Counting Problem of *PureCircuit*

Christos Demetriou *Department of Computer Science*
University of Warwick
Coventry, UK
u2018918@live.warwick.ac.uk, u2018918

Abstract

Test.Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris. Nam arcu libero, nonummy eget, consectetuer id, vulputate a, magna. Donec vehicula augue eu neque. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Mauris ut leo. Cras viverra metus rhoncus sem. Nulla et lectus vestibulum urna fringilla ultrices. Phasellus eu tellus sit amet tortor gravida placerat. Integer sapien est, iaculis in, pretium quis, viverra ac, nunc. Praesent eget sem vel leo ultrices bibendum. Aenean faucibus. Morbi dolor nulla, malesuada eu, pulvinar at, mollis ac, nulla. Curabitur auctor semper nulla. Donec varius orci eget risus. Duis nibh mi, congue eu, accumsan eleifend, sagittis quis, diam. Duis eget orci sit amet orci dignissim rutrum.

I. Introduction

Over the last couple of years, there has been a revolutionary initiative in the field of combinatronics. Combinatronics has been a field of study in mathematics that primarily focused on the notion of counting objects with certain properties. Over time, this notion has shifted, especially in the subfield of algebraic combinatorics, where there is no clear notion of the object that we are counting, and the numbers express something more abstract [1]. This gave a need to be able to assign a combinatorial interpretation to such numbers, or more simply, do these numbers correspond to some counting over a set of objects. Being able to find such definitions or interpretations can be very important, it allows us to utilise tools from combinatorics as well as allow us to understand and reveal hidden structures and properties for such numbers [1]. Moreover, there are several problems or numbers such as *Kronecker coefficients*, whose combinatorial interpretation, would give a step towards the resolution of the $P \neq NP$ conjecture [2].

To reiterate the previous statement, we can understand combinatorial interpretation as the process of: given a sequence of numbers $\{a_x\}$, find a set of combinatorial objects A_x such that $|A_x| = a_x$ To formalise the current idea, Igor Pak et al. has concluded that $f \in$ *#P*, implies that f has a combinatorial interpretation [1], [2]. We will explore this idea in much greater detail in the upcoming section, but the main benefit is the ability to use a very expressive but formal language that encapsulates this notion of a combinatorial interpretation.

In our current work, we focus on extending the work done by Ikenmeyer et al., where they focused on the creation of frameworks that determine whether $f \in^?$ **#P**, by looking at the complexity class of **#***TFNP -1*. This is a class of problems that are guaranteed to have a solution and their solutions are verifiable in polynomial time. In their paper, they were able to show that for a subclass of problems, also known as *PPAD* $\subseteq$ *TFNP*, different *PPAD-complete* problems, may or may not have a combinatorial interpretation. Our contribution, comes to the analysis of a specific problem, known as PureCircuit, which utilises Kleene logic, to find satisfying assignments in sequential circuits. We hope to demonstrate that such problem could help us bound, the counting complexity limits of #PureCircuit $- 1$.

A. Project objectives

Below we will present our table of objectives. We will denote updated objectives with (*), new objectives with (!), completed objectives with (✓), deleted objectives with (-).

R.1) (✓) Find a parsimonious reduction from the *EndOfLine* to *EndOfLine*.
R.2) (!) Improve the combinatorial bound between the reduction from EndOfLine to PureCircuit
R.3) (!) Demonstrate that #PPAD$(PureCircuit) - 1 \not\subseteq$ **#P**
R.4) (*) Prove or disprove the following $\forall n \in \mathbb{N}_{\geq 2}$:

$$\exists c \in \mathbb{N} : \#\textsc{SourceOrExcess}(n, 1) \subseteq^c \#\textsc{PureCircuit}$$

Below we will be representing the development portion of the project.

S.1) Visualise a pure circuit
S.2) Generate a solution
S.3) Count number of solutions for smaller scales

We will compile the rest of the report, based on our current findings, how we modified our objectives to the current ones as well as general reflections.

B. Research Question

Our main objectives lead to the following conjecture:

Conjecture I.1: #PPAD $-$ 1 **hardness**

We conjecture that proposition that every language in #PPAD $-$ 1 can be parsimoniously reduced up to some polynomial factor to #PureCircuit $-$ 1. We express that idea more formally as such:

$$\forall L \in \text{PPAD}, \exists f \in n^{O(1)} : \#L - 1 \subseteq^f \#\text{PureCircuit} - 1$$

The conjecture I.1 captures the essence of our work. Being able to finding such relations will allow us to understand in greater depth the complexities behind #TFNP $-$ 1 class.

II. Preliminaries and Background review

A. Important Notation

To avoid ambiguity, we introduce the following notation conventions used throughout the paper. For any $n \in \mathbb{N}$, we write $[n] \triangleq \{1, \ldots, n\}$ and $[n]_0 \triangleq [n] \cup \{0\}$. When using the notation $\bar{n}$, we refer to the unary representation of the number n; the plain symbol n denotes the binary representation by default. Given sets A and B, we denote by $A^B \triangleq \{f \mid f : A \to B\}$ the set of all functions from B to A. We define the Boolean domain as $\mathbb{B} \triangleq \{0, 1\}$, the natural numbers as $\mathbb{N} \triangleq \mathbb{Z}_{\geq 1}$, and the non-negative integers as $\mathbb{N}_0 \triangleq \mathbb{Z}_{\geq 0}$.

B. Background overview

As mentioned in the introduction, the current project works as an interaction of three different fields: 1. counting complexity and combinatorial interpretation, 2. total search problems and PPAD and 3. Kleene logic.

1) Counting Complexity and Combinatorial Interpretations: As we previously mentioned, we say that an object or a structure f has a combinatorial interpretation if $f \in$ **#P**. **#P** is a complexity class created by Valiant [3], to define a formal combinatorial framework in complexity theory.

Definition II.1: #P Complexity Class

#P is a class of functions $f : \{0, 1\}^* \to \mathbb{N}$ such that: there exists a polynomial-time deterministic TM M, and $p : \mathbb{N} \to \mathbb{N}$ such that $p \in n^{O(1)}$, we have:

$$f(w) = \left| \left\{ v \in \{0, 1\}^{p(|w|)} \mid M(w, v) = 1 \right\} \right|$$

#P was initially created by [3], to demonstrate, that even if we have a problem $L \in P$, $\#L$ can be computationally hard to compute, by providing an example of computing the permanent of a 01-matrix, with number of perfect matchings.

As we can see, **#P** allows us to define a set of objects, whose cardinality equals $f(w)$. Core reasoning for choosing **#P** to define combinatorial objects is mainly for the following two reasons [4]:

1) By polynomially bounding words, we avoid cases such as: $f(w) = \{1, \ldots, f(w)\}$
2) Current framework allows us to work with $f(\cdot)$, whose direct computation can be computationally hard

The current framework was used in several papers such as [2] and [4] where they were able to use tools from complexity theory to show that many structures do or do not have a combinatorial interpretation. For the purposes of the current project, we are focusing on [2], where they demonstrated how several TFNP problems, change in complexity as we ignore one of its solutions.

Any class of problems in complexity theory utilises the notion of reduction to indicate the similarity between problems. For counting complexity this is referred to as *parsimonious reductions* II.2.

Definition II.2: Parsimonious reductions

Let R, R' be search problems and let M be a Karp reduction of $S_R = \{x \mid R(x) \neq \emptyset\}$ to $S_{R'} = \{x \mid R'(x) \neq \emptyset\}$. We say f is **parsimonious** if:

$$\forall x \in S_R : |R(x)| = |R'(f(x))|$$

2) Total Search Problems and PPAD: When talking about search problems, we are using the following definition:

Definition II.3: Search Problems and Total Search Problems

Search problems can be defined as relations $R \subseteq \{0, 1\}^* \times \{0, 1\}^*$, where given $x \in \{0, 1\}^*$, we want to find $y \in \{0, 1\}^*$ such that xRy.

