COMPUTING ALGEBRAIC INVARIANTS OF TENSORS AND THEIR APPLICATION TO TENSOR DECOMPOSITION

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ABSTRACT. This dissertation proposal outlines two areas of study. First, on algorithms for faster computation of algebraic invariants of tensors. Second, an investigation into a class of tensors built up as tensor products on the space of multilinear maps.

CONTENTS

1. INTRODUCTION

Tensors encapsulate multilinear maps. Often given as a multiway array of numbers, they are used across various disciplines of mathematics to record information for some fixed reference frame. As such, they are studied from many complementary perspectives [Bro97] [KB09] [Lan12] [RS18] [DLDMV00] [Tuc66].

Throughout, we fix a field K. A tensor space T is a vector space equipped with a multilinear interpretation $\langle\cdot| : T \hookrightarrow \mathrm{Mult}(U_n, \dots, U_1; U_0)$ for U_i each a K-vector space. A tensor t is an element of a tensor space T, and we write $\langle t| : U_n \times \cdots \times U_1 \rightarrowtail U_0$ to indicate $\langle t|$ is a multilinear function. The spaces $\{U_0, \dots, U_n\}$ are the frame of tensor, the size of the frame $(n+1)$ its valence, and $\{0, \dots, n\}$, the labels on the vector spaces, its axes. For $|u\rangle = |u_n, \dots, u_1\rangle$, write $\langle t|u\rangle \in U_0$ to mean evaluating $\langle t|$ at $|u\rangle$. With this definition, given some $a \times b$ matrix of numbers $[t_{ij}]$, interpret it as a multilinear map $\langle t| : K^a \times K^b \rightarrowtail K$ where $\langle t|e_i, e_j\rangle = t_{ij}$. This extends to cubes of numbers and so on, so our definition interoperates with the popular model of tensors as dense grids of numbers.

Example 1.1. *Let* $[t_{ij}] \in K^{2\times 3} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. *Then* t *can be interpreted as a bilinear form* $\langle t| : K^2 \times K^3 \rightarrowtail K$ *where* $\langle t|e_i, e_j\rangle = 1$ *for* $(i,j) \in \{(1,1),(1,3),(2,2)\}$.

The perspective taken in this dissertation propsal is to study tensors as distributive products using the tools of algebra. Fix a 3-tensor (bimap) $* : U \times V \rightarrowtail W$. Existing work such as [BW14],

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and [Wil16] highlights the role of the commutative centroid algebra

$$\mathrm{Cen}(t) := \{\sigma \in \mathrm{End}(U) \times \mathrm{End}(V) \times \mathrm{End}(W) : \sigma u * v = u * \sigma v = \sigma(u * v) \ \ \forall u \in U, v \in V\},$$

associative adjoint algebra

$$\mathrm{Adj}(t) := \{\sigma \in \mathrm{End}(U)^{\mathrm{op}} \times \mathrm{End}(V) : u\sigma * v = u * \sigma v\},$$

and Lie derivation algebra

$$\mathrm{Der}(t) := \{\delta \in \mathrm{End}(U) \times \mathrm{End}(V) \times \mathrm{End}(W) : \delta u * v + u * \delta v = \delta(u * v)\}.$$

Existing work using this approach include discovering basis independent cluster pattern in tensors [BKW24], decomposing p-groups [Wil09a], finding direct product decomposition of groups [Wil12], and advances in isomorphism testing [BMW17] [BW12] [BMW22].

Two avenues of study are proposed for my dissertation. First to find faster algorithms to compute these algebras, and second to prove structure and recognition theorems for specific classes of tensors using these algebras.

1.1. **Work on faster algorithms.** The computation of algebras $\mathrm{Adj}(t)$, $\mathrm{Cen}(t)$, and $\mathrm{Der}(t)$ are given by linear equations. For fixed bases, each is cubic in the number of variables by standard methods. For tensors with each frame of dimension n, this is at minimum $O(n^6)$ operations. In collaboration with James Wilson and Joshua Maglione, we have preliminary results for an asymptotically faster algorithm in computing $\mathrm{Adj}(t)$ and $\mathrm{Cen}(t)$ in $O(n^3)$ operations, inspired by analogous results for matricies known as the Bartels-Stewart algorithm [BS72]. **My proposed work is to explore asymptotic speedups for the computation of $\mathrm{Der}(t)$ in the 3-tensor case, and to extend to higher valence tensors in general.**

1.2. **Finding structural decompositions by X-raying tensors.** My proposed work is to prove structure and recognition theorems for tensors combined by what we call *soldering*, a generalization of the tensor product of multilinear maps. Drawing naming inspiration from medical imaging, we call our techniques *X-ray*ing tensors.

1.3. **Prior work.** We now survey recent work which informs the class of tensors we wish to investigate. Wilson in [Wil16] [Wil12] [Wil09a] [Wil09b] proves for bimaps the algebras $\mathrm{Adj}(t)$ and $\mathrm{Cen}(t)$ control direct sum decompositions and automorphisms of t, using them to prove properties for the originating algebraic structures. Recent work generalizing from bimaps finds a long exact sequence linking the various nuclei (generalized adjoints), centroids, and derivations of a higher valence tensor [BMW20]. Further work by First, Maglione, and Wilson [FMW20] defines a ternary Galois connection between tensors, operators, and polynomial ideals. Alongside it, they define a generalized tensor product $◖ U_1, \dots, U_n ◗_{\Omega}^{P}$, parametrized by operators $\Omega \subset \prod_i \mathrm{End}(U_i)$ and polynomials $P \subset K[x_1, \dots, x_n]$.

Let **d** be the polynomial $x_n + \cdots + x_1$. It is proven for a tensor t, the $(\mathbf{d}, \mathrm{Der}(t))$-tensor product is universally the smallest among the (P, Ω)-products that t factors through, for which $P \subset K[x_1, \dots, x_n]$ is an ideal generated by linear homogeneous polynomials. This motivates studies of $\mathrm{Der}(t)$ and the associated $(\mathbf{d}, \mathrm{Der}(t))$-tensor product space.

For the remainder of the introduction, let $U_0 = k$. Then as investigated in [BMW22], the vector subspace denoted $(\!| t |\!)$ (**derivation closure** of t) consisting of tensors t' whose derivation algebra contains the derivation algebra of t, may be identified with the $(\mathbf{d}, \mathrm{Der}(t))$-tensor product space $◖ U_1, \dots, U_n ◗_{\mathrm{Der}(t)}^{\mathbf{d}}$ and thus is universally the smallest search space for solving tensor isomorphism questions involving t. In [BMW22], an infinite family of tensors with 1 dimensional derivation closures are constructed. However, little else is known about $(\!| t |\!)$. By Theorem B of [FMW20], a basis for the space $(\!| t |\!)$ is computable in polynomial time, so computational examples are available in practice.

