Lemma 1. Suppose $\lfloor x \rfloor = \lfloor n + \psi \rfloor$ where $n \in \mathbb{Z}$ such that $n \leq x < n+1$, and $\psi \in \mathbb{R}$ such that $0 \leq \psi < 1$. Then $\lfloor x \rfloor = n + \lfloor \psi \rfloor = n + 0 = n$.

Theorem 1. Let x be a real number. Then $\lfloor 3x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor$.

Proof. There are three cases under consideration for this proof.

$(i) : 0 \leq \psi < \frac{1}{3}$.

In this case we find that $0 \leq 3\psi < 1$, $\frac{1}{3} \leq \psi + \frac{1}{3} < \frac{2}{3}$, and $\frac{2}{3} \leq \psi + \frac{2}{3} < 1$. Thus, $\lfloor 3\psi \rfloor = 0$, $\lfloor \psi + \frac{1}{3} \rfloor = 0$, and $\lfloor \psi + \frac{2}{3} \rfloor = 0$. Consequently, $\lfloor 3x \rfloor = 3n + 0$, $\lfloor x + \frac{1}{3} \rfloor = n + 0$, and $\lfloor x + \frac{2}{3} \rfloor = n + 0$. By Lemma 1, $\lfloor x \rfloor = n$. Since $3n + 0 = n + (n+0) + (n+0)$, it follows that $\lfloor 3x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor$, if $0 \leq \psi < \frac{1}{3}$.

$(ii) : \frac{1}{3} \leq \psi < \frac{2}{3}$.

In this case we find that $1 \leq 3\psi < 2$, $\frac{2}{3} \leq \psi + \frac{1}{3} < 1$, and $1 \leq \psi + \frac{2}{3} < 1\frac{1}{3}$. Thus, $\lfloor 3\psi \rfloor = 1$, $\lfloor \psi + \frac{1}{3} \rfloor = 0$, and $\lfloor \psi + \frac{2}{3} \rfloor = 1$. Consequently, $\lfloor 3x \rfloor = 3n + 1$, $\lfloor x + \frac{1}{3} \rfloor = n + 0$, and $\lfloor x + \frac{2}{3} \rfloor = n + 1$. By Lemma 1, $\lfloor x \rfloor = n$. Since $3n + 1 = n + (n+0) + (n+1)$, it follows that $\lfloor 3x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor$, if $\frac{1}{3} \leq \psi < \frac{2}{3}$.

$(iii) : \frac{2}{3} \leq \psi < 1$.

In this case we find that $2 \leq 3\psi < 3$, $1 \leq \psi + \frac{1}{3} < 1\frac{1}{3}$, and $1\frac{1}{3} \leq \psi + \frac{2}{3} < 1\frac{2}{3}$. Thus, $\lfloor 3\psi \rfloor = 2$, $\lfloor \psi + \frac{1}{3} \rfloor = 1$, and $\lfloor \psi + \frac{2}{3} \rfloor = 1$. Consequently, $\lfloor 3x \rfloor = 3n + 2$, $\lfloor x + \frac{1}{3} \rfloor = n + 1$, and $\lfloor x + \frac{2}{3} \rfloor = n + 1$. By Lemma 1, $\lfloor x \rfloor = n$. Since $3n + 2 = n + (n+1) + (n+1)$, it follows that $\lfloor 3x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor$, if $\frac{2}{3} \leq \psi < 1$.

The theorem is true $\forall \psi \in [0, 1)$ $\therefore$ If $x = n + \psi \in \mathbb{R}$, then $\lfloor 3x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor$. ■