Introduction

Tensors encapsulate multilinear maps. Often given as a multiway array of numbers, they are used across various disciplines of mathematics to record information for some fixed reference frame. As such, they are studied from many complementary perspectives [4] [16] [17] [20] [12] [21]. (**TODO: lengthen and contexualize as necessary**)

Throughout, we fix a field K. A tensor t is an element of a tensor space T that admit interpretation as a multilinear map $\langle t| : U_n \times \cdots \times U_1 \rightarrowtail U_0$ where each U_i is a K-vector space. We say the spaces $\{U_0, \dots, U_n\}$ are the frame of tensor, and n the valence. For $|u\rangle = |u_n, \dots, u_1\rangle$, write $\langle t|u\rangle \in U_0$ to mean evaluating $\langle t|$ at $|u\rangle$. With this definition, given some $a \times b$ matrix of numbers $[t_{ij}] \in \mathbb{M}_{a\times b}(K)$, interpret the matrix as a multilinear map $\langle t| : K^a \times K^b \rightarrowtail K$ where $\langle t|e_a^{(i)}, e_b^{(j)}| = t_{ij}$. This extends to cubes of numbers and so on, so our definition interoperates with the popular model of tensors as dense grids of numbers.

The perspective taken in this dissertation propsal is to study tensors as distributive products using the tools of algebra. Fix a trivalent tensor (bimap) $* : U \times V \rightarrowtail W$. Existing work such as [10], and [26] highlights the role of the commutative centroid algebra ($\mathrm{Cen}(t) := \{\sigma \in \mathrm{End}(U) \times \mathrm{End}(V) \times \mathrm{End}(W) : \sigma u * v = u * \sigma v = \sigma(u * v) \forall u \in U, v \in V\}$), associative adjoint algebra ($\mathrm{Adj}(t) := \{\sigma \in \mathrm{End}(U)^{op} \times \mathrm{End}(V) : u\sigma * v = u * \sigma v\}$), and Lie derivation algebra ($\mathrm{Der}(t) := \{\delta \in \mathrm{End}(U) \times \mathrm{End}(V) \times \mathrm{End}(W) : \delta u * v + u * \delta v = \delta(u * v)\}$). Existing work using this approach include discovering basis independent cluster pattern in tensors [5], decomposing p-groups [23], finding direct product decomposition of groups [24], and advances in isomorphism testing [7] [11] [9].

Two avenues of study are proposed for my dissertation. On the algorithmic front, the computation of algebras $\mathrm{Adj}(t)$, $\mathrm{Cen}(t)$, and $\mathrm{Der}(t)$, although each being given by linear equations, are bottlenecks for existing tensor methods due to them being cubic in the number of variables. For roughly cubic tensors of dimension n for each vector space in the frame, this is at minimum $O(n^6)$ operations. In collaboration with James Wilson and Joshua Maglione, we have preliminary results for an asymptotically faster algorithm in computing $\mathrm{Adj}(t)$ and $\mathrm{Cen}(t)$ in $O(n^3)$ operations, inspired by analogous results for matricies known as the Bartels-Stewart algorithm [1]. My proposed work is to explore asymptotic speedups for the computation of $\mathrm{Der}(t)$ in the trivalent case, and to extend to higher valence tensors in general.

On the theory front, our goal is to extend our understanding of how algebraic invariants of tensors constrain the structure of the tensors themselves. We begin with a survey of existing work. Wilson in [26] [24] [23] [25] proves for bimaps the algebric invariants of $\mathrm{Adj}(t)$ and $\mathrm{Cen}(t)$ control direct sum decompositions and automorphisms of t, utilizing these results to prove properties for the originating algebraic structures.

Recent work generalizing from bimaps [8] finds a long exact sequence linking the various nuclei (generalized adjoints), centroids, and derivations of a higher valence tensor. Examples in Section 6 of [8] are given of tensors whose derivation algebra suggests a decomposition. Further work by First, Maglione, and Wilson [13] defines a ternary Galois connection between tensors, operators, and polynomial ideals. Alongside it, they define a multivalent generalization of the Whitney tensor product, parametrized by operators $\Omega \subset \prod_i \mathrm{End}(U_i)$ and polynomials $P \subset K[x_0, \dots, x_n]$. The notation $◖\, U_1, \dots, U_n\, ◗_{\Omega}^{P}$ is used to indicate the (P, Ω)-tensor product on vector spaces. It is proven for a tensor t, the $(\mathbf{d}, \mathrm{Der}(t))$-tensor product for $\mathbf{d}$ the derivation polynomial $x_n + \cdots + x_0$ is universally the smallest amongst the (P, Ω)-products for which $P \subset K[x_1, \dots, x_n]$ is an ideal generated by linear homogeneous polynomials. This motivates future studies of $\mathrm{Der}(t)$ and its associated $(\mathbf{d}, \mathrm{Der}(t))$-tensor product space.

For the remainder of the introduction, let $U_0 = k$. Then as investigated in [9], the vector space of tensors denoted $(\!|t|\!)$ (derivation closure of t) consisting of tensors t' whose derivation algebra contains the derivation algebra of t, may be identified with $◖\, U_1, \dots, U_n\, ◗_{\mathrm{Der}(t)}^{\mathbf{d}}$ and thus is universally the smallest search space for solving tensor isomorphism questions involving t. In [9], an infinite family of tensors $\dim (\!|t|\!) = 1$ in [9] is constructed. However, little else is known about $(\!|t|\!)$.

By Theorem B of [13], a basis for the space $(\!|t|\!)$ is computable in polynomial time, so computational examples are available in practice.

