DISCRETE MATHEMATICS

Book of Proofs

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Contents

Chapter 1

Introduction to Proofs

1.1 The sum of two odd integers is even.

Theorem 1. *Let $\boldsymbol{\chi}$ and $\boldsymbol{\zeta}$ be integers. If $\boldsymbol{\chi}$ and $\boldsymbol{\zeta}$ are odd, then $\boldsymbol{\chi} + \boldsymbol{\zeta}$ is even.*

Proof. By the definition for odd numbers, there exists integers $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ such that $\boldsymbol{\chi} = \mathbf{2}\boldsymbol{\mu}+\mathbf{1}$ and $\boldsymbol{\zeta} = \mathbf{2}\boldsymbol{\nu} + \mathbf{1}$. Hence,

$$\chi + \zeta = \Big[\big\langle 2\mu + 1\big\rangle + \big\langle 2\nu + 1\big\rangle\Big] = \Big[2\big\langle \mu + \nu + 1\big\rangle\Big]$$

Integers are closed under addition. Thus, the factor $\big\langle \boldsymbol{\mu} + \boldsymbol{\nu} + \mathbf{1}\big\rangle$ is an integer. It follows that $\boldsymbol{\chi} + \boldsymbol{\zeta}$ is even, by the definition for even numbers. ∎

1.2 The sum of two even integers is even.

Theorem 2. *Let $\boldsymbol{\chi}$ and $\boldsymbol{\zeta}$ be integers. If $\boldsymbol{\chi}$ and $\boldsymbol{\zeta}$ are even, then $\boldsymbol{\chi} + \boldsymbol{\zeta}$ is even.*

Proof. By the definition for even numbers, there exist integers $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ such that $\mathbf{2}\boldsymbol{\mu} = \boldsymbol{\chi}$ and $\mathbf{2}\boldsymbol{\nu} = \boldsymbol{\zeta}$. Hence,

$$\chi + \zeta = \Big[\big\langle 2\mu\big\rangle + \big\langle 2\nu\big\rangle\Big] = \Big[2\big\langle \mu + \nu\big\rangle\Big]$$

Integers are closed under addition. Thus, the factor $\big\langle \boldsymbol{\mu} + \boldsymbol{\nu}\big\rangle$ is an integer. It follows that $\boldsymbol{\chi} + \boldsymbol{\zeta}$ is even, by the definition for even numbers. ∎

1.3 The square of an even number is even.

Theorem 3. *If $\boldsymbol{\chi}$ is an even integer, then $\boldsymbol{\chi^2}$ is an even integer.*

Proof. By the definition for even numbers, there exists an integer $\boldsymbol{\eta}$ such that $\boldsymbol{\chi = 2\eta}$. Hence,

$$\left\langle 2\eta \right\rangle^2 = 4\eta^2 = 2\left\langle 2\eta^2 \right\rangle$$

Integers are closed under multiplication. Thus, the factor $\boldsymbol{\left\langle 2\eta^2 \right\rangle}$ is an integer. It follows that $\boldsymbol{\chi^2}$ is even, by the definition for even numbers. ■

1.4 The additive inverse of an even number.

Theorem 4. *The additive inverse of an even number is an even number.*

Proof. Let $\boldsymbol{\chi}$ be an even number. There exists an integer $\boldsymbol{\eta}$ such that $\boldsymbol{\chi = 2\eta}$, by the definition for even numbers. The additive inverse for $\boldsymbol{\chi}$ is,

$$-1\left\langle \chi \right\rangle = -1\left\langle 2\eta \right\rangle$$

By commutativity of multiplication that is,

$$-1\left\langle 2\eta \right\rangle = 2\left\langle -\eta \right\rangle$$

Since integers are closed under multiplication, the factor $\boldsymbol{\left\langle -\eta \right\rangle}$ is an integer. It follows that the additive inverse of $\boldsymbol{\chi}$ is an even number, by the definition for even numbers. ■

1.5 Special even parity.

Theorem 5. *Let $\boldsymbol{\mu}$, $\boldsymbol{\zeta}$, and $\boldsymbol{\pi}$ be integers. If $\boldsymbol{\mu}+\boldsymbol{\zeta}$ and $\boldsymbol{\zeta}+\boldsymbol{\pi}$ are even, then $\boldsymbol{\mu}+\boldsymbol{\pi}$ is even.*

Proof. By the hypothesis, there exist integers $\boldsymbol{\sigma}$ and $\boldsymbol{\epsilon}$ such that $\boldsymbol{\mu}+\boldsymbol{\zeta} = \mathbf{2}\boldsymbol{\sigma}$, and $\boldsymbol{\zeta}+\boldsymbol{\pi} = \mathbf{2}\boldsymbol{\epsilon}$. Hence,

$$\left\langle \mu + \zeta \right\rangle + \left\langle \zeta + \pi \right\rangle = 2\sigma + 2\epsilon$$

Subtracting $\mathbf{2}\boldsymbol{\zeta}$ from both sides, by the subtraction property of equality for equations, produces

$$\Big\langle \mu + \pi \Big\rangle = \Big\langle 2\sigma + 2\epsilon - 2\zeta \Big\rangle = \Big[2\big\langle \sigma + \epsilon - \zeta \big\rangle \Big]$$

$\boldsymbol{\sigma}$ and $\boldsymbol{\epsilon}$ are integers, by the definition for even numbers, and $\boldsymbol{\zeta}$ is an integer by the hypothesis. Since addition and subtraction are closed on integers, the factor $\left\langle \boldsymbol{\sigma} + \boldsymbol{\epsilon} - \boldsymbol{\zeta} \right\rangle$ is an integer. It follows that $\boldsymbol{\mu} + \boldsymbol{\pi}$ is an even, by the definition for even numbers. ∎

1.6 The product of two odd numbers is odd.

Theorem 6. *The product of two odd numbers is odd.*

Proof. Suppose that $\boldsymbol{\mu}$ and $\boldsymbol{\zeta}$ are odd numbers. By the definition for odd numbers, there exist integers $\boldsymbol{\sigma}$ and $\boldsymbol{\epsilon}$ such that $\boldsymbol{\mu} = \mathbf{2}\boldsymbol{\sigma} + \mathbf{1}$ and $\boldsymbol{\zeta} = \mathbf{2}\boldsymbol{\epsilon} + \mathbf{1}$. Thus, the product of odd numbers $\boldsymbol{\mu}\boldsymbol{\zeta}$ is,

$$\mu\zeta = \Big[\big\langle 2\sigma + 1 \big\rangle \big\langle 2\epsilon + 1 \big\rangle \Big] = \Big[2\sigma 2\epsilon + 2\sigma + 2\epsilon + 1 \Big] = \Big[2\big\langle \sigma\epsilon + \sigma + \epsilon \big\rangle + 1 \Big]$$

