COMPUTING ALGEBRAIC INVARIANTS OF TENSORS AND THEIR APPLICATION TO PRODUCT DECOMPOSITIONS

CHRIS LIU

ABSTRACT. This dissertation proposal outlines two areas of study. First, on algorithms for efficient algorithms of algebraic invariants of tensors. Second, an investigation into decomposing tensors as products of smaller tensors.

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1. INTRODUCTION

Tensors encapsulate multilinear maps. Often given as a multiway array of numbers, tensors are used across various disciplines within mathematics and sciences to record information for some fixed reference frame. As such, they are studied from many complementary perspectives [Bro97] [KB09] [Lan12] [RS18] [DLDMV00] [Tuc66].

Throughout, we fix a field K, e.g $K = \mathbb{R}$. Given vector spaces U, V, and W, a function $f : U \times V \to W$ is K-bilinear if $f(ku, v) = f(u, kv) = kf(u, v)$ for all k in K, u in U, and v in V. We write $f : U \times V \rightarrowtail W$ ($\rightarrowtail$ for bilinear). When context is clear, we avoid the prefix K. The above extends to explain K-trilinear and K-multilinear in general. The space of K-multilinear maps from $U_n \times \cdots \times U_1$ to U_0 is denoted $\mathrm{Mult}(U_n, \ldots, U_1; U_0)$.

A **tensor space** T is a K-vector space equipped with a K-multilinear interpretation $\langle \cdot | : T \hookrightarrow \mathrm{Mult}(U_n, \ldots, U_1; U_0)$ for U_i each a K-vector space. A **tensor** t is an element of a tensor space T, and we write $\langle t| : U_n \times \cdots \times U_1 \rightarrowtail U_0$ as its multilinear interpretation. The spaces $\{U_0, \ldots, U_n\}$ are the frame of tensor, the size of the frame $(n+1)$ its valence, and the labels on the vector spaces (in this case $\{0, \ldots, n\}$), its axes. For $|u\rangle = |u_n, \ldots, u_1\rangle$, write $\langle t|u\rangle \in U_0$ to mean evaluating $\langle t|$ at $|u\rangle$.

This definition accommodates the common existing understanding of tensors as a multiway grid $\Gamma \in K^{d_1 \times \cdots \times d_n}$ of numbers. For example,

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Example 1.1. Given the (2×3) matrix

$$M = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

If M is to be interpreted as a bilinear form (a bilinear map into the underlying field), then $\langle M| : \mathbb{R}^2 \times \mathbb{R}^3 \rightarrowtail \mathbb{R}$ where $\langle M|u, v\rangle = u^T M v$.

For vector spaces U and V, the **tensor product** of U and V is the vector space $U \otimes V$ with the canonical map $\varphi : U \times V \rightarrowtail U \otimes V$, such that for every bilinear map f with domain $U \times V$, there is a unique induced linear map $\hat{f}$ satisfying $f = \hat{f} \circ \varphi$. For vectors $s \in U$ and $t \in V$, $s \otimes t$ is the image of (s,t) under φ. The tensor product space $U \otimes V$ is not a tensor space, but commonly a fixed isomorphism to $(U \otimes V)^*$ is given and $U \otimes V$ is understood to be a tensor space by this interpretation.

The perspective taken in this dissertation proposal is to study tensors as distributive products using the tools of algebra. Fix a 3-tensor (bimap) $* : U \times V \rightarrowtail W$.

Let $\mathrm{End}(U) = \mathrm{Hom}(U,U)$ be the endomorphisms of U. Existing work such as [Jac10], [Mya90], [BW14], and [Wil16] highlights the role of the centroid algebra

$$\mathrm{Cen}(*) := \{\sigma \in \mathrm{End}(U) \times \mathrm{End}(V) \times \mathrm{End}(W) : \sigma u * v = u * \sigma v = \sigma(u * v) \;\; \forall u \in U, v \in V\}.$$

Indeed, $\mathrm{Cen}(*)$ is a commutative algebra as we'll demonstrate closure under composition for $\sigma, \tau \in \mathrm{Cen}(t)$, and all u, v,

$$(\sigma\tau)(u * v) = ((\sigma\tau)u) * v = (\sigma(\tau(u))) * v = \tau(u) * \sigma(v) = \tau(u * \sigma(v)) = (\sigma\tau)(u * v).$$

Similarly, the adjoint algebra

$$\mathrm{Adj}(*) := \{\sigma \in \mathrm{End}(U)^{\mathrm{op}} \times \mathrm{End}(V) : \sigma u * v = u * \sigma v\},$$

is an associative algebra as we demonstrate closure under composition. For $(\sigma_u, \sigma_v), (\tau_u, \tau_v)$ in $\mathrm{Adj}(*)$, we require $(\sigma_u, \sigma_v) \cdot (\tau_u, \tau_v) = (\tau_u\sigma_u, \sigma_v\tau_v)$ to be in $\mathrm{Adj}(*)$:

$$(\tau_u\sigma_u)u * v = \tau_u(\sigma_u(u)) * v = (\sigma_u(u)) * \tau_u(v) = (\sigma_u\tau_u)(v)$$

And the derivation algebra

$$\mathrm{Der}(*) := \{\delta \in \mathrm{End}(U) \times \mathrm{End}(V) \times \mathrm{End}(W) : \delta u * v + u * \delta v = \delta(u * v)\},$$

itself is a Lie algebra under the product $[\delta, \sigma] = \delta\sigma - \sigma\delta$, where composition is the Lie bracket component wise. We demonstrate closure by showing $[(\delta_U, \delta_V, \delta_W), (\sigma_U, \sigma_V, \sigma_W)] \in \mathrm{Der}(*)$:

$$\begin{aligned}
&[\delta_U, \sigma_U]u * v + u * [\delta_V, \sigma_V] \\
&= \delta_U(\sigma_U(u)) * v - \sigma_U(\delta_U(u)) * v + u * \delta_V(\sigma_V(v)) - u * \sigma_V(\delta_V(v)) \\
&= (\delta_W(\sigma_U(u) * v) - \sigma_U(u) * \delta_V(v)) - (\sigma_W(\delta_U(u) * v) - \delta_U(u) * \sigma_V(v)) \\
&\qquad + (\delta_W(u * \sigma_V(v)) - \delta_U(u) * \sigma_V(v)) - (\sigma_W(u * \delta_V(v)) - \sigma_U(u) * \delta_V(v)) \\
&= \delta_W(\sigma_U(u) * v) - \sigma_W(\delta_U(u) * v) + \delta_W(u * \sigma_V(v)) - \sigma_W(u * \delta_V(v)) \\
&= [\delta_W, \sigma_W](u * v)
\end{aligned}$$

Results using these algebras include discovering basis independent cluster pattern in tensors [BKW24], decomposing p-groups [Wil09a], computing endomorphisms of modules [BL08], and advances in isomorphism testing [IQ19], [BMW17] [BW12] [BMW22].

Two avenues of study are proposed for my dissertation. First to find efficient algorithms to compute these algebras, and second to study product decomposition of tensors using these algebras.

1.1. **Efficient algorithms.** The description of algebras $\mathrm{Adj}(t)$, $\mathrm{Cen}(t)$, and $\mathrm{Der}(t)$ are given by linear equations. For fixed bases, each takes operations cubic in the number of variables by standard linear system solving methods. For tensors with each frame of dimension n, this is at minimum $O(n^6)$ operations. In collaboration with Joshua Maglione and James B. Wilson, we have preliminary results for an asymptotically lower complexity algorithm for computing $\mathrm{Adj}(t)$ and $\mathrm{Cen}(t)$ in $O(n^3)$ operations, inspired by analogous results for matrices known as the Bartels-Stewart algorithm [BS72].

