COMPUTING ALGEBRAIC INVARIANTS OF TENSORS AND THEIR APPLICATION TO PRODUCT DECOMPOSITIONS

CHRIS LIU

ABSTRACT. This dissertation proposal outlines two areas of study. First, on algorithms for faster computation of algebraic invariants of tensors. Second, an investigation into decomposing tensors as products of smaller tensors.

CONTENTS

1. INTRODUCTION

Tensors encapsulate multilinear maps. Often given as a multiway array of numbers, they are used across various disciplines of mathematics to record information for some fixed reference frame. As such, they are studied from many complementary perspectives [Bro97] [KB09] [Lan12] [RS18] [DLDMV00] [Tuc66].

Throughout, we fix a field K. A tensor space T is a vector space equipped with a multilinear interpretation $\langle\cdot| : T \hookrightarrow \mathrm{Mult}(U_n, \ldots, U_1; U_0)$ for U_i each a K-vector space. A tensor t is an element of a tensor space T, and we write $\langle t| : U_n \times \cdots \times U_1 \rightarrowtail U_0$ to indicate $\langle t|$ is a multilinear function. The spaces $\{U_0, \ldots, U_n\}$ are the frame of tensor, the size of the frame $(n+1)$ its valence, and $\{0, \ldots, n\}$, the labels on the vector spaces, its axes. For $|u\rangle = |u_n, \ldots, u_1\rangle$, write $\langle t|u\rangle \in U_0$ to mean evaluating $\langle t|$ at $|u\rangle$.

This definition accomodates the common existing understanding of tensors as a multiway grid of numbers. For example,

Example 1.1. *Let* $[t_{ij}] \in \mathbb{R}^{2\times 3} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. *Then* t *can be interpreted as a bilinear form* $\langle t| : \mathbb{R}^2 \times \mathbb{R}^3 \rightarrowtail \mathbb{R}$ *where* $\langle t|e_i, e_j\rangle = 1$ *for* $(i,j) \in \{(1,1),(1,3),(2,2)\}$.

The perspective taken in this dissertation proposal is to study tensors as distributive products using the tools of algebra. Fix a 3-tensor (bimap) $* : U \times V \rightarrowtail W$. Let $\mathrm{End}(U) = \mathrm{Hom}(U,U)$ be

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the endomorphisms of U. Existing work such as [BW14], and [Wil16] highlights the role of the commutative centroid algebra

$$\mathrm{Cen}(t) := \{\sigma \in \mathrm{End}(U) \times \mathrm{End}(V) \times \mathrm{End}(W) : \sigma u * v = u * \sigma v = \sigma(u * v) \ \ \forall u \in U, v \in V\},$$

associative adjoint algebra

$$\mathrm{Adj}(t) := \{\sigma \in \mathrm{End}(U)^{\mathrm{op}} \times \mathrm{End}(V) : u\sigma * v = u * \sigma v\},$$

and Lie derivation algebra

$$\mathrm{Der}(t) := \{\delta \in \mathrm{End}(U) \times \mathrm{End}(V) \times \mathrm{End}(W) : \delta u * v + u * \delta v = \delta(u * v)\}.$$

Results using these algebras include discovering basis independent cluster pattern in tensors [BKW24], decomposing p-groups [Wil09a], finding direct product decomposition of groups [Wil12], and advances in isomorphism testing [BMW17] [BW12] [BMW22].

Two avenues of study are proposed for my dissertation. First to find faster algorithms to compute these algebras, and second to study product decomposition of tensors using these algebras.

1.1. **Work on faster algorithms.** The computation of algebras $\mathrm{Adj}(t)$, $\mathrm{Cen}(t)$, and $\mathrm{Der}(t)$ are given by linear equations. For fixed bases, each is cubic in the number of variables by standard methods. For tensors with each frame of dimension n, this is at minimum $O(n^6)$ operations. In collaboration with James Wilson and Joshua Maglione, we have preliminary results for an asymptotically faster algorithm in computing $\mathrm{Adj}(t)$ and $\mathrm{Cen}(t)$ in $O(n^3)$ operations, inspired by analogous results for matrices known as the Bartels-Stewart algorithm [BS72].

> My proposed work is to find asymptotic speedups for the computation of $\mathrm{Der}(t)$ in the 3-tensor case, and to extend to higher valence tensors in general.

1.2. **X-raying tensors to find product decompositions.** Given a tensor t, there is rich literature on decomposing a tensor *additively*, for instance, via the CP-decomposition [Hit27]. We look for decompositions of a tensor as a *product* of smaller tensors. X-rays in medical imaging produces images of the body's internal structure, which we take as naming inspiration. We call finding these product decompositions *X-raying*.

> My proposed work is to prove structure and recognition theorems for product decomposition of tensors.

1.3. **Prior work.** We now survey recent work which informs the class of tensors we wish to investigate. Wilson in [Wil16] [Wil12] [Wil09a] [Wil09b] proves for bimaps the algebras $\mathrm{Adj}(t)$ and $\mathrm{Cen}(t)$ control direct sum decompositions and automorphisms of t, using them to prove properties for the originating algebraic structures. Recent work generalizing from bimaps finds a long exact sequence linking the various nuclei (generalized adjoints), centroids, and derivations of a higher valence tensor [BMW20]. Further work by First, Maglione, and Wilson [FMW20] defines a ternary Galois connection between tensors, operators, and polynomial ideals. Alongside it, they define a generalized (P, Ω)-tensor product of vector spaces $U_1, \dots, U_n$, for any $\Omega \subset \prod_i \mathrm{End}(U_i)$ and polynomials $P \subset K[x_1, \dots, x_n]$.

Let $\mathbf{d}$ be the polynomial $x_n + \cdots + x_1$. It is proven for a tensor t, the $(\mathbf{d}, \mathrm{Der}(t))$-tensor product is universally the smallest among the (P, Ω)-products that t factors through, for which $P \subset K[x_1, \dots, x_n]$ is an ideal generated by linear homogeneous polynomials. This motivates studies of $\mathrm{Der}(t)$ and the associated $(\mathbf{d}, \mathrm{Der}(t))$-tensor product space.

For the remainder of the introduction, let $U_0 = k$. Then as investigated in [BMW22], the vector subspace denoted $(\!|t|\!)$ (**derivation closure** of t) consisting of tensors t' whose derivation algebra contains the derivation algebra of t, may be identified with the $(\mathbf{d}, \mathrm{Der}(t))$-tensor product space and thus is universally the smallest search space for solving tensor isomorphism questions involving t.

In [BMW22], an infinite family of tensors with 1 dimensional derivation closures are constructed. However, little else is known about $(\!|t|\!)$. By Theorem B of [FMW20], a basis for the space $(\!|t|\!)$ is computable in polynomial time, so computational examples are available in practice.