The factor $\left\langle \boldsymbol{\sigma}\boldsymbol{\epsilon} + \boldsymbol{\sigma} + \boldsymbol{\epsilon} \right\rangle$ is an integer because $\boldsymbol{\sigma}$ and $\boldsymbol{\epsilon}$ are integers by definition, and integers are closed on addition. Therefore, $\boldsymbol{\mu}\boldsymbol{\zeta}$ is odd by the definition for odd numbers. ∎

1.7 Two plus a perfect square is not perfect.

Theorem 7. *If $\boldsymbol{\eta}$ is a perfect square, then $\boldsymbol{\eta+2}$ is not a perfect square.*

Proof. Let $\boldsymbol{\eta}$ be a perfect square. Assume $\boldsymbol{\eta+2}$ is a perfect square for the purpose of contradiction. By the definition of perfect square, $\sqrt{\boldsymbol{\eta}}$ has to be an integer, and by our assumption there exists an integer $\boldsymbol{\zeta}$ such that $\boldsymbol{\zeta^2=\eta+2}$. So the equivalence $\boldsymbol{\zeta^2-\left\langle\sqrt{\eta}\right\rangle^2=2}$ must be the difference of squares $\boldsymbol{\left\langle\zeta+\sqrt{\eta}\right\rangle\left\langle\zeta-\sqrt{\eta}\right\rangle=2}$. Since integers are closed on addition and subtraction, it follows that the factors of $\boldsymbol{2}$, $\boldsymbol{\left\langle\zeta+\sqrt{\eta}\right\rangle}$ and $\boldsymbol{\left\langle\zeta-\sqrt{\eta}\right\rangle}$, have to be integers. Because $\boldsymbol{2}$ is prime, those integer factors can only be elements in the set $\boldsymbol{\{-2,-1,1,2\}}$. Thus, there are exactly two possibilities:

$\boldsymbol{(i)}$ $\zeta^2-\left\langle\sqrt{\eta}\right\rangle^2=\left\langle 2\right\rangle\left\langle 1\right\rangle$,
or $\boldsymbol{(ii)}$ $\zeta^2-\left\langle\sqrt{\eta}\right\rangle^2=\left\langle -1\right\rangle\left\langle -2\right\rangle$.

In case $\boldsymbol{(i)}$, without loss of generality, we have a system of linear equations in two variables $\boldsymbol{\zeta}$ and $\sqrt{\boldsymbol{\eta}}$:

$$\zeta+\sqrt{\eta}=2$$
$$\zeta-\sqrt{\eta}=1$$

The matrix of coefficients $\boldsymbol{\Psi}=\left[\begin{smallmatrix}\mathbf{1}&\mathbf{1}\\ \mathbf{1}&\mathbf{-1}\end{smallmatrix}\right]$, the inverse for which is $\boldsymbol{\Psi^{-1}}=\left[\begin{smallmatrix}\mathbf{0.5}&\mathbf{0.5}\\ \mathbf{0.5}&\mathbf{-0.5}\end{smallmatrix}\right]$. The product of $\boldsymbol{\Psi^{-1}}$ and the matrix of solutions yields $\boldsymbol{\zeta=1.5}$, which is not in $\mathbb{Z}$; contradicting the assumption that $\boldsymbol{\zeta^2}$ was a perfect square.

In case $\boldsymbol{(ii)}$, we are presented with a similar system of linear equations. The only difference in this system compared to $\boldsymbol{(i)}$ is the matrix of solutions $\boldsymbol{\Phi}=\left[\begin{smallmatrix}\mathbf{-1}\\ \mathbf{-2}\end{smallmatrix}\right]$. $\boldsymbol{\Psi^{-1}\Phi}$ yields $\boldsymbol{\zeta=-1.5}$, which is not in $\mathbb{Z}$, a contradiction. Thus, the assumption that $\boldsymbol{\zeta^2}$ was a perfect square must be false in this case, as well.

Since the assumption proves false in all possible cases, it is not possbile that both $\boldsymbol{\eta+2}$, and $\boldsymbol{\eta}$ are perfect squares. ∎

1.8 The sum of irrational and rational is irrational.

Theorem 8. *The sum of an irrational number and a rational number is irrational.*

Proof. By contradiction. Suppose that $\boldsymbol{\mu}$ and $\boldsymbol{\zeta}$ are rational numbers, and let $\boldsymbol{\chi}$ be an irrational number. For the purpose of contradiction, assume the negation of the hypothesis. That is, the proposition

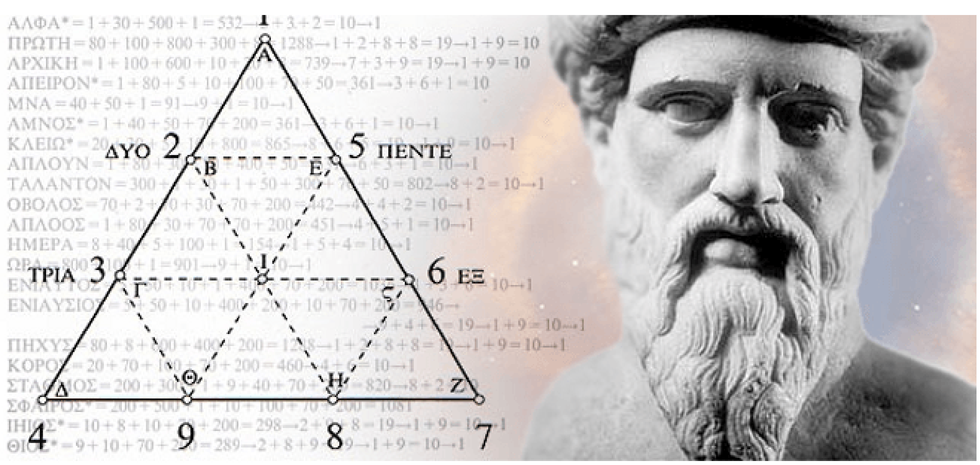
$\neg p$: *the sum of an irrational number and a rational number is rational.*

