O}$ . By [Cox13, Theorem 11.1], we have  $[\mathbf{Q}(j(E)) : \mathbf{Q}] = [K(j(E)) : K] = h(\mathcal{O})$ , where  $h(\mathcal{O})$  is the class number. There are only finitely many  $K$  of a given class number by [Hei34] and each order is associated to only finitely many CM  $j$ -invariants by [Cox13, Corollary 10.20]. These results imply the following well-known theorem.

**Theorem 2.1.** *For every  $d \in \mathbf{N}$ , there are only finitely many CM  $j$ -invariants in number fields of degree  $d$ .*

**2.3. Profinite constructions.** For a prime  $\ell \in \mathbf{N}$ , the ring of  $\ell$ -adic integers, denoted  $\mathbf{Z}_{\ell}$ , is the inverse limit  $\varprojlim \mathbf{Z}/\ell^n \mathbf{Z}$  taken over  $n \in \mathbf{N}$  with homomorphisms the evaluation maps  $\mathbf{Z}/\ell^j \mathbf{Z} \rightarrow \mathbf{Z}/\ell^i \mathbf{Z}$  for  $i \leq j$ . The profinite integers, denoted  $\widehat{\mathbf{Z}}$ , are the inverse limit  $\varprojlim \mathbf{Z}/n \mathbf{Z}$  taken over the directed set consisting of natural numbers ordered by divisibility. There is a ring isomorphism

$$\widehat{\mathbf{Z}} = \varprojlim \mathbf{Z}/n \mathbf{Z} \simeq \prod_{\substack{\ell \in \mathbf{N} \\ \text{prime}}} \varprojlim \mathbf{Z}/\ell^n \mathbf{Z} = \prod_{\ell} \mathbf{Z}_{\ell}.$$

For an elliptic curve  $E$  over a number field  $k$  and a prime  $\ell \in \mathbf{N}$ , we define the  $\ell$ -adic Tate module of  $E$  to be the inverse limit  $T_{\ell}(E) := \varprojlim E[\ell^n]$  taken over  $n \in \mathbf{N}$  with homomorphisms the maps  $[\ell^{j-i}] : E[\ell^j] \rightarrow E[\ell^i]$  for  $i \leq j$ . As a  $\mathbf{Z}_{\ell}$ -module,  $T_{\ell}(E)$  has the following structure:

$$T_{\ell}(E) = \varprojlim E[\ell^n] \simeq \varprojlim (\mathbf{Z}/\ell^n \mathbf{Z} \times \mathbf{Z}/\ell^n \mathbf{Z}) = \varprojlim (\mathbf{Z}/\ell^n \mathbf{Z}) \times \varprojlim (\mathbf{Z}/\ell^n \mathbf{Z}) = \mathbf{Z}_{\ell} \times \mathbf{Z}_{\ell}.$$

Following a similar construction to  $\widehat{\mathbf{Z}}$ , we also define the adelic Tate module of  $E$  to be the inverse limit  $T(E) := \varprojlim E[n]$  so that

$$T(E) \simeq \prod_{\ell} T_{\ell}(E) \simeq \prod_{\ell} \mathbf{Z}_{\ell} \times \mathbf{Z}_{\ell} \simeq \widehat{\mathbf{Z}} \times \widehat{\mathbf{Z}}.$$

We will also study the profinite group  $\mathbf{GL}_2(\widehat{\mathbf{Z}}) \simeq \varprojlim \mathbf{GL}_2(\mathbf{Z}/n \mathbf{Z})$ . For each  $n \in \mathbf{N}$ , there exists a natural projection map  $\pi_n : \mathbf{GL}_2(\widehat{\mathbf{Z}}) \rightarrow \mathbf{GL}_2(\mathbf{Z}/n \mathbf{Z})$  induced by the inverse limit. Similarly, by abuse of notation, for a prime number  $\ell$ , we denote  $\pi_{\ell^n}$  to be the natural projection map  $\mathbf{GL}_2(\mathbf{Z}_{\ell}) \rightarrow \mathbf{GL}_2(\mathbf{Z}/\ell^n \mathbf{Z})$ . For a subgroup  $H \leq \mathbf{GL}_2(\widehat{\mathbf{Z}})$ , we denote  $H(n)$  to be  $\pi_n(H)$ . In general, we have  $H \leq \pi_n^{-1}(H(n))$ . If  $H \leq \mathbf{GL}_2(\widehat{\mathbf{Z}})$  is an open subgroup, then there exists a positive integer  $n \in \mathbf{N}$  such that  $\ker \pi_n \leq H$ . We define the level of  $H$  to be the least such positive integer. If  $H$  has level  $n$ , then  $H = \pi_n^{-1}(H(n))$  and the index of  $H$  in  $\mathbf{GL}_2(\widehat{\mathbf{Z}})$  (which is finite because  $H$  is open) is equal to the index of  $H(n)$  in  $\mathbf{GL}_2(\mathbf{Z}/n \mathbf{Z})$ .

In Section 5, we will consider specific subgroups  $H \leq \mathbf{GL}_2(\widehat{\mathbf{Z}})$  with  $\det(H) = \widehat{\mathbf{Z}}^{\times}$ . We will refer to these subgroups using the notation of Rouse, Sutherland, and Zureick-Brown [RSZ22, Section 2.4], which assigns to each subgroup  $H$  a label

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where  $N$  is the level of  $H$ ,  $i$  is the index of  $H$  in  $\mathbf{GL}_2(\widehat{\mathbf{Z}})$ ,  $g$  is the genus of the modular curve  $X_H$ , and  $n$  is an integer which specifies  $H$  among other subgroups having the same level, index, and genus.

**2.4. Galois representations.** For every  $\sigma \in \text{Gal}_k$  and  $P \in E$ , we denote by  $P^\sigma$  the image of the pair  $(P, \sigma)$  under the natural (right) action  $E \times \text{Gal}_k \rightarrow E$ . For every  $P \in E[n]$ , we have  $[n]P = O$ , where  $O$  is the base point of  $E$ . For every  $\sigma \in \text{Gal}_k$ ,

$$[n](P^\sigma) = ([n]P)^\sigma = O^\sigma = O,$$

so the natural action of  $\text{Gal}_k$  on  $E[n]$  is well-defined. This action defines a representation

$$\rho_{E,n} : \text{Gal}_k \rightarrow \text{Aut}(E[n])$$

called the **mod- $n$  Galois representation associated to  $E$** . Let  $\alpha : E[n] \rightarrow (\mathbf{Z}/n\mathbf{Z}) \times (\mathbf{Z}/n\mathbf{Z})$  be an isomorphism of  $(\mathbf{Z}/n\mathbf{Z})$ -modules. The **mod- $n$  Galois representation associated to  $E$  and  $\alpha$**  is the composition of the above representation with the isomorphism  $\text{Aut}(E[n]) \simeq \mathbf{GL}_2(\mathbf{Z}/n\mathbf{Z})$  induced by  $\alpha$  and is denoted

$$\rho_{E,n,\alpha} : \text{Gal}_k \rightarrow \text{Aut}(E[n]) \xrightarrow{\simeq} \mathbf{GL}_2(\mathbf{Z}/n\mathbf{Z}).$$

If  $\beta : E[n] \rightarrow (\mathbf{Z}/n\mathbf{Z}) \times (\mathbf{Z}/n\mathbf{Z})$  is another isomorphism of groups, then  $\rho_{E,n,\alpha}(\text{Gal}_k)$  and  $\rho_{E,n,\beta}(\text{Gal}_k)$  are conjugate in  $\mathbf{GL}_2(\mathbf{Z}/n\mathbf{Z})$ .

Suppose  $\ell \in N$  is a prime and  $\alpha : T_\ell(E) \rightarrow \mathbf{Z}_\ell \times \mathbf{Z}_\ell$  is an isomorphism. The action of  $\text{Gal}_k$  on  $E[\ell^n]$  for every  $n \in N$  induces an action on  $T_\ell(E)$ . This defines a representation

$$\rho_{E,\ell^\infty,\alpha} : \text{Gal}_k \rightarrow \text{Aut}(T_\ell(E)) \xrightarrow{\simeq} \mathbf{GL}_2(\mathbf{Z}_\ell)$$

called the  **$\ell$ -adic Galois representation associated to  $E$  and  $\alpha$** . Similarly, the action of  $\text{Gal}_k$  on  $E[n]$  for every  $n \in N$  extends to an action on  $T(E)$ . If  $\alpha : T(E) \rightarrow \widehat{\mathbf{Z}} \times \widehat{\mathbf{Z}}$  is an isomorphism, then this action induces a representation

$$\rho_{E,\alpha} : \text{Gal}_k \rightarrow \text{Aut}(T(E)) \xrightarrow{\simeq} \mathbf{GL}_2(\widehat{\mathbf{Z}})$$

called the **adelic Galois representation associated to  $E$  and  $\alpha$** . Just as with the mod- $n$  Galois representation, we may not keep track of the isomorphisms  $\alpha$  if we only care about the image of Galois up to conjugation.