1.3.1. *Products of tensors.* For vector spaces U and V, the tensor product of U and V is the vector space $U \otimes V$ alongside the canonical map $\varphi : U \times V \rightarrowtail U \otimes V$, such that for every bilinear map f with domain $U \times V$, there is a unique induced linear map $\hat{f}$ satisfying $f = \hat{f} \circ \varphi$. For tensors $s \in U$ and $t \in V$, $s \otimes t$ is the image of (s,t) under φ. The tensor product is functorial in each argument meaning for linear maps $f : U \to U', g : V \to V'$, there is an induced linear map $f \otimes g : U \otimes V \to U' \otimes V'$ defined by $(u \otimes v) \mapsto f(u) \otimes g(v)$.

For unital associative algebras A and B, the tensor product of A and B is their tensor product as a vector space, with multiplication given by

$$(a \otimes b)(c \otimes d) = ac \otimes bd. \tag{1.1}$$

Let $\mu_A : A \otimes A \to A$ and $\mu_B : B \otimes B \to B$ be the linear structure maps of A and B. Then the induced linear map on the structure maps, gives the structure map of $A \otimes B$: $\mu_{A\otimes B} : (A \otimes B) \otimes (A \otimes B) \to A \otimes B$. [Gre12, Section 2.2]. The corresponding bilinear map is the tensor product of the multiplication tensors.

We now generalize the above construction. Let s and t be tensors with interpretations $\langle s| : \prod_{i=1}^{n} U_i \rightarrowtail U_0$ and $\langle t| : \prod_{i=1}^{n} V_i \rightarrowtail V_0$. The tensor product of multilinear maps is a well-known ([Gre12, Section 1.21] describes the bilinear case). Interpret $s \otimes t$ as $\langle s \otimes t| : \prod_{i=1}^{n}(U_i \otimes V_i) \rightarrowtail U_0 \otimes V_0$, where $\langle s \otimes t | u_1 \otimes v_1, \ldots, u_n \otimes v_n\rangle = \langle s | u_1, \ldots, u_n\rangle \otimes \langle t | v_1, \ldots, v_n\rangle$. To avoid the tongue twister "tensor product of tensors", we call this the $\otimes$-product ("otimes product") of s and t.

Example 1.2. *Let $\langle s| : K^2 \times K \rightarrowtail K^2$ be the right scalar action tenesor, and $\langle t| : K \times K^3 \rightarrowtail K^3$ be the left scalar action tensor. The tensor $\langle s \otimes t| : (K^2 \otimes K) \times (K \otimes K^3) \rightarrowtail (K^2 \otimes K^3)$ is defined as*

$$\langle s \otimes t | u \otimes k_1, k_2 \otimes v\rangle = \langle s | u, k_2\rangle \otimes \langle t | k_1, v\rangle = k_2 u \otimes k_1 v = k_1 k_2 (u \otimes v) \tag{1.2}$$

Let f be an isomorphism of $K^2 \otimes K$ with K^2 and g an isomorphism of $K \otimes K^3$ with K^3. Then $\langle s \otimes t|$ is identified with the outer product tensor $\langle r| : K^2 \times K^3 \rightarrowtail K^2 \otimes K^3$ by mapping the first input via f and the second via g. This is called an isotopism of tensors. [Wil16]

The diagram scheme in Figure 1 illustrates the $\otimes$-product of tensors. In it, the 3-valent tensors s and t are drawn as shapes with 3 wires indicating 3 axes, with orientation given to the wire to indicate input and output. This notation is known as tensor network diagrams [BB17]. When $\otimes$-producting two tensors, our notation is for wires to be combined by the $\otimes$ symbol. This is non-standard in tensor network diagram literature. Part of our work will be to extend and adapt tensor network diagram to products.

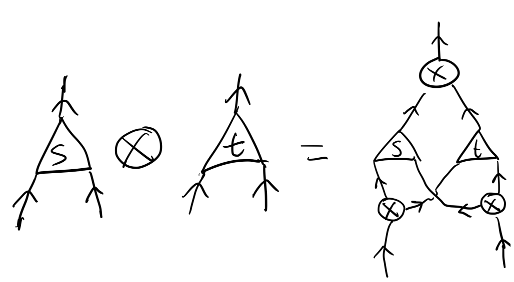

FIGURE 1. Pictorial illustration of soldering s and t

For preliminary results, we have proven $(\!|s \otimes t|\!) = (\!|s|\!) \otimes (\!|t|\!)$. This suggests for certain r, computing $(\!|r|\!)$ detects $\otimes$-decompositions - for instance if $\dim(\!|r|\!) = 1$ and $(\!|r|\!) \cong (\!|s|\!)$ for a known tensor with 1 dimensional derivation closure sitting in a smaller space, then there exists some t such that $r \cong s \otimes t$.

1.3.2. *Related Works.* In related works, the physics community uses techniques like the Density Matrix Renormalization Algorithm [Whi93] attempts to uncover the structure of a high valence tensor by factoring it as a contracted product of 3-tensors called a Matrix Product State. This iterative optimization technique is for complex-valued tensors, relying on the Singular Value Decomposition. Techniques such as Tensor-Train decompositions [Ose11] and Tucker decompositions [Tuc66] are similar but assumes different fixed internal structure.

In the next two sections, we describe in detail the proposed problems we are investigating and contributions we forsee as part of this dissertation.

2. Faster algorithms for algebraic invariants of tensors

We wish to compute $\mathrm{Adj}(t)$, $\mathrm{Cen}(t)$, and $\mathrm{Der}(t)$ for a bimap t.

TODO: Prove closure under composition of one of these algebras. Mention algebra structure is advantageous for small generating sets and structure theory.

For fixed bases, each of the algebras are specified by linear systems of equations, and thus can be computed in a number of steps polynomial to the sum of dimensions. But the naive solution has takes operations cubic in the number of variables. We aim to do better in the general case.

We report on results in collaboration with James Wilson and Joshua Maglione. First, the problem of computing adjoints of bimaps is stated in coordinates. Next, we describe our approach, which is to translate the system to a coordinate free formulation, solve a smaller subproblem, and propogate the subproblem solution to a full solution.

2.1. Simultaneous Sylvester System - Coordinatized.

We solve the following

Given: arrays $R \in K^{r\times b\times c}$, $S \in K^{a\times s\times c}$, and $T \in K^{a\times b\times c}$

Return: matrices $X \in K^{a\times r}$ and $Y \in K^{s\times b}$ such that

$$(\forall i)(XR_i + S_iY = T_i). \tag{2.1}$$

Expressed as list of matrix equation, Equation (2.1) is the natural extension of the Sylvester Equation, which asks for X satisfying the matrix equation $XA + BX = C$. For R and S filled in from a tensor with a fixed basis, and T to be all zero, solving an instance of this problem finds the adjoint algebra of the tensor.

Example 2.1. *Let t be the tensor in $K^{2\times 2\times 2}$ given by a pair of 2×2 matrices ("a system of forms")*

$$t_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, t_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The algebra $\mathrm{Adj}(t)$ is computed by solving for variables x_{ij}, y_{ij} satisfying

$$\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} t_1 = t_1 \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \text{ and } \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} t_2 = t_2 \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}$$

Below gives a coordinate free description.