Hence, $\boldsymbol{\chi} + \boldsymbol{\mu} = \boldsymbol{\zeta}$, by the assumption $\neg \boldsymbol{p}$. Thus, $\boldsymbol{\chi} = \boldsymbol{\zeta} + \big\langle -\boldsymbol{\mu}\big\rangle$, by the additive equality property for equations. But rational numbers are closed under addition by the closure property for rational numbers. So $\neg \boldsymbol{p}$ implies $\boldsymbol{\chi}$ is rational, and $\boldsymbol{\chi}$ is irrational; a contradiction. ∎

1.9 The product of two rational numbers is rational.

Theorem 9. *The product of two rational numbers is rational.*

Proof. Let $\boldsymbol{\mu}$ and $\boldsymbol{\zeta}$ be rational numbers. By the definition for rational numbers, there exist integers $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, and $\boldsymbol{\delta}$ such that $\boldsymbol{\mu} = \frac{\boldsymbol{\alpha}}{\boldsymbol{\beta}}$ and $\boldsymbol{\zeta} = \frac{\boldsymbol{\gamma}}{\boldsymbol{\delta}}$. The product of $\boldsymbol{\mu}$ and $\boldsymbol{\zeta}$ is $\frac{\boldsymbol{\alpha\gamma}}{\boldsymbol{\beta\delta}}$. Since integers are closed under multiplcation, $\boldsymbol{\alpha\gamma}$ and $\boldsymbol{\beta\delta}$ are integers. Thus $\boldsymbol{\mu\zeta}$ is rational by definition. ∎



1.10 The product of irrational and rational is irrational.

Theorem 10. *The product of a nonzero rational number and an irrational number is irrational.*

Proof. For the purpose of contradiction, assume the negation of the hypothesis; the proposition

$\neg p$: *the product of a nonzero rational number and an irrational number is rational*

Let $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, and $\boldsymbol{\delta}$ be integers such that $\boldsymbol{\alpha \neq 0}$, and let $\boldsymbol{\chi}$ be an irrational number. Then the proposition $\boldsymbol{\neg p}$ states

$$\left(\frac{\alpha}{\beta} \cdot \chi\right) = \left(\frac{\gamma}{\delta}\right)$$

By the multiplicative equality property for equations, that is

$$\left(\chi\right) = \left(\frac{\gamma}{\delta} \cdot \frac{\beta}{\alpha}\right) = \left(\frac{\gamma\beta}{\delta\alpha}\right)$$

By Theorem 9 (the closure property for multplication on rational numbers,) $\boldsymbol{\chi}$ is rational. Thus, $\boldsymbol{\neg p}$ implies $\boldsymbol{\chi}$ is rational and irrational. ■

1.11 The multiplicative inverse of an irrational number.

Theorem 11. *If* $\boldsymbol{\chi}$ *is an irrational number, then* $\boldsymbol{\frac{1}{\chi}}$ *is irrational.*

Proof. By the contrapositive. Suppose that $\boldsymbol{\frac{1}{\chi}}$ is a rational number. By the definition for rational numbers, there exist integers $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$ such that $\boldsymbol{\frac{1}{\chi} = \frac{\alpha}{\gamma}}$. Note that $\boldsymbol{\alpha}$ is nonzero (because $\boldsymbol{\frac{1}{\chi}}$ is nonzero.) By the multiplicative property of equality for equations,

$$\left\{\left(\chi \cdot \frac{1}{\chi}\right) = \left(\chi \cdot \frac{\alpha}{\gamma}\right)\right\} \equiv \left\{\left(\frac{\chi}{\chi} \cdot \frac{\gamma}{\alpha}\right) = \left(\frac{\chi\alpha}{\gamma} \cdot \frac{\gamma}{\alpha}\right)\right\} \equiv \left\{\frac{\gamma}{\alpha} = \chi\right\}$$

$\boldsymbol{\frac{\gamma}{\alpha} = \chi}$ is rational, by definition. Thus, if $\boldsymbol{\frac{1}{\chi}}$ is rational, then $\boldsymbol{\chi}$ is rational. ■

1.12 The multiplicative inverse of a rational number.

Theorem 12. *If $\boldsymbol{\chi}$ is a rational number and $\boldsymbol{\chi \neq 0}$, then $\frac{\mathbf{1}}{\boldsymbol{\chi}}$ is rational.*

Proof. Let $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$ be nonzero integers. $\boldsymbol{\chi = \frac{\alpha}{\gamma}}$, by the definition for rational numbers. By the multiplicative property of equality for equations

$$\left\{\left(\frac{1}{\chi}\cdot\chi\right)=\left(\frac{1}{\chi}\cdot\frac{\alpha}{\gamma}\right)\right\}\equiv\left\{\left(\frac{\chi}{\chi}\cdot\frac{\gamma}{\alpha}\right)=\left(\frac{\alpha}{\chi\gamma}\cdot\frac{\gamma}{\alpha}\right)\right\}\equiv\left\{\frac{\gamma}{\alpha}=\frac{1}{\chi}\right\}$$

$\boldsymbol{\frac{\gamma}{\alpha} = \frac{1}{\chi}}$ is rational, by definition. Thus, if $\boldsymbol{\chi}$ is a rational number and $\boldsymbol{\chi \neq 0}$, then $\frac{\mathbf{1}}{\boldsymbol{\chi}}$ is rational. ■

1.13 A special corollary from additive compatibility.

Theorem 13. *Let $\boldsymbol{\chi}$ and $\boldsymbol{\zeta}$ be real numbers. If $\boldsymbol{\chi + \zeta \geq 2}$, then $\langle \boldsymbol{\chi \geq 1} \rangle \vee \langle \boldsymbol{\zeta \geq 1} \rangle$.*

Proof. By the contrapositive. Suppose the negation of the consequent:

$$\langle \chi < 1 \rangle \wedge \langle \zeta < 1 \rangle$$

By additive compatibility,

$$\Big\langle \chi + \zeta \Big\rangle < \Big\langle 1 + 1 \Big\rangle = \Big\langle 2 \Big\rangle$$

This is the logical negation of the direct hypothesis. Thus concludes the proof. ■

1.14 Divisors of an even number.

Theorem 14. *Let $\boldsymbol{\mu}$ and $\boldsymbol{\zeta}$ be integers. If the product $\boldsymbol{\mu\zeta}$ is even, then $\boldsymbol{\mu}$ is even or $\boldsymbol{\zeta}$ is even.*

