Theorem (2.3.70a)**.** *Let x be a real number.* $\lfloor \lceil x \rceil \rfloor = \lceil x \rceil$.

Proof. Let n be the integer such that $n - 1 < x \leq n$. By the properties for ceiling functions, $\lceil x \rceil = n$. So $\lfloor \lceil x \rceil \rfloor = \lfloor n \rfloor$. Since $n \leq n < n + 1$ is a tautology, by the properties for floor functions it must be the case that $\lfloor n \rfloor = n$. But $n = \lceil x \rceil$. So $\lfloor \lceil x \rceil \rfloor = \lceil x \rceil$. ■