Theorem (2.3.67b)**.** *Let A, and B be sets with universal set U. Let $f_{A\cup B}$ be the characteristic function $f_{A\cup B} : U \implies \{0,1\}$. Let f_A be the characteristic function $f_A : U \implies \{0,1\}$. Let f_B be the characteristic function*
$f_B : U \implies \{0,1\}$.
$f_{A\cup B}(x) = f_A(x) + f_B(x) - f_A(x) \times f_B(x)$.

Proof. First suppose that x were not an element in $A \cup B$. It follows from the definition for characteristic functions that $f_{A\cup B}(x) = 0$. Also, by the definition for set union x is in neither A nor B, so $f_A(x) = 0$, and $f_B(x) = 0$. Thus, $f_A(x) + f_B(x) - f_A(x) \times f_B(x) = 0 + 0 - 0 \times 0 = 0$. Therefore, $f_{A\cup B}(x) = f_A(x) + f_B(x) - f_A(x) \times f_B(x)$.

Now suppose it were the case that x was an element in $A \cup B$. It follows from the definition for characteristic functions that $f_{A\cup B}(x) = 1$. Also, by the definition for set union $(x \in A) \vee (x \in B)$. Hence, there are three cases to consider here.

(i) Suppose $(x \in A)$ and $(x \notin B)$. By the definition for characteristic functions we have $f_A(x) = 1$ and $f_B(x) = 0$. Thus,
$f_A(x) + f_B(x) - f_A(x) \times f_B(x) = 1 + 0 - 1 \times 0 = 1$.

(ii) Suppose $(x \notin A)$ and $(x \in B)$. Without loss of generality this case has the same result as case (i).

(iii) If x is in the intersection of A and B we have
$f_A(x) + f_B(x) - f_A(x) \times f_B(x) = 1 + 1 - 1 \times 1 = 1$.

Since $f_{A\cup B}(x) = f_A(x) + f_B(x) - f_A(x) \times f_B(x) = 1$ for all three possible cases, thus concludes the proof. ■