CS907 Interim Report

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Abstract

In the world of combinatronics, one often looks for the combinatronic interpretation between numbers. Recently, through the works of Igor Pak and other researchers, people have decided to combine the worlds of counting complexity theory with combinatronics to give formal definition to this idea [1]–[3]. Our project focuses on analysing the combinatorial properties of PPAD $-$ 1 using the PURECIRCUIT problem as well as developing a visualisation tool for it. In this report, we give a progress report as to the work that we have done as well as talk about notable results such as parsimonious reductions between Sperner problems to PURECIRCUIT up to some constant bound.

I. INTRODUCTION

Over the last couple of years, there has been a revolutionary initiative in the field of combinatronics. Combinatronics has been a field of study in mathematics that primarily focused on the notion of counting objects with certain properties. Over time, this notion has shifted, especially in the subfield of algebraic combinatorics, where there is no clear notion of the object that we are counting, and the numbers express something more abstract [3]. This gave a need to be able to assign a combinatorial interpretation to such numbers. Being able to find such definitions or interpretations can be very important as it allows us to utilise tools from combinatorics and allow us to understand and reveal hidden structures and properties [3]. Moreover, there are several problems or numbers such as *Kronecker coefficients* [4], whose combinatorial interpretation, would give a step towards the resolution of the $P \neq NP$ conjecture [1], [5].

In our current work, we focus on extending the work done by Ikenmeyer et al., where they focused on the creation of frameworks that determines whether $f \in^? $ **#P**, by looking at the subclasses of #TFNP problems. In our work we focus specifically on the PPAD class of problems, under the lens of PureCircuit, which utilises sequential Kleene-based circuits, to find satisfying assignments. We hope to uncover many insights of the #PURECIRCUIT $-$ 1 problem as well as the limits of #PPAD $-$ 1.

A. Project objectives

Below we will present our table of objectives. We will denote updated objectives with (*), new objectives with (!), completed objectives with (✓).

R.1) Find a parsimonious reduction from the *EndOfLine* to *PureCircuit*.
R.2) Improve the combinatorial bound between the reduction from EndOfLine to PureCircuit
R.3) Demonstrate that #PPAD$(PureCircuit) - 1 \not\subseteq$ **#P**
R.4) Prove or disprove the following

$$\forall n \in \mathbb{N}_{\geq 2} \exists c \in \mathbb{N} : \#\textsc{SourceOrExcess}(n, 1) \subseteq^c \#\textsc{PureCircuit}$$

Below we will be representing the development portion of the project.

S.1) Visualise and verify a PURECIRCUIT instance.
S.2) Generate a solution given a PURECIRCUIT instance for a small number of nodes.
S.3) Count the number of solutions of a PURECIRCUIT instance for a small number of nodes.

We will compile the rest of the report, based on our current findings, how we modified our objectives to the current ones as well as general reflections.

B. Research Question

Our main objectives revolve around the conjecture in I.1, where we investigate the boundaries of #PPAD $-$ 1 with the help of the PURECIRCUIT problem.

Conjecture I.1: #PPAD $-$ 1 **hardness**

Every lanaguage in PPAD can be parsimoniously reduced up to some polynomial factor, to the pure circuit problem, or more formally:

$$\forall L \in \text{PPAD}, \exists f \in n^{O(1)} : \#L \subseteq^f \#\textsc{PureCircuit}$$

II. Preliminaries and Background review

A. Important Notation

To avoid ambiguity, we introduce the following notation conventions used throughout the paper. For any $n \in \mathbb{N}$, we write $[n] \triangleq \{1, \ldots, n\}$ and $[n]_0 \triangleq [n] \cup \{0\}$. We define the Boolean domain as $\mathbb{B} \triangleq \{0, 1\}$ and three value domain $\mathbb{T} \triangleq \{0, 1, \perp\}$.

B. Background overview

1) Counting Complexity and Combinatorial Interpretations: Counting complexity looks into complexity of counting solutions using notions such as **#P**. **#P** was created by [6], to demonstrate the difficulty of counting the number of solutions to a problem, even they are polynomially verifiable.

Definition II.1: #P Complexity Clas [6]

#P is a class of functions $f : \{0,1\}^* \to \mathbb{N}$ such that: there exists a polynomial-time deterministic TM M, and $p : \mathbb{N} \to \mathbb{N}$ such that $p \in n^{O(1)}$, we have:

$$f(w) = \left|\left\{v \in \{0,1\}^{p(|w|)} \mid M(w, v) = 1\right\}\right|$$

As we can see, **#P** allows us to define a set of objects, whose cardinality equals $f(w)$. The reason for choosing **#P** to define combinatorial interpretations is [2]:

1) By polynomially bounding words, we avoid cases such as: $f(w) = |\{1, \ldots, f(w)\}|$.
2) Current framework allows us to work with $f(\cdot)$, whose direct computation can be computationally hard.

The current framework was used in several papers such as [1] and [2] where they were able to use tools from complexity theory to show that many structures do or do not have a combinatorial interpretation. For the purposes of the current project, we are focusing on [1], and TFNP problems. Lastly we introduce the idea of correlating two counting problems with the help of *parsimonious reductions* II.2.

Definition II.2: Parsimonious reductions

Let R, R' be search problems and let f be a reduction of $S_R = \{x \mid R(x) \neq \emptyset\}$ to $S_{R'} = \{x \mid R'(x) \neq \emptyset\}$. We say f is **parsimonious** if:

$$\forall x \in S_R : |R(x)| = |R'(f(x))|$$

Below we are introducing a variant of parimonious reductions that allows for one-to-many reductions.

