Theorem (2.3.71b)**.** *Let x be a positive real number.* $\lceil \sqrt{\lceil x \rceil} \rceil = \lceil \sqrt{x} \rceil$.

Proof. By the properties for floor functions,
$\lceil \sqrt{x} \rceil \iff n - 1 < \sqrt{x} \leq n$. Squaring the inequalities we can determine the value for the floor of x. Thus, there are two cases under consideration (i) $\lceil x \rceil = n^2 - 2n$, or (ii) $\lceil x \rceil = n^2$.

(i) Suppose that $\lceil x \rceil = n^2 - 2n$. It follows that,
$\lceil \sqrt{\lceil x \rceil} \rceil = \lceil \sqrt{n^2 - 2n} \rceil = \lceil \sqrt{n^2 - 2n} - \sqrt{1} + \sqrt{1} \rceil =$
$\lceil \sqrt{n^2 - 2n - 1} + 1 \rceil = \lceil \sqrt{(n-1)^2} + 1 \rceil = \lceil (n-1) + 1 \rceil = n$. Since $n = \lceil \sqrt{x} \rceil$, in this case it is proved that $\lceil \sqrt{\lceil x \rceil} \rceil = \lceil \sqrt{x} \rceil$.

(ii) Suppose that $\lceil x \rceil = n^2$. It follows that $\lceil \sqrt{\lceil x \rceil} \rceil = \lceil \sqrt{n^2} \rceil = \lceil n \rceil$. Since n is an integer, n is the smallest integer that is greater than or equal to n. So $\lceil n \rceil = n$, by the definition for ceiling functions. Because $n = \lceil \sqrt{x} \rceil$, in this case it is proved that $\lceil \sqrt{\lceil x \rceil} \rceil = \lceil \sqrt{x} \rceil$. ■