2.2. Simultaneous Sylvester System - Basis independent.

Given: Elements $r \in R\otimes B\otimes C$, $s \in A\otimes S\otimes C$, $t \in A\otimes B\otimes C$, and isomorphisms identifying each vector space with its dual.

Return: Elements $x \in \mathrm{Hom}(A, R)$ and $y \in \mathrm{Hom}(B, S)$ such that $(x\otimes I_B\otimes I_C)(r) + (I_A\otimes y\otimes I_C)(s) = t$.

Below we describe how to compute x and y without solving a full system of linear equations.

Preliminaries:

First, by composing the isomorphism between R and its dual followed by the natural isomorphism $R^*\otimes B\otimes C \cong \mathrm{Hom}(R, B\otimes C)$, we view r as an element of $\mathrm{Hom}(R, B\otimes C)$. Similarly, view s as an element of $\mathrm{Hom}(S, A\otimes C)$. We now look for subspaces $B' \leq B$ and $A' \leq A$ such that r and s have left inverses after post-composing with projections. That is, we want $r_{B'} := (\pi_{B'}\otimes I_C)\circ r$ and $s_{A'} := (\pi_{A'}\otimes I_C)\circ s$ to have left inverses. Denote these left inverses as $r^{\#}_{B'}$ and $s^{\#}_{A'}$. We also need

B' and A' to have an induced isomorphism to their respective duals, meaning $B \cong B^*$ restricts an isomorphism $B' \cong (B')^*$, and similar for $A \cong A^*$ restricting to $A' \cong (A')^*$.

Solving a smaller subproblem:
To assist in calculation, let $\pi_{A',B',C} := \pi_{A'} \otimes \pi_{B'} \otimes I_C$, and $xr := (x \otimes I_B \otimes I_C)(r)$, and $ys := (I_A \otimes y \otimes I_C)(s)$.
Projecting to the spaces A' and B', we compute

$$\begin{aligned}
&\pi_{A',B',C}(xr + ys) = \pi_{A',B',C}(t) \\
\iff\ &\pi_{A',B',C}(xr) + \pi_{A',B',C}(ys) = \pi_{A',B',C}(t) \\
\iff\ &x_{A'}(\pi_{R,B',C}(r)) + y_{B'}(\pi_{A',S,C}(s)) = \pi_{A',B',C}(t)
\end{aligned}$$

Propagating solution to full problem:
Solving for $x_{A'}$ and $y_{B'}$ proceed by standard linear algebra, but as $\dim x_{A'} = \dim A' \cdot \dim R$ and $\dim y_{B'} = \dim B' \cdot \dim S$, this smaller system have unknowns of considerably lower dimension if the subspaces A' and B' are lower dimensional compared to A and B.

After solving for $x_{A'}$ and $y_{B'}$ by conventional methods, our algorithm proceeds by finding complementary subspaces $A = A' \oplus U$, and $B = B' \oplus V$. Let r_V, s_U, x_U, and y_V be defined analogously to above. Then projecting to the subspaces A' and V, we require $\pi_{A',V,C}(xr + ys) = \pi_{A',V,C}(t)$.

Using the fixed isomorphism between vector spaces and their duals, as well as the natural isomorphisms above, we see that

$$\begin{aligned}
&\pi_{A',V,C}(xr + ys) = \pi_{A',V'C}(t) \\
\iff\ &\pi_{A',V,C}(xr) + \pi_{A',V,C}(ys) = \pi_{A',V,C}(t) \\
\iff\ &x_{A'}(\pi_{R,V,C}(r)) + y_V(\pi_{A',S,C}(s)) = \pi_{A',V,C}(t) \\
\iff\ &(x_{A'} \otimes I_C) \circ r \circ \pi_V + (\pi_{A'} \otimes I_C) \circ s \circ y_V = (\pi_{A'} \otimes I_C) \circ t \circ \pi_V \qquad A' \otimes V \otimes C \cong \mathrm{Hom}(V, A' \otimes C) \\
\iff\ &\tilde{r} + s_{A'} \circ y_V = \tilde{t} \qquad\qquad \text{Where } \tilde{r} := (x_{A'} \otimes I_C) \circ r_V \text{ and } \tilde{t} := (\pi_{A'} \otimes I_C) \circ t_V
\end{aligned}$$

The only unknown in the last equation is y_V as $x_{A'}$ was previously solved for. Hence precomposing with $s^{\#}_{A'}$ to both sides gives a unique y_V solution. Analogously projecting to the subspaces U and B' followed by precomposing $r^{\#}_{B'}$ solves for a unique x_U given a $y_{B'}$. Finally, projecting to the subspaces U and V verifies the overall equation is satisfied for the pair x_U, y_V.

Remark 2.2. *Our algorithm requires the user to provide the subspace A' and B' with the desired invertibility properties of $r_{B'}$ and $s_{A'}$. In coordinates, the linear transformation $r \in \mathrm{Hom}(R, B \otimes C)$ has $bc := \dim B \otimes C$ rows and $r := \dim R$ columns, thus projecting to a dim $\lceil r/c \rceil$ subspace of B have a high probability of maintaining the left invertibility of r.*

For example, if $\dim A, \dim B, \dim C, \dim R, \dim S$ are all $O(n)$ then $\lceil r/c \rceil$ is $O(1)$ hence the naive $O(n^6)$ operations necessary to solve the system is reduced to the $O(n^3)$ operations needed to solve the smaller subproblem.

2.3. **Derivation System.** First to state the problem in coordinates, we need

Definition 2.3 (Outer action). *Given an array $T \in K^{a \times b \times d}$, let $[T_1, \ldots, T_d]$ be a list of $K^{a \times b}$ matrices corresponding to unfolding this array along the third index,. Then for a matrix Z of size $d \times c$, define T^Z, the outer action of Z on T, as the $K^{a \times b \times c}$ array satisfying $(T^Z)_j = \sum_{i=1}^{d} T_i Z_{ij}$.*

Now we are ready to state the problem of solving the derivation system in coordinates.

Problem A (Derivation system - Coordinatized).
Given: *arrays $R \in K^{r \times b \times c}$, $S \in K^{a \times s \times c}$, and $T \in K^{a \times b \times t}$*
Return: *matrices $X \in K^{a \times r}$, $Y \in K^{s \times b}$, and $Z \in K^{t \times c}$ such that*

$$(\forall i) X R_i + S_i Y + (T^Z)_i = 0. \tag{2.2}$$

The above equation is no longer a list of matrix equations due to the outer action by Z. But it is exactly the equation satisfied by the derivation algebra of a tensor when R, S, T are all filled in from a tensor in a fixed basis.

Example 2.4. *Let t be the tensor as in $Example$ 2.1.*
The computation of $\mathrm{Der}(t)$ is to solve for variables x_{ij}, y_{ij}, z_{ij} satisfying

$$\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} t_1 + t_1 \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} = t_1 z_{11} + t_2 z_{21} \quad \textit{and}$$
$$\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} t_1 + t_1 \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} = t_1 z_{12} + t_2 z_{22}$$

The problem has a coordinate free version of the problem written below.

