2.3 Functions

Theorem (2.3.20). *Let f be the function f: $\mathbb{R} \implies \mathbb{R}$, such that $\forall x((x \in \mathbb{R}) \implies (f(x) > 0))$. Let g be the function $g : \mathbb{R} \implies \mathbb{R}$ defined by g(x) = 1/f(x). f(x) is strictly increasing if and only if g(x) is strictly decreasing.*

Proof. Suppose there exist real numbers x and y such that $x < y$, and suppose that $f(x) < f(y)$. f is a strictly increasing real-valued function by definition. It follows that $g(x) = 1/f(x) > g(y) = 1/f(y)$, which is the definition for strictly decreasing real-valued functions.

Conversely, suppose there exist real numbers x and y such that $x < y$, and suppose that $g(x) > g(y)$. g is a strictly decreasing real-valued function by definition. It follows that $f(x) = 1/g(x) < f(y) = 1/f(y)$, which is the definition for strictly increasing real-valued functions.

Thus, $f(x)$ is strictly increasing if and only if $g(x)$ is strictly decreasing. ∎

Theorem (2.3.21). *Let f be the function f: $\mathbb{R} \implies \mathbb{R}$, such that $\forall x((x \in \mathbb{R}) \implies (f(x) > 0))$. Let g be the function $g : \mathbb{R} \implies \mathbb{R}$ defined by g(x) = 1/f(x). f(x) is strictly decreasing if and only if g(x) is strictly increasing.*

Proof. Suppose there exist real numbers x and y such that $x < y$, and suppose that $f(x) > f(y)$. f is a strictly decreasing real-valued function by definition. It follows that $g(x) = 1/f(x) < g(y) = 1/f(y)$, which is the definition for strictly increasing real-valued functions.

Conversely, suppose there exist real numbers x and y such that $x < y$, and suppose that $g(x) < g(y)$. g is a strictly increasing real-valued function by definition. It follows that $f(x) = 1/g(x) > f(y) = 1/f(y)$, which is the definition for strictly decreasing real-valued functions.

Thus, $f(x)$ is strictly decreasing if and only if $g(x)$ is strictly increasing. ∎

Theorem (2.3.24). *Let f be the function $f : \mathbb{R} \implies \mathbb{R}$ defined by $f(x) = e^x$. $f(x)$ is not invertible.*

Proof. The inverse function of $f(x) = e^x$ is $f(x)^{-1} = \log_e x$. But logarithmic functions are undefined for negative-valued domains. Thus, $f(x)$ is not bijective, and $f(x)$ is not invertible. ■

Theorem (2.3.25). *Let f be a function $f : \mathbb{R} \implies \mathbb{R}$ defined by $f(x) = |x|$. $f(x)$ is not invertible.*

Proof. Let x be a postive real number. $f(x) = y$ and $f(-x) = y$. If f had an inverse then $f^{-1}(y) = x$ or $f^{-1}(y) = -x$, so f^{-1} is not a function by definition. Which concludes the proof. ■

Theorem (2.3.29a). *Let f be a function $f : B \implies C$, and let g be a function $g : A \implies B$. If both f and g are injective, then $f \circ g$ is injective.*

Proof. By the contrapositive. Let the domain of discourse be A. Suppose it were not the case that $(f \circ g)$ were injective. Then by the definition for injective functions, the following universally quantified statement is true, $\neg\forall a \forall b((f \circ g)(a) = (f \circ g)(b) \implies (a = b))$. Note that the composition of functions $(f \circ g)(x)$ is defined by $f(g(x))$. Thus, we have the equivalent universal quantification $\neg\forall a \forall b(f(g(a)) = f(g(b)) \implies (a = b))$. In other words, it is not the case that f is injective, by the definition for injective functions. Also, because $f = f$, $\neg\forall a \forall b(g(a) = g(b) \implies (a = b))$ is a logically equivalent universal quantification. That is, it is not the case that g is injective, by the definition for injective functions. Since the contrapositive follows directly from the negation of the conclusion, it is necessarily the case that if both f and g are injective, then $f \circ g$ is injective. ■

Theorem (2.3.29b). *Let f be a function $f : B \implies C$, and let g be a function $g : A \implies B$. If both f and g are surjective, then $f \circ g$ is surjective.*

Proof. Let C be the domain of discourse. By the hypothesis, and by the definition for surjective functions, the following universally quantified statement must be true, $\forall c \exists b(f(b) = c)$. Note that $g(x)$ is in the domain of f, for every x in the domain of g, by the definition of g. It immediately follows from the general definition for functions that $\forall c \exists a(f(g(a)) = c)$ must be a logically equivalent universal quantification. Since the composition of functions $(f \circ g)(x)$ is defined by $f(g(x))$, it follows directly from the hypothesis that $f \circ g$ is surjective, by the definition for surjective functions. ■

Theorem (2.3.30). *Let f and $f \circ g$ be injective functions. g is injective.*

Proof. By the contrapositive. Suppose that g were not injective. Then by the definition for injective functions we have the following universally quantified statement, with the domain of discourse being the domain of g, $\neg \forall a \forall b((g(a) = g(b)) \implies (a = b))$. Because $f = f$, this statement is logically equivalent to $\neg \forall a \forall b((f(g(a)) = f(g(b))) \implies (a = b))$. By the definition for the compositions of functions we can also draw this equivalence, $\neg \forall a \forall b((f \circ g)(a) = (f \circ g)(b)) \implies (a = b))$. That is, it is not the case that $f \circ g$ is injective, by the definition for injective functions. So it follows directly from the negation of the statement "g is injective," that $f \circ g$ is not injective. Thus, if f and $(f \circ g)$ are injective functions, then g is indeed injective. ■

