CS419 - Assignment1

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1 Problem 1

1.1 Part a

$$\begin{aligned} Y &= i\,|1\rangle\langle 0| - i\,|0\rangle\langle 1| \\ &= i\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} - i\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix} \\ &= \begin{pmatrix}0 & 0\\ i & 0\end{pmatrix} + \begin{pmatrix}0 & -i\\ 0 & 0\end{pmatrix} \\ &= \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \end{aligned}$$

$$\begin{aligned} Z &= |0\rangle\langle 0| - |1\rangle\langle 1| \\ &= \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix} \\ &= \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} - \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} \\ &= \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \end{aligned}$$

1.2 Part b

1.2.1 X matrix

$$\begin{aligned} X\,|0\rangle &= (|0\rangle\langle 1| + |1\rangle\langle 0|)\,|0\rangle \\ &= |0\rangle\,\langle 1|0\rangle + |1\rangle\,\langle 0|0\rangle \\ &= |0\rangle * 0 + |1\rangle * 1 = |1\rangle \\ X\,|1\rangle &= (|0\rangle\langle 1| + |1\rangle\langle 0|)\,|1\rangle \\ &= |0\rangle\,\langle 1|1\rangle + |1\rangle\,\langle 0|1\rangle = |0\rangle \end{aligned}$$

$$X\,|+\rangle = \frac{1}{\sqrt{2}}X(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}(X\,|0\rangle + X\,|1\rangle) = \frac{1}{\sqrt{2}}(|1\rangle + |0\rangle) = |+\rangle$$

$$X\,|-\rangle = \frac{1}{\sqrt{2}}X(|0\rangle - |1\rangle) = \frac{1}{\sqrt{2}}(X\,|0\rangle - X\,|1\rangle) = \frac{1}{\sqrt{2}}(|1\rangle - |0\rangle) = -\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = -\,|-\rangle$$

1.2.2 Y matrix

$$\begin{aligned}
Y|0\rangle &= i(|1\rangle\langle 0| - |0\rangle\langle 1|)|0\rangle \\
&= i(|1\rangle\langle 0|0\rangle - |0\rangle\langle 1|0\rangle) = i|1\rangle
\end{aligned}$$

$$\begin{aligned}
Y|1\rangle &= i(|1\rangle\langle 0| - |0\rangle\langle 1|)|1\rangle \\
&= i(|1\rangle\langle 0|1\rangle - |0\rangle\langle 1|1\rangle) = -i|0\rangle
\end{aligned}$$

$$Y|+\rangle = \frac{1}{\sqrt{2}}Y(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}(Y|0\rangle + Y|1\rangle) = \frac{1}{\sqrt{2}}(i|1\rangle - i|0\rangle) = -i|-\rangle$$

$$Y|-\rangle = \frac{1}{\sqrt{2}}Y(|0\rangle - |1\rangle) = \frac{1}{\sqrt{2}}(Y|0\rangle - Y|1\rangle) = \frac{1}{\sqrt{2}}(i|1\rangle + i|0\rangle) = i|+\rangle$$

1.2.3 Z matrix

$$\begin{aligned}
Z|0\rangle &= (|0\rangle\langle 0| - |1\rangle\langle 1|)|0\rangle \\
&= |0\rangle\langle 0|0\rangle - |1\rangle\langle 1|0\rangle = |0\rangle
\end{aligned}$$

$$\begin{aligned}
Z|1\rangle &= (|0\rangle\langle 0| - |1\rangle\langle 1|)|1\rangle \\
&= |0\rangle\langle 0|1\rangle - |1\rangle\langle 1|1\rangle = -|1\rangle
\end{aligned}$$

$$Z|+\rangle = \frac{1}{\sqrt{2}}Z(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}(Z|0\rangle + Z|1\rangle) = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = |-\rangle$$

$$Z|-\rangle = \frac{1}{\sqrt{2}}Z(|0\rangle - |1\rangle) = \frac{1}{\sqrt{2}}(Z|0\rangle - Z|1\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle$$