The proposed work is to find asymptotic speedups for the computation of $\mathrm{Der}(t)$ in the 3-tensor case, targeting an improvement from $O(n^6)$ to $O(n^{4.5})$.

1.2. **Product decompositions.** The following is a product decomposition of bimaps. Given $r : W_2 \times W_1 \rightarrowtail W_0$, a product decomposition consists of vector spaces $W_i \cong U_i \otimes V_i$, $i = 0, 1, 2$, and maps $s : U_2 \times U_1 \rightarrowtail U_0$, $t : V_2 \times V_1 \rightarrowtail V_0$ such that $s \otimes t \cong r$ as tensor product of bimaps. This requires for pure tensors $u_2 \otimes v_2 = w_2$ and $u_1 \otimes v_1 = w_1$, we have $\langle s|u_2, u_1\rangle \otimes \langle t|v_2, v_1\rangle \cong \langle r|w_2, w_1\rangle$. We call this a *Kronecker decomposition* of r.

We have seen rich literature on decomposing a tensor *additively*, meaning writing a tensor t as a sum of tensors $\{t_i\}_{i\in I}$. For instance, the CP-decomposition [Hit27] decomposes $t \in (U_n \otimes \cdots \otimes U_1)^*$ as a sum of rank 1 tensors. Other variants include PARAFAC2 [Har72], block decompositions [DL08], and decompositions subject to symmetry [Rob15]. But on products, from what we see in Section 1.3.2, effort have been focused on assuming specific internal product structure, and using optimization techniques specific to working over $\mathbb{C}$ to get approximate answers.

The proposed work is to understand which properties of a tensor control product decompositions, targeting computable characterizations of when a tensor is product indecomposable.

1.3. **Prior work.** We build on a number of methods which we briefly summarize below. Myasnikov, Wilson, and others in [Mya90], [Wil16], [Wil12], [MM10], [Wil09a] proves for the bimap t the algebras $\mathrm{Adj}(t)$ and $\mathrm{Cen}(t)$ control direct sum decompositions and automorphisms of t, using them to prove properties for the originating algebraic structures. In recent work [BKW24], it's shown that derivations also controls the original tensor's cluster patterns, limiting the subspaces on which the tensor is non-zero.

Recent work generalizing from bimaps finds a long exact sequence linking the various generalized adjoints, centroids, and derivations of a higher valence tensor [BMW20]. Further work by First, Maglione, and Wilson [FMW20] defines a ternary Galois connection between tensors, operators, and polynomial ideals. Alongside it, they define a generalized (P, Ω)-tensor product of vector spaces $U_1, \ldots, U_n$, for any $\Omega \subset \prod_i \mathrm{End}(U_i)$ and polynomials $P \subset K[x_1, \ldots, x_n]$.

For the remainder of this section, let $U_0 = k$. Let $\mathbf{d}$ be the polynomial $x_n + \cdots + x_1$. It is proven for a tensor t, the $(\mathbf{d}, \mathrm{Der}(t))$-tensor product is universally the smallest among the (P, Ω)-space that t factors through, for which $P \subset K[x_1, \ldots, x_n]$ is an ideal generated by linear homogeneous polynomials. This motivates studies of $\mathrm{Der}(t)$ and the associated $(\mathbf{d}, \mathrm{Der}(t))$-tensor product space.

Then as investigated in [BMW22], the **derivation closure** of t, consisting of tensors t' whose derivation algebra contains the derivation algebra of t is of interest for isomorphism testing. Specifically, tensors whose derivation closure is 1-dimensional admit polynomial time isomorphism tests.

In [BMW22], an infinite family of tensors with 1 dimensional derivation closures are constructed. However, little else is known about $(\!|t|\!)$. By Theorem B of [FMW20], a basis for the space $(\!|t|\!)$ is computable in polynomial time. The Multilinear Algebra library in the Computer Algebra System Magma implements this functionality, and I have utilized it to compute examples in practice.

1.3.1. *Products of tensors.* For unital associative K-algebras A and B, the tensor product of A and B is their tensor product as a vector space, with multiplication given by

$$(a \otimes b)(c \otimes d) = ac \otimes bd. \tag{1.1}$$

Let $\mu_A : A \otimes A \to A$ and $\mu_B : B \otimes B \to B$ be the linear structure maps of A and B. Then the unique induced linear map on the structure maps is the structure map of $A \otimes B$: $\mu_{A\otimes B} : (A \otimes B) \otimes (A \otimes B) \to A \otimes B$. [Gre12, Section 2.2]. The corresponding bilinear map is the tensor product of the multiplication maps.

We now generalize the above construction. Let s and t be tensors with interpretations $\langle s| : \prod_{i=1}^{n} U_i \rightarrowtail U_0$ and $\langle t| : \prod_{i=1}^{n} V_i \rightarrowtail V_0$. Interpret $s \otimes t$ as $\langle s \otimes t| : \prod_{i=1}^{n}(U_i \otimes V_i) \rightarrowtail U_0 \otimes V_0$, where $\langle s \otimes t | u_1 \otimes v_1, \dots, u_n \otimes v_n\rangle = \langle s|u_1, \dots, u_n\rangle \otimes \langle t|v_1, \dots, v_n\rangle$. This tensor product of multilinear maps is a generalization of the bilinear case ([Gre12, Section 1.21]). Throughout, we use the terminology "tensor product of tensors" interchangably with "tensor product of multilinear maps".

Example 1.2. Let $\langle s| : K^2 \times K \rightarrowtail K^2$ be the right scalar action map, and $\langle t| : K \times K^3 \rightarrowtail K^3$ be the left scalar action map. The map $\langle s \otimes t| : (K^2 \otimes K) \times (K \otimes K^3) \rightarrowtail (K^2 \otimes K^3)$ is defined as

$$\langle s \otimes t|u \otimes k_1, k_2 \otimes v\rangle = \langle s|u, k_2\rangle \otimes \langle t|k_1, v\rangle = k_2 u \otimes k_1 v = k_1 k_2 (u \otimes v) \tag{1.2}$$

Let f be an isomorphism of $K^2 \otimes K$ with K^2 and g an isomorphism of $K \otimes K^3$ with K^3. Then $\langle s \otimes t|$ is identified with the outer product tensor $\langle r| : K^2 \times K^3 \rightarrowtail K^2 \otimes K^3$ by mapping the first input via f and the second via g. This is called an *isotopism* of tensors. [Wil16]

The diagram scheme in Figure 1 illustrates the tensor product of tensors. In it, the tri-valent tensors s and t are drawn as shapes with 3 wires indicating 3 axes, with orientation given to the wire to indicate input and output. This notation is known as *tensor network diagrams* [BB17]. When tensor producting two tensors, our notation is for wires at the same position to be combined by the $\otimes$ symbol. This merging of frames is non-standard in tensor network diagram literature. Part of our work will be to extend and adapt tensor network diagram to products.

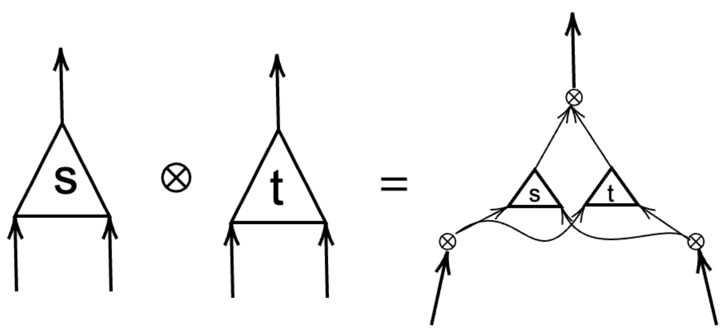

FIGURE 1. Pictorial illustration of the tensor product of s and t satisfying formula $\langle s \otimes t|\rangle = \langle s|u\rangle \otimes \langle t|v\rangle$

For preliminary results, I have proven $(\!|s \otimes t|\!) = (\!|s|\!) \otimes (\!|t|\!)$. This with Theorem 1.4 of [BMW22] allows for isomorphism testing of any tensor r with $s \otimes s$ for $\dim(\!|s|\!) = 1$.

