Find a formula for $\sum_{k=0}^{m} \lfloor \sqrt{k} \rfloor$, when m is a positive integer.

Theorem 1. $\sum_{k=0}^{m} \lfloor \sqrt{k} \rfloor = \lfloor \sqrt{m} \rfloor [\frac{1}{6}(\lfloor \sqrt{m} \rfloor - 1)(4\lfloor \sqrt{m} \rfloor + 1) + (m - \lfloor \sqrt{m} \rfloor^2 + 1)]$

Proof. Whenever $\sum_{k=0}^{m} \lfloor \sqrt{k} \rfloor$, $\lfloor \sqrt{k} \rfloor = n$ if and only if $n \leq \sqrt{k} < n$. It follows that $n^2 \leq k < n^2 + 2n + 1$. This means there are $2\lfloor \sqrt{k} \rfloor + 1$ integers $\lfloor \sqrt{k} \rfloor$, for each unique $\lfloor \sqrt{k} \rfloor < \lfloor \sqrt{m} \rfloor$ in $\sum_{k=0}^{m} \lfloor \sqrt{k} \rfloor$. In other words, $\left(\sum_{k=0}^{m} \lfloor \sqrt{k} \rfloor \right) - f(m) = \lfloor \sqrt{0} \rfloor (2\lfloor \sqrt{0} \rfloor + 1) + \lfloor \sqrt{1} \rfloor (2\lfloor \sqrt{1} \rfloor + 1) + \lfloor \sqrt{2} \rfloor (2\lfloor \sqrt{2} \rfloor + 1) + \ldots + (\lfloor \sqrt{m} \rfloor - 1)[2(\lfloor \sqrt{m} \rfloor - 1) + 1)$, where $f(m)$ is the function expressing the summation of values $\lfloor \sqrt{k} \rfloor$ for all k in $\sum_{k=0}^{m} \lfloor \sqrt{k} \rfloor$ if and only if $\lfloor \sqrt{k} \rfloor = \lfloor \sqrt{m} \rfloor$. Hence, $\left(\sum_{k=0}^{m} \lfloor \sqrt{k} \rfloor \right) - f(m) = \sum_{k=0}^{\lfloor \sqrt{m} \rfloor - 1} k(2k+1)$.

$f(m) = \lfloor \sqrt{m} \rfloor (m - \lfloor \sqrt{m} \rfloor^2 + 1)$. To see this, note that $\lfloor \sqrt{m} \rfloor$ is the value that occurs everywhere in the sequence predicated of $\sum_{k=0}^{m} \lfloor \sqrt{k} \rfloor$, whenever $\lfloor \sqrt{k} \rfloor = \lfloor \sqrt{m} \rfloor$. $(m - \lfloor \sqrt{m} \rfloor^2 + 1)$ is the number of times that value $\lfloor \sqrt{m} \rfloor$ occurs. So we have the value $\lfloor \sqrt{m} \rfloor$ in the sequence $(m - \lfloor \sqrt{m} \rfloor^2 + 1)$ times, for all k if and only if $\lfloor \sqrt{k} \rfloor = \lfloor \sqrt{m} \rfloor$. We add 1 to $m - \lfloor \sqrt{m} \rfloor^2$ because if m is a perfect square, then $m - \lfloor \sqrt{m} \rfloor^2 = 0$; but $\lfloor \sqrt{m} \rfloor$ is guaranteed to occur at least once, for an upper limit m.

$\left(\sum_{k=0}^{\lfloor \sqrt{m} \rfloor - 1} 2k^2 + k \right) + f(m) = 2 \left(\sum_{k=0}^{\lfloor \sqrt{m} \rfloor - 1} k^2 \right) + \left(\sum_{k=0}^{\lfloor \sqrt{m} \rfloor - 1} k \right) + f(m)$, a summation of squares, and the summation of integers in $\lfloor \sqrt{m} \rfloor - 1$. By definition, then $\sum_{k=0}^{m} \lfloor \sqrt{k} \rfloor = 2 \left(\frac{\lfloor \sqrt{m} \rfloor (\lfloor \sqrt{m} \rfloor - 1)[2(\lfloor \sqrt{m} \rfloor - 1) + 1]}{6} \right) + \frac{\lfloor \sqrt{m} \rfloor (\lfloor \sqrt{m} \rfloor - 1)}{2} + f(m)$.

Performing some algebra, we can derive the equation.

$\sum_{k=0}^{m} \lfloor \sqrt{k} \rfloor = \frac{2}{6} \lfloor \sqrt{m} \rfloor (\lfloor \sqrt{m} \rfloor - 1)[2(\lfloor \sqrt{m} \rfloor - 1) + 1] + \frac{3}{6} \lfloor \sqrt{m} \rfloor (\lfloor \sqrt{m} \rfloor - 1) + f(m) =$

$\frac{1}{6} \lfloor \sqrt{m} \rfloor (\lfloor \sqrt{m} \rfloor - 1)[2(2\lfloor \sqrt{m} \rfloor - 2 + 1) + 3] + f(m)$. Because $f(m) = m\lfloor \sqrt{m} \rfloor - \lfloor \sqrt{m} \rfloor^3 + \lfloor \sqrt{m} \rfloor$, factoring out $\lfloor \sqrt{m} \rfloor$ and simplifying, yields the equation $\sum_{k=0}^{m} \lfloor \sqrt{k} \rfloor = \lfloor \sqrt{m} \rfloor [\frac{1}{6}(\lfloor \sqrt{m} \rfloor - 1)(4\lfloor \sqrt{m} \rfloor + 1) + (m - \lfloor \sqrt{m} \rfloor^2 + 1)]$ ■