COMPUTING ALGEBRAIC INVARIANTS OF TENSORS AND THEIR APPLICATION TO PRODUCT DECOMPOSITIONS

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ABSTRACT. This dissertation proposal outlines two areas of study. First, on algorithms for efficient algorithms of algebraic invariants of tensors. Second, an investigation into decomposing tensors as products of smaller tensors.

CONTENTS

1. INTRODUCTION

Tensors encapsulate multilinear maps. Often given as a multiway array of numbers, tensors are used across various disciplines within mathematics and sciences to record information for some fixed reference frame. As such, they are studied from many complementary perspectives [Bro97] [KB09] [Lan12] [RS18] [DLDMV00] [Tuc66].

Throughout, we fix a field K, e.g $K = \mathbb{R}$. Given vector spaces U, V, W, a function $f : U \times V \to W$ is K-bilinear if $f(ku, v) = f(u, kv) = kf(u, v)$ for all k in K. We write $f : U \times V \rightarrowtail W$ ($\rightarrowtail$ for bilinear). When context is clear, we avoid the prefix K. The above extends to explain K-trilinear and K-multilinear in general. The space of K-multilinear maps from $U_n \times \cdots \times U_1$ to U_0 is denoted $\mathrm{Mult}(U_n, \ldots, U_1; U_0)$.

A **tensor space** T is a K-vector space equipped with a K-multilinear interpretation $\langle \cdot | : T \hookrightarrow \mathrm{Mult}(U_n, \ldots, U_1; U_0)$ for U_i each a K-vector space. A **tensor** t is an element of a tensor space T, and we write $\langle t| : U_n \times \cdots \times U_1 \rightarrowtail U_0$ to indicate $\langle t|$ is a multilinear function. The spaces $\{U_0, \ldots, U_n\}$ are the frame of tensor, the size of the frame $(n + 1)$ its valence, and $\{0, \ldots, n\}$, the labels on the vector spaces, its axes. For $|u\rangle = |u_n, \ldots, u_1\rangle$, write $\langle t|u\rangle \in U_0$ to mean evaluating $\langle t|$ at $|u\rangle$.

This definition accomodates the common existing understanding of tensors as a multiway grid $\Gamma \in K^{d_1 \times \cdots \times d_n}$ of numbers. For example,

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Example 1.1. *Given* $M = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. *If* M *is to be interpreted as a bilinear form (a bilinear map into the underlying field), then* $\langle M| : \mathbb{R}^2 \times \mathbb{R}^3 \rightarrowtail \mathbb{R}$ *where* $\langle M|u, v\rangle = u^T M v$.

For vector spaces U and V, the **tensor product** of U and V is the vector space $U \otimes V$ with the canonical map $\varphi : U \times V \rightarrowtail U \otimes V$, such that for every bilinear map f with domain $U \times V$, there is a unique induced linear map $\hat{f}$ satisfying $f = \hat{f} \circ \varphi$. For vectors $s \in U$ and $t \in V$, $s \otimes t$ is the image of (s, t) under φ. The tensor product space $U \otimes V$ is not a tensor space, but commonly a fixed isomorphism to $(U \otimes V)^*$ is given and $U \otimes V$ is understood to be a tensor space by this interpretation. With this in mind, we call elements of $U \otimes V$ tensors as well.

The perspective taken in this dissertation proposal is to study tensors as distributive products using the tools of algebra. Fix a 3-tensor (bimap) $* : U \times V \rightarrowtail W$.

Let $\mathrm{End}(U) = \mathrm{Hom}(U, U)$ be the endomorphisms of U. Existing work such as [Jac10], [Mya90], [BW14], and [Wil16] highlights the role of the centroid algebra

$$\mathrm{Cen}(*) := \{\sigma \in \mathrm{End}(U) \times \mathrm{End}(V) \times \mathrm{End}(W) : \sigma u * v = u * \sigma v = \sigma(u * v) \;\; \forall u \in U, v \in V\}.$$

Indeed, $\mathrm{Cen}(*)$ is a commutative algebra because for $\sigma, \tau \in \mathrm{Cen}(t)$ and all u, v,

$$\sigma\tau(u * v) = \sigma(\tau u * v) = \tau u * \sigma v = \sigma(\tau u * v) = \tau\sigma(u * v).$$

The only problem that may arise are zero divisors $U^\perp := \{v \in V : U * v = 0\}$ and $V^\perp := \{u \in U : u * V = 0\}$, which can be factored out before analyzing the bimap.

Similarly, the adjoint algebra

$$\mathrm{Adj}(*) := \{\sigma \in \mathrm{End}(U)^{\mathrm{op}} \times \mathrm{End}(V) : u\sigma * v = u * \sigma v\},$$

and derivation algebra

$$\mathrm{Der}(*) := \{\delta \in \mathrm{End}(U) \times \mathrm{End}(V) \times \mathrm{End}(W) : \delta u * v + u * \delta v = \delta(u * v)\}$$

play key roles.

Results using these algebras include discovering basis independent cluster pattern in tensors [BKW24], decomposing p-groups [Wil09a], and advances in isomorphism testing [IQ19], [BMW17] [BW12] [BMW22].

Two avenues of study are proposed for my dissertation. First to find efficient algorithms to compute these algebras, and second to study product decomposition of tensors using these algebras.

1.1. **Efficient algorithms.** The description of algebras $\mathrm{Adj}(t)$, $\mathrm{Cen}(t)$, and $\mathrm{Der}(t)$ are given by linear equations. For fixed bases, each is cubic in the number of variables by standard linear system solving methods. For tensors with each frame of dimension n, this is at minimum $O(n^6)$ operations for all 3 algebras. In collaboration with James Wilson and Joshua Maglione, we have preliminary results for an asymptotically lower complexity algorithm for computing $\mathrm{Adj}(t)$ and $\mathrm{Cen}(t)$ in $O(n^3)$ operations, inspired by analogous results for matrices known as the Bartels-Stewart algorithm [BS72].

> The proposed work is to find asymptotic speedups for the computation of $\mathrm{Der}(t)$ in the 3-tensor case, targeting an improvement from $O(n^6)$ to $O(n^{4.5})$.

1.2. **Product decompositions.** There is rich literature on decomposing a tensor *additively*, meaning writing a tensor t as $\sum_i t_i$. For instance, the CP-decomposition [Hit27] decomposes $t \in (U_n \otimes \cdots \otimes U_1)^*$ as a sum of rank 1 tensors. Other variants include PARAFAC2 [Har72], block decompositions [BKW24], and decompositions subject to symmetry [Rob15]. We look for decompositions of a tensor as a *product* of smaller tensors. We elaborate on the bimap case: for $r : W_2 \times W_1 \rightarrowtail W_0$, a product decomposition consists of vector spaces $W_i \cong U_i \otimes V_i$, $i = 0, 1, 2$, and maps $s : U_2 \times U_1 \rightarrowtail U_0$, $t : V_2 \times V_1 \rightarrowtail V_0$ such that $s \otimes t \cong r$ as tensor product of bimaps. This requires for pure tensors $u_2 \otimes v_2 = w_2$ and $u_1 \otimes v_1 = w_1$, we have $\langle s|u_2, u_1\rangle \otimes \langle t|v_2, v_1\rangle \cong \langle r|w_2, w_1\rangle$. We call the the decomposition a *Kronecker decomposition*.

The proposed work is to study the properties of product decompositions of tensors, targeting a computable characterizations of when a tensor is product indecomposable.

1.3. **Prior work.** We build on a number of methods which we briefly summarize below. Myasnikov, Wilson, and others in [Mya90], [Wil16], [Wil12], [MM10], [Wil09a] proves for the bimap t the algebras $\mathrm{Adj}(t)$ and $\mathrm{Cen}(t)$ control direct sum decompositions and automorphisms of t, using them to prove properties for the originating algebraic structures. Recent work generalizing from bimaps finds a long exact sequence linking the various generalized adjoints, centroids, and derivations of a higher valence tensor [BMW20]. Further work by First, Maglione, and Wilson [FMW20] defines a ternary Galois connection between tensors, operators, and polynomial ideals. Alongside it, they define a generalized (P, Ω)-tensor product of vector spaces $U_1, \dots, U_n$, for any $\Omega \subset \prod_i \mathrm{End}(U_i)$ and polynomials $P \subset K[x_1, \dots, x_n]$.

Let **d** be the polynomial $x_n + \cdots + x_1$. It is proven for a tensor t, the $(\mathbf{d}, \mathrm{Der}(t))$-tensor product is universally the smallest among the (P, Ω)-space that t factors through, for which $P \subset K[x_1, \dots, x_n]$ is an ideal generated by linear homogeneous polynomials. This motivates studies of $\mathrm{Der}(t)$ and the associated $(\mathbf{d}, \mathrm{Der}(t))$-tensor product space.