Proof. For the purpose of contraposition, suppose the negation of the consequent $\boldsymbol{q}$

$$\neg q : \mu \text{ is odd and } \zeta \text{ is odd.}$$

By definition, there exist integers $\boldsymbol{\sigma}$ and $\boldsymbol{\epsilon}$ such that $\boldsymbol{\mu = 2\sigma + 1}$ and $\boldsymbol{\zeta = 2\epsilon + 1}$. Thus,

$$\mu\zeta = \Big[\big\langle 2\sigma + 1\big\rangle\big\langle 2\epsilon + 1\big\rangle\Big] = \Big[2\big\langle \sigma\epsilon + \sigma + \epsilon\big\rangle + 1\Big]$$

The factor $\big\langle \boldsymbol{\sigma\epsilon + \sigma + \epsilon} \big\rangle$ is an integer, because integers are closed under addition and multiplication. Thus, the product $\boldsymbol{\mu\zeta}$ is odd, by definition. ■

1.15 Odd integers of the form $\boldsymbol{\zeta^3 + 5}$.

Theorem 15. *Let $\boldsymbol{\zeta}$ be an integer. If $\boldsymbol{\zeta^3 + 5}$ is odd, then $\boldsymbol{\zeta}$ is even.*

Proof. By the contrapositive. Suppose that $\boldsymbol{\zeta}$ were odd. By the definition for odd numbers, there exists an integer $\boldsymbol{\gamma}$ such that $\boldsymbol{\zeta = 2\gamma + 1}$. By the Binomial Theorem,

$$\left\{\big\langle 2\gamma + 1\big\rangle^3 + 5\right\} = \left\{5 + \sum_{\iota=0}^{3}\binom{3}{\iota} 2\gamma^{\langle 3-\iota\rangle}\right\} = \left\{2\big\langle 4\gamma^3 - 6\gamma^2 + 3\gamma + 3\big\rangle\right\}$$

The factor $\big\langle \boldsymbol{4\gamma^3 - 6\gamma^2 + 3\gamma + 3} \big\rangle$ is an integer because integers are closed on addition and multiplication. Thus, $\boldsymbol{\zeta^3 + 5}$ is even, by definition. ■

1.16 Even numbers of the form $3\gamma + 2$.

Theorem 16. *Let $\boldsymbol{\gamma}$ be an integer. If $\mathbf{3}\boldsymbol{\gamma} + \mathbf{2}$ is even, then $\boldsymbol{\gamma}$ is even.*

Proof. By the contrapositive. Suppose $\boldsymbol{\gamma}$ were odd. By the definition of odd numbers, there exist an integer $\boldsymbol{\mu}$ such that $\boldsymbol{\gamma} = \mathbf{2}\boldsymbol{\mu} + \mathbf{1}$. Thus,

$$\Big\langle 3\big[2\mu + 1\big] + 2 \Big\rangle = \Big\langle 6\mu + 5 \Big\rangle = \Big\langle 6\mu + 4 + 1 \Big\rangle = \Big\langle 2\big[3\mu + 2\big] + 1 \Big\rangle$$

The factor $\big[\mathbf{3}\boldsymbol{\mu} + \mathbf{2}\big]$ is an integer, since integers are closed on addition and multiplcation. Thus, $\mathbf{3}\boldsymbol{\gamma} + \mathbf{2}$ is odd, by definition. ■

1.17 ρ does not exist.

Theorem 17. *There does not exist a rational number $\boldsymbol{\rho}$ such that $\boldsymbol{\rho}^{\mathbf{3}} + \boldsymbol{\rho} + \mathbf{1} = \mathbf{0}$.*

Proof. For the purpose of contradiction, assume that there exists a rational number $\boldsymbol{\rho}$ satisfying the equation $\boldsymbol{\rho}^{\mathbf{3}} + \boldsymbol{\rho} + \mathbf{1} = \mathbf{0}$. By the definition for rational numbers, there exist integers $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ ($\boldsymbol{\beta}$ is nonzero,) such that

$$\big(\rho^3 + \rho + 1\big) = \left(\frac{\alpha^3}{\beta^3} + \frac{\alpha}{\beta} + 1\right) = 0$$

By the additive equality property for equations, that is

$$\frac{\alpha^3}{\beta^3} = \left(-1 - \frac{\alpha}{\beta}\right)$$

It is possible to derive $\boldsymbol{\rho}^{\mathbf{2}}$ from $\boldsymbol{\rho}^{\mathbf{3}}$ by multiplying $\boldsymbol{\rho}^{\mathbf{3}}$ by the multiplicative inverse for $\boldsymbol{\rho}$. By the multiplicative equality property for equations,

$$\frac{\alpha^3}{\beta^3} \cdot \frac{\beta}{\alpha} = \left(-1 - \frac{\alpha}{\beta}\right) \cdot \frac{\beta}{\alpha} = \left\{\frac{-\beta}{\alpha} - \frac{\alpha\beta}{\beta\alpha}\right\}$$

Thus, by the field axioms, $\boldsymbol{\rho}^{\mathbf{2}}$ is

$$\left\{\frac{-\beta}{\alpha} - \frac{\alpha\beta}{\beta\alpha}\right\} = \left(\frac{-\beta - \alpha}{\alpha}\right) = -1 \cdot \left(\frac{\beta + \alpha}{\alpha}\right)$$

Applying the square root to $\boldsymbol{\rho}^{\mathbf{2}}$ gives the identity for $\boldsymbol{\rho}$

$$\sqrt{\frac{\alpha^2}{\beta^2}} = \sqrt{-1 \cdot \left(\frac{\beta + \alpha}{\alpha}\right)} = i \cdot \sqrt{\left(\frac{\beta + \alpha}{\alpha}\right)}$$

$\boldsymbol{\rho}$ is imaginary and rational. Thus, the negation of the hypothesis implies a contradiction. In other words, $\boldsymbol{\rho}$ does not exist. ■

Chapter 2

Set Operations

Theorem (2205). *Let* $\mathbf{\Lambda}$ *be a subset of* $\mathbf{\Omega}$*.* $\overline{\overline{\mathbf{\Lambda}}} = \mathbf{\Lambda}$*.*

Proof. Suppose there exists an element $\boldsymbol{\chi}$ such that $\boldsymbol{\chi}$ is a member of $\overline{\overline{\mathbf{\Lambda}}}$. By the definition for set complementation, and by the defintion for set membership, that is

$$\left\langle \chi \in \overline{\overline{\Lambda}} \right\rangle \equiv \left\langle \chi \notin \overline{\Lambda} \right\rangle \equiv \neg \left\langle \chi \in \overline{\Lambda} \right\rangle \equiv \neg \left\langle \chi \notin \Lambda \right\rangle \equiv \neg \left\langle \neg \langle \chi \in \Lambda \rangle \right\rangle$$