Definition II.3: Poly-Function Bounded Parsimonious Reductions

Given two counting problems $A, B : \{0,1\}* \to \mathbb{N}$ and a function $f : 0, 1^* \to \mathbb{N}$ such that $f \in n^{O(1)}$, we say that:

$$A \subseteq^f B$$

If for input $w \in \{0,1\}^*$, if a represent the number of solutions for problem A and b the number of solutions for problem B, we have:

$$a \leq b \leq f(|w|) \cdot a$$

If $\forall x : f(x) = c$ for $c \in \mathbb{N}$ then we will just say

$$A \subseteq^c B$$

2) Total Search Problems and PPAD: We give a brief overview of **TFNP** and **PPAD**.

Definition II.4: Search Problems and Total Search Problems

Search problems can be defined as relations $R \subseteq \{0,1\}^* \times \{0,1\}^*$, where given $x \in \{0,1\}^*$, we want to find $y \in \{0,1\}^*$ such that xRy.
Total Search problems are search problems such that:

$$\forall x \in \{0,1\}^*, \exists y \in \{0,1\}^* : xRy$$

Definition II.5: FNP **and** TFNP

FNP are *search problems* such that there exists poly-time TM $M : \{0,1\}^* \to \{0,1\}$ and a poly function $p : \mathbb{N} \to \mathbb{N}$ such that:

$$\forall x \in \{0,1\}^*, y \in \{0,1\}^{p(|x|)} : xRy \iff M(x, y) = 1$$

Lastly $\textbf{TFNP} = \{L \in \textbf{FNP} \mid L \text{ is total}\}$

Our current work focuses on a specific subclass of **TFNP** problems which is defined as follows:

Definition II.6: *EndOfLine* problem [7]

Given circuits $S, P \in \{0,1\}^n \to \{0,1\}^n$ such that $S, P \in n^{O(1)}$ we define a directed graph $G = (V, E)$, such that $V = \{0,1\}^n$ and E defined as:

$$E = \{(x, y) \in V^2 : S(x) = y \wedge P(y) = x\}$$

We define source or sinks $\forall v \in V : deg(v) = (0, 1)$ or $deg(v) = (1, 0)$, respectively. We also syntactially ensure that the 0^n node is always a source, meaning $S(P(0^n) \neq 0 \wedge P(S(0^n)) = 0^n$. A node $v \in V$ is a solution if and only if $deg(v)$ is either $(0, 1)$ or $(1, 0)$.

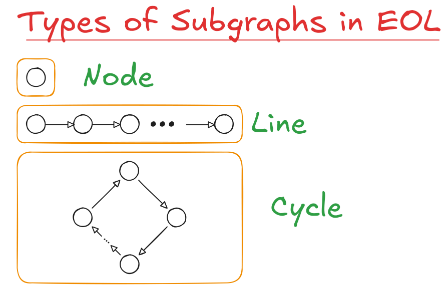


Fig. 1: Types of subgraphs in EndOfLine

An illustrative example of an instance can be seen in the figure 1. Using the EndOfLine problem, we define the PPAD complexity class II.7

Definition II.7: PPAD complexity class [7]

PPAD is defined as the set of search problems that are reducible to the EndOfLine problem II.6.

PPAD has been created by Papadimitriou [7] to demonstrate a subset of problems in **NP** that are guaranteed to have a solution but can be very difficult to find. In the next section we will introduce the relevant of PPAD problems.

a) The PureCircuit problem: The PureCircuit was created by Deligkas et al. [8] to demonstrate the hardness of approximating PPAD problems. it uses kleene-logic based circuits and with continuity of arguments, they demonstrated PPAD-completeness.

Definition II.8: PureCiruit Problem Definition [8]

An instance of *PureCircuit* is given by vertex set $V = [n]$ and gate set G such that $\forall g \in G : g = (T, u, v, w)$ where $u, v, w \in V$ and $T \in \{\text{NOR}, \text{Purify}\}$. Each gate is interpreted as:

1) *NOR*: Takes as input u, v and outputs w
2) *Purify*: Takes as input u and outputs v, w

And each vertex is ensured to have in-deg$(v) \leq 1$. A solution to input instance (V, G) is denoted as an assignment $\mathbf{x} : V \to \{0, \perp, 1\}$ such that for all nodes we have:

1) if v is the output of a $(\textit{NOR}, u, v, w)$ gate:

$$\mathbf{x}[u] = \mathbf{x}[v] = 0 \implies \mathbf{x}[w] = 1$$
$$(\mathbf{x}[u] = 1 \vee \mathbf{x}[v] = 1) \implies \mathbf{x}[w] = 0$$
$$\text{otherwise} \implies \perp$$

2) *Purify*:

$$\forall b \in \{0,1\} : \mathbf{x}[u] = b \implies \mathbf{x}[v] = b \wedge \mathbf{x}[w] = b$$
$$\mathbf{x}[u] = \perp \implies \{\mathbf{x}[v] \cup \mathbf{x}[w]\} \cap \{0, 1\} \neq \emptyset$$

The definition of PureCircuit that we will be using for the standard set of gates $\{\wedge, \vee, \neg\}$, is based Kleene's three-valued strong logic of indeterminancy which extends the traditional $\mathbb{B}$ logic [9]. Their behaviour can be described in the tables shown in I In addition to that, we will make use of the *Copy* gate which we can define as $\textit{Copy}(x) = \neg(\neg x)$. This allows our circuits to be robust and easier to work with.