For the remainder of this section, let $U_0 = k$. Then as investigated in [BMW22], the vector subspace denoted $(\!|t|\!)$ (**derivation closure** of t) consisting of tensors t' whose derivation algebra contains the derivation algebra of t, may be identified with the $(\mathbf{d}, \mathrm{Der}(t))$-tensor product space and thus is universally the smallest search space for solving tensor isomorphism questions involving t.

In [BMW22], an infinite family of tensors with 1 dimensional derivation closures are constructed. However, little else is known about $(\!|t|\!)$. By Theorem B of [FMW20], a basis for the space $(\!|t|\!)$ is computable in polynomial time. The Multilinear Algebra library in the Computer Algebra system Magma implements some of this functionality, and I have utilized it to compute examples in practice.

1.3.1. *Products of tensors.* For unital associative K-algebras A and B, the tensor product of A and B is their tensor product as a vector space, with multiplication given by

$$(a \otimes b)(c \otimes d) = ac \otimes bd. \tag{1.1}$$

Let $\mu_A : A \otimes A \to A$ and $\mu_B : B \otimes B \to B$ be the linear structure maps of A and B. Then the unique induced linear map on the structure maps is the structure map of $A \otimes B$: $\mu_{A\otimes B} : (A \otimes B) \otimes (A \otimes B) \to A \otimes B$. [Gre12, Section 2.2]. The corresponding bilinear map is the tensor product of the multiplication maps.

We now generalize the above construction. Let s and t be tensors with interpretations $\langle s| : \prod_{i=1}^n U_i \rightarrowtail U_0$ and $\langle t| : \prod_{i=1}^n V_i \rightarrowtail V_0$. Interpret $s \otimes t$ as $\langle s \otimes t| : \prod_{i=1}^n (U_i \otimes V_i) \rightarrowtail U_0 \otimes V_0$, where $\langle s \otimes t | u_1 \otimes v_1, \dots, u_n \otimes v_n\rangle = \langle s|u_1, \dots, u_n\rangle \otimes \langle t|v_1, \dots, v_n\rangle$. This tensor product of multilinear maps is a generalization of the bilinear case ([Gre12, Section 1.21]). Throughout, we use the terminology "tensor product of tensors" interchangably with "tensor product of multilinear maps".

Example 1.2. *Let $\langle s| : K^2 \times K \rightarrowtail K^2$ be the right scalar action map, and $\langle t| : K \times K^3 \rightarrowtail K^3$ be the left scalar action map. The map $\langle s \otimes t| : (K^2 \otimes K) \times (K \otimes K^3) \rightarrowtail (K^2 \otimes K^3)$ is defined as*

$$\langle s \otimes t | u \otimes k_1, k_2 \otimes v\rangle = \langle s|u, k_2\rangle \otimes \langle t|k_1, v\rangle = k_2 u \otimes k_1 v = k_1 k_2 (u \otimes v) \tag{1.2}$$

Let f be an isomorphism of $K^2 \otimes K$ with K^2 and g an isomorphism of $K \otimes K^3$ with K^3. Then $\langle s \otimes t|$ is identified with the outer product tensor $\langle r| : K^2 \times K^3 \rightarrowtail K^2 \otimes K^3$ by mapping the first input via f and the second via g. This is called an isotopism of tensors. [Wil16]

The diagram scheme in Figure 1 illustrates the tensor product of tensors. In it, the 3-valent tensors s and t are drawn as shapes with 3 wires indicating 3 axes, with orientation given to the wire to indicate input and output. This notation is known as *tensor network diagrams* [BB17]. When tensor producting two tensors, our notation is for wires to be combined by the $\otimes$ symbol.

This is non-standard in tensor network diagram literature. Part of our work will be to extend and adapt tensor network diagram to products.

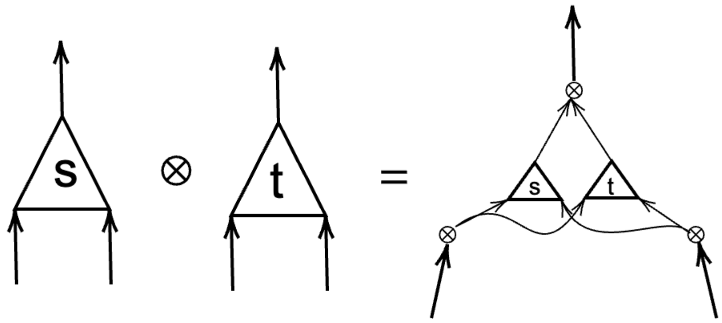

FIGURE 1. Pictorial illustration of soldering s and t

For preliminary results, I have proven $(\!|s \otimes t|\!) = (\!|s|\!) \otimes (\!|t|\!)$. This with Theorem 1.4 of [BMW22] allows for isomorphism testing of any tensor r with $s \otimes s$ for $\dim(\!|s|\!) = 1$.

1.3.2. *Related Works.* In related works, the physics community uses techniques like the Density Matrix Renormalization Algorithm [Whi93] to uncover the structure of a high valence tensor by factoring it as a contracted product of 3-tensors called a Matrix Product State. This iterative optimization technique is for complex-valued tensors, relying on the Singular Value Decomposition. Techniques such as Tensor-Train decompositions [Ose11] and Tucker decompositions [Tuc66], also known as HOSVD - Higher Order Singular Value Decompositions [DLDMV00], are similar but assumes different fixed internal structure.

In the next two sections, we describe in detail the proposed problems we are investigating and contributions we forsee as part of this dissertation.

2. EFFICIENT ALGORITHMS FOR ALGEBRAIC INVARIANTS OF TENSORS

We wish to compute $\mathrm{Adj}(t)$, $\mathrm{Cen}(t)$, and $\mathrm{Der}(t)$ for a bimap t.

For fixed bases, each of the algebras are specified by linear systems of equations, and thus can be computed in a number of steps polynomial to the sum of dimensions. The naive solution takes operations cubic in the number of variables. We aim to do better in the general case.

We report on results in collaboration with James Wilson and Joshua Maglione. First, the problem of computing adjoints of bimaps is stated in coordinates. Next, we describe our approach, which is to translate the system to a basis independent formulation, solve a smaller subproblem, and propogate the subproblem solution to a full solution.

2.1. **Simultaneous Sylvester System.** We solve the following

Given: arrays $R \in K^{r\times b\times c}$, $S \in K^{a\times s\times c}$, and $T \in K^{a\times b\times c}$
Return: matrices $X \in K^{a\times r}$ and $Y \in K^{s\times b}$ such that

$$(\forall i)(XR_i + S_iY = T_i). \tag{2.1}$$

Expressed as list of matrix equation, Equation (2.1) is the natural extension of the Sylvester Equation, which asks for X satisfying the matrix equation $XA+BX = C$. For R and S coordinates of a tensor with a fixed basis, and T to be all zero, solving an instance of this problem finds the adjoint algebra of the tensor. When asked to solve this system in Magma, we convert the equation to matrix-vector form.

Next we give a basis independent statement of the problem and an algorithm to solve the system without

2.2. Simultaneous Sylvester System - Basis independent.

Given: Elements $r \in R \otimes B \otimes C$, $s \in A \otimes S \otimes C$, $t \in A \otimes B \otimes C$, and isomorphisms identifying each vector space with its dual.

Return: Elements $x \in \mathrm{Hom}(R, A)$ and $y \in \mathrm{Hom}(S, B)$ such that $(x \otimes I_B \otimes I_C)(r) + (I_A \otimes y \otimes I_C)(s) = t$.

Below we describe how to compute x and y without solving a full system of linear equations.

Preliminaries:

First, by composing the isomorphism between R and its dual followed by the isomorphism $R^* \otimes B \otimes C \cong \mathrm{Hom}(R, B \otimes C)$, we view r as an element of $\mathrm{Hom}(R, B \otimes C)$. Similarly, view s as an element of $\mathrm{Hom}(S, A \otimes C)$. We now look for subspaces $B' \leq B$ and $A' \leq A$ such that r and s have left inverses after post-composing with projections. That is, we want $r_{B'} := (\pi_{B'} \otimes I_C) \circ r$ and $s_{A'} := (\pi_{A'} \otimes I_C) \circ s$ to have left inverses. Denote these left inverses as $r^{\#}_{B'}$ and $s^{\#}_{A'}$. We also need B' and A' to have an induced isomorphism to their respective duals, meaning $B \cong B^*$ restricts an isomorphism $B' \cong (B')^*$, and similar for $A \cong A^*$ restricting to $A' \cong (A')^*$.