1.3.2. *Related Works.* In related works, the physics community uses techniques like the Density Matrix Renormalization Algorithm [Whi93] to uncover the structure of a high valence tensor by factoring it as a contracted product of 3-tensors called a Matrix Product State, see Figure 2 for a pictorial illustration of this technique. This iterative optimization technique is for complex-valued tensors, relying on the Singular Value Decomposition.

Techniques such as Tensor-Train decompositions [Ose11] and Tucker decompositions [Tuc66], also known as HOSVD - Higher Order Singular Value Decompositions [DLDMV00], are similar but assumes different fixed internal structure.

In the next two sections, we describe in detail the proposed problems we are investigating and contributions we forsee as part of this dissertation.

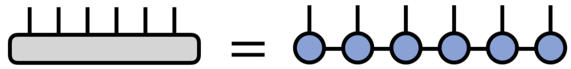

FIGURE 2. A tensor network diagram illustrating a matrix product state decomposition. Here, the objective is to approximate the grey tensor on the left by the blue tensors on the right, which has a fixed internal structure - the two tensors at the end have 2 indices, while all interior tensors have 3 indicies contracted in a specific way.

2. Efficient algorithms for algebraic invariants of tensors

We wish to compute $\mathrm{Adj}(t)$, $\mathrm{Cen}(t)$, and $\mathrm{Der}(t)$ for a bimap t.

For fixed bases, each of the algebras are specified by linear systems of equations, and thus can be computed in a number of steps polynomial to the sum of dimensions. The naive solution takes operations cubic in the number of variables. We aim to do better in the general case.

We report on results in collaboration with James Wilson and Joshua Maglione. First, the problem of computing adjoints of bimaps is stated in coordinates. Next, we describe our approach, which is to translate the system to a basis independent formulation, solve a smaller subproblem, and propagate the subproblem solution to a full solution.

2.1. Simultaneous Sylvester System.

We solve the following

Given: arrays $R \in K^{r\times b\times c}$, $S \in K^{a\times s\times c}$, and $T \in K^{a\times b\times c}$

Return: matrices $X \in K^{a\times r}$ and $Y \in K^{s\times b}$ such that

$$(\forall i)(XR_i + S_iY = T_i). \tag{2.1}$$

Expressed as list of matrix equation, Equation (2.1) is the natural extension of the Sylvester Equation, which asks for X satisfying the matrix equation $XA + BX = C$. For R and S coordinates of a tensor with a fixed basis, and T to be all zero, solving an instance of this problem finds the adjoint algebra of the tensor. When asked to solve this system in Magma, we convert the equation to matrix-vector form.

Next we give a basis independent statement of the problem and an algorithm to solve the system without

2.2. Simultaneous Sylvester System - Basis independent.

Given: Trilinear maps $r : R \times B \times C \rightarrowtail K$, $s : A \times S \times C \rightarrowtail K$, $t : A \times B \rightarrowtail K$, and isomorphisms identifying each vector space with its dual.

Return: Linear maps $x : A \to R$ and $y : B \to S$ such that for all $(a, b, c) \in A \times B \times C$,

$$r(x(a), b, c) + s(a, y(b), c) = t(a, b, c)$$

We see this abstracts the matricies of Equation (2.1) as linear maps, and the cubes of data as trilinear maps.

Below we describe how to compute x and y while only solving a smaller linear system.

Notation:

Given a function $f : A \times B \times C \rightarrowtail K$, its restriction onto a subspace $A' \leq A$, written $f|_{A'\times B\times C}$, will be shortened to $f|_{A'}$. Restriction onto a pair of subspaces, such as $f|_{A'\times B'\times C}$ will be shortened to $f|_{A',B'}$.

We will also write $r_x : A \times B \times C \rightarrowtail K$ as the function defined by $r_x(a, b, c) = r(x(a), b, c)$. The same defines the function s_y.

Preliminaries:

By the fixed dual isomorphism, r can be identified as an element of $R^* \otimes B \otimes C \cong \mathrm{Hom}(R, B \otimes C)$. Similarly, s can be identified as an element of $\mathrm{Hom}(S, A \otimes C)$. We now look for subspaces $B' \leq B$ and $A' \leq A$ such that r and s under these identifications have left inverses after post-composing with projections. That is, we want $r_{B'} := (\pi_{B'} \otimes I_C) \circ r$ and $s_{A'} := (\pi_{A'} \otimes I_C) \circ s$ to have left inverses. Denote these left inverses as $r^{\#}_{B'}$ and $s^{\#}_{A'}$. We also need B' and A' to have an induced isomorphism to their respective duals, meaning $B \cong B^*$ restricts an isomorphism $B' \cong (B')^*$, and similar for $A \cong A^*$ restricting to $A' \cong (A')^*$.

Solving a small subproblem:
We solve the smaller subsystem $(r_x + s_y)|_{A',B'} = t|_{A',B'}$. Denote the unknowns $x_{A'} := x|_{A'}$ and $y_{B'} := y|_{B'}$. Solving for $x_{A'}$ and $y_{B'}$ proceed by standard linear algebra, but as $\dim x_{A'} = \dim A' \cdot \dim R$ and $\dim y_{B'} = \dim B' \cdot \dim S$, this smaller system have unknowns of considerably lower dimension if the subspaces A' and B' are lower dimensional compared to A and B.

Propagating solution to full problem:
After solving for $x_{A'}$ and $y_{B'}$ by conventional methods, our algorithm proceeds by finding complementary subspaces $A = A' \oplus U$, and $B = B' \oplus V$. The objective is to find $x_U : U \to R$ and $y_V : V \to S$ which are the unknown parts of x and y.

For x_U, we solve $(r_x + s_y)|_{U,B'} = t|_{U,B'}$. The only unknown in this equation is $x_U : U \to R$. To solve for it, view both sides of the equation as elements of $\mathrm{Hom}(U, B' \otimes C)$ and rearrange to isolate $r_x|_{U,B'}$. Precompose with the left inverse $r^{\#}_{B'} : \mathrm{Hom}(B' \otimes C, R)$ gives the unique element of $\mathrm{Hom}(U, R)$ satisfying the above equation, hence it must be x_U.

The analogous technique applied to $s_y|_{A',V}$ solves uniquely for x_V.

For correctness, a full verification that $(r_x + s_y)|_{U,V} = t|_{U,V}$ is needed. But for implementation considerations, probabilistic methods such as only randomly selecting a small portion of data to check can greatly speed up computing candidate solutions.

Remark 2.1. Our algorithm requires the user to provide the subspace A' and B' with the desired invertibility properties of $r_{B'}$ and $s_{A'}$. As an implementation consideration, the linear transformation $r \in \mathrm{Hom}(R, B \otimes C)$ has $bc := \dim B \otimes C$ rows and $r := \dim R$ columns as a matrix. Thus projecting to a dim $\lceil r/c \rceil$ subspace of B have a high probability of maintaining the left invertibility of r.

For example, if $\dim A, \dim B, \dim C, \dim R, \dim S$ are all $O(n)$ then $\lceil r/c \rceil$ is $O(1)$ hence the naive $O(n^6)$ operations necessary to solve the system is reduced to the $O(n^3)$ operations needed to solve the smaller subproblem.

