Theorem (2.3.71a)**.** *Let x be a positive real number.* $\lfloor \sqrt{\lfloor x \rfloor} \rfloor = \lfloor \sqrt{x} \rfloor$.

Proof. By the properties for floor functions,
$\lfloor \sqrt{x} \rfloor = n \iff n \leq \sqrt{x} < n+1$. Squaring the inequalities we can determine the value for the floor of x. So there are two cases under consideration,
(i) $\lfloor x \rfloor = n^2$, or (ii) $\lfloor x \rfloor = n^2 + 2n$.

(i) Suppose that $\lfloor x \rfloor = n^2$. It follows that $\lfloor \sqrt{\lfloor x \rfloor} \rfloor = \lfloor \sqrt{n^2} \rfloor = \lfloor n \rfloor$. Since n is an integer, n is the largest integer less than or equal to n. So $\lfloor n \rfloor = n$, by the definition for floor functions. Because $n = \lfloor \sqrt{x} \rfloor$, in this case it is proved that $\lfloor \sqrt{\lfloor x \rfloor} \rfloor = \lfloor \sqrt{x} \rfloor$.

(ii) Suppose that $\lfloor x \rfloor = n^2 + 2n$. Then,
$\lfloor \sqrt{\lfloor x \rfloor} \rfloor = \lfloor \sqrt{n^2 + 2n} \rfloor = \lfloor \sqrt{n^2 + 2n} + \sqrt{1} - \sqrt{1} \rfloor =$
$\lfloor \sqrt{n^2 + 2n + 1} - 1 \rfloor = \lfloor \sqrt{(n+1)^2} + 1 \rfloor = \lfloor (n+1) - 1 \rfloor = n$. Since $n = \lfloor \sqrt{x} \rfloor$, in this case it is proved that $\lfloor \sqrt{\lfloor x \rfloor} \rfloor = \lfloor \sqrt{x} \rfloor$. ■