Total Search problems are search problems such that:

$$\forall x \in \{0, 1\}^*, \exists y \in \{0, 1\}^* : xRy$$

Using the above, we can define the following complexity classes

Definition II.4: FNP and TFNP

FNP are *search problems* such that there exists poly-time TM $M : \{0,1\}^* \to \{0,1\}$ and a poly function $p : \mathbb{N} \to \mathbb{N}$ such that:

$$\forall x \in \{0,1\}^*, y \in \{0,1\}^{p(|x|)} : xRy \iff M(x,y) = 1$$

Lastly **TFNP** $= \{L \in$ **FNP** $\mid L$ is total$\}$

Definition II.5: Levin Reductions

Given a pair of search problems R_A, R_B, a pair of computable time functions (f,g) is called a Levin reduction from $R_A \to R_B$

$$S_R \triangleq \{x \mid \exists y : xRy\}$$
$$R(x) \triangleq \{y \mid xRy\}$$
$$\forall x \in S_{R_A}, y_b \in R(f(x)) : (x, g(x, y_b)) \in R_A$$

Levin reductions are the equivalent of *Kerp reductions* for search problems. Our current work focuses on a specific subclass of **TFNP** problems which is defined as follows:

Definition II.6: *EndOfLine* problem [5]

Given circuits $S, P \in \{0,1\}^n \to \{0,1\}^n$ such that $S, P \in n^{O(1)}$ we define a directed graph $G = (V, E)$, such that $V = \{0,1\}^n$ and E defined as:

$$E = \{(x,y) \in V^2 : S(x) = y \wedge P(y) = x\}$$

We define source or sinks $\forall v \in V : deg(v) = (0,1)$ or $deg(v) = (1,0)$, respectively. We also syntactially ensure that the 0^n node is always a source, meaning $S(P(0^n) \neq 0 \wedge P(S(0^n)) = 0^n$. A node $v \in V$ is a solution if and only if $deg(v)$ is either $(0,1)$ or $(1,0)$.

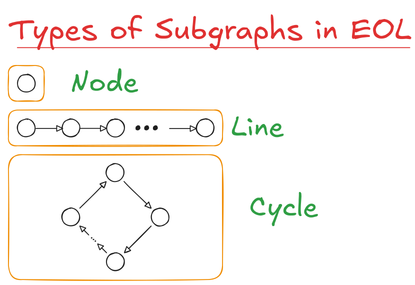


Fig. 1: Types of subgraphs in EndOfLine

An illustrative example of an instance can be seen in the figure 1. A solution to such problem instances are all the directed leafs and to ensure that a solution always exists, we ensure that the 0^n. Using the EndOfLine problem, we define the PPAD complexity class II.7

Definition II.7: PPAD complexity class [5]

PPAD is defined as the set of search problems that are *Levin reducible* II.5 to the EndOfLine problem II.6.

PPAD has been created by Papadimitriou [5] to demonstrate a subset of problems in **NP** that are guaranteed to have a solution but can be very difficult to find. In the next section we will introduce the core set of PPAD problems that we are working with.

a) The PureCircuit problem: Below we will introduce the PureCircuit, which was create initially by Deligkas et al. [6] to demonstrate the hardness of approximating PPAD problems. This problem uses a kleene-logic based circuit construction with the help of continuity of arguments, they demonstrated PPAD-completeness.

Definition II.8: PURECIRUIT Problem Definition [6]

An instance of *PureCircuit* is given by vertex set $V = [n]$ and gate set G such that $\forall g \in G : g = (T, u, v, w)$ where $u, v, w \in V$ and $T \in \{\text{NOR}, \text{Purify}\}$. Each gate is interpreted as:

1) *NOR*: Takes as input u, v and outputs w
2) *Purify*: Takes as input u and outputs v, w

And each vertex is ensured to have in-deg$(v) \leq 1$. A solution to input instance (V, G) is denoted as an assignment $\mathbf{x} : V \to \{0, \perp, 1\}$ such that for all nodes we have:

1) if v is the output of a $(\textit{NOR}, u, v, w)$ gate:

$$\mathbf{x}[u] = \mathbf{x}[v] = 0 \implies \mathbf{x}[w] = 1$$
$$(\mathbf{x}[u] = 1 \vee \mathbf{x}[v] = 1) \implies \mathbf{x}[w] = 0$$
$$\text{otherwise} \implies \perp$$

2) *Purify*:

$$\forall b \in \{0, 1\} : \mathbf{x}[u] = b \implies \mathbf{x}[v] = b \wedge \mathbf{x}[w] = b$$
$$\mathbf{x}[u] = \perp \implies \{\mathbf{x}[v] \cup \mathbf{x}[w]\} \cap \{0, 1\} \neq \emptyset$$

The definition of PURECIRCUIT that we will be using for the standard set of gates $\{\wedge, \vee, \neg\}$, is based Kleene's three-valued strong logic of indeterminancy which extends the traditional $\mathbb{B}$ logic [7]. In addition to that, we will make use of the *Copy* gate but adding the robustness constraint such that if $(\textit{Copy}, u, v) \in G \implies x[u] = x[v]$. This allows our circuits to be robust and easier to work with.

For the *Purify* gate, Deligkas et al. showed that the only solutions that are essential for *Purify* are $\{(0, \perp), (\perp, 1), (0, 1)\}$ with the help of continuity arguments [6]. Adding more solutions does not change the complexity as he showed in the original variant of the problem. We acknowledge that the solution set will be different than the original one but we can easily observe that #PURECIRCUIT-SIMPLIFIED $\subseteq$ #PURECIRCUIT, and therefore any proposition or argument of the sort: #A $\subseteq$ #PURECIRCUIT-SIMPLIFIED still holds true. For the purposes of the report, any solution change will be made explicit and we will refer to all such simplified variants as #PURECIRCUIT to avoid confusion.

Definition II.9: SOURCEOREXCESS problem

We define as *SourceOrExcess*$(k, 1)$ for $k \in \mathbb{N}_{\geq 2}$ the search problem as such: Given a poly-sized successor circuit $S : \{0, 1\}^n$ and a set of predecssor poly-sized circuits $\{P_i\}_{i \in [k]}$, we define the graph $G = (V, E)$ such that, $V = \{0, 1\}^n$ and E as:

$$\forall x, y \in V : (x, y) \in E \iff (S(x) = y) \wedge \bigvee_{i \in [k]} P_i(y) = x$$

We ensure that 0^n is as sink, meaning deg$(0^n) = (0, 1)$. A valid solution is a vertex v such that *in-deg*$(v) \neq$ *out-deg*(v)

b) Sperner problems: Below we will refer to the notion of Sperner problems which involve the idea of using the topology of a problem and a colouring scheme to ensure that a substructure is panchromatic. There two variants of colouring scheme that are used: one of them will be referred to as the **linear** colouring where for dimension d, assign $d + 1$ distinct colours to each point [8], [9]. Below we will refer to **bipolar** colouring, which has been used grid-like topologies of the Sperner property [6], [10], [11]. It has to be noted that these are not their official names, but we have decided to use this naming scheme for clarity.

Definition II.10: Bipolar colouring

Given dimension d, we refer to the bipolar colouring C of a point $v \in S^d$ where S is some arbitrary set in d dimensionality the following:

$$\forall j \in [d] : [C(v)]_j \in \{-1, 1\}$$

Essentially a point is a associate with a d dimensionaly binary vector. We say that a set of points $A \subseteq S^d$ **cover all the labels** if:

$$\forall i \in [d], \ell \in \{-1, +1\}, \exists x \in A : [\lambda(x)]_i = \ell$$

Definition II.11: StrongSperner problem

Input: A boolean circuit that computes a bipolar labelling $\lambda : [M]^N \to \{-1, 1\}^N$ II.10 satisfying the following boundary conditions $\forall i \in [N]$:

- if $x_i = 1 \implies [\lambda(x)]_i = +1$
- if $x_i = M \implies [\lambda(x)]_i = -1$

Output: A set of points $\{x^{(i)}\}_{i \in [N]} \subseteq [M]^{[N]}$, such that:

- *Closessness condition*: $\forall i, j \in [N] : \|x^{(i)} - x^{(j)}\|_\infty \leq 1$
- *Covers all labels* as defined in II.10

The above is a generalised variant of the tradtional Sperner problem to a grid of dimensions N and width of M. Throughout literature the same variants of the problem or specifications have been defined using sperner or discrete brouwer [6], [8]–[10]. For the sake of clarity, we will stick to StrongSperner.