Theorem (2.3.36a). *Let f be the function $f : A \implies B$. Let S, and T be subsets of A. $f(S \cup T) = f(S) \cup f(T)$.*

Proof. By the definition for the image of a set $(S \cup T)$ under the function f we have $f(S \cup T) = \{t | \exists s \in S \cup T(t = f(s))\} \equiv \{f(s) | s \in (S \cup T)\}$. Since $\{f(s) | s \in (S \cup T)\}$ is a set, from this we can write $f(S \cup T) \equiv \{f(s) | (s \in S)\} \cup \{f(s) | (s \in T)\}$. The right-hand side of this equivalence is the set $f(S) \cup f(T)$, by the definition for the image of a set S or T under the function $f \therefore f(S \cup T) = f(S) \cup f(T)$. ■

Theorem (2.2.36b). *Let f be the function $f : A \implies B$. Let S, and T be subsets of A. $f(S \cap T) \subseteq f(S) \cap f(T)$.*

Proof. Let a be an element in A such that $f(a) \in f(S \cap T)$. Hence, by the definition for the image of $(S \cap T)$ under the function f, $a \in (S \cap T)$. The set intersection is defined as $(a \in S) \wedge (a \in T)$. Of course $[f(a) \in f(S)] \wedge [f(a) \in f(T)]$. That is, $f(a) \in [f(S) \cap f(T)]$, and indeed $f(S \cap T) \subseteq f(S) \cap f(T)$. ■

Theorem (2.3.40a). *Let f be the function $f : A \implies B$. Let S, and T be subsets of B. $f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$.*

Proof. By the definition for the inverse image of the set $(S \cup T)$ under the function f^{-1}, we have $f^{-1}(S \cup T) = \{a \in A | f(a) \in (S \cup T)\}$. Then equivalently, $f^{-1}(S \cup T) \equiv \{a \in A | f(a) \in S\} \cup \{a \in A | f(a) \in T\}$. This is the formal definition for $f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$. ■

Theorem (2.3.40b). *Let f be the function $f : A \implies B$. Let S, and T be subsets of B. $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$.*

Proof. By the definition for the inverse image of the set $(S \cap T)$ under the function f^{-1}, we have $f^{-1}(S \cap T) = \{a \in A | f(a) \in (S \cap T)\}$. Then equivalently, $f^{-1}(S \cap T) \equiv \{a \in A | f(a) \in S\} \cap \{a \in A | f(a) \in T\}$. This is the formal definition for $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$. ■

Theorem (2.3.41). *Let f be the function $f : A \implies B$. Let S be a subset of B. $f^{-1}(\overline{S}) = \overline{f^{-1}(S)}$.*

Proof. By the definition for the inverse image of $\overline{S}$ under the function f^{-1}, we have $f^{-1}(\overline{S}) \equiv \{a \in A | f(a) \in \overline{S}\}$. Factoring the complementation out from the right-hand side of the equivalence, $f^{-1}(\overline{S}) \equiv \overline{\{a \in A | f(a) \in S\}}$. But this statement is the negation of the formal definition for the inverse image of S under the function f^{-1}. In other words, $f^{-1}(\overline{S}) = \overline{f^{-1}(S)}$. ■

Theorem (2.3.42). *Let x be a real number. $\lfloor x + \frac{1}{2} \rfloor$ is the closest integer to x, except when x is midway between two integers, when it is the larger of these two integers.*

Proof. By cases. Let n be the integer such that $n \leq x < n + 1$ and $\lfloor x + \frac{1}{2} \rfloor = \lfloor n + \epsilon + \frac{1}{2} \rfloor$. ϵ is the decimal part of x.

(i) If $\epsilon \geq \frac{1}{2}$, then $\epsilon + \frac{1}{2} \geq \frac{1}{2} + \frac{1}{2}$. That is, if $\epsilon + \frac{1}{2} \geq 1$, then $\lfloor x + \frac{1}{2} \rfloor \geq \lfloor (x - \epsilon) + 1 \rfloor = n + 1$.

(ii) If $\epsilon < \frac{1}{2}$, then $\epsilon + \frac{1}{2} < 1$. That is, if $\epsilon + \frac{1}{2} < 1$, then $\lfloor x + \frac{1}{2} \rfloor = \lfloor n + (\epsilon + \frac{1}{2}) \rfloor = n$. ∎

Theorem (2.3.43). *Let x be a real number. $\lceil x - \frac{1}{2} \rceil$ is the closest integer to x, except when x is midway between two integers, when it is the smaller of these two integers.*

Proof. By cases. Let n be the integer such that $n \leq x < n + 1$ and $\lceil x - \frac{1}{2} \rceil = \lceil (n + \epsilon) - \frac{1}{2}) \rceil$. ϵ is the decimal part of x.

(i) If $\epsilon > \frac{1}{2}$, then $\epsilon - \frac{1}{2} > \frac{1}{2} - \frac{1}{2} = 0$. So $\lceil (n + \epsilon) - \frac{1}{2} \rceil = n + 1$.