Solving a smaller subproblem:

To assist in calculation, let $\pi_{A',B',C} := \pi_{A'} \otimes \pi_{B'} \otimes I_C$, and $xr := (x \otimes I_B \otimes I_C)(r)$, and $ys := (I_A \otimes y \otimes I_C)(s)$.

Projecting to the spaces A' and B', we compute

$$\begin{aligned}
&\pi_{A',B',C}(xr + ys) = \pi_{A',B',C}(t)\\
\iff\ &\pi_{A',B',C}(xr) + \pi_{A',B',C}(ys) = \pi_{A',B',C}(t)\\
\iff\ &x_{A'}(\pi_{R,B',C}(r)) + y_{B'}(\pi_{A',S,C}(s)) = \pi_{A',B',C}(t)
\end{aligned}$$

Propagating solution to full problem:

Solving for $x_{A'}$ and $y_{B'}$ proceed by standard linear algebra, but as $\dim x_{A'} = \dim A' \cdot \dim R$ and $\dim y_{B'} = \dim B' \cdot \dim S$, this smaller system have unknowns of considerably lower dimension if the subspaces A' and B' are lower dimensional compared to A and B.

After solving for $x_{A'}$ and $y_{B'}$ by conventional methods, our algorithm proceeds by finding complementary subspaces $A = A' \oplus U$, and $B = B' \oplus V$. Let r_V, s_U, x_U, and y_V be defined analogously to above. Then projecting to the subspaces A' and V, we require $\pi_{A',V,C}(xr + ys) = \pi_{A',V,C}(t)$.

Using the fixed isomorphism between vector spaces and their duals, as well as the natural isomorphisms above, we see that

$$\begin{aligned}
&\pi_{A',V,C}(xr + ys) = \pi_{A',V,C}(t)\\
\iff\ &\pi_{A',V,C}(xr) + \pi_{A',V,C}(ys) = \pi_{A',V,C}(t)\\
\iff\ &x_{A'}(\pi_{R,V,C}(r)) + y_V(\pi_{A',S,C}(s)) = \pi_{A',V,C}(t)\\
\iff\ &(x_{A'} \otimes I_C) \circ r \circ \pi_V + (\pi_{A'} \otimes I_C) \circ s \circ (y_V)^{\dagger} = (\pi_{A'} \otimes I_C) \circ t \circ \pi_V \qquad A' \otimes V \otimes C \cong \mathrm{Hom}(V, A' \otimes C)\\
\iff\ &\tilde{r} + s_{A'} \circ (y_V)^{\dagger} = \tilde{t} \qquad \text{Where } \tilde{r} := (x_{A'} \otimes I_C) \circ r_V \text{ and } \tilde{t} := (\pi_{A'} \otimes I_C) \circ t_V
\end{aligned}$$

The only unknown in the last equation is $(y_V)^{\dagger}$ as $x_{A'}$ was previously solved for. Hence precomposing with $s^{\#}_{A'}$ to both sides gives a unique $(y_V)^{\dagger}$ solution, which by the isomorphism of vector spaces to their duals gives a unique y_V. Analogously projecting to the subspaces U and B' followed by precomposing $r^{\#}_{B'}$ solves for a unique x_U given a $y_{B'}$. Finally, projecting to the subspaces U and V verifies the overall equation is satisfied for the pair x_U, y_V.

Remark 2.1. *Our algorithm requires the user to provide the subspace A' and B' with the desired invertibility properties of $r_{B'}$ and $s_{A'}$. In coordinates, the linear transformation $r \in \mathrm{Hom}(R, B \otimes C)$ has $bc := \dim B \otimes C$ rows and $r := \dim R$ columns, thus projecting to a dim $\lceil r/c \rceil$ subspace of B have a high probability of maintaining the left invertibility of r.*

For example, if $\dim A, \dim B, \dim C, \dim R, \dim S$ *are all* $O(n)$ *then* $\lceil r/c \rceil$ *is* $O(1)$ *hence the naive* $O(n^6)$ *operations necessary to solve the system is reduced to the* $O(n^3)$ *operations needed to solve the smaller subproblem.*

2.3. **Derivation System.** First to state the problem in coordinates, we need

Definition 2.2 (Outer action). *Given an array* $T \in K^{a\times b\times d}$*, let* $[T_1, \dots, T_d]$ *be a list of* $K^{a\times b}$ *matrices corresponding to unfolding this array along the third index,. Then for a matrix* Z *of size* $d \times c$*, define* T^Z*, the outer action of* Z *on* T*, as the* $K^{a\times b\times c}$ *array satisfying* $(T^Z)_j = \sum_{i=1}^d T_i Z_{ij}$*.*

Now we are ready to state the problem of solving the derivation system in coordinates.

Problem A (Derivation System - Coordinatized).
Given: *arrays* $R \in K^{r\times b\times c}$*,* $S \in K^{a\times s\times c}$*, and* $T \in K^{a\times b\times t}$
Return: *matrices* $X \in K^{a\times r}$*,* $Y \in K^{s\times b}$*, and* $Z \in K^{t\times c}$ *such that*

$$(\forall i) X R_i + S_i Y + (T^Z)_i = 0. \tag{2.2}$$

The above equation is no longer a list of matrix equations due to the outer action by Z. But it is exactly the equation satisfied by the derivation algebra of a tensor t when R, S are coordinates of that tensor in a fixed basis, and T its negative. Next we describe a basis independent formulation.

2.4. **Derivation system - Basis independent.**

Given: Elements $r \in R \otimes B \otimes C$, $s \in A \otimes S \otimes C$, $t \in A \otimes B \otimes T$, and isomorphisms identifying the vector spaces and their duals.

Return: Elements $x \in \mathrm{Hom}(R, A)$, $y \in \mathrm{Hom}(S, B)$, and $Z \in \mathrm{Hom}(T, C)$ such that $(x \otimes I_B \otimes I_C)(r) + (I_A \otimes y \otimes I_C)(s) + (I_A \otimes I_B \otimes z)(t) = 0$.

Preliminary investigations suggest a similar approach to Simultaneous Sylvester Systems but with 3 subspaces, $A' \le A$, $B' \le B$, and $C' \le C$. If the dimension of each space is $O(\sqrt{n})$ then the number of variables to solve in the dense system is $O(n^{1.5})$, giving the $O(n^{4.5})$ target. Additional work is necessary.

Thus far we have only been concerned about bimaps. As a challenge upon resolving Problem A, we propose extending the above ideas to higher valence tensors. We frame the question as solving for algebraic invariants of tensors.

We first give notation for an endomorphism acting on a specific axis of the tensor.

Definition 2.3. *Let* $\langle t| : \prod_i U_i \rightarrowtail K$*. Let* $\sigma_a \in \mathrm{End}(U_a)$*. Then define the tensor* $\langle t|\sigma_a$ *as*

$$\langle t|\sigma_a|u\rangle = \langle t|\sigma_a u_a, u_{\overline{a}}\rangle$$

Where $u = (u_a, u_{\overline{a}})$ *splits* $u \in \prod_i U_i$ *as an element of* $U_a \times \prod_{i\neq a} U_i$*.*

Let $\langle t| : \prod_{i\in I} U_i \rightarrowtail K$. For a 2-element subset $\{a, b\} \subset I$, we define the ab-nucleus of t as

$$\mathrm{Nuc}_{ab}(t) := \{(\sigma_a, \sigma_b) \in \mathrm{End}(U_a)^{\mathrm{op}} \times \mathrm{End}(U_b) : \langle t|\sigma_a u_a, u_{\overline{a}}\rangle = \langle t|\sigma_b u_b, u_{\overline{b}}\rangle\}. \tag{2.3}$$

For $J \subset I$, define the J-centroid as

$$\mathrm{Cen}_J(t) := \left\{ (\sigma_j)_{j\in J} \in \prod_j \mathrm{End}(U_j) : \langle t|\sigma_j u_j, u_{\overline{j}}\rangle = \langle t|\sigma_k u_k, u_{\overline{k}}\rangle \ \forall j, k \in J \right\}. \tag{2.4}$$

Similary, define the J-derivation as

$$\mathrm{Der}_J(t) := \left\{ (\delta_j)_{j\in J} \in \prod_j \mathfrak{gl}(U_j) : \sum_j \langle t|\delta_j u_j, u_{\overline{j}}\rangle = 0 \right\}. \tag{2.5}$$

These spaces are computed by linear equations, and gives higher valence tensors the analogue of the centroid, adjoint, and derivation algebra of 3-tensors. Now we ask

Challenge A. *Can the higher valence nuclei, centroid, and derivation algebras be computed in operations fewer than cubic in the number of variables?*

3. Product decompositions of tensors

3.1. **Preliminaries.** As described in the introduction, we wish to understand product decompositions of tensors. To state the problem, we follow the exposition and notation in [FMW20].