Definition II.12: nD-StrongSperner problem

Input: A tuple $(\lambda, 0^k)$ of a StrongSperner instance but for only n dimensions, such that $\lambda : (\{0, 1\}^k)^n \to \{-1, +1\}^n$.
Output: A point $\alpha = (a_1, \ldots, a_n) \in A^n$, where $A = \{0, 1\}^k \setminus \{1^k\}$ such that:

$$\{\alpha + x \mid x \in \{0, 1\}^n\} \text{ cover all the labels II.10}$$

We use A to avoid edge cases. We assume dimensionality $n \geq 2$.

Chen et al. [10] demonstrated that all nD-StrongSperner are PPAD-complete. We mainly use this to show counting arguments with respect to the EndOfLine as people have indicated a parsimonious reduction between 2D-StrongSperner and 3D-StrongSperner to the EndOfLine problem but with *linear* colouring. The authors of these papers use these problems indifferently when talking about the reductions and therefore we can assume that either colouring will create a correct reduction.

c) Other PPAD problems: Counting complexity of PPAD

These problems have found connections with various other problems such as Nash Equilbria, Fixed point theorems or Sperner Lemmas. The current project looks on the counting complexity of such problems as despite two problems being PPAD-complete, Ikenmeyer et al. [1] has showed several examples where their counting difficulty can differ vastly. Examples of such statements can be summarised as:

$$\textbf{\#PPAD}(\textsc{SourceOrSink}) - 1/2 = \textbf{\#P}$$
$$\textbf{\#P}^A \not\subseteq \textbf{\#PPAD}(\textsc{SourceOrExcess}(2,1))^A - 1$$

Currently we are mainly focused into the study of a certain **PPAD**-*complete* problem, known as PureCircuit

3) Kleene Logic and Hazard-Free Circuits: Kleene logic, is the type of system that extends the traditional binary system to: $\mathbb{T} = \{0, \perp, 1\}$. Study of such systems has an impact to both the development of robust systems and from a theoritical point of view, people developed systems such as *Alternative Turing mahcines*, as well as showing important correlations between the complexity of monotone circuits and the robustness of hazard free circuits.

We want to introduce the following concepts:

Definition II.1 (Kleene Resolutions). *Given $x \in \mathbb{T}^n$, we define the resolutions of x, using the following notation:*

$$\textsc{Res}(x) \triangleq \left\{ y \in \mathbb{B}^n \mid \forall j \in [n] : x_i \in \mathbb{B} \implies x_i = y_i \right\}$$

Definition II.2 (Hazard circuit). *A circuit C, on n inputs has **hazard** at $x \in \mathbb{T}^n \iff C(x) = \perp$ and $\exists b \in \mathbb{B}, \forall r \in \textsc{Res}(x) : C(r) = b$*

Definition II.3 (Natural function). *A function $F : \mathbb{T}^n \to \mathbb{T}$ can be computed by stable values and is monotone with respect to $\preceq$*

Definition II.4 (K-bit Hazard). *For $k \in \mathbb{N}$ at $x \in \mathbb{T}^n \iff C$ has a hazard at x and $\perp$ appears at most k times in x*

III. Current Progress

A. Current Timetable

B. Theory Progress

1) EndOfLine to PureCircuit Constant Parsimonious Reduction: We will define the necessary prerequisites for the current section:

Definition III.1 (Poly-Function Bounded Parsimonious Reductions)**.** *Given two counting problems $A, B : \{0,1\}* \to \mathbb{N}$ and a function $f : 0, 1^* \to \mathbb{N}$ such that $f \in n^{O(1)}$, we say that:*

$$A \subseteq^f B$$

If for input $w \in \{0,1\}^*$, if a represent the number of solutions for problem A and b the number of solutions for problem B, we have:

$$a \leq b \leq f(|w|) \cdot a$$

If $\forall x : f(x) = c$ for $c \in \mathbb{N}$ then we will just say

$$A \subseteq^c B$$

Proposition III.1 (EndOfLine to StrongSperner Parsimonious reductions)**.**

$$\begin{aligned} \#\text{ENDOFLINE} - 1 &\subseteq \#\text{3D-STRONGSPERNER} - 1 \\ &\subseteq \#\text{2D-STRONGSPERNER} - 1 \end{aligned}$$

Daskalakis et al. [8] showed that we can find a parsimonious reduction between ENDOFLINE and 3D-BROUWER, which is a variant of 3D-STRONGSPERNER and is based on the same principal. Later on Chen et al. [9] was able to show that it suffices to use 2 dimensions by reducing the problem to a 2D planar graph variant of ENDOFLINE. Our contribution comes from the ability to parsimoniously constant reduce every ND-STRONGSPERNER problem to PURECIRCUIT. We will demonstrate for 3 dimensions and then we will generalise.

Theorem III.2 (3D-StrongnSperner to PureCircuit)**.** *Given $f(\cdot) = 19$, we argue*

$$\#\text{3D-STRONGSPERNER} \subseteq^f \#\text{PURECIRCUIT}$$

Where 3D-STRONGSPERNER is represented as a tuple $(\lambda, 0^n)$ where n corresponds to grid size of 2^n.

To show the above holds true we will first show the construction and prove its correctness. To simplify the counting argument, we modify a solution of 3D-STRONGSPERNER as such: Assuming $S \subseteq \{0,1\}^n$ is the set of all solutions, such that we define S as such:

$$\begin{aligned} S \triangleq \Big\{&(i,j,k) \in (\{0,1\}^n)^3 \mid \\ &\{\lambda(i + x_1, j + x_2, k + x_3) \mid x \in \{0,1\}^3\}\Big\} \text{ covers all labels} \end{aligned}$$

We will create $\Lambda : (\{0,1\}^{(n+1)})^3 \to \{-1,+1\}^3$ such that

$$\forall (i,j,k) \in (\{0,1\}^{(n+1)})^3 : \Lambda(i,j,k) \triangleq \lambda\left(\left\lfloor \frac{i+1}{2} \right\rfloor, \left\lfloor \frac{j+1}{2} \right\rfloor, \left\lfloor \frac{k+1}{2} \right\rfloor\right)$$

Conversely we can say that we map from our original domain to our new domain as such

$$\forall x \in \{0,1\}^{3n}, i \subseteq \{0,1,2\} : \lambda(x_{-i}, x_i) \to \Lambda(2x_i, 2x_{-i} - 1)$$

The above transformation can be visualised using the following figure

Claim III.3 (Transformation claim)**.** *We claim that $|S_{even}| = |S|$ where S_{even} is defined as:*

$$S_{even} \triangleq \left\{(i0, j0, k0) \in (\{0,1\}^{(n+1)})^3 \mid \{\Lambda(2i + x_1, 2j + x_2, 2k + x_3) \mid x \in \{0,1\}^3\}\right\} \textit{ covers all labels}$$