Serre showed [Ser72, Théorème 3] that for a non-CM elliptic curve  $E/k$ , the adelic representation of Galois,  $\rho_E : \text{Gal}_k \rightarrow \mathbf{GL}_2(\widehat{\mathbf{Z}})$  has open image in  $\mathbf{GL}_2(\widehat{\mathbf{Z}})$ . Note that since the coset of an open subgroup of a profinite group is open,  $\rho_{E,\alpha}(\text{Gal}_k)$  is open regardless of the isomorphism  $\alpha$ , as this only changes the image of Galois up to conjugation.

**2.5. Modular curves.** In essence, for  $H$  an open subgroup of  $\mathbf{GL}_2(\mathbf{Z}/n\mathbf{Z})$  (resp.,  $\mathbf{GL}_2(\mathbf{Z}_\ell)$ ;  $\mathbf{GL}_2(\widehat{\mathbf{Z}})$ ), the modular curve  $X_H$  is a curve whose non-cuspidal points parametrize elliptic curves having mod- $n$  (resp.,  $\ell$ -adic; adelic) image of Galois contained in  $H$ . We now give a more precise definition.

Let  $H$  be a subgroup of  $\mathbf{GL}_2(\mathbf{Z}/n\mathbf{Z})$  and let  $E$  be an elliptic curve defined over a number field  $k$ . We say two isomorphisms  $\alpha, \alpha' : E[n] \rightarrow (\mathbf{Z}/n\mathbf{Z}) \times (\mathbf{Z}/n\mathbf{Z})$  are  **$H$ -equivalent**, denoted  $\alpha \sim_H \alpha'$  if there exists  $h \in H$  such that  $\alpha = h \circ \alpha'$ . Let  $E$  be an elliptic curve over a number field  $k$ . An  **$H$ -level structure on  $E$**  is an equivalence class  $[\alpha]_H$  of isomorphisms  $\alpha : E[n] \rightarrow (\mathbf{Z}/n\mathbf{Z}) \times (\mathbf{Z}/n\mathbf{Z})$  under  $H$ -equivalence.

We define the modular curve  $Y_H$  (resp.  $X_H$ ) to be the coarse moduli space of the stack  $\mathcal{M}_H^0$  (resp.  $\mathcal{M}_H$ ), which parametrizes elliptic curves (resp. generalized elliptic curves) with  $H$ -level structure. The modular curve  $Y_H$  is an affine subscheme of  $X_H$ . We call the  $s$  of  $X_H - Y_H$  **cusps** and the closed points of  $Y_H$  **non-cuspidal points**. The curve  $X_H$  is a smooth, projective, integral curve over  $\mathbf{Q}$  and is geometrically integral if and only if  $H$  has full determinant.

If  $H$  is an open subgroup of  $\mathbf{GL}_2(\widehat{\mathbf{Z}})$  or  $\mathbf{GL}_2(\mathbf{Z}_\ell)$  of level  $n$ , then we define the modular curve  $X_H$  to be the modular curve  $X_{H(n)}$ . If  $m$  is any positive integer divisible by  $n$ , then the modular curve  $X_H = X_{H(n)}$  is isomorphic to the modular curve  $X_{H(m)}$ . Even if  $H$  does not contain  $-I$ , we always have that the modular curves  $X_H$  and  $X_{\pm H}$  are isomorphic as curves. Because of this, we will only consider subgroups  $H$  containing  $-I$ .

We give a precise description of the geometric non-cuspidal points of  $X_H$ . The set  $Y_H(\bar{k})$  consists of equivalence classes of pairs  $(E, [\alpha]_H)$ , where  $E/k$  is an elliptic curve and  $[\alpha]_H$  is an  $H$ -level structure on  $E$ . We say two pairs  $(E, [\alpha]_H)$  and  $(E', [\alpha']_H)$  are equivalent if there exists an isomorphism  $\phi : E \rightarrow E'$  such that the induced isomorphism  $\phi : E[n] \rightarrow E'[n]$  satisfies  $\alpha \sim_H \alpha' \circ \phi$ , i.e.  $\alpha = h \circ \alpha' \circ \phi$ , for some  $h \in H$ .

The absolute Galois group  $\text{Gal}_k$  admits a right action on  $Y_H(\bar{k})$  as follows: For an automorphism  $\sigma \in \text{Gal}_k$  and a point  $[(E, [\alpha]_H)] \in Y_H(\bar{k})$ , we define

$$[(E, [\alpha]_H)] \cdot \sigma = [(E^\sigma, [\alpha \circ \sigma^{-1}]_H)].$$

This action is well-defined: Indeed, if  $(E', [\alpha']_H), [(E, [\alpha]_H)] \in Y_H(\bar{k})$  and  $\sigma \in \text{Gal}_k$ , then there exists an isomorphism  $\phi : E[n] \rightarrow E'[n]$  and an element  $h \in H$  such that  $\alpha = h \circ \alpha' \circ \phi$ . This implies

$$\alpha \circ \sigma^{-1} = (h \circ \alpha' \circ \phi) \circ \sigma^{-1} = h \circ (\alpha' \circ \sigma^{-1}) \circ (\sigma \circ \phi \circ \sigma^{-1}).$$

The map  $\sigma \circ \phi \circ \sigma^{-1} : E^\sigma[n] \rightarrow E'^\sigma[n]$  given by  $P \mapsto \phi(P^{\sigma^{-1}})^\sigma$  is an isomorphism, so

$$((E', [\alpha']_H) \cdot \sigma) = (E'^\sigma, [\alpha' \circ \sigma^{-1}]_H) \in [(E^\sigma, [\alpha \circ \sigma^{-1}]_H)] = [(E, [\alpha]_H) \cdot \sigma].$$

A **closed point** of  $X_H$  is a point  $x \in X_H$  such that  $\{x\}$  is Zariski closed in  $X$  and the **degree** (over  $\mathbf{Q}$ ) of the closed point, denoted  $\deg(x)$ , is the degree of the residue field of  $x$  over  $\mathbf{Q}$ . By [Poo17, Proposition 2.4.6], there is a one-to-one correspondence between closed points of  $X_H$  and  $\text{Gal}_k$ -orbits of points in  $X_H(\bar{k})$  and the degree of a closed point corresponds to the size of the  $\text{Gal}_k$ -orbit. We will often refer to such a  $\text{Gal}_k$ -orbit as a closed point.

We say a point in  $X_H(\bar{k})$  is  $k$ -**rational** if it is fixed by every element of  $\text{Gal}_k$ . Equivalently, a  $k$ -rational point is a closed point of degree 1.

We say a pair  $(E, [\alpha]_H)$ , consisting of an elliptic curve  $E/\mathbf{Q}(j(E))$  and an  $H$ -level structure  $[\alpha]_H$  on  $E$ , is a **minimal representative** for a closed point  $x \in X_H$  if  $x$  corresponds to the  $\text{Gal}_{\mathbf{Q}}$ -orbit of  $[(E, [\alpha]_H)] \in X_H(\bar{\mathbf{Q}})$ . A minimal representative for a non-cuspidal closed point always exists by [Ter24, Lemma 4.9].