(ii) If $\epsilon \leq \frac{1}{2}$, then $\epsilon - \frac{1}{2} \leq 0$. So $(n-1) \leq [(n+\epsilon) - \frac{1}{2}] < n$, and $\lceil x - \frac{1}{2} \rceil = n$. ∎

Theorem (2.3.44). *Let x be a real number. $\lceil x \rceil - \lfloor x \rfloor = 1$, if $x \notin \mathbb{Z}$. $\lceil x \rceil - \lfloor x \rfloor = 0$, if $x \in \mathbb{Z}$.*

Proof. By cases.

(i) Suppose $x \notin \mathbb{Z}$. $\lceil x \rceil = \lfloor x \rfloor + 1$. Thus, $\lceil x \rceil - \lfloor x \rfloor = (\lfloor x \rfloor + 1) - \lfloor x \rfloor = 1$.

(ii) Suppose $x \in \mathbb{Z}$. $\lceil x \rceil = \lfloor x \rfloor = x$. Thus, $\lceil x \rceil - \lfloor x \rfloor = x - x = 0$. ∎

Theorem (2.3.45). *Let x be a real number.*
$(x-1) < \lfloor x \rfloor \leq x \leq \lceil x \rceil < (x+1)$.

Proof. Notice that $\epsilon = x - \lfloor x \rfloor$, so $(0 \leq \epsilon < 1)$. It is important to note too, that multiplying this inequality by -1 on every side yields $(0 \geq -\epsilon > -1) = (-1 < -\epsilon \leq 0)$. Finally, note that $\sigma = \lceil x \rceil - x$, so $(0 \leq \sigma < 1)$. But these inequalities together state that $-1 < -\epsilon \leq 0 \leq \sigma < 1$. Since this inequality is true, by adding x to every side we find that the following statement is also true: $(x-1) < \lfloor x \rfloor \leq x \leq \lceil x \rceil < (x+1)$. ∎

Theorem (2.3.46). *Let x be a real number, and let m be an integer.*
$\lceil x + m \rceil = \lceil x \rceil + m$.

Proof. By the definition of the ceiling function, we have the follow tautology.
$\lceil x \rceil = \lceil x \rceil \iff (\lceil x \rceil - 1) < x \leq \lceil x \rceil$.
Adding the integer m to every side of this inequality gives the following resultant tautology, by definition,
$\lceil x + m \rceil = \lceil x \rceil + m \iff (\lceil x \rceil + m) - 1 < x + m \leq \lceil x \rceil + m$. ∎

Theorem (2.3.47a). *Let x be a real number, and let n be an integer.*
$x < n \iff \lfloor x \rfloor < n$.

Proof. $\lfloor x \rfloor \leq x$, by the properties of the floor function. So if $x < n$, then $\lfloor x \rfloor \leq x < n$, and of course $\lfloor x \rfloor < n$.

Proving the converse, suppose $\lfloor x \rfloor < n$. Since $\lfloor x \rfloor$ and n are integers, $\lfloor x \rfloor + 1 \leq n$. Now, by the properties of the floor function, we have the following tautology, $\lfloor x \rfloor = \lfloor x \rfloor \iff \lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$. Since we know that $\lfloor x \rfloor + 1 \leq n$, it must be that $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1 \leq n$. This statement says that $x < n$. ∎

Theorem (2.3.47b). *Let x be a real number, and let n be an integer.*
$n < x \iff n < \lceil x \rceil$.

Proof. $x \leq \lceil x \rceil$, by the properties of the ceiling function. So if $n < x$, then $n < x \leq \lceil x \rceil$, and $n < \lceil x \rceil$.

Proving the converse, suppose $n < \lceil x \rceil$. Since n and $\lceil x \rceil$ are integers, $n \leq \lceil x \rceil - 1$. By the properties of ceiling functions we have the following tautology, $\lceil x \rceil = \lceil x \rceil \iff \lceil x \rceil - 1 < x \leq \lceil x \rceil$. Combining these two inequalities yields $n \leq \lceil x \rceil - 1 < x \leq \lceil x \rceil$. Thus, $n < x$. ■

Theorem (2.3.48a). *Let x be a real number, and let n be an integer.*
$x \leq n \iff \lceil x \rceil \leq n$.

Proof. Direct form by the contrapositive. Suppose $\lceil x \rceil > n$. Since $\lceil x \rceil$ and n are integers, $\lceil x \rceil - 1 \geq n$. By the properties of ceiling functions we have the following tautology, $\lceil x \rceil = \lceil x \rceil \iff \lceil x \rceil \geq x > \lceil x \rceil - 1$. Combining these two inequalities yields $\lceil x \rceil \geq x > \lceil x \rceil - 1 \geq n$. This says, $x > n$. Since this statement following from the negation of the direct consequent is itself the negation of the direct hypothesis, $x \leq n \implies \lceil x \rceil \leq n$, is true.

Converse form by the contrapositive. Suppose $x > n$. Note that $\lceil x \rceil \geq x$, by the properties of the ceiling function. So if $x > n$, then $\lceil x \rceil \geq x > n$, and $\lceil x \rceil > n$. Since this statement is the negation of the converse hypothesis following directly from negation of the converse consequent,
$x \leq n \impliedby \lceil x \rceil \leq n$, is true.

Thus proves, the biconditional statement $x \leq n \iff \lceil x \rceil \leq n$. ■

Theorem (2.3.48b). *Let x be a real number, and let n be an integer.*
$n \leq x \iff n \leq \lfloor x \rfloor$.