Proof. We argue that we have a bijective transformation $S \leftrightarrow S_{\text{even}}$. To do that we will first show that every point in $x \in S$ is mapped to a single point in $x' \in S_{\text{even}}$. Assume $(i,j,k) = x$ for some $i,j,k \in \{0,1\}^n$. For a point to be in S_{even} has to be the case all 3 coordinates are even. From our transformation, there is only one mapping that achieves that, which is when $j = \{1,2,3\}$. We have to verify that this point corresponds to the same set of points or when $x' = (2i, 2j, 2k)$. We can observe its neighbourhood set:

$$\left\{\Lambda(2i + x_1, 2j + x_2, 2k + x_3) \mid x \in \{0,1\}^3\right\}$$

We can observe that WLOG: $\Lambda(2i+1, \cdot, \cdot) = \lambda(i+1, \cdot, \cdot)$. This implies that the set above corresponds to the neighbourhood:

$$\left\{\lambda(i + x_i, j + x_j, k + x_k) \mid x \in \{0,1\}^3\right\}$$

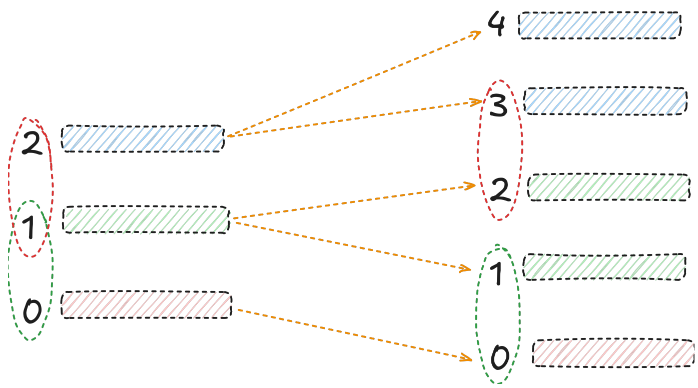


Fig. 2: The current figure allows depicts how solutions or pairs of solutions are mapped. For example for cube starting at $i = 0$ and $i = 1$, we now just have to look at $i = 0$ and $i = 2$.

And therefore the points match. We can conversely use the same argument to show the opposite direction and therefore we can conclude that $|S| = |S_{\text{even}}|$ ☐

For simplicity's sake I will rename $n + 1$ as n to keep the notation consistent. Given the above we provide a main sketch of our tranformation as such:

1) **Input bits**: We initialize input nodes s_1, s_2, s_3
2) **Bit generator**: We apply the *Bit Generator gadget* $\hat{P}$ for each s_i, to create numbers $u^1, u^2, u^3 \in \{0, 1\}^k$
3) **Circuit application**: We apply circuit modified circuit $\bar{\Lambda}$ to u^1, u^2, u^3 to create output values o_1, o_2, o_3
4) **Validation**: Copy the results back to s_1, s_2, s_3

The goal of the above transformation is to show that if $o_1 = o_2 = o_3 = \perp$, then we have a solution to the original instance. The current construction allows us to match a single box of the original problem to up to 19 solutions of the PureCircuit one. We can visualise it as such:

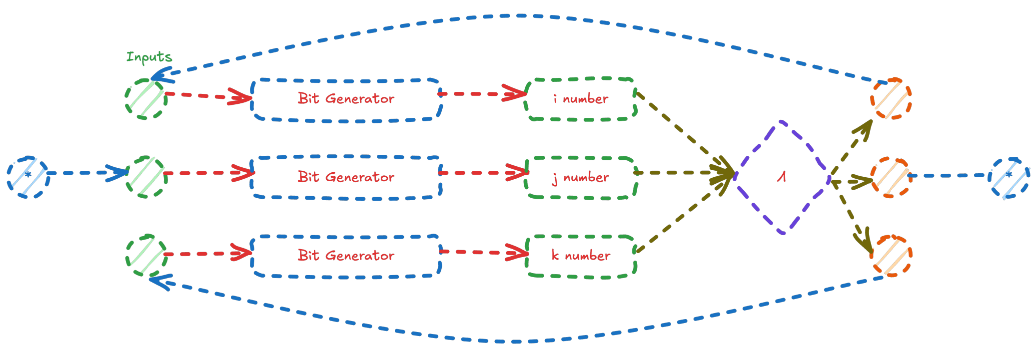


Fig. 3: Testing

Before we look at the construction, we will refer to the standard set of gates $\{*, +, \neg\}$. These can be trivially build using NOR gates. Lastly we only accept the following as valid outputs of the *Purify* gate: $(0, \perp), (1, \perp), (\perp, 1), (0, 1)$. It is easy to see that for these set of outputs, *PureCircuit* remains **PPAD**-complete, by using the continuity argument.

a) Bit generator: In order to generate our number we use the construction shown in the figure 4,

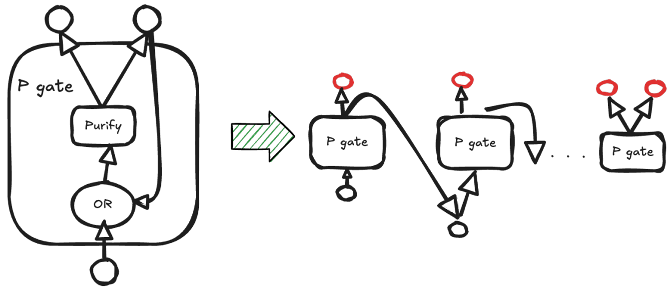


Fig. 4: Our bit generator bit is stacked version of a lot of P circuits. By stacking them as shown in the figure and extracting the red nodes, we get our binary number.

Given that construction we make the following lemma

Lemma III.4 (Bit Generator Lemma)**.** *The following hold true in our construction:*

1) *if* $s_i = b \implies u^i = b^n$, *given* $b \in \mathbb{B}$
2) *if* $s_i = \perp \implies \forall j \in [n-1] : u^i_j \in \mathbb{B}$ *and* $u^i_n \in \{1, \perp\}$

The above translates to: if we have $\perp$ *in our number, it can only be found in the LSB.*

Proof. The first part follows trivially from the defintion of the PURIFY gate, therefore if our input is a pure bit, we are copying it to all the inputs. To prove the second part of lemma, we use assume contradiction. Assume $\exists j \in [n-1]$. That would imply that one of the purify gates had an output of $(\perp, 1)$. But due to our OR gate, that would force the input to be 1 which would imply that the output is $1, 1$. This implies that the only bit that can be $\perp$ is the last bit. □

b) Circuit Application: In the current phase, we will modify the circuit Λ to be hazard-free using the construction by Ikenmeyer et al. [12] *Corollary 10*, where they showed that we can ensure a k-bit hazard-free circuit with the following properties:

$$\text{size}(C) = \left(\frac{ne}{k}\right)^{2k} (|C| + 6) + O(n^{2.71k})$$
$$\text{depth}(C) = D + 8k + O(k \log n)$$

Using the bit generator lemma III.4, we can observe that we can have at most 3 $\perp$ values in our input. This implies that we can create a circuit of polynomial size such that it is 3-bit-hazard-free. Using that property we create the lemma below:

Lemma III.5 (Circuit Application Lemma)**.** *If* $o_1 = o_2 = o_3 = \perp$ *implies that we covered all the labels and found a solution in* $(i0, j0, k0) \in S_{even}$

h!. First part is to understand what our $u^{(1)}, u^{(2)}, u^{(3)}$ represent. To do that, we use the following visualisation on the figure 5. The first $n-1$ bits denote the i, j, k indexes of our original solution. The cube essentially represents all possible realisations

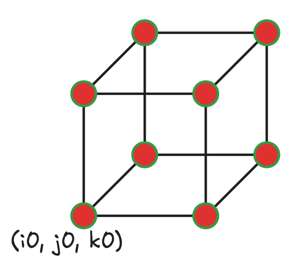


Fig. 5: Representation of our solutions

of $i\perp, j\perp, k\perp$. We know that the neighbouring corners correspond to vertices in our original instance. When all LSBs are $\perp$, we can observe that we are calculating our $\Lambda(\cdot)$ for all corners of the cube simultaneously. If $(i0, j0, k0)$ is a solution that implies:

$$\forall i \in \{1, 2, 3\}, b \in \{-1, 1\} \exists c \in \text{RES}(i\perp, j\perp, k\perp) : [\bar{\Lambda}(c)]_i = b$$

Since $\bar{\Lambda}$ is 3-bit hazard free, if we outputs $\perp$ in dimension j, then found two points with different labels in j. We can therefore argue that if the output it $\perp^3$, then we covered all the labels. □

The above section conclude our construction. We will now demonstrate the correctness argument:

Lemma III.6 (Correctness lemma)**.** *We argue that every correct assingment of the above circuit, corresponds to a point in* S_{even}

From the transformation claim III.3, we know that if our circuit can find all corresponding points in P_{even}, we can find the all the solutions of the original circuit.