Throughout this section, all tensor spaces will be the space of multilinear maps. The interpretation map will be the identity map. As a result the word "tensor" is used interchangably with "multilinear map".

Definition 3.1. *(Ternary Galois Connection of Tensors, Ideals, and Operators)*

We define evaluating a multivariable polynomial, with operators substituting for the indeterminantes. For $p = \sum_e \lambda_e X^e \in K[x_0, \ldots, x_n] =: K[X]$ and $\omega \in \prod_i \mathrm{End}(U_i)$, let

$$p(\omega) := \sum_e \lambda_e \left(\omega_0^{e_0}, \ldots, \omega_n^{e_n}\right) \in \prod_i \mathrm{End}(U_i).$$

Let $S \subset \mathrm{Mult}(U_n, \ldots, U_1; U_0)$. It is evidently a tensor space with the identity interpretation map. For all $t \in S$, define $\langle t|p(\omega)$ where for any $(u_1, \ldots, u_n)$,

$$\langle t|p(\omega)|u\rangle = \sum_e \lambda_e \omega_0^{e_0} \langle t|\omega_1^{e_1} u_1, \ldots, \omega_n^{e_n} u_n\rangle.$$

Now fix a polynomial p and operator ω. Define the set

$$\boldsymbol{T}(p, \omega) := \{t \in \mathrm{Mult}(U_n, \ldots, U_1; U_0) : \langle t \mid p(\omega) = 0\}.$$

Extend this definition to subsets P and Ω via

$$\boldsymbol{T}(P, \Omega) := \bigcap_{p \in P} \bigcap_{\omega \in \Omega} \boldsymbol{T}(p, \omega).$$

Similarly, for fixed polynomial p and tensor t define the set

$$\boldsymbol{Z}(t, p) := \{\omega \in \prod_u \mathrm{End}(U_i) : \langle t|p(\omega) = 0\},$$

and extend to subsets $\boldsymbol{Z}(S, P)$.

Fixing $P \subset K[X]$, there is an inclusion reversing Galois connection between subsets of $\mathrm{Mult}(U_n, \ldots, U_1; U_0)$ and subsets of $\prod_i \mathrm{End}(U_i)$ given by

$$S \subset \boldsymbol{T}(P, \Omega) \iff \Omega \subset \boldsymbol{Z}(S, P) \tag{3.1}$$

From [FMW20] the set $\boldsymbol{T}(P, \Omega)$ is a vector subspace and $\boldsymbol{Z}(S, \boldsymbol{d})$ is a Lie algebra for $\boldsymbol{d} = x_n + \cdots + x_0$.

Definition 3.2. *(Derivation closure) Let $t \in \mathrm{Mult}(U_n, \ldots, U_1; U_0)$. Then $(\!|t|\!)$, the derivation closure of t, is the vector subspace consisting of all s such that $\mathrm{Der}(t) \subset \mathrm{Der}(s)$. Hence $(\!|t|\!) := \boldsymbol{T}(\boldsymbol{d}, \boldsymbol{Z}(\boldsymbol{d}, t))$.*

Example 3.3. *Let t be the matrix multiplication tensor for 2×3 and 3×4 rectangular matrices. That is, $\langle t| : K^{2\times 3} \times K^{3\times 4} \rightarrowtail K^{2\times 4}$ by $\langle t|M, N\rangle := MN$. Then by Corollary 8.4.4 of [FMW20], we have $(\!|t|\!)$ as a 1-dimensional vector subspace spanned by Kt.*

Definition 3.4. *(Tensor product of multilinear maps)*

Let $s \in \mathrm{Mult}(U_n, \ldots, U_1; U_0)$ and $t \in \mathrm{Mult}(V_n, \ldots, V_1; V_0)$. Define $s \otimes t \in \mathrm{Mult}(U_n \otimes V_n, \ldots, U_1 \otimes V_1; U_0 \otimes V_0)$ as the tensor product of multilinear maps, with interpretation $\langle s \otimes t| : \prod_i U_i \otimes V_i \rightarrowtail U_0 \otimes V_0$ given by $\langle s \otimes t|u_1 \otimes v_1, \ldots, u_n \otimes v_n\rangle = \langle s|u_1, \ldots, u_n\rangle \otimes \langle t|v_1, \ldots, v_n\rangle$. We say $s \otimes t$ is the tensor product of s and t.

Remark 3.5. *Writing $s \otimes t$ to mean a multilinear map is an abuse of notation similar to a similar notion in the tensor product of linear maps. Since $s \in \mathrm{Mult}(U_n, \dots, U_1; U_0)$ and $t \in \mathrm{Mult}(V_n, \dots, V_1; V_0)$ are vectors in vector spaces, $s \otimes t$ is definitionally the $\varphi(s,t)$, image of the tensor product of vector spaces.*

However, by the following sequence of natural isomorphisms, $\varphi(s,t)$ is identified with an element of $\mathrm{Mult}(U_n \otimes V_n, \dots, U_1 \otimes V_1; U_0 \otimes V_0)$. Starting with $\varphi(s,t) \in \mathrm{Mult}(U_n, \dots, U_1; U_0) \otimes \mathrm{Mult}(V_n, \dots, V_1; V_0)$:

$$\begin{aligned}
\mathrm{Mult}(U_n, \dots, U_1; U_0) \otimes \mathrm{Mult}(V_n, \dots, V_1; V_0) &\cong (U_n^* \otimes \cdots \otimes U_1^* \otimes U_0) \otimes (V_n^* \otimes \cdots \otimes V_1^* \otimes V_0) \\
&\cong (U_n^* \otimes V_n^*) \otimes \cdots \otimes (U_1^* \otimes V_1^*) \otimes (U_0 \otimes V_0) \\
&\cong \mathrm{Mult}(U_n \otimes V_n, \dots, U_1 \otimes V_1; U_0 \otimes V_0)
\end{aligned}$$

Example 3.6. *The Kronecker product of matrices is a case of the tensor product of multilinear maps. Let $M \in \mathrm{Mult}(K^2, K^2; K)$ and $N \in \mathrm{Mult}(K^2, K^2; K)$ be given as 2×2 matrices.*

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \qquad N = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$$

Then

$$M \otimes N = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \otimes \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} = \left(\begin{array}{cc|cc} 3 & 4 & \multicolumn{2}{c}{\multirow{2}{*}{$\mathbf{0}$}} \\ 5 & 6 & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{$\mathbf{0}$}} & 6 & 8 \\ & & 10 & 12 \end{array}\right)$$

The 4×4 matrix $M \otimes N$ is given as an element of $\mathbb{M}_2(\mathbb{M}_2(K))$. It has interpretation $\langle M \otimes N| : (K^2 \otimes K^2), (K^2 \otimes K^2) \rightarrowtail K$ given by mapping basis element $e_i \otimes e_j$ to the $2(i-1)+j$th row or column of the matrix. Specifically, $\langle M \otimes N | e_i \otimes e_j, e_k \otimes e_l \rangle$ is row $2i - 1 + j$, column $2k - 1 + l$ of $\mathrm{Kron}(M, N)$, giving the product of the (i,k)th row of M with the (j,l)th row of N.

3.2. **Existing techniques.** Next we describe some existing techniques that analyze tensors admitting product decompositions.

Example 3.7 (Algebras). *Consider the isomorphism of $\mathbb{R}$-algebras $\mathbb{M}_2(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{M}_2(\mathbb{C})$ given by*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes z \mapsto \begin{bmatrix} az & bz \\ cz & dz \end{bmatrix} \tag{3.2}$$

Let $r : \mathbb{M}_2(\mathbb{C}) \times \mathbb{M}_2(\mathbb{C}) \rightarrowtail \mathbb{M}_2(\mathbb{C})$, $s : \mathbb{M}_2(\mathbb{R}) \times \mathbb{M}_2(\mathbb{R}) \rightarrowtail \mathbb{M}_2(\mathbb{R})$, and $t : \mathbb{C} \times \mathbb{C} \rightarrowtail \mathbb{C}$ be their respective multiplication tensors. The isomorphism extends to the multiplication tensors, meaning $r \cong s \otimes t$.