**Theorem 2.2** ([Ter24, Theorem 4.24]). *Let  $H \leq \mathbf{GL}_2(\widehat{\mathbf{Z}})$  be an open subgroup and let  $x \in X_H$  be a non-cuspidal closed point with minimal representative  $(E, [\alpha]_H)$  defined over  $k := \mathbf{Q}(j(E))$ . Let  $A_{E,\alpha} \leq \mathbf{GL}_2(\widehat{\mathbf{Z}})$  be the subgroup  $\{\alpha \circ \phi \circ \alpha^{-1} \mid \phi \in \text{Aut}(E_{\bar{k}})\}$ . Then*

$$\deg(x) = [\mathbf{Q}(j(E)) : \mathbf{Q}][\rho_{E,\alpha}(\text{Gal}_k)A_{E,\alpha} : \rho_{E,\alpha}(\text{Gal}_k)A_{E,\alpha} \cap A_{E,\alpha}H].$$

If  $j(x) \neq 0, 1728$ —in particular, if  $E$  is non-CM—then  $\text{Aut}(E_{\bar{k}}) = \{\pm 1\}$ , so  $A_{E,\alpha} = \{\pm I\}$ . Since we are assuming  $H$  contains  $-I$ , we get that  $A_{E,\alpha}H = \{\pm I\}H = H$ . This yields:

**Corollary 2.3.** *Let  $H \leq \mathbf{GL}_2(\widehat{\mathbf{Z}})$  be an open subgroup containing  $-I$  and let  $x \in X_H$  be a non-cuspidal closed point with minimal representative  $(E, [\alpha]_H)$  defined over  $k := \mathbf{Q}(j(E))$  such that  $j(E) \neq 0, 1728$ . Then*

$$\deg(x) = [\mathbf{Q}(j(E)) : \mathbf{Q}][\rho_{E,\alpha}(\mathrm{Gal}_k) : \rho_{E,\alpha}(\mathrm{Gal}_k) \cap H].$$

If  $H_1$  and  $H_2$  are subgroups of  $\mathbf{GL}_2(\mathbf{Z}/n\mathbf{Z})$  containing  $-I$  with  $H_1 \leq H_2$ , then there is a natural inclusion map  $X_{H_1} \rightarrow X_{H_2}$ . Indeed, if  $(E, [\alpha]_{H_1})$  is a representative for a non-cuspidal point in  $Y_{H_1}(\bar{k})$ , then  $[\alpha]_{H_1}$  defines an  $H_2$ -level structure on  $E$  because  $\alpha \sim_{H_1} \alpha'$  implies  $\alpha \sim_{H_2} \alpha'$ . We summarize a result from [DS05], which describes the degree of this inclusion map.

**Theorem 2.4** ([DS05, Page 66]). *Let  $H_1, H_2 \leq \mathbf{GL}_2(\widehat{\mathbf{Z}})$  be open subgroups containing  $-1$ . Suppose  $H_1 \leq H_2$  and denote  $f$  to be the natural map  $X_{H_1} \rightarrow X_{H_2}$  of curves over  $\mathbf{Q}$ . Then  $\deg(f) = [H_2 : H_1]$ .*

**2.6. Isolated points.** Let  $C$  be a smooth, projective curve over a number field  $k$ . Let  $\mathbf{Pic}_{C/k}$  be the Picard scheme of  $C$  and let  $\mathbf{Pic}_{C/k}^0$  be the connected component of the identity. Denote  $\mathbf{Div}_{C/k}$  to be the divisor scheme of  $C$ . Denote by  $\mathbf{A}_{C/k} : \mathbf{Div}_{C/k} \rightarrow \mathbf{Pic}_{C/k}$  the Abel map, and by  $\mathbf{W}_{C/k}$  its image. We refer the reader to [Ter24, Section 2] for more details.

Let  $x \in C$  be a closed point. By taking  $x$  to be a sum of  $\mathrm{Gal}_k$ -conjugates, we can view  $x$  as an element of  $\mathbf{Div}_{C/k}(k)$ .

- (1) We say  $x$  is **P<sup>1</sup>-parametrized** if there exists  $x' \in \mathbf{Div}_{C/k}(k)$  with  $x' \neq x$  such that  $\mathbf{A}_{C/k}(x) = \mathbf{A}_{C/k}(x')$ . We say  $x$  is **P<sup>1</sup>-isolated** if it is not P<sup>1</sup>-parametrized.
- (2) We say  $x$  is **AV-parametrized** if there exists a positive rank abelian subvariety  $A \subseteq \mathbf{Pic}_C^0$  such that  $\mathbf{A}_{C/k}(x) + A \subseteq \mathbf{W}_{C/k}$ . We say  $x$  is **AV-isolated** if it is not AV-parametrized.
- (3) We say  $x$  is **isolated** if it is both P<sup>1</sup>-isolated and AV-isolated.
- (4) We say  $x$  is **sporadic** if there are finitely many points  $y \in C$  with  $\deg(y) \leq \deg(x)$ .

The study of isolated points is motivated by Faltings's theorem [Fal83] which says that if  $C/k$  has genus strictly greater than 1, then  $C(k)$  is finite and hence every point in  $C(k)$  is sporadic. Indeed, it was shown in [Bou+19, Theorem 4.2] that on  $C/k$ , every sporadic point is isolated and there are only finitely many isolated points. So, if  $C/k$  has genus strictly greater than 1, the study of isolated points is a natural extension of the study of rational points.

The following theorem generalizes [Ejd22, Lemma 2.3] and will be particularly useful to us in Section 5.

**Theorem 2.5** ([Ter24, Theorem 2.17]). *Let  $C$  be a smooth, projective curve over a number field  $k$  and let  $K := k(C) \cap \bar{k}$ . Let  $r = [K : k]$  be the number of geometric components of  $C$  and let  $g$  be the genus of  $C_K$ . If  $x \in C$  is a closed point with  $\deg(x) > rg$ , then  $x$  is P<sup>1</sup>-parametrized.*

For a finite locally free map of curves, we have the following inequality:

**Lemma 2.6.** *Let  $C$  and  $D$  be smooth, projective curves over a number field  $k$  and let  $f : C \rightarrow D$  be a finite locally free map. If  $x \in C$  is a closed point, then*

$$\deg(x) \leq \deg(f) \deg(f(x)).$$

*Proof.* Let  $x \in C$  be a closed point. Let  $f^* : \mathbf{Div}_{D/k} \rightarrow \mathbf{Div}_{C/k}$  be the pullback map induced by  $f$  and let  $f_* : \mathbf{Div}_{C/k} \rightarrow \mathbf{Div}_{D/k}$  be the pushforward map induced by  $f$ . By [Gro67, Proposition 21.10.4], we have

$$\deg(f^*(f(x))) = \sum_{x' \in f^{-1}(f(x))} e_f(x') \deg(x'),$$

where  $e_f(x')$  is the ramification index of  $f$  at  $x'$ . By [Gro67, Proposition 21.10.18], we have  $f_*(f^*(f(x))) = \deg(f)f(x)$ , so

$$\deg(f^*(f(x))) = \deg(f_*(f^*(f(x)))) = \deg(f) \deg(f(x)).$$

This implies

$$\deg(x) \leq \sum_{x' \in f^{-1}(f(x))} e_f(x') \deg(x') = \deg(f) \deg(f(x)). \quad \blacksquare$$

If  $f : C \rightarrow D$  is a finite map of curves  $x \in C$  is an isolated point, it is not true in general that the image  $f(x)$  is an isolated point of  $D$ . However, if the degree is as large as possible—that is, if  $\deg(x) = \deg(f) \deg(f(x))$ —then we can in fact conclude that  $f(x)$  is isolated. This result was first proved for nice curves in [Bou+19, Theorem 4.3] and then generalized by Terao to the setting of smooth, projective curves.

**Theorem 2.7** ([Ter24, Theorem 2.15]). *Let  $f : C \rightarrow D$  be a finite locally free map of smooth, projective curves over a number field  $k$ . If  $x \in C$  and  $y \in D$  are closed points such that  $y = f(x)$  and  $\deg(x) = \deg(y) \deg(f)$ , then the following hold:*

- (1) *If  $x$  is  $\mathbf{P}^1$ -isolated, then  $y$  is  $\mathbf{P}^1$ -isolated.*
- (2) *If  $x$  is AV-isolated, then  $y$  is AV-isolated.*

In particular, if  $x$  is isolated, then  $y$  is also isolated.

### 3. MODULAR CURVES OF PRIME-POWER LEVEL

In this section we prove Theorem 1.1. Our strategy is to bound the level of modular curves on which a non-CM isolated point with  $j$ -invariant of fixed degree must appear.

