1.6 Introduction to Proofs

Theorem (1.6.1). *Let x and y be integers. If x and y are odd, then $x + y$ is even.*

Proof. By definition, there exists integers m and n such that $x = 2m+1$ and $y = 2n+1$. $2m+1+2n+1 = 2(m+n+1)$. $m+n+1$ is an integer k because the sum of integers is an integer $\therefore$ $x + y = 2k$ is even by definition. $\blacksquare$

Theorem (1.6.2). *Let x and y be integers. If x and y are even then $x + y$ is even.*

Proof. By definition, there exist integers m and n such that $2m = x$ and $2n = y$. $2m + 2n = 2(m + n)$. $m + n$ is an integer k because the sum of integers is an integer. Thus, $x + y = 2k$ is even, by definition. $\blacksquare$

Theorem (1.6.3). *If n is an even integer, then n^2 is an even integer.*

Proof. By definition, there exists an integer k such that $n = 2k$.
$(2k)^2 = 4k^2 = 2(2k^2)$. $2k^2$ is an integer $\therefore$ n^2 is even, by definition. $\blacksquare$

Theorem (1.6.4). *The additive inverse of an even number is an even number.*

Proof. Let n be an even number. There exists an integer k such that $n = 2k$, by definition. The additive inverse of n is $-n = -2k$. By commutativity of multiplication, $-2k = 2(-k)$, and $-k$ is an integer because the product of integers is an integer $\therefore$ $-n$ is an even number by definition. $\blacksquare$

Theorem (**1.6.5**). *Let m, n, and p be integers. If $m + n$ and $n + p$ are even integers, then $m + p$ is even.*

Proof. By the hypothesis there exist integers k and j such that $m+n = 2k$, and $n+p = 2j$. So $m+n+n+p = 2k+2j$. Subtracting $2n$ from both sides produces $m+p = 2k+2j-2n = 2(k+j-n)$. Since $k+j-n$ is an integer, $m+p$ is an integer and even by definition. ■

Theorem (**1.6.6**). *The product of two odd numbers is odd.*

Proof. Suppose that x and y are odd numbers. By definition, there exist integers m and n such that $x = 2m+1$ and $y = 2n+1$. $xy = (2m+1)(2n+1) = 2m2n+2m+2n+1 = 2(mn+m+n)+1$. $mn+m+n$ is an integer because the sum of integers is an integer. Thus, xy is odd by definition. ■

Theorem (1.6.8). *If n is a perfect square, then $n + 2$ is not a perfect square.*

Proof. Let n be a perfect square. Assume $n + 2$ is a perfect square for the purpose of contradiction. By the definition of perfect square, $\sqrt{n}$ has to be an integer, and by our assumption there exists an integer m such that $m^2 = n + 2$. So the equivalence $m^2 - (\sqrt{n})^2 = 2$ must be the difference of squares $(m + \sqrt{n})(m - \sqrt{n}) = 2$. Since the sum or difference of integers is an integer it follows that the factors of 2, $(m + \sqrt{n})$ and $(m - \sqrt{n})$, have to be integers. Because 2 is prime those integer factors can only be elements in $\{-2, -1, 1, 2\}$. Thus, there are exactly two possibilities:

(*i*) $m^2 - (\sqrt{n})^2 = (2)(1)$,
or (*ii*) $m^2 - (\sqrt{n})^2 = (-1)(-2)$.

In case (*i*), without loss of generality, we have a system of linear equations in two variables m and $\sqrt{n}$:

$$\begin{aligned} m + \sqrt{n} &= 2 \\ m - \sqrt{n} &= 1 \end{aligned}$$

The matrix of coefficients $A = \left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$, the inverse for which is $A^{-1} = \left[\begin{smallmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{smallmatrix}\right]$. The product of A^{-1} and the matrix of solutions yields $m = 1.5$, which is not in $\mathbb{Z}$; contradicting the assumption that m^2 was a perfect square.

In case (*ii*), we are presented with a similar system of linear equations. The only difference in this system compared to (*i*) is the matrix of solutions $S = \left[\begin{smallmatrix} -1 \\ -2 \end{smallmatrix}\right]$. $A^{-1}S$ yields $m = -1.5$, which is not in $\mathbb{Z}$, a contradiction. The assumption that m^2 was a perfect square must be false in this case, as well.