Proof. By the direct form contrapositive. Suppose $n > \lfloor x \rfloor$. Since n and $\lfloor x \rfloor$ are integers, $n \geq \lfloor x \rfloor + 1$. Now, by the properties of the floor function, we have the following tautology, $\lfloor x \rfloor = \lfloor x \rfloor \iff \lfloor x \rfloor + 1 > x \geq \lfloor x \rfloor$. Combining these two inequalities yields $n \geq \lfloor x \rfloor + 1 > x \geq \lfloor x \rfloor$ This statement says that $n > x$.

Proving the converse form by the contrapositive. Note that $x \geq \lfloor x \rfloor$, by the properties of the floor function. So if $n > x$, then $n > x \geq \lfloor x \rfloor$, and of course $n > \lfloor x \rfloor$.

$\therefore n \leq x \iff n \leq \lfloor x \rfloor$ ■

Theorem (2.3.49). *Let n be an integer. If n is even, then $\lfloor \frac{n}{2} \rfloor = \frac{n}{2}$. If n is odd, then $\lfloor \frac{n}{2} \rfloor = \frac{(n-1)}{2}$.*

Proof. By cases.

(i) Since n is even, there exists an integer k such that $n = 2k$.
$\lfloor \frac{n}{2} \rfloor = \lfloor \frac{2k}{2} \rfloor = \lfloor k \rfloor = k$. Also, $\frac{n}{2} = \frac{2k}{2} = k$. So $k = \lfloor \frac{n}{2} \rfloor = \frac{n}{2}$.

(ii) Since n is odd, there exists an integer k such that $n = 2k + 1$.
$\lfloor \frac{n}{2} \rfloor = \lfloor \frac{2k+1}{2} \rfloor = \lfloor k + \frac{1}{2} \rfloor = k$. Also, $\frac{(n-1)}{2} = \frac{[(2k+1)-1]}{2} = \frac{2k}{2} = k$.
So, $k = \lfloor \frac{n}{2} \rfloor = \frac{(n-1)}{2}$. ■

Theorem (2.3.50). *Let x be a real number.*
$\lfloor -x \rfloor = -\lceil x \rceil$, and $\lceil -x \rceil = -\lfloor x \rfloor$.

Proof. By the properties of ceiling functions,
$\lceil x \rceil = n \iff (n-1) < x \leq n$. Multiplying every side of this inequality by -1 yields $(-n+1) > -x \geq -n$. By the properties of floor functions, this means that $\lfloor -x \rfloor = -n$. And of course since $-1 \times \lceil x \rceil = -n$, we have $-n = \lfloor -x \rfloor = -\lceil x \rceil$.

By the properties of floor functions,
$\lfloor x \rfloor = n \iff n \leq x < (n+1)$. Multiplying every side of this inequality by -1 yields $-n \geq -x > (-n-1)$. By the properties of ceiling functions, this means that $\lceil -x \rceil = -n$. And of course since $-1 \times \lfloor x \rfloor = -n$, we have $-n = \lceil -x \rceil = -\lfloor x \rfloor$. ■

Theorem (2.3.66). *Let f be the invertible function $f : Y \implies Z$, and let g be the invertible function $g : X \implies Y$. The inverse of the composition $f \circ g$ is given by $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.*

Proof. By Theorem 2.3.29a and Theorem 2.3.29b, and by the definition for bijective functions, $f \circ g$ is invertible. Thus, $(f \circ g)^{-1} \circ (f \circ g) = \iota_X$.

What remains to be determined is whether $(g^{-1} \circ f^{-1}) \circ (f \circ g) = \iota_X$. Let x be an element in the domain of g such that $((g^{-1} \circ f^{-1}) \circ (f \circ g))(x) = x$. By the definition for the composition of functions, that is $g^{-1}(f^{-1}(f(g(x)))) = x$. Clearly, $(g^{-1} \circ f^{-1}) \circ (f \circ g) = \iota_X$.

Thus, the inverse of the composition $f \circ g$ is indeed given by
$(f \circ g)^{-1} = g^{-1} \circ f^{-1}$. ■

Theorem (2.3.67a). *Let A, and B be sets with universal set U. Let $f_{A \cap B}$ be the characteristic function $f_{A \cap B} : U \implies \{0, 1\}$. Let f_A be the characteristic function $f_A : U \implies \{0, 1\}$. Let f_B be the characteristic function $f_B : U \implies \{0, 1\}$. $f_{A \cap B}(x) = f_A(x) \times f_B(x)$.*

Proof. Let x be an element in $A \cap B$. By the definition for characteristic functions, $f_{A \cap B}(x) = 1$. Since the definition for set intersection says that $(x \in A) \wedge (x \in B)$, we know by the definition for characteristic functions that $f_A(x) = f_B(x) = 1$. Thus, it follows immediately by the multiplicative identity law from the field axioms that $f_{A \cap B}(x) = f_A(x) \times f_B(x)$.

Suppose it were not the case that x were an element in $A \cap B$. That is, $x \notin (A \cap B) \equiv [(x \notin A) \vee (x \notin B)]$, by DeMorgans law. By the definition for characteristic functions, $f_{A \cap B}(x) = 0$. Also, again by the definition for characteristic functions we know that $(f_A(x) = 0) \vee (f_B(x) = 0)$. Without loss of generality we can suppose $f_A(x) = 0$. It follows immediately from the multiplicative property of zero that $f_{A \cap B}(x) = 0 \times f_B(x) = 0$. Thus, $f_{A \cap B}(x) = f_A(x) \times f_B(x)$. ■

Theorem (2.3.67b). *Let A, and B be sets with universal set U. Let $f_{A\cup B}$ be the characteristic function $f_{A\cup B} : U \implies \{0,1\}$. Let f_A be the characteristic function $f_A : U \implies \{0,1\}$. Let f_B be the characteristic function*
$f_B : U \implies \{0,1\}$.
$f_{A\cup B}(x) = f_A(x) + f_B(x) - f_A(x) \times f_B(x)$.

