**Theorem** (2.3.69c)**.** *Let $x$ and $y$ be real numbers.*
$\lceil x \rceil + \lceil y \rceil - \lceil x + y \rceil = 0$, *or* 1.

*Proof.* By cases. There are two possible cases to take into consideration.
$(i)$ $x$ or $y$ (or both) are integers in real numbers, or $(ii)$ neither $x$ nor $y$ is an integer.

$(i)$ Suppose $x$ or $y$ (or both) are integers in real numbers. Since at least one of these numbers $x$ or $y$ must be an integer, and because addition is commutative, without loss of generality it can be supposed that $y$ is certainly an integer. Then since $y$ is an integer, the smallest integer greater than or equal to $y$, is $y$. So by the definition for ceiling functions, $\lceil y \rceil = y$. By that fact, and by Theorem 2.3.46, $\lceil x \rceil + \lceil y \rceil - \lceil x + y \rceil = \lceil x \rceil + \lceil y \rceil - \lceil x \rceil + \lceil y \rceil = 0$.

$(ii)$ Suppose that neither $x$ nor $y$ is an integer. Then let $\epsilon$ and $\sigma$ be real numbers such that $\lceil x \rceil - x = \epsilon$, and $\lceil y \rceil - y = \sigma$. By Theorem 2.3.44, $\lceil x \rceil = \lfloor x \rfloor + 1$, and $\lceil y \rceil = \lfloor y \rfloor + 1$. Naturally, $x = \lfloor x \rfloor + (1 - \epsilon)$, and $y = \lfloor y \rfloor + (1 - \sigma)$. Thus, $\lceil x \rceil + \lceil y \rceil - \lceil x + y \rceil = (\lfloor x \rfloor + 1) + (\lfloor y \rfloor + 1) - \lceil \lfloor x \rfloor + (1 - \epsilon) + \lfloor y \rfloor + (1 - \sigma) \rceil$. Rearranging these terms according to the usual rules for arithmetic yields $(\lfloor x \rfloor + \lfloor y \rfloor + 2) - \lceil \lfloor x \rfloor + \lfloor y \rfloor + [2 - (\epsilon + \sigma)] \rceil$. Now, there are two possible sub cases to consider regarding this expression. Either $(a)$ $\epsilon + \sigma \geq 1$, or $(b)$ $\epsilon + \sigma < 1$.

> $(a)$ Suppose $\epsilon + \sigma \geq 1$. This means that $1 \geq 2 - (\epsilon + \sigma)$. By Theorem 2.3.69a we get the following equation,
> $(\lfloor x \rfloor + \lfloor y \rfloor + 2) - \lceil \lfloor x \rfloor + \lfloor y \rfloor + [2 - (\epsilon + \sigma)] \rceil =$
> $(\lfloor x \rfloor + \lfloor y \rfloor + 2) - (\lfloor x \rfloor + \lfloor y \rfloor + 1) = 1$.
>
> $(b)$ Suppose $\epsilon + \sigma < 1$. This means that $1 < 2 - (\epsilon + \sigma)$. By Theorem 2.3.69a we get the following equation,
> $(\lfloor x \rfloor + \lfloor y \rfloor + 2) - \lceil \lfloor x \rfloor + \lfloor y \rfloor + [2 - (\epsilon + \sigma)] \rceil =$
> $(\lfloor x \rfloor + \lfloor y \rfloor + 2) - (\lfloor x \rfloor + \lfloor y \rfloor + 2) = 0$.

$\therefore$ $\lceil x \rceil + \lceil y \rceil - \lceil x + y \rceil = 0$, or 1, whenever $x$ and $y$ are real numbers. ■