Theorem (2.4.22). *The sum of squares from 1 to n is*

$$\frac{n(n+1)(2n+1)}{6}$$

.

Proof. Let $\{a_n\}$ be the sequence of integers from 1 to n. The formula for the summation of squares from 1 to n can be derived from the cube of n. By theorem 2.4.19,

$$n^3 = \sum_{k=1}^{n} k^3 - (k-1)^3$$

This summation is telescopic, and thus collapses to $n^3 - (1-1)^3 = n^3$. The expansion for $(k-1)^3$ in that term of summation is $k^3 - 3k^2 + 3k - 1$, by the Binomial Theorem. Thus, yielding the algebraic identity

$$k^3 - (k-1)^3 = 3k^2 - 3k + 1$$

Hence, $n^3 = \sum_{k=1}^{n} 3k^2 - 3k + 1$, and by the field axioms,

$$n^3 = \left(3\sum_{k=1}^{n} k^2\right) - \left(3\sum_{k=1}^{n} k\right) + \left(\sum_{k=1}^{n} 1\right)$$

Note that $(\sum_{k=1}^{n} 1) = n(1)$, and by Theorem 2.4.21b, $(3\sum_{k=1}^{n} k) = 3(\frac{n(n+1)}{2})$. Thus,

$$n^3 + 3\frac{n(n+1)}{2} - n = 3\sum_{k=1}^{n} k^2$$

Eliminating the coefficient 3 from the right-hand side by division gives us the sum of squares in terms of an equation,

$$\frac{1}{3}\left(n^3 + 3\frac{n(n+1)}{2} - n\right) = \sum_{k=1}^{n} k^2$$

All that is left to do is to simplify the left-hand side $\frac{1}{3}[n^3 + 3\frac{n(n+1)}{2} - n] = \frac{2n^3+3n^2+3n-2n}{6}$. Factoring $\frac{1}{6}n$ gives $\frac{1}{6}n(2n^2+3n+3-2) = \frac{1}{6}n(2n^2+2n+n+1)$. Factoring $2n$ out of the first two terms in the sum, $\frac{1}{6}n[2n(n+1)+(n+1)]$. The simplification process is complete by factoring $(n+1)$ out of the sum, $\frac{1}{6}n(n+1)(2n+1)$. Thus, the sum of squares

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

■