Counting Problem of *PureCircuit*

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***Abstract*—Test.Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris. Nam arcu libero, nonummy eget, consectetuer id, vulputate a, magna. Donec vehicula augue eu neque. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Mauris ut leo. Cras viverra metus rhoncus sem. Nulla et lectus vestibulum urna fringilla ultrices. Phasellus eu tellus sit amet tortor gravida placerat. Integer sapien est, iaculis in, pretium quis, viverra ac, nunc. Praesent eget sem vel leo ultrices bibendum. Aenean faucibus. Morbi dolor nulla, malesuada eu, pulvinar at, mollis ac, nulla. Curabitur auctor semper nulla. Donec varius orci eget risus. Duis nibh mi, congue eu, accumsan eleifend, sagittis quis, diam. Duis eget orci sit amet orci dignissim rutrum.**

***Index Terms*—Complexity Theory, Counting Complexity, Hazard Free Circuits, TFNP, PPAD, Search Problems, Kleene Logic**

I. Introduction

Over the last couple of years, there has been a revolutionary initiative in the field of combinatronics. Combinatronics has been a field of study in mathematics that primarly focused on the notion of counting objects with certain properties. Over time, this notion has shifted, especially in the subfield of algebraic combinatronics, where there is no clear notion of the object that we are counting and the numbers express something more abstract [1]. This gave a need to be able to assign a combinatorial interpretation to such numbers, or more simply, do these numbers correpsond to some counting over a set of objects. Being able to find such definitions or interpretations can be very important, it allows us to utilise tools from combinatronics as well as allow us to understand and reveal hidden structures and properties for such numbers [1]. Moreover there are several problems or numbers such as *Kronecker coefficients*, whose combinatorial interpretation, would give a step towards the resolution of the $P \neq NP$ conjecture [2].

To reiterate the previous statement, we can understand combinatorial interpretation as the process of: given a sequence of numbers $\{a_x\}$, find a set of combinatorial objects A_x such that $|A_x| = a_x$ To formalise the current idea, Igor Pak et al. has concluded that $f \in$ *#P*, implies that f has a combinatorial interpretation [1], [2]. We will explore this idea in much greater detail in the upcoming section, but the main benefit is the ability to use a very expressive but formal language that encapsulates this notion of a combinatorial interpretation.

In our current work, we focus on extending the work done by Ikenmeyer et al., where they focused on the creation of frameworks that determine whether $f \in^{?}$ **#P**, by looking at the complexity class of #*TFNP -1*. This is a class of problems that are guaranteed to have a solution and their solutions are verifiable in polynomial time. In their paper, they were able to show that for a subclass of problems, also known as *PPAD* $\subseteq$ *TFNP*, different *PPAD-complete* problems, may or may not have a combinatorial interpretation. Our contribution, comes to the analysis of a specific problem, known as PureCircuit, which utilises Kleene logic, to find satisfying assingments in sequential circuits. We hope to demonstrate that such problem could help us bound, the counting complexity limits of #PureCircuit $-\ 1$.

A. Project objectives

Below we will present our table of objectives. We will denote updated objectives with *, new objectives with (!), completed objectives with (✓), deleted objectives with (-).

R.1) (✓) Find a parsimonious reduction from the *EndOfLine* to *EndOfLine*.
R.2) (!) Improve the combinatorial bound between the reduction from EndOfLine to PureCircuit
R.3) (!) Demonstrate that #PPAD$(PureCircuit) - 1 \nsubseteq$ **#P**
R.4) (*) Prove or disprove the following $\forall n \in \mathbb{N}_{\geq 2}$:

$$\exists c \in \mathbb{N} : \#\textsc{SourceOrExcess}(n, 1) \subseteq^{c} \#\textsc{PureCircuit}$$

R.5) (*) Prove or disprove the following claim: $\forall L \in$ **PPAD**

$$\exists c \in \mathbb{N} : \#L \subseteq^{c} \#\textsc{PureCircuit}$$

Below we will be representing the development portion of the project.

S.1) Visualise a pure circuit
S.2) Generate a solution
S.3) Count number of solutions for smaller scales

We will compile the rest of the report, based on our current findings, how we modified our objectives to the current ones as well as general reflections.

B. Core set of results

Below we will present our main list of findings up to this point:

Definition 1.1 (Poly-Function Bounded Parsimonious Reductions): Given two counting problems $A, B : \{0, 1\}* \to \mathbb{N}$ and a function $f : 0, 1^* \to \mathbb{N}$ such that $f \in n^{O(1)}$, we say that:

$$A \subseteq^{f} B$$

If for input $w \in \{0,1\}^*$, if a represent the number of solutions for problem A and b the number of solutions for problem B, we have:

$$a \leq b \leq f(|w|) \cdot a$$

Theorem 1.1 (ND-Brouwer to PureCircuit): Given $f(x) \triangleq 20$

$$\#\textsc{Brouwer}^{f_3} \subset^f \#\textsc{PureCircuit}$$

Corollary 1.1.1: For any $d \in \mathbb{N}_{\geq 2}$, we can define $f(\cdot) = \sum_{i=1}^{d-1} \binom{d}{i} 2^i$ such that:

$$\#\textsc{Brouwer}^{d} \subset^f \#\textsc{PureCircuit}$$

Theorem 1.1: For any $d \in \mathbb{N}_{\geq 2}$, we can define $f(\cdot) = 5$ such that:

$$\#\textsc{Brouwer}^{d} \subset^f \#\textsc{PureCircuit}$$

Throughout our search, we stumbled upon various variants of the problem as well as reductions and claims from other subclasses of PPAD or Hazard-Free logic

Proposition 1.2: Weaker variants of PureCircuit can result to parsimonious reductions to the EndOfLine problem as such:

$$\#\textsc{AcyclicBPureCircuit} \subseteq \#\textsc{EndOfLine}$$
$$\#\textsc{PermutationFreeBPureCircuit} \subseteq \#\textsc{EndOfLine}$$

In addition, we were able to show that for different promise problems, we can find reductions to the PureCircuit problem

Proposition 1.3:

$$\#\textsc{PUnsatHazard} \subseteq \#\textsc{PureCircuit}$$

Lastly we introduce the following proposition

Proposition 1.4: Given function $F : \mathbb{T}^n \to \mathbb{T}^n$ where $\mathbb{T} \triangleq \{0, 1, \perp\}$ and F is a **natural** function:

$$\exists x^\star \in \mathbb{T}^n : F(x^\star) = x^\star$$

A simpler version of the proposition above has been demonstrated in other works such as **??**

II. Preliminaries and Background review

As mentioned in the introduction, the current project works as an interaction of three different fields: 1. counting complexity and combinatorial interpretation, 2. total search problems and PPAD and 3. Kleene logic.

III. Research Methodology

IV. Project Plan

V. Conclusion

References

[1] I. Pak, "What is a combinatorial interpretation?" Sep. 2022.
[2] C. Ikenmeyer and I. Pak, "What is in #P and what is not?" in *2022 IEEE 63rd Annual Symposium on Foundations of Computer Science (FOCS)*, Oct. 2022, pp. 860–871.