2.4. **Derivation system - Basis independent.**

Given: Elements $r \in R \otimes B \otimes C$, $s \in A \otimes S \otimes C$, $t \in A \otimes B \otimes T$, and isomorphisms identifying the vector spaces and their duals.

Return: Elements $x \in \mathrm{Hom}(A, R)$, $y \in \mathrm{Hom}(B, S)$, and $Z \in \mathrm{Hom}(C, T)$ such that $(x \otimes I_B \otimes I_C)(r) + (I_A \otimes y \otimes I_C)(s) + (I_A \otimes I_B \otimes z)(t) = 0$.

Preliminary investigations suggest a similar approach to Simultaneous Sylvester Systems but with 3 subspaces, $A' \leq A$, $B' \leq B$, and $C' \leq C$. Additional work is necessary.

Thus far we have only been concerned about bimaps. As a challenge upon resolving Problem A, we propose extending the above ideas to higher valence tensors. We frame the question as solving for algebraic invariants of tensors.

We first give notation for an endomorphism acting on a specific axis of the tensor.

Definition 2.5. *Let $\langle t| : \prod_i U_i \rightarrowtail K$. Let $\sigma \in \mathrm{End}(U_a)$. Then define the tensor $\langle t|\sigma$ as*

$$\langle t|\sigma|u\rangle = \langle t|\sigma u_a, u_{\overline{a}}\rangle$$

Where $u = (u_a, u_{\overline{a}})$ splits $u \in \prod_i U_i$ as an element of $U_a \times \prod_{i \neq a} U_i$.

Let $\langle t| : \prod_{i \in I} U_i \rightarrowtail K$. For a 2-element subset $\{a, b\} \subset I$, we define the ab-nucleus of t as

$$\mathrm{Nuc}_{ab}(t) := \{(\sigma_a, \sigma_b) \in \mathrm{End}(U_a) \times \mathrm{End}(U_b) : \langle t|\sigma_a u_a, u_{\overline{a}}\rangle = \langle t|\sigma_b u_b, u_{\overline{b}}\rangle\}. \tag{2.3}$$

For $J \subset I$, define the J-centroid as

$$\mathrm{Cen}_J(t) := \left\{ (\sigma_j)_{j \in J} \in \prod_j \mathrm{End}(U_j) : \langle t|\sigma_j u_j, u_{\overline{j}}\rangle = \langle t|\sigma_k u_k, u_{\overline{k}}\rangle \;\forall j, k \in J \right\}. \tag{2.4}$$

Similary, define the J-derivation as

$$\mathrm{Der}_J(t) := \left\{ (\delta_j)_{j \in J} \in \prod_j \mathrm{End}(U_j) : \sum_j \langle t|\delta_j u_j, u_{\overline{j}}\rangle = 0 \right\}. \tag{2.5}$$

These spaces are computed by linear equations, and gives higher valence tensors the analogue of the centroid, adjoint, and derivation algebra of 3-tensors. Now we ask

> **Challenge A.** *Can the higher valence nuclei, centroid, and derivation algebras be computed in operations fewer than cubic in the number of variables?*

3. X-RAYING TENSORS FOR PRODUCT DECOMPOSITIONS OF TENSORS

3.1. **Preliminaries.** As described in the introduction, we wish to understand product decompositions of tensors through *X-raying*. To state the problem, we follow the exposition and notation in [FMW20].

Definition 3.1. *(Ternary Galois Connection of Tensors, Ideals, and Operators)*

We define evaluating a multivariable polynomial, with operators substituting for the indeterminantes. For $p = \sum_e \lambda_e X^e \in K[x_0, \dots, x_n] =: K[X]$ *and* $\omega \in \prod_i \mathrm{End}(U_i)$*, let*

$$p(\omega) := \sum_e \lambda_e \left(\omega_0^{e_0}, \dots, \omega_n^{e_n}\right) \in \prod_i \mathrm{End}(U_i).$$

Let $S \subset \mathrm{Mult}(U_n, \dots, U_1; U_0)$*. It is evidently a tensor space with the identity interpretation map. For all* $t \in S$*, define* $\langle t|p(\omega)$ *where for any* $(u_1, \dots, u_n)$*,*

$$\langle t|p(\omega)|u\rangle = \sum_e \lambda_e \omega_0^{e_0} \langle t|\omega_1^{e_1} u_1, \dots, \omega_n^{e_n} u_n\rangle.$$

Now fix a polynomial p *and operator* ω*. Define the set* $\boldsymbol{T}(p, \omega) := \{t \in \mathrm{Mult}(U_n, \dots, U_1; U_0) : \langle t \mid p(\omega) = 0\}$*. Extend this definition to subsets* P *and* Ω *via*

$$\boldsymbol{T}(P, \Omega) := \bigcap_{p \in P} \bigcap_{\omega \in \Omega} \boldsymbol{T}(p, \omega).$$

Similarly, for fixed polynomial p *and tensor* t *define the set* $\boldsymbol{Z}(t, p) := \{\omega \in \prod_u \mathrm{End}(U_i) : \langle t|p(\omega) = 0\}$ *and extend to subsets* $\boldsymbol{Z}(S, P)$*.*

Fixing $P \subset K[X]$*, there is an inclusion reversing Galois connection between subsets of* $\mathrm{Mult}(U_n, \dots, U_1; U_0)$ *and subsets of* $\prod_i \mathrm{End}(U_i)$ *given by*

$$S \subset \boldsymbol{T}(P, \Omega) \iff \Omega \subset \boldsymbol{Z}(S, P) \tag{3.1}$$

From [FMW20] the set $\boldsymbol{T}(P, \Omega)$ *is a vector subspace and* $\boldsymbol{Z}(S, \boldsymbol{d})$ *is a Lie algebra for* $\boldsymbol{d} = x_n + \cdots + x_0$*.*

Definition 3.2. *(Derivation closure) Let* $t \in \mathrm{Mult}(U_n, \dots, U_1; U_0)$*. Then* $(\!|t|\!)$*, the derivation closure of* t*, is the vector subspace consisting of all* s *such that* $\mathrm{Der}(t) \subset \mathrm{Der}(s)$*. Hence* $(\!|t|\!) := \boldsymbol{T}(\boldsymbol{d}, \boldsymbol{Z}(\boldsymbol{d}, t))$*.*

Example 3.3. *Let* t *be the matrix multiplication tensor for* 2×3 *and* 3×4 *rectangular matrices. That is,* $\langle t| : K^{2\times 3} \times K^{3\times 4} \rightarrowtail K^{2\times 4}$ *by* $\langle t|M, N\rangle := MN$*. Then by Corollary 8.4.4 of [FMW20], we have* $(\!|t|\!)$ *as a 1-dimensional vector subspace spanned by* Kt*.*

