Theorem (2.3.70e)**.** *Let x, and y be real numbers.* $\lfloor x \rfloor + \lfloor y \rfloor + \lfloor x + y \rfloor \leq \lfloor 2x \rfloor + \lfloor 2y \rfloor$.

Proof. There are four possible cases that could be considered in this proof, but only two of those cases require consideration. Case (i) demonstrating the minimum possible amount occurring on the right-hand side will establish truth for equality expressed by the theorem. Case (ii) demonstrating the maximum possible amount occurring on the right-hand side will establish truth for inequality expressed by the theorem. In all cases the left-hand side remains relatively constant.

First we establish the necessary preliminary facts. Let ϵ and σ be real numbers such that $x - \lfloor x \rfloor = \epsilon$, and $y - \lfloor y \rfloor = \sigma$. Of course, $x - \epsilon = \lfloor x \rfloor$. By the properties for floor functions we know that $x - \epsilon \leq x < (x - \epsilon) + 1$. And, again by the properties for floor functions, $2(x - \epsilon) \leq 2x < 2[(x - \epsilon) + 1] \implies \lfloor 2x \rfloor$. This means that ($i$) $\lfloor 2x \rfloor = 2x - 2\epsilon$, or (logical) ($ii$) $\lfloor 2x \rfloor = 2x - 2\epsilon + 2$. Without loss of generality, all of these equations remain true whenever predicated of y and σ. Also, note that the left-hand side has a constant form, $\lfloor x \rfloor + \lfloor y \rfloor + \lfloor x + y \rfloor = 2(x + y) - 2(\epsilon + \sigma)$.

> (i) Suppose it were the case that $\lfloor 2x \rfloor = 2x - 2\epsilon$, and $\lfloor 2y \rfloor = 2y - 2\sigma$. These are the least amounts possible for the right-hand side of the inequality expressed by the theorem. The sum being $2(x + y) - 2(\epsilon + \sigma)$, equal to the left-hand side. In this case (i) the theorem proves true.
>
> (ii) Suppose it were the case that $\lfloor 2x \rfloor = 2x - 2\epsilon + 2$, and $\lfloor 2y \rfloor = 2y - 2\sigma + 2$. These are the greatest amounts possible for the right-hand side of the inequality expressed by the theorem. The sum being $2(x + y + 2) - 2(\epsilon + \sigma)$. This is clearly greater than the left-hand side. In this case (ii) the theorem proves true.

Since the entire range of all possible values are covered by cases (i) and (ii), and the statement remains true throughout, it is proven that $\lfloor x \rfloor + \lfloor y \rfloor + \lfloor x + y \rfloor \leq \lfloor 2x \rfloor + \lfloor 2y \rfloor$. ∎