Prove that $\lfloor x \rfloor + \lfloor y \rfloor + \lfloor x+y \rfloor \leq \lfloor 2x \rfloor + \lfloor 2y \rfloor$, for all real numbers x and y.

Proof. Let f be a function, $f : \mathbb{R} \times \mathbb{R} \to \mathbb{Z}$ such that $f(x,y) = \lfloor x \rfloor + \lfloor y \rfloor + \lfloor x+y \rfloor$, and let g be a function, $g : \mathbb{R} \times \mathbb{R} \to \mathbb{Z}$ such that $g(x,y) = \lfloor 2x \rfloor + \lfloor 2y \rfloor$.

Suppose $\lfloor x \rfloor = \lfloor m + \epsilon \rfloor$ where $m \in \mathbb{Z}$ such that $m \leq x < m+1$, and $\epsilon \in \mathbb{R}$ such that $0 \leq \epsilon < 1$.
Also, suppose $\lfloor y \rfloor = \lfloor n + \sigma \rfloor$ where $n \in \mathbb{Z}$ such that $n \leq y < n+1$, and $\sigma \in \mathbb{R}$ such that $0 \leq \sigma < 1$.

It follows that $\lfloor x \rfloor = m + \lfloor \epsilon \rfloor = m$, because $\lfloor \epsilon \rfloor = 0$. $\lfloor x+y \rfloor = m + n + \lfloor \epsilon + \sigma \rfloor$. And, $\lfloor 2x \rfloor = \lfloor 2(m+\epsilon) \rfloor = \lfloor 2m + 2\epsilon \rfloor = 2m + \lfloor 2\epsilon \rfloor$.

There are two cases to consider in this proof.

(i) If $\epsilon + \sigma \geq 1$, then $\lfloor \epsilon + \sigma \rfloor = 1$. In this case $\lfloor x+y \rfloor = m+n+1$, and it follows that $f(x,y) = 2m + 2n + 1$. Whenever $\epsilon + \sigma \geq 1$, at least one of the following statements is true: $\frac{1}{2} \leq \epsilon < 1$, or $\frac{1}{2} \leq \sigma < 1$. While it is possible that both of these statements are true we need only consider the case where at most one of these statements is true (because the value at $g(x,y)$ in the former exceeds the value of $g(x,y)$ in the latter, and the proof is satisfied by the latter.) So let $\frac{1}{2} \leq \epsilon < 1$. Then $\lfloor 2\epsilon \rfloor = 1$, and $\lfloor 2x \rfloor = 2m+1$. This means that $g(x,y)$ is at least $2m + 2n + 1 = f(x,y)$.

(ii) If $\epsilon + \sigma < 1$, then $\lfloor \epsilon + \sigma \rfloor = 0$. So $\lfloor x+y \rfloor = m+n$, and it follows that $f(x,y) = 2m + 2n$. In this case, not both of, but only one of $\frac{1}{2} \leq \epsilon < 1$, or $\frac{1}{2} \leq \sigma < 1$ can be true, or neither statement can be true. If either one of these statements are true, then $g(x,y) = 2m+2n+1 > 2m+2n = f(x,y)$ If neither of those statements are true, then $g(x,y) = 2m + 2n = f(x,y)$.

$\therefore$ $\lfloor x \rfloor + \lfloor y \rfloor + \lfloor x+y \rfloor \leq \lfloor 2x \rfloor + \lfloor 2y \rfloor$, for all real numbers x and y. ∎