We now describe the class of tensors to be studied. For tensors s and t with interpretations $\langle s| : \prod_{i=1}^{n} U_i \rightarrowtail K$ and $\langle t| : \prod_{i=1}^{n} V_i \rightarrowtail K$, define the tensor product of K-tensors s and t as $\langle s \otimes t| : \prod_{i=1}^{n}(U_i \otimes V_i) \rightarrowtail K$, where $\langle s \otimes t|u \otimes v\rangle = \langle s|u\rangle\langle t|v\rangle$. This we call *soldering* the tensors s and t together, due to the output tensor having the same valence as the inputs. This construction is used an example admitting non-associative tensor decomposition in [8]. We view it as generalizing the tensor product of algebras to the heterogeneous case. For instance, given $\langle s| : K^2 \times K^2 \rightarrowtail K$ the dot product on K, and $\langle t| : L \times L \rightarrowtail L$ the multiplication tensor of some extension field L of K, the soldered tensor is the dot product on L as $\langle s \otimes t| : (K^2 \otimes L) \times (K^2 \otimes L) \rightarrowtail (K \otimes L) \cong L^2 \times L^2 \rightarrowtail L$. This is a specific case of extension of scalars - a special case of tensor product of algebras - on the dot product of vector spaces. Computationally, we observe for soldered tensors that $\dim(\!|s|\!) \otimes (\!|t|\!) = \dim(\!|s \otimes t|\!)$ for all known examples that would compute in a reasonable time. It is important to clarify the $\otimes$ on the left hand side is the Whitney tensor product of vector spaces, while $\otimes$ on the right is soldering the tensors s and t.

Fix isomorphisms between the vector spaces U_i and its dual, as well as V_i and its dual. Then s, being a multilinear map $\prod_i U_i \rightarrowtail K$, can be identified with an element of $(\bigotimes_i U_i)^*$. Because of the fixed isomorphisms to the duals, we can further identify s as an element of $U := \bigotimes_i U_i$. Similarly, t may be identified as an element of $V := \bigotimes_i V_i$. Let $\eta : U \otimes V \to \bigotimes_{i=1}^{n}(U_i \otimes V_i)$ be the natural isomorphism arising from the natural isomorphism $(A \otimes B) \otimes C \to A \otimes (B \otimes C)$.

Treating s and t as vectors and denoting $s \otimes_K t$ as the image of (s, t) under the Whitney tensor product, we see $\eta(s \otimes_K t)$ is precisely the tensor arising from soldering s and t, which we denote $s \otimes t$ without the K subscript. Furthermore, the soldering of tensors is equivalent to $\eta ◖ s, t ◗^{\emptyset}_{\emptyset}$. This entails a study of soldered tensors as generalized (P, Ω)-tensor products.

In particular, the motivating question is given some tensor r are the algebraic invariants of r, together with to-be-proven structure and recognition theorems, sufficient to recover in a basis-independent manner information about how r was constructed? As X-rays are able to create images of internal structures for diagnostic purposes, we call our work *x-ray*ing tensors. Related and necessary to resolve is an understanding of the structure of $\mathrm{Der}(s \otimes t)$, a so far unexplored question.

In related works, the physics community uses techniques like the Density Matrix Renormalization Algorithm [22] attempts to uncover the structure of a high valence tensor by factoring it as a contracted product of trivalent tensors called a Matrix Product State. This iterative optimization technique is for complex-valued tensors, relying on the Singular Value Decomposition. Techniques such as Tensor-Train decompositions and Tucker decompositions are similar but assumes different fixed internal structure. [16] Other tensor factorization techniques such as CP-decomposition [15] attempts to decompose a tensor as a sum of smaller tensors, whereas the soldering of tensors is more akin to a product.

Problems and Methodology

In this section, we outline the proposed problems we are investigating and contributions we forsee as part of this dissertation.

Algorithms Contribution Proposal

We wish to compute $\mathrm{Adj}(t)$, $\mathrm{Cen}(t)$, and $\mathrm{Der}(t)$ when studying the bimap t. For fixed bases, each of the 3 algebras are specified by linear systems of equations, and thus can be computed in a polynomial number of steps. But the naive solution has complexity cubic in the number of variables. We aim to do better in the general case.

Simultaneous Sylvester Equations

We report on results in collaboration with James Wilson and Joshua Maglione. First, the problem of computing adjoints of bimaps is stated in coordinates. Next, we describe our approach, which is to translate the problem to a coordinate free formulation, solving a smaller subproblem, and propogate the subproblem solution to a full solution.

Problem 0.1 (Simultaneous Sylvester System - Coordinatized)**.**

Given *arrays $R \in K^{r\times b\times c}$, $S \in K^{a\times s\times c}$, and $T \in K^{a\times b\times c}$*

Return *matrices $X \in K^{a\times r}$ and $Y \in K^{s\times b}$ such that*

$$(\forall i)(XR_i + S_iY = T_i). \tag{0.1}$$

Expressed as list of matrix equation, it is the natural extension of the Sylvester Equation, which asks for a single matrix equation $XA + BX = C$. For R and S filled in from a tensor with a fixed basis, and T to be all zero, solving an instance of this problem finds the adjoint algebra of the tensor. Below gives a coordinate free description.

Problem 0.2 (Simultaneous Sylvester System - Coordinate Free)**.**

Given *Elements $r \in R \otimes B \otimes C$, $s \in A \otimes S \otimes C$, $t \in A \otimes B \otimes C$, and isomorphisms identifying each vector space with its dual.*

Return *Elements $x \in \mathrm{Hom}(A, R)$ and $y \in \mathrm{Hom}(B, S)$ such that $(x \otimes I_B \otimes I_C)(r) + (I_A \otimes y \otimes I_C)(s) = t$.*

First, by composing the isomorphism between R and its dual followed by the natural isomorphism $R^* \otimes B \otimes C \cong \mathrm{Hom}(R, B \otimes C)$, we view r as an element of $\mathrm{Hom}(R, B \otimes C)$. Similarly, view s as an element of $\mathrm{Hom}(S, A \otimes C)$. We now look for subspaces $B' \leq B$ and $A' \leq A$ such that r and s have left inverses after post-composing with projections. We also need B' and A' to have an induced isomorphism to their respective duals. That is, we want $r_{B'} := (\pi_{B'} \otimes I_C) \circ r$ and $s_{A'} := (\pi_{A'} \otimes I_C) \circ s$ to have left inverses. Denote these left inverses as $r^{\#}_{B'}$ and $s^{\#}_{A'}$. For the second condition, the isomorphism $B \cong B^*$ needs to restrict an isomorphism $B' \cong (B')^*$, and similar for $A \cong A^*$ restricting to $A' \cong (A')^*$.