By the logical law of double negation, $\boldsymbol{\chi} \in \mathbf{\Lambda}$. Since logical equivalence is biconditional by definition, this sequence of equivalencies proves both, that

$$\left\langle \overline{\overline{\Lambda}} \subseteq \Lambda \right\rangle \wedge \left\langle \Lambda \subseteq \overline{\overline{\Lambda}} \right\rangle$$

$\therefore$ $\overline{\overline{\mathbf{\Lambda}}} = \mathbf{\Lambda}$; the complementation law for sets. ■

Theorem (2206a). *Let* $\mathbf{\Xi}$ *be a set. The set identity for* $\mathbf{\Xi}$ *is* $\mathbf{\Xi} \cup \varnothing = \mathbf{\Xi}$*.*

Proof. Suppose there exists an element $\boldsymbol{\zeta}$ such that $\boldsymbol{\zeta}$ is a member of $\mathbf{\Xi} \cup \varnothing$. By the definition of set union, that is

$$\left\langle \zeta \in \Xi \right\rangle \vee \left\langle \zeta \in \varnothing \right\rangle$$

The logical identity for the statement $\boldsymbol{\zeta} \in \varnothing$ is trivially $\perp$, because the empty set contains no members. Thus, by that identity, and by the identity law for logical disjunction,

$$\left\{ \left\langle \zeta \in \Xi \right\rangle \vee \left\langle \zeta \in \varnothing \right\rangle \right\} \equiv \left\{ \left\langle \zeta \in \Xi \right\rangle \vee \left\langle \perp \right\rangle \equiv \left\langle \zeta \in \Xi \right\rangle \right\} \equiv$$

$$\left\{ \left\langle \zeta \in \Xi \right\rangle \vee \left\langle \zeta \in \varnothing \right\rangle \equiv \left\langle \zeta \in \Xi \right\rangle \right\}$$

$\therefore$ by the definition for set union, the set identity for $\mathbf{\Xi}$ is $\mathbf{\Xi} \cup \varnothing = \mathbf{\Xi}$. ■

Theorem (2206b). *Let* $\boldsymbol{\Xi}$ *be a set with universal set* $\boldsymbol{\Omega}$*. The set identity for* $\boldsymbol{\Xi}$ *is* $\boldsymbol{\Xi} \cap \boldsymbol{\Omega} = \boldsymbol{\Xi}$*.*

Proof. Suppose there exists an element $\boldsymbol{\zeta}$ such that $\boldsymbol{\zeta}$ is a member of $\boldsymbol{\Xi} \cap \boldsymbol{\Omega}$. By the definition for set intersection, that is

$$\left\langle \zeta \in \Xi \right\rangle \wedge \left\langle \zeta \in \Omega \right\rangle$$

The logical identity for the statement $\boldsymbol{\zeta} \in \boldsymbol{\Omega}$ is trivially $\top$, because $\boldsymbol{\Omega}$ is the universe. Thus, by that identity, and by the identity law for logical conjunction,

$$\left\{\left\langle \zeta \in \Xi \right\rangle \wedge \left\langle \zeta \in \Omega \right\rangle\right\} \equiv \left\{\left\langle \zeta \in \Xi \right\rangle \wedge \left\langle \top \right\rangle \equiv \left\langle \zeta \in \Xi \right\rangle\right\} \equiv$$

$$\left\{\left\langle \zeta \in \Xi \right\rangle \wedge \left\langle \zeta \in \Omega \right\rangle \equiv \left\langle \zeta \in \Xi \right\rangle\right\}$$

$\therefore$ by the definition for the intersection of sets, the set identity for $\boldsymbol{\Xi}$ is $\boldsymbol{\Xi} \cap \boldsymbol{\Omega} = \boldsymbol{\Xi}$. ■

Theorem (2207a). *Let* $\boldsymbol{\Xi}$ *be a set with universal set* $\boldsymbol{\Omega}$*.* $\boldsymbol{\Omega}$ *dominates set union such that* $\boldsymbol{\Xi} \cup \boldsymbol{\Omega} = \boldsymbol{\Omega}$*.*

Proof. Suppose there exists an element $\boldsymbol{\zeta}$ such that $\boldsymbol{\zeta}$ is a member of $\boldsymbol{\Xi} \cup \boldsymbol{\Omega}$. By the definition for set union, that is

$$\left\langle \zeta \in \Xi \right\rangle \vee \left\langle \zeta \in \Omega \right\rangle$$

The logical identity for the statement $\boldsymbol{\zeta} \in \boldsymbol{\Omega}$ is trivially $\top$, since $\boldsymbol{\Omega}$ is the universe. Thus, by that identity, and by the domination law for logical disjunction,

$$\left\{\left\langle \zeta \in \Xi \right\rangle \vee \left\langle \zeta \in \Omega \right\rangle\right\} \equiv \left\{\left\langle \zeta \in \Xi \right\rangle \vee \left\langle \top \right\rangle \equiv \left\langle \zeta \in \Omega \right\rangle\right\} \equiv$$

$$\left\{\left\langle \zeta \in \Xi \right\rangle \vee \left\langle \zeta \in \Omega \right\rangle \equiv \left\langle \zeta \in \Omega \right\rangle\right\}$$

$\therefore$ by the definition for the union of sets, $\boldsymbol{\Omega}$ dominates set union such that $\boldsymbol{\Xi} \cup \boldsymbol{\Omega} = \boldsymbol{\Omega}$. ■

Theorem (**2207b**)**.** *Let* $\mathbf{\Xi}$ *be a set. The empty set dominates set intersection such that* $\mathbf{\Xi} \cap \boldsymbol{\varnothing} = \boldsymbol{\varnothing}$.

