**Theorem (2.4.25).** *Let $m$ be a positive integer. The closed form formula for $\sum_{k=0}^{m}\lfloor\sqrt{k}\rfloor$ is $\lfloor\sqrt{m}\rfloor[\frac{1}{6}(\lfloor\sqrt{m}\rfloor-1)(4\lfloor\sqrt{m}\rfloor+1)+(m-\lfloor\sqrt{m}\rfloor^2+1)]$.*

*Proof.* By the properties for floor functions, there exists an integer $n=\lfloor\sqrt{k}\rfloor$ if and only if $n^2\leq k<n^2+2n+1$. Thus, each integer value $n<\lfloor\sqrt{m}\rfloor$ occurs exactly $2n+1$ times, in the terms of summation. The value $n=\lfloor\sqrt{m}\rfloor$ occurs exactly $(m-\lfloor\sqrt{m}\rfloor^2+1)$ times. Subtracting those terms $n=\lfloor\sqrt{m}\rfloor$ from $\sum_{k=0}^{m}\lfloor\sqrt{k}\rfloor$ produces the sequence

$$\lfloor\sqrt{0}\rfloor(2\lfloor\sqrt{0}\rfloor+1)+\cdots+(\lfloor\sqrt{m}\rfloor-1)[2(\lfloor\sqrt{m}\rfloor-1)+1]$$

Summarily expressed as

$$\sum_{n=0}^{\lfloor\sqrt{m}\rfloor-1} n(2n+1)$$

Thus,

$$\sum_{k=0}^{m}\lfloor\sqrt{k}\rfloor=\left(\sum_{n=0}^{\lfloor\sqrt{m}\rfloor-1} n(2n+1)\right)+\lfloor\sqrt{m}\rfloor(m-\lfloor\sqrt{m}\rfloor^2+1)$$

That is, the summation of squares, and integers

$$\sum_{k=0}^{m}\lfloor\sqrt{k}\rfloor=\left(2\sum_{n=0}^{\lfloor\sqrt{m}\rfloor-1} n^2\right)+\left(\sum_{n=0}^{\lfloor\sqrt{m}\rfloor-1} n\right)+[\lfloor\sqrt{m}\rfloor(m-\lfloor\sqrt{m}\rfloor^2+1)]$$

By Theorem 2.4.22, and by Theorem 2.4.21b

$$\sum_{k=0}^{m}\lfloor\sqrt{k}\rfloor=\left\{\frac{2}{6}\lfloor\sqrt{m}\rfloor(\lfloor\sqrt{m}\rfloor-1)[2(\lfloor\sqrt{m}\rfloor-1)+1]\right\}+$$

$$\left\{\frac{3}{6}\lfloor\sqrt{m}\rfloor(\lfloor\sqrt{m}\rfloor-1)\right\}+\lfloor\sqrt{m}\rfloor(m-\lfloor\sqrt{m}\rfloor^2+1)$$

Factoring $\frac{1}{6}\lfloor\sqrt{m}\rfloor(\lfloor\sqrt{m}\rfloor-1)$ out of the first two terms yields

$$\frac{1}{6}\lfloor\sqrt{m}\rfloor(\lfloor\sqrt{m}\rfloor-1)\{2[2(\lfloor\sqrt{m}\rfloor-1)+1]+3\}+\lfloor\sqrt{m}\rfloor(m-\lfloor\sqrt{m}\rfloor^2+1)$$

And by arithmetic simplification that is

$$\frac{1}{6}\lfloor\sqrt{m}\rfloor(\lfloor\sqrt{m}\rfloor-1)(4\lfloor\sqrt{m}\rfloor+1)+\lfloor\sqrt{m}\rfloor(m-\lfloor\sqrt{m}\rfloor^2+1)$$

Factoring $\lfloor\sqrt{m}\rfloor$ from the outer sum completes the derivation

$$\sum_{k=0}^{m}\lfloor\sqrt{k}\rfloor=\lfloor\sqrt{m}\rfloor\left[\frac{1}{6}(\lfloor\sqrt{m}\rfloor-1)(4\lfloor\sqrt{m}\rfloor+1)+(m-\lfloor\sqrt{m}\rfloor^2+1)\right]$$

. ■