Definition 3.4. *($\otimes$-product of tensors)*

Let $s \in \mathrm{Mult}(U_n, \dots, U_1; U_0) =: S$ *and* $t \in \mathrm{Mult}(V_n, \dots, V_1; V_0) =: T$*. Define* $s \otimes t \in \mathrm{Mult}(U_n \otimes V_n, \dots, U_1 \otimes V_1; U_0 \otimes V_0)$ *as the tensor product of multilinear maps, with interpretation* $\langle s \otimes t| : \prod_i U_i \otimes V_i \rightarrowtail U_0 \otimes V_0$ *given by* $\langle s \otimes t|u_1 \otimes v_1, \dots, u_n \otimes v_n\rangle = \langle s|u_1, \dots, u_n\rangle \otimes \langle t|v_1, \dots, v_n\rangle$*. We say* $s \otimes t$ *is the* $\otimes$*-product of* s *and* t*.*

Remark 3.5. *Writing* $s \otimes t$ *to mean a multilinear map is an abuse of notation inherited from a similar notion in the tensor product of linear maps. Since* $s \in \mathrm{Mult}(U_n, \dots, U_1; U_0)$ *and* $t \in \mathrm{Mult}(V_n, \dots, V_1; V_0)$ *are both vectors of vector spaces,* $s \otimes t$ *strictly speaking is the* $\varphi(s, t)$*, image of the tensor product of vector spaces.*

However, by the following sequence of natural isomorphisms, $\varphi(s, t)$ *is identified with an element of* $\mathrm{Mult}(U_n \otimes V_n, \dots, U_1 \otimes V_1; U_0 \otimes V_0)$*. Starting with* $\varphi(s, t) \in \mathrm{Mult}(U_n, \dots, U_1; U_0) \otimes \mathrm{Mult}(V_n, \dots, V_1; V_0)$*:*

$$\begin{aligned}
\mathrm{Mult}(U_n, \dots, U_1; U_0) \otimes \mathrm{Mult}(V_n, \dots, V_1; V_0) &\cong (U_n^* \otimes \cdots \otimes U_1^* \otimes U_0) \otimes (V_n^* \otimes \cdots \otimes V_1^* \otimes V_0) \\
&\cong (U_n^* \otimes V_n^*) \otimes \cdots \otimes (U_1^* \otimes V_1^*) \otimes (U_0 \otimes V_0) \\
&\cong \mathrm{Mult}(U_n \otimes V_n, \dots, U_1 \otimes V_1; U_0 \otimes V_0)
\end{aligned}$$

Example 3.6. *The Kronecker product of matrices is a case of soldering tensors. Let* $M \in \mathrm{Mult}(K^2, K^2; K)$ *and* $N \in \mathrm{Mult}(K^2, K^2; K)$ *be given as* 2×2 *matrices.*

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \qquad N = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$$

Let $\mathrm{Kron}(M, N)$ *be the Kronecker product of matrices* M *and* N.

$$\mathrm{Kron}(M,N) = \left(\begin{array}{cc|cc} 3 & 4 & \multicolumn{2}{c}{\multirow{2}{*}{$\mathbf{0}$}} \\ 5 & 6 & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{$\mathbf{0}$}} & 6 & 8 \\ & & 10 & 12 \end{array}\right)$$

The 4×4 *matrix* $\mathrm{Kron}(M, N)$ *is given as an element of* $\mathbb{M}_2(\mathbb{M}_2(K))$. *It has interpretation* $\langle M \otimes N| : (K^2 \otimes K^2), (K^2 \otimes K^2) \rightarrowtail K$ *given by mapping basis element* $e_i \otimes e_j$ *to the* $2(i-1)+j$*th row or column of the matrix. Specifically,* $\langle M \otimes N | e_i \otimes e_j, e_k \otimes e_l\rangle$ *is row* $2i - 1 + j$*, column* $2k - 1 + l$ *of* $\mathrm{Kron}(M, N)$*, giving the product of the* (i, k)*th row of* M *with the* (j, l)*th row of* N.

3.2. **Existing techniques.** Next we describe some existing techniques that analyze tensors admitting $\otimes$-decompositions.

Example 3.7 (Algebras)**.** *Consider the isomorphism of* $\mathbb{R}$*-algebras* $\mathbb{M}_2(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{M}_2(\mathbb{C})$ *given by*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes z \mapsto \begin{bmatrix} az & bz \\ cz & dz \end{bmatrix} \tag{3.2}$$

Let $r : \mathbb{M}_2(\mathbb{C}) \times \mathbb{M}_2(\mathbb{C}) \rightarrowtail \mathbb{M}_2(\mathbb{C})$, $s : \mathbb{M}_2(\mathbb{R}) \times \mathbb{M}_2(\mathbb{R}) \rightarrowtail \mathbb{M}_2(\mathbb{R})$*, and* $t : \mathbb{C} \times \mathbb{C} \rightarrowtail \mathbb{C}$ *be their respective multiplication tensors. The isomorphism extends to the multiplication tensors, meaning* $r \cong s \otimes t$.

But also $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{M}_2(\mathbb{C})$ *for* $\mathbb{H}$ *the real Quaternions via*

$$i \otimes 1 \mapsto \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad j \otimes 1 \mapsto \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad k \otimes 1 \mapsto \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \tag{3.3}$$

This example illustrate challenges with having unique $\otimes$*-decompositions.*

Example 3.8 (Tensor over centroid)**.** *Let* $s \in \mathbb{F}_3^{2\times2\times2}$ *be the tensor given as a system of forms*

$$s_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad s_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

with interpretation $\langle s| : \mathbb{F}_3^2 \times \mathbb{F}_3^2 \rightarrowtail \mathbb{F}_3^2$. *This is the multiplication table of the quotient ring* $\mathbb{F}_3[x]/(x^2 + 1) \cong \mathbb{F}_9$*, which is a field. Hence* $\langle s|$ *has centroid isomorphic to* $\mathbb{F}_9$.

Let t *be the* 2×2 *identity matrix, interpreted as the bilinear form* $\langle t| : \mathbb{F}_3^2 \times \mathbb{F}_3^2 \rightarrowtail \mathbb{F}_3$. *Let* $r = s \otimes t$ *be the* $\otimes$*-product of* s *and* t. *Computing,* $\mathrm{Cen}(r) \cong \mathbb{F}_9$. *Viewing* r *as framed by* $\mathbb{F}_9$ *vector spaces, we recover that* r *is bilinear over* $\mathbb{F}_9$.

TODO: Write up extracting the field multiplication tensor.