Proof. Let $\boldsymbol{\zeta}$ be an element in $\mathbf{\Xi} \cap \boldsymbol{\varnothing}$. By the definition for set intersection, that is

$$\Big\langle \zeta \in \Xi \Big\rangle \wedge \Big\langle \zeta \in \varnothing \Big\rangle$$

The logical identity for the statement $\boldsymbol{\zeta} \in \boldsymbol{\varnothing}$ is trivially $\perp$, since the empty set contains no members. Thus, by that identity, and by the domination law for logical conjunction,

$$\Bigg\{ \Big\langle \zeta \in \Xi \Big\rangle \wedge \Big\langle \zeta \in \varnothing \Big\rangle \Bigg\} \equiv \Bigg\{ \Big\langle \zeta \in \Xi \Big\rangle \wedge \Big\langle \perp \Big\rangle \equiv \Big\langle \zeta \in \varnothing \Big\rangle \Bigg\} \equiv$$

$$\Bigg\{ \Big\langle \zeta \in \Xi \Big\rangle \wedge \Big\langle \zeta \in \varnothing \Big\rangle \equiv \Big\langle \zeta \in \varnothing \Big\rangle \Bigg\}$$

$\therefore$ by the definition for the intersection of sets, the empty set dominates set intersection such that $\mathbf{\Xi} \cap \boldsymbol{\varnothing} = \boldsymbol{\varnothing}$. ∎

Theorem (**2208a**)**.** *Let* $\mathbf{\Lambda}$ *be a set.* $\mathbf{\Lambda}$ *is idempotent such that* $\mathbf{\Lambda} \cup \mathbf{\Lambda} = \mathbf{\Lambda}$.

Proof. Let $\boldsymbol{\mu}$ be an element in $\mathbf{\Lambda} \cup \mathbf{\Lambda}$. By the definition of set union, that is

$$\Big\langle \mu \in \Lambda \Big\rangle \vee \Big\langle \mu \in \Lambda \Big\rangle$$

Thus, by the idempotent law for logical disjunction,

$$\Big\langle \mu \in \Lambda \Big\rangle \vee \Big\langle \mu \in \Lambda \Big\rangle \equiv \Big\langle \mu \in \Lambda \Big\rangle$$

$\therefore$ by the defintion for set union, $\mathbf{\Lambda}$ is idempotent such that $\mathbf{\Lambda} \cup \mathbf{\Lambda} = \mathbf{\Lambda}$. ∎

Theorem (**2208b**)**.** *Let* $\mathbf{\Lambda}$ *be a set.* $\mathbf{\Lambda}$ *is idempotent such that* $\mathbf{\Lambda} \cap \mathbf{\Lambda} = \mathbf{\Lambda}$.

Proof. Let $\boldsymbol{\mu}$ be an element in $\mathbf{\Lambda} \cap \mathbf{\Lambda}$. By the definition for set intersection,

$$\Big\langle \mu \in \Lambda \Big\rangle \wedge \Big\langle \mu \in \Lambda \Big\rangle$$

Thus, by the idempotent law for logical conjunction,

$$\Big\langle \mu \in \Lambda \Big\rangle \wedge \Big\langle \mu \in \Lambda \Big\rangle \equiv \Big\langle \mu \in \Lambda \Big\rangle$$

$\therefore$ by the definition for the intersection of sets, $\mathbf{\Lambda}$ is idempotent such that $\mathbf{\Lambda} \cap \mathbf{\Lambda} = \mathbf{\Lambda}$. ∎

Theorem (2209a). *Let* $\boldsymbol{\Psi}$ *be a set with universal set* $\boldsymbol{\Omega}$*.* $\boldsymbol{\Psi} \cup \overline{\boldsymbol{\Psi}} = \boldsymbol{\Omega}$*.*

Proof. Let $\boldsymbol{\sigma}$ be an element in $\boldsymbol{\Psi} \cup \overline{\boldsymbol{\Psi}}$. By the definition for set union,

$$\Big\langle \sigma \in \Psi \Big\rangle \vee \Big\langle \sigma \in \overline{\Psi} \Big\rangle$$

The right-hand side of this disjunction is equivalent to $\boldsymbol{\sigma} \in \boldsymbol{\Omega} - \boldsymbol{\Psi}$, by the definition for set complementation. By Theorem 2219, $\boldsymbol{\sigma} \in \boldsymbol{\Omega} \cap \overline{\boldsymbol{\Psi}}$, which is defined as $\langle \boldsymbol{\sigma} \in \boldsymbol{\Omega} \rangle \wedge \langle \boldsymbol{\sigma} \notin \boldsymbol{\Psi} \rangle$, by the definitions for set intersection and set complementation. Thus, the original disjunction is the same as

$$\Big\langle \sigma \in \Psi \Big\rangle \vee \left[\Big\langle \sigma \in \Omega \Big\rangle \wedge \Big\langle \sigma \notin \Psi \Big\rangle \right]$$

We must distribute the left-hand side of this disjunction over the conjunction occurring in the right-hand side. We get

$$\left[\Big\langle \sigma \in \Psi \Big\rangle \vee \Big\langle \sigma \in \Omega \Big\rangle \right] \wedge \left[\Big\langle \sigma \in \Psi \Big\rangle \vee \Big\langle \sigma \notin \Psi \Big\rangle \right]$$

By the logical law of negation, the identity for the right-hand side of this conjunction is $\top$. The left-hand side of this conjunction is dominated by $\boldsymbol{\Omega}$, according to Theorem 2207a. Therefore, the statement $\boldsymbol{\sigma} \in \boldsymbol{\Psi} \cup \overline{\boldsymbol{\Psi}}$ can be equivalently stated as $\langle \boldsymbol{\sigma} \in \boldsymbol{\Omega} \rangle \wedge \top$; the logical identity for which is $\boldsymbol{\sigma} \in \boldsymbol{\Omega}$. The converse trivially follows from the fact of logical equivalence. Thus, proves the set complement law for the union of sets, $\boldsymbol{\Psi} \cup \overline{\boldsymbol{\Psi}} = \boldsymbol{\Omega}$. ∎

Theorem (2209b). *Let* $\boldsymbol{\Xi}$ *be a set.* $\boldsymbol{\Xi} \cap \overline{\boldsymbol{\Xi}} = \boldsymbol{\varnothing}$*.*

Proof. Let $\boldsymbol{\zeta}$ be an element in $\boldsymbol{\Xi} \cap \overline{\boldsymbol{\Xi}}$. By the definition for the intersection of sets, that is

$$\Big\langle \zeta \in \Xi \Big\rangle \wedge \Big\langle \zeta \in \overline{\Xi} \Big\rangle$$

According to the definitions for set complementation and set membership, and by the negation law of logic, that is

$$\left\{ \Big\langle \zeta \in \Xi \Big\rangle \wedge \Big\langle \zeta \in \overline{\Xi} \Big\rangle \right\} \equiv \left\{ \Big\langle \zeta \in \Xi \Big\rangle \wedge \neg \Big\langle \zeta \in \Xi \Big\rangle \equiv \Big\langle \bot \Big\rangle \right\}$$

