Theorem (2.3.72)**.** *Let x be a real number.* $\lfloor 3x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor$.

Proof. By cases. Let ϵ be a real number such that $x - \lfloor x \rfloor = \epsilon$. Clearly, $3x - \lfloor 3x \rfloor = 3\epsilon$. There are three possible cases that must be proved,
(i) $0 \leq \epsilon < \frac{1}{3}$, (ii) $\frac{1}{3} \leq \epsilon < \frac{2}{3}$, and (iii) $\frac{2}{3} \leq \epsilon < 1$.

(i) Suppose $0 \leq \epsilon < \frac{1}{3}$. Since $\epsilon + \frac{2}{3} < 1$, every term on the right-hand side is equal to $x - \epsilon$. That is $3x - 3\epsilon$. But $\lfloor 3x \rfloor = 3x - 3\epsilon$. Therefore, both sides of the equation are equal in this case.

(ii) Suppose $\frac{1}{3} \leq \epsilon < \frac{2}{3}$. We know that $\frac{1}{3} + \epsilon < 1$, and $\frac{2}{3} + \epsilon < \frac{4}{3}$. So the first two terms on the right-hand side must be equal to $x - \epsilon$, and the last term equals $(x-\epsilon)+1$. That is $(3x-3\epsilon)+1$. Now, $\lfloor 3x \rfloor = \lfloor (3x-3\epsilon)+3\epsilon \rfloor$. By the inequality, $1 \leq 3\epsilon < 2$. This means that $\lfloor 3x \rfloor = (3x - 3\epsilon) + 1$. Hence, both sides of the equation are equal in this case.

(iii) Suppose $\frac{2}{3} \leq \epsilon < 1$. We know that $1 \leq \frac{1}{3} + \epsilon < \frac{1}{3} + 1$, and
$\frac{4}{3} \leq \frac{2}{3} + \epsilon < \frac{2}{3} + 1$. Clearly the first term in the right-hand side of the equation is equal to $x - \epsilon$ since ϵ is less than 1. The remaining two terms are equal to $(x - \epsilon) + 1$, since $2(\frac{2}{3}) \leq 2\epsilon < 2$. Now, $\lfloor 3x \rfloor = \lfloor (3x - 3\epsilon) + 3\epsilon) \rfloor$, and we know that $2 \leq 3\epsilon < 3$, by the inequality. So $\lfloor 3x \rfloor = (3x - 3\epsilon) + 2$, which is exactly the same as the right-hand side of the equation. ∎