Example 3.9. *[Wil16, Chapter 6][Condensation by full idempotents]*

Let $\langle s| : U \times K \rightarrowtail U$ *be defined as* $\langle s|u, k\rangle = uk$*, scalar multiplication by* K *on the right. Let* $\langle t| : K \times V \rightarrowtail V$ *be scalar multiplication by* K *on the left. The product* $\langle s \otimes t| : (U \otimes K) \times (K \otimes V) \rightarrowtail (U \otimes V)$ *is isotopic to* $\varphi : U \times V \rightarrowtail U \otimes V$*, the canonical map into the tensor product.*

Let $\mathrm{Nuc}_{20}(\varphi) := \mathcal{L}$ *be defined as per Equation* (2.3)*, consisting of all* $(\lambda_2, \lambda_0) \in \mathrm{End}(U) \times \mathrm{End}(U \otimes V)$ *satisfying* $\langle \varphi|\lambda_2 u, v\rangle = \lambda_0\langle\varphi|u, v\rangle$. *Suppose* e *is a full idempotent of* $\mathcal{L}$*, meaning* $e^2 = e$ *and* $\mathcal{L}e\mathcal{L} = \mathcal{L}$. *Instead of finding an* $\otimes$*-decomposition, we condense the bimap* φ *as* $\tilde{\varphi} : eU \times V \rightarrowtail eW$ *where* $\mathrm{Nuc}_{20}(\tilde{*}) = e\mathcal{L}e$. *This is of interest because* $e\mathcal{L}e$ *is Morita equivalent to* $\mathcal{L}$ *but smaller.*

When U *and* V *are given coordinates,* $\mathcal{L} = (\mathbb{M}_n(K), \mathbb{M}_n(K) \otimes I_m(K))$ *so* $e = (E_{11}, E_{11} \otimes I_m(K))$ *is a full idempotent and condenses* φ *down to scalar multiplication on the left. In [Wil16] this technique is used to prove a property of automorphisms of bimaps.*

Applying the same technique for the algebras $\mathrm{Adj}(t)$ *and* $\mathrm{Nuc}_{10}(t)$ *to Example 3.3 compresses the rectangular matrix multiplication tensor down to multiplication of the underlying field.*

3.3. **X-raying.** We are interested in the derivation closure of the $\otimes$-product of tensors as the first step towards understanding them. That is, given $s \in \mathrm{Mult}(U_n, \ldots, U_1; U_0)$ and $t \in \mathrm{Mult}(V_n, \ldots, V_1; V_0)$, do $(\!|s|\!)$ and $(\!|t|\!)$ relate to $(\!|s \otimes t|\!)$? We have recently resolved this question. Below is a preliminary lemma.

Lemma 3.10. *For $s \in \mathrm{Mult}(U_n, \ldots, U_1; U_0)$ and $t \in \mathrm{Mult}(V_n, \ldots, V_1; V_0)$, there is an embedding $\iota_s : \mathrm{Der}(s) \hookrightarrow \mathrm{Der}(s \otimes t)$ given by $(\sigma_i)_{i\in[n]} \mapsto (\sigma_i \otimes I_{V_i})_{i\in[n]}$ and an embedding $\iota_t : \mathrm{Der}(t) \hookrightarrow \mathrm{Der}(s \otimes t)$ given by $(\tau_i)_{i\in[n]} \mapsto (I_{U_i} \otimes \tau_i)_{i\in[n]}$.*

Proof. The map ι_s is injective on each factor as tensoring with the identity morphism is injective. The endomorphism $(\sigma_i \otimes I_{V_i})_{i\in[n]}$ is in $\mathrm{Der}(s \otimes t)$ as $\langle s \otimes t|\mathbf{d}((\sigma_i \otimes I_{V_i})_{i\in[n]}) = \sum_{i=1}^{n}\langle s|\sigma_i \otimes \langle t|I_{V_i} = \langle s|\mathbf{d}((\sigma_i)_{i\in[n]}) \otimes \langle t| = 0$. Lastly, we need to demonstrate ι_i is a map of Lie algebras. This follows by the calculation $\iota(\delta + \rho) = \iota((\delta_i + \rho_i)_{i\in[n]}) = ((\delta_i + \rho_i) \otimes I_{V_i})_{i\in[n]} = (\delta_i \otimes I_{V_i})_{i\in[n]} + (\rho_i \otimes I_{V_i})_{i\in[n]} = \iota(\delta) + \iota(\rho)$. The case for ι_t is analogous. □

Theorem 3.11. *Let $s \in \mathrm{Mult}(U_n, \ldots, U_1; U_0) =: U$ and $t \in \mathrm{Mult}(V_n, \ldots, V_1; V_0) =: V$. Then $(\!|s|\!) \otimes (\!|t|\!) \cong (\!|s \otimes t|\!)$.*

Proof. Our strategy is to show $(\!|s \otimes t|\!) \subset (\!|s|\!) \otimes (\!|t|\!)$ and $(\!|s|\!) \otimes (\!|t|\!) \subset (\!|s \otimes t|\!)$.

$(\!|s \otimes t|\!) \subset (\!|s|\!) \otimes (\!|t|\!)$:

By Lemma 3.10, there are embeddings $\iota_s : \mathrm{Der}(s) \hookrightarrow \mathrm{Der}(s \otimes t)$ and $\iota_t : \mathrm{Der}(t) \hookrightarrow \mathrm{Der}(s \otimes t)$. The inclusion reversing nature of the antitone Galois connection in Definition 3.1 implies $(\!|s \otimes t|\!) = T(\mathbf{d}, \mathrm{Der}(s \otimes t)) \subset T(\mathbf{d}, \iota_s(\mathrm{Der}(s)))$ and $(\!|s \otimes t|\!) \subset T(\mathbf{d}, \iota_t(\mathrm{Der}(t)))$. Thus $(\!|s \otimes t|\!)$ is in their intersection. We shall prove $T(\mathbf{d}, \iota_s(\mathrm{Der}(s))) = (\!|s|\!) \otimes V$ and $T(\mathbf{d}, \iota_j(\mathrm{Der}(t))) = U \otimes (\!|t|\!)$. The conclusion follows as $(\!|s \otimes t|\!) \subset (\!|s|\!) \otimes V \cap U \otimes (\!|t|\!) = (\!|s|\!) \otimes (\!|t|\!)$.

The statement to prove is $T(\mathbf{d}, \iota_s(\mathrm{Der}(s))) = (\!|s|\!) \otimes V$.

We first show the direction $(\!|s|\!) \otimes V \subset T(\mathbf{d}, \iota_s(\mathrm{Der}(s)))$. As $(\!|s|\!) \otimes V$ is generated by $\acute{s} \otimes t$ for $\acute{s} \in (\!|s|\!)$ and $t \in V$, it suffices to show $\acute{s} \otimes t \in T(\mathbf{d}, \iota_s(\mathrm{Der}(S)))$. This follows as $\langle \acute{s} \otimes t|$ satisfies $\langle \acute{s} \otimes t|\mathbf{d}(\delta) = 0$ for all $\delta = (\sigma_i \otimes I_{V_i})_{i\in[n]} \in \iota_s(\mathrm{Der}(s))$ since $(\sigma_i)_{i\in[n]} \in \mathrm{Der}(s)$.