Now considering a projection to the spaces A' and B', we see $(\pi_{A'} \otimes \pi_{B'} \otimes I_C)((x \otimes I_B \otimes I_C)(r) + (I_A \otimes y \otimes I_C)(s)) = (\pi_{A'} \otimes \pi_{B'} \otimes I_C)(t)$. Setting $x_{A'} := \pi_{A'} \circ x$ and $y_{B'} := \pi_{B'} \circ y$, we have a smaller equation of the form $(x_{A'} \otimes \pi_{B'} \otimes I_C)(r) + (\pi_{A'} \otimes y_{B'} \otimes I_C)(s) = (\pi_{A'} \otimes \pi_{B'} \otimes I_C)(t)$. Solving for $x_{A'}$ and $y_{B'}$ proceed by standard linear algebra, but as $\dim x_{A'} = \dim A' \cdot \dim R$ and $\dim y_{B'} = \dim B' \cdot \dim S$, this smaller system have unknowns of considerably lower dimension if the subspaces A' and B' are lower dimensional compared to A and B.

After solving for $x_{A'}$ and $y_{B'}$ by conventional methods, our algorithm proceeds by finding complementary subspaces $A = A' \oplus U$, and $B = B' \oplus V$. Let r_V, s_U, x_U, and y_V be defined analogously to above. Then projecting to the subspaces A' and V, we require $(\pi_{A'} \otimes \pi_V \otimes I_C)((x \otimes I_B \otimes I_C)(r) + (I_A \otimes y \otimes I_C)(s)) = (\pi_{A'} \otimes \pi_V \otimes I_C)(t)$.

Using the fixed isomorphism between vector spaces and their duals, as well as the natural isomorphisms above, we see that

$$\begin{aligned}
&(\pi_{A'} \otimes \pi_V \otimes I_C)((x \otimes I_B \otimes I_C)(r) + (I_A \otimes y \otimes I_C)(s)) = (\pi_{A'} \otimes \pi_V \otimes I_C)(t) \\
&\iff (\pi_{A'} \otimes \pi_V \otimes I_C)((x \otimes I_B \otimes I_C)(r)) + (\pi_{A'} \otimes \pi_V \otimes I_C)((I_A \otimes y \otimes I_C)(s)) = (\pi_{A'} \otimes \pi_V \otimes I_C)(t) \\
&\iff (x_{A'} \otimes \pi_V \otimes I_C)(r) + (\pi_{A'} \otimes y_V \otimes I_C)(s) = (\pi_{A'} \otimes \pi_V \otimes I_C)(t) \\
&\iff (x_{A'} \otimes I_C) \circ r \circ \pi_V + (\pi_{A'} \otimes I_C) \circ s \circ y_V = (\pi_{A'} \otimes I_C) \circ t \circ \pi_V \quad \text{because } A' \otimes V \otimes C \cong \mathrm{Hom}(V, A' \otimes C) \\
&\iff (x_{A'} \otimes I_C) \circ r_V + s_{A'} \circ y_V = (\pi_{A'} \otimes I_C) \circ t \circ \pi_V
\end{aligned}$$

Given $x_{A'}$, the only unknown in the last equation is y_V. Hence precomposing with $s^{\#}_{A'}$ to both sides gives a unique y_V solution. Analogously projecting to the subspaces U and B' followed by precomposing $r^{\#}_{B'}$ solves for a unique x_U given a $y_{B'}$. Finally, projecting to the subspaces U and V verifies the overall equation is satisfied for the pair x_U, y_V.

In summary, our algorithm requires the user to provide in addition to the data of the problem the subspace A' and B' with the desired invertibility properties of $r_{B'}$ and $s_{A'}$. In coordinates, the linear transformation $r \in \mathrm{Hom}(R, B \otimes C)$ has $bc := \dim B \otimes C$ rows and $r := \dim R$ columns, thus projecting to a dim $\lceil r/c \rceil$ subspace of B have a high probability of maintaining the left invertibility of r.

TODO: Talk about how a naive guess of A' and B' will succeed in most cases and speed up the "common" case of the Sylvester equations

Derivation System

Extending the above approach to computing derivations of bimaps is a part of my dissertation proposal. First to state the problem in coordinates, we need

Definition 0.3 (Outer action)**.** *Given an array $T \in K^{a\times b\times d}$, let $[T_1, \dots, T_d]$ be a list of $K^{a\times b}$ matricies corresponding to unfolding this array along the third index,. Then for a matrix Z of size $d \times c$, define T^Z, the outer action of Z on T, as the $K^{a\times b\times c}$ array satisfying $(T^Z)_j = \sum_{i=1}^{d} T_i Z_{ij}$.*

Problem 0.4 (Derivation system - Coordinatized)**.**

Given *arrays $R \in K^{r\times b\times c}$, $S \in K^{a\times s\times c}$, and $T \in K^{a\times b\times t}$*

Return *matrices $X \in K^{a\times r}$, $Y \in K^{s\times b}$, and $Z \in K^{t\times c}$ such that*

$$(\forall i) X R_i + S_i Y + (T^Z)_i = 0. \tag{0.2}$$

The above equation is no longer a list of matrix equations due to the outer action by Z. But it is exactly the equation satisfied by the derivation algebra of a tensor when R, S, T are all filled in from a tensor in a fixed basis. We propose studying the coordinate free version of the problem below.

Problem 0.5 (Derivation system - Coordinate Free)**.**

Given *Elements $r \in R \otimes B \otimes C$, $s \in A \otimes S \otimes C$, $t \in A \otimes B \otimes T$, and isomorphisms identifying the vector spaces and their duals.*

Return *Elements $x \in \mathrm{Hom}(A, R)$, $y \in \mathrm{Hom}(B, S)$, and $Z \in \mathrm{Hom}(C, T)$ such that $(x \otimes I_B \otimes I_C)(r) + (I_A \otimes y \otimes I_C)(s) + (I_A \otimes I_B \otimes z)(t) = 0$.*

Preliminary investigations suggest a similar approach to Problem 0.1 but with 3 subspaces, $A' \leq A$, $B' \leq B$, and $C' \leq C$. Additional work is needed to prove a faster solution exists, analyze its complexity, and implement in code.

