Theorem (2.3.67d)**.** *Let A, and B be sets with universal set U. Let $f_{A\oplus B}$ be the characteristic function $f_{A\oplus B} : U \implies \{0,1\}$. Let f_A be the characteristic function $f_A : U \implies \{0,1\}$. Let f_B be the characteristic function $f_B : U \implies \{0,1\}$. $f_{A\oplus B}(x) = f_A(x) + f_B(x) - 2f_A(x)f_B(x)$.*

Proof. There are two major cases to consider, each consisting of two sub cases. The major cases are where x is an element in $A \oplus B$, and the negation of that statement.

(i) Let x be an element in $A \oplus B$. By the definition for characteristic functions, $f_{A\oplus B}(x) = 1$. Since the definition for set symmetric difference says $[(x \in A) \wedge (x \notin B)] \vee [(x \notin A) \wedge (x \in B)]$, there are two sub cases that need to be taken under consideration.

(a) Suppose $(x \in A) \wedge (x \notin B)$. By the definition for characteristic functions $f_A(x) = 1$ and $f_B(x) = 0$. This means that $f_A(x) + f_B(x) - 2f_A(x)f_B(x) = 1 + 0 - 2(1)(0) = 1$.

(b) Suppose $(x \notin A) \wedge (x \in B)$. Without loss of generality we arrive at the same result as that of case (a).

Thus, if x is an element in $A \oplus B$, $f_{A\oplus B}(x) = f_A(x) + f_B(x) - 2f_A(x)f_B(x)$.

(ii) Suppose it were not the case that x were an element in $A \oplus B$. Then (c) x must either be an element in the intersection of A and B, or (d) x must be in the universe minus $A \cup B$.

(c) Suppose $x \in (A \cap B)$. By the definition for characteristic functions $f_{A\oplus B}(x) = 0$, $f_A(x) = 1$ and $f_B(x) = 1$. Thus, $f_A(x) + f_B(x) - 2f_A(x)f_B(x) = 1 + 1 - 2(1)(1) = 0$.

(d) Suppose $x \in [U - (A \cup B)$. In this case, by the definition for characteristic functions, $f_{A\oplus B}(x) = 0$, $f_A(x) = 0$ and $f_B(x) = 0$. So, $f_A(x) + f_B(x) - 2f_A(x)f_B(x) = 0 + 0 - 2(0)(0) = 0$.

Thus, if x is not an element in $A \oplus B$, $f_{A\oplus B}(x) = f_A(x) + f_B(x) - 2f_A(x)f_B(x)$ is still a true statement; concludes the proof. ∎