Theorem (2.2.40)**.** *Let A, B, and C be sets. The symmetric difference for sets is associative such that $(A \oplus B) \oplus C = A \oplus (B \oplus C)$.*

Proof. Let x be an element in $(A \oplus B) \oplus C$. By the definition for the symmetric difference of sets, $x \in [(A \oplus B) \cap \overline{C}] \cup [\overline{(A \oplus B)} \cap C]$. By Theorem 2.2.36, x is an element in $\{[(A - B) \cup (B - A)] \cap \overline{C}\} \cup \{\overline{[(A - B) \cup (B - A)]} \cap C\}$. And by Theorem 2.2.19, $\{[(A \cap \overline{B}) \cup (B \cap \overline{A})] \cap \overline{C}\} \cup \{[\overline{(A \cap \overline{B}) \cup (B \cap \overline{A})}] \cap C]\}$. Applying DeMorgans law for sets to the subset that is the right-hand side of the union in this superset, twice, produces the following logical superset equivalence, $x \in \{[(A \cap \overline{B}) \cup (B \cap \overline{A})] \cap \overline{C}\} \cup \{[(\overline{A} \cup B) \cap (\overline{B} \cup A)] \cap C]\}$. Then, distributing the terms in the subset that is the right-hand side of the union of this superset gives the logical subset equivalence, $[(\overline{A} \cup B) \cap (\overline{B} \cup A)] \cap C \equiv \{[\overline{A} \cap (\overline{B} \cup A)] \cup [B \cap (\overline{B} \cup A)]\} \cap C$. The subset terms must be distributed further, $\{[(\overline{A} \cap \overline{B}) \cup (\overline{A} \cap A)] \cup [(B \cap \overline{B}) \cup (B \cap A)]\} \cap C$. By the negation, and identity laws, $[(\overline{A} \cup B) \cap (\overline{B} \cup A)] \cap C \equiv [(\overline{A} \cap \overline{B}) \cup (B \cap A)] \cap C$. Distributing C, $[(\overline{A} \cup B) \cap (\overline{B} \cup A)] \cap C \equiv (C \cap \overline{A} \cap \overline{B}) \cup (C \cap B \cap A)$. Finally, carrying out distribution on the subset that is the left-hand side of the union of the superset gives us the logically equivalent superset statement
$(A \cap \overline{B} \cap \overline{C}) \cup (B \cap \overline{A} \cap \overline{C}) \cup (C \cap \overline{A} \cap \overline{B}) \cup (C \cap B \cap A)$.

Now let x be an element in $A \oplus (B \oplus C)$. By the definition for the symmetric difference of sets, $x \in [A \cap \overline{(B \oplus C)}] \cup [\overline{A} \cap (B \oplus C)]$. By Theorem 2.2.36, x is an element in $\{A \cap \overline{[(B - C) \cup (C - B)]}\} \cup \{\overline{A} \cap [(B - C) \cup (C - B)]\}$. And by Theorem 2.2.19, $\{A \cap \overline{[(B \cap \overline{C}) \cup (C \cap \overline{B})]}\} \cup \{\overline{A} \cap [(B \cap \overline{C}) \cup (C \cap \overline{B})]\}$. Applying DeMorgans laws for sets to the subset that is the left-hand side of the union of this superset, twice, produces the following logical superset equivalence, $x \in \{A \cap [(\overline{B} \cup C) \cap (\overline{C} \cup B)]\} \cup \{\overline{A} \cap [(B \cap \overline{C}) \cup (C \cap \overline{B})]\}$. Then, distributing the terms in the subset that is the left-hand side of the union of this superset gives the logical subset equivalence $A \cap [(\overline{B} \cup C) \cap (\overline{C} \cup B)] \equiv A \cap \{[\overline{B} \cap (\overline{C} \cup B)] \cup [C \cap (\overline{C} \cup B)]\}$. The subset terms must be distributed further, $A \cap \{[(\overline{B} \cap \overline{C}) \cup (\overline{B} \cap B)] \cup [(C \cap \overline{C}) \cup (C \cap B)]\}$. By the negation, and identity laws, $A \cap [(\overline{B} \cup C) \cap (\overline{C} \cup B)] \equiv A \cap [(\overline{B} \cap \overline{C}) \cup (C \cap B)]$. Distributing A, $A \cap [(\overline{B} \cup C) \cap (\overline{C} \cup B)] \equiv (A \cap \overline{B} \cap \overline{C}) \cup (A \cap C \cap B)$. Finally, carrying out distribution on the subset that is the right-hand side of the union of the superset gives us the logically equivalent superset statement
$(A \cap \overline{B} \cap \overline{C}) \cup (A \cap C \cap B) \cup (\overline{A} \cap B \cap \overline{C}) \cup (\overline{A} \cap C \cap \overline{B})$.

Because $x \in [(A \cap \overline{B} \cap \overline{C}) \cup (A \cap C \cap B) \cup (\overline{A} \cap B \cap \overline{C}) \cup (\overline{A} \cap C \cap \overline{B})]$, whenever x is an element in $(A \oplus B) \oplus C$ and x is an element in $A \oplus (B \oplus C)$, it follows that $(A \oplus B) \oplus C = A \oplus (B \oplus C)$ $\therefore$ the symmetric difference for sets is indeed, associative. ■