Thus far we have only been concerned about bimaps. We propose extending the above ideas to higher valence tensors. The problem to solve is framed in context of solving for algebraic invariants of tensors as they do not clearly translate to matrix equations.

We need a preliminary definition for an endomorphism acting on a specific axis of the tensor.

Definition 0.6. *Let $t : \prod_i U_i \rightarrowtail K$. Let $\sigma \in \mathrm{End}(U_a)$. Then define the tensor $\langle t|\sigma$ as*

$$\langle t|\sigma|u\rangle = \langle t|\sigma u_a, u_{\overline{a}}\rangle$$

Where $u = (u_a, u_{\overline{a}})$ splits $u \in \prod_i U_i$ as $U_a \times \prod_{i\neq a} U_i$.

Problem 0.7. *Let $t : \prod_{i\in I} U_i \rightarrowtail K$. Let $\{a, b\} \subset I$.*

Generalizing the adjoint, define the ab-nucleus of t as $\mathrm{Nuc}_{ab}(t) := \{(\sigma_a, \sigma_b) \in \mathrm{End}(U_a)\times\mathrm{End}(U_b) : \langle t|\sigma_a u_a, u_{\overline{a}}\rangle = \langle t|\sigma_b u_b, u_{\overline{b}}\rangle\}$. For $J \subset I$, define the J-centroid as

$$\mathrm{Cen}_J(t) := \left\{ (\sigma_j)_{j\in J} \in \prod_j \mathrm{End}(U_j) : \langle t|\sigma_j u_j, u_{\overline{j}}\rangle = \langle t|\sigma_k u_k, u_{\overline{k}}\rangle \ \forall j,k \in J \right\}.$$

Similary, define the J-derivation as

$$\mathrm{Der}_J(t) := \left\{ (\delta_j)_{j\in J} \in \prod_j \mathrm{End}(U_j) : \sum_j \langle t|\delta_j u_j, u_{\overline{j}}\rangle = 0 \right\}.$$

These spaces are computed by linear equations of a similar flavor to the adjoint and derivation equation for 3-tensors. We ask if the equations can be computed faster than cubic in the number of variables?

Theoretical Contribution Proposal

The high level goal is to better understand tensors through their algebraic invariants. We call our approach *X-ray*ing tensors. To start, we study a specific class of tensors, the ones *soldered together*. To state the problem, we define, following the exposition and notation in [13].

Definition 0.8. *(Ternary Galois Connection of Tensors, Ideals, and Operators) Fix a space of tensors $\mathrm{Hom}(U_n \otimes \cdots \otimes U_1, U_0)$, a space of polynomials $K[X] := K[x_0, \dots, x_n]$ and a space of operators $E[U_0, \dots, U_n] := \mathrm{End}(U_n) \times \cdots \mathrm{End}(U_0)$. Note a tensor in t in $\mathrm{Hom}(U_n \otimes \cdots \otimes U_1, U_0)$ has the interpretation as a multilinear map $\langle t| : \prod_{i=1}^n U_i \rightarrowtail U_0$.*

Then for $p = \sum_e \lambda_e X^e \in K[X]$ and $\omega \in E[U_0, \dots, U_n]$, define $p(\omega) \in E[U_0, \dots, U_n]$ as $\sum_e \lambda_e\,(\omega_0^{e_0}, \dots, \omega_n^{e_n})$. For all $t \in \mathrm{Hom}(U_n\otimes\cdots\otimes U_1, U_0)$, the tensor $\langle t|p(\omega)$ is defined as $\langle t|p(\omega)|u\rangle = \sum_e \lambda_e \omega_0^{e_0}\langle t|\omega_1^{e_1}u_1, \dots, \omega_n^{e_n}u_n\rangle$.

Now fix a polynomial p and operator ω. Define the fiber $T(p,\omega) := \{s \in \mathrm{Hom}(U_n \otimes \cdots \otimes U_1, U_0) : \langle s \mid p(\omega) = 0\}$. Extend this definition to subsets P and Ω via $T(P,\Omega) := \cap_{p\in P} \cap_{\omega\in\Omega} T(p,\omega)$. Similarly, for fixed polynomial p and tensor t define the fiber $Z(t,p) := \{\omega \in E[U_n, \dots, U_0] : \langle t|p(\omega) = 0\}$ and extend to subsets $Z(T,P)$.

Now fixing $P \subset K[X]$, there is an inclusion reversing Galois connection between subsets of tensors and operators

$$S \subset T(P,\Omega) \iff \Omega \subset Z(S,P) \tag{0.3}$$

Lastly, from [13] we also know $T(P,\Omega)$ is a vector subspace of $\mathrm{Hom}(U_n \otimes \cdots \otimes U_1, U_0)$ and when $P = \boldsymbol{d}$ the derivation polynomial, $Z(S, \boldsymbol{d})$ is a Lie algebra.

Definition 0.9. *(Derivation closure) Let $s \in \mathrm{Hom}(U_n \otimes \cdots \otimes U_1, U_0)$. Then $(\!|s|\!)$, the derivation closure of s, is the vector subspace consisting of all tensors t such that $\mathrm{Der}(s) \subset \mathrm{Der}(t)$. Hence $(\!|s|\!) := T(\boldsymbol{d}, Z(\boldsymbol{d}, s))$.*

Definition 0.10. *(Soldering of tensors) Let $\langle s| : \prod_i U_i \rightarrowtail U_0$ and $\langle t| : \prod_i V_i \rightarrowtail V_0$. The soldering of s and t is the tensor $s \otimes t$ is the multilinear map $\langle s \otimes t| : \prod_i U_i \otimes V_i \rightarrowtail U_0 \otimes V_0$ defined by $\langle s \otimes t|u \otimes v = \langle s|u \otimes \langle t|v$.*

