Theorem (2.4.25)**.** *Let m be a positive integer. The closed form formula for* $\sum_{k=0}^{m}\lfloor\sqrt{k}\rfloor$ *is* $\lfloor\sqrt{m}\rfloor[\frac{1}{6}(\lfloor\sqrt{m}\rfloor-1)(4\lfloor\sqrt{m}\rfloor+1)+(m-\lfloor\sqrt{m}\rfloor^2+1)]$.

Proof. By the properties for floor functions, there exists an integer $n=\lfloor\sqrt{k}\rfloor$ such that $n^2\leq k<n^2+2n+1$. This means that each term less than $\lfloor\sqrt{m}\rfloor$ in the summation occurs exactly $2\lfloor\sqrt{k}\rfloor+1$ times. Since the last term of summation occurs $(m-\lfloor\sqrt{m}\rfloor^2+1)$ times, $(\sum_{k=0}^{m}\lfloor\sqrt{k}\rfloor)-\lfloor\sqrt{m}\rfloor(m-\lfloor\sqrt{m}\rfloor^2+1)=$ $\lfloor\sqrt{0}\rfloor(2\lfloor\sqrt{0}\rfloor+1)+\cdots+(\lfloor\sqrt{m}\rfloor-1)[2(\lfloor\sqrt{m}\rfloor-1)+1]$. By the pattern in the terms of that summation the following equation holds, $\sum_{k=0}^{m}\lfloor\sqrt{k}\rfloor=$ $(\sum_{k=0}^{\lfloor\sqrt{m}\rfloor-1}k(2k+1))+\lfloor\sqrt{m}\rfloor(m-\lfloor\sqrt{m}\rfloor^2+1)$. The summation on the right-hand side is the summation of squares, and the summation of integers. Thus, $\sum_{k=0}^{m}\lfloor\sqrt{k}\rfloor=(2\sum_{k=0}^{\lfloor\sqrt{m}\rfloor-1}k^2)+(\sum_{k=0}^{\lfloor\sqrt{m}\rfloor-1}k)+[\lfloor\sqrt{m}\rfloor(m-\lfloor\sqrt{m}\rfloor^2+1)]$. By Theorem 2.4.22, and by Theorem 2.4.21b, the formula is immediately derived $\sum_{k=0}^{m}\lfloor\sqrt{k}\rfloor=\frac{2}{6}\lfloor\sqrt{m}\rfloor(\lfloor\sqrt{m}\rfloor-1)[2(\lfloor\sqrt{m}\rfloor-1)+1]+\frac{3}{6}\lfloor\sqrt{m}\rfloor(\lfloor\sqrt{m}\rfloor-1)+$ $\lfloor\sqrt{m}\rfloor(m-\lfloor\sqrt{m}\rfloor^2+1)$. Factoring $\frac{1}{6}\lfloor\sqrt{m}\rfloor(\lfloor\sqrt{m}\rfloor-1)$ out of the first two terms yields $\sum_{k=0}^{m}\lfloor\sqrt{k}\rfloor=\frac{1}{6}\lfloor\sqrt{m}\rfloor(\lfloor\sqrt{m}\rfloor-1)\{2[2(\lfloor\sqrt{m}\rfloor-1)+1]+3\}+$ $\lfloor\sqrt{m}\rfloor(m-\lfloor\sqrt{m}\rfloor^2+1)$. Simplifying that is, $\sum_{k=0}^{m}\lfloor\sqrt{k}\rfloor=$ $\frac{1}{6}\lfloor\sqrt{m}\rfloor(\lfloor\sqrt{m}\rfloor-1)(4\lfloor\sqrt{m}\rfloor+1)+\lfloor\sqrt{m}\rfloor(m-\lfloor\sqrt{m}\rfloor^2+1)$. Factoring $\lfloor\sqrt{m}\rfloor$, $\sum_{k=0}^{m}\lfloor\sqrt{k}\rfloor=\lfloor\sqrt{m}\rfloor[\frac{1}{6}(\lfloor\sqrt{m}\rfloor-1)(4\lfloor\sqrt{m}\rfloor+1)+(m-\lfloor\sqrt{m}\rfloor^2+1)]$. ■