Proof. To prove our lemma, it suffices to show that $o_1 = o_2 = o_3 = \perp$. Let's assume by contradiction that $\exists i \in \{1, 2, 3\}$ such that label i is not covered, meaning $o_i \neq \perp$. WLOG assume that $o_i = 0$. Our verification stage, will copy 0 onto s_i. By our bit generator lemma III.4, we have that $u^{(i)} = 1^n$. From the boundary conditions of our circuits and the k-bit hazard-freeness construction, we know that $\Lambda(*, 0^n, *) = \{*, 1, *\}$. But that implies $o_i = 1$ which leads to a contradiction. We can make a similar argument to when $o_i = 1$. Therefore, by III.5, if $o_1 = o_2 = \perp$, we have a found a polychromatic cube. $\square$

Counting Argument: To find how many solutions correspond to a single solution of the original instance, we can make the following observation: Looking at our cube in figure 5, if a side of the cube contains all labels or if an edge contains all labels, we are counting these as additional solutions. We can find an upper bound by making the following observation:

$$\underbrace{\binom{3}{1} \cdot 2}_{\substack{\text{One of the sides is odd} \\ \text{and covers all labels}}} + \underbrace{\binom{3}{2} \cdot 2^2}_{\substack{\text{One of the edges is odd} \\ \text{and covers all labels}}} + \underbrace{1}_{\text{All LSBs are } \perp} = 19$$

Therefore, we can bound the number of solutions of 3D-StrongSperner by a factor of 19. Core breakthorugh came from the utilisation of the hazard-freeness from [12], as well as studying some problems in EOPL and UEOPL or more specifically the One-Permutation Discrete map. Athough in the end we were not able to extract any significantlly useful results out of it, its key observations were enough.

Useful Corollaries: We can easily observe that our reduction above can work with any dimensionality. In fact for any $\forall n \in \mathbb{N}_{\geq 2}$, Chen et al. showed that nD-StrongSperner is still *PPAD-Hard* in [10]. Additionally based on our proof, we can make the following corollary:

Corollary III.6.1. *Given* nD-StrongSperner*, we define* f *as:*

$$f(\cdot) = 1 + \sum_{i=1}^{n-1} \binom{n}{i} 2^i = 3^n - 2^n = a_n$$

Such that we have the following relation

$$\#\text{nD-StrongSperner} \subseteq^{a_n} \#\text{PureCircuit}$$

To complete our last part of the theorem, it suffices to make use of the parsimonious reduction EndOfLine to 2D-StrongSperner that Chen et al. [10]. It has to be denoted that a similar reduction was made by Daskalakis et al. [8], where he reduced EndOfLine to 3D-StrongSperner parsimoniously with similar ideas. For our theorem we will refer to the 2D-StrongSperner reduction and conclude with the theorem.

Theorem III.7 (From EndOfLine to PureCircuit)**.**

$$\#\text{EndOfLine} - 1 \subseteq \#\text{2D-StrongSperner} - 1 \subseteq^5 \#\text{PureCircuit} - 1$$

2) Hazard-Free Logic Findings: Whilst trying to tackle the main objectives of the dissertation, we stumbled upon several interest relations across Hazard. We will use the notion of promise problems to show the following relation. First we will define a variant of UnSAT hazard as explained in [12].

Definition III.2. *Given a circuit* C *that computes* $f : \mathbb{B}^n \to \mathbb{B}$ *such that* $\forall x \in \mathbb{B}^n : f(x) = 0$*, find* $\bar{x} \in \mathbb{T}^n$ *such that* $C(\bar{x}) = \perp$*. We assume that* $\forall x \in \mathbb{B}^n : C(x) = 0$*.*

Given the idea above we propose the following:

Proposition III.8.

$$\#\text{PromiseUnsatHazard} \subseteq \#\text{PureCircuit} - 1$$

For the purposes of the current reduction, we will use the same Purify values $(0, 1), (0, \perp), (1, \perp), (\perp, 1)$

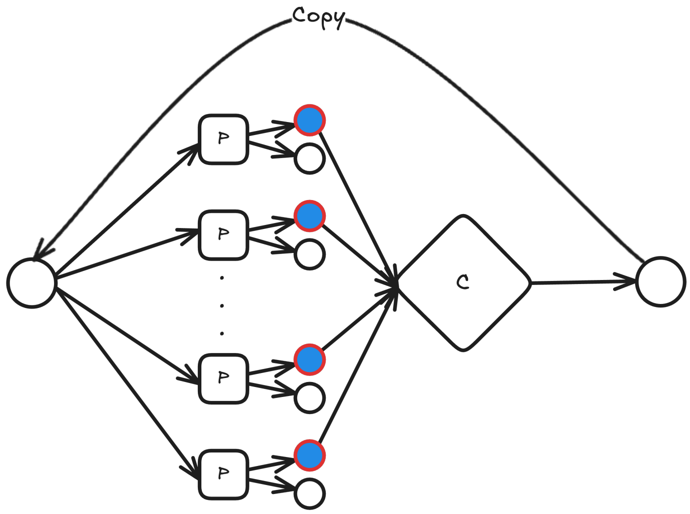


Fig. 6: Pure Circuit construction

a) Construction: We start with a single node s. We create n copies of PURIFY, where all use s as the input. From each purify gate we use the left output and create a vector $\hat{x} \in \mathbb{T}^n$ and pass them onto C which will output onto a node o. We copy the output onto s. The above description can be summarised in the following figure 6.

Proof. To prove our counting argument, we can make the following observation: If $o = 0$, we are computing $C(0^n) = 0$, which is the one guaranteed solution. We know that $o \neq 1$, therefore the only other possible solutions are $\bar{x} \in \mathbb{T}^n : C(\bar{x}) = \bot$. We also know from our construction of our PURIFY gate values, that our highlighted nodes can take values $\{0, 1, \bot\}$. Therefore, given that $\hat{x} \in \mathbb{T}^n$, our construction can find all possible solutions. □

3) Hazard Circuits and Tarski: The following finding was derived when looking into alternative fixed point problems. In the following proposition we found a pretty interesting connection between *natural* functions and TARSKI. More specifically, TARSKI is a complexity problem in CLS. First it is essential to define monotone functions:

Definition III.3 (Monotone functions)**.** *Given two posets $(L_1, \preceq_{L_1})$ and $(L_2, \preceq_{L_2})$, a function $f : L_1 \to L_2$ is* ***monotone*** *if and only if:*

$$\forall x, y \in L_1 : x \preceq_{L_1} y \implies f(x) \preceq_{L_2} f(y)$$

Definition III.4 (TARSKI problem definition)**.** *Given a lattice $(L, \wedge, \vee)$, and a monotone function $f : L \to L$, we define solutions to the problem as:*

1) *Find $x \in L : f(x) = x$*
2) *Find $x, y \in L$ such that $x \preceq y$ and $f(x) \not\preceq f(y)$*

We can use the information ordering to impose a partial order on $(\mathbb{T}, \preceq)$ as such:

$$\forall x, y \in \mathbb{T}^n : x \preceq y \implies \forall j \in [n] : x_j \in \mathbb{B} \implies x_j = y_j$$

Therefore we create the following subclass of TARSKI

Definition III.5 (KLEENETARSKI problem definition)**.** *Given $F : \mathbb{T}^n \to \mathbb{T}^n$, where F is a natural function, we want to find the set of points* ***Fix**** *such that*

$$\forall x \in \textbf{\textit{Fix}}^\star : F(x) = x$$

We assume F will be represented as a circuit, that uses $\{*, +, \neg, \nvdash, \nVdash\}$.