Proof. First suppose that x were not an element in $A \cup B$. It follows from the definition for characteristic functions that $f_{A\cup B}(x) = 0$. Also, by the definition for set union x is in neither A nor B, so $f_A(x) = 0$, and $f_B(x) = 0$. Thus, $f_A(x) + f_B(x) - f_A(x) \times f_B(x) = 0 + 0 - 0 \times 0 = 0$. Therefore, $f_{A\cup B}(x) = f_A(x) + f_B(x) - f_A(x) \times f_B(x)$.

Now suppose it were the case that x was an element in $A \cup B$. It follows from the definition for characteristic functions that $f_{A\cup B}(x) = 1$. Also, by the definition for set union $(x \in A) \vee (x \in B)$. Hence, there are three cases to consider here.

(i) Suppose $(x \in A)$ and $(x \notin B)$. By the definition for characteristic functions we have $f_A(x) = 1$ and $f_B(x) = 0$. Thus,
$f_A(x) + f_B(x) - f_A(x) \times f_B(x) = 1 + 0 - 1 \times 0 = 1$.

(ii) Suppose $(x \notin A)$ and $(x \in B)$. Without loss of generality this case has the same result as case (i).

(iii) If x is in the intersection of A and B we have
$f_A(x) + f_B(x) - f_A(x) \times f_B(x) = 1 + 1 - 1 \times 1 = 1$.

Since $f_{A\cup B}(x) = f_A(x) + f_B(x) - f_A(x) \times f_B(x) = 1$ for all three possible cases, thus concludes the proof. ∎

Theorem (2.3.67c). *Let A be a set with universal set U. Let f_A be the characteristic function $f_{\overline{A}} : U \implies \{0,1\}$. $f_{\overline{A}}(x) = 1 - f_A(x)$.*

Proof. Let x be an element in A. Then clearly $x \notin \overline{A}$. By the definition for characteristic functions $f_{\overline{A}}(x) = 0$, and $f_A(x) = 1$. It follows immediately that $f_{\overline{A}}(x) = 1 - f_A(x)$.

Now suppose $(x \notin A) \wedge (x \in \overline{A})$ By the definition for characteristic functions that is $f_{\overline{A}}(x) = 1$, and $f_A(x) = 0$. It follows immediately that $f_{\overline{A}}(x) = 1 - f_A(x)$. ∎

Theorem (2.3.67d). *Let A, and B be sets with universal set U. Let $f_{A\oplus B}$ be the characteristic function $f_{A\oplus B} : U \implies \{0,1\}$. Let f_A be the characteristic function $f_A : U \implies \{0,1\}$. Let f_B be the characteristic function $f_B : U \implies \{0,1\}$. $f_{A\oplus B}(x) = f_A(x) + f_B(x) - 2f_A(x)f_B(x)$.*

Proof. There are two major cases to consider, each consisting of two sub cases. The major cases are where x is an element in $A \oplus B$, and the negation of that statement.

(i) Let x be an element in $A \oplus B$. By the definition for characteristic functions, $f_{A\oplus B}(x) = 1$. Since the definition for set symmetric difference says $[(x \in A) \wedge (x \notin B)] \vee [(x \notin A) \wedge (x \in B)]$, there are two sub cases that need to be taken under consideration.

(a) Suppose $(x \in A) \wedge (x \notin B)$. By the definition for characteristic functions $f_A(x) = 1$ and $f_B(x) = 0$. This means that $f_A(x) + f_B(x) - 2f_A(x)f_B(x) = 1 + 0 - 2(1)(0) = 1$.

(b) Suppose $(x \notin A) \wedge (x \in B)$. Without loss of generality we arrive at the same result as that of case (a).

Thus, if x is an element in $A \oplus B$, $f_{A\oplus B}(x) = f_A(x) + f_B(x) - 2f_A(x)f_B(x)$.

(ii) Suppose it were not the case that x were an element in $A \oplus B$. Then (c) x must either be an element in the intersection of A and B, or (d) x must be in the universe minus $A \cup B$.

(c) Suppose $x \in (A \cap B)$. By the definition for characteristic functions $f_{A\oplus B}(x) = 0$, $f_A(x) = 1$ and $f_B(x) = 1$. Thus, $f_A(x) + f_B(x) - 2f_A(x)f_B(x) = 1 + 1 - 2(1)(1) = 0$.

(d) Suppose $x \in [U - (A \cup B)$. In this case, by the definition for characteristic functions, $f_{A\oplus B}(x) = 0$, $f_A(x) = 0$ and $f_B(x) = 0$. So, $f_A(x) + f_B(x) - 2f_A(x)f_B(x) = 0 + 0 - 2(0)(0) = 0$.

Thus, if x is not an element in $A \oplus B$, $f_{A\oplus B}(x) = f_A(x) + f_B(x) - 2f_A(x)f_B(x)$ is still a true statement; concludes the proof. ∎

Theorem (2.3.68). *Let f be a function $f : A \implies B$, where A and B are finite sets, and $|A| = |B|$. f is injective if and only if f is surjective.*

Proof. Direct form by the contrapositive. Suppose the negation of the statement given by the definition for surjective functions, $\exists y \forall x(f(x) \neq y)$. This statement can only be true if either $|A| < |B|$ (contradicting the hypothesis,) or $\exists x \exists y((f(x) = f(y)) \wedge (x \neq y))$. Since contradiction is $\perp$ by the law for logical negation, by the identity law for logical disjunction, f is defined as not injective. Thus, if f is injective, then f is surjective.