But also $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{M}_2(\mathbb{C})$ for $\mathbb{H}$ the real Quaternions via

$$1 \otimes 1 \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad i \otimes 1 \mapsto \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad j \otimes 1 \mapsto \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad k \otimes 1 \mapsto \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \tag{3.3}$$

This example illustrate challenges with having unique product decompositions.

Example 3.8 (Tensor over centroid). *Let $s \in \mathbb{F}_3^{2\times 2\times 2}$ be the tensor given as a system of forms*

$$s_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad s_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

with interpretation $\langle s| : \mathbb{F}_3^2 \times \mathbb{F}_3^2 \rightarrowtail \mathbb{F}_3^2$. This is the multiplication table of the quotient ring $\mathbb{F}_3[x]/(x^2+1) \cong \mathbb{F}_9$, which is a field. Hence $\langle s|$ has centroid isomorphic to $\mathbb{F}_9$.

Let t be the 2×2 identity matrix, interpreted as the bilinear form $\langle t| : \mathbb{F}_3^2 \times \mathbb{F}_3^2 \rightarrowtail \mathbb{F}_3$. Let $r = s \otimes t$ be the tensor product of s and t. Computing, $\mathrm{Cen}(r)$ has basis of endomorphisms $\{1, \alpha\}$ with $\alpha^2 = -1$, hence $\mathrm{Cen}(r) \cong \mathbb{F}_9$. With the isomorphism $\mathbb{F}_3^2 \cong \mathbb{F}_9$, we recover t from r.

In the backwards direction, suppose we have a tensor $r : \mathbb{F}_3^4 \times \mathbb{F}_3^4 \rightarrowtail \mathbb{F}_3^2$ satisfying $\mathrm{Cen}(r) \cong \mathbb{F}_9$ with basis $\{1, \alpha\}$. With the isomorphism $\mathbb{F}_3^4 \cong \mathbb{F}_9 \otimes \mathbb{F}_3^2$, and $\mathbb{F}_3^2 \cong \mathbb{F}_9 \otimes \mathbb{F}_3$, there's an isotopic tensor $\tilde{r} : (\mathbb{F}_9 \otimes \mathbb{F}_3^2) \times (\mathbb{F}_9 \otimes \mathbb{F}_3^2) \rightarrowtail \mathbb{F}_9 \otimes \mathbb{F}_3$. We define the tensor $t : \mathbb{F}_3^2 \times \mathbb{F}_3^2 \rightarrowtail \mathbb{F}_3$ via $t(u,v) = \tilde{r}(1_{\mathbb{F}_9} \otimes u, 1_{\mathbb{F}_9} \otimes v)$. Definitionally $t \otimes \mathrm{Cen}(r) \cong r$. This functionality has been implemented as the TensorOverCentroid operation in the multlinear algebra library in Magma.

Example 3.9. *[Wil16, Chapter 6][Condensation by full idempotents]*

Similar to Example 1.2, let $\langle s| : U \times K \rightarrowtail U$ be defined as $\langle s|u, k\rangle = uk$, scalar multiplication by K on the right. Let $\langle t| : K \times V \rightarrowtail V$ be scalar multiplication by K on the left. The product $\langle s \otimes t| : (U \otimes K) \times (K \otimes V) \rightarrowtail (U \otimes V)$ is isotopic to $\varphi : U \times V \rightarrowtail U \otimes V$, the canonical map into the tensor product.

Let $\mathrm{Nuc}_{20}(\varphi) := \mathcal{L}$ be defined as per Equation (2.3), consisting of all $(\lambda_2, \lambda_0) \in \mathrm{End}(U) \times \mathrm{End}(U \otimes V)$ satisfying $\langle \varphi|\lambda_2 u, v\rangle = \lambda_0\langle\varphi|u, v\rangle$. Suppose e is a full idempotent of $\mathcal{L}$, meaning $e^2 = e$ and $\mathcal{L}e\mathcal{L} = \mathcal{L}$. Instead of finding a product decomposition, we condense the bimap φ as $\tilde{\varphi} : eU \times V \rightarrowtail eW$ where $\mathrm{Nuc}_{20}(\tilde{*}) = e\mathcal{L}e$. This is of interest because $e\mathcal{L}e$ is Morita equivalent to $\mathcal{L}$ but smaller.

When U and V are given coordinates, $\mathcal{L} = (\mathbb{M}_n(K), \mathbb{M}_n(K) \otimes I_m(K))$ so $e = (E_{11}, E_{11} \otimes I_m(K))$ is a full idempotent and condenses φ down to scalar multiplication on the left. In [Wil16] this technique is used to prove a property of automorphisms of bimaps.

Applying the same technique for the algebras $\mathrm{Adj}(t)$ and $\mathrm{Nuc}_{10}(t)$ to Example 3.3 compresses the rectangular matrix multiplication tensor down to multiplication of the underlying field.

3.3. **Preliminary results on product decompositions.** The first question is how derivation closure of the product of tensors relate to the derivation closure of the individual tensors. That is, given $s \in \mathrm{Mult}(U_n, \dots, U_1; U_0)$ and $t \in \mathrm{Mult}(V_n, \dots, V_1; V_0)$, do $(\!|s|\!)$ and $(\!|t|\!)$ relate to $(\!|s \otimes t|\!)$? I have recently resolved this question. Below is a preliminary lemma.

Lemma 3.10. *For $s \in \mathrm{Mult}(U_n, \dots, U_1; U_0)$ and $t \in \mathrm{Mult}(V_n, \dots, V_1; V_0)$, there is an embedding $\iota_s : \mathrm{Der}(s) \hookrightarrow \mathrm{Der}(s\otimes t)$ given by $(\sigma_i)_{i\in[n]} \mapsto (\sigma_i \otimes 1_{V_i})_{i\in[n]}$ and an embedding $\iota_t : \mathrm{Der}(t) \hookrightarrow \mathrm{Der}(s\otimes t)$ given by $(\tau_i)_{i\in[n]} \mapsto (1_{U_i} \otimes \tau_i)_{i\in[n]}$.*

Proof. The map ι_s is injective on each factor as tensoring with the identity morphism is injective. The endomorphism $(\sigma_i \otimes 1_{V_i})_{i\in[n]}$ is in $\mathrm{Der}(s \otimes t)$ as $\langle s \otimes t|\mathbf{d}((\sigma_i \otimes 1_{V_i})_{i\in[n]}) = \sum_{i=1}^{n}\langle s|\sigma_i \otimes \langle t|1_{V_i} = \langle s|\mathbf{d}((\sigma_i)_{i\in[n]}) \otimes \langle t| = 0$. Lastly, we need to demonstrate ι_i is a map of Lie algebras. This follows by the calculation $\iota(\delta + \rho) = \iota((\delta_i + \rho_i)_{i\in[n]}) = ((\delta_i + \rho_i) \otimes 1_{V_i})_{i\in[n]} = (\delta_i \otimes 1_{V_i})_{i\in[n]} + (\rho_i \otimes 1_{V_i})_{i\in[n]} = \iota(\delta) + \iota(\rho)$. The case for ι_t is analogous. □

Theorem 3.11. *Let $s \in \mathrm{Mult}(U_n, \dots, U_1; U_0) =: U$ and $t \in \mathrm{Mult}(V_n, \dots, V_1; V_0) =: V$. Then $(\!|s|\!) \otimes (\!|t|\!) = (\!|s \otimes t|\!)$.*

Proof. Our strategy is to show $(\!|s \otimes t|\!) \subset (\!|s|\!) \otimes (\!|t|\!)$ and $(\!|s|\!) \otimes (\!|t|\!) \subset (\!|s \otimes t|\!)$.

$(\!|s \otimes t|\!) \subset (\!|s|\!) \otimes (\!|t|\!)$:

By Lemma 3.10, there are embeddings $\iota_s : \mathrm{Der}(s) \hookrightarrow \mathrm{Der}(s \otimes t)$ and $\iota_t : \mathrm{Der}(t) \hookrightarrow \mathrm{Der}(s \otimes t)$. The inclusion reversing nature of the antitone Galois connection in Definition 3.1 implies $(\!|s \otimes t|\!) = T(\mathbf{d}, \mathrm{Der}(s\otimes t)) \subset T(\mathbf{d}, \iota_s(\mathrm{Der}(s)))$ and $(\!|s\otimes t|\!) \subset T(\mathbf{d}, \iota_t(\mathrm{Der}(t)))$. Thus $(\!|s\otimes t|\!)$ is in their intersection. We shall prove $T(\mathbf{d}, \iota_s(\mathrm{Der}(s))) = (\!|s|\!) \otimes V$ and $T(\mathbf{d}, \iota_j(\mathrm{Der}(t))) = U \otimes (\!|t|\!)$. The conclusion follows as $(\!|s \otimes t|\!) \subset (\!|s|\!) \otimes V \cap U \otimes (\!|t|\!) = (\!|s|\!) \otimes (\!|t|\!)$.