$\langle \bot \rangle$ is trivially the logical identity for the statement $\boldsymbol{\zeta} \in \boldsymbol{\varnothing}$, since the empty set contains no members. Thus, by that identity, and following from the series of equivalencies from above,

$$\Big\langle \zeta \in \Xi \Big\rangle \wedge \Big\langle \zeta \in \overline{\Xi} \Big\rangle \equiv \Big\langle \zeta \in \varnothing \Big\rangle$$

$\therefore$ the complement law for sets, $\boldsymbol{\Xi} \cap \overline{\boldsymbol{\Xi}} = \boldsymbol{\varnothing}$, follows immediately from the definition for the intersection of sets. ∎

Theorem (2210a). *Let* $\mathbf{\Xi}$ *be a set.* $\mathbf{\Xi} - \boldsymbol{\varnothing} = \mathbf{\Xi}$.

Proof. Suppose there exists an element $\boldsymbol{\zeta}$ such that $\boldsymbol{\zeta}$ is a member of $\mathbf{\Xi} - \boldsymbol{\varnothing}$. By the definition for set difference, that is

$$\Big\langle \zeta \in \Xi \Big\rangle \wedge \Big\langle \zeta \notin \varnothing \Big\rangle$$

It is trivial that the logical identity for the statement $\boldsymbol{\zeta} \notin \boldsymbol{\varnothing}$ is $\top$, since the empty set contains no members. Thus, by that identity, and by the identity law for logical conjunction,

$$\left\{ \Big\langle \zeta \in \Xi \Big\rangle \wedge \Big\langle \zeta \notin \varnothing \Big\rangle \right\} \equiv \left\{ \Big\langle \zeta \in \Xi \Big\rangle \wedge \Big\langle \top \Big\rangle \equiv \Big\langle \zeta \in \Xi \Big\rangle \right\} \equiv$$

$$\left\{ \Big\langle \zeta \in \Xi \Big\rangle \wedge \Big\langle \zeta \notin \varnothing \Big\rangle \equiv \Big\langle \zeta \in \Xi \Big\rangle \right\}$$

$\therefore$ by the definition for set difference, $\mathbf{\Xi} - \boldsymbol{\varnothing} = \mathbf{\Xi}$. ■

Theorem (2210b). *Let* $\mathbf{\Xi}$ *be a set.* $\boldsymbol{\varnothing} - \mathbf{\Xi} = \boldsymbol{\varnothing}$.

Proof. Let $\boldsymbol{\zeta}$ be an element in $\boldsymbol{\varnothing} - \mathbf{\Xi}$. By the definition for set difference,

$$\Big\langle \zeta \in \varnothing \Big\rangle \wedge \Big\langle \zeta \notin \Xi \Big\rangle$$

The logical identity for the statement $\boldsymbol{\zeta} \in \boldsymbol{\varnothing}$ is trivially $\perp$, since the empty set contains no members. Thus, by that identity, and by the domination law for logical conjunction,

$$\left\{ \Big\langle \zeta \in \varnothing \Big\rangle \wedge \Big\langle \zeta \notin \Xi \Big\rangle \right\} \equiv \left\{ \Big\langle \perp \Big\rangle \wedge \Big\langle \zeta \notin \Xi \Big\rangle \equiv \Big\langle \perp \Big\rangle \right\} \equiv$$

$$\left\{ \Big\langle \zeta \in \varnothing \Big\rangle \wedge \Big\langle \zeta \notin \Xi \Big\rangle \equiv \Big\langle \zeta \in \varnothing \Big\rangle \right\}$$

$\therefore$ by the definition for set difference, $\boldsymbol{\varnothing} - \mathbf{\Xi} = \boldsymbol{\varnothing}$ ■

Theorem (2211a). *Let* $\mathbf{A}$ *and* $\mathbf{\Lambda}$ *be sets. The union of* $\mathbf{A}$ *and* $\mathbf{\Lambda}$ *is commutative.*

Proof. Let $\boldsymbol{\lambda}$ be an element in $\mathbf{A} \cup \mathbf{\Lambda}$. By the definition for set union,

$$\Big\langle \lambda \in \mathrm{A} \Big\rangle \vee \Big\langle \lambda \in \Lambda \Big\rangle$$

Because logical disjunction is commutative, that is

$$\left[\Big\langle \lambda \in \mathrm{A} \Big\rangle \vee \Big\langle \lambda \in \Lambda \Big\rangle \right] \equiv \left[\Big\langle \lambda \in \Lambda \Big\rangle \vee \Big\langle \lambda \in \mathrm{A} \Big\rangle \right]$$

$\therefore$ $\mathbf{A} \cup \mathbf{\Lambda} = \mathbf{\Lambda} \cup \mathbf{A}$, and the union of $\mathbf{A}$ and $\mathbf{\Lambda}$ is indeed commutative. ■

Theorem (2211b). *Let* $\mathbf{A}$ *and* $\mathbf{\Lambda}$ *be sets. The intersection of* $\mathbf{A}$ *and* $\mathbf{\Lambda}$ *is commutative.*

Proof. Let $\boldsymbol{\lambda}$ be an element in $\mathbf{A} \cap \mathbf{\Lambda}$. By the definition for set intersection,

$$\Big\langle \lambda \in \mathrm{A} \Big\rangle \wedge \Big\langle \lambda \in \Lambda \Big\rangle$$

Because logical conjunction is commutative, that is

$$\Big[\Big\langle \lambda \in \mathrm{A} \Big\rangle \wedge \Big\langle \lambda \in \Lambda \Big\rangle \Big] \equiv \Big[\Big\langle \lambda \in \Lambda \Big\rangle \wedge \Big\langle \lambda \in \mathrm{A} \Big\rangle \Big]$$

$\therefore$ $\mathbf{A} \cap \mathbf{\Lambda} = \mathbf{\Lambda} \cap \mathbf{A}$, and indeed the intersection of $\mathbf{A}$ and $\mathbf{\Lambda}$ is commutative. ∎

Theorem (2212). *Let* $\mathbf{A}$ *and* $\mathbf{\Lambda}$ *be sets.* $\mathbf{A} \cup \big\langle \mathbf{A} \cap \mathbf{\Lambda} \big\rangle = \mathbf{A}$.

