Theorem (3.2.5)**.** *Let f be the function defined by f(x)* $= \frac{x^2+1}{x+1}$. *f(x) is* $\mathcal{O}(x)$.

Proof. Let g be the function defined by $g(x) = x$. $\frac{x^2+1}{x+1}$ is a sum of functions which can be found by the equation

$$\frac{x^2+1}{x+1} = \frac{x^2-1+2}{x+1} = \frac{x^2-1}{x+1} + \frac{2}{x+1} = \frac{(x+1)(x-1)}{x+1} + \frac{2}{x+1} = (x-1) + \frac{2}{x+1}.$$

Now clearly $(x-1) \leq g(x)$ for all $x \in \mathbb{R}$. So the inequality $|(x-1)| \leq |g(x)|$ holds, and it follows that $(x-1)$ is $\mathcal{O}(g(x))$ with constant witnesses $C = 1$, and any $k \in \mathbb{R}$. The other term in the sum of functions, $\frac{2}{x+1}$, is a decreasing function with respect to $g(x)$, for all $x \geq 1$. So if $x > 1$, then $\frac{2}{x+1} \leq x$. This means that $|\frac{2}{x+1}| \leq |g(x)|$, for all $x > 1$, and the function $\frac{2}{x+1}$ is $\mathcal{O}(g(x))$ with constant witnesses $C = 1$, and $k = 1$. By the corollary that follows from the theorem stating that the bounding function for a sum of functions is the maximum of the bounding functions for each of those functions, it follows that $f(x)$ is $\mathcal{O}(x)$ with constant witnesses $C = 1$, and $k = 1$. ■