Theorem (2.3.71a). *Let x be a positive real number.* $\lfloor\sqrt{\lfloor x\rfloor}\rfloor = \lfloor\sqrt{x}\rfloor$.

Proof. By the properties for floor functions,
$\lfloor\sqrt{x}\rfloor = n \iff n \leq \sqrt{x} < n+1$. Squaring the inequalities we can determine the value for the floor of x. So there are two cases under consideration, (i) $\lfloor x\rfloor = n^2$, or (ii) $\lfloor x\rfloor = n^2+2n$.

(i) Suppose that $\lfloor x\rfloor = n^2$. It follows that $\lfloor\sqrt{\lfloor x\rfloor}\rfloor = \lfloor\sqrt{n^2}\rfloor = \lfloor n\rfloor$. Since n is an integer, n is the largest integer less than or equal to n. So $\lfloor n\rfloor = n$, by the definition for floor functions. Because $n = \lfloor\sqrt{x}\rfloor$, in this case it is proved that $\lfloor\sqrt{\lfloor x\rfloor}\rfloor = \lfloor\sqrt{x}\rfloor$.

(ii) Suppose that $\lfloor x\rfloor = n^2+2n$. Then,
$\lfloor\sqrt{\lfloor x\rfloor}\rfloor = \lfloor\sqrt{n^2+2n}\rfloor = \lfloor\sqrt{n^2+2n}+\sqrt{1}-\sqrt{1}\rfloor =$
$\lfloor\sqrt{n^2+2n+1}-1\rfloor = \lfloor\sqrt{(n+1)^2}+1\rfloor = \lfloor(n+1)-1\rfloor = n$. Since $n = \lfloor\sqrt{x}\rfloor$, in this case it is proved that $\lfloor\sqrt{\lfloor x\rfloor}\rfloor = \lfloor\sqrt{x}\rfloor$. ∎