Theorem (2.3.71b)**.** *Let x be a positive real number.* $\lceil\sqrt{\lceil x\rceil}\rceil = \lceil\sqrt{x}\rceil$.

Proof. By the properties for floor functions,
$\lceil\sqrt{x}\rceil \iff n-1 < \sqrt{x} \leq n$. Squaring the inequalities we can determine the value for the floor of x. Thus, there are two cases under consideration (i) $\lceil x\rceil = n^2 - 2n$, or (ii) $\lceil x\rceil = n^2$.

(i) Suppose that $\lceil x\rceil = n^2 - 2n$. It follows that,
$\lceil\sqrt{\lceil x\rceil}\rceil = \lceil\sqrt{n^2-2n}\rceil = \lceil\sqrt{n^2-2n} - \sqrt{1} + \sqrt{1}\rceil =$
$\lceil\sqrt{n^2-2n-1} + 1\rceil = \lceil\sqrt{(n-1)^2} + 1\rceil = \lceil(n-1)+1\rceil = n$. Since $n = \lceil\sqrt{x}\rceil$, in this case it is proved that $\lceil\sqrt{\lceil x\rceil}\rceil = \lceil\sqrt{x}\rceil$.

(ii) Suppose that $\lceil x\rceil = n^2$. It follows that $\lceil\sqrt{\lceil x\rceil}\rceil = \lceil\sqrt{n^2}\rceil = \lceil n\rceil$. Since n is an integer, n is the smallest integer that is greater than or equal to n. So $\lceil n\rceil = n$, by the definition for ceiling functions. Because $n = \lceil\sqrt{x}\rceil$, in this case it is proved that $\lceil\sqrt{\lceil x\rceil}\rceil = \lceil\sqrt{x}\rceil$. ■