1.3.1. *Soldered tensors.* We now describe the class of tensors to be studied. For vector spaces U and V, the tensor product of U and V is the vector space $U \otimes V$ alongside the canonical map $\varphi : U \times V \rightarrowtail U \otimes V$, such that for every bilinear map f out of $U \times V$, there is a unique induced linear map $\hat{f}$ satisfying $f = \hat{f} \circ \varphi$. For tensors $s \in U$ and $t \in V$, $s \otimes t$ is the image of (s,t) under φ.

Because s and t are tensors, they have multilinear interpretations $\langle s| : \prod_{i=1}^{n} U_i \rightarrowtail K$ and $\langle t| : \prod_{i=1}^{n} V_i \rightarrowtail K$. We interpret $s \otimes t$ as $\langle s \otimes t| : \prod_{i=1}^{n}(U_i \otimes V_i) \rightarrowtail K$, where $\langle s \otimes t | u_1 \otimes v_1, \ldots, u_n \otimes v_n\rangle = \langle s|u_1, \ldots, u_n\rangle \otimes \langle t|v_1, \ldots, v_n\rangle$. This we call *soldering* the tensors s and t together. This construction is well-known, and is for example, specified for the bilinear case in [Gre12, Section 1.21] as the tensor product of bilinear maps. In [BMW20], soldering is used to construct an example of a tensor with decomposition detectable by the derivation algebra.

The diagram scheme in Figure 1 illustrates soldering. In it, the 3-valent tensors s and t are drawn as shapes with 3 wires indicating 3 axes, with orientation given to the wire to indicate input and output. This notation is known as tensor network diagrams. [BB17] When soldering two tensors, our notation is for wires to be combined by a $\otimes$ symbol. This is non-standard in tensor network diagram literature. Part of our work will be to extend and adapt tensor network diagram to soldering.

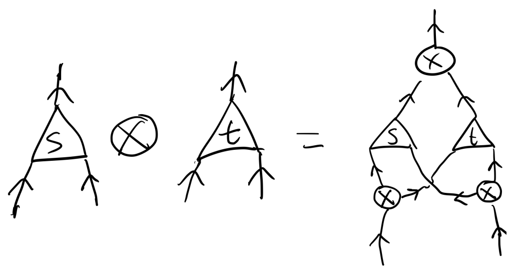

FIGURE 1. Pictorial illustration of soldering s and t

Example 1.2. *Given $\langle s| : K^2 \times K^2 \rightarrowtail K$ a bilinear form on K, and $\langle t| : L \times L \rightarrowtail L$ the multiplication tensor of some extension field L of K, the soldered tensor $\langle s \otimes t| : (K^2 \otimes L) \times (K^2 \otimes L) \rightarrowtail (K \otimes L)$ is defined as*

$$\langle s \otimes t | u \otimes l_1, v \otimes l_2\rangle = \langle s|u, v\rangle \otimes \langle t|l_1, l_2\rangle = u \cdot v \otimes l_1 l_2$$

Let f be an isomorphism of $K^2 \otimes L$ with L^2 and g an isomorphism of $K \otimes L$ with L. Then $\langle s \otimes t|$ is identified with a bilinear form $\langle r| : L^2 \times L^2 \rightarrowtail L$ by mapping inputs via f and the output via g. This is called an isotopism of tensors. [Wil16]

Computationally, we observe for soldered tensors that $\dim(\!|s|\!) \otimes (\!|t|\!) = \dim(\!|s \otimes t|\!)$ for all known examples that would compute in a reasonable time. **We wish to investigate this relationship further.**

1.3.2. *Related Works.* In related works, the physics community uses techniques like the Density Matrix Renormalization Algorithm [Whi93] attempts to uncover the structure of a high valence tensor by factoring it as a contracted product of 3-tensors called a Matrix Product State. This iterative optimization technique is for complex-valued tensors, relying on the Singular Value Decomposition. Techniques such as Tensor-Train decompositions and Tucker decompositions are similar but assumes different fixed internal structure. [KB09] Other tensor factorization techniques such as CP-decomposition [Hit27] attempts to decompose a tensor as a sum of smaller tensors, whereas the soldering of tensors is a product of tensors.

In the next two sections, we outline the proposed problems we are investigating and contributions we forsee as part of this dissertation.

2. FASTER ALGORITHMS FOR ALGEBRAIC INVARIANTS OF TENSORS

We wish to compute $\mathrm{Adj}(t)$, $\mathrm{Cen}(t)$, and $\mathrm{Der}(t)$ for a bimap t. For fixed bases, each of the algebras are specified by linear systems of equations, and thus can be computed in a number of steps polynomial to the sum of dimensions. But the naive solution has takes operations cubic in the number of variables. We aim to do better in the general case.

We report on results in collaboration with James Wilson and Joshua Maglione. First, the problem of computing adjoints of bimaps is stated in coordinates. Next, we describe our approach, which is to translate the system to a coordinate free formulation, solve a smaller subproblem, and propogate the subproblem solution to a full solution.

2.1. **Simultaneous Sylvester System - Coordinatized.** We solve the following

Given: arrays $R \in K^{r\times b\times c}$, $S \in K^{a\times s\times c}$, and $T \in K^{a\times b\times c}$
Return: matrices $X \in K^{a\times r}$ and $Y \in K^{s\times b}$ such that

$$(\forall i)(XR_i + S_iY = T_i). \tag{2.1}$$

Expressed as list of matrix equation, Equation (2.1) is the natural extension of the Sylvester Equation, which asks for X satisfying the matrix equation $XA + BX = C$. For R and S filled in from a tensor with a fixed basis, and T to be all zero, solving an instance of this problem finds the adjoint algebra of the tensor.

Example 2.1. *Let t be the tensor in $K^{2\times2\times2}$ given by a pair of 2×2 matricies ("a system of forms")*

$$t_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, t_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The algebra $\mathrm{Adj}(t)$ is computed by solving for variables x_{ij}, y_{ij} satisfying

$$\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} t_1 = t_1 \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \text{ and } \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} t_2 = t_2 \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}$$

Below gives a coordinate free description.

2.2. **Simultaneous Sylvester System - Basis independent.**

Given: Elements $r \in R\otimes B\otimes C$, $s \in A\otimes S\otimes C$, $t \in A\otimes B\otimes C$, and isomorphisms identifying each vector space with its dual.
Return: Elements $x \in \mathrm{Hom}(A, R)$ and $y \in \mathrm{Hom}(B, S)$ such that $(x\otimes I_B\otimes I_C)(r) + (I_A\otimes y\otimes I_C)(s) = t$.