2.3. Derivation System.
First to state the problem in coordinates, we need

Definition 2.2 (Outer action). *Given an array $T \in K^{a \times b \times d}$, let $[T_1, \ldots, T_d]$ be a list of $K^{a \times b}$ matrices corresponding to unfolding this array along the third index,. Then for a matrix Z of size $d \times c$, define T^Z, the outer action of Z on T, as the $K^{a \times b \times c}$ array satisfying $(T^Z)_j = \sum_{i=1}^{d} T_i Z_{ij}$.*

Now we are ready to state the problem of solving the derivation system in coordinates.

Problem A (Derivation System - Coordinatized).
Given: *arrays $R \in K^{r \times b \times c}$, $S \in K^{a \times s \times c}$, and $T \in K^{a \times b \times t}$*
Return: *matrices $X \in K^{a \times r}$, $Y \in K^{s \times b}$, and $Z \in K^{t \times c}$ such that*

$$(\forall i) X R_i + S_i Y + (T^Z)_i = 0. \tag{2.2}$$

The above equation is no longer a list of matrix equations due to the outer action by Z. But it is exactly the equation satisfied by the derivation algebra of a tensor t when R, S are coordinates of that tensor in a fixed basis, and T its negative. Next we describe a basis independent formulation.

2.4. Derivation system - Basis independent.

Given: Trilinear maps $r : R \times B \times C \rightarrowtail K$, $s : A \times S \times C \rightarrowtail K$, $t : A \times B \times T \rightarrowtail K$, and isomorphisms identifying the vector spaces and their duals.

Return: Linear maps $x : A \to R$, $y : B \to S$, and $z : C \to T$ such that for all $(a, b, c) \in A \times B \times C$,

$$r(x(a), b, c) + s(a, y(b), c) + t(a, b, z(c)) = 0$$

Preliminary investigations suggest a similar approach to Simultaneous Sylvester Systems but with 3 subspaces, $A' \leq A$, $B' \leq B$, and $C' \leq C$. If the dimension of each space is $O(\sqrt{n})$ then the number of variables to solve in the dense system is $O(n^{1.5})$, giving the $O(n^{4.5})$ target. Additional work is necessary.

Thus far we have only been concerned about bimaps. As a challenge upon resolving Problem A, we propose extending the above ideas to higher valence tensors. We frame the question as solving for algebraic invariants of tensors.

We first give notation for an endomorphism acting on a specific axis of the tensor.

Definition 2.3. *Let $\langle t| : \prod_i U_i \rightarrowtail K$. Let $\sigma_a \in \mathrm{End}(U_a)$. Then define the tensor $\langle t|\sigma_a$ as*

$$\langle t|\sigma_a|u\rangle = \langle t|\sigma_a u_a, u_{\overline{a}}\rangle$$

Where $u = (u_a, u_{\overline{a}})$ splits $u \in \prod_i U_i$ as an element of $U_a \times \prod_{i \neq a} U_i$.

Let $\langle t| : \prod_{i\in I} U_i \rightarrowtail K$. For a 2-element subset $\{a, b\} \subset I$, we define the ab-nucleus of t as

$$\mathrm{Nuc}_{ab}(t) := \{(\sigma_a, \sigma_b) \in \mathrm{End}(U_a)^{\mathrm{op}} \times \mathrm{End}(U_b) : \langle t|\sigma_a u_a, u_{\overline{a}}\rangle = \langle t|\sigma_b u_b, u_{\overline{b}}\rangle\}. \tag{2.3}$$

For $J \subset I$, define the J-centroid as

$$\mathrm{Cen}_J(t) := \left\{ (\sigma_j)_{j\in J} \in \prod_j \mathrm{End}(U_j) : \langle t|\sigma_j u_j, u_{\overline{j}}\rangle = \langle t|\sigma_k u_k, u_{\overline{k}}\rangle \ \forall j, k \in J \right\}. \tag{2.4}$$

Similary, define the J-derivation as

$$\mathrm{Der}_J(t) := \left\{ (\delta_j)_{j\in J} \in \prod_j \mathfrak{gl}(U_j) : \sum_j \langle t|\delta_j u_j, u_{\overline{j}}\rangle = 0 \right\}. \tag{2.5}$$

These spaces are computed by linear equations, and gives higher valence tensors the analogue of the centroid, adjoint, and derivation algebra of 3-tensors. Now we ask

> **Challenge A.** *Can the higher valence nuclei, centroid, and derivation algebras be computed in operations fewer than cubic in the number of variables?*
>
> *Specifically for a m-valent tensor framed by n-dimensional vector spaces, we target $O(n^3)$ to find nuclei and less than $O((mn^3)^2)$ for the full derivation algebra.*

3. Product decompositions of tensors

3.1. **Preliminaries.** As described in the introduction, we wish to understand product decompositions of tensors. To state the problem, we follow the exposition and notation in [FMW20].

Throughout this section, all tensor spaces will be the space of multilinear maps. The interpretation map will be the identity map. As a result the word "tensor" is used interchangeably with "multilinear map".

Definition 3.1. *(Ternary Galois Connection of Tensors, Ideals, and Operators)*

We define evaluating a multivariable polynomial, with operators substituting for the indeterminantes. For $p = \sum_e \lambda_e X^e \in K[x_0, \dots, x_n] =: K[X]$ and $\omega \in \prod_i \mathrm{End}(U_i)$, let

$$p(\omega) := \sum_e \lambda_e \left(\omega_0^{e_0}, \dots, \omega_n^{e_n}\right) \in \prod_i \mathrm{End}(U_i).$$

Let $S \subset \mathrm{Mult}(U_n, \dots, U_1; U_0)$. It is evidently a tensor space with the identity interpretation map. For all $t \in S$, define $\langle t|p(\omega)$ where for any $(u_1, \dots, u_n)$,

$$\langle t|p(\omega)|u\rangle = \sum_e \lambda_e \omega_0^{e_0} \langle t|\omega_1^{e_1} u_1, \dots, \omega_n^{e_n} u_n\rangle.$$

Now fix a polynomial p and operator ω. Define the set

$$\boldsymbol{T}(p,\omega) := \{t \in \mathrm{Mult}(U_n,\ldots,U_1;U_0) : \langle t \mid p(\omega) = 0\}.$$

Extend this definition to subsets P and Ω via

$$\boldsymbol{T}(P,\Omega) := \bigcap_{p\in P}\bigcap_{\omega\in\Omega} \boldsymbol{T}(p,\omega).$$

Similarly, for fixed polynomial p and tensor t define the set

$$\boldsymbol{Z}(t,p) := \{\omega \in \prod_u \mathrm{End}(U_i) : \langle t|p(\omega) = 0\},$$

and extend to subsets $\boldsymbol{Z}(S,P)$.

Fixing $P \subset K[X]$, there is an inclusion reversing Galois connection between subsets of $\mathrm{Mult}(U_n,\ldots,U_1;U_0)$ *and subsets of $\prod_i \mathrm{End}(U_i)$ given by*

$$S \subset \boldsymbol{T}(P,\Omega) \iff \Omega \subset \boldsymbol{Z}(S,P) \tag{3.1}$$

From [FMW20] the set $\boldsymbol{T}(P,\Omega)$ is a vector subspace and $\boldsymbol{Z}(S,\boldsymbol{d})$ is a Lie algebra for $\boldsymbol{d} = x_n+\cdots+x_0$.

Definition 3.2. *(Derivation closure) Let $t \in \mathrm{Mult}(U_n,\ldots,U_1;U_0)$. Then $(\!|t|\!)$, the derivation closure of t, is the vector subspace consisting of all s such that $\mathrm{Der}(t) \subset \mathrm{Der}(s)$. Hence $(\!|t|\!) := \boldsymbol{T}(\boldsymbol{d},\boldsymbol{Z}(\boldsymbol{d},t))$.*

Example 3.3. Let t be the matrix multiplication tensor for 2×3 and 3×4 rectangular matrices. That is, $\langle t| : K^{2\times 3}\times K^{3\times 4} \rightarrowtail K^{2\times 4}$ by $\langle t|M,N\rangle := MN$. Then by Corollary 8.4.4 of [FMW20], we have $(\!|t|\!)$ as a 1-dimensional vector subspace spanned by Kt.