2 Problem 2

Given state $|\psi\rangle = \sqrt{\frac{3}{5}}|0\rangle + \sqrt{\frac{2}{5}}|1\rangle$. I will use $x = \sqrt{\frac{3}{5}}$ and $y = \sqrt{\frac{2}{5}}$ to simplify the equations. In addition, since $x, y \in \mathbb{R}$, $\bar{x} = x, \bar{y} = y$

2.0.1 Part a

$$\begin{aligned}
\langle\psi|\psi\rangle &= (x\langle 0| + y\langle 1|)(x|0\rangle + y|1\rangle) \\
&= x^2\langle 0|0\rangle + xy\langle 0|1\rangle + yx\langle 1|0\rangle + y^2\langle 1|1\rangle \\
&= x^2 + y^2 = \frac{3}{5} + \frac{2}{5} = 1
\end{aligned}$$

2.0.2 Part b

$$\begin{aligned}
|\psi\rangle\langle\psi| &= (x|0\rangle + y|1\rangle)(x\langle 0| + y\langle 1|) \\
&= x^2|0\rangle\langle 0| + xy|0\rangle\langle 1| + yx|1\rangle\langle 0| + y^2|1\rangle\langle 1| \\
&= \begin{pmatrix} x^2 & xy \\ xy & y^2 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 3 & \sqrt{6} \\ \sqrt{6} & 2 \end{pmatrix}
\end{aligned}$$

2.0.3 Part c

$$\begin{aligned} Pr[0] &= |\langle 0|\psi\rangle|^2 = |\langle 0|(x|0\rangle + y|1\rangle)|^2 \\ &= |(x\langle 0|0\rangle + y\langle 0|1\rangle)|^2 \\ &= |x|^2 = x^2 = \frac{3}{5} \end{aligned}$$

$$\begin{aligned} Pr[1] &= |\langle 1|\psi\rangle|^2 = |\langle 1|(x|0\rangle + y|1\rangle)|^2 \\ &= |(x\langle 1|0\rangle + y\langle 1|1\rangle)|^2 \\ &= |y|^2 = y^2 = \frac{2}{5} \end{aligned}$$

2.0.4 Part d

$$\begin{aligned} Pr[+] &= |\langle +|\psi\rangle|^2 = |\langle +|(x|0\rangle + y|1\rangle)|^2 \\ &= |(x\langle +|0\rangle + y\langle +|1\rangle)|^2 \\ &= |\frac{1}{\sqrt{2}}(x+y)|^2 \\ &= \frac{(x+y)^2}{2} = \frac{(\sqrt{2}+\sqrt{3})^2}{10} \\ &= \frac{5+2\sqrt{6}}{10} \end{aligned}$$

$$\begin{aligned} Pr[-] &= |\langle -|\psi\rangle|^2 = |\langle -|(x|0\rangle + y|1\rangle)|^2 \\ &= |(x\langle -|0\rangle + y\langle -|1\rangle)|^2 \\ &= |\frac{1}{\sqrt{2}}(x-y)|^2 \\ &= \frac{(x-y)^2}{2} = \frac{(\sqrt{3}-\sqrt{2})^2}{10} \\ &= \frac{5-2\sqrt{6}}{10} \end{aligned}$$

3 Problem 3

3.1 Part a

Given $R_\theta|0\rangle = \cos\theta|0\rangle + \sin\theta|1\rangle$

$$\begin{aligned} Pr[0] &= |\langle 0|R_\theta|0\rangle|^2 \\ &= |\langle 0|(\cos\theta|0\rangle + \sin\theta|1\rangle)|^2 = \cos\theta^2 \end{aligned}$$

$$\begin{aligned} Pr[1] &= |\langle 1|R_\theta|0\rangle|^2 \\ &= |\langle 1|(\cos\theta|0\rangle + \sin\theta|1\rangle)|^2 = \sin\theta^2 \end{aligned}$$