Below we describe how to compute x and y without solving a full system of linear equations.

Preliminaries:

First, by composing the isomorphism between R and its dual followed by the natural isomorphism $R^*\otimes B\otimes C \cong \mathrm{Hom}(R, B\otimes C)$, we view r as an element of $\mathrm{Hom}(R, B\otimes C)$. Similarly, view s as an element of $\mathrm{Hom}(S, A\otimes C)$. We now look for subspaces $B' \leq B$ and $A' \leq A$ such that r and s have left inverses after post-composing with projections. That is, we want $r_{B'} := (\pi_{B'}\otimes I_C)\circ r$ and $s_{A'} := (\pi_{A'}\otimes I_C)\circ s$ to have left inverses. Denote these left inverses as $r^{\#}_{B'}$ and $s^{\#}_{A'}$. We also need B' and A' to have an induced isomorphism to their respective duals, meaning $B \cong B^*$ restricts an isomorphism $B' \cong (B')^*$, and similar for $A \cong A^*$ restricting to $A' \cong (A')^*$.

Solving a smaller subproblem:

To assist in calculation, let $\pi_{A',B',C} := \pi_{A'}\otimes\pi_{B'}\otimes I_C$, and $xr := (x\otimes I_B\otimes I_C)(r)$, and $ys := (I_A\otimes y\otimes I_C)(s)$.

Projecting to the spaces A' and B', we compute

$$\begin{aligned} &\pi_{A',B',C}(xr + ys) = \pi_{A',B',C}(t) \\ &\iff \pi_{A',B',C}(xr) + \pi_{A',B',C}(ys) = \pi_{A',B',C}(t) \\ &\iff x_{A'}(\pi_{R,B',C}(r)) + y_{B'}(\pi_{A',S,C}(s)) = \pi_{A',B',C}(t) \end{aligned}$$

Propagating solution to full problem:

Solving for $x_{A'}$ and $y_{B'}$ proceed by standard linear algebra, but as $\dim x_{A'} = \dim A' \cdot \dim R$ and $\dim y_{B'} = \dim B' \cdot \dim S$, this smaller system have unknowns of considerably lower dimension if the subspaces A' and B' are lower dimensional compared to A and B.

After solving for $x_{A'}$ and $y_{B'}$ by conventional methods, our algorithm proceeds by finding complementary subspaces $A = A' \oplus U$, and $B = B' \oplus V$. Let r_V, s_U, x_U, and y_V be defined analogously to above. Then projecting to the subspaces A' and V, we require $\pi_{A',V,C}(xr + ys) = \pi_{A',V,C}(t)$.

Using the fixed isomorphism between vector spaces and their duals, as well as the natural isomorphisms above, we see that

$$\begin{aligned}
&\pi_{A',V,C}(xr + ys) = \pi_{A',V'C}(t) \\
&\iff \pi_{A',V,C}(xr) + \pi_{A',V,C}(ys) = \pi_{A',V,C}(t) \\
&\iff x_{A'}(\pi_{R,V,C}(r)) + y_V(\pi_{A',S,C}(s)) = \pi_{A',V,C}(t) \\
&\iff (x_{A'} \otimes I_C) \circ r \circ \pi_V + (\pi_{A'} \otimes I_C) \circ s \circ y_V = (\pi_{A'} \otimes I_C) \circ t \circ \pi_V \qquad A' \otimes V \otimes C \cong \mathrm{Hom}(V, A' \otimes C) \\
&\iff \tilde{r} + s_{A'} \circ y_V = \tilde{t} \qquad \text{Where } \tilde{r} := (x_{A'} \otimes I_C) \circ r_V \text{ and } \tilde{t} := (\pi_{A'} \otimes I_C) \circ t_V
\end{aligned}$$

The only unknown in the last equation is y_V as $x_{A'}$ was previously solved for. Hence precomposing with $s^{\#}_{A'}$ to both sides gives a unique y_V solution. Analogously projecting to the subspaces U and B' followed by precomposing $r^{\#}_{B'}$ solves for a unique x_U given a $y_{B'}$. Finally, projecting to the subspaces U and V verifies the overall equation is satisfied for the pair x_U, y_V.

Remark 2.2. *Our algorithm requires the user to provide the subspace A' and B' with the desired invertibility properties of $r_{B'}$ and $s_{A'}$. In coordinates, the linear transformation $r \in \mathrm{Hom}(R, B \otimes C)$ has $bc := \dim B \otimes C$ rows and $r := \dim R$ columns, thus projecting to a dim $\lceil r/c \rceil$ subspace of B have a high probability of maintaining the left invertibility of r.*

For example, if $\dim(A), \dim(B), \dim(C), \dim(R), \dim(S)$ are all $O(n)$ then $\lceil r/c \rceil$ is $O(1)$ hence the naive $O(n^6)$ operations necessary to solve the system is reduced to the $O(n^3)$ operations needed to solve the smaller subproblem.

Extending this approach to computing derivations of bimaps is a part of my dissertation proposal.

2.3. **Derivation System.** First to state the problem in coordinates, we need

Definition 2.3 (Outer action)**.** *Given an array $T \in K^{a \times b \times d}$, let $[T_1, \dots, T_d]$ be a list of $K^{a \times b}$ matricies corresponding to unfolding this array along the third index,. Then for a matrix Z of size $d \times c$, define T^Z, the outer action of Z on T, as the $K^{a \times b \times c}$ array satisfying $(T^Z)_j = \sum_{i=1}^{d} T_i Z_{ij}$.*

Now we are ready to state the problem of solving the derivation system in coordinates.

Problem A (Derivation system - Coordinatized)**.**

Given: *arrays $R \in K^{r \times b \times c}$, $S \in K^{a \times s \times c}$, and $T \in K^{a \times b \times t}$*
Return: *matrices $X \in K^{a \times r}$, $Y \in K^{s \times b}$, and $Z \in K^{t \times c}$ such that*

$$(\forall i) X R_i + S_i Y + (T^Z)_i = 0. \tag{2.2}$$

The above equation is no longer a list of matrix equations due to the outer action by Z. But it is exactly the equation satisfied by the derivation algebra of a tensor when R, S, T are all filled in from a tensor in a fixed basis.