Definition 3.4. *(Tensor product of multilinear maps)*

Let $s \in \mathrm{Mult}(U_n,\ldots,U_1;U_0)$ and $t \in \mathrm{Mult}(V_n,\ldots,V_1;V_0)$. Define $s\otimes t \in \mathrm{Mult}(U_n\otimes V_n,\ldots,U_1\otimes V_1;U_0\otimes V_0)$ as the tensor product of multilinear maps, with interpretation $\langle s\otimes t| : \prod_i U_i\otimes V_i \rightarrowtail U_0\otimes V_0$ given by $\langle s\otimes t|u_1\otimes v_1,\ldots,u_n\otimes v_n\rangle = \langle s|u_1,\ldots,u_n\rangle\otimes\langle t|v_1,\ldots,v_n\rangle$. We say $s\otimes t$ is the tensor product of s and t.

Remark 3.5. Writing $s\otimes t$ to mean a multilinear map is an abuse of notation similar to a similar notion in the tensor product of linear maps. Since $s \in \mathrm{Mult}(U_n,\ldots,U_1;U_0)$ and $t \in \mathrm{Mult}(V_n,\ldots,V_1;V_0)$ are vectors in vector spaces, $s\otimes t$ is definitionally the $\varphi(s,t)$, image of the tensor product of vector spaces.

However, by the following sequence of natural isomorphisms, $\varphi(s,t)$ is identified with an element of $\mathrm{Mult}(U_n\otimes V_n,\ldots,U_1\otimes V_1;U_0\otimes V_0)$. Starting with $\varphi(s,t) \in \mathrm{Mult}(U_n,\ldots,U_1;U_0)\otimes \mathrm{Mult}(V_n,\ldots,V_1;V_0)$:

$$\begin{aligned}\mathrm{Mult}(U_n,\ldots,U_1;U_0)\otimes\mathrm{Mult}(V_n,\ldots,V_1;V_0) &\cong (U_n^*\otimes\cdots\otimes U_1^*\otimes U_0)\otimes(V_n^*\otimes\cdots\otimes V_1^*\otimes V_0)\\ &\cong (U_n^*\otimes V_n^*)\otimes\cdots\otimes(U_1^*\otimes V_1^*)\otimes(U_0\otimes V_0)\\ &\cong \mathrm{Mult}(U_n\otimes V_n,\ldots,U_1\otimes V_1;U_0\otimes V_0)\end{aligned}$$

Example 3.6. The Kronecker product of matrices is a case of Definition 3.4, being the the coordinate matrix of the tensor product of linear maps. Let $M \in \mathrm{Mult}(K^2;K^2) = \mathrm{End}(K^2)$ and $N \in \mathrm{End}(K^2)$ be given as 2×2 matrices.

$$M = \begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix} \qquad N = \begin{bmatrix}3 & 4\\ 5 & 6\end{bmatrix}$$

Then

$$M\otimes N = \begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}\otimes\begin{bmatrix}3 & 4\\ 5 & 6\end{bmatrix} = \left(\begin{array}{cc|cc} 3 & 4 & \multicolumn{2}{c}{\multirow{2}{*}{$\mathbf{0}$}}\\ 5 & 6 & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{$\mathbf{0}$}} & 6 & 8\\ & & 10 & 12\end{array}\right)$$

The 4×4 matrix $M\otimes N$ is given as an element of $\mathbb{M}_2(\mathbb{M}_2(K)) \cong \mathbb{M}_4(K)$. It has interpretation $\langle M\otimes N| : (K^2\otimes K^2) \rightarrowtail (K^2\otimes K^2)$ given by mapping basis element $e_i\otimes e_j$ (dual basis $\epsilon_i\otimes\epsilon_j$) to the $2(i-1)+j$th column (row) of the matrix. Specifically, $(\epsilon_k\otimes\epsilon_l)\langle M\otimes N|e_i\otimes e_j\rangle$ is row $2k-1+l$, column $2i-1+j$ of $M \otimes N$, given by the product of the (k,i)th entry of M with the (l,j)th entry of N.

3.2. **Existing techniques.** Next we describe some existing techniques that relate to product decompositions.

Example 3.7 (Algebras)**.** Consider the isomorphism of $\mathbb{R}$-algebras $\mathbb{M}_2(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{M}_2(\mathbb{C})$ given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes z \mapsto \begin{bmatrix} az & bz \\ cz & dz \end{bmatrix} \tag{3.2}$$

Let $r : \mathbb{M}_2(\mathbb{C}) \times \mathbb{M}_2(\mathbb{C}) \rightarrowtail \mathbb{M}_2(\mathbb{C})$, $s : \mathbb{M}_2(\mathbb{R}) \times \mathbb{M}_2(\mathbb{R}) \rightarrowtail \mathbb{M}_2(\mathbb{R})$, and $t : \mathbb{C} \times \mathbb{C} \rightarrowtail \mathbb{C}$ be their respective multiplication tensors. The isomorphism extends to the multiplication tensors, meaning $r \cong s \otimes t$.

But also $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{M}_2(\mathbb{C})$ for $\mathbb{H}$ the real Quaternions via

$$1 \otimes 1 \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad i \otimes 1 \mapsto \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad j \otimes 1 \mapsto \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad k \otimes 1 \mapsto \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \tag{3.3}$$

This example illustrate challenges with having unique product decompositions.

Example 3.8 (Tensor over centroid)**.** Let $K = \mathbb{F}_{13}$. The tensor $r : K^4 \times K^4 \rightarrowtail K^2$ is given by

$$r_1 = \begin{bmatrix} 7 & 10 & 4 & 2 \\ 10 & 12 & 2 & 5 \\ 1 & 7 & 11 & 12 \\ 7 & 11 & 12 & 4 \end{bmatrix} \quad r_2 = \begin{bmatrix} 5 & 6 & 1 & 9 \\ 6 & 1 & 9 & 8 \\ 10 & 12 & 6 & 2 \\ 12 & 2 & 2 & 9 \end{bmatrix}$$

A computation gives the centroid of r as spanned by $\{1, \alpha\}$ where $\alpha^2 + 8\alpha + 11 = 0$. Thus $\mathrm{Cen}(r) \cong K[x]/(x^2+8x+11) \cong \mathbb{F}_{13^2} =: L$. The isomorphism $K^2 \cong L$ means r may be viewed as a L-bilinear map $L^2 \times L^2 \rightarrowtail L$. The isomorphism $L^2 \cong K^2 \otimes L$ and $L \cong K \otimes L$ gives r an interpretation as $K^2 \otimes L \times K^2 \otimes L \rightarrowtail K \otimes L$.

By L-bilinearity, define $\langle s| : K^2 \times K^2 \rightarrowtail K$ via $\langle s|u, v\rangle = k$ when $\langle r|u \otimes \ell, v \otimes 1\rangle = k \otimes 1$, ranging over all u, v in K^2 and for some fixed $\ell \in L$. Then $\langle r|u \otimes m, v \otimes n\rangle = mn\ell^{-1}\langle r|u \otimes \ell, v \otimes n\rangle \cong \langle s|u, v\rangle \otimes \langle t|m, n\rangle$ for $\langle t|$ a tensor that's a ℓ-shifted version of the L-multiplication tensor. This successfully decomposes r as $s \otimes t$.