Proposition III.9 (Parsimonious Reduction between KLEENETARSKI and PURECIRCUIT)**.**

$$\#\text{KLEENETARSKI-1} \subseteq \#\text{PURECIRCUIT} - 1$$

The above reduction can be done trivially using the following three steps:

1) Initiate a vector of nodes s
2) Pass the inputs into a circuit C that computes the input into an output vector o
3) Copy the results back into s

We can observe that for any valid assignment, $\mathbf{x}[s] = \mathbf{x}[o]$. Therefore, our construction can find all fix points for Kleene Circuits.

4) Reductions to the EndOfLine: One can also attempt to make some reductions to the ENDOFLINE. It has to be noted, that we assume that there are no parsimonious reductions to the ENDOFLINE, but we were able to construct some variants that allows to make there reductions. They key insights for this comes from the PPAD-inclusion proof by Deligkas et al. [6], where they showed that we can reduce PureCircuit, to *Brouwer* problem by constructing a continuous function $F : [0,1]^n \to [0,1]^n$ as such: For each $v \in V$, we create continuous functions $f_i(\cdot) : [0,1]^n \to [0,1]$ where we take thne input of vector of all the nodes and output the value of the current node

1) If $v \in V$ is the output of a NOR gate with inputs x_1, x_2, then we have:

$$f_v(\mathbf{x}) := (1 - \mathbf{x}_{x_1}) \cdot (1 - \mathbf{x}_{x_2})$$

2) If y_1, y_2 are the outputs of a PURIFY gate with z as input, then we have:

$$f_{y_1}(\mathbf{x}) := \max\{2 \cdot \mathbf{x}_z - 1, 0\}$$
$$f_{y_2}(\mathbf{x}) := \min\{2 \cdot \mathbf{x}_z, 1\}$$

If $F(x) = x$, then we also found a solution for PURECIRCUIT by:

$$\forall v \in V : \mathbf{x}[v] = \begin{cases} 0 & \mathbf{x}[v] = 0 \\ \perp & 0 < \mathbf{x}[v] < 1 \\ 1 & \mathbf{x}[v] = 1 \end{cases}$$

Given that as an assumption we can restrict our *Purify* gate to outputs $(0, \perp), (0, 1), (\perp, 1)$. Using that we can define two types of variants of PURECIRCUIT that are parsimoniously reducible to SOURCEORPRESINK. We first define SOURCEORPRESINK as:

Definition III.6 (SOURCEORPRESINK definition)**.** *A* SOURCEORPRESINK *is defined by the same construction as the* ENDOFLINE *but the solutions are sources and predecessors of sinks. If a node is both a source and a presink, then we count it once.*

We will annotate our *Purify* gates with an additional bit $b \in \mathbb{B}$ such that

$$\forall x \in \mathbb{T}, y_1, y_2 \in \mathbb{T} : \text{PURIFY}^b(x) = (y_1, y_2) \implies \text{PURIFY}^{\neg b}(x) = (y_2, y_1)$$

We can create two types of problems based on the annotation:

1) Acyclic PURECIRCUIT: Our instance is acyclic
2) Permutation-free PURECIRCUIT: The outputs of a pure circuit, are connected to a circuit such that any permutation of the inputs retains the result. More formally:

$$\forall x \in \mathbb{T}^n, \sigma \in \mathcal{A} : C(\sigma(x)) = C(x)$$

Where $\mathcal{A}$ is the set of automorphisms of $[n] \to [n]$. The above description can be depicted in the following figure: (ADD FIGURE)

We can show that above variants we can create the following proposition (ref). To prove the reduction we can need to argue that we can go from one solution to another. The core idea of these problems is: if we find a solution given in a set of configurations $\gamma \in \{0,1\}^p$ where p are the number of purify gates, we can match it with an additional solution where our set of configurations is $\neg\gamma$. This can be visualised as such: If the flip does not affect the inputs then we can create the following graph:

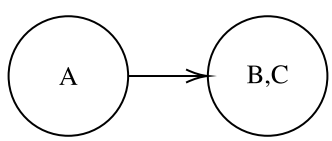


Fig. 7: Equal solution conversion

Otherwise for two distinct solutions we can use the following

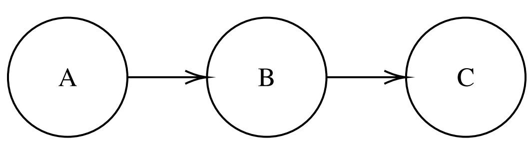


Fig. 8: Not equal solution

5) Next steps: Our next steps will focus mainly on two directions. One is lowering the bound as we declare in our conjecture with the usage of snake embeddings. We believe that each dimension can be parsimoniously reduced with another. This implies that, if we can show that $\forall n \in \mathbb{N}_{\geq 3}$: #nD-StrongSperner $\subseteq$ #2D-StrongSperner we get to prove our conjecture.

With regards to, our second direction, we want to emphasize on the hardness of PureCircuit. Or more specifically we will investigate as to if any of the following statements hold true.

$$\exists n, \alpha \in \mathbb{N}_{\geq 2} : \#\textsc{SourceOrExcess}(\alpha, 1) - 1 \subseteq \#n\textsc{D-StrongSperner} - 1$$
$$\exists n \in \mathbb{N}_{\geq 2} : \#\textsc{SourceOrExcess}(n, 1) - 1 \subseteq \#\textsc{PureCircuit} - 1$$

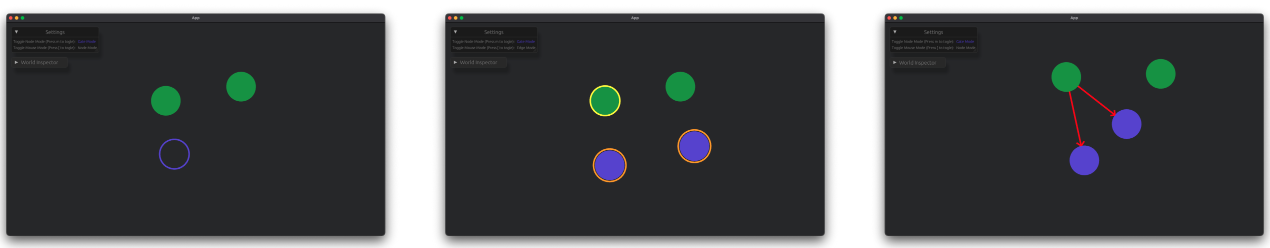
We believe the above two should be our next steps in order to get close to the main question of our project. Ideally generalising, the above reductions to the infinity hierarchy of #SourceOrExcess$(\cdot, 1)$ would be ideal as it may lead to finding an overall upper bound towards the entire #PPAD $- 1$ class.

C. Software Progress

In our project we are also focused on developing a visualisation tool for PureCircuit instances. In our introduction we referred to three main objectives. As of time of writing w are close to the completion of the first one. We will provide a table of objectives that have been achieved, as well as the remaining requirements needed to complete the first milestone in II. Lastly, we provide a figure of our current visualisation in 9

Accomplished	**Finished**
Ability to add and move nodes	Add values to value nodes
Toggle between value nodes and gate nodes	Specify the type of gate to add
Add edges. Ensure that edges can only be added between heterogeneous nodes	Add indicators as to how many edges are excess/remaining for the gate to be valid
Move whole graph	Inclusion of a status bar as to whether the current assignment is correct, wrong or syntactically incorrect
Create panel to show current state as well as some indicator and guides	

TABLE II: Finished and remaining issues



(a) Add nodes (b) Add edges (c) Final state

Fig. 9: Screenshots of different states of the visualisation tool

IV. Next Steps

A. Theory Crafting Next steps

Our next steps will focus mainly on two directions. One is lowering the bound as we declare in our conjecture with the usage of snake embeddings. We believe that each dimension can be parsimoniously reduced with another. This implies that, if we can show that $\forall n \in \mathbb{N}_{\geq 3}$: #nD-StrongSperner $\subseteq$ #2D-StrongSperner we get to prove our conjecture.