Example 2.4. *Let t be the tensor as in $Example$ 2.1.*

The computation of $\mathrm{Der}(t)$ is to solve for variables x_{ij}, y_{ij}, z_{ij} satisfying

$$\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} t_1 + t_1 \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} = t_1 z_{11} + t_2 z_{21} \quad \textit{and}$$
$$\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} t_1 + t_1 \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} = t_1 z_{12} + t_2 z_{22}$$

The problem has a coordinate free version of the problem written below.

2.4. **Derivation system - Basis independent.**

Given: Elements $r \in R \otimes B \otimes C$, $s \in A \otimes S \otimes C$, $t \in A \otimes B \otimes T$, and isomorphisms identifying the vector spaces and their duals.

Return: Elements $x \in \mathrm{Hom}(A, R)$, $y \in \mathrm{Hom}(B, S)$, and $Z \in \mathrm{Hom}(C, T)$ such that $(x \otimes I_B \otimes I_C)(r) + (I_A \otimes y \otimes I_C)(s) + (I_A \otimes I_B \otimes z)(t) = 0$.

Preliminary investigations suggest a similar approach to Simultaneous Sylvester Systems but with 3 subspaces, $A' \leq A$, $B' \leq B$, and $C' \leq C$. Additional work is necessary.

Thus far we have only been concerned about bimaps. As a challenge upon resolving Problem A, we propose extending the above ideas to higher valence tensors. We frame the question as solving for algebraic invariants of tensors.

We first give notation for an endomorphism acting on a specific axis of the tensor.

Definition 2.5. *Let $\langle t| : \prod_i U_i \rightarrowtail K$. Let $\sigma \in \mathrm{End}(U_a)$. Then define the tensor $\langle t|\sigma$ as*

$$\langle t|\sigma|u\rangle = \langle t|\sigma u_a, u_{\overline{a}}\rangle$$

Where $u = (u_a, u_{\overline{a}})$ splits $u \in \prod_i U_i$ as an element of $U_a \times \prod_{i \neq a} U_i$.

Challenge A.
Let $\langle t| : \prod_{i \in I} U_i \rightarrowtail K$. Let $\{a, b\} \subset I$.
Generalizing the adjoint, define the ab-nucleus of t as

$$\mathrm{Nuc}_{ab}(t) := \{(\sigma_a, \sigma_b) \in \mathrm{End}(U_a) \times \mathrm{End}(U_b) : \langle t|\sigma_a u_a, u_{\overline{a}}\rangle = \langle t|\sigma_b u_b, u_{\overline{b}}\rangle\}. \tag{2.3}$$

For $J \subset I$, define the J-centroid as

$$\mathrm{Cen}_J(t) := \left\{ (\sigma_j)_{j \in J} \in \prod_j \mathrm{End}(U_j) : \langle t|\sigma_j u_j, u_{\overline{j}}\rangle = \langle t|\sigma_k u_k, u_{\overline{k}}\rangle \ \forall j, k \in J \right\}. \tag{2.4}$$

Similary, define the J-derivation as

$$\mathrm{Der}_J(t) := \left\{ (\delta_j)_{j \in J} \in \prod_j \mathrm{End}(U_j) : \sum_j \langle t|\delta_j u_j, u_{\overline{j}}\rangle = 0 \right\}. \tag{2.5}$$

These spaces are computed by linear equations, and generalizes the adjoint and derivation equation of 3-tensors to higher valence tensors. ***Can these algebras can be computed in operations less than cubic in the number of variables, similar to the bimap case?***

3. X-RAYING TENSORS

3.1. **Preliminaries.**

As described in the introduction, we wish to better understand tensors through *X-ray*ing them using algebra, starting with soldered tensors. To state the problem, we follow the exposition and notation in [FMW20].

Definition 3.1. *(Ternary Galois Connection of Tensors, Ideals, and Operators)*
We define evaluating a multivariable polynomial, with operators substituting for the indeterminants. For $p = \sum_e \lambda_e X^e \in K[x_0, \ldots, x_n] =: K[X]$ and $\omega \in \prod_i \mathrm{End}(U_i)$, let

$$p(\omega) := \sum_e \lambda_e \, (\omega_0^{e_0}, \ldots, \omega_n^{e_n}) \in \prod_i \mathrm{End}(U_i).$$

Let $S \subset \mathrm{Mult}(U_n, \ldots, U_1; U_0)$. It is evidently a tensor space with the identity interpretation map. For all $t \in S$, define $\langle t|p(\omega)$ where for any $(u_1, \ldots, u_n)$,

$$\langle t|p(\omega)|u\rangle = \sum_e \lambda_e \omega_0^{e_0} \langle t|\omega_1^{e_1} u_1, \ldots, \omega_n^{e_n} u_n\rangle.$$

Now fix a polynomial p and operator ω. Define the set $\boldsymbol{T}(p,\omega) := \{t \in \mathrm{Mult}(U_n,\dots,U_1;U_0) : \langle t \mid p(\omega) = 0\}$. Extend this definition to subsets P and Ω via

$$\boldsymbol{T}(P,\Omega) := \bigcap_{p\in P}\bigcap_{\omega\in\Omega}\boldsymbol{T}(p,\omega).$$

Similarly, for fixed polynomial p and tensor t define the set $\boldsymbol{Z}(t,p) := \{\omega \in \prod_u \mathrm{End}(U_i) : \langle t|p(\omega) = 0\}$ and extend to subsets $\boldsymbol{Z}(S,P)$.

Fixing $P \subset K[X]$, there is an inclusion reversing Galois connection between subsets of $\mathrm{Mult}(U_n,\dots,U_1;U_0)$ and $\prod_i \mathrm{End}(U_i)$ given by

$$S \subset \boldsymbol{T}(P,\Omega) \iff \Omega \subset \boldsymbol{Z}(S,P) \tag{3.1}$$

From [FMW20] the set $\boldsymbol{T}(P,\Omega)$ is a vector subspace and $\boldsymbol{Z}(S,\boldsymbol{d})$ is a Lie algebra for $\boldsymbol{d} = x_n + \cdots + x_0$.