`not`	
0	1
$\perp$	$\perp$
1	0

(a) `not` gate

`and`	0	$\perp$	1
0	0	0	0
$\perp$	0	$\perp$	$\perp$
1	0	$\perp$	1

(b) `and` gate

`or`	0	$\perp$	1
0	0	$\perp$	1
$\perp$	$\perp$	$\perp$	1
1	1	1	1

(c) `or` gate

TABLE I: Three-valued logic [9]

For the *Purify* gate, Deligkas et al. showed that the only solutions that are essential for *Purify* are $\{(0, \perp), (\perp, 1), (0, 1)\}$ with the help of continuity arguments [8]. Adding more solutions does not change the complexity as he showed in the original variant of the problem. We acknowledge that the solution set will be different from the original one but, one can observe that #PureCircuit-simplified $\subseteq$ #PureCircuit, and therefore any proposition or argument of the sort: #A $\subseteq$

#PURECIRCUIT-SIMPLIFIED $\implies$ #A $\subseteq$ #PURECIRCUIT. For the purposes of the report, any solution change will be made explicit and we will refer to all such simplified variants as #PURECIRCUIT to avoid confusion.

Definition II.9: SOURCEOREXCESS problem

We define as *SourceOrExcess*$(k, 1)$ for $k \in \mathbb{N}_{\geq 2}$ the search problem as such: Given a poly-sized successor circuit $S : \{0,1\}^n$ and a set of predecssor poly-sized circuits $\{P_i\}_{i \in [k]}$, we define the graph $G = (V, E)$ such that, $V = \{0,1\}^n$ and E as:

$$\forall x, y \in V : (x, y) \in E \iff (S(x) = y) \wedge \bigvee_{i \in [k]} P_i(y) = x$$

We ensure that 0^n is as sink, meaning $\deg(0^n) = (0, 1)$. A valid solution is a vertex v such that *in-deg*$(v) \neq$ *out-deg*(v)

b) Sperner problems: Below we will refer to the notion of Sperner problems which involve the idea of using the topology of a problem and a colouring scheme to ensure that a substructure is panchromatic. There two variants of colouring scheme that are used: one of them will be referred to as the **linear** colouring where for dimension d, assign $d + 1$ distinct colours to each point [10], [11]. Below we will refer to **bipolar** colouring, which has been used grid-like topologies of the Sperner property [8], [12], [13]. It has to be noted that these are not their official names, but we have decided to use this naming scheme for clarity.

Definition II.10: Bipolar colouring

Given dimension d, we refer to the bipolar colouring C of a point $v \in S^d$ where S is some arbitrary set in d dimensionality the following:

$$\forall j \in [d] : [C(v)]_j \in \{-1, 1\}$$

Essentially a point is a associate with a d dimensionaly binary vector. We say that a set of points $A \subseteq S^d$ **cover all the labels** if:

$$\forall i \in [d], \ell \in \{-1, +1\}, \exists x \in A : [\lambda(x)]_i = \ell$$

Definition II.11: STRONGSPERNER problem

Input: A boolean circuit that computes a bipolar labelling $\lambda : [M]^N \to \{-1, 1\}^N$ II.10 satisfying the following boundary conditions $\forall i \in [N]$:

- if $x_i = 1 \implies [\lambda(x)]_i = +1$
- if $x_i = M \implies [\lambda(x)]_i = -1$

Output: A set of points $\{x^{(i)}\}_{i \in [N]} \subseteq [M]^{[N]}$, such that:

- *Closessness condition*: $\forall i, j \in [N] : \|x^{(i)} - x^{(j)}\|_\infty \leq 1$
- *Covers all labels* as defined in II.10

The above is a generalised variant of the tradtional Sperner problem to a grid of dimensions N and width of M. Throughout literature the same variants of the problem or specifications have been defined using sperner or discrete brouwer [8], [10]–[12]. For the sake of clarity, we will stick to STRONGSPERNER.

Definition II.12: ND-STRONGSPERNER problem

Input: A tuple $(\lambda, 0^k)$ of a STRONGSPERNER instance but for only n dimensions, such that $\lambda : (\{0,1\}^k)^n \to \{-1, +1\}^n$.
Output: A point $\alpha = (a_1, \ldots, a_n) \in A^n$, where $A = \{0,1\}^k \setminus \{1^k\}$ such that:

$$\{\alpha + x \mid x \in \{0,1\}^n\} \text{ cover all the labels II.10}$$

We use A to avoid edge cases. We assume dimensionality $n \geq 2$.

Chen et al. [12] demonstrated that all ND-STRONGSPERNER are PPAD-COMPLETE. We mainly use this to show counting arguments with respect to the ENDOFLINE as people have indicated a parsimonious reduction between 2D-STRONGSPERNER and 3D-STRONGSPERNER to the EndOfLine problem but with *linear* colouring. The authors of these papers use these problems indifferently when talking about the reductions and therefore we can assume that either colouring will create a correct reduction.

c) Other PPAD problems: We will define several problems in PPAD that are related with our project and demonstrate the counting complexity of PURECIRCUIT.

Definition II.13: SOURCEOREXCESS problem

We define as *SourceOrExcess*$(k, 1)$ for $k \in \mathbb{N}_{\geq 2}$ the search problem as such: Given a poly-sized successor circuit $S : \{0,1\}^n$ and a set of predecessor poly-sized circuits $\{P_i\}_{i\in[k]}$, we define the graph $G = (V, E)$ such that, $V = \{0,1\}^n$ and E as:

$$\forall x, y \in V : (x, y) \in E \iff (S(x) = y) \wedge \bigvee_{i\in[k]} P_i(y) = x$$

We ensure that 0^n is as sink, meaning $\deg(0^n) = (0, 1)$. A valid solution is a vertex v such that *in-deg*$(v) \neq$ *out-deg*(v)

Lastly we will introduce TARSKI **??**, which is a problem in PLS $\cap$ PPAD where PLS is a class of problems based on the idea of local search [14] and uses the Knaster-Tarski fixed point theorem II.1. We will use this problem in later sections to connect Kleene algebra with PPAD.