**Proposition 3.1.** *Fix positive integers  $\ell, m \in \mathbf{N}$  with  $\ell$  prime. Let  $H \leq \mathbf{GL}_2(\mathbf{Z}_\ell)$  be an open subgroup containing  $-I$  of level  $\ell^n$  with  $n \geq m$ . Suppose  $x \in X_H$  is a non-cuspidal closed point with minimal representative  $(E, [\alpha]_H)$  such that  $E$  is non-CM and  $\rho_{E, \alpha, \ell^\infty}(\mathrm{Gal}_{\mathbf{Q}(j(E))}) \leq \mathbf{GL}_2(\mathbf{Z}_\ell)$  has level dividing  $\ell^m$ . Let  $H' := \pi_{\ell^m}^{-1}(H(\ell^m))$  and let  $f$  denote the natural map  $X_H \rightarrow X_{H'}$ . Then  $\deg(x) = \deg(f) \deg(f(x))$ .*

**3.1. Preliminary results.** We shall need the following lemma.

**Lemma 3.2.** *Let  $H$  be a subgroup of a group  $G$  and let  $f : G \rightarrow G'$  be a surjective homomorphism of groups.*

- (1) *If  $H$  has finite index in  $G$ , then  $[G : H] \geq [f(G) : f(H)]$ .*
- (2) *If  $H' := f^{-1}(f(H))$  and  $K := \ker(f)$ , then  $H' = HK$ .*

*Proof.*

- (1) Let  $G/H$  denote the set of all cosets of  $H$  in  $G$ . Define a function  $\phi : G/H \rightarrow f(G)/f(H)$  by  $xH \mapsto f(x)f(H)$ . First, we show that  $\phi$  is well-defined. Suppose  $xH = yH$ . Then  $y^{-1}x \in H$ , so  $f(y^{-1}x) \in f(H)$ . Then

$$f(H) = f(y^{-1}x)f(H) = f(y)^{-1}f(x)f(H),$$

which implies

$$\phi(xH) = f(x)f(H) = f(y)f(H) = \phi(yH).$$

Thus,  $\phi$  is well-defined. Moreover,  $\phi$  is surjective because  $f$  is surjective. Hence, we conclude

$$[G : H] = \#(G/H) \geq \#(f(G)/f(H)) = [f(G) : f(H)].$$

- (2) First, we show  $HK \subseteq H'$ . Let  $x \in HK$  be given and suppose  $x = hk$  for some  $h \in H$  and  $k \in K$ . Then

$$f(x) = f(hk) = f(h)f(k) = f(h) \in f(H).$$

Thus,  $x \in H'$ .

Now we show  $H' \subseteq HK$ . Let  $x \in H'$  be given. Then  $f(x) \in f(H)$ , so there exists  $h \in H$  with  $f(x) = f(h)$ . Notice

$$1 = f(x)f(h)^{-1} = f(xh^{-1}),$$

so  $xh^{-1} \in K$ . Thus,  $x \in KH = HK$ . ■

### 3.2. Proof of Proposition 3.1.

*Proof.* By Lemma 2.6, we know  $\deg(x) \leq \deg(f) \deg(f(x))$ , so it suffices to show

$$(1) \quad \frac{\deg(x)}{\deg(f(x))} \geq \deg(f).$$

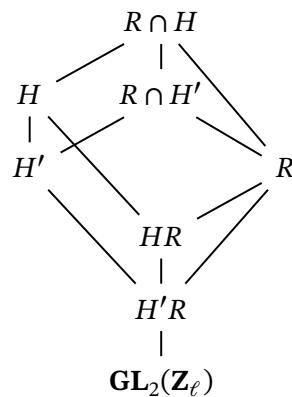
Let  $\pi_{\ell^\infty}$  denote the projection

$$\mathrm{GL}_2(\widehat{\mathbf{Z}}) \xrightarrow{\sim} \prod_p \mathrm{GL}_2(\mathbf{Z}_p) \rightarrow \mathrm{GL}_2(\mathbf{Z}_\ell).$$

Since  $H$  contains  $-I$ , Theorem 2.4 implies  $\deg(f) = [\pi_{\ell^\infty}^{-1}(H') : \pi_{\ell^\infty}^{-1}(H)] = [H' : H]$ . Then

$$\begin{aligned} \frac{\deg(x)}{\deg(f(x))} &= \frac{[\mathbf{Q}(j(E)) : \mathbf{Q}][\mathrm{im} \rho_{E,\alpha} : \mathrm{im} \rho_{E,\alpha} \cap \pi_{\ell^\infty}^{-1}(H)]}{[\mathbf{Q}(j(E)) : \mathbf{Q}][\mathrm{im} \rho_{E,\alpha} : \mathrm{im} \rho_{E,\alpha} \cap \pi_{\ell^\infty}^{-1}(H')]} && \text{(By Corollary 2.3)} \\ &= [\mathrm{im} \rho_{E,\alpha} \cap \pi_{\ell^\infty}^{-1}(H') : \mathrm{im} \rho_{E,\alpha} \cap \pi_{\ell^\infty}^{-1}(H)] \\ &\geq [\pi_{\ell^\infty}(\mathrm{im} \rho_{E,\alpha} \cap \pi_{\ell^\infty}^{-1}(H')) : \pi_{\ell^\infty}(\mathrm{im} \rho_{E,\alpha} \cap \pi_{\ell^\infty}^{-1}(H))] && \text{(By Lemma 1.3.2)} \\ &= [\mathrm{im} \rho_{E,\alpha,\ell^\infty} \cap H' : \mathrm{im} \rho_{E,\alpha,\ell^\infty} \cap H]. \end{aligned}$$

Denote  $R := \rho_{E,\alpha,\ell^\infty}(\mathrm{Gal}_{\mathbf{Q}(j(E))})$  and consider the following subset lattice:



The set  $HR$  need not be a group. However,  $HR$  is a union of left cosets of  $R$ , the number of which we denote by  $[HR : R]$ . By assumption,  $R$  and  $H$  both have finite index in  $\mathbf{GL}_2(\mathbf{Z}_\ell)$ , so all subgroups in the above lattice have finite index. The map  $H/(R \cap H) \rightarrow HR/R$  defined by  $h(R \cap H) \mapsto hR$  is a well-defined bijection of sets, so  $[H : R \cap H] = [HR : R]$ . Similarly, we have  $[H' : R \cap H'] = [H'R : R]$ .

Notice that Lemma 3.2.2 implies  $H' = H(\ker \pi_{\ell^m})$ . Moreover,  $R$  contains  $\ker \pi_{\ell^m}$  because  $R$  has level dividing  $\ell^m$ . This implies  $H'R = H(\ker \pi_{\ell^m})R = HR$ . Then,

$$\begin{aligned} [R \cap H' : R \cap H] &= [H' : H] \frac{[H : R \cap H]}{[H' : R \cap H']} \\ &= [H' : H] \frac{[HR : R]}{[H'R : R]} \\ &= [H' : H] \frac{[HR : R]}{[HR : R]} \\ &= [H' : H]. \end{aligned}$$

Thus, we have verified inequality (1). ■

**3.3. Theorem 1.1.** The proof of Theorem 1.1 relies on a result of Cadoret and Tamagawa [CT13, Theorem 1.1], which, for a fixed prime  $\ell$  and a fixed positive integer  $d$ , gives a uniform bound on the level of  $\text{im } \rho_{E, \ell^\infty}$ , as  $E$  varies over all non-CM elliptic curves defined over number fields of degree  $d$ .

**Theorem 3.3.** *Fix a prime  $\ell \in \mathbf{N}$  and define*

$$\mathcal{I} := \{x \in X_H \mid x \text{ is isolated and } H \leq \mathbf{GL}_2(\mathbf{Z}_\ell) \text{ an open subgroup}\}.$$

*For every  $d \in \mathbf{N}$ , there are finitely many  $j$ -invariants of degree  $d$  in  $j(\mathcal{I})$ .*

*Proof.* Let  $x \in \mathcal{I}$  be given. Then  $x \in X_H$  is isolated for some subgroup  $H \leq \mathbf{GL}_2(\mathbf{Z}_\ell)$  of level  $\ell^n$ . Since  $X_H$  and  $X_{\pm H}$  are isomorphic as curves, we can assume  $H$  contains  $-I$ . Define  $k := \mathbf{Q}(j(x))$  and  $d := [k : \mathbf{Q}]$ . Let  $(E, [\alpha]_H)$  be a minimal representative for  $x$ . By Theorem 2.1, there are only finitely many CM  $j$ -invariants in number fields of degree  $d$ . As such, it suffices to consider the case where  $E$  is non-CM.

By [CT13, Theorem 1.1], there exists a number  $m \in \mathbf{N}$  such that for every degree  $d$  number field  $k'$  and every non-CM elliptic curve  $E'/k'$ , the image of the  $\ell$ -adic Galois representation associated to  $E'/k'$ ,

$$\rho_{E', \ell^\infty} : \text{Gal}_{k'} \rightarrow \text{Aut}(T_\ell(E)) \simeq \mathbf{GL}_2(\mathbf{Z}_\ell)$$

has level dividing  $\ell^m$ .

For each  $n' \leq m$ , there exist finitely many subgroups of  $\mathbf{GL}_2(\mathbf{Z}/\ell^{n'}\mathbf{Z})$  and thus finitely many modular curves of level  $\ell^{n'}$ , each of which has finitely many isolated points by [Bou+19, Theorem 4.2]. So, we may assume  $n > m$ .

Let  $H' \leq \mathbf{GL}_2(\mathbf{Z}_\ell)$  be the subgroup  $\pi_{\ell^m}^{-1}(H(\ell^m))$ . Let  $f : X_H \rightarrow X_{H'}$  denote the natural inclusion map. By Proposition 3.1,

$$\deg(x) = \deg(f) \deg(f(x)).$$

Then Theorem 2.7 implies  $f(x)$  is isolated.