Definition 3.2. *(Derivation closure) Let $t \in \mathrm{Mult}(U_n,\dots,U_1;U_0)$. Then $(\!|t|\!)$, the derivation closure of t, is the vector subspace consisting of all s such that $\mathrm{Der}(t) \subset \mathrm{Der}(s)$. Hence $(\!|t|\!) := \boldsymbol{T}(\boldsymbol{d},\boldsymbol{Z}(\boldsymbol{d},t))$.*

Example 3.3. *Let t be the matrix multiplication tensor for 2×3 and 3×4 rectangular matricies. That is, $\langle t| : K^{2\times 3} \times K^{3\times 4} \rightarrowtail K^{2\times 4}$ by $\langle t|M,N\rangle := MN$. Then by Corollary 8.4.4 of [FMW20], we have $(\!|t|\!)$ as a 1-dimensional vector subspace spanned by Kt.*

Definition 3.4. *(Soldering of tensors)*

Let $s \in \mathrm{Mult}(U_n,\dots,U_1;U_0) =: S$ and $t \in \mathrm{Mult}(V_n,\dots,V_1;V_0) =: T$. Define the multilinear interpretation on $S\otimes T$ as $\langle s\otimes t| : \prod_i U_i \otimes V_i \rightarrowtail U_0\otimes V_0$ where $\langle s\otimes t|u_1\otimes v_1,\dots,u_n\otimes v_n\rangle = \langle s|u_1,\dots,u_n\rangle \otimes \langle t|v_1,\dots,v_n\rangle$. We say $s\otimes t$ is the soldering of tensors s and t.

Remark 3.5. *We wish to compute $(\!|s\otimes t|\!)$, which requires $s\otimes t$ to be a part of the Galois connection of Definition 3.1. For that we need $s\otimes t \in \mathrm{Mult}(W_n,\dots,W_1;W_0)$ for some vector spaces $W_n,\dots,W_0$. This is accomplished by the following sequence of natural isomorphisms, starting with $s\otimes t \in \mathrm{Mult}(U_n,\dots,U_1;U_0)\otimes\mathrm{Mult}(V_n,\dots,V_1;V_0)$:*

$$\begin{aligned}\mathrm{Mult}(U_n,\dots,U_1;U_0)\otimes\mathrm{Mult}(V_n,\dots,V_1;V_0) &\cong (U_n^*\otimes\cdots\otimes U_1^*\otimes U_0)\otimes(V_n^*\otimes\cdots\otimes V_1^*\otimes V_0)\\ &\cong (U_n^*\otimes V_n^*)\otimes\cdots\otimes(U_1^*\otimes V_1^*)\otimes(U_0\otimes V_0)\\ &\cong \mathrm{Mult}(U_n\otimes V_n,\dots,U_1\otimes V_1;U_0\otimes V_0)\end{aligned}$$

Example 3.6. *The Kronecker product of matricies is a case of soldering tensors. Let $M \in \mathrm{Mult}(K^2,K^2;K)$ and $N \in \mathrm{Mult}(K^2,K^2;K)$ each be given as 2×2 matricies.*

$$M = \begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix} \qquad N = \begin{bmatrix}3 & 4\\ 5 & 6\end{bmatrix}$$

Let $\mathrm{Kron}(M,N)$ be the Kronecker product of matricies M and N.

$$\mathrm{Kron}(M,N) = \left(\begin{array}{cc|cc} 3 & 4 & \multicolumn{2}{c}{\multirow{2}{*}{$\mathbf{0}$}}\\ 5 & 6 & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{$\mathbf{0}$}} & 6 & 8\\ & & 10 & 12\end{array}\right)$$

The 4×4 matrix $\mathrm{Kron}(M,N)$ is an element of $\mathrm{Mult}(K^2,K^2;K)\otimes\mathrm{Mult}(K^2,K^2;K)$ given as an element of $\mathbb{M}_2(\mathbb{M}_2(K))$. It has interpretation $\langle M\otimes N| : (K^2\otimes K^2),(K^2\otimes K^2) \rightarrowtail K$ given by mapping basis element $e_i\otimes e_j$ to the $2(i-1)+j$th row or column of the matrix. Specifically, $\langle M\otimes N|e_i\otimes e_j, e_k\otimes e_l\rangle$ is row $2i-1+j$, column $2k-1+l$ of $\mathrm{Kron}(M,N)$, giving the product of the (i,k)th row of M with the (j,l)th row of N.

Remark 3.7. *In Definition 3.4 the tensors s and t had to have the same valence, and that axis U_i of s is matched with axis V_i of t in the soldered output.*

Both are convinent but not necessary for soldering. A tensor can be padded by axes consisting of K due to the isomorphism $A\otimes K \cong A$. Not matching U_i with V_i brings combinatorial explosion to soldering. We defer further investigations to future work, except to say we do not rule out this possibility when X-raying soldered tensors.

3.2. **Existing techniques.** Next we describe some existing techniques that analyze soldered tensors. Exposition follows [Wil16].

Example 3.8 (Tensor over centroid). *Let $s \in \mathbb{F}_3^{2\times2\times2}$ be the tensor given as a system of forms*

$$s_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad s_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

with interpretation $\langle s| : \mathbb{F}_3^2 \times \mathbb{F}_3^2 \rightarrowtail \mathbb{F}_3^2$. This is the multiplication table of the quotient ring $\mathbb{F}_3[x]/(x^2 + 1) \cong \mathbb{F}_9$, and has centroid isomorphic to $\mathbb{F}_9$.

Let t be the 2×2 identity matrix, interpreted as the bilinear form $\langle t| : \mathbb{F}_3^2 \times \mathbb{F}_3^2 \rightarrowtail \mathbb{F}_3$. Let $r = s \otimes t$ be the soldering of s and t. Computing, $\mathrm{Cen}(r) \cong \mathbb{F}_9$. Viewing r as framed by $\mathbb{F}_9$ vector spaces, we recover that r is a bilinear over $\mathbb{F}_9$.

Example 3.9 (Condensation by full idempotents). *Let $\langle s| : U \times K \rightarrowtail U$ be defined as $\langle s|u, k\rangle = uk$, scalar multiplication by K on the right. Let $\langle t| : K \times V \rightarrowtail V$ be scalar multiplication by K on the left. The soldering $\langle s \otimes t| : (U \otimes K) \times (K \otimes V) \rightarrowtail (U \otimes V)$ is isotopic to $\varphi : U \times V \rightarrowtail U \otimes V$, the canonical map into the tensor product.*

Let $\mathrm{Nuc}_{20}(\varphi) := \mathcal{L}$ be defined as per Equation (2.3)*, consisting of all $(\lambda_2, \lambda_0) \in \mathrm{End}(U) \times \mathrm{End}(U \otimes V)$ satisfying $\langle \varphi|\lambda_2 u, v\rangle = \lambda_0\langle \varphi|u, v\rangle$. Suppose e is a full idempotent of $\mathcal{L}$, meaning $e^2 = e$ and $\mathcal{L}e\mathcal{L} = \mathcal{L}$. Then the bimap $*$ condenses as $\tilde{*} : eU \times V \rightarrowtail eW$ where $\mathrm{Nuc}_{20}(\tilde{*}) = e\mathcal{L}e$. This is of interest because $e\mathcal{L}e$ is Morita equivalent to $\mathcal{L}$ but smaller. When U and V are given coordinates, $\mathcal{L} = (\mathbb{M}_n(K), \mathbb{M}_n(K) \otimes I_m(K))$ so $e = (E_{11}, E_{11} \otimes I_m(K))$ is a full idempotent and condenses φ down to scalar multiplication on the left.*

In [Wil16] this technique is used to prove a property of automorphisms of bimaps. Applying the same technique for the algebras $\mathrm{Adj}(t)$ and $\mathrm{Nuc}_{10}(t)$ to Example 3.3 compresses the rectangular matrix multiplication tensor multiplication of the underlying field.