Converse form by the contrapositive. Suppose the negation of the statement given by the definition for injective functions, $\exists x \exists y((f(x) = f(y)) \wedge (x \neq y))$. This statement can only be true if either $|A| > |B|$ (contradicting the hypothesis,) or $\exists y \forall x(f(x) \neq y)$. Since contradiction is $\perp$ by the law for logical negation, by the identity law for logical disjunction, f is defined as not surjective. Thus, if f is surjective, then f is injective.

$\therefore \forall x \forall y((f(x) = f(y)) \implies (x = y)) \iff \forall y \exists x(f(x) = y)$, whenever f is a function $f : A \implies B$, where A and B are finite sets, and $|A| = |B|$. ∎

Theorem (2.3.69a). *Let x be a real number. $\lceil \lfloor x \rfloor \rceil = \lfloor x \rfloor$.*

Proof. Let n be the integer such that $n \leq x < n + 1$. By the properties for floor functions, $\lfloor x \rfloor = n$. So $\lceil \lfloor x \rfloor \rceil = \lceil n \rceil$. Since $n - 1 < n \leq n$ is a tautology, by the properties for ceiling functions it must be the case that $\lceil n \rceil = n$. But $n = \lfloor x \rfloor$, so $\lceil \lfloor x \rfloor \rceil = \lfloor x \rfloor$. ∎

Theorem (2.3.69c). *Let x and y be real numbers.*
$\lceil x \rceil + \lceil y \rceil - \lceil x+y \rceil = 0$, *or* 1.

Proof. By cases. There are two possible cases to take into consideration.
(i) x or y (or both) are integers in real numbers, or (ii) neither x nor y is an integer.

(i) Suppose x or y (or both) are integers in real numbers. Since at least one of these numbers x or y must be an integer, and because addition is commutative, without loss of generality it can be supposed that y is certainly an integer. Then since y is an integer, the smallest integer greater than or equal to y, is y. So by the definition for ceiling functions, $\lceil y \rceil = y$. By that fact, and by Theorem 2.3.46, $\lceil x \rceil + \lceil y \rceil - \lceil x+y \rceil = \lceil x \rceil + \lceil y \rceil - \lceil x \rceil + \lceil y \rceil = 0$.

(ii) Suppose that neither x nor y is an integer. Then let ϵ and σ be real numbers such that $\lceil x \rceil - x = \epsilon$, and $\lceil y \rceil - y = \sigma$. By Theorem 2.3.44, $\lceil x \rceil = \lfloor x \rfloor + 1$, and $\lceil y \rceil = \lfloor y \rfloor + 1$. Naturally, $x = \lfloor x \rfloor + (1-\epsilon)$, and $y = \lfloor y \rfloor + (1-\sigma)$. Thus, $\lceil x \rceil + \lceil y \rceil - \lceil x+y \rceil = (\lfloor x \rfloor + 1) + (\lfloor y \rfloor + 1) - \lceil \lfloor x \rfloor + (1-\epsilon) + \lfloor y \rfloor + (1-\sigma) \rceil$. Rearranging these terms according to the usual rules for arithmetic yields $(\lfloor x \rfloor + \lfloor y \rfloor + 2) - \lceil \lfloor x \rfloor + \lfloor y \rfloor + [2 - (\epsilon + \sigma)] \rceil$. Now, there are two possible sub cases to consider regarding this expression. Either (a) $\epsilon + \sigma \geq 1$, or (b) $\epsilon + \sigma < 1$.

> (a) Suppose $\epsilon + \sigma \geq 1$. This means that $1 \geq 2 - (\epsilon + \sigma)$. By Theorem 2.3.69a we get the following equation,
> $(\lfloor x \rfloor + \lfloor y \rfloor + 2) - \lceil \lfloor x \rfloor + \lfloor y \rfloor + [2 - (\epsilon + \sigma)] \rceil =$
> $(\lfloor x \rfloor + \lfloor y \rfloor + 2) - (\lfloor x \rfloor + \lfloor y \rfloor + 1) = 1$.
>
> (b) Suppose $\epsilon + \sigma < 1$. This means that $1 < 2 - (\epsilon + \sigma)$. By Theorem 2.3.69a we get the following equation,
> $(\lfloor x \rfloor + \lfloor y \rfloor + 2) - \lceil \lfloor x \rfloor + \lfloor y \rfloor + [2 - (\epsilon + \sigma)] \rceil =$
> $(\lfloor x \rfloor + \lfloor y \rfloor + 2) - (\lfloor x \rfloor + \lfloor y \rfloor + 2) = 0$.

$\therefore \lceil x \rceil + \lceil y \rceil - \lceil x+y \rceil = 0$, or 1, whenever x and y are real numbers. ∎

Theorem (2.3.70a). *Let x be a real number.* $\lfloor \lceil x \rceil \rfloor = \lceil x \rceil$.

Proof. Let n be the integer such that $n - 1 < x \leq n$. By the properties for ceiling functions, $\lceil x \rceil = n$. So $\lfloor \lceil x \rceil \rfloor = \lfloor n \rfloor$. Since $n \leq n < n + 1$ is a tautology, by the properties for floor functions it must be the case that $\lfloor n \rfloor = n$. But $n = \lceil x \rceil$. So $\lfloor \lceil x \rceil \rfloor = \lceil x \rceil$. ■

Theorem (2.3.70c). *Let x be a real number.* $\lceil \lceil \frac{x}{2} \rceil \div 2 \rceil = \lceil \frac{x}{4} \rceil$.