The statement to prove is $T(\mathbf{d}, \iota_s(\mathrm{Der}(s))) = (\!|s|\!) \otimes V$.

We first show the direction $(\!|s|\!) \otimes V \subset T(\mathbf{d}, \iota_s(\mathrm{Der}(s)))$. As $(\!|s|\!) \otimes V$ is generated by $\acute{s} \otimes t$ for $\acute{s} \in (\!|s|\!)$ and $t \in V$, it suffices to show $\acute{s} \otimes t \in T(\mathbf{d}, \iota_s(\mathrm{Der}(S)))$. This follows as $\langle \acute{s} \otimes t|$ satisfies $\langle \acute{s} \otimes t|\mathbf{d}(\delta) = 0$ for all $\delta = (\sigma_i \otimes 1_{V_i})_{i\in[n]} \in \iota_s(\mathrm{Der}(s))$ since $(\sigma_i)_{i\in[n]} \in \mathrm{Der}(s)$.

In the opposite direction, $T(\mathbf{d}, \iota_s(\mathrm{Der}(s)))$ is a subspace of $U \otimes V$. We shall show in fact it is the subspace $(\!|s|\!) \otimes V$ by showing every element in $T(\mathbf{d}, \iota_s(\mathrm{Der}(s)))$ is the sum of pure tensors $\acute{s} \otimes t$ for $\acute{s} \in (\!|s|\!), t \in V$.

Let $r = \sum_{i=1}^{m} s_i \otimes t_i \in U \otimes V$ be an element of $T(\mathbf{d}, \iota_s(\mathrm{Der}(s)))$, with all t_i's linearly independent. Showing each s_i is in $(\!|s|\!)$ concludes r is in $(\!|s|\!) \otimes V$. By definition r satisfies $\langle r|\mathbf{d}(\iota_s(\sigma)) = 0$ for all $\sigma \in \mathrm{Der}(s)$.

Let $\sigma \in \mathrm{Der}(s)$ and $\iota_s(\sigma) = (\sigma_j \otimes 1_{V_j})_{j\in[n]}$. Computing,

$$\begin{aligned} 0 &= \langle r|\mathbf{d}(\iota_s(\sigma)) \\ &= \left\langle \sum_i s_i \otimes t_i \middle| \mathbf{d}(\iota_s(\sigma)) \right. \end{aligned}$$

$$
\begin{aligned}
&= \sum_i \langle s_i \otimes t_i | \mathbf{d}(\iota_s(\sigma)) \qquad \text{linearity of tensor evaluation} \\
&= \sum_i \left(\langle s_i \otimes t_i | \left(\sum_{j=1}^{n} \sigma_j \otimes 1_{V_j} \right) \right) \\
&= \sum_i \left(\sum_{j=1}^{n} \left(\langle s_i \otimes t_i | \sigma_j \otimes 1_{V_j} \right) \right) \\
&= \sum_i \left(\sum_{j=1}^{n} \left(\langle s_i | \sigma_j \otimes t_i \right) \right) \\
&= \sum_i \left(\sum_j \langle s_i | \sigma_j \right) \otimes t_i \\
&= \sum_i \langle s_i | \mathbf{d}(\sigma) \otimes t_i
\end{aligned}
$$

Let $\{F_b\}_{b \in \mathcal{B}}$ be a basis of V. Then expanding each t_i in this basis,

$$
\begin{aligned}
0 &= \sum_i \left(\langle s_i | \mathbf{d}(\sigma) \otimes \left(\sum_{b \in \mathcal{B}} \lambda_{ib} F_b \right) \right) \\
&= \sum_{b \in \mathcal{B}} \left(\sum_i \left(\lambda_{ib} \langle s_i | \mathbf{d}(\sigma) \right) \otimes F_b \right)
\end{aligned}
$$

Since F_b are linearly independent, $\sum_i \lambda_{ib} \langle s_i | \mathbf{d}(\sigma) = 0$. Since t_is are linearly independent, the m by $|\mathcal{B}|$ matrix $[\lambda_{ib}]$ has full row rank. Thus the only way $\sum_i \lambda_{ib} \langle s_i | \mathbf{d}(\sigma) = 0$ is if $\langle s_i | \mathbf{d}(\sigma) = 0$ for all i. Thus $s_i \in (\!| s |\!)$ for all i. This concludes the proof of $T(\mathbf{d}, \iota_s(\mathrm{Der}(s))) = (\!| s |\!) \otimes V$.

The statement $T(\mathbf{d}, \iota_t(\mathrm{Der}(t))) = U \otimes (\!| t |\!)$ is proven analogously.

$(\!| s |\!) \otimes (\!| t |\!) \subset (\!| s \otimes t |\!)$:

The strategy will be to first show for $\acute{s} \in (\!| s |\!)$, that $\acute{s} \otimes t \in (\!| s \otimes t |\!)$, and secondly, show if $\acute{s} \otimes t \in (\!| s \otimes t |\!)$ for all s, then for all $\acute{t} \in (\!| t |\!)$, that $\acute{s} \otimes \acute{t} \in (\!| s \otimes t |\!)$. The proof concludes as $(\!| s |\!) \otimes (\!| t |\!)$ is generated by $\acute{s} \otimes \acute{t}$ for $\acute{s} \in (\!| s |\!)$ and $\acute{t} \in (\!| t |\!)$, and

To show $\acute{s} \otimes t \in (\!| s \otimes t |\!)$, let $\delta \in \mathrm{Der}(s \otimes t)$. Since δ is an element of $\prod_i \mathfrak{gl}(U_i \otimes V_i) \cong \prod_i (\mathfrak{gl}(U_i) \otimes \mathfrak{gl}(V_i))$, write δ as $\left(\sum_{j=1}^{R_i} (\sigma_j \otimes \tau_j) \right)_{i \in [n]}$. By construction $\langle s \otimes t | \mathbf{d}(\delta) = 0$. Calculating,

$$
\begin{aligned}
0 &= \langle s \otimes t | \mathbf{d}(\delta) \\
&= \langle s \otimes t | \sum_{i=1}^{n} \left(\sum_{j=1}^{R_i} (\sigma_j \otimes \tau_j) \right) \\
&= \langle s \otimes t | \sum_{a \in A} (\sigma_a \otimes \tau_a) \qquad \text{grouping into one indexing set} \\
&= \sum_{a \in A} \langle s | \sigma_a \otimes \langle t | \tau_a \\
&= \sum_{a \in A} \langle s | \sigma_a \otimes \left(\sum_{b \in \mathcal{B}} \lambda_{ab} F_b \right) \qquad \text{For } F_b \text{ basis of } V \\
&= \sum_{b \in \mathcal{B}} \left(\sum_{a \in A} \lambda_{ab} \langle s | \sigma_a \right) \otimes F_b
\end{aligned}
$$

As $\{F_b\}_{b\in\mathcal{B}}$ is a basis of V, $\sum_{a\in A}\lambda_{ab}\langle s|\sigma_a = 0$ for each b. Regrouping and combining the terms in the indexing set A by axes, we have $\langle s|\sigma = 0$, meaning $\sigma \in \mathrm{Der}(s)$. Thus $\langle \acute{s}|\sigma = 0$ as well. Substituting $\acute{s}$ in place of s in the above equation also equals 0, concluding $\langle \acute{s}\otimes t|\mathbf{d}(\delta) = 0$.

The proof that $\acute{s}\otimes\acute{t} \in (\!|s\otimes t|\!)$, assuming $\acute{s}\otimes t$ is in $(\!|s\otimes t|\!)$ is analogous.

□

Example 3.12. *Let $\langle s| : \mathfrak{sl}_2 \times \mathfrak{sl}_2 \rightarrowtail \mathfrak{sl}_2$ be the multiplication tensor of $\mathfrak{sl}_2$. We compute $(\!|s|\!) = Ks$. Let $\langle t| : \mathbb{F}_3^2 \times \mathbb{F}_3^2 \rightarrowtail \mathbb{F}_3^2$ be the multplication table of $\mathbb{F}_3[x]/(x^2+1)$ as given in Example 3.8. We compute $(\!|t|\!)$ as spanned by t and the tensor r given by the system of forms $\left[\begin{bmatrix}0 & 2\\ 2 & 0\end{bmatrix}, \begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}\right]$.*

As expected, $(\!|s|\!)\otimes(\!|t|\!)$ and $(\!|s\otimes t|\!)$ are both 2 dimensional. By definition $(\!|s|\!)\otimes(\!|t|\!)$ is spanned by $s\otimes t$ and $s\otimes r$, while computation verifies $(\!|s\otimes t|\!)$ is spanned by these same tensors.

