Theorem (2.4.41). *The union of a countable number of countable sets is countable.*

Proof. Let A_i be a countable set, for integers $i = 0$ to $n \leq \infty$ such that

$$S = \bigcup_{i=0}^{n} A_i$$

The function $f : \mathbb{N} \to A_i$ is the sequence $\{a_{ij}\} = a_{i0}, a_{i1}, a_{i2}, \ldots$. Thus, by f, all elements a_{ij} in S can be listed in the second dimension

$$a_{00}, a_{01}, a_{02}, \ldots$$

$$a_{10}, a_{11}, a_{12}, \ldots$$

$$a_{20}, a_{21}, a_{22}, \ldots$$

$$\vdots$$

By tracing the diagonal path along the two dimensional listing for S we get the countable order

$$a_{00}, a_{01}, a_{10}, a_{20}, a_{11}, a_{02}, \ldots$$

$\therefore |S| \leq \aleph_0$, and indeed the union of a countable number of countable sets is countable. ∎