Thus,  $j(x)$  is the  $j$ -invariant of an isolated point on a modular curve of level dividing  $\ell^m$ . There are only finitely many such modular curves and each has finitely many isolated points.

Since  $m$  was dependent only on  $d$ , we conclude  $j(\mathcal{J})$  contains finitely many  $j$ -invariants of degree  $d$ .  $\blacksquare$

It is natural to ask if Theorem 3.3 holds as  $\ell$  varies over all primes. This is in fact a stronger claim than an affirmative answer to Serre's uniformity problem. Indeed, suppose the set  $\mathcal{J}$  of rational isolated  $j$ -invariants on modular curves of prime-power level is finite. In particular, this implies there are only finitely many rational  $j$ -invariants corresponding to isolated points on  $X_{C_{\text{ns}}^+(\ell)}$ , as  $\ell$  ranges over all prime numbers. For each non-CM  $j \in \mathcal{J}$ , there exists a non-CM elliptic curve  $E/\mathbf{Q}$  with  $j(E) = j$ . Serre's open image theorem [Ser72, Théorème 3] ensures that for sufficiently large primes  $\ell$ , the mod- $\ell$  representation of Galois associated to  $E$  is surjective. So, there are only finitely-many primes  $\ell$  with  $\rho_{E,\ell}(\text{Gal}_{\mathbf{Q}})$  conjugate to a subgroup of  $C_{\text{ns}}^+(\ell)$ . This implies that  $j$  arises as an isolated point on only finitely many modular curves  $X_{C_{\text{ns}}^+(\ell)}$ , each of which has finitely many isolated points.

#### 4. INTERMEDIATE MODULAR CURVES

Let  $n$  be a positive integer and let  $\Delta$  be a subgroup of  $(\mathbf{Z}/n\mathbf{Z})^\times$ . We define a subgroup of  $\mathbf{GL}_2(\mathbf{Z}/n\mathbf{Z})$  as follows:

$$B_\Delta(n) := \left\{ \begin{pmatrix} \delta & a \\ 0 & b \end{pmatrix} \mid \delta \in \Delta, a \in \mathbf{Z}/n\mathbf{Z}, b \in (\mathbf{Z}/n\mathbf{Z})^\times \right\}.$$

Moreover, we denote  $B_1(n)$  (resp.  $B_{\pm 1}(n)$ ;  $B_0(n)$ ) to be  $B_\Delta(n)$ , with  $\Delta = \{1\}$  (resp.  $\Delta = \{\pm 1\}$ ;  $\Delta = (\mathbf{Z}/n\mathbf{Z})^\times$ ).

We will make frequent use of the following formulae.

**Proposition 4.1.** *Let  $n \in \mathbf{N}$  have prime factorization  $p_1^{a_1} \cdots p_r^{a_r}$  and let  $\Delta$  be a subgroup of  $(\mathbf{Z}/n\mathbf{Z})^\times$ .*

- (1)  $\#\mathbf{GL}_2(\mathbf{Z}/n\mathbf{Z}) = \prod_{i=1}^r p_i^{4a_i-3}(p_i^2 - 1)(p_i - 1)$ .
- (2)  $\#B_\Delta(n) = (\#\Delta) \prod_{i=1}^r p_i^{2a_i-1}(p_i - 1)$ .
- (3)  $[\mathbf{GL}_2(\mathbf{Z}/n\mathbf{Z}) : B_\Delta(n)] = \frac{1}{\#\Delta} \prod_{i=1}^r p_i^{2a_i-2}(p_i^2 - 1)$ .

*Proof.* Formula (1) follows from the short exact sequence

$$1 \rightarrow \mathbf{SL}_2(\mathbf{Z}/n\mathbf{Z}) \hookrightarrow \mathbf{GL}_2(\mathbf{Z}/n\mathbf{Z}) \xrightarrow{\det} (\mathbf{Z}/n\mathbf{Z})^\times \rightarrow 1,$$

formula (2) follows from the definition of  $B_\Delta(n)$ , and formula (3) follows from Lagrange's theorem.  $\blacksquare$

It will occasionally be useful to rewrite Proposition 4.1.3 as

$$[\mathbf{GL}_2(\mathbf{Z}/n\mathbf{Z}) : B_\Delta(n)] = \frac{\phi(n)}{\#\Delta} [\mathbf{GL}_2(\mathbf{Z}/n\mathbf{Z}) : B_0(n)].$$

Let  $n$  be a positive integer and let  $\Delta$  be a subgroup of  $(\mathbf{Z}/n\mathbf{Z})^\times$  be given. We define the **modular curve**  $X_\Delta(n)$  to be the modular curve  $X_{B_\Delta(n)}$ . If  $\Delta$  is such that  $\{\pm 1\} \subsetneq \Delta \subsetneq (\mathbf{Z}/n\mathbf{Z})^\times$ , we call  $X_\Delta(n)$  an **intermediate modular curve**, since it is between  $X_1(n)$  and  $X_0(n)$ .

We briefly describe another construction of  $X_\Delta(n)$ . Define a congruence subgroup associated to  $\Delta$  as follows:

$$\Gamma_\Delta(n) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}_2(\mathbf{Z}) \mid c \equiv 0 \pmod{n}, (a \pmod{n}) \in \Delta \right\}.$$

The group  $\Gamma_\Delta(n)$  inherits from  $\mathbf{SL}_2(\mathbf{Z})$  the action on the upper half plane  $\mathbf{H}$  by linear fractional transformations. This action can be extended to  $\mathbf{H}^* := \mathbf{H} \cup \mathbf{P}^1(\mathbf{Q})$ , the extended upper half plane, and the quotient space  $\mathbf{H}^*/\Gamma_\Delta(n)$  is a Riemann surface. There exists a smooth projective curve  $X(\Gamma_\Delta(n))/\mathbf{Q}$  and a complex analytic isomorphism  $\mathbf{H}^*/\Gamma_\Delta(n) \rightarrow X(\Gamma_\Delta(n))(\mathbf{C})$  (c.f. [Sil09, Remark C.13.2]). Moreover, the curves  $X(\Gamma_\Delta(n))$  and  $X_\Delta(n)$  are isomorphic over  $\mathbf{Q}$ . Note that  $\mathbf{H}^*$  is fixed under the action by  $-I \in \mathbf{SL}_2(\mathbf{Z})$ , so for any subgroup  $\Delta < (\mathbf{Z}/n\mathbf{Z})^\times$ , the curves  $X_\Delta(n)$  and  $X_{\pm\Delta}(n)$  are isomorphic. For this reason, in this paper we only consider subgroups  $\Delta$  containing  $-1$ .

For a point  $P$  of order  $n$  on an elliptic curve  $E$ , we define  $\Delta P$  to be the set  $\{\delta P \mid \delta \in \Delta\}$  and  $\mathbf{Q}(\Delta P)$  to be the fixed field<sup>2</sup> of  $\{\sigma \in \text{Gal}_{\mathbf{Q}} \mid \sigma(\Delta P) = \Delta P\}$ . Then the non-cuspidal points of  $X_\Delta(n)(\overline{\mathbf{Q}})$  parametrize equivalence classes of pairs  $(E, \Delta P)$ , where  $E$  is an elliptic curve and  $P \in E$  is a point of order  $n$ . Moreover, with this construction, every non-cuspidal  $k$ -rational point of  $X_\Delta(n)$  is of the form  $[(E, \Delta P)]$ , with  $E/k$  an elliptic curve and  $P \in E(k)$  a point of order  $n$  such that  $\Delta P$  is fixed by  $\text{Gal}_k$ .