For the above tensor r, we get $r = s \otimes t$ for

$$s = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \qquad t_1 = \begin{bmatrix} 10 & 5 \\ 5 & 6 \end{bmatrix} \quad t_2 = \begin{bmatrix} 9 & 3 \\ 3 & 7 \end{bmatrix},$$

where t is given by the system of forms $[t_1, t_2]$ and corresponds to the algebra $K[x]/(x^2 + 8x + 11)$ in the $\{3 + 4x, 7 + 10x\}$ basis. This example illustrates a special case of the product decomposition.

3.3. **Preliminary results on product decompositions.** The first question is how derivation closure of the product of tensors relate to the derivation closure of the individual tensors. That is, given $s \in \mathrm{Mult}(U_n, \ldots, U_1; U_0)$ and $t \in \mathrm{Mult}(V_n, \ldots, V_1; V_0)$, do $(\!|s|\!)$ and $(\!|t|\!)$ relate to $(\!|s \otimes t|\!)$? I have recently resolved this question. Below is a preliminary lemma.

Lemma 3.9. *For $s \in \mathrm{Mult}(U_n, \ldots, U_1; U_0)$ and $t \in \mathrm{Mult}(V_n, \ldots, V_1; V_0)$, there is an embedding $\iota_s : \mathrm{Der}(s) \hookrightarrow \mathrm{Der}(s \otimes t)$ given by $(\sigma_i)_{i\in[n]} \mapsto (\sigma_i \otimes 1_{V_i})_{i\in[n]}$ and an embedding $\iota_t : \mathrm{Der}(t) \hookrightarrow \mathrm{Der}(s \otimes t)$ given by $(\tau_i)_{i\in[n]} \mapsto (1_{U_i} \otimes \tau_i)_{i\in[n]}$.*

Proof. The map ι_s is injective on each factor as tensoring with the identity morphism is injective. The endomorphism $(\sigma_i \otimes 1_{V_i})_{i\in[n]}$ is in $\mathrm{Der}(s \otimes t)$ as $\langle s \otimes t|\mathbf{d}((\sigma_i \otimes 1_{V_i})_{i\in[n]}) = \sum_{i=1}^{n} \langle s|\sigma_i \otimes \langle t|1_{V_i} = \langle s|\mathbf{d}((\sigma_i)_{i\in[n]}) \otimes \langle t| = 0$. Lastly, we need to demonstrate ι_i is a map of Lie algebras. This follows by the calculation $\iota(\delta + \rho) = \iota((\delta_i + \rho_i)_{i\in[n]}) = ((\delta_i + \rho_i) \otimes 1_{V_i})_{i\in[n]} = (\delta_i \otimes 1_{V_i})_{i\in[n]} + (\rho_i \otimes 1_{V_i})_{i\in[n]} = \iota(\delta) + \iota(\rho)$. The case for ι_t is analogous. ∎

Theorem 3.10. *Let $s \in \mathrm{Mult}(U_n, \ldots, U_1; U_0) =: U$ and $t \in \mathrm{Mult}(V_n, \ldots, V_1; V_0) =: V$. Then $(\!|s|\!) \otimes (\!|t|\!) = (\!|s \otimes t|\!)$.*

Proof. Our strategy is to show $(\!|s \otimes t|\!) \subset (\!|s|\!) \otimes (\!|t|\!)$ and $(\!|s|\!) \otimes (\!|t|\!) \subset (\!|s \otimes t|\!)$.

$(\!|s \otimes t|\!) \subset (\!|s|\!) \otimes (\!|t|\!)$:

By Lemma 3.9, there are embeddings $\iota_s : \mathrm{Der}(s) \hookrightarrow \mathrm{Der}(s \otimes t)$ and $\iota_t : \mathrm{Der}(t) \hookrightarrow \mathrm{Der}(s \otimes t)$. The inclusion reversing nature of the antitone Galois connection in Definition 3.1 implies $(\!|s \otimes t|\!) = T(\mathbf{d}, \mathrm{Der}(s \otimes t)) \subset T(\mathbf{d}, \iota_s(\mathrm{Der}(s)))$ and $(\!|s \otimes t|\!) \subset T(\mathbf{d}, \iota_t(\mathrm{Der}(t)))$. Thus $(\!|s \otimes t|\!)$ is in their intersection. We shall prove $T(\mathbf{d}, \iota_s(\mathrm{Der}(s))) = (\!|s|\!) \otimes V$ and $T(\mathbf{d}, \iota_j(\mathrm{Der}(t))) = U \otimes (\!|t|\!)$. The conclusion follows as $(\!|s \otimes t|\!) \subset (\!|s|\!) \otimes V \cap U \otimes (\!|t|\!) = (\!|s|\!) \otimes (\!|t|\!)$.

The statement to prove is $T(\mathbf{d}, \iota_s(\mathrm{Der}(s))) = (\!|s|\!) \otimes V$.

We first show the direction $(\!|s|\!) \otimes V \subset T(\mathbf{d}, \iota_s(\mathrm{Der}(s)))$. As $(\!|s|\!) \otimes V$ is generated by $\acute{s} \otimes t$ for $\acute{s} \in (\!|s|\!)$ and $t \in V$, it suffices to show $\acute{s} \otimes t \in T(\mathbf{d}, \iota_s(\mathrm{Der}(S)))$. This follows as $\langle \acute{s} \otimes t|$ satisfies $\langle \acute{s} \otimes t|\mathbf{d}(\delta) = 0$ for all $\delta = (\sigma_i \otimes 1_{V_i})_{i \in [n]} \in \iota_s(\mathrm{Der}(s))$ since $(\sigma_i)_{i \in [n]} \in \mathrm{Der}(s)$.

In the opposite direction, $T(\mathbf{d}, \iota_s(\mathrm{Der}(s)))$ is a subspace of $U \otimes V$. We shall show in fact it is the subspace $(\!|s|\!) \otimes V$ by showing every element in $T(\mathbf{d}, \iota_s(\mathrm{Der}(s)))$ is the sum of pure tensors $\acute{s} \otimes t$ for $\acute{s} \in (\!|s|\!), t \in V$.

Let $r = \sum_{i=1}^m s_i \otimes t_i \in U \otimes V$ be an element of $T(\mathbf{d}, \iota_s(\mathrm{Der}(s)))$, with all t_i's linearly independent. Showing each s_i is in $(\!|s|\!)$ concludes r is in $(\!|s|\!) \otimes V$. By definition r satisfies $\langle r|\mathbf{d}(\iota_s(\sigma)) = 0$ for all $\sigma \in \mathrm{Der}(s)$.

Let $\sigma \in \mathrm{Der}(s)$ and $\iota_s(\sigma) = (\sigma_j \otimes 1_{V_j})_{j \in [n]}$. Computing,

$$\begin{aligned}
0 &= \langle r|\mathbf{d}(\iota_s(\sigma)) \\
&= \left\langle \sum_i s_i \otimes t_i \right| \mathbf{d}(\iota_s(\sigma)) \\
&= \sum_i \langle s_i \otimes t_i|\mathbf{d}(\iota_s(\sigma)) && \text{linearity of tensor evaluation} \\
&= \sum_i \left(\langle s_i \otimes t_i| \left(\sum_{j=1}^n \sigma_j \otimes 1_{V_j} \right) \right) \\
&= \sum_i \left(\sum_{j=1}^n \left(\langle s_i \otimes t_i|\sigma_j \otimes 1_{V_j} \right) \right) \\
&= \sum_i \left(\sum_{j=1}^n \langle s_i|\sigma_j \otimes t_i \right) && \text{Apply } \sigma_j \text{ to } s_i \text{ and } 1_{V_j} \text{ to } t_i \\
&= \sum_i \left(\sum_{j=1}^n \langle s_i|\sigma_j \right) \otimes t_i \\
&= \sum_i \langle s_i|\mathbf{d}(\sigma) \otimes t_i
\end{aligned}$$

