Theorem (2.3.69c). *Let x and y be real numbers.*
$\lceil x \rceil + \lceil y \rceil - \lceil x + y \rceil = 0$, *or* 1.

Proof. By cases. There are two possible cases to take into consideration. (i) x or y (or both) are integers in real numbers, or (ii) neither x nor y is an integer.

(i) Suppose x or y (or both) are integers in real numbers. Since at least one of these numbers x or y must be an integer, and because addition is commutative, without loss of generality it can be supposed that y is certainly an integer. Then since y is an integer, the smallest integer greater than or equal to y, is y. So by the definition for ceiling functions, $\lceil y \rceil = y$. By that fact, and by Theorem 2.3.46, $\lceil x \rceil + \lceil y \rceil - \lceil x + y \rceil = \lceil x \rceil + \lceil y \rceil - \lceil x \rceil + \lceil y \rceil = 0$.

(ii) Suppose that neither x nor y is an integer. Then let ϵ and σ be real numbers such that $\lceil x \rceil - x = \epsilon$, and $\lceil y \rceil - y = \sigma$. By Theorem 2.3.44, $\lceil x \rceil = \lfloor x \rfloor + 1$, and $\lceil y \rceil = \lfloor y \rfloor + 1$. Naturally, $x = \lfloor x \rfloor + (1 - \epsilon)$, and $y = \lfloor y \rfloor + (1 - \sigma)$. Thus, $\lceil x \rceil + \lceil y \rceil - \lceil x + y \rceil = (\lfloor x \rfloor + 1) + (\lfloor y \rfloor + 1) - \lceil \lfloor x \rfloor + (1 - \epsilon) + \lfloor y \rfloor + (1 - \sigma) \rceil$. Rearranging these terms according to the usual rules for arithmetic yields $(\lfloor x \rfloor + \lfloor y \rfloor + 2) - \lceil \lfloor x \rfloor + \lfloor y \rfloor + [2 - (\epsilon + \sigma)] \rceil$. Now, there are two possible sub cases to consider regarding this expression. Either (a) $\epsilon + \sigma \geq 1$, or (b) $\epsilon + \sigma < 1$.

(a) Suppose $\epsilon + \sigma \geq 1$. This means that $1 \geq 2 - (\epsilon + \sigma)$. By Theorem 2.3.69a we get the following equation,
$(\lfloor x \rfloor + \lfloor y \rfloor + 2) - \lceil \lfloor x \rfloor + \lfloor y \rfloor + [2 - (\epsilon + \sigma)] \rceil =$
$(\lfloor x \rfloor + \lfloor y \rfloor + 2) - (\lfloor x \rfloor + \lfloor y \rfloor + 1) = 1$.

(b) Suppose $\epsilon + \sigma < 1$. This means that $1 < 2 - (\epsilon + \sigma)$. By Theorem 2.3.69a we get the following equation,
$(\lfloor x \rfloor + \lfloor y \rfloor + 2) - \lceil \lfloor x \rfloor + \lfloor y \rfloor + [2 - (\epsilon + \sigma)] \rceil =$
$(\lfloor x \rfloor + \lfloor y \rfloor + 2) - (\lfloor x \rfloor + \lfloor y \rfloor + 2) = 0$.

$\therefore \lceil x \rceil + \lceil y \rceil - \lceil x + y \rceil = 0$, or 1, whenever x and y are real numbers. ■