Proof. Let n be an integer satisfying the properties for ceiling functions with respect to x such that $\lceil \frac{x}{4} \rceil = n$. Thus establishes the fact, $4n - 4 < x \leq 4n$. We shall proceed by analyzing the statement $\lceil \lceil \frac{x}{2} \rceil \div 2 \rceil = n$. If $\lceil \lceil \frac{x}{2} \rceil \div 2 \rceil = \lceil \frac{x}{4} \rceil$ is true, then $\lceil \lceil \frac{x}{2} \rceil \div 2 \rceil = n$ will be defined, by the properties of ceiling functions, as $4n - 4 < x \leq 4n$; since this is the case for $\lceil \frac{x}{4} \rceil = n$.

First $\lceil \lceil \frac{x}{2} \rceil \div 2 \rceil = n$ says that $2n - 2 < \lceil \frac{x}{2} \rceil \leq 2n$. By the properties for ceiling functions, this is equivalently stated as (i) $\lceil \frac{x}{2} \rceil = 2n - 1$, or (logical) (ii) $\lceil \frac{x}{2} \rceil = 2n$.

(i) $\lceil \frac{x}{2} \rceil = 2n - 1$, by the properties for ceiling functions, states that $4n - 4 < x \leq 4n - 2$.

(ii) $\lceil \frac{x}{2} \rceil = 2n$, by the properties for ceiling functions, states that $4n - 2 < x \leq 4n$.

The statement $4n-4 < x \leq 4n-2$ or (logical) $4n-2 < x \leq 4n$ is the same as $4n-4 < x \leq 4n$. Thus, $\lceil \lceil \frac{x}{2} \rceil \div 2 \rceil = n$, is indeed defined by $4n-4 < x \leq 4n$. Because both sides of the equation have the same definition, the statement $\lceil \lceil \frac{x}{2} \rceil \div 2 \rceil = \lceil \frac{x}{4} \rceil$, is true. ■

Theorem (2.3.70e). *Let x, and y be real numbers.* $\lfloor x \rfloor + \lfloor y \rfloor + \lfloor x+y \rfloor \leq \lfloor 2x \rfloor + \lfloor 2y \rfloor$.

Proof. There are four possible cases that could be considered in this proof, but only two of those cases require consideration. Case (i) demonstrating the minimum possible amount occurring on the right-hand side will establish truth for equality expressed by the theorem. Case (ii) demonstrating the maximum possible amount occurring on the right-hand side will establish truth for inequality expressed by the theorem. In all cases the left-hand side remains relatively constant.

First we establish the necessary preliminary facts. Let ϵ and σ be real numbers such that $x - \lfloor x \rfloor = \epsilon$, and $y - \lfloor y \rfloor = \sigma$. Of course, $x - \epsilon = \lfloor x \rfloor$. By the properties for floor functions we know that $x - \epsilon \leq x < (x-\epsilon)+1$. And, again by the properties for floor functions, $2(x-\epsilon) \leq 2x < 2[(x-\epsilon)+1] \implies \lfloor 2x \rfloor$. This means that ($i$) $\lfloor 2x \rfloor = 2x - 2\epsilon$, or (logical) ($ii$) $\lfloor 2x \rfloor = 2x - 2\epsilon + 2$. Without loss of generality, all of these equations remain true whenever predicated of y and σ. Also, note that the left-hand side has a constant form, $\lfloor x \rfloor + \lfloor y \rfloor + \lfloor x+y \rfloor = 2(x+y) - 2(\epsilon+\sigma)$.

(i) Suppose it were the case that $\lfloor 2x \rfloor = 2x - 2\epsilon$, and $\lfloor 2y \rfloor = 2y - 2\sigma$. These are the least amounts possible for the right-hand side of the inequality expressed by the theorem. The sum being $2(x+y) - 2(\epsilon+\sigma)$, equal to the left-hand side. In this case (i) the theorem proves true.

(ii) Suppose it were the case that $\lfloor 2x \rfloor = 2x - 2\epsilon + 2$, and $\lfloor 2y \rfloor = 2y - 2\sigma + 2$. These are the greatest amounts possible for the right-hand side of the inequality expressed by the theorem. The sum being $2(x+y+2) - 2(\epsilon+\sigma)$. This is clearly greater than the left-hand side. In this case (ii) the theorem proves true.

Since the entire range of all possible values are covered by cases (i) and (ii), and the statement remains true throughout, it is proven that $\lfloor x \rfloor + \lfloor y \rfloor + \lfloor x+y \rfloor \leq \lfloor 2x \rfloor + \lfloor 2y \rfloor$. ■

Theorem (2.3.71a). *Let x be a positive real number.* $\lfloor\sqrt{\lfloor x\rfloor}\rfloor = \lfloor\sqrt{x}\rfloor$.

Proof. By the properties for floor functions,
$\lfloor\sqrt{x}\rfloor = n \iff n \le \sqrt{x} < n+1$. Squaring the inequalities we can determine the value for the floor of x. So there are two cases under consideration, (i) $\lfloor x\rfloor = n^2$, or (ii) $\lfloor x\rfloor = n^2 + 2n$.