In the opposite direction, $T(\mathbf{d}, \iota_s(\mathrm{Der}(s)))$ is a subspace of $U \otimes V$. We shall show in fact it is the subspace $(\!|s|\!) \otimes V$ by showing every element in $T(\mathbf{d}, \iota_s(\mathrm{Der}(s)))$ is the sum of pure tensors $\acute{s} \otimes t$ for $\acute{s} \in (\!|s|\!), t \in V$.

Let $r = \sum_{i=1}^{m} s_i \otimes t_i \in U \otimes V$ be an element of $T(\mathbf{d}, \iota_s(\mathrm{Der}(s)))$, with all t_is linearly independent. Showing each s_i is in $(\!|s|\!)$ concludes r is in $(\!|s|\!) \otimes V$. By definition r satisfies $\langle r|\mathbf{d}(\iota_s(\sigma)) = 0$ for all $\sigma \in \mathrm{Der}(s)$.

Let $\sigma \in \mathrm{Der}(s)$ and $\iota_s(\sigma) = (\sigma_j \otimes I_{V_j})_{j\in[n]}$. Computing,

$$
\begin{aligned}
0 &= \langle r|\mathbf{d}(\iota_s(\sigma)) \\
&= \left\langle \sum_i s_i \otimes t_i \right| \mathbf{d}(\iota_s(\sigma)) \\
&= \sum_i \langle s_i \otimes t_i|\mathbf{d}(\iota_s(\sigma)) \qquad \text{linearity of tensor evaluation} \\
&= \sum_i \left(\langle s_i \otimes t_i| \left(\sum_{j=1}^{n} \sigma_j \otimes I_{V_j} \right) \right) \\
&= \sum_i \left(\sum_{j=1}^{n} (\langle s_i \otimes t_i|\sigma_j \otimes I_{V_j}) \right) \\
&= \sum_i \left(\sum_{j=1}^{n} (\langle s_i|\sigma_j \otimes t_i) \right) \\
&= \sum_i \left(\sum_j \langle s_i|\sigma_j \right) \otimes t_i \\
&= \sum_i \langle s_i|\mathbf{d}(\sigma) \otimes t_i
\end{aligned}
$$

Let $\{F_b\}_{b\in\mathcal{B}}$ be a basis of V. Then expanding each t_i in this basis,

$$\begin{aligned}0 &= \sum_i \left(\langle s_i|\mathbf{d}(\sigma) \otimes \left(\sum_{b\in\mathcal{B}} \lambda_{ib}F_b\right)\right)\\ &= \sum_{b\in\mathcal{B}} \left(\sum_i (\lambda_{ib}\langle s_i|\mathbf{d}(\sigma)) \otimes F_b\right)\end{aligned}$$

Since F_b are linearly independent, $\sum_i \lambda_{ib}\langle s_i|\mathbf{d}(\sigma) = 0$. Since t_is are linearly independent, the m by $|\mathcal{B}|$ matrix $[\lambda_{ib}]$ has full row rank. Thus the only way $\sum_i \lambda_{ib}\langle s_i|\mathbf{d}(\sigma) = 0$ is if $\langle s_i|\mathbf{d}(\sigma) = 0$ for all i. Thus $s_i \in (\!|s|\!)$ for all i. This concludes the proof of $T(\mathbf{d}, \iota_s(\mathrm{Der}(s))) = (\!|s|\!) \otimes V$.

The statement $T(\mathbf{d}, \iota_t(\mathrm{Der}(t))) = U \otimes (\!|t|\!)$ is proven analogously.

$(\!|s|\!) \otimes (\!|t|\!) \subset (\!|s \otimes t|\!)$:

The strategy will be to first show for $\acute{s} \in (\!|s|\!)$, that $\acute{s}\otimes t \in (\!|s\otimes t|\!)$, and secondly, show if $\acute{s}\otimes t \in (\!|s\otimes t|\!)$ for all s, then for all $\acute{t} \in (\!|t|\!)$, that $\acute{s} \otimes \acute{t} \in (\!|s \otimes t|\!)$. The proof concludes as $(\!|s|\!) \otimes (\!|t|\!)$ is generated by $\acute{s} \otimes \acute{t}$ for $\acute{s} \in (\!|s|\!)$ and $\acute{t} \in (\!|t|\!)$, and

To show $\acute{s} \otimes t \in (\!|s \otimes t|\!)$, let $\delta \in \mathrm{Der}(s \otimes t)$. Since δ is an element of $\prod_i \mathfrak{gl}(U_i \otimes V_i) \cong \prod_i (\mathfrak{gl}(U_i) \otimes \mathfrak{gl}(V_i))$, write δ as $\left(\sum_{j=1}^{R_i} (\sigma_j \otimes \tau_j)\right)_{i\in[n]}$. By construction $\langle s \otimes t|\mathbf{d}(\delta) = 0$. Calculating,

$$\begin{aligned}0 &= \langle s \otimes t|\mathbf{d}(\delta)\\ &= \langle s \otimes t| \sum_{i=1}^{n} \left(\sum_{j=1}^{R_i} (\sigma_j \otimes \tau_j)\right)\\ &= \langle s \otimes t| \sum_{a\in A} (\sigma_a \otimes \tau_a) && \text{grouping into one indexing set}\\ &= \sum_{a\in A} \langle s|\sigma_a \otimes \langle t|\tau_a\\ &= \sum_{a\in A} \langle s|\sigma_a \otimes \left(\sum_{b\in\mathcal{B}} \lambda_{ab}F_b\right) && \text{For } F_b \text{ basis of } V\\ &= \sum_{b\in\mathcal{B}} \left(\sum_{a\in A} \lambda_{ab}\langle s|\sigma_a\right) \otimes F_b\end{aligned}$$

As $\{F_b\}_{b\in\mathcal{B}}$ is a basis of V, $\sum_{a\in A} \lambda_{ab}\langle s|\sigma_a = 0$ for each b. Regrouping and combining the terms in the indexing set A by axes, we have $\langle s|\sigma = 0$, meaning $\sigma \in \mathrm{Der}(s)$. Thus $\langle \acute{s}|\sigma = 0$ as well. Substituting $\acute{s}$ in place of s in the above equation also equals 0, concluding $\langle \acute{s} \otimes t|\mathbf{d}(\delta) = 0$.

The proof that $\acute{s} \otimes \acute{t} \in (\!|s \otimes t|\!)$, assuming $\acute{s} \otimes t$ is in $(\!|s \otimes t|\!)$ is analogous.

□

Example 3.12. *Let $\langle s| : \mathfrak{sl}_2 \times \mathfrak{sl}_2 \rightarrowtail \mathfrak{sl}_2$ be the multiplication tensor of $\mathfrak{sl}_2$. We compute $(\!|s|\!) = Ks$. Let $\langle t| : \mathbb{F}_3^2 \times \mathbb{F}_3^2 \rightarrowtail \mathbb{F}_3^2$ be the multplication table of $\mathbb{F}_3[x]/(x^2+1)$ as given in Example 3.8. We compute $(\!|t|\!)$ as spanned by t and the tensor r given by the system of forms $\left[\begin{bmatrix}0 & 2\\ 2 & 0\end{bmatrix}, \begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}\right]$.*

As expected, $(\!|s|\!) \otimes (\!|t|\!)$ and $(\!|s \otimes t|\!)$ are both 2 dimensional. By definition $(\!|s|\!) \otimes (\!|t|\!)$ is spanned by $s \otimes t$ and $s \otimes r$, while computation verifies $(\!|s \otimes t|\!)$ is spanned by these same tensors.

