Theorem (2.4.26)**.** *Let m be a positive integer. The closed form formula for* $\sum_{k=0}^{m}\lfloor\sqrt[3]{k}\rfloor$ *is* $\lfloor\sqrt[3]{m}\rfloor[\frac{1}{4}(\lfloor\sqrt[3]{m}\rfloor^2-\lfloor\sqrt[3]{m}\rfloor)(3\lfloor\sqrt[3]{m}\rfloor+1)+(m-\lfloor\sqrt[3]{m}\rfloor^3+1)]$.

Proof. By the properties for floor functions there exists an integer $n=\lfloor\sqrt[3]{k}\rfloor$ such that $n^3\le k<n^3+3n^2+3n+1$. This means that each term less than $\lfloor\sqrt[3]{m}\rfloor$ in the summation occurs exactly $3\lfloor\sqrt[3]{k}\rfloor^2+3\lfloor\sqrt{k}\rfloor+1$ times. Since the last term of summation occurs $(m-\lfloor\sqrt[3]{m}\rfloor^3+1)$ times, $(\sum_{k=0}^{m}\lfloor\sqrt[3]{k}\rfloor)-\lfloor\sqrt[3]{m}\rfloor(m-\lfloor\sqrt[3]{m}\rfloor^3+1)=$ $\lfloor\sqrt[3]{0}\rfloor(3\lfloor\sqrt[3]{0}\rfloor^2+3\lfloor\sqrt{0}\rfloor+1)+\lfloor\sqrt[3]{1}\rfloor(3\lfloor\sqrt[3]{1}\rfloor^2+3\lfloor\sqrt{1}\rfloor+1)+\cdots+$ $(\lfloor\sqrt[3]{m}\rfloor-1)[(3\lfloor\sqrt[3]{m}\rfloor^2-1)+3(\lfloor\sqrt[3]{m}\rfloor-1)+1)]$. By the pattern in the terms of that summation the following equation holds, $\sum_{k=0}^{m}\lfloor\sqrt[3]{k}\rfloor=$ $[\sum_{k=0}^{\lfloor\sqrt[3]{m}\rfloor-1}k(3k^2+3k+1)]+\lfloor\sqrt[3]{m}\rfloor(m-\lfloor\sqrt[3]{m}\rfloor^3+1)$. The summation on the right-hand side is the summation of cubes, the summation of squares, and the summation of integers. Thus, $\sum_{k=0}^{m}\lfloor\sqrt[3]{k}\rfloor=$ $(3\sum_{k=0}^{\lfloor\sqrt[3]{m}\rfloor-1}k^3)+(3\sum_{k=0}^{\lfloor\sqrt[3]{m}\rfloor-1}k^2)+(\sum_{k=0}^{\lfloor\sqrt[3]{m}\rfloor-1}k)+\lfloor\sqrt[3]{m}\rfloor(m-\lfloor\sqrt[3]{m}\rfloor^3+1)$. By the closed form formulas for each individual summation we have, $\sum_{k=0}^{m}\lfloor\sqrt[3]{k}\rfloor=$ $\frac{3}{4}\lfloor\sqrt[3]{m}\rfloor^2(\lfloor\sqrt[3]{m}\rfloor-1)^2+\frac{2}{4}\lfloor\sqrt[3]{m}\rfloor(\lfloor\sqrt[3]{m}\rfloor-1)[2(\lfloor\sqrt[3]{m}\rfloor-1)+1]+$ $\frac{2}{4}\lfloor\sqrt[3]{m}\rfloor(\lfloor\sqrt[3]{m}\rfloor-1)+\lfloor\sqrt[3]{m}\rfloor(m-\lfloor\sqrt[3]{m}\rfloor^3+1)$. Algebraic simplification yields, $\sum_{k=0}^{m}\lfloor\sqrt[3]{k}\rfloor=\lfloor\sqrt[3]{m}\rfloor[\frac{1}{4}(\lfloor\sqrt[3]{m}\rfloor^2-\lfloor\sqrt[3]{m}\rfloor)(3\lfloor\sqrt[3]{m}\rfloor+1)+(m-\lfloor\sqrt[3]{m}\rfloor^3+1)]$. ■