3.2 Part b

Given $R_\theta \left|1\right\rangle = \cos\theta \left|1\right\rangle - \sin\theta \left|0\right\rangle$

$$
\begin{aligned}
Pr[+] &= |\left\langle +\right| R_\theta \left|0\right\rangle|^2 \\
&= |\left\langle +\right| (\cos\theta \left|1\right\rangle - \sin\theta \left|0\right\rangle)|^2 \\
&= |\cos\theta \left\langle +|1\right\rangle - \sin\theta \left\langle +|0\right\rangle|^2 \\
&= \left|\frac{\cos\theta - \sin\theta}{\sqrt{2}}\right|^2 \\
&= \frac{\cos\theta^2 - 2\sin\theta\cos\theta + \sin\theta^2}{2} \\
&= \frac{1 - 2\sin\theta\cos\theta}{2} = \frac{1 - \sin 2\theta}{2}
\end{aligned}
$$

$$
\begin{aligned}
Pr[-] &= |\left\langle -\right| R_\theta \left|0\right\rangle|^2 \\
&= |\left\langle -\right| (\cos\theta \left|1\right\rangle - \sin\theta \left|0\right\rangle)|^2 \\
&= |\cos\theta \left\langle -|1\right\rangle - \sin\theta \left\langle -|0\right\rangle|^2 \\
&= \left|-\frac{\cos\theta + \sin\theta}{\sqrt{2}}\right|^2 \\
&= \frac{\cos\theta^2 + 2\sin\theta\cos\theta + \sin\theta^2}{2} \\
&= \frac{1 + 2\sin\theta\cos\theta}{2} = \frac{1 + \sin 2\theta}{2}
\end{aligned}
$$

4 Problem 5

4.1 Part a

To find the relation, it is important to make the observation:

$$
\begin{pmatrix} I & 0 \\ 0 & Z \end{pmatrix}
$$

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. In addition, we can make the following observations :

$$
\begin{aligned}
HXH &= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = Z
\end{aligned}
$$

And

$$
\begin{aligned}
|0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes Z &= \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\\
&= \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}\\
&= cZ
\end{aligned}
$$

Based on this, we can find the following relation

$$
\begin{aligned}
cZ &= |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes Z\\
&= |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes HXH\\
&= (I \otimes H)(|0\rangle\langle 0| \otimes H + |1\rangle\langle 1| \otimes XH)
\end{aligned}
$$

This is true because:

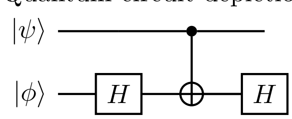
$$
\begin{aligned}
(I \otimes H)(|0\rangle\langle 0| \otimes H) &= (I\,|0\rangle\langle 0|) \otimes HH\\
&= |0\rangle\langle 0| \otimes I \text{ ,true since } H \text{ is unary}
\end{aligned}
$$

And :

$$
\begin{aligned}
(I \otimes H)(|1\rangle\langle 1| \otimes XH) &= (I\,|1\rangle\langle 1|) \otimes HXH\\
&= |1\rangle\langle 1| \otimes HXH
\end{aligned}
$$

$$
\begin{aligned}
cZ &= (I \otimes H)(|0\rangle\langle 0| \otimes H + |1\rangle\langle 1| \otimes XH)\\
&= (I \otimes H)(|0\rangle\langle 0| \otimes HH + |1\rangle\langle 1| \otimes X)(I \otimes H)\\
&= (I \otimes H)(|0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X)(I \otimes H)\\
&= (I \otimes H)\ \text{CNOT}\ (I \otimes H)
\end{aligned}
$$

Quantum circuit depiction:



4.2 Part b

Will be using the same line of reasoning as above. CNOT can be rewritten as:

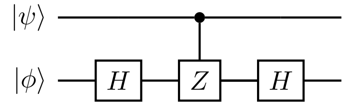
$$
\text{CNOT} = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X
$$

In addition, there is the additional relation:

$$
\begin{aligned}
HZH &= \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\\
&= \frac{1}{2}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\\
&= \frac{1}{2}\begin{pmatrix}0 & 2\\ 2 & 0\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = X
\end{aligned}
$$

With this in mind, we can use the same line of reasoning and reduce CNOT to $(I \otimes H)\mathrm{cZ}(I \otimes H)$.