With regards to, our second direction, we want to emphasize on the hardness of PureCircuit. Or more specifically we will investigate as to if any of the following statements hold true.

$$\exists n, \alpha \in \mathbb{N}_{\geq 2} : \#\textsc{SourceOrExcess}(\alpha, 1) - 1 \subseteq \#n\textsc{D-StrongSperner} - 1$$
$$\exists n \in \mathbb{N}_{\geq 2} : \#\textsc{SourceOrExcess}(n, 1) - 1 \subseteq \#\textsc{PureCircuit} - 1$$

We believe the above two should be our next steps in order to get close to the main question of our project. Ideally generalising, the above reductions to the infinity hierarchy of #SOURCEOREXCESS$(\cdot, 1)$ would be ideal as it may lead to finding an overall upper bound towards the entire #PPAD $- 1$ class.

B. Software Next Steps

In the next steps we aim to complete the tasks that were mentioned in the above table, as well as finding a solution or even better finding all possible solutions. Of course due to the nature of the problem, finding a single or all solutions for big instances is computationally difficult. We aim to investigate a method introduced by Eichelberger, where he utilised repeated applications of the circuit to detect oscillations and replace them with $\perp$ values. The usage of that procedure was to detect hazard but aim to utilise that procedure as a heuristical approach to reach a satisfying assignment. We do not primarily aim to fully optimise these algorithms, as the project focuses more on analysing the theoretical findings of PURECIRCUIT.

Alternatively, one can use some facts of PURECIRCUIT showed by Deligkas et al. [6], were under specific gate sets, one can find a PURECIRCUIT instance in polynomial time. Additionally we were able to find instances or variants of PURECIRCUIT that are much easier. Conditions like acyclicity make our problem much easier to deal with or permutation free circuit, make the problem easier to solve. But just because a solution can be easy to find, does not imply that finding the total number of solutions is easy, as [3] indicated, or from a more direct example by [13], where

$$\text{HAMILTONIANCYCLE} \leq_P \#\text{CYCLE}$$

V. APPRAISALS

A. Reflections

Over the last couple of months I made a lot of mistakes whilst working with the project. Was just not focusing enough or spending enough time to do the necessary research needed to comprehend the topic. Moreover, I spent time focusing on the wrong aspects of my assignment. An example of that is referred to the sections were we showed variants of our pure circuit that have a combinatorial interpretation. Another core issue I faced was the lack of experience when dealing with more theoretical based project. This led me to trying to tackle the problem head on without exploring alternative avenues. For example we can observe that our main reduction from the *EndOfLine* came when using topological problems instead.

B. Lessons Learned

Up to the current phase of our project, we were able to gather several important takeaways. Exploring the literature and trying to work the problem under different perspectives may lead to potential breakthroughs. The main catalyst to our core finding, came when looking into problems such as EOPL or more specifically the OPDC problem, described by Fernley et al. [14]. OPDC can be described as a simpler version of the traditonal n-dimensional Brouwer problem where instead, we have a unqiue fixed point and our function will point towards the unique solution. Moreover, deeper look into Kleene-logic or more specifically *Hazard-free* circuits allowed us to exploit the $\perp$ value to compute multiple input simultaneously [12]. Using the two aforementioned observations as well as looking at the original reduction again allowed us to achieve the desired breakthrough.

In order to accomplish the above research, we have also learned how to research more effectively. Exploitation of LLMs such as `ChatGPT`, `Perplexity` and more, we are able to approach the problem from multiple perspectives. Moreover, thought organisation and cataloguing with `Obsidian` also came crucial when handling a large volume of ideas and information over a long period of time, as explained in the previous sections. Last but not least, utilisation of books, related research and textbooks also came important when it came to theory crafting and idea extraction.

Last but not least, the general approach to the project changed over the course of time. Understanding that the journey of discovery as important as the discovery itself, made the project more enjoyable. Having the freedom to explore areas and expand the horizons of what we know, is what keeps this project exhilarating. Ultimately, this mindset helped the project stay engaging and allowed for continuous learning throughout.

VI. PROJECT MANAGEMENT

In order to refer to our project management, we will talk about the toolings and methods we used, as summarised in the figure 10.

Our project also includes the development of a visualisation tool for creation and verification of several PURECIRCUIT instances. We opted with *Rust*, as the main language of development, due to its high and low level features. From the one hand, *Rust* can efficiently handle memory allocation safely with its clever usage of the Borrower-Ownership framework. Conversely, it implements a strongly typed system with help of generics, associated types and algebraic types allowing us to create a versatile and compact library. Given the above, we utilise techniques such as `Proptest`, where we create strategic randomized tests that check whether function follows an expected property. On the other hand we make use of `Petgraph`, which is a sophisticated library that handles graph-like structures in *Rust*.

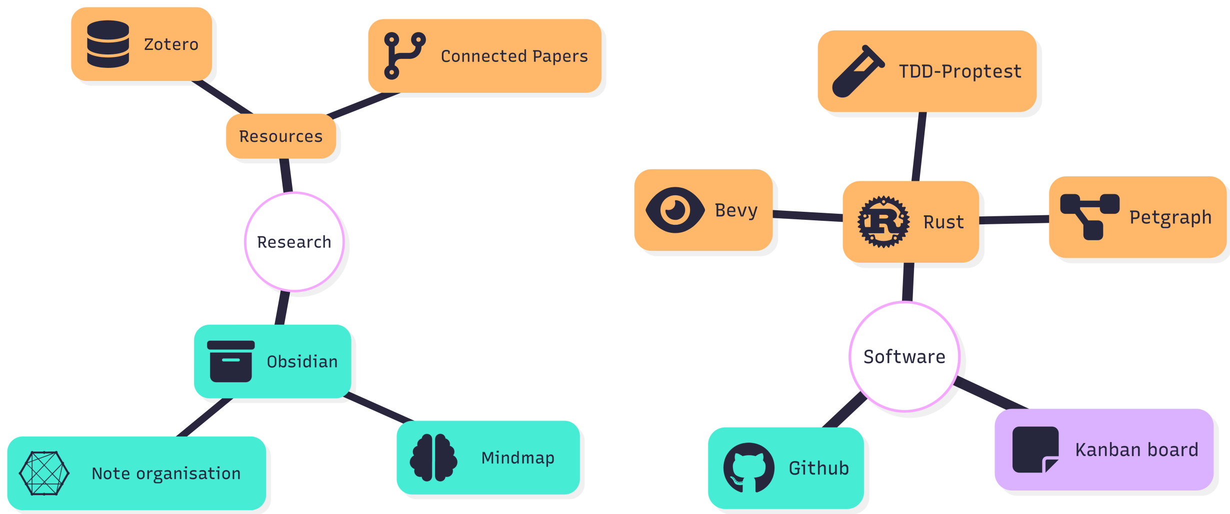


Fig. 10: Tooling

For visualisation we decided to go with *Bevy*. *Bevy* is a game engine that uses the *ECS* software architecture. A general workflow of an *ECS* system can be described as such: a state contains entities, each of which is composed of components or properties. A system is a specialised method that gathers entities based on their components and describes an interaction between them. This whole process can be visualised in the figure 11