Before describing our main problem, we address potential complications in finding $\otimes$-decompositions.

3.3.1. *Non-canonical choices in $\otimes$-decompositions.* In Definition 3.4 the tensors s and t have the same valence, and there is a matching of axis U_i of s with V_i of t.

Both are convenient but not necessary when looking for $\otimes$-decompositions. A tensor can be padded by axes consisting of K. For instance, a linear transformation $K^2 \to K^2$ is isotopic to the tensor $K^2 \times K \rightarrowtail K^2$ via the isomorphism $t \mapsto \tilde{t}$ where $\langle \tilde{t}|u, k\rangle = k\langle t|u\rangle$. Notice there's combinatoral explosion of possibilities of which axes to pad when one tensor has fewer axes than the other. Not matching U_i with V_i brings combinatorial explosion to potential $\otimes$-decompositions. We defer their investigations to future work, except to say we do not rule out this possibility when X-raying.

Now we describe the main problem under consideration.

Problem B (Product decompositions of tensors).
Given tensor $r \in \mathrm{Mult}(W_n, \dots, W_1; W_0)$***, what are necessary and sufficient conditions to concludes*** r ***is*** $\otimes$***-indecomposable or prove there exists non-trivial*** s ***and*** t ***such that*** $r = s \otimes t$***?***

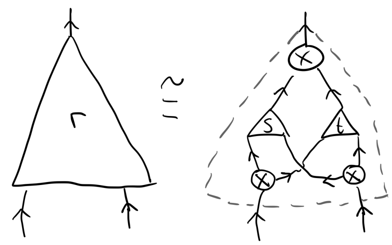

FIGURE 2. Pictorial illustration of Problem B. We are given is the tensor r. We'd like to find non-trivial s, t such that $r \cong s \otimes t$. The dashed lines indicate if successful, while we don't get to view the internals of r, we do understand how r is built.

The non-trivial stipulation avoids the situation of $r \cong r \otimes \mu$ for μ the field multiplication tensor $K \times \cdots \times K \rightarrowtail K$. Examples in Section 2.1 give some partial answers for special cases. We'd like to formalize these results and extend to cases beyond the associative case, with the derivation algebra as our primary tool in Theorem 3.11.

As part of answering this problem, we plan to analyze the derivation algebras themselves to understand the tensor. That is, **given** $s \in \mathrm{Mult}(U_n, \dots, U_1; U_0)$ **and** $t \in \mathrm{Mult}(V_n, \dots, V_1; V_0)$**, is** $\mathrm{Der}(s \otimes t)$ **completely determined by** $\mathrm{Der}(s)$ **and** $\mathrm{Der}(t)$**?**

This question has an affirmative answer for the case of adjoints, as $\mathrm{Adj}(s \otimes t) = \mathrm{Adj}(s) \otimes \mathrm{Adj}(t)$ [Wil09a]. Nothing is known for $\mathrm{Der}(s \otimes t)$ other than Lemma 3.10 above. Preliminary computations suggests the embeddings of $\mathrm{Der}(s)$ and $\mathrm{Der}(t)$ do not behave like the adjoint case, as the dimension of $\mathrm{Der}(s \otimes t)$ in some computed examples is not the product of $\dim \mathrm{Der}(s)$ and $\dim \mathrm{Der}(t)$.

We want to allow for soldering using the (P, Ω)**-tensor product, which we define below.**

Definition 3.13 ((P, Ω)-tensor product). *[FMW20]*
Let $U_1, \dots, U_n$ *be* K*-vector spaces,* $P \subset K[x_1, \dots, x_n]$ *and* $\Omega \subset \mathrm{End}(U_1) \times \cdots \times \mathrm{End}(U_n)$. *Define the following subspace of* $U_1 \otimes \cdots \otimes U_n$

$$\Xi(P, \Omega) := \left\langle \sum_e \lambda_e \omega_1^{e_1} u_1 \otimes \cdots \otimes \omega_n^{e_n} u_n \,\middle|\, \omega \in \Omega, \sum_e \lambda_e x_1{}^{e_1} \cdots x_n^{e_n} \in P, u_i \in U_i \right\rangle.$$

Define the (P, Ω)*-tensor product space as the quotient space*

$$\text{◖}\, U_1, \dots, U_n \,\text{◗}_\Omega^P := (U_1 \otimes \cdots \otimes U_n)/\Xi(P, \Omega),$$

together with a K*-multilinear map* $\text{◖} \cdots \text{◗} : U_1 \times \cdots \times U_n \rightarrowtail \text{◖}\, U_1, \dots, U_n \,\text{◗}_\Omega^P$*, where* $\text{◖}\, u_1, \dots, u_n \,\text{◗} = u_1 \otimes \cdots \otimes u_n + \Xi(P, \Omega)$.

Notice $\text{◖}\, U_1, \dots, U_n \,\text{◗}_\emptyset^\emptyset$ is the usual tensor product of vector spaces. Let $P \subset K[x_1, \dots, x_n]$ and $\Omega \subset \prod_i \mathrm{End}(U_i)$. Suppose each U_i is a tensor space $\mathrm{Mult}(V_i^m, \dots, V_i^1; K) \cong \left(\bigoplus_j V_i^j\right)^*$, and an isomorphism $V_i^j \cong (V_i^j)^*$ is specified. Let $(s_1, \ldots, s_n) \in \prod_i U_i$ be a tuple of tensors. Then the image of $(s_1, \ldots, s_n)$ under a (P, Ω)-tensor product, denoted $\langle\!\langle s_1, \ldots, s_n \rangle\!\rangle =: r$, is an element of a quotient space $\bigotimes_i U_i / \Xi(P, \Omega)$.

Let $\tilde{r}$ be an element of $\bigotimes_i U_i$ identified to r by fixing a complementary subspace to $\Xi(P, \Omega)$ in $\bigotimes_i U_i$. As $\bigotimes_i U_i = \bigotimes_i \bigotimes_j V_i^j \cong \bigotimes_j \bigotimes_i V_i^j$, $\tilde{r}$ can be identified with a multilinear map in $\mathrm{Mult}(\bigotimes_i V_i^1, \ldots, \bigotimes_i V_i^m; K)$. This gives a multilinear interpretation $\langle r| : \prod_{j=1}^m \bigotimes_i V_i^j \rightarrowtail K$.

Challenge B. *Let $r \in \mathrm{Mult}(U_n, \ldots, U_1; U_0)$ be a tensor. For what (P, Ω) does r admit a (P, Ω)-product decomposition?*

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