Before describing our main problem, we address potential complications in finding product decompositions.

3.3.1. *Non-canonical choices.* In Definition 3.4 the tensors s and t have the same valence, and there is a matching of axis U_i of s with V_i of t.

Both are convenient but not necessary when looking for product decompositions. A tensor can be padded by axes consisting of K. For instance, a linear transformation $K^2 \to K^2$ is isotopic to the tensor $K^2 \times K \rightarrowtail K^2$ via the isomorphism $t \mapsto \tilde{t}$ where $\langle \tilde{t}|u,k\rangle = k\langle t|u\rangle$. Notice there's combinatoral explosion of possibilities of which axes to pad when one tensor has fewer axes than the other. Not matching U_i with V_i brings combinatorial explosion to potential decompositions. We do not rule out these possibilities and consider them as an expanded form of products of tensors for future invesigation.

Now we describe the main problem under consideration.

Definition 3.13. *(Kronecker decomposition) A tensor $r \in \mathrm{Mult}(W_n,\dots,W_1;W_0)$ has a* ***Kronecker decomposition*** *into a finite set $\mathcal{S}$ if*

$$r \cong \bigotimes_{s\in\mathcal{S}} s \qquad \text{for } s : V_{s,n}\times\cdots\times V_{s,1} \rightarrowtail V_{s,0} \tag{3.4}$$

r is ***Kronecker indecomposable*** *of $r \cong \bigotimes_{s\in\mathcal{S}} s$ implies $\mathcal{S} \subset \{r,\mu\}$ where μ is the K-multiplication tensor.*

Problem B (Kronecker decompositions of tensors).
Given tensor $r \in \mathrm{Mult}(W_n,\dots,W_1;W_0)$, what criterion guarantee r is Kronecker indecomposable? Can we compute these criterion and the decomposition efficiently?

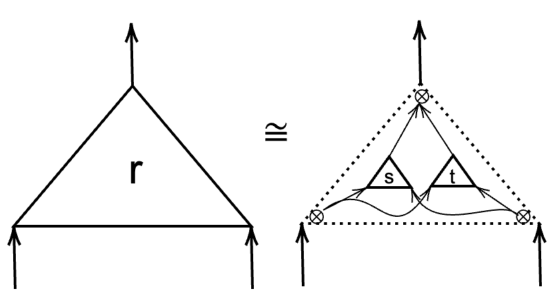

FIGURE 2. Pictorial illustration of Problem B. We are given is the tensor r. We'd like to find non-trivial s, t such that $r \cong s\otimes t$. The dashed lines indicate if successful, while we don't get to view the internals of r, we do understand how r is built.

Examples in Section 2.1 give some partial answers for special cases. We'd like to formalize these results and extend to cases beyond the associative case, with the derivation algebra as our primary tool in Theorem 3.11.

As part of answering this problem, we plan to analyze the derivation algebras themselves to understand the tensor. That is, **given** $s \in \mathrm{Mult}(U_n, \dots, U_1; U_0)$ **and** $t \in \mathrm{Mult}(V_n, \dots, V_1; V_0)$**, is** $\mathrm{Der}(s \otimes t)$ **completely determined by** $\mathrm{Der}(s)$ **and** $\mathrm{Der}(t)$**?**

This question has an affirmative answer for the case of adjoints, as $\mathrm{Adj}(s \otimes t) = \mathrm{Adj}(s) \otimes \mathrm{Adj}(t)$ [Wil09a]. Nothing is known for $\mathrm{Der}(s \otimes t)$ other than Lemma 3.10 above. Preliminary computations suggests the embeddings of $\mathrm{Der}(s)$ and $\mathrm{Der}(t)$ do not behave like the adjoint case, as the dimension of $\mathrm{Der}(s \otimes t)$ in some computed examples is not the product of $\dim \mathrm{Der}(s)$ and $\dim \mathrm{Der}(t)$.

Lastly, we look to parametrized products of tensors. The existence of (P, Ω)**-products means the possibility of a** (P, Ω)**-product decomposition.**

Definition 3.14 ((P,Ω)-tensor product)**.** *[FMW20]*

Let $U_1, \dots, U_n$ *be* K*-vector spaces,* $P \subset K[x_1, \dots, x_n]$ *and* $\Omega \subset \mathrm{End}(U_1) \times \cdots \times \mathrm{End}(U_n)$*. Define the following subspace of* $U_1 \otimes \cdots \otimes U_n$

$$\Xi(P, \Omega) := \left\langle \sum_e \lambda_e \omega_1^{e_1} u_1 \otimes \cdots \otimes \omega_n^{e_n} u_n \middle| \omega \in \Omega, \sum_e \lambda_e x_1 e^{e_1} \cdots x_n^{e_n} \in P, u_i \in U_i \right\rangle.$$

Define the (P, Ω)*-tensor product space as the quotient space*

$$\lbag U_1, \dots, U_n \rbag_\Omega^P := (U_1 \otimes \cdots \otimes U_n)/\Xi(P, \Omega),$$

together with a K*-multilinear map* $\lbag \cdots \rbag : U_1 \times \cdots \times U_n \rightarrowtail \lbag U_1, \dots, U_n \rbag_\Omega^P$*, where* $\lbag u_1, \dots, u_n \rbag = u_1 \otimes \cdots \otimes u_n + \Xi(P, \Omega)$*.*

Notice $\lbag U_1, \dots, U_n \rbag_\emptyset^\emptyset$ is the usual tensor product of vector spaces. Let $P \subset K[x_1, \dots, x_n]$ and $\Omega \subset \prod_i \mathrm{End}(U_i)$. Suppose each U_i is a tensor space $\mathrm{Mult}(V_i^m, \dots, V_i^1; K) \cong \left(\bigotimes V_i^j\right)^*$, and an isomorphism $V_i^j \cong (V_i^j)^*$ is specified. Let $(s_1, \dots, s_n) \in \prod_i U_i$ be a tuple of tensors. Then the image of $(s_1, \dots, s_n)$ under the (P, Ω)-tensor product, denoted $\lbag s_1, \dots, s_n \rbag =: r$, is an element of a quotient space $\bigotimes_i U_i / \Xi(P, \Omega)$.

Fix a complementary subspace W to $\Xi(P, \Omega)$ in $\bigotimes_i U_i$ so $\Xi(P, \Omega) \oplus W = \bigotimes_i U_i$.Let $\tilde{r} \in W$ be the element of $\bigotimes_i U_i$ identified to r. As $\bigotimes_i U_i = \bigotimes_i \bigotimes_j V_i^j \cong \bigotimes_j \bigotimes_i V_i^j$, $\tilde{r}$ can be identified with a multilinear map in $\mathrm{Mult}(\bigotimes_i V_i^1, \dots, \bigotimes_i V_i^m; K)$. This gives a multilinear interpretation $\langle r| : \prod_{j=1}^m \left(\bigotimes_i V_i^j\right) \rightarrowtail K$.

> **Challenge B.** *Let* $r \in \mathrm{Mult}(U_n, \dots, U_1; U_0)$ *be a tensor. For what* (P, Ω) *does* r *admit a* (P, Ω)*-product decomposition?*

References

[BB17] Jacob Biamonte and Ville Bergholm. *Tensor Networks in a Nutshell*. 2017. arXiv: 1708.00006 [quant-ph]. URL: https://arxiv.org/abs/1708.00006.

[BC17] Jacob C Bridgeman and Christopher T Chubb. "Hand-waving and interpretive dance: an introductory course on tensor networks". In: *Journal of Physics A: Mathematical and Theoretical* 50.22 (May 2017), p. 223001. ISSN: 1751-8121. DOI: 10.1088/1751-8121/aa6dc3. URL: http://dx.doi.org/10.1088/1751-8121/aa6dc3.

[BKW24] Peter A. Brooksbank, Martin D. Kassabov, and James B. Wilson. *Detecting cluster patterns in tensor data*. 2024. arXiv: 2408.17425 [math.NA]. URL: https://arxiv.org/abs/2408.17425.

[BL08] Peter A. Brooksbank and Eugene M. Luks. "Testing isomorphism of modules". In: *Journal of Algebra* 320.11 (2008), pp. 4020–4029.