3.3. **X-raying soldered tensors.** We are interested in the derivation closure of a soldered tensor as the first step towards understanding them. That is, given $s \in \mathrm{Mult}(U_n, \dots, U_1; U_0)$ and $t \in \mathrm{Mult}(V_n, \dots, V_1; V_0)$, do $(\!|s|\!)$ and $(\!|t|\!)$ relate to $(\!|s \otimes t|\!)$? We have recently resolved this question. Below is a preliminary lemma.

Lemma 3.10. *For $s \in \mathrm{Mult}(U_n, \dots, U_1; U_0)$ and $t \in \mathrm{Mult}(V_n, \dots, V_1; V_0)$, there are embedding $\iota_s : \mathrm{Der}(s) \hookrightarrow \mathrm{Der}(s \otimes t)$ given by $(\sigma_i)_{i\in[n]} \mapsto (\sigma_i \otimes I_{V_i})_{i\in[n]}$ and $\iota_t : \mathrm{Der}(t) \hookrightarrow \mathrm{Der}(s \otimes t)$ given by $(\tau_i)_{i\in[n]} \mapsto (I_{U_i} \otimes \tau_i)_{i\in[n]}$.*

Proof. The map ι_i is injective on each factor as tensoring with the identity morphism is injective. The endomorphism $(\sigma_i \otimes I_{V_i})_{i\in[n]}$ is in $\mathrm{Der}(s \otimes t)$ as $\langle s \otimes t|\mathbf{d}((\sigma_i \otimes I_{V_i})_{i\in[n]}) = \sum_{i=1}^{n}\langle s|\sigma_i \otimes \langle t|I_{V_i} = \langle s|\mathbf{d}((\sigma_i)_{i\in[n]}) \otimes \langle t| = 0$. Lastly, we need to demonstrate ι_i is a map of Lie algebras. This follows by the calculation $\iota(\delta + \rho) = \iota((\delta_i + \rho_i)_{i\in[n]}) = ((\delta_i + \rho_i) \otimes I_{V_i})_{i\in[n]} = (\delta_i \otimes I_{V_i})_{i\in[n]} + (\rho_i \otimes I_{V_i})_{i\in[n]} = \iota(\delta) + \iota(\rho)$. The case for ι_j is analogous. □

Theorem 3.11. *Let $s \in \mathrm{Mult}(U_n, \dots, U_1; U_0) =: U$ and $t \in \mathrm{Mult}(V_n, \dots, V_1; V_0) =: V$. Then $(\!|s|\!) \otimes (\!|t|\!) \cong (\!|s \otimes t|\!)$.*

Proof. Our strategy is to show $(\!|s \otimes t|\!) \subset (\!|s|\!) \otimes (\!|t|\!)$ and $(\!|s|\!) \otimes (\!|t|\!) \subset (\!|s \otimes t|\!)$.

$(\!|s \otimes t|\!) \subset (\!|s|\!) \otimes (\!|t|\!)$:

By Lemma 3.10, there are embeddings $\iota_s : \mathrm{Der}(s) \hookrightarrow \mathrm{Der}(s \otimes t)$ and $\iota_t : \mathrm{Der}(t) \hookrightarrow \mathrm{Der}(s \otimes t)$. The inclusion reversing nature of the antitone Galois connection in Definition 3.1 implies $(\!|s \otimes t|\!) = T(\mathbf{d}, \mathrm{Der}(s\otimes t)) \subset T(\mathbf{d}, \iota_s(\mathrm{Der}(s)))$ and $(\!|s\otimes t|\!) \subset T(\mathbf{d}, \iota_t(\mathrm{Der}(t)))$. Thus $(\!|s\otimes t|\!)$ is in their intersection. We shall prove $T(\mathbf{d}, \iota_s(\mathrm{Der}(s))) = (\!|s|\!) \otimes V$ and $T(\mathbf{d}, \iota_j(\mathrm{Der}(t))) = U \otimes (\!|t|\!)$. The conclusion follows as $(\!|s \otimes t|\!) \subset (\!|s|\!) \otimes V \cap U \otimes (\!|t|\!) = (\!|s|\!) \otimes (\!|t|\!)$.

The statement to prove is $T(\mathbf{d}, \iota_s(\mathrm{Der}(s))) = (\!|s|\!) \otimes V$.

We first show the direction $(\!|s|\!) \otimes V \subset T(\mathbf{d}, \iota_s(\mathrm{Der}(s)))$. As $(\!|s|\!) \otimes V$ is generated by $\acute{s} \otimes t$ for $\acute{s} \in (\!|s|\!)$ and $t \in V$, it suffices to show $\acute{s} \otimes t \in T(\mathbf{d}, \iota_s(\mathrm{Der}(S)))$. This follows as $\langle \acute{s} \otimes t|$ satisfies $\langle \acute{s} \otimes t|\mathbf{d}(\delta) = 0$ for all $\delta = (\sigma_i \otimes I_{V_i})_{i\in[n]} \in \iota_s(\mathrm{Der}(s))$ since $(\sigma_i)_{i\in[n]} \in \mathrm{Der}(s)$.

In the opposite direction, $T(\mathbf{d}, \iota_s(\mathrm{Der}(s)))$ is a subspace of $U \otimes V$. We shall show in fact it is the subspace $(\!|s|\!) \otimes V$ by showing every element in $T(\mathbf{d}, \iota_s(\mathrm{Der}(s)))$ is the sum of pure tensors $\acute{s} \otimes t$ for $\acute{s} \in (\!|s|\!), t \in V$.

Let $r = \sum_{i=1}^{m} s_i \otimes t_i \in U \otimes V$ be an element of $T(\mathbf{d}, \iota_s(\mathrm{Der}(s)))$, with all t_is linearly independent. Showing each s_i is in $(\!|s|\!)$ concludes r is in $(\!|s|\!) \otimes V$. By definition r satisfies $\langle r|\mathbf{d}(\iota_s(\sigma)) = 0$ for all $\sigma \in \mathrm{Der}(s)$.

