Theorem (2.4.40)**.** *The union of two countable sets is countable.*

Proof. By cases. Let A, and B be countable sets. There are three cases that must be considered. (i) A and B are finite, (ii) exclusively A or B is finite and the other is countably infinite, (iii) A and B are both countably infinite.

(i) Suppose A and B are finite. There exist natural numbers m, and n such that $|A| = m$ and $|B| = n$. The maximum cardinality for $A \cup B$ occurs when A and B are disjoint, where the cardinality is $m + n$. $m + n$ is a natural number less than $\aleph_0$. Thus, $A \cup B$ is finite and countable by definition.

(ii) Without loss of generality suppose A is finite with cardinality n, and B is countably infinite. It must be that a sequence exists $\{a_i\} = \{a_0, a_1, \ldots a_n\}$ containing all elements in A. Since a bijection exists between B and $\mathbb{N}$ by the definition for countability, a sequence exists $\{b_i\} = \{b_0, b_1, b_2, \ldots\}$ containing all elements in B. Clearly, for the union of A and B a sequence exists $\{c_i\} = \{a_0, a_1, \ldots, a_n, b_0, b_1, b_2, \ldots\}$. Infinite sequences are countable by definition, so $A \cup B$ is a countably infinite set.

(iii) Suppose A and B are infinitely countable sets. Since the set cardinalities are $\aleph_0$, A and B are bijective with $\mathbb{N}$. Thus, the elements in A can be ordered by the sequence $\{a_i\} = \{a_0, a_1, a_2, \ldots\}$, and the elements in B can be ordered by the sequence $\{b_i\} = \{b_0, b_1, b_2, \ldots\}$. The union of A and B can be ordered by the sequence $\{c_i\} = \{a_0, b_0, a_1, b_1, a_2, b_2, \ldots\}$. Thus a bijection exists between the union of A and B and the cardinality of the union is $\aleph_0$. ∎