Quantum circuit depiction:



4.3 Part c

Important to note : $\mathrm{CNOT}^{(i)}$ denotes the CNOT gate with the ith qubit being the control bit.

We know the following truth table holds.

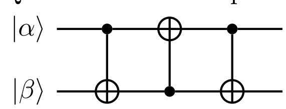
$\lvert q\rangle$	$\mathrm{CNOT}^{(0)}\lvert q\rangle$	$\mathrm{CNOT}^{(1)}\lvert q\rangle$
$\lvert 00\rangle$	$\lvert 00\rangle$	$\lvert 00\rangle$
$\lvert 01\rangle$	$\lvert 01\rangle$	$\lvert 11\rangle$
$\lvert 10\rangle$	$\lvert 11\rangle$	$\lvert 10\rangle$
$\lvert 11\rangle$	$\lvert 10\rangle$	$\lvert 01\rangle$

Based on the previous truth table, we can deduce the following:

$\lvert q\rangle$	$\mathrm{CNOT}^{(0)}\lvert q\rangle$	$\mathrm{CNOT}^{(1)}\mathrm{CNOT}^{(0)}\lvert q\rangle$	$\mathrm{CNOT}^{(0)}\mathrm{CNOT}^{(1)}\mathrm{CNOT}^{(0)}\lvert q\rangle$
$\lvert 00\rangle$	$\lvert 00\rangle$	$\lvert 00\rangle$	$\lvert 00\rangle$
$\lvert 01\rangle$	$\lvert 01\rangle$	$\lvert 11\rangle$	$\lvert 10\rangle$
$\lvert 10\rangle$	$\lvert 11\rangle$	$\lvert 01\rangle$	$\lvert 01\rangle$
$\lvert 11\rangle$	$\lvert 10\rangle$	$\lvert 10\rangle$	$\lvert 11\rangle$

Based on that, we can deduce that $\mathrm{SWAP} = \mathrm{CNOT}^{(0)}\mathrm{CNOT}^{(1)}\mathrm{CNOT}^{(0)}$

Quantum circuit depiction:



4.4 Part d

Suppose exist unitary matrices $U_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$, $U_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$ such that $\mathrm{CNOT} = U_1 \otimes U_2$. This implies

$$\begin{aligned}
\begin{pmatrix} I & 0 \\ 0 & X \end{pmatrix} &= \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \otimes \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \\
&= \begin{pmatrix} a_1 \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} & b_1 \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \\ c_1 \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} & d_1 \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \end{pmatrix}
\end{aligned}$$

From that we can deduce the following equations

$$\begin{aligned}
I &= a_1 \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \\
0 &= b_1 \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \\
0 &= c_1 \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \\
X &= d_1 \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}
\end{aligned}$$

Based on the simultaneous equations we know the following.

- From the first equation: $a_1 = \lambda$ and $U_2 = \begin{pmatrix} \frac{1}{\lambda} & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}$ where λ must be equal to 1 due to being a unary matrix.
- From the 4th equation: $d_1 = \lambda$ and $U_2 = \begin{pmatrix} 0 & \frac{1}{\lambda} \\ \frac{1}{\lambda} & 0 \end{pmatrix}$ using the same reasoning as the previous observation.

But $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Therefore, CNOT cannot be constructed using a tensor product of single-qubit unaries.