Let $\mathcal{B}$ be a basis of V. Then expanding each t_i in this basis,

$$\begin{aligned}
0 &= \sum_i \left(\langle s_i|\mathbf{d}(\sigma) \otimes \left(\sum_{b \in \mathcal{B}} \lambda_{ib} b \right) \right) \\
&= \sum_{b \in \mathcal{B}} \left(\sum_i \left(\lambda_{ib} \langle s_i|\mathbf{d}(\sigma) \right) \otimes b \right)
\end{aligned}$$

Since $\mathcal{B}$ is a basis, $\sum_i \lambda_{ib} \langle s_i|\mathbf{d}(\sigma) = 0$. Since t_i's are linearly independent, the m by $|\mathcal{B}|$ matrix $[\lambda_{ib}]$ has full row rank. Thus the only way $\sum_i \lambda_{ib} \langle s_i|\mathbf{d}(\sigma) = 0$ is if $\langle s_i|\mathbf{d}(\sigma) = 0$ for all i. Thus $s_i \in (\!|s|\!)$ for all i. This concludes the proof of $T(\mathbf{d}, \iota_s(\mathrm{Der}(s))) = (\!|s|\!) \otimes V$.

The statement $T(\mathbf{d}, \iota_t(\mathrm{Der}(t))) = U \otimes (\!|t|\!)$ is proven analogously.

$(\!|s|\!) \otimes (\!|t|\!) \subset (\!|s \otimes t|\!)$:

The strategy will be to first show for $\acute{s} \in (\!|s|\!)$, that $\acute{s}\otimes t \in (\!|s\otimes t|\!)$, and secondly, show if $\acute{s}\otimes t \in (\!|s\otimes t|\!)$ for all s, then for all $\acute{t} \in (\!|t|\!)$, that $\acute{s} \otimes \acute{t} \in (\!|s \otimes t|\!)$. The proof concludes as $(\!|s|\!) \otimes (\!|t|\!)$ is generated by $\acute{s} \otimes \acute{t}$ for $\acute{s} \in (\!|s|\!)$ and $\acute{t} \in (\!|t|\!)$, and

To show $\acute{s} \otimes t \in (\!|s \otimes t|\!)$, let $\delta \in \mathrm{Der}(s \otimes t)$. Since δ is an element of $\prod_i \mathfrak{gl}(U_i \otimes V_i) \cong \prod_i (\mathfrak{gl}(U_i) \otimes \mathfrak{gl}(V_i))$, write δ as $\left(\sum_{j=1}^{R_i} (\sigma_j \otimes \tau_j)\right)_{i\in[n]}$. By construction $\langle s \otimes t|\mathbf{d}(\delta) = 0$. Calculating,

$$
\begin{aligned}
0 &= \langle s \otimes t|\mathbf{d}(\delta) \\
&= \langle s \otimes t| \sum_{i=1}^{n} \left(\sum_{j=1}^{R_i} (\sigma_j \otimes \tau_j) \right) \\
&= \langle s \otimes t| \sum_{a\in A} (\sigma_a \otimes \tau_a) \qquad \text{grouping into one indexing set} \\
&= \sum_{a\in A} \langle s|\sigma_a \otimes \langle t|\tau_a \qquad (3.4) \\
&= \sum_{a\in A} \langle s|\sigma_a \otimes \left(\sum_{b\in\mathcal{B}} \lambda_{ab} b \right) \qquad \text{For } \mathcal{B} \text{ basis of } V \\
&= \sum_{b\in\mathcal{B}} \left(\sum_{a\in A} \lambda_{ab} \langle s|\sigma_a \right) \otimes b
\end{aligned}
$$

As $\mathcal{B}$ is a basis of V, $\sum_{a\in A} \lambda_{ab}\langle s|\sigma_a = 0$ for each b in $\mathcal{B}$. Regrouping and combining the terms in the indexing set A by axes, we have $\langle s|\sigma = 0$, meaning $\sigma \in \mathrm{Der}(s)$. Thus $\langle \acute{s}|\sigma = 0$ as well. Substituting $\acute{s}$ in place of s in Equation (3.4) also equals 0, concluding $\langle \acute{s} \otimes t|\mathbf{d}(\delta) = 0$.

The proof that $\acute{s} \otimes \acute{t} \in (\!|s \otimes t|\!)$, assuming $\acute{s} \otimes t$ is in $(\!|s \otimes t|\!)$ is analogous.

□

Example 3.11. Let $\langle s| : \mathfrak{sl}_2 \times \mathfrak{sl}_2 \rightarrowtail \mathfrak{sl}_2$ be the multiplication tensor of $\mathfrak{sl}_2$. We compute $(\!|s|\!) = Ks$. Let $\langle t| : \mathbb{F}_3^2 \times \mathbb{F}_3^2 \rightarrowtail \mathbb{F}_3^2$ be the multiplication table of $\mathbb{F}_3[x]/(x^2+1)$ as given in Example 3.8. We compute $(\!|t|\!)$ as spanned by t and the tensor r given by the system of forms $\left[\begin{bmatrix}0 & 2\\ 2 & 0\end{bmatrix}, \begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}\right]$.

As expected, $(\!|s|\!) \otimes (\!|t|\!)$ and $(\!|s \otimes t|\!)$ are both 2 dimensional. By definition $(\!|s|\!) \otimes (\!|t|\!)$ is spanned by $s \otimes t$ and $s \otimes r$, while computation verifies $(\!|s \otimes t|\!)$ is spanned by these same tensors.

Before describing our main problem, we address potential complications in finding product decompositions.

3.3.1. *Non-canonical choices.* In Definition 3.4 the tensors s and t have the same valence, and there is a matching of axis U_i of s with V_i of t.

Both are convenient but not necessary when looking for product decompositions. A tensor can be padded by axes consisting of K. For instance, a linear transformation $K^2 \to K^2$ is isotopic to the tensor $K^2 \times K \rightarrowtail K^2$ via the isomorphism $t \mapsto \tilde{t}$ where $\langle \tilde{t}|u, k\rangle = k\langle t|u\rangle$. Notice there's combinatoral explosion of possibilities of which axes to pad when one tensor has fewer axes than the other. Not matching U_i with V_i brings combinatorial explosion to potential decompositions. We do not rule out these possibilities and consider them as an expanded form of products of tensors for future invesigation.

Now we describe the main problem under consideration.

Definition 3.12. *(Kronecker decomposition) A tensor* $r \in \mathrm{Mult}(W_n, \dots, W_1; W_0)$ *has a* ***Kronecker decomposition*** *into a finite set* $\mathcal{S}$ *if*

$$
r \cong \bigotimes_{s\in\mathcal{S}} s \qquad \textit{for } s : V_{s,n} \times \cdots \times V_{s,1} \rightarrowtail V_{s,0}. \tag{3.5}
$$

We say r is ***Kronecker indecomposable*** *if $r \cong \bigotimes_{s\in\mathcal{S}} s$ implies $\mathcal{S} \subset \{r, \mu\}$ where μ is the K-multiplication tensor.*

Problem B (Kronecker decompositions of tensors).
Given tensor $r \in \mathrm{Mult}(W_n, \dots, W_1; W_0)$, what are computable criterions that guarantee r is Kronecker indecomposable or finds a decomposition, should one exist?

