Theorem (2.3.48a)**.** *Let x be a real number, and let n be an integer.* $x \leq n \iff \lceil x \rceil \leq n$.

Proof. Direct form by the contrapositive. Suppose $\lceil x \rceil > n$. Since $\lceil x \rceil$ and n are integers, $\lceil x \rceil - 1 \geq n$. By the properties of ceiling functions we have the following tautology, $\lceil x \rceil = \lceil x \rceil \iff \lceil x \rceil \geq x > \lceil x \rceil - 1$. Combining these two inequalities yields $\lceil x \rceil \geq x > \lceil x \rceil - 1 \geq n$. This says, $x > n$. Since this statement following from the negation of the direct consequent is itself the negation of the direct hypothesis, $x \leq n \implies \lceil x \rceil \leq n$, is true.

Converse form by the contrapositive. Suppose $x > n$. Note that $\lceil x \rceil \geq x$, by the properties of the ceiling function. So if $x > n$, then $\lceil x \rceil \geq x > n$, and $\lceil x \rceil > n$. Since this statement is the negation of the converse hypothesis following directly from negation of the converse consequent, $x \leq n \impliedby \lceil x \rceil \leq n$, is true.

Thus proves, the biconditional statement $x \leq n \iff \lceil x \rceil \leq n$. ■