Remark 0.11. *Let $\langle s| : \prod_i U_i \rightarrowtail K$ and $\langle t| : \prod_i V_i \rightarrowtail K$. Define $U := \bigotimes_i U_i$ and $V := \bigotimes_i V_i$. Let $\hat{s} \in U$, $\hat{t} \in V$ be the elements identified with s and t using the Whitney tensor product universal property and dual space isomorphism. When not ambiguous, we shall say $s \in U$. Similarly, the soldered multilinear map $\langle s \otimes t| : \prod_i U_i \otimes V_i \rightarrowtail K$ is identified with $\widehat{s \otimes t} \in \bigotimes_i (U_i \otimes V_i)$, and we shall also say $s \otimes t \in \bigotimes_i (U_i \otimes V_i)$ when not ambiguous.*

We now pause to resolve an ambiguity in $s \otimes t$. From above, $s \otimes t$ may be shorthand for $\widehat{s \otimes t} \in \bigotimes_i (U_i \otimes V_i)$, which is naturally identified with $\langle s \otimes t|$. Alternatively, identifying s with $\hat{s} \in U$ and t with $\hat{t} \in V$ as vectors, $s \otimes t \in U \otimes V$ is the image of $(\hat{s}, \hat{t})$ by the Whitney tensor product. This ambiguity is resolved up to the natural isomorphism $\eta : U \otimes V \to \bigotimes_{i=1}^n (U_i \otimes V_i)$. When dismambiguation is needed, the notation $s \otimes_K t$ will refer to the Whitney tensor product of s and t, thought of as vectors. The above discussion extends to the sum $\sum_{i=1}^n u_i \otimes v_i$ for $u_i \in U$ and $v_i \in V$.

Below we summarize the theoretical results we plan on working towards, listed in order of concreteness. Perhaps not surprisingly, these problems are also ordered by the current amount of insight I have towards a solution to each.

We begin with a case we believe we have a handle on, namely analyzing the derivation closure of tensors that are soldered together.

Problem 0.12 (Soldered tensors derivation closure)**.**

Given $\langle s| : \prod_{i=1}^n U_i \rightarrowtail K$ and $\langle t| : \prod_{i=1}^n V_i \rightarrowtail K$, how does $(\!| s |\!)$ and $(\!| t |\!)$ relate to $(\!| s \otimes t |\!)$?

I have recently resolved this first problem. Let $U, V, \hat{s}, \hat{t}, \widehat{s \otimes t}$ be defined as above. Below is a preliminary lemma.

Lemma 0.13. *For $\langle s| : \prod_{i=1}^n U_i \rightarrowtail K$, $\langle t| : \prod_{i=1}^n V_i \rightarrowtail K$, there are embedding $\iota_s : \mathrm{Der}(s) \hookrightarrow \mathrm{Der}(s \otimes t)$ given by $(\sigma_i)_{i\in[n]} \mapsto (\sigma_i \otimes I_{V_i})_{i\in[n]}$ and $\iota_t : \mathrm{Der}(t) \hookrightarrow \mathrm{Der}(s \otimes t)$ given by $(\tau_i)_{i\in[n]} \mapsto (I_{U_i} \otimes \tau_i)_{i\in[n]}$.*

Proof. The map ι_i is injective on each factor as tensoring with the identity morphism is injective. The endomorphism $(\sigma_i \otimes I_{V_i})_{i\in[n]}$ is in $\mathrm{Der}(s \otimes t)$ as $\langle s \otimes t|\mathbf{d}((\sigma_i \otimes I_{V_i})_{i\in[n]}) = \sum_{i=1}^n \langle s|\sigma_i \otimes \langle t|I_{V_i} = \langle s|\mathbf{d}((\sigma_i)_{i\in[n]}) \otimes \langle t| = 0$. Lastly, we need to demonstrate ι_i is a map of Lie algebras. This follows by the calculation $\iota(\delta + \rho) = \iota((\delta_i + \rho_i)_{i\in[n]}) = ((\delta_i + \rho_i) \otimes I_{V_i})_{i\in[n]} = (\delta_i \otimes I_{V_i})_{i\in[n]} + (\rho_i \otimes I_{V_i})_{i\in[n]} = \iota(\delta) + \iota(\rho)$. The case for ι_j is analogous. This concludes the proof. □

Theorem 0.14. *There is a natural isomorphism between the vector spaces $(\!| s |\!) \otimes (\!| t |\!) \leq U \otimes V$ and $(\!| s \otimes t |\!) \leq \bigotimes_{i=1}^n (U_i \otimes V_i)$, given by the restriction of the natural isomorphism $\eta : U \otimes V \to \bigotimes_{i=1}^n (U_i \otimes V_i)$.*

Proof. Our strategy is to show $\eta((\!| s |\!) \otimes (\!| t |\!)) \subset (\!| s \otimes t |\!)$ and $(\!| s \otimes t |\!) \subset \eta((\!| s |\!) \otimes (\!| t |\!))$.

$(\!| s \otimes t |\!) \subset \eta((\!| s |\!) \otimes (\!| t |\!))$:

By Lemma 0.13, there are embeddings $\iota_s : \mathrm{Der}(s) \hookrightarrow \mathrm{Der}(s \otimes t)$ and $\iota_t : \mathrm{Der}(t) \hookrightarrow \mathrm{Der}(s \otimes t)$. The inclusion reversing nature of the antitone Galois connection in Definition 0.8 implies $(\!| s \otimes t |\!) = T(\mathbf{d}, \mathrm{Der}(s \otimes t)) \subset T(\mathbf{d}, \iota_s(\mathrm{Der}(s)))$ and $(\!| s \otimes t |\!) \subset T(\mathbf{d}, \iota_t(\mathrm{Der}(t)))$. Thus $(\!| s \otimes t |\!)$ is in their intersection. We shall prove $T(\mathbf{d}, \iota_s(\mathrm{Der}(s))) = \eta((\!| s |\!) \otimes V)$ and $T(\mathbf{d}, \iota_j(\mathrm{Der}(t))) = \eta(U \otimes (\!| t |\!))$. The conclusion follows as $(\!| s \otimes t |\!) \subset \eta((\!| s |\!) \otimes V) \cap \eta(U \otimes (\!| t |\!)) = \eta((\!| s |\!) \otimes (\!| t |\!))$.