[BMW17] Peter A. Brooksbank, Joshua Maglione, and James B. Wilson. "A fast isomorphism test for groups whose Lie algebra has genus 2". In: *J. Algebra* 473 (2017), pp. 545–590. ISSN: 0021-8693. DOI: 10.1016/j.jalgebra.2016.12.007. URL: https://doi.org/10.1016/j.jalgebra.2016.12.007.

[BMW20] Peter A. Brooksbank, Joshua Maglione, and James B. Wilson. "Exact sequences of inner automorphisms of tensors". In: *Journal of Algebra* 545 (Mar. 2020), 43–63. ISSN: 0021-8693. DOI: 10.1016/j.jalgebra.2019.07.006. URL: http://dx.doi.org/10.1016/j.jalgebra.2019.07.006.

[BMW22] Peter A. Brooksbank, Joshua Maglione, and James B. Wilson. "Tensor isomorphism by conjugacy of Lie algebras". In: *Journal of Algebra* 604 (Aug. 2022), 790–807. ISSN: 0021-8693. DOI: 10.1016/j.jalgebra.2022.04.014. URL: http://dx.doi.org/10.1016/j.jalgebra.2022.04.014.

[Bro97] Rasmus Bro. "PARAFAC. Tutorial and applications". In: *Chemometrics and Intelligent Laboratory Systems* 38.2 (1997), pp. 149–171.

[BS72] R. H. Bartels and G. W. Stewart. "Algorithm 432 [C2]: Solution of the matrix equation AX + XB = C [F4]". In: *Commun. ACM* 15.9 (Sept. 1972), 820–826. ISSN: 0001-0782. DOI: 10.1145/361573.361582. URL: https://doi.org/10.1145/361573.361582.

[BW12] Peter A. Brooksbank and James B. Wilson. "Intersecting two classical groups". In: *J. Algebra* 353 (2012), pp. 286–297. ISSN: 0021-8693. DOI: 10.1016/j.jalgebra.2011.12.004. URL: https://doi.org/10.1016/j.jalgebra.2011.12.004.

[BW14] Peter A. Brooksbank and James B. Wilson. "Groups acting on tensor products". In: *Journal of Pure and Applied Algebra* 218.3 (2014), pp. 405–416. ISSN: 0022-4049. DOI: https://doi.org/10.1016/j.jpaa.2013.06.011. URL: https://www.sciencedirect.com/science/article/pii/S0022404913001382.

[DL08] Lieven De Lathauwer. "Decompositions of a Higher-Order Tensor in Block Terms—Part I: Lemmas for Partitioned Matrices". In: *SIAM Journal on Matrix Analysis and Applications* 30.3 (2008), pp. 1022–1032.

[DLDMV00] Lieven De Lathauwer, Bart De Moor, and Joos Vandewalle. "A Multilinear Singular Value Decomposition". In: *SIAM Journal on Matrix Analysis and Applications* 21.4 (2000), pp. 1253–1278.

[FMW20] Uriya First, Joshua Maglione, and James B. Wilson. *A spectral theory for transverse tensor operators*. 2020. arXiv: 1911.02518 [math.SP]. URL: https://arxiv.org/abs/1911.02518.

[GG13] Joachim von zur Gathen and Jürgen Gerhard. *Modern computer algebra*. Third. Cambridge University Press, Cambridge, 2013, pp. xiv+795. ISBN: 978-1-107-03903-2. DOI: 10.1017/CBO9781139856065. URL: https://doi.org/10.1017/CBO9781139856065.

[Gre12] W.H. Greub. *Multilinear Algebra*. Grundlehren der mathematischen Wissenschaften. Springer Berlin Heidelberg, 2012. ISBN: 9783662007952. URL: https://books.google.com/books?id=jlvoCAAAQBAJ.

[Har72] R. A. Harshman. "PARAFAC2: Mathematical and technical notes". In: *UCLA Working Papers in Phonetics* 22 (1972b), pp. 30–44.

[Hit27] Frank Lauren Hitchcock. "The Expression of a Tensor or a Polyadic as a Sum of Products". In: *Journal of Mathematics and Physics* 6 (1927), pp. 164–189. URL: https://api.semanticscholar.org/CorpusID:124183279.

[IQ19] Gábor Ivanyos and Youming Qiao. "Algorithms Based on *-Algebras, and Their Applications to Isomorphism of Polynomials with One Secret, Group Isomorphism, and Polynomial Identity Testing". In: *SIAM Journal on Computing* 48.3 (2019), pp. 926–963.

[Jac10] Nathan Jacobson. "Abstract derivation and Lie algebras". In: *Transactions of the American Mathematical Society* 42.2 (2010), pp. 206–224. DOI: 10.1090/s0002-9947-1937-1501922-7. URL: https://doi.org/10.1090/s0002-9947-1937-1501922-7.

[KB09] Tamara G. Kolda and Brett W. Bader. “Tensor Decompositions and Applications”. In: *SIAM Review* 51.3 (2009), pp. 455–500.

[Lan12] J.M. Landsberg. *Tensors: Geometry and Applications*. Graduate studies in mathematics. American Mathematical Society, 2012. ISBN: 9780821884812. URL: https://books.google.com/books?id=JTjv3DTvxZIC.

[LL00] George F. Leger and Eugene M. Luks. “Generalized Derivations of Lie Algebras”. In: *Journal of Algebra* 228.1 (2000), pp. 165–203. ISSN: 0021-8693. DOI: https://doi.org/10.1006/jabr.1999.8250. URL: https://www.sciencedirect.com/science/article/pii/S0021869399982509.

[MM10] Takanori Maehara and Kazuo Murota. “A numerical algorithm for block-diagonal decomposition of matrix *-algebras with general irreducible components”. In: *Japan Journal of Industrial and Applied Mathematics* 27 (2010), pp. 263–293. URL: https://api.semanticscholar.org/CorpusID:7116115.

[Mya90] A. G. Myasnikov. “Definable invariants of bilinear mappings”. In: *Siberian Mathematical Journal* 31.1 (1990), pp. 89–99.

[Ose11] I. V. Oseledets. “Tensor-Train Decomposition”. In: *SIAM Journal on Scientific Computing* 33.5 (2011), pp. 2295–2317. DOI: 10.1137/090752286. eprint: https://doi.org/10.1137/090752286. URL: https://doi.org/10.1137/090752286.

[Rob15] Elina Robeva. *Orthogonal Decomposition of Symmetric Tensors*. 2015. arXiv: 1409.6685 [math.AG]. URL: https://arxiv.org/abs/1409.6685.

[RS18] Elina Robeva and Anna Seigal. “Duality of graphical models and tensor networks”. In: *Information and Inference: A Journal of the IMA* 8.2 (June 2018), pp. 273–288. ISSN: 2049-8772. DOI: 10.1093/imaiai/iay009. eprint: https://academic.oup.com/imaiai/article-pdf/8/2/273/28864933/iay009.pdf. URL: https://doi.org/10.1093/imaiai/iay009.

[Tuc66] L. R. Tucker. “Some mathematical notes on three-mode factor analysis”. In: *Psychometrika* 31 (1966c), pp. 279–311.

[Whi38] Whitney. “Tensor products of Abelian groups”. In: (1938). DOI: 10.1215/s0012-7094-38-00442-9.

[Whi93] Steven R. White. “Density-matrix algorithms for quantum renormalization groups”. In: *Phys. Rev. B* 48.12 (1993), pp. 10312–10315. DOI: 10.1103/PhysRevB.48.10312.

[Wil09a] James B. Wilson. “Decomposing p-groups via Jordan algebras”. In: *Journal of Algebra* 322.8 (2009), pp. 2642–2679.

[Wil09b] James B. Wilson. “Finding central decompositions of p-groups”. In: *Journal of Group Theory* 12.6 (Jan. 2009). ISSN: 1435-4446. DOI: 10.1515/jgt.2009.015. URL: http://dx.doi.org/10.1515/JGT.2009.015.

[Wil12] James B. Wilson. “Existence, algorithms, and asymptotics of direct product decompositions, I”. In: *Groups - Complexity - Cryptology* 4.1 (Jan. 2012). ISSN: 1869-6104. DOI: 10.1515/gcc-2012-0007. URL: http://dx.doi.org/10.1515/gcc-2012-0007.

[Wil16] James B. Wilson. “On automorphisms of groups, rings, and algebras”. In: *Communications in Algebra* 45.4 (Oct. 2016), 1452–1478. ISSN: 1532-4125. DOI: 10.1080/00927872.2016.1175617. URL: http://dx.doi.org/10.1080/00927872.2016.1175617.