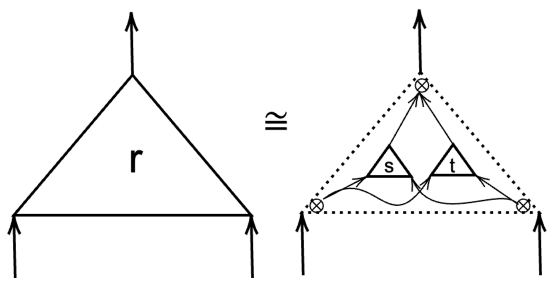

FIGURE 3. Pictorial illustration of Problem B. We are given is the tensor r. The goal is to find criterion that indicates $r \cong s \otimes t$, or that r is indecomposable, without brute force techniques such as iterating over all possible subspace combinations.

Examples in Section 2.1 give some partial answers for special cases. We'd like to formalize these results and extend to cases beyond the associative case, with the derivation algebra as our primary tool in Theorem 3.10.

As part of answering this problem, we plan to analyze the derivation algebras themselves to understand the tensor. That is, **given** $s \in \mathrm{Mult}(U_n, \dots, U_1; U_0)$ **and** $t \in \mathrm{Mult}(V_n, \dots, V_1; V_0)$**, is** $\mathrm{Der}(s \otimes t)$ **completely determined by** $\mathrm{Der}(s)$ **and** $\mathrm{Der}(t)$**?**

This question has an affirmative answer for the case of adjoints, as $\mathrm{Adj}(s \otimes t) = \mathrm{Adj}(s) \otimes \mathrm{Adj}(t)$ [Wil09a]. Nothing is known for $\mathrm{Der}(s \otimes t)$ other than Lemma 3.9 above. Preliminary computations suggests the embeddings of $\mathrm{Der}(s)$ and $\mathrm{Der}(t)$ do not behave like the adjoint case, as the dimension of $\mathrm{Der}(s \otimes t)$ in some computed examples is not the product of $\dim \mathrm{Der}(s)$ and $\dim \mathrm{Der}(t)$.

Example 3.13 (Bowtie tensor)**.** Let $K = \mathbb{Q}$, and $r : K^5 \times K^5 \rightarrowtail K$ be given by $\langle r | e_i, e_j, e_k \rangle$ having entry 1 when (i, j, k) is a permutation of $(1, 2, 3)$ or $(1, 4, 5)$, and having entry 0 otherwise.

As a system of forms, r has data

$$R_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & \mathbf{1} & 0 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad R_3 = \begin{bmatrix} 0 & \mathbf{1} & 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad R_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 & 0 \end{bmatrix}, R_5 = \begin{bmatrix} 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.6}$$

Using Theorem 3.10 we prove r is Kronecker indecomposable. We do not know of an alternative technique for proving this other than exhaustive search.

By computation, $\dim (\!| r |\!) = 4$. So if $r = s \otimes t$, then $(\!| r |\!) = (\!| s |\!) \otimes (\!| t |\!)$, and s, t have to be tensors whose derivation closure dimensions multiply to 4.

Dimensions multiply in a tensor product and 5 is prime so either r or s is the K-multiplication tensor $K \times K \rightarrowtail K$ or both s and t have a 1 dimensional axis. After identifying a K^a with its dual, both s and t are of the form $K^a \times K^b \rightarrowtail K$. It now suffices to prove tensors of this form have 1 dimensional derivation closure. This is a known result, which we sketch briefly below.

Let $m : K^a \times K^b \rightarrowtail K$ be a tensor given in reduced row and column basis

$$M = \begin{bmatrix} I_r & 0_{b-r\times r} \\ 0_{a-r\times r} & 0a - r \times b - r \end{bmatrix}.$$

An element δ is in $\mathrm{Der}(m)$ if $\delta = (X, Y, z)$ satisfies

$$XM + MY = zM,$$

for $X \in \mathbb{M}_{a\times a}(K), Y \in \mathbb{M}_{b\times b}(K), z \in K$. For $n \in (\!|m|\!)$ for N, the matrix N in the same basis has to satisfy the same equations. Block matrix multiplication of matricies of the below form

$$X = \begin{bmatrix} \operatorname{diag}(\lambda)_r & * \\ 0 & * \end{bmatrix} \quad Y = \begin{bmatrix} \operatorname{diag}(\mu)_r & 0 \\ * & * \end{bmatrix} \quad z = \lambda + \mu$$

concludes they are all in $\mathrm{Der}(m)$. As the $*$ values are arbitrary, if N is not a scalar multiple, there will be some (X, Y, z) in $\mathrm{Der}(m)$ for which $XN + NY \neq cN$, so N must be a scalar multiple of M. This gives the desired conclusion.

Lastly, we look to parametrized products of tensors. The existence of (P, Ω)-products means the possibility of a (P, Ω)-product decomposition.

Definition 3.14 ((P,Ω)-tensor product)**.** *[FMW20]*
Let $U_1, \dots, U_n$ be K-vector spaces, $P \subset K[x_1, \dots, x_n]$ and $\Omega \subset \mathrm{End}(U_1) \times \cdots \times \mathrm{End}(U_n)$. Define the following subspace of $U_1 \otimes \cdots \otimes U_n$

$$\Xi(P,\Omega) := \left\langle \sum_e \lambda_e \omega_1^{e_1} u_1 \otimes \cdots \otimes \omega_n^{e_n} u_n \,\middle|\, \omega \in \Omega, \sum_e \lambda_e x_1 e^{e_1} \cdots x_n^{e_n} \in P, u_i \in U_i \right\rangle.$$

Define the (P,Ω)-tensor product space as the quotient space

$$\blacktriangleleft U_1, \dots, U_n \blacktriangleright_\Omega^P := (U_1 \otimes \cdots \otimes U_n)/\Xi(P,\Omega),$$

together with a K-multilinear map $\blacktriangleleft \cdots \blacktriangleright : U_1 \times \cdots \times U_n \rightarrowtail \blacktriangleleft U_1, \dots, U_n \blacktriangleright_\Omega^P$, where $\blacktriangleleft u_1, \dots, u_n \blacktriangleright = u_1 \otimes \cdots \otimes u_n + \Xi(P,\Omega)$.

Notice $\blacktriangleleft U_1, \dots, U_n \blacktriangleright_\emptyset^\emptyset$ is the usual tensor product of vector spaces. Let $P \subset K[x_1, \dots, x_n]$ and $\Omega \subset \prod_i \mathrm{End}(U_i)$. Suppose each U_i is a tensor space $\mathrm{Mult}(V_i^m, \dots, V_i^1; K) \cong \left(\bigotimes V_i^j\right)^*$, and an isomorphism $V_i^j \cong (V_i^j)^*$ is specified. Let $(s_1, \dots, s_n) \in \prod_i U_i$ be a tuple of tensors. Then the image of $(s_1, \dots, s_n)$ under the (P,Ω)-tensor product, denoted $\blacktriangleleft s_1, \dots, s_n \blacktriangleright =: r$, is an element of a quotient space $\bigotimes_i U_i/\Xi(P,\Omega)$.

Fix a complementary subspace W to $\Xi(P,\Omega)$ in $\bigotimes_i U_i$ so $\Xi(P,\Omega) \oplus W = \bigotimes_i U_i$.Let $\tilde{r} \in W$ be the element of $\bigotimes_i U_i$ identified to r. As $\bigotimes_i U_i = \bigotimes_i \bigotimes_j V_i^j \cong \bigotimes_j \bigotimes_i V_i^j$, $\tilde{r}$ can be identified with a multilinear map in $\mathrm{Mult}(\bigotimes_i V_i^1, \dots, \bigotimes_i V_i^m; K)$. This gives a multilinear interpretation $\langle r| : \prod_{j=1}^m \left(\bigotimes_i V_i^j\right) \rightarrowtail K$.

Challenge B. *Let $r \in \mathrm{Mult}(U_n, \dots, U_1; U_0)$ be a tensor. For what (P,Ω) does r admit a (P,Ω)-product decomposition?*

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