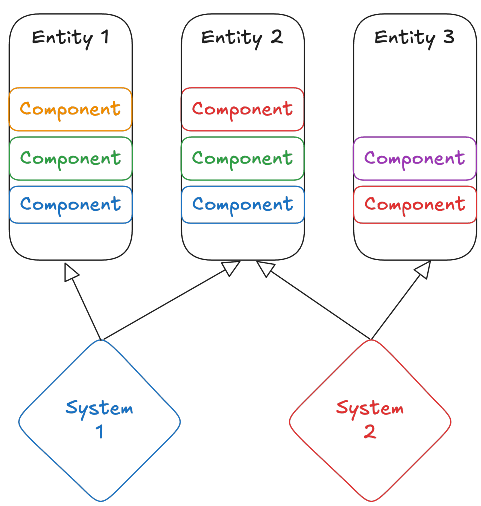
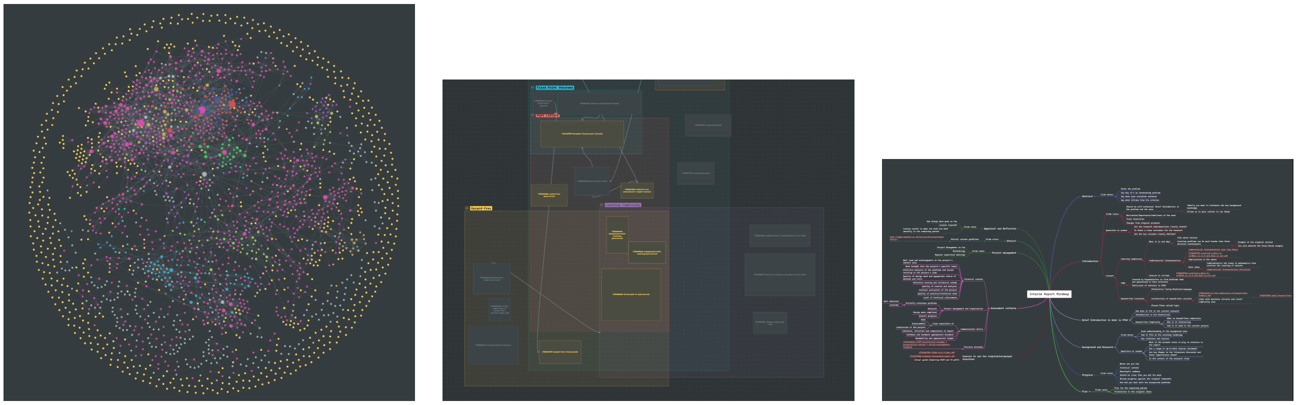


Fig. 11: ECS workflow visualisation

With regards to the our research management, our core tool came through the usage of obsidian. As we can see in the figure 10, `Obsidian` beyond its traditional usage of note taking, it comes with several handy tools such as note organisation and mind-mapping. These can be seen in the figure 12, where we utilised connections across notes as well as its drawings or other thought organisation tools in order experiment and manage ideas.

A. Risk Management

With regards to risks and mitigations, the biggest risk we face is the inability to resolve our main question. Due to the limited literature surrounding PURECIRCUIT and its counting nature as well as the recency in the development of TFNP – 1, our question could go into many directions. The discovery of our reduction allowed us to step closer to our question and in our table IV, we will analyze this is greater depth. The general mitigation strategy we can do is to keep researching, keep finding connections until we can connect all the necessary pieces to do the jump.



(a) Obsidian Graph (b) Obsidian Canvas (c) Obsidian mindmap

Fig. 12: Usages of Obsidian

Severity	Probability	Mitigation	Mitigation	Address
High	Medium	Software may not be feasible within the remaining time frame.	Focus on the the first objective where the project. Restrict to solution finding or counting when the number of nodes is small.	Not yet
Low	Medium	Software not identifying a correct solution	Usage of TDD techniques and property testing to ensure correctness. Comparison with hand-made instances	Not yet
High	High	Inability to extend the reductions of the PureCircuit problem to SourceOrExcess	Find a path of parsimonious reductions between the two problems. If we cannot ensure a one-to-one relation between the solutions, we will focus on $1 - c$ for some $c \in \mathbb{N}$	✓
Medium	High	Inability to prove the main conjecture	We can make some heuristical arguments or find reductions between other problems. We argue that if we are able to unravel enough correlations, we will hope to at least bridge the problems. Develop useful gadgets one can use with PureCircuit	Not yet
High	High	Incorrect proofs or reductions.	Analyse the problem under different constraints. Apply the duck method, where attempt to explain the solution to a person which not necesserily an expert. Validate proof with supervisor	Not yet
High	Medium	Develop combinatorial friendly variants of PureCircuit	Apply robustness on the gate set of PureCircuit. Develop new gates or variants are easier to work with.	✓

TABLE IV: Risk managment table.

From our table, it is worth expanding on some of the points. The best method we found when tackling this problem is when we try to uncover reductions between other PPAD problems or when trying to incorporate gadgets from Kleene theory. We hope that by expereminting enough we will be able to get close to resolve our conjectures. In addition, for the last point, we made several efforts to ensure our problem is PPAD-complete and combinatorial friendly, by trying to restrict the PURIFY gate. We soon realised that any methods that detect the $\perp$ value or try to eliminate ultimately fail. The observation can be based on a continuity argument that Marino created [15] and in general any possible extensions or gates we can add seem to adhere continuity. (TODO)

VII. CONCLUSION

REFERENCES

[1] I. Pak, "What is a combinatorial interpretation?" Sep. 2022.

[2] C. Ikenmeyer and I. Pak, "What is in #P and what is not?" in *2022 IEEE 63rd Annual Symposium on Foundations of Computer Science (FOCS)*, Oct. 2022, pp. 860–871.

[3] L. G. Valiant, "The complexity of computing the permanent," *Theoretical Computer Science*, vol. 8, no. 2, pp. 189–201, Jan. 1979.

[4] C. Ikenmeyer, I. Pak, and G. Panova, "Positivity of the Symmetric Group Characters Is as Hard as the Polynomial Time Hierarchy," *International Mathematics Research Notices*, vol. 2024, no. 10, pp. 8442–8458, May 2024.

[5] C. H. Papadimitriou, "On the complexity of the parity argument and other inefficient proofs of existence," *Journal of Computer and System Sciences*, vol. 48, no. 3, pp. 498–532, Jun. 1994.

[6] A. Deligkas, J. Fearnley, A. Hollender, and T. Melissourgos, "Pure-Circuit: Tight Inapproximability for PPAD," *J. ACM*, vol. 71, no. 5, pp. 31:1–31:48, Oct. 2024.

[7] S. Kleene and M. Beeson, *Introduction to Metamathematics*. Ishi Press International, 2009.

[8] C. Daskalakis, P. Goldberg, and C. Papadimitriou, *The Complexity of Computing a Nash Equilibrium*. Association for Computing Machinery, Jan. 2006, vol. 39.

[9] X. Chen and X. Deng, "On the complexity of 2D discrete fixed point problem," *Theoretical Computer Science*, vol. 410, no. 44, pp. 4448–4456, Oct. 2009.

[10] X. Chen, X. Deng, and S.-H. Teng, "Settling the complexity of computing two-player Nash equilibria," *J. ACM*, vol. 56, no. 3, pp. 14:1–14:57, May 2009.

[11] C. Daskalakis, S. Skoulakis, and M. Zampetakis, "The complexity of constrained min-max optimization," in *Proceedings of the 53rd Annual ACM SIGACT Symposium on Theory of Computing*, ser. STOC 2021. New York, NY, USA: Association for Computing Machinery, Jun. 2021, pp. 1466–1478.

[12] C. Ikenmeyer, B. Komarath, C. Lenzen, V. Lysikov, A. Mokhov, and K. Sreenivasaiah, "On the complexity of hazard-free circuits," *Journal of the ACM*, vol. 66, no. 4, pp. 1–20, Aug. 2019.

[13] S. Arora and B. Barak, *Computational Complexity: A Modern Approach*. Cambridge: Cambridge University Press, 2009.

[14] J. Fearnley, S. Gordon, R. Mehta, and R. Savani, "Unique end of potential line," *Journal of Computer and System Sciences*, vol. 114, pp. 1–35, Dec. 2020.

[15] L. R. Marino, "General theory of metastable operation," *IEEE Transactions on Computers*, vol. C-30, no. 2, pp. 107–115, Feb. 1981.