The statement to prove is $T(\mathbf{d}, \iota_s(\mathrm{Der}(s))) = \eta((\!| s |\!) \otimes V)$.

We first show the direction $\eta((\!| s |\!) \otimes V) \subset T(\mathbf{d}, \iota_s(\mathrm{Der}(s)))$. As $(\!| s |\!) \otimes V$ is generated by $\acute{s} \otimes_K t$ for $\acute{s} \in (\!| s |\!)$ and $t \in V$, it suffices to show $\eta(\acute{s} \otimes_K t) \in T(\mathbf{d}, \iota_s(\mathrm{Der}(S)))$. This follows as $\langle \acute{s} \otimes t|$ satisfies $\langle \acute{s} \otimes t|\mathbf{d}(\delta) = 0$ for all $\delta = (\sigma_i \otimes I_{V_i})_{i\in[n]} \in \iota_s(\mathrm{Der}(s))$ since $(\sigma_i)_{i\in[n]} \in \mathrm{Der}(s)$.

In the opposite direction, by definition $T(\mathbf{d}, \iota_s(\mathrm{Der}(s)))$ is a subspace of $\eta(U \otimes V)$. We shall show in fact it is the subspace $\eta((\!| s |\!) \otimes V)$ by showing every element in $T(\mathbf{d}, \iota_s(\mathrm{Der}(s)))$ is the image of sum of pure tensors $\acute{s} \otimes t$ under η for $\acute{s} \in (\!| s |\!), t \in V$.

Let r be an element of $T(\mathbf{d}, \iota_s(\mathrm{Der}(s)))$. Using η, identify r as $\sum_{i=1}^m s_i \otimes_K t_i \in U \otimes V$. Showing each s_i is in $(\!| s |\!)$ concludes r is in $\eta((\!| s |\!) \otimes V)$. By definition r satisfies $\langle r|\mathbf{d}(\iota_s(\sigma)) = 0$ for all $\sigma \in \mathrm{Der}(s)$. We shall assume all t_is linearly independent by collecting terms.

Let $\sigma \in \mathrm{Der}(s)$ and $\iota_s(\sigma) = (\sigma_j \otimes I_{V_j})_{j\in[n]}$. Computing,

$$\begin{aligned} 0 &= \langle r|\mathbf{d}(\iota_s(\sigma)) \\ &= \left\langle \sum_i s_i \otimes t_i \right| \mathbf{d}(\iota_s(\sigma)) \end{aligned}$$

$$
\begin{aligned}
&= \sum_i \langle s_i \otimes t_i | \mathbf{d}(\iota_s(\sigma)) \qquad \text{linearity of tensor evaluation} \\
&= \sum_i \left(\langle s_i \otimes t_i | \left(\sum_{j=1}^{n} \sigma_j \otimes I_{V_j} \right) \right) \\
&= \sum_i \left(\sum_{j=1}^{n} \left(\langle s_i \otimes t_i | \sigma_j \otimes I_{V_j} \right) \right) \qquad \text{linearity of } \mathbf{d} \\
&= \sum_i \left(\sum_{j=1}^{n} \left(\langle s_i | \sigma_j \otimes t_i \right) \right) \\
&= \sum_i \left(\sum_j \langle s_i | \sigma_j \right) \otimes t_i \\
&= \sum_i \left(\langle s_i | \mathbf{d}(\sigma) \otimes t_i \right)
\end{aligned}
$$

Let $\{F_b\}_{b\in\mathcal{B}}$ be a basis of V. Then expanding each t_i in this basis,

$$
\begin{aligned}
0 &= \sum_i \left(\langle s_i | \mathbf{d}(\sigma) \otimes \left(\sum_{b\in\mathcal{B}} \lambda_{ib} F_b \right) \right) \\
&= \sum_{b\in\mathcal{B}} \left(\sum_i \left(\lambda_{ib} \langle s_i | \mathbf{d}(\sigma) \right) \otimes F_b \right)
\end{aligned}
$$

Since F_b are linearly independent, $\sum_i \lambda_{ib} \langle s_i | \mathbf{d}(\sigma) = 0$. Since t_is are linearly independent, the m by $|\mathcal{B}|$ matrix $[\lambda_{ib}]$ has full row rank. Thus the only way $\sum_i \lambda_{ib} \langle s_i | \mathbf{d}(\sigma) = 0$ is if $\langle s_i | \mathbf{d}(\sigma) = 0$ for all i. Thus $s_i \in (\!|s|\!)$ for all i. This concludes the proof of $T(\mathbf{d}, \iota_s(\mathrm{Der}(s))) = \eta((\!|s|\!) \otimes V)$.

The statement $T(\mathbf{d}, \iota_t(\mathrm{Der}(t))) = \eta(U \otimes (\!|t|\!))$ is proven analogously.

$\eta((\!|s|\!) \otimes (\!|t|\!)) \subset (\!|s \otimes t|\!)$:

The strategy will be to first show for $\acute{s} \in (\!|s|\!)$, that $\acute{s} \otimes t \in (\!|s \otimes t|\!)$, and secondly, show if $\acute{s} \otimes t \in (\!|s \otimes t|\!)$ for all s, then for all $\acute{t} \in (\!|t|\!)$, that $\acute{s} \otimes \acute{t} \in (\!|s \otimes t|\!)$, or $\eta(\acute{s} \otimes_K \acute{t}) \in (\!|s \otimes t|\!)$. The proof concludes as $(\!|s|\!) \otimes (\!|t|\!)$ is generated by $\acute{s} \otimes_K \acute{t}$ for $\acute{s} \in (\!|s|\!)$ and $\acute{t} \in (\!|t|\!)$.