Definition II.14: Monotone functions

Given two posets $(L_1, \preceq_{L_1})$ and $(L_2, \preceq_{L_2})$, a function $f : L_1 \to L_2$ is **monotone** if and only if:

$$\forall x, y \in L_1 : x \preceq_{L_1} y \implies f(x) \preceq_{L_2} f(y)$$

Theorem II.1: Knaster Tarski Fixed point theorem [15], [16]

Given a lattice $(L, \wedge, \vee)$ and a *monotone* $f : L \to L$ II.14

$$\exists c \in L : f(c) = c$$

Using the Tarski theorem, Fearnley et al. [16] created a TFNP variant of the problem which finds fixed points or identifies points that break the monotone argument. We will be using Tarski with Kleene logic to connect the two.

d) Counting complexity of PPAD: The current project looks on the counting complexity of such problems as despite two problems being PPAD-COMPLETE, Ikenmeyer et al. [3] has showed examples where their counting difficulty can differ vastly as it can be seen in the figure 2.

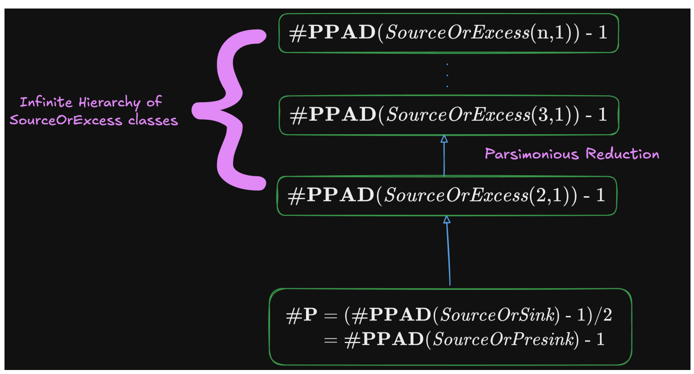


Fig. 2: Hierarchy graph between #PPAD problems. Figure from [1]

3) Kleene Logic and Hazard-Free Circuits: Kleene logic uses the boolean system with an additional *unstable* value [9]. Study of Kleene logic aided in the development of robust physical systems [17], as well as giving definitions to *Alternative Turing mahcines* [18], and showing correlations complexity of monotone circuits sizes [19]–[22].

Our current report will focus on specific concepts with Kleene logic as PURECIRCUIT instances use the underlying logic directly. Below we introduce important concepts [23].

Definition II.15: Kleene value ordering [23]

The instability ordering for $\leq^u$ is defined as: $\perp \leq^u 0, 1$. The n-dimensional extension of the ordering is:

$$\forall x, y \in \mathbb{T}^n : x \leq^u y \implies \forall j \in [n] : (x_j \in \mathbb{B} \implies x_i = y_i)$$

Definition II.16: Kleene Resolutions [20], [23]

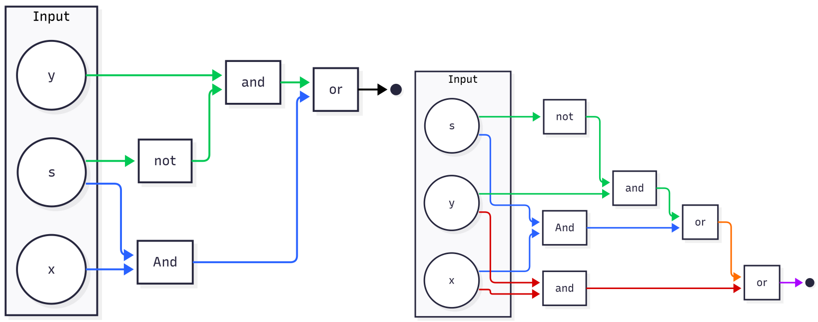
Given $x \in \mathbb{T}^n$, we define the *resolutions* of x, using the following notation:

$$\textsc{Res}(x) \triangleq \left\{ y \in \mathbb{B}^n \mid x \leq^u y \right\}$$

Detecting hazards is a concept that is analysed heavily when talking about Kleene logic. The idea is ensuring robustness of our circuits, meaning if all resolutions of $x \in \mathbb{T}^n$ give the same value, then we should expect circuit to behave the same way II.17. An example of hazard values can be seen in the figure 3.

Definition II.17: Hazard [19], [20]

A *circuit* C, on n inputs has **hazard** at $x \in \mathbb{T}^n \iff C(x) = \perp$ and $\exists b \in \mathbb{B}, \forall r \in \textsc{Res}(x) : C(r) = b$. If such value does not exists then we say that C is hazard-free.



(a) Multiplexer with hazard at $(1, 1, \perp)$ (b) Hazard free multiplexer

Fig. 3: Hazard circuit and hazard-free circuit. Figure by [20]

Definition II.1 (K-bit Hazard)**.** *For $k \in \mathbb{N}$ at $x \in \mathbb{T}^n \iff C$ has a hazard at x and $\perp$ appears at most k times in x*

III. Current Progress

A. Current Timetable

B. Theory Progress

In the current section we will be focusing on progress made. We experimented with various approach such as investigating easier variants of PureCircuit to find reductions to the EndOfLine. Moreover, we looked into using the Purify gate or a variant where we have more control over the $\perp$ value but we realised that under Marino et al. [24] continuity arguments, such variants are not trivial to find. Our core breakthrough, came when experimenting with Hazard-Free circuits and EOPL problems. Specifically, we looked at a problem by Fearnley et al. [25] where they introduced a variant brouwer problems known as OPDC, which is a local descent problem with fixed point topologies. Combining ideas of the two lead to the main find of our project in III-B1

1) ND-StrongSperner Constant Parsimonious Reduction: Deligkas et al. [8] proved PPAD-hardness by reducing from StrongSperner II.11, but with unbounded dimensions of polynomial width. Our contribution parsimoniously constant reduces II.3 every ND-StrongSperner problem to PureCircuit. We will demonstrate our reduction for 3 dimensions and then we will generalise.