(i) Suppose that $\lfloor x\rfloor = n^2$. It follows that $\lfloor\sqrt{\lfloor x\rfloor}\rfloor = \lfloor\sqrt{n^2}\rfloor = \lfloor n\rfloor$. Since n is an integer, n is the largest integer less than or equal to n. So $\lfloor n\rfloor = n$, by the definition for floor functions. Because $n = \lfloor\sqrt{x}\rfloor$, in this case it is proved that $\lfloor\sqrt{\lfloor x\rfloor}\rfloor = \lfloor\sqrt{x}\rfloor$.

(ii) Suppose that $\lfloor x\rfloor = n^2 + 2n$. Then,
$\lfloor\sqrt{\lfloor x\rfloor}\rfloor = \lfloor\sqrt{n^2+2n}\rfloor = \lfloor\sqrt{n^2+2n} + \sqrt{1} - \sqrt{1}\rfloor =$
$\lfloor\sqrt{n^2+2n+1} - 1\rfloor = \lfloor\sqrt{(n+1)^2} + 1\rfloor = \lfloor(n+1) - 1\rfloor = n$. Since $n = \lfloor\sqrt{x}\rfloor$, in this case it is proved that $\lfloor\sqrt{\lfloor x\rfloor}\rfloor = \lfloor\sqrt{x}\rfloor$. ∎

Theorem (2.3.71b). *Let x be a positive real number.* $\lceil\sqrt{\lceil x\rceil}\rceil = \lceil\sqrt{x}\rceil$.

Proof. By the properties for floor functions,
$\lceil\sqrt{x}\rceil \iff n - 1 < \sqrt{x} \le n$. Squaring the inequalities we can determine the value for the floor of x. Thus, there are two cases under consideration (i) $\lceil x\rceil = n^2 - 2n$, or (ii) $\lceil x\rceil = n^2$.

(i) Suppose that $\lceil x\rceil = n^2 - 2n$. It follows that,
$\lceil\sqrt{\lceil x\rceil}\rceil = \lceil\sqrt{n^2-2n}\rceil = \lceil\sqrt{n^2-2n} - \sqrt{1} + \sqrt{1}\rceil =$
$\lceil\sqrt{n^2-2n-1} + 1\rceil = \lceil\sqrt{(n-1)^2} + 1\rceil = \lceil(n-1) + 1\rceil = n$. Since $n = \lceil\sqrt{x}\rceil$, in this case it is proved that $\lceil\sqrt{\lceil x\rceil}\rceil = \lceil\sqrt{x}\rceil$.

(ii) Suppose that $\lceil x\rceil = n^2$. It follows that $\lceil\sqrt{\lceil x\rceil}\rceil = \lceil\sqrt{n^2}\rceil = \lceil n\rceil$. Since n is an integer, n is the smallest integer that is greater than or equal to n. So $\lceil n\rceil = n$, by the definition for ceiling functions. Because $n = \lceil\sqrt{x}\rceil$, in this case it is proved that $\lceil\sqrt{\lceil x\rceil}\rceil = \lceil\sqrt{x}\rceil$. ∎

Theorem (2.3.72). *Let x be a real number.* $\lfloor 3x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor$.

Proof. By cases. Let ϵ be a real number such that $x - \lfloor x \rfloor = \epsilon$. Clearly, $3x - \lfloor 3x \rfloor = 3\epsilon$. There are three possible cases that must be proved,
(i) $0 \leq \epsilon < \frac{1}{3}$, (ii) $\frac{1}{3} \leq \epsilon < \frac{2}{3}$, and (iii) $\frac{2}{3} \leq \epsilon < 1$.

(i) Suppose $0 \leq \epsilon < \frac{1}{3}$. Since $\epsilon + \frac{2}{3} < 1$, every term on the right-hand side is equal to $x - \epsilon$. That is $3x - 3\epsilon$. But $\lfloor 3x \rfloor = 3x - 3\epsilon$. Therefore, both sides of the equation are equal in this case.

(ii) Suppose $\frac{1}{3} \leq \epsilon < \frac{2}{3}$. We know that $\frac{1}{3} + \epsilon < 1$, and $\frac{2}{3} + \epsilon < \frac{4}{3}$. So the first two terms on the right-hand side must be equal to $x - \epsilon$, and the last term equals $(x - \epsilon) + 1$. That is $(3x - 3\epsilon) + 1$. Now, $\lfloor 3x \rfloor = \lfloor (3x - 3\epsilon) + 3\epsilon \rfloor$. By the inequality, $1 \leq 3\epsilon < 2$. This means that $\lfloor 3x \rfloor = (3x - 3\epsilon) + 1$. Hence, both sides of the equation are equal in this case.

(iii) Suppose $\frac{2}{3} \leq \epsilon < 1$. We know that $1 \leq \frac{1}{3} + \epsilon < \frac{1}{3} + 1$, and
$\frac{4}{3} \leq \frac{2}{3} + \epsilon < \frac{2}{3} + 1$. Clearly the first term in the right-hand side of the equation is equal to $x - \epsilon$ since ϵ is less than 1. The remaining two terms are equal to $(x - \epsilon) + 1$, since $2(\frac{2}{3}) \leq 2\epsilon < 2$. Now, $\lfloor 3x \rfloor = \lfloor (3x - 3\epsilon) + 3\epsilon) \rfloor$, and we know that $2 \leq 3\epsilon < 3$, by the inequality. So $\lfloor 3x \rfloor = (3x - 3\epsilon) + 2$, which is exactly the same as the right-hand side of the equation. ∎