Since the assumption proves false in all possible cases, it is not possbile that both $m^2 = n + 2$, and n are perfect squares. ∎

Theorem (**1.6.9**). *The sum of an irrational number and a rational number is irrational.*

Proof. By contradiction. Suppose that m and n are rational numbers. By definition, there exist integers a, b, c, and d such that $m = \frac{a}{b}$ and $n = \frac{c}{d}$. Let x be an irrational number such that the sum of a rational number and an irrational number can be expressed as $m + x = n$ and $n + (-m) = x$. In terms of a, b, c, and d we have $\frac{-a}{b} + \frac{c}{d} = \frac{-ad}{bd} + \frac{cb}{bd} = \frac{-ad+cb}{bd} = x$. Note that the sum of products of integers is an integer. But this is impossible because x is irrational; thus a contradiction. ■

Theorem (**1.6.10**). *The product of two rational numbers is rational.*

Proof. Let m and n be rational numbers. By definition there exist integers a, b, c, and d such that $m = \frac{a}{b}$ and $n = \frac{c}{d}$. The product of m and n is $\frac{ac}{bd}$. Since the product of integers is an integer, ac and bd are integers. Thus mn is rational by definition. ■

Theorem (**1.6.12**). *The product of a nonzero rational number and an irrational number is irrational.*

Proof. For the purpose of contradiction, suppose that the product of a nonzero rational number and an irrational number is rational. This can be expressed as $\frac{a}{b} \cdot x = \frac{c}{d}$, where a, b, c, and d are integers and x is irrational. Since $a \neq 0$ we equivalently have $x = \frac{c}{d} \cdot \frac{b}{a} = \frac{cb}{da}$. A contradiction. ■

Theorem (**1.6.13**). *If x is an irrational number, then $\frac{1}{x}$ is irrational.*

Proof. By the contrapositive. Suppose that $\frac{1}{x}$ is a rational number. By definition there exist integers a and b such that $\frac{1}{x} = \frac{a}{b}$. Logical equivalence has it that $x = \frac{b}{a}$, thus rational. ■

Theorem (1.6.14). *If x is a rational number and $x \neq 0$, then $\frac{1}{x}$ is rational.*

Proof. It is trivial to express x as $x = \frac{x}{1}$. Since x is rational, by the definition of ration numbers there exist integers a and b such that $\frac{x}{1} = \frac{a}{b}$. By equivalence we have $\frac{b}{a} = \frac{1}{x}$, so $\frac{1}{x}$ is rational by definition whenever x is a nonzero rational number. ■

Theorem (1.6.15). *Let x and y be real numbers. If $x + y \geq 2$, then $(x \geq 1) \vee (y \geq 1)$.*

Proof. By the contrapositive. Suppose it were the case that $(x < 1) \wedge (y < 1)$. We can simply add the inequalities: $x + y < 1 + 1 = 2$. This is the logical negation for the direct form hypothesis, by DeMorgans law. Thus concludes the proof. ■

Theorem (1.6.16). *Let m and n be integers. If the product mn is even, then m is even or n is even.*

Proof. By the contrapositive. Suppose the negation of the consequent; that is, m is odd and n is odd. By definition, there exist integers k and j such that $m = 2k+1$ and $n = 2j+1$. Thus, $mn = (2k+1)(2j+1) = 2(kj+k+j)+1$. The factor $kj+k+j$ is an integer, and so the product mn is odd by definition. ■

Theorem (1.6.17). *Let n be an integer. If $n^3 + 5$ is odd, then n is even.*

Proof. By the contrapositive. Suppose that n is odd. By definition there exists and integer k such that $n = 2k + 1$. By the Binomial Theorem, $(2k+1)^3 + 5 = 5 + \sum_{i=0}^{3} \binom{3}{i} 2k^{(3-i)} = 2(4k^3 - 6k^2 + 3k + 3)$. That is an integer factor with a coefficient of 2, even by definition. ■

Theorem (1.6.18). *Let n be an integer. If $3n + 2$ is even, then n is even.*

Proof. By the contrapositive. Suppose n is odd. By the definition of odd numbers there exist an integer k such that $n = 2k + 1$. We have $3(2k + 1) + 2 = 2(3k + 2) + 1$. Since $3k + 2$ is an integer $3n + 2$ is odd by definition. ∎

Theorem (1.6.25). *There does not exist a rational number r such that $r^3 + r + 1 = 0$.*

Proof. By contradiction. Assume that there exists a rational number r satisfying the equation $r^3+r+1 = 0$. By definition there exist integers a and b (b is nonzero,) such that $\frac{a^3}{b^3}+\frac{a}{b}+1 = a^3+ab^2+b^3 = 0$. Clearly $a^3 = -(ab^2+b^3)$ and $b^3 = -(a^3+ab^2)$. So we have $-(ab^2+b^3)+ab^2-(a^3+ab^2) = 0$. Simplifying we find that $-a^3 - ab^2 - b^3 = a^3 + ab^2 + b^3$. This can only happen when $b = 0$, but $b = 0$ is a contradiction because b is a divisor in r. ∎