Theorem III.1: 3D-StrongSperner to PureCircuit

$$\#\text{3D-StrongSperner} \subseteq^7 \#\text{PureCircuit}$$

When referring to the fact that x is a solution, we imply the definition described in II.12. Additionally we will restrict the solution set of the *Purify* gate of PureCircuit to $(0, \perp), (\perp, 1), (0, 1), (1, \perp)$. We explain in II-B2a why this does not affect the PPAD-hardness of the problem nor any counting arguments.

Proof. The first step of the proof requires constructing a colouring function $\Lambda : (\{0,1\}^{n+1})^3$ such that:

$$\forall (i,j,k) \in (\{0,1\}^{(n+1)})^3 : \Lambda(i,j,k) \triangleq \lambda\left(\left\lfloor \frac{i+1}{2} \right\rfloor, \left\lfloor \frac{j+1}{2} \right\rfloor, \left\lfloor \frac{k+1}{2} \right\rfloor\right)$$

Conversely we can say that we map from our original domain to our new domain as such

$$\forall x \in \{0,1\}^{3n}, i \subseteq \{0,1,2\} : \lambda(x_{-i}, x_i) \rightarrow \Lambda(2x_i, 2x_{-i} - 1) \tag{1}$$

The transformation can be visualised in the figure 4. The reasoning for doubling the dimensions will be explained in the later sections.

Given that S is the original set of solution, we define the claim III.1.

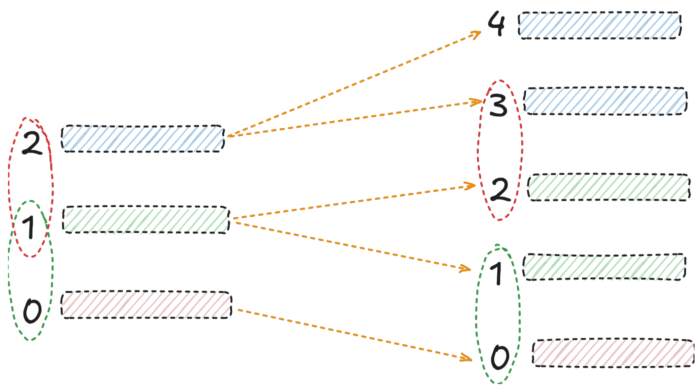


Fig. 4: Transformation of solutions. In the original cube we have as solutions $(0, i, j)$ and $(1, i, j)$. In our new mapping, these solutions correspond to $(0, i, j), (2, i, j)$

Claim III.1: Transformation claim

Given S_{even} the set of even solutions or more formally:

$$S_{\text{even}} \triangleq \left\{(i0, j0, k0) \in (\{0,1\}^{(n+1)})^3 \mid (i0, j0, k0) \text{ covers all labels by II.10}\right\}$$

We claim $|S| = |S_{\text{even}}|$

Proof. We argue that we can bijectively map between $S \leftrightarrow S_{\text{even}}$. To show the left to right direction, assume $(i, j, k) \in S$. Using equation 1, only one transformation maps all points to even coordinates. Additonally, we can observe its neighbourhood set:

$$N = \left\{\Lambda(2i + x_1, 2j + x_2, 2k + x_3) \mid x \in \{0,1\}^3\right\}$$

We can observe that WLOG $\Lambda(2i + 1, \cdot, \cdot) = \lambda(i + 1, \cdot, \cdot)$, which implies:

$$N = \left\{\lambda(i + x_i, j + x_j, k + x_k) \mid x \in \{0,1\}^3\right\}$$

And therefore the the point is also a solution. Conversely we can use the previous neighbourhood argument to show the opposite direction. □

We will redefine $n+1$ as n to keep the notation consistent. Given the above we provide a main sketch of our tranformation as such:

1) **Input bits**: We initialize input nodes s_1, s_2, s_3
2) **Bit generator**: We apply the *Bit Generator gadget* $\hat{P}$ for each s_i, to create numbers $u^1, u^2, u^3 \in \{0,1\}^k$
3) **Circuit application**: We apply circuit $\bar{\Lambda}(u^1, u^2, u^3)$ to create output values o_1, o_2, o_3, where $\bar{\Lambda}$ is a 3-bit hazard-free variant of Λ.
4) **Validation**: Copy the results to s_1, s_2, s_3

The transformation can be summarised in the figure 5. The goal of the above transformation is to show that if $o_1 = o_2 = o_3 = \perp$, then we have a solution to the original instance.

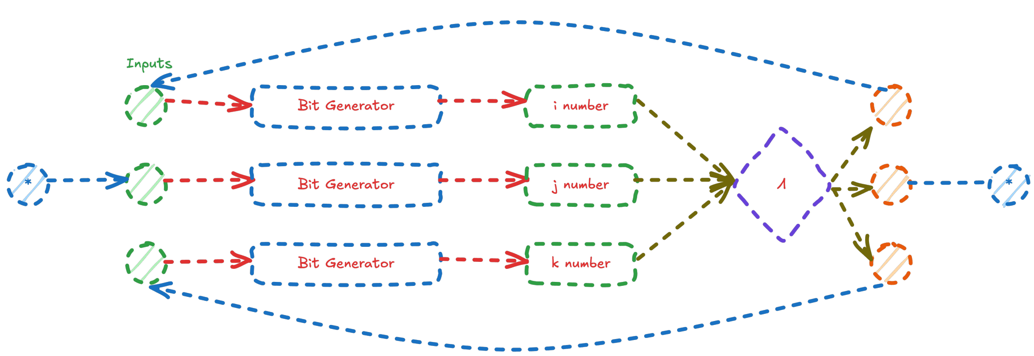


Fig. 5: Testing

a) Bit generator: We create the gadget described in the figure 6.