Let $\sigma \in \mathrm{Der}(s)$ and $\iota_s(\sigma) = (\sigma_j \otimes I_{V_j})_{j\in[n]}$. Computing,

$$
\begin{aligned}
0 &= \langle r|\mathbf{d}(\iota_s(\sigma)) \\
&= \left\langle \sum_i s_i \otimes t_i \right| \mathbf{d}(\iota_s(\sigma)) \\
&= \sum_i \langle s_i \otimes t_i|\mathbf{d}(\iota_s(\sigma)) \qquad \text{linearity of tensor evaluation} \\
&= \sum_i \left(\langle s_i \otimes t_i| \left(\sum_{j=1}^{n} \sigma_j \otimes I_{V_j} \right) \right) \\
&= \sum_i \left(\sum_{j=1}^{n} \left(\langle s_i \otimes t_i|\sigma_j \otimes I_{V_j} \right) \right) \\
&= \sum_i \left(\sum_{j=1}^{n} \left(\langle s_i|\sigma_j \otimes t_i \right) \right) \\
&= \sum_i \left(\sum_j \langle s_i|\sigma_j \right) \otimes t_i \\
&= \sum_i \langle s_i|\mathbf{d}(\sigma) \otimes t_i
\end{aligned}
$$

Let $\{F_b\}_{b\in\mathcal{B}}$ be a basis of V. Then expanding each t_i in this basis,

$$
\begin{aligned}
0 &= \sum_i \left(\langle s_i|\mathbf{d}(\sigma) \otimes \left(\sum_{b\in\mathcal{B}} \lambda_{ib} F_b \right) \right) \\
&= \sum_{b\in\mathcal{B}} \left(\sum_i \left(\lambda_{ib} \langle s_i|\mathbf{d}(\sigma) \right) \otimes F_b \right)
\end{aligned}
$$

Since F_b are linearly independent, $\sum_i \lambda_{ib}\langle s_i|\mathbf{d}(\sigma) = 0$. Since t_is are linearly independent, the m by $|\mathcal{B}|$ matrix $[\lambda_{ib}]$ has full row rank. Thus the only way $\sum_i \lambda_{ib}\langle s_i|\mathbf{d}(\sigma) = 0$ is if $\langle s_i|\mathbf{d}(\sigma) = 0$ for all i. Thus $s_i \in (\!|s|\!)$ for all i. This concludes the proof of $T(\mathbf{d}, \iota_s(\mathrm{Der}(s))) = (\!|s|\!) \otimes V$.

The statement $T(\mathbf{d}, \iota_t(\mathrm{Der}(t))) = U \otimes (\!|t|\!)$ is proven analogously.

$(\!|s|\!) \otimes (\!|t|\!) \subset (\!|s \otimes t|\!)$:

The strategy will be to first show for $\acute{s} \in (\!|s|\!)$, that $\acute{s}\otimes t \in (\!|s\otimes t|\!)$, and secondly, show if $\acute{s}\otimes t \in (\!|s\otimes t|\!)$ for all s, then for all $\acute{t} \in (\!|t|\!)$, that $\acute{s} \otimes \acute{t} \in (\!|s \otimes t|\!)$. The proof concludes as $(\!|s|\!) \otimes (\!|t|\!)$ is generated by $\acute{s} \otimes \acute{t}$ for $\acute{s} \in (\!|s|\!)$ and $\acute{t} \in (\!|t|\!)$, and

To show $\acute{s} \otimes t \in (\!|s \otimes t|\!)$, let $\delta \in \mathrm{Der}(s \otimes t)$. Since δ is an element of $\prod_i \mathfrak{gl}(U_i \otimes V_i) \cong \prod_i (\mathfrak{gl}(U_i) \otimes \mathfrak{gl}(V_i))$, write δ as $\left(\sum_{j=1}^{R_i} (\sigma_j \otimes \tau_j)\right)_{i\in[n]}$. By construction $\langle s \otimes t|\mathbf{d}(\delta) = 0$. Calculating,

$$
0 = \langle s \otimes t|\mathbf{d}(\delta)
$$

$$
\begin{aligned}
&= \langle s \otimes t | \sum_{i=1}^{n} \left(\sum_{j=1}^{R_i} (\sigma_j \otimes \tau_j) \right) \\
&= \langle s \otimes t | \sum_{a \in A} (\sigma_a \otimes \tau_a) \qquad \text{grouping into one indexing set} \\
&= \sum_{a \in A} \langle s | \sigma_a \otimes \langle t | \tau_a \\
&= \sum_{a \in A} \langle s | \sigma_a \otimes \left(\sum_{b \in \mathcal{B}} \lambda_{ab} F_b \right) \qquad \text{For } F_b \text{ basis of } V \\
&= \sum_{b \in \mathcal{B}} \left(\sum_{a \in A} \lambda_{ab} \langle s | \sigma_a \right) \otimes F_b
\end{aligned}
$$

As $\{F_b\}_{b\in\mathcal{B}}$ is a basis of V, $\sum_{a\in A} \lambda_{ab}\langle s|\sigma_a = 0$ for each b. Regrouping and combining the terms in the indexing set A by axes, we have $\langle s|\sigma = 0$, meaning $\sigma \in \mathrm{Der}(s)$. Thus $\langle \acute{s}|\sigma = 0$ as well. Substituting $\acute{s}$ in place of s in the above equation also equals 0, concluding $\langle \acute{s} \otimes t|\mathbf{d}(\delta) = 0$.

The proof that $\acute{s} \otimes \acute{t} \in (\!|s \otimes t|\!)$, assuming $\acute{s} \otimes t$ is in $(\!|s \otimes t|\!)$ is analogous.

□

Example 3.12. *Let $\langle s| : \mathfrak{sl}_2 \times \mathfrak{sl}_2 \rightarrowtail \mathfrak{sl}_2$ be the multiplication tensor of $\mathfrak{sl}_2$. We compute $(\!|s|\!) = Ks$. Let $\langle t| : \mathbb{F}_3^2 \times \mathbb{F}_3^2 \rightarrowtail \mathbb{F}_3^2$ be the multplication table of $\mathbb{F}_3[x]/(x^2+1)$ as given in Example 3.8. We compute $(\!|t|\!)$ as spanned by t and the tensor r given by the system of forms $\left[\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\right]$.*

As expected, $(\!|s|\!) \otimes (\!|t|\!)$ and $(\!|s \otimes t|\!)$ are both 2 dimensional. By definition $(\!|s|\!) \otimes (\!|t|\!)$ is spanned by $s \otimes t$ and $s \otimes r$, while computation verifies $(\!|s \otimes t|\!)$ is spanned by these same tensors.