**Lemma 4.2.** *Let  $n \in \mathbf{N}$  and  $\{\pm 1\} \leq \Delta \leq (\mathbf{Z}/n\mathbf{Z})^\times$  be given. Let  $E/\mathbf{Q}(j(E))$  be a non-CM elliptic curve and let  $P \in E$  be a point of order  $n$ . If  $x \in X_\Delta(n)$  is the closed point associated to  $(E, \Delta P)$ , then*

$$\deg(x) = [\mathbf{Q}(j(E), \Delta P) : \mathbf{Q}].$$

*Proof.* Let  $\mathcal{S} \leq \text{Gal}_{\mathbf{Q}}$  denote the stabilizer of the action of  $\text{Gal}_{\mathbf{Q}}$  on  $[(E, \Delta P)]$ . Since  $E$  is defined over  $\mathbf{Q}(j(E))$ , we have  $\sigma E = E$  for all  $\sigma \in \text{Gal}_{\mathbf{Q}(j(E))}$ . Moreover, if  $\sigma \in \text{Gal}_{\mathbf{Q}}$  fixes  $E$ , then there exists an isomorphism  $\phi : E \rightarrow \sigma E$  defined over  $\overline{\mathbf{Q}}$ . This implies  $j(E) = j(\sigma E)$ , so  $\sigma \in \text{Gal}_{\mathbf{Q}(j(E))}$ . We then have

$$\begin{aligned} \mathcal{S} &= \{\sigma \in \text{Gal}_{\mathbf{Q}} \mid \sigma[(E, \Delta P)] = [(E, \Delta P)]\} \\ &= \{\sigma \in \text{Gal}_{\mathbf{Q}} \mid \sigma E = E\} \cap \{\sigma \in \text{Gal}_{\mathbf{Q}} \mid \sigma(\Delta P) = \Delta P\} \\ &= \text{Gal}_{\mathbf{Q}(j(E))} \cap \text{Gal}_{(\mathbf{Q}(\Delta P))}. \end{aligned}$$

By the orbit-stabilizer theorem and Galois theory, we conclude

$$[\mathbf{Q}(x) : \mathbf{Q}] = [\text{Gal}_{\mathbf{Q}} : \mathcal{S}] = [\mathbf{Q}^{\mathcal{S}} : \mathbf{Q}] = [\mathbf{Q}(j(E))\mathbf{Q}(\Delta P) : \mathbf{Q}] = [\mathbf{Q}(j(E), \Delta P) : \mathbf{Q}]. \quad \blacksquare$$

## 5. INTERMEDIATE MODULAR CURVES OF PRIME-POWER LEVEL

**5.1. Rational isolated  $j$ -invariants on  $X_\Delta(\ell^n)$ .** In this section, we consider the rational  $j$ -invariants that arise from isolated points on intermediate modular curves of prime-power level. Ejder [Ejd22] proved that there are finitely many rational  $j$ -invariants arising from isolated points on  $X_1(\ell^n)$ , ranging over all primes  $\ell > 7$ , and gave a partial classification of which rational  $j$ -invariants may appear. Bourdon and Ejder [BE25] completed this classification for  $X_1(\ell^n)$  and proved an analogous result for the family of modular curves  $X_0(\ell^n)$ .

**Theorem 5.1** ([BE25, Theorems 1 and 2.]). *Let  $\ell$  be a prime, let  $n$  be a positive integer, and let  $j \in \mathbf{Q}$  be a rational number.*

- (1) *There exists an isolated point  $x \in X_1(\ell^n)$  with  $j(x) = j$  if and only if  $j$  is a CM  $j$ -invariant,  $-7 \cdot 11^3$ , or  $-7 \cdot 137^3 \cdot 2083^3$ . Moreover, the non-CM  $j$ -invariants occur if and only if  $\ell = 37$ .*

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<sup>2</sup>Note  $\mathbf{Q}(\Delta P)$  is not the adjunction of the coordinates of points in  $\Delta P$ .

- (2) There exists an isolated point  $x \in X_0(\ell^n)$  with  $j(x) = j$  if and only if  $j$  is a CM  $j$ -invariant,  $-11 \cdot 131^3$ ,  $-11^2$ ,  $-17^2 \cdot 101^3/2$ ,  $-17 \cdot 373^3/2^{17}$ ,  $-7 \cdot 11^3$ , or  $-7 \cdot 137^3 \cdot 2083^3$ . Moreover, the non-CM  $j$ -invariants in this list correspond to isolated rational points on  $X_0(\ell)$ .

Our goal for this section is to consider the above theorem in the setting of intermediate modular curves of prime-power level. Notice that in the above theorem, the  $j$ -invariants arising from  $X_1(\ell^n)$  all appear as  $j$ -invariants arising from  $X_0(\ell)$ . We will see that any rational  $j$ -invariant corresponding to an isolated point on a modular curve  $X_\Delta(\ell^n)$  must appear in the list of rational isolated  $j$ -invariants arising from  $X_0(\ell)$ .

**5.2. Possible  $\ell$ -adic images of Galois.** We shall need a classification by Rouse, Sutherland, and Zureick-Brown [RSZ22] of the possible  $\ell$ -adic images of Galois associated to non-CM elliptic curves over  $\mathbf{Q}$ . The classification builds on work of Sutherland and Zywnina [SZ17] and of Rouse and Zureick-Brown [RZ15].

**Theorem 5.2** ([RSZ22, Theorem 1.1.6], [BE25, Proposition 1]). *Let  $E/\mathbf{Q}$  be a non-CM elliptic curve, let  $\ell$  be a prime, and denote  $R := \rho_{E,\ell^\infty}(\text{Gal}_{\mathbf{Q}})$ . Then exactly one of the following is true:*

- (a) *The modular curve  $X_R$  is isomorphic to  $\mathbf{P}^1$  or a rank one elliptic curve.*
- (b) *The modular curve  $X_R$  has an exceptional rational point with known image.*
- (c)  *$R$  is conjugate to a subgroup of  $C_{ns}^+(3^3), C_{ns}^+(5^2), C_{ns}^+(7^2), C_{ns}^+(11^2)$  or  $C_{ns}^+(\ell)$  for  $\ell \geq 19$ .*
- (d)  *$R$  is conjugate to a subgroup of  $49 \cdot 147 \cdot 9 \cdot 1$  or  $49 \cdot 196 \cdot 9 \cdot 1$ .*

For a non-CM  $j$ -invariant attached to an elliptic curve  $E/\mathbf{Q}$ , Theorem 5.2 gives us four cases to consider. If  $R$  is as in case (a), the possible subgroups that can arise as  $R$  are all known. Specifically, [SZ17, Corollary 1.6] says that for  $\ell = 2, 3, 5, 7, 11, 13$ , there are 1201, 47, 23, 15, 2, 11 subgroups of  $\mathbf{GL}_2(\mathbf{Z}_\ell)$  that can arise as  $R$ , respectively. And for  $\ell > 13$ , the only possible subgroup is  $R = \mathbf{GL}_2(\mathbf{Z}_\ell)$ . If  $R$  is as in case (b), there are 23 possible known images for  $R$ , which have levels  $2^4, 2^3, 5^2, 7, 11, 13, 17, 37$ . It is conjectured [RSZ22, Conjecture 1.1.5] that these known subgroups are all of the exceptional groups of prime-power level. If  $R$  is as in case (d) and is conjugate to a subgroup of  $49 \cdot 196 \cdot 1$ , then Bourdon and Ejder [BE25] showed that  $R$  must be conjugate to  $49 \cdot 196 \cdot 9 \cdot 1$ .

**Proposition 5.3** ([Ejd22, Proposition 3.1]). *Let  $E/\mathbf{Q}$  be a non-CM elliptic curve. Suppose  $\text{im } \rho_{E,\ell}$  is equal to the normalizer of a nonsplit Cartan subgroup for some prime  $\ell > 13$ . For  $P \in E(\overline{\mathbf{Q}})$  of order  $\ell^n$ , let  $x = [(E, P)] \in X_1(\ell^n)$  be the associated closed point on the modular curve. Then*

$$\deg(x) = \frac{1}{2}(\ell^2 - 1)\ell^{2n-2} = \deg(X_1(\ell^n) \rightarrow X(1)).$$

As noted in [BE25], there was in an error in a result used to prove [Ejd22, Proposition 3.1], so the proof is given in [BE25, Proposition 4]. From the above proposition, we immediately obtain the following corollary.

**Corollary 5.4.** *Let  $E/\mathbf{Q}$  be a non-CM elliptic curve. Suppose  $\text{im } \rho_{E,\ell}$  is equal to the normalizer of a nonsplit Cartan subgroup for some prime  $\ell > 13$ . For  $P \in E(\overline{\mathbf{Q}})$  of order  $\ell^k$  and  $\{\pm 1\} \leq \Delta \leq (\mathbf{Z}/\ell^n\mathbf{Z})^\times$ , let  $x = [(E, \Delta P)] \in X_\Delta(\ell^n)$  be the associated closed point on the modular curve. Then*

$$\deg(x) = \frac{1}{\#\Delta}(\ell^2 - 1)\ell^{2n-2} = \deg(X_\Delta(\ell^n) \rightarrow X(1)).$$

We will make use of the data from an algorithm given in [BE25], which we will refer to as the  $\Gamma_1$ -algorithm. As explained in the proof of [BE25, Theorem 4] the  $\Gamma_1$ -algorithm takes as input an image  $\text{im } \rho_{E,\ell^\infty}$  attached to a non-CM elliptic curve  $E/\mathbf{Q}$  and returns a finite list of pairs  $\{\langle \ell^{a_1}, d_1 \rangle, \dots, \langle \ell^{a_r}, d_r \rangle\}$  such that there exists some isolated point  $x \in X_1(\ell^n)$  having  $j(x) = j(E)$  if and only if there is some  $i \in \{1, \dots, r\}$  and some isolated point  $x_i \in X_1(\ell^{a_i})$  of degree  $d_i$  and with  $j(x_i) = j(E)$ . The algorithm removes any pairs  $\langle \ell^{a_i}, d_i \rangle$  with  $d_i > \text{genus}(X_1(\ell^{a_i}))$  since the corresponding point cannot be isolated by Theorem 2.5. In particular, if the  $\Gamma_1$ -algorithm returns an empty list for a particular subgroup  $\text{im } \rho_{E,\ell^\infty}$ , then there are no isolated points above  $j(E)$  on a modular curve  $X_1(\ell^n)$ .