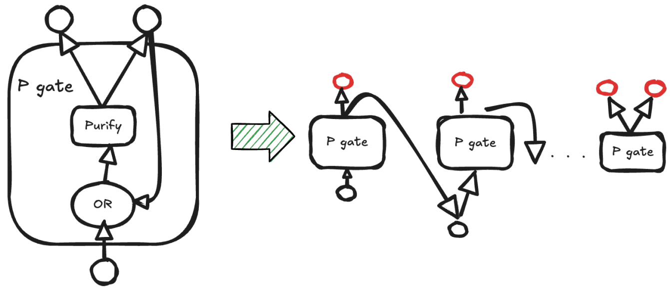


Fig. 6: Bit generator gadget which is defined as a linear stack of P gates.

Given that construction we make the following lemma

Lemma III.1 (Bit Generator Lemma)**.** *The following hold true in our construction:*

1) if $s_i = b \implies u^i = b^n$*, given* $b \in \mathbb{B}$
2) if $s_i = \perp \implies \forall j \in [n-1] : u^i_j \in \mathbb{B}$ *and* $u^i_n \in \{1, \perp\}$

The above translates to: if we have $\perp$ *in our number, it can only be found in the LSB.*

Proof. The first part follows trivially from the defintion of the PURIFY gate. To prove the second part of lemma, WLOG we choose u^i out of the three outputs and assume $\exists j \in [n-1] : [u^i]_j = \perp$. It implies that one of the purify gates had an output of $(\perp, 1)$. But due to our OR gate, the input would be 1 which implies the output is $1, 1$, which leads to a contradiction. □

b) Circuit Application: In the current phase, we apply a polynomial transformation to Λ to be 3-bit hazard-free using the construction by Ikenmeyer et al. [20] *Corollary 10* or by [22] *Corollary 1.9*. By lemma III.1, we can have at most 3 $\perp$ values in our input. Using that property we create the lemma below:

Lemma III.1: Circuit Application Lemma

Given $o_1 = o_2 = o_3 = \perp$ it implies u^1, u^2, u^3 correspond to a solution $(i0, j0, k0) \in S_{\text{even}}$

Proof. First we argue that if $i\perp, j\perp, k\perp = u^1, u^2, u^3$, then its its resolutions can represent hypercubes as shown in figure 7.

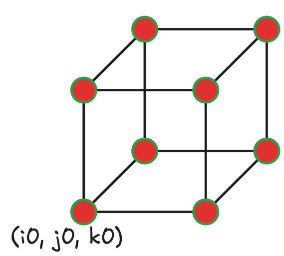


Fig. 7: Resolutions of $i\perp, j\perp, k\perp$

If $(i0, j0, k0)$ is a solution that implies:

$$\forall i \in \{1,2,3\}, b \in \{-1,1\} \exists c \in \textsc{Res}(i\perp, j\perp, k\perp) : [\bar{\Lambda}(c)]_i = b$$

Since $\bar{\Lambda}$ is 3-bit hazard free, if all outputs are $\perp$, we satisfy the colouring criteria. □

The above proof also demonstrates why the domain transformation we did in our first step is needed as we only capture cubes with even smallest corners. We can observe by fixing the LSB of the input to some value, we extract a specific edge or face of the cube.

Lemma III.2: Correctness lemma

A valid assingment x of the current instace, corresponds to a point in S_{even}

Proof. It suffices to show that a valid assingment only occurs when $o_1 = o_2 = o_3 = \perp$. Assume $\exists i \in \{1,2,3\}$ such that label i is not covered, meaning $o_i \neq \perp$. WLOG assume that $o_i = 0$. Our verification stage, will copy 0 onto s_i. By lemma III.1, $u^{(i)} = 1^n$. From the boundary conditions of our circuits and the k-bit hazard-freeness construction, we know that $\Lambda(*, 0^n, *) = \{*, 1, *\}$. But that implies $o_i = 1$ which leads to a contradiction. We can make a similar argument to when

$o_i = 1$. Therefore, by III.1, if $o_1 = o_2 = \perp$, we have a found a panchromatic cube. Lastly, every assignment u^1, u^2, u^3k corresponds to a cube with even corners. □

□

Counting Argument: We observe that the LSB can only be $1, \perp$. Moreover, we double count any edges or faces of the cube that cover all labels, which gives the equation:

$$\underbrace{\binom{3}{1}}_{\substack{\text{One of the sides is odd}\\ \text{and covers all labels}}} + \underbrace{\binom{3}{2}}_{\substack{\text{One of the edges is odd}\\ \text{and covers all labels}}} + \underbrace{1}_{\text{All LSBs are } \perp} = 2^3 - 1 = 7$$

Therefore, we can bound the number of solutions of 3D-STRONGSPERNER by a factor of 7.

ENDOFLINE problem parsimoniously reduces to 2D-STRONGSPERNER and 3D-STRONGSPERNER under **linear** colouring [10], [11], but more work has to been done in order to determine whether this still holds for bipolar colouring.

The reduction described above can work with any dimensionality. and we know $foralln \in \mathbb{N}_{\geq 2}$, Chen et al. showed that nD-STRONGSPERNER is still *PPAD-Hard* in [12]. Therefore the following corollary holds III.1

Corollary III.1: nD-STRONGSPERNER parsimonious reduction bounds

$$f(\cdot) = 1 + \sum_{i=1}^{n-1} \binom{n}{i} = 2^n - 1 = a_n$$

$$\#\text{nD-STRONGSPERNER} \subseteq^{a_n} \#\text{PURECIRCUIT}$$

2) Hazard Circuits and Tarski: We define the following subclass of TARSKI **??**:

Definition III.1: KLEENETARSKI problem definition

Given $F : \mathbb{T}^n \to \mathbb{T}^n$, where F is a *natural* function, we want to find the set of points **Fix*** such that

$$\forall x \in \textbf{Fix}^\star : F(x) = x$$

We assume F will be represented as a circuit, that uses $\{*, +, \neg, \mathbf{0}, \mathbf{1}\}$.

