Theorem (2.4.26). *Let m be a positive integer. The closed form formula for $\sum_{k=0}^{m}\lfloor\sqrt[3]{k}\rfloor$ is $\lfloor\sqrt[3]{m}\rfloor[\frac{1}{4}(\lfloor\sqrt[3]{m}\rfloor^2-\lfloor\sqrt[3]{m}\rfloor)(3\lfloor\sqrt[3]{m}\rfloor+1)+(m-\lfloor\sqrt[3]{m}\rfloor^3+1)]$.*

Proof. By the properties for floor functions there exists an integer $n_k=\lfloor\sqrt[3]{k}\rfloor$ such that $n_k^3\le k<n_k^3+3n_k^2+3n_k+1$. This means that each value less than $\lfloor\sqrt[3]{m}\rfloor$ in the terms of summation occurs exactly $3\lfloor\sqrt[3]{k}\rfloor^2+3\lfloor\sqrt[3]{k}\rfloor+1$ times. The maximum value in the terms of summation occurs $(m-\lfloor\sqrt[3]{m}\rfloor^3+1)$ times. Subtracting those terms consisting of the maximum value produces the sequence

$$n_0(3n_0^2+3n_0+1)+n_1(3n_1^2+3n_1+1)+\cdots+n_{\lfloor\sqrt[3]{m}\rfloor-1}(3n^2_{\lfloor\sqrt[3]{m}\rfloor-1}+3_{\lfloor\sqrt[3]{m}\rfloor-1}+1)$$

Summarily expressed as

$$\sum_{n=0}^{\lfloor\sqrt[3]{m}\rfloor-1} n(3n^2+3n+1)$$

Thus,

$$\sum_{k=0}^{m}\lfloor\sqrt[3]{k}\rfloor=\left(\sum_{n=0}^{\lfloor\sqrt[3]{m}\rfloor-1} n(3n^2+3n+1)\right)+\lfloor\sqrt[3]{m}\rfloor(m-\lfloor\sqrt[3]{m}\rfloor^3+1)$$

That is, the summation of cubes, squares, and integers

$$\sum_{k=0}^{m}\lfloor\sqrt[3]{k}\rfloor=\left(3\sum_{n=0}^{\lfloor\sqrt[3]{m}\rfloor-1} n^3\right)+\left(3\sum_{n=0}^{\lfloor\sqrt[3]{m}\rfloor-1} n^2\right)+\left(\sum_{n=0}^{\lfloor\sqrt[3]{m}\rfloor-1} n\right)+$$
$$\lfloor\sqrt[3]{m}\rfloor(m-\lfloor\sqrt[3]{m}\rfloor^3+1)$$

By the closed form formula for each individual summation we have,

$$\sum_{k=0}^{m}\lfloor\sqrt[3]{k}\rfloor=\left\{\frac{3}{4}\lfloor\sqrt[3]{m}\rfloor^2(\lfloor\sqrt[3]{m}\rfloor-1)^2\right\}+$$
$$\left\{\frac{2}{4}\lfloor\sqrt[3]{m}\rfloor(\lfloor\sqrt[3]{m}\rfloor-1)[2(\lfloor\sqrt[3]{m}\rfloor-1)+1]\right\}+\left\{\frac{2}{4}\lfloor\sqrt[3]{m}\rfloor(\lfloor\sqrt[3]{m}\rfloor-1)\right\}+$$
$$\lfloor\sqrt[3]{m}\rfloor(m-\lfloor\sqrt[3]{m}\rfloor^3+1)$$

Algebraic simplification completes the derivation

$$\sum_{k=0}^{m}\lfloor\sqrt[3]{k}\rfloor=\lfloor\sqrt[3]{m}\rfloor\left[\frac{1}{4}(\lfloor\sqrt[3]{m}\rfloor^2-\lfloor\sqrt[3]{m}\rfloor)(3\lfloor\sqrt[3]{m}\rfloor+1)+(m-\lfloor\sqrt[3]{m}\rfloor^3+1)\right]$$

∎