**Proposition 5.5.** *Let  $\ell \in \mathbf{N}$  be prime and  $n \in \mathbf{N}$  a positive integer. Let  $E/\mathbf{Q}$  be a non-CM elliptic curve and suppose  $\rho_{E,\ell^\infty}(\text{Gal}_{\mathbf{Q}})$  is conjugate to a known image, as in cases (a) or (b), not equal to  $\mathbf{GL}_2(\mathbf{Z}_\ell)$ . If  $\ell \neq 11, 17, 37$ , then there are no isolated points above  $j(E)$  on a modular curve  $X_\Delta(\ell^n)$ .*

*Proof.* We are assuming that  $\text{im } \rho_{E,\ell^\infty}$  is conjugate to a subgroup not equal to  $\mathbf{GL}_2(\mathbf{Z}_\ell)$  which appears in [SZ17, Tables 1-4] (case (a)) or in [RSZ22, Table 1] (case (b)). When run on all of these known images, the output of the  $\Gamma_1$ -algorithm is empty except for the subgroups 17.72.1.2, 37.114.4.1, and 37.114.4.2.

Suppose  $x \in X_1(\ell^n)$  is a closed point lying above  $j(E)$  and let  $\ell^m$  be the level of  $\rho_{E,\ell^\infty}(\text{Gal}_{\mathbf{Q}})$  and recall that  $\ell^m$  is in the set

$$\{1, 2, 4, 8, 16, 32, 3, 9, 27, 5, 25, 7, 11, 13, 17\}.$$

We are assuming that  $\ell \neq 11, 17, 37$ . Moreover, there are no intermediate modular curves of levels 1, 2, 4, 8, 3, 9, 5, 11, so it suffices to consider levels  $\ell^m$  in the set

$$\mathcal{L} := \{13, 16, 25, 27, 32\}.$$

Let  $\Delta \leq (\mathbf{Z}/\ell^n\mathbf{Z})^\times$  be a subgroup containing  $-1$ . For each non-negative integer  $j \leq n$ , we have a projection map  $\pi_j : X_1(\ell^j) \rightarrow X_\Delta(\ell^j)$  and natural maps  $f_j : X_1(\ell^n) \rightarrow X_1(\ell^j)$  and  $g_j : X_\Delta(\ell^n) \rightarrow X_\Delta(\ell^j)$ . By Proposition 3.1, we have

$$\deg(x) = \deg(f_m) \deg(f_m(x)) \text{ and } \deg(\pi_n(x)) = \deg(g_m) \deg(g_m(\pi_n(x))).$$

Let  $a$  and  $b$  be the least non-negative integers such that

$$\deg(x) = \deg(f_a) \deg(f_a(x)) \text{ and } \deg(\pi_n(x)) = \deg(g_b) \deg(g_b(\pi_n(x)))$$

and notice that  $m \geq a \geq b$ .

First, suppose  $a > b$ . Examining  $\mathcal{L}$ , we see the only possibility is  $\ell^a = 32$  and  $\ell^b = 16$ . Define  $\Delta_1 := \{1, 7, 9, 15\}$  and note that  $X_{\Delta_1}(16)$  is the only intermediate modular curve of level 16. However,  $\text{genus}(X_{\Delta_1}(16)) = 0$ , so there are no isolated points on  $X_{\Delta_1}(16)$ .

Next, suppose  $a = b$ . Observe that if  $\deg(f_a(x)) > \text{genus}(X_1(\ell^a))$ , then we have  $\deg(\pi_n(x)) > (2/\#\Delta) \text{genus}(X_1(\ell^a))$ . In particular, if  $\text{genus}(X_\Delta(\ell^a)) < (2/\#\Delta) \text{genus}(X_1(\ell^a))$ , then there are no isolated points on  $X_\Delta(\ell^a)$  by Theorem 2.5. The data in Table 1 shows that for modular curves  $X_\Delta(\ell^a)$  with  $\ell^a \in \mathcal{L}$ , we have  $\text{genus}(X_\Delta(\ell^a)) \geq (2/\#\Delta) \text{genus}(X_1(\ell^a))$  if and only if  $X_\Delta(\ell^a) = X_0(17)$ . ■

We now consider the situation when  $R := \text{im } \rho_{E,\ell^\infty}$  is as in case (c) of Theorem 5.2. That is, we have  $R \leq C_{\text{ns}}^+(3^3), C_{\text{ns}}^+(5^2), C_{\text{ns}}^+(7^2), C_{\text{ns}}^+(11^2)$  or  $C_{\text{ns}}^+(\ell)$  for  $\ell \geq 19$ . For this, a key ingredient is work of Furio [Fur25, Theorem 1.9], which characterizes the possibilities for  $R$  in this case.

$\ell^a$	$\#\Delta$	genus( $X_\Delta(\ell^a)$ )	$(2/\#\Delta)\text{genus}(X_1(\ell^a))$
13	4	0	1
13	6	0	2/3
13	12	0	1/3
16	4	0	1
16	8	0	1/2
25	4	4	6
25	10	0	12/5
25	20	0	6/5
27	6	1	13/3
27	18	1	13/9
32	4	5	17/2
32	8	1	17/4
32	16	1	17/8

TABLE 1. Data for  $X_\Delta(\ell^a)$  (excluding  $X_1(\ell^a)$ ) with  $\ell^a \in \mathcal{L}$ .

**Proposition 5.6.** *Let  $\ell$  be an odd prime and  $n \in \mathbf{N}$ . Let  $E/\mathbf{Q}$  be a non-CM elliptic curve and  $\Delta$  a subgroup  $\{\pm 1\} < \Delta < (\mathbf{Z}/\ell^n\mathbf{Z})^\times$ . If  $\text{im } \rho_{E,\ell}$  is contained in the normalizer of a non-split Cartan subgroup, then there is no isolated point  $x \in X_\Delta(\ell^n)$  with  $j(x) = j(E)$ .*

*Proof.* Let  $x = [(E, \Delta P)] \in X_\Delta(\ell^n)$  be such that  $j(x) = j(E)$ . By Proposition 5.5, we may assume  $\text{im } \rho_{E,\ell^\infty}$  is not a known image. By [Fur25, Theorem 1.9], either  $\text{im } \rho_{E,\ell^\infty}$  has level  $\ell^d$  and is conjugate to  $C_{\text{ns}}^+(\ell^d)$  or  $\text{im } \rho_{E,\ell^\infty}$  has level  $\ell^2$  and

$$\text{im } \rho_{E,\ell^2} \simeq C_{\text{ns}}^+(\ell) \ltimes \left\{ I + \ell \begin{pmatrix} a & \varepsilon b \\ -b & c \end{pmatrix} \right\},$$

with the semidirect product defined by the conjugation action.

Suppose  $\text{im } \rho_{E,\ell^\infty}$  has level  $\ell^d$  and is conjugate to  $C_{\text{ns}}^+(\ell^d)$ . The proof of [BE25, Theorem 6], which proves the claim for the case of  $X_\Delta(\ell^d) = X_1(\ell^d)$ , shows that  $[(E, P)] \in X_1(\ell^n)$  has degree  $\ell^{2n-2}(\ell^2 - 1)/2$ , so  $\deg(x) = \ell^{2n-1}(\ell^2 - 1)/(\#\Delta)$ . But then Theorem 2.7 implies that the image of  $x$  under the natural map  $X_\Delta(\ell^n) \rightarrow X(1)$  is isolated, which is absurd, since  $X(1)$  has genus 0.