Proposition III.2 (Parsimonious Reduction between KLEENETARSKI and PURECIRCUIT)**.**

$$\#\text{KLEENETARSKI} \subseteq \#\text{PURECIRCUIT}$$

Proof. Given an input circuit C, consturct PURECIRCUIT instance:

1) Initiate a vector of nodes s
2) Apply $C(x)$ and store result on output vector o.
3) Copy the results back into s

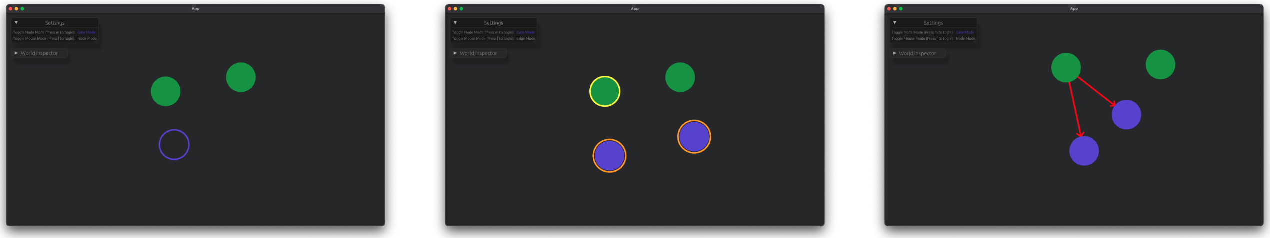
We can observe that for any valid assignment, $\mathbf{x}[s] = \mathbf{x}[o]$. Therefore, our construction can find all fix points for Kleene Circuits. □

C. Software Progress

In our project we are also focused on developing a visualisation tool for PURECIRCUIT instances. In our introduction we referred to three main objectives. As of time of writing w are close to the completion of the first one. We will provide a table of objectives that have been achieved, as well as the remaining requirements needed to complete the first milestone in II. Lastly, we provide a figure of our current visualisation in 8

Accomplished	Finished
Ability to add and move nodes	Add values to value nodes
Toggle between value nodes and gate nodes	Specify the type of gate to add
Add edges. Ensure that edges can only be added between heterogeneous nodes	Add indicators as to how many edges are excess/remaining for the gate to be valid
Move whole graph	Inclusion of a status bar as to whether the current assignment is correct, wrong or syntactically incorrect
Create panel to show current state as well as some indicator and guides	

TABLE II: Finished and remaining issues



(a) Add nodes (b) Add edges (c) Final state

Fig. 8: Screenshots of different states of the visualisation tool

IV. Next Steps

A. Theory Crafting Next steps

For the remaining duration, we will focus on three main objectives: One will be to demonstrate that, the *bipolar* colouring of nD-StrongSperner will keep the reduction parsimonious. Second, will be to demonstrate hardness over #P class either by findning reductions from the SourceOrExcess problem. And lastly, we will be to generalise our reduction to all $\#PPAD$ problems.

B. Software Next Steps

In the next steps we aim to complete the rest of the milestones. We believe that the usage of methods by Eichelberger et al. [19] and or polynomial algorithm for specific cases by [8] will allow us to identify solutions. With regards to counting, we can only hope to count solutions for small instances.

C. Timetable

The figure 9 depicts how the remaining project will progress with regards to the milestones we aim to achieve. Overall we modified our timeline to prioritise theory crafting and therefore we hope this adjusted timeline will capture accurately the remaining duration of the project.

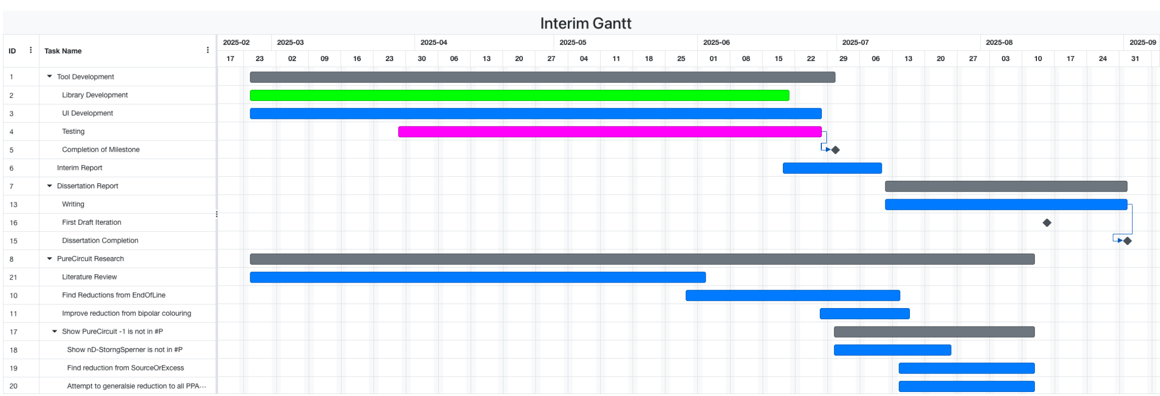


Fig. 9: Updated gantt chart

V. Project Management

To manage our software we are using *Rust*, as the main language of development, due to its high expressibility and low level management. We utilise methods such as Test-Driven-Developmen with the help of `Proptest`, for property testing and kanban boards to accurately track our progress as seen in the figure **??**. Lastly we are making use of *Bevy* which is a game engine that use the Entitiy-Component-System for render interactinos between entities as seen in 10.