To show $\acute{s} \otimes t \in (\!|s \otimes t|\!)$, let $\delta \in \mathrm{Der}(s \otimes t)$. Since δ is an element of $\prod_i \mathfrak{gl}(U_i \otimes V_i) \cong \prod_i (\mathfrak{gl}(U_i) \otimes \mathfrak{gl}(V_i))$, write δ as $\left(\sum_{j=1}^{R_i} (\sigma_j \otimes \tau_j) \right)_{i\in[n]}$. By construction $\langle s \otimes t | \mathbf{d}(\delta) = 0$. Calculating,

$$
\begin{aligned}
0 &= \langle s \otimes t | \mathbf{d}(\delta) \\
&= \langle s \otimes t | \sum_{i=1}^{n} \left(\sum_{j=1}^{R_i} (\sigma_j \otimes \tau_j) \right) \\
&= \langle s \otimes t | \sum_{a\in A} (\sigma_a \otimes \tau_a) \qquad \text{grouping into one indexing set} \\
&= \sum_{a\in A} \langle s | \sigma_a \otimes \langle t | \tau_a \\
&= \sum_{a\in A} \langle s | \sigma_a \otimes \left(\sum_{b\in\mathcal{B}} \lambda_{ab} F_b \right) \qquad \text{For } F_b \text{ basis of } V \\
&= \sum_{b\in\mathcal{B}} \left(\sum_{a\in A} \lambda_{ab} \langle s | \sigma_a \right) \otimes F_b
\end{aligned}
$$

As $\{F_b\}_{b\in\mathcal{B}}$ is a basis of V, $\sum_{a\in A}\lambda_{ab}\langle s|\sigma_a = 0$ for each b. Regrouping and combining the terms in the indexing set A by axes, we have $\langle s|\sigma = 0$, meaning $\sigma \in \mathrm{Der}(s)$. Thus $\langle \acute{s}|\sigma = 0$ as well. Substituting $\acute{s}$ in place of s in the above equation also equals 0, concluding $\langle \acute{s}\otimes t|\mathbf{d}(\delta) = 0$.

The proof that $\acute{s}\otimes\acute{t} \in (\!|s\otimes t|\!)$, assuming $\acute{s}\otimes t$ is in $(\!|s\otimes t|\!)$ is analogous. This concludes the proof. $\square$

Two generalizations of the above problem will also be investigated. Firstly, being given a list of tensors $s_1,\dots,s_n$, and secondly, soldering using a (P,Ω) tensor product producing, producing the tensor $\eta\blacktriangleleft s_1,\dots,s_n \blacktriangleright^P_\Omega$.

Problem 0.15 (Soldered tensors derivations).

Given $s : \prod_i U_i \rightarrowtail U_0$ and $t : \prod_i V_i \rightarrowtail V_0$, how does $\mathrm{Der}(s\otimes t)$ relate to $\mathrm{Der}(s)$ and $\mathrm{Der}(t)$?

We know for the case of adjoints, $\mathrm{Adj}(s\otimes t) = \mathrm{Adj}(s)\otimes\mathrm{Adj}(t)$ [23], but nothing is known for Der other than the observation that both $\mathrm{Der}(s)$ and $\mathrm{Der}(t)$ embed into $\mathrm{Der}(s\otimes t)$. Preliminary computations and dimension counting suggests the embeddings of $\mathrm{Der}(s)$ and $\mathrm{Der}(t)$ do not interact, as the dimension of $\mathrm{Der}(s\otimes t)$ in the computed examples are close to the dimension of $\mathrm{Der}(s)\cdot\mathrm{Cen}(t)+\mathrm{Der}(t)\cdot\mathrm{Cen}(s)$.

Once again, two generalizations of the above problem will be investigated once the original problem is resolved. Firstly, given a list of tensors $s_1,\dots,s_n$ rather than a pair s and t, and secondly, soldering using a (P,Ω)-tensor product, producing the tensor $\eta\blacktriangleleft s_1,\dots,s_n \blacktriangleright^P_\Omega$.

Problem 0.16 (X-raying soldered tensors).

Given $r : \prod_{i=1}^n W_i \rightarrowtail W_0$, what are necessary and sufficient conditions to for $r = s\otimes t$ for some $s : \prod_{i=1}^n U_i \rightarrowtail U_0$ and $t : \prod_{i=1}^n V_i \rightarrowtail V_0$?

Here, the problem is to understand for which classes of soldered tensors does X-raying succeed? The tools at our disposal for this problem are the algebraic invariants of the tensor r, in addition to results on the previous problems and additional hypotheses on the structure of r.

Lastly, if there is enough time and the above problems come to a satisfactory conclusion, we'd like to investigate X-raying in a more general context. [8] produces a long exact sequence between the restricted centroids. We plan on following and extending this idea to understand how the various restricted derivation algebras interact and restrict a given tensor. An ideal outcome in this investigation is to find more general necessary and sufficient conditions for successful X-raying of tensors.

Definition 0.17. *TODO: Define the glued restricted derivation algebras $\bigoplus_A \mathrm{Der}_A(t)$*

Problem 0.18 (X-raying general tensors).

Given $t : \prod_i U_i \rightarrowtail U_0$, what are the necessary and sufficient conditions to . . . ***TODO***

Applications

In addition to the application mentioned above to the study of tensors, we list some other uses to solutions of Problem 0.1.

TODO: This can be written last. The Sylvester equation applications are already written. Derivation applications have been mentioned, and some blurbs about X-raying can go here

Conclusion

TODO: This can also be written last.

References

[1] R. H. Bartels and G. W. Stewart. “Algorithm 432 [C2]: Solution of the matrix equation AX + XB = C [F4]”. In: *Commun. ACM* 15.9 (Sept. 1972), 820–826. ISSN: 0001-0782. DOI: 10.1145/361573.361582. URL: https://doi.org/10.1145/361573.361582.

[2] Jacob Biamonte and Ville Bergholm. *Tensor Networks in a Nutshell*. 2017. arXiv: 1708.00006 [quant-ph]. URL: https://arxiv.org/abs/1708.00006.

[3] Jacob C Bridgeman and Christopher T Chubb. “Hand-waving and interpretive dance: an introductory course on tensor networks”. In: *Journal of Physics A: Mathematical and Theoretical* 50.22 (May 2017), p. 223001. ISSN: 1751-8121. DOI: 10.1088/1751-8121/aa6dc3. URL: http://dx.doi.org/10.1088/1751-8121/aa6dc3.

[4] Rasmus Bro. “PARAFAC. Tutorial and applications”. In: *Chemometrics and Intelligent Laboratory Systems* 38.2 (1997), pp. 149–171.