Now suppose  $\text{im } \rho_{E,\ell^\infty}$  has level  $\ell^2$  and is isomorphic to the semidirect product above. The proof of [BE25, Theorem 6] shows that  $[E, P] \in X_1(\ell^n)$  has degree at least  $\ell(\ell^2 - 1)/2$ , so  $\deg(x) \geq \ell(\ell^2 - 1)/\#\Delta$ . If  $\ell > 13$ , then Corollary 5.4 and Theorem 2.7 imply there are no isolated points on  $X_\Delta(\ell^n)$ . For primes  $\ell \leq 13$ , the data in Table 2 combined with Theorem 2.5 shows that  $x$  is not isolated. ■

By the previous proposition, it remains to consider the case when  $\text{im } \rho_{E,\ell^\infty}$  is conjugate to 49.196.9.1.

**Proposition 5.7.** *Let  $E/\mathbf{Q}$  be a non-CM elliptic curve and suppose  $\text{im } \rho_{E,7^\infty}$  is conjugate to 49.196.9.1. Then there are no isolated points above  $j(E)$  on a modular curve  $X_\Delta(7^n)$ .*

*Proof.* Let  $E/\mathbf{Q}$  be a non-CM elliptic curve and suppose  $\text{im } \rho_{E,7^\infty}$  is conjugate to 49.196.9.1. A Magma computation shows that if  $x = [(E, \langle P \rangle)] \in X_0(49)$  is a closed point with  $j(x) = j(E)$ , then  $\deg(x) = 56$ . For each intermediate modular curve  $X_\Delta(49)$ , we have  $\text{genus}(X_\Delta(49)) < 56$ , so  $[(E, \Delta P)] \in X_\Delta(49)$  is not isolated by Theorem 2.5. ■

$\ell$	$\#\Delta$	genus( $X_\Delta(\ell^2)$ )	$\ell(\ell^2 - 1)/\#\Delta$
5	4	4	30
5	10	0	12
7	6	19	56
7	14	3	24
11	10	106	132
11	22	26	60
13	4	516	546
13	6	340	364
13	12	164	182
13	26	50	84
13	52	24	42
13	78	16	28

TABLE 2. Data for  $X_\Delta(\ell^2)$  with  $\ell \leq 13$ .

5.3. **Proof of Theorem 1.3.** We restate Theorem 1.3 below for convenience.

**Theorem 5.8.** *Let  $\ell \in \mathbf{N}$  be a prime, let  $n \in \mathbf{N}$  a positive number, and suppose  $\Delta \leq (\mathbf{Z}/\ell^n\mathbf{Z})^\times$  is a subgroup containing  $-1$ . If  $x \in X_\Delta(\ell^n)$  is a non-cuspidal isolated point with non-CM rational  $j$ -invariant  $j \in \mathbf{Q}$ , then  $j$  appears as the  $j$ -invariant of an isolated point on  $X_0(\ell)$ , for some  $\ell \in \{11, 17, 37\}$ .*

*Proof.* Let  $E/\mathbf{Q}$  be a non-CM elliptic curve and let  $\ell \in \mathbf{N}$  be prime. By Theorem 5.2, there are four cases for the image  $R := \rho_{E,\ell^\infty}(\text{Gal}_{\mathbf{Q}}) \leq \text{GL}_2(\mathbf{Z}_\ell)$ . If  $R$  is in case (a) or (b) and  $\ell \neq 11, 17, 37$ , then Proposition 5.5 implies that  $j(E)$  is not the  $j$ -invariant of an isolated point on a modular curve  $X_\Delta(\ell^n)$ . Similarly, if  $R$  is in case (c) or case (d), then Proposition 5.6 and Proposition 5.7 imply that  $j(E)$  does not appear as the  $j$ -invariant of an isolated point on a modular curve  $X_\Delta(\ell^n)$ .

It remains to consider the case when  $\ell = 11, 17, 37$  and  $R$  is a known image. Because  $\phi(11)/2$  is prime, there are no intermediate modular curves of level 11. The  $j$ -invariants of the known images of levels 17 and 37 all arise from rational points on  $X_0(17)$  or  $X_0(37)$ , as desired.  $\blacksquare$

## 6. REFERENCES

- [BE25] Abbey Bourdon and Özlem Ejder. Rational isolated  $j$ -invariants from  $X_1(\ell^n)$  and  $X_0(\ell^n)$ . 2025. arXiv: 2506.19560 [math.NT].
- [Bou+19] Abbey Bourdon, Özlem Ejder, Yuan Liu, Frances Odumodu, and Bianca Viray. “On the level of modular curves that give rise to isolated  $j$ -invariants”. In: Adv. Math. 357 (2019), pp. 106824, 33.
- [Bou+25] Abbey Bourdon, Sachi Hashimoto, Timo Keller, Zev Klagsbrun, David Lowry-Duda, et al. “Towards a classification of isolated  $j$ -invariants”. In: Math. Comp. 94.351 (2025). With an appendix by Maarten Derickx and Mark van Hoeij, pp. 447–473.
- [CT13] Anna Cadoret and Akio Tamagawa. “A uniform open image theorem for  $\ell$ -adic representations, II”. In: Duke Math. J. 162.12 (2013), pp. 2301–2344.

- [Cox13] David A. Cox. Primes of the form  $x^2 + ny^2$ . Second. Pure and Applied Mathematics (Hoboken). Fermat, class field theory, and complex multiplication. John Wiley & Sons, Inc., Hoboken, NJ, 2013, pp. xviii+356.
- [DS05] Fred Diamond and Jerry Shurman. A first course in modular forms. Vol. 228. Graduate Texts in Mathematics. Springer-Verlag, New York, 2005, pp. xvi+436.
- [Ejd22] Özlem Ejder. “Isolated points on  $X_1(\ell^n)$  with rational  $j$ -invariant”. In: Res. Number Theory 8.1 (2022), Paper No. 16, 7.
- [Fal83] G. Faltings. “Endlichkeitssätze für abelsche Varietäten über Zahlkörpern”. In: Invent. Math. 73.3 (1983), pp. 349–366.
- [Fur25] Lorenzo Furio. Effective bounds for adelic Galois representations attached to elliptic curves over the rationals. 2025. arXiv: 2412.10340 [math.NT].
- [Gro67] A. Grothendieck. “Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV”. In: Inst. Hautes Études Sci. Publ. Math. 32 (1967), p. 361.
- [Hei34] Hans Heilbronn. “On the class-number in imaginary quadratic fields”. In: The Quarterly Journal of Mathematics os-5 (1 1934), pp. 150–160.
- [Lee25] Meghan Lee. Isolated  $j$ -invariants arising from the modular curve  $X_0(n)$ . 2025. arXiv: 2507.19462 [math.NT].
- [Maz77] B. Mazur. “Rational points on modular curves”. In: Modular functions of one variable, V (Proc. Second Internat. Conf., Univ. Bonn, Bonn, 1976). Vol. Vol. 601. Lecture Notes in Math. Springer, Berlin-New York, 1977, pp. 107–148.
- [Men22] Zonia Menendez. Images of sporadic points on the family of modular curves  $X_0(n)$ . 2022.
- [Poo17] Bjorn Poonen. Rational points on varieties. Vol. 186. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2017, pp. xv+337.
- [RSZ22] Jeremy Rouse, Andrew V. Sutherland, and David Zureick-Brown. “ $\ell$ -adic images of Galois for elliptic curves over  $\mathbb{Q}$  (and an appendix with John Voight)”. In: Forum Math. Sigma 10 (2022). With an appendix with John Voight, Paper No. e62, 63.
- [RZ15] Jeremy Rouse and David Zureick-Brown. “Elliptic curves over  $\mathbb{Q}$  and 2-adic images of Galois”. In: Res. Number Theory 1 (2015), Paper No. 12, 34.
- [Ser72] Jean-Pierre Serre. “Propriétés galoisiennes des points d’ordre fini des courbes elliptiques”. In: Invent. Math. 15.4 (1972), pp. 259–331.
- [Sil09] Joseph H. Silverman. The arithmetic of elliptic curves. Second. Vol. 106. Graduate Texts in Mathematics. Springer, Dordrecht, 2009.
- [Sut16] Andrew V. Sutherland. “Computing images of Galois representations attached to elliptic curves”. In: Forum Math. Sigma 4 (2016), Paper No. e4, 79.
- [SZ17] Andrew V. Sutherland and David Zywina. “Modular curves of prime-power level with infinitely many rational points”. In: Algebra Number Theory 11.5 (2017), pp. 1199–1229.
- [Ter24] Kenji Terao. Maps between isolated points on modular curves. 2024. arXiv: 2412.13108 [math.NT].
- [Zyw24] David Zywina. Explicit open images for elliptic curves over  $\mathbb{Q}$ . 2024. arXiv: 2206.14959 [math.NT].

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