Now we describe the main problem under consideration.

Problem B (X-raying soldered tensors).
Given** $r \in \mathrm{Mult}(W_n, \dots, W_1; W_0)$**, what are necessary and sufficient conditions to concludes** $r \cong s \otimes t$**, the soldering of two tensors in a non-trivial manner?

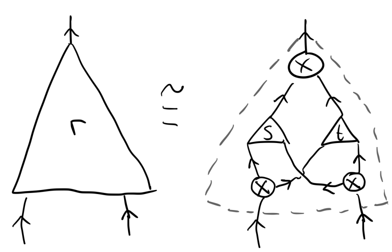

FIGURE 2. Pictorial illustration of Problem B. We are given is the tensor r. We'd like to find non-trivial s, t such that $r \cong s \otimes t$. The dashed lines indicate if successful, while we don't get to view the internals of r, we do understand how r is built.

Non-trivial means when $r \cong s \otimes t$, for $s \in \mathrm{Mult}(U_n, \dots, U_1; U_0) =: U$ and $t \in \mathrm{Mult}(V_n, \dots, V_1; V_0) =: V$, then we require $\dim U > 1$, and $\dim V > 1$. Otherwise, as discussed in Remark 3.7, for $\langle s| : K \times \cdots K \rightarrowtail K$ is the n-tensor for field multiplication, then $r \cong s \otimes r$. Examples in Section 2.1 give some partial answers using the centroid, adjoint algebra, and nuclei. We'd like to formalize these results and extend to cases beyond the associative case, with the derivation algebra as our primary tool using tools like Theorem 3.11.

One thing deserves further clarification. For $s \in \mathrm{Mult}(U_n, \dots, U_1; U_0) =: U$ and $t \in \mathrm{Mult}(V_n, \dots, V_1; V_0) =: V$, by Definition 3.4, $s \otimes t$ is an element in $U \otimes V$. To say $r \cong s \otimes t$ necessitates a map $U \otimes V \to$

$\mathrm{Mult}(W_n, \dots, W_1; W_0)$. Computationally this runs into combinatorial explosion of matching axes. But we would like to defer resolving this to after proving certain structure theorems and needing to implement them in algorithms.

As part of answering this problem, we plan to analyze the derivation algebras themselves to understand the tensor. That is, **given** $s \in \mathrm{Mult}(U_n, \dots, U_1; U_0)$ **and** $t \in \mathrm{Mult}(V_n, \dots, V_1; V_0)$**, is** $\mathrm{Der}(s \otimes t)$ **completely determined by** $\mathrm{Der}(s)$ **and** $\mathrm{Der}(t)$**?**

This question has an affirmative answer for the case of adjoints, as $\mathrm{Adj}(s \otimes t) = \mathrm{Adj}(s) \otimes \mathrm{Adj}(t)$ [Wil09a]. Nothing is known for $\mathrm{Der}(s \otimes t)$ other than Lemma 3.10 above. Preliminary computations suggests the embeddings of $\mathrm{Der}(s)$ and $\mathrm{Der}(t)$ do not behave like the adjoint case, as the dimension of $\mathrm{Der}(s \otimes t)$ in some computed examples is not the product of $\dim \mathrm{Der}(s)$ and $\dim \mathrm{Der}(t)$.

We want to allow for soldering using the (P, Ω)-tensor product, which we define below.

Definition 3.13. *((P, Ω)-tensor product) [FMW20]*

Let $U_1, \dots, U_n$ be K-vector spaces, $P \subset K[x_1, \dots, x_n]$ and $\Omega \subset \mathrm{End}(U_1) \times \cdots \times \mathrm{End}(U_n)$. Define the following subspace of $U_1 \otimes \cdots \otimes U_n$

$$\Xi(P, \Omega) := \left\langle \sum_e \lambda_e \omega_1^{e_1} u_1 \otimes \cdots \otimes \omega_n^{e_n} u_n \,\middle|\, \omega \in \Omega, \sum_e \lambda_e x_1 e^{e_1} \cdots x_n^{e_n} \in P, u_i \in U_i \right\rangle.$$

Define the (P, Ω)-tensor product space as the quotient space

$$\blacktriangleleft U_1, \dots, U_n \blacktriangleright_\Omega^P := (U_1 \otimes \cdots \otimes U_n)/\Xi(P, \Omega),$$

together with a K-multilinear map $\blacktriangleleft \cdots \blacktriangleright : U_1 \times \cdots \times U_n \rightarrowtail \blacktriangleleft U_1, \dots, U_n \blacktriangleright_\Omega^P$, where $\blacktriangleleft u_1, \dots, u_n \blacktriangleright = u_1 \otimes \cdots \otimes u_n + \Xi(P, \Omega)$.

Notice $\blacktriangleleft U_1, \dots, U_n \blacktriangleright_\emptyset^\emptyset$ is the usual tensor product of vector spaces. Let $P \subset K[x_1, \dots, x_n]$ and $\Omega \subset \prod_i \mathrm{End}(U_i)$. Suppose each U_i is a tensor space $\mathrm{Mult}(V_i^m, \dots, V_i^1; K) \cong \left(\bigoplus_j V_i^j\right)^*$, and an isomorphism $V_i^j \cong (V_i^j)^*$ is specified. Let $(s_1, \dots, s_n) \in \prod_i U_i$ be a tuple of tensors. Then the image of $(s_1, \dots, s_n)$ under a (P, Ω)-tensor product, denoted $\blacktriangleleft s_1, \dots, s_n \blacktriangleright =: r$, is an element of a quotient space $\bigotimes_i U_i / \Xi(P, \Omega)$.

Let $\tilde{r}$ be an element of $\bigotimes_i U_i$ identified to r by fixing a complementary subspace to $\Xi(P, \Omega)$ in $\bigotimes_i U_i$. As $\bigotimes_i U_i = \bigotimes_i \bigotimes_j V_i^j \cong \bigotimes_j \bigotimes_i V_i^j$, $\tilde{r}$ can be identified with a multilinear map in $\mathrm{Mult}(\bigotimes_i V_i^1, \dots, \bigotimes_i V_i^m; K)$. This gives a multilinear interpretation $\langle r| : \prod_{j=1}^m \bigotimes_i V_i^j \rightarrowtail K$.

Challenge B. *The above generalizes of the soldering of tensors s and t to (P, Ω)-tensor products. We wish to investigate its properties and prove analogous statements for (P, Ω)-soldered tensors.*

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