Theorem (2.3.69a)**.** *Let x be a real number.* $\lceil \lfloor x \rfloor \rceil = \lfloor x \rfloor$.

Proof. Let n be the integer such that $n \leq x < n + 1$. By the properties for floor functions, $\lfloor x \rfloor = n$. So $\lceil \lfloor x \rfloor \rceil = \lceil n \rceil$. Since $n - 1 < n \leq n$ is a tautology, by the properties for ceiling functions it must be the case that $\lceil n \rceil = n$. But $n = \lfloor x \rfloor$, so $\lceil \lfloor x \rfloor \rceil = \lfloor x \rfloor$. ■