[5] Peter A. Brooksbank, Martin D. Kassabov, and James B. Wilson. *Detecting cluster patterns in tensor data*. 2024. arXiv: 2408.17425 [math.NA]. URL: https://arxiv.org/abs/2408.17425.

[6] Peter A. Brooksbank and Eugene M. Luks. “Testing isomorphism of modules”. In: *Journal of Algebra* 320.11 (2008), pp. 4020–4029.

[7] Peter A. Brooksbank, Joshua Maglione, and James B. Wilson. “A fast isomorphism test for groups whose Lie algebra has genus 2”. In: *J. Algebra* 473 (2017), pp. 545–590. ISSN: 0021-8693. DOI: 10.1016/j.jalgebra.2016.12.007. URL: https://doi.org/10.1016/j.jalgebra.2016.12.007.

[8] Peter A. Brooksbank, Joshua Maglione, and James B. Wilson. “Exact sequences of inner automorphisms of tensors”. In: *Journal of Algebra* 545 (Mar. 2020), 43–63. ISSN: 0021-8693. DOI: 10.1016/j.jalgebra.2019.07.006. URL: http://dx.doi.org/10.1016/j.jalgebra.2019.07.006.

[9] Peter A. Brooksbank, Joshua Maglione, and James B. Wilson. “Tensor isomorphism by conjugacy of Lie algebras”. In: *Journal of Algebra* 604 (Aug. 2022), 790–807. ISSN: 0021-8693. DOI: 10.1016/j.jalgebra.2022.04.014. URL: http://dx.doi.org/10.1016/j.jalgebra.2022.04.014.

[10] Peter A. Brooksbank and James B. Wilson. “Groups acting on tensor products”. In: *Journal of Pure and Applied Algebra* 218.3 (2014), pp. 405–416. ISSN: 0022-4049. DOI: https://doi.org/10.1016/j.jpaa.2013.06.011. URL: https://www.sciencedirect.com/science/article/pii/S0022404913001382.

[11] Peter A. Brooksbank and James B. Wilson. “Intersecting two classical groups”. In: *J. Algebra* 353 (2012), pp. 286–297. ISSN: 0021-8693. DOI: 10.1016/j.jalgebra.2011.12.004. URL: https://doi.org/10.1016/j.jalgebra.2011.12.004.

[12] Lieven De Lathauwer, Bart De Moor, and Joos Vandewalle. “A Multilinear Singular Value Decomposition”. In: *SIAM Journal on Matrix Analysis and Applications* 21.4 (2000), pp. 1253–1278.

[13] Uriya First, Joshua Maglione, and James B. Wilson. *A spectral theory for transverse tensor operators*. 2020. arXiv: 1911.02518 [math.SP]. URL: https://arxiv.org/abs/1911.02518.

[14] Joachim von zur Gathen and Jürgen Gerhard. *Modern computer algebra*. Third. Cambridge University Press, Cambridge, 2013, pp. xiv+795. ISBN: 978-1-107-03903-2. DOI: 10.1017/CBO9781139856065. URL: https://doi.org/10.1017/CBO9781139856065.

[15] Frank Lauren Hitchcock. "The Expression of a Tensor or a Polyadic as a Sum of Products". In: *Journal of Mathematics and Physics* 6 (1927), pp. 164–189. URL: https://api.semanticscholar.org/CorpusID:124183279.

[16] Tamara G. Kolda and Brett W. Bader. "Tensor Decompositions and Applications". In: *SIAM Review* 51.3 (2009), pp. 455–500.

[17] J.M. Landsberg. *Tensors: Geometry and Applications*. Graduate studies in mathematics. American Mathematical Society, 2012. ISBN: 9780821884812. URL: https://books.google.com/books?id=JTjv3DTvxZIC.

[18] George F. Leger and Eugene M. Luks. "Generalized Derivations of Lie Algebras". In: *Journal of Algebra* 228.1 (2000), pp. 165–203. ISSN: 0021-8693. DOI: https://doi.org/10.1006/jabr.1999.8250. URL: https://www.sciencedirect.com/science/article/pii/S0021869399982509.

[19] Elina Robeva. *Orthogonal Decomposition of Symmetric Tensors*. 2015. arXiv: 1409.6685 [math.AG]. URL: https://arxiv.org/abs/1409.6685.

[20] Elina Robeva and Anna Seigal. "Duality of graphical models and tensor networks". In: *Information and Inference: A Journal of the IMA* 8.2 (June 2018), pp. 273–288. ISSN: 2049-8772. DOI: 10.1093/imaiai/iay009. eprint: https://academic.oup.com/imaiai/article-pdf/8/2/273/28864933/iay009.pdf. URL: https://doi.org/10.1093/imaiai/iay009.

[21] L. R. Tucker. "Some mathematical notes on three-mode factor analysis". In: *Psychometrika* 31 (1966c), pp. 279–311.

[22] Steven R. White. "Density-matrix algorithms for quantum renormalization groups". In: *Phys. Rev. B* 48.12 (1993), pp. 10312–10315. DOI: 10.1103/PhysRevB.48.10312.

[23] James B. Wilson. "Decomposing p-groups via Jordan algebras". In: *Journal of Algebra* 322.8 (2009), pp. 2642–2679.

[24] James B. Wilson. "Existence, algorithms, and asymptotics of direct product decompositions, I". In: *Groups - Complexity - Cryptology* 4.1 (Jan. 2012). ISSN: 1869-6104. DOI: 10.1515/gcc-2012-0007. URL: http://dx.doi.org/10.1515/gcc-2012-0007.

[25] James B. Wilson. "Finding central decompositions of p-groups". In: *Journal of Group Theory* 12.6 (Jan. 2009). ISSN: 1435-4446. DOI: 10.1515/jgt.2009.015. URL: http://dx.doi.org/10.1515/JGT.2009.015.

[26] James B. Wilson. "On automorphisms of groups, rings, and algebras". In: *Communications in Algebra* 45.4 (Oct. 2016), 1452–1478. ISSN: 1532-4125. DOI: 10.1080/00927872.2016.1175617. URL: http://dx.doi.org/10.1080/00927872.2016.1175617.