With regards to the our research management, we revolved our work around obsidian. As we can see in the figure 11, `Obsidian` beyond its traditional usage of note taking, it comes with several handy tools such as note organisation and mind-mapping.

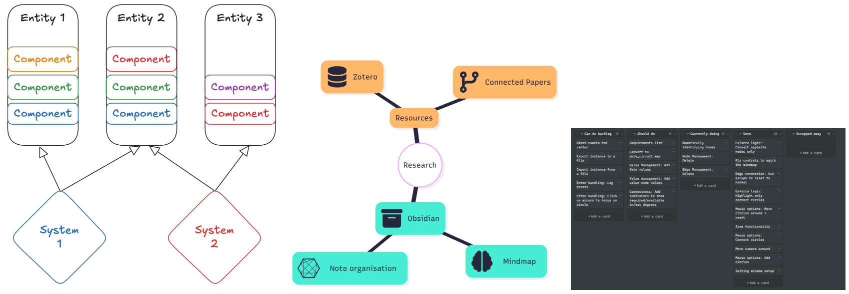
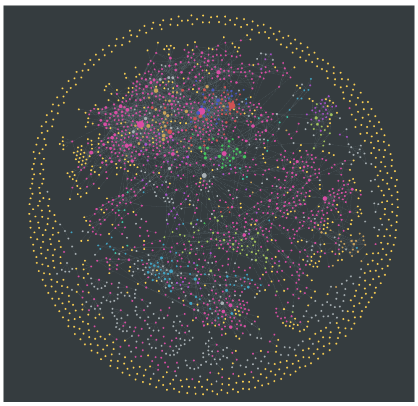
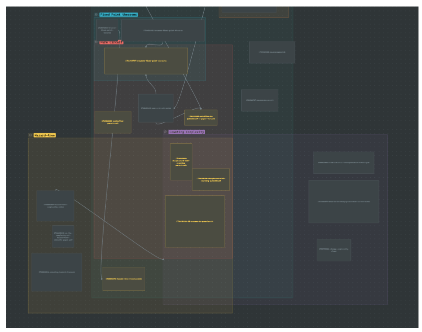


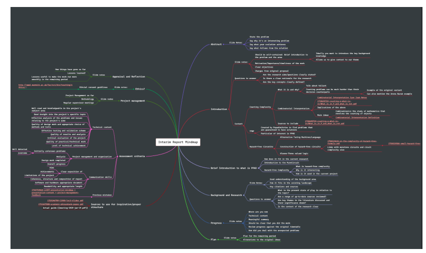
Fig. 10: ECS workflow visualisation



(a) Obsidian Graph



(b) Obsidian Canvas



(c) Obsidian mindmap

Fig. 11: Usages of Obsidian

A. *Risk Management*

With regards to risks and mitigations, the biggest risk we face is the inability to resolve our main question. Due to the limited literature surrounding PURECIRCUIT and its counting nature as well as the recency in the development of TFNP $-$ 1, our question could go into many directions. The discovery of our reduction allowed us to step closer to our question and in our table III, we will analyze this is greater depth. The general mitigation strategy we can do is to keep researching, keep finding connections until we can connect all the necessary pieces to do the jump.

Severity	**Probability**	**Description**	**Mitigation**	**Address**
High	Medium	Software may not be feasible within the remaining time frame.	Focus on the the first objective where the project. Restrict to solution finding or counting when the number of nodes is small.	Not yet
Low	Medium	Software not identifying a correct solution	Usage of TDD techniques and property testing to ensure correctness. Comparison with hand-made instances	Not yet
High	High	Inability to extend the reductions of the PureCircuit problem to SourceOrExcess	Find a path of parsimonious reductions between the two problems. If we cannot ensure a one-to-one relation between the solutions, we will focus on $1 - c$ for some $c \in \mathbb{N}$	✓
Medium	High	Inability to prove the main conjecture	We can make some heuristical arguments or find reductions between other problems. We argue that if we are able to unravel enough correlations, we will hope to at least bridge the problems. Develop useful gadgets one can use with PureCircuit	Not yet

High	High	Incorrect proofs or reductions.	Analyse the problem under different constraints. Apply the duck method, where attempt to explain the solution to a person which not necesserily an expert. Validate proof with supervisor	Not yet
High	Medium	Develop combinatorial friendly variants of PURECIRCUIT	Apply robustness on the gate set of PURECIRCUIT. Develop new gates or variants are easier to work with.	✓

TABLE III: Risk managment table.

From our table, it is worth expanding on some of the points. The best method we found when tackling this problem is when we try to uncover reductions between other PPAD problems or when trying to incorporate gadgets from Kleene theory. We hope that by experemínting enough we will be able to get close to resolve our conjectures.

VI. APPRAISALS

A. *Reflections*

Overall, the current project progressed as anticipated. We managed to keep consistent progress in both theory and development aspect. In addition, due to the theoretical overlap the project has with other fields, we were able to experiment with great variety of problems as well as develop research methodologies for tackling these problems.

Of course understanding how to conduct such research came with a lot of trials and error. We quickly realised that such projects require a much more extensive literature review than anticipated. Moreover, when it comes to theorising and proof development, we faced many failures. Often, we spent an excessive amount of time working on a problem without being methodical. From that, we understood that when trying to prove statements, it is often more beneficial exploring the field as a whole, tackling various problems and gathering ideas from each. Moreover, each failed attempt allows us to understand a problem in greater depth, how it behaves and how to formulate our attempt to increase our chances. This can be observed in our canvas board 11b, where we used “whiteboard” approach to keep track of the project from a bird’s-eye view. We hope to use this exploratory methodology to uncover even more mysteries that surround the complexity of counting problems.

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