Proof. Let $\boldsymbol{\lambda}$ be an element in $\mathbf{A} \cup \langle \mathbf{A} \cap \mathbf{\Lambda} \rangle$. By the definitions for the union of sets and the intersection of sets, that is

$$\Big\langle \lambda \in \mathrm{A} \Big\rangle \vee \Big[\Big\langle \lambda \in \mathrm{A} \Big\rangle \wedge \Big\langle \lambda \in \Lambda \Big\rangle \Big]$$

It follows immediately from the laws of logical absorption that

$$\Big\{ \Big\langle \lambda \in \mathrm{A} \Big\rangle \vee \Big[\Big\langle \lambda \in \mathrm{A} \Big\rangle \wedge \Big\langle \lambda \in \Lambda \Big\rangle \Big] \Big\} \equiv \Big\langle \lambda \in \mathrm{A} \Big\rangle$$

$\therefore$ by the definitions for set union and set intersection, $\mathbf{A} \cup \big\langle \mathbf{A} \cap \mathbf{\Lambda} \big\rangle = \mathbf{A}$; the absorption law for set union over intersection. ∎

Theorem (2213). *Let* $\mathbf{A}$ *and* $\mathbf{\Lambda}$ *be sets.* $\mathbf{A} \cap \big\langle \mathbf{A} \cup \mathbf{\Lambda} \big\rangle = \mathbf{A}$.

Proof. Let $\boldsymbol{\lambda}$ be an element in $\mathbf{A} \cap \langle \mathbf{A} \cup \mathbf{\Lambda} \rangle$. By the definitions for set union and set intersection, that is

$$\Big\langle \lambda \in \mathrm{A} \Big\rangle \wedge \Big[\Big\langle \lambda \in \mathrm{A} \Big\rangle \vee \Big\langle \lambda \in \Lambda \Big\rangle \Big]$$

It follows immediately from the laws of logical absorption that

$$\Big\{ \Big\langle \lambda \in \mathrm{A} \Big\rangle \wedge \Big[\Big\langle \lambda \in \mathrm{A} \Big\rangle \vee \Big\langle \lambda \in \Lambda \Big\rangle \Big] \Big\} \equiv \Big\langle \lambda \in \mathrm{A} \Big\rangle$$

$\therefore$ by the definitions for set intersection and set union, $\mathbf{A} \cap \big\langle \mathbf{A} \cup \mathbf{\Lambda} \big\rangle = \mathbf{A}$; the absorption law for set intersection over union. ∎

Theorem (2215). *Let* $\mathbf{A}$ *and* $\boldsymbol{\Lambda}$ *be sets.* $\overline{\mathbf{A} \cup \boldsymbol{\Lambda}} = \overline{\mathbf{A}} \cap \overline{\boldsymbol{\Lambda}}$.

Proof. Let $\boldsymbol{\lambda}$ be an element in $\overline{\mathbf{A} \cup \boldsymbol{\Lambda}}$. By the definitions for set complementation, and set membership, that is $\neg\Big[\boldsymbol{\lambda} \in \langle \mathbf{A} \cup \boldsymbol{\Lambda} \rangle\Big]$. Hence, by the definition of set union,

$$\neg\Big[\Big\langle \lambda \in \mathrm{A} \Big\rangle \vee \Big\langle \lambda \in \Lambda \Big\rangle\Big]$$

By DeMorgans law (from logic), and by the definitions for set complementation and set membership

$$\Bigg\{\neg\Big[\Big\langle \lambda \in \mathrm{A} \Big\rangle \vee \Big\langle \lambda \in \Lambda \Big\rangle\Big]\Bigg\} \equiv \Bigg\{\neg\Big\langle \lambda \in \mathrm{A} \Big\rangle \wedge \neg\Big\langle \lambda \in \Lambda \Big\rangle\Bigg\} \equiv$$

$$\Bigg\{\Big\langle \lambda \in \overline{\mathrm{A}} \Big\rangle \wedge \Big\langle \lambda \in \overline{\Lambda} \Big\rangle\Bigg\}$$

$\therefore$ by the definition of set intersection, $\overline{\mathbf{A} \cup \boldsymbol{\Lambda}} = \overline{\mathbf{A}} \cap \overline{\boldsymbol{\Lambda}}$; DeMorgans law for sets. ∎

Theorem (2216a). *Let* $\mathbf{A}$ *and* $\boldsymbol{\Lambda}$ *be sets.* $\langle \mathbf{A} \cap \boldsymbol{\Lambda} \rangle \subseteq \mathbf{A}$.

Proof. Let $\boldsymbol{\lambda}$ be an element in $\mathbf{A} \cap \boldsymbol{\Lambda}$. By the definition for set intersection,

$$\Big\langle \lambda \in \mathrm{A} \Big\rangle \wedge \Big\langle \lambda \in \Lambda \Big\rangle$$

It trivially follows from the simplification rule of inference that

$$\Big[\Big\langle \lambda \in \mathrm{A} \Big\rangle \wedge \Big\langle \lambda \in \Lambda \Big\rangle\Big] \rightarrow \Big\langle \lambda \in \mathrm{A} \Big\rangle$$

$\therefore$ $\langle \mathbf{A} \cap \boldsymbol{\Lambda} \rangle \subseteq \mathbf{A}$, by the definition of subsets. ∎

Theorem (2216b). *Let* $\mathbf{A}$ *and* $\boldsymbol{\Lambda}$ *be sets.* $\mathbf{A} \subseteq \langle \mathbf{A} \cup \boldsymbol{\Lambda} \rangle$.

Proof. Suppose there existed and element $\boldsymbol{\lambda}$ such that $\boldsymbol{\lambda}$ were a member of $\mathbf{A}$. It follows from the addition rule of inference that

$$\Big\langle \lambda \in \mathrm{A} \Big\rangle \rightarrow \Big[\Big\langle \lambda \in \mathrm{A} \Big\rangle \vee \Big\langle \lambda \in \Lambda \Big\rangle\Big]$$

$\therefore$ by the definitons for set union and subsets, $\mathbf{A} \subseteq \langle \mathbf{A} \cup \boldsymbol{\Lambda} \rangle$. ∎

Theorem (2216c). *Let* $\mathbf{A}$ *and* $\mathbf{\Lambda}$ *be sets.* $\langle \mathbf{A} - \mathbf{\Lambda} \rangle \subseteq \mathbf{A}$.

Proof. Let $\boldsymbol{\lambda}$ be an element in $\mathbf{A} - \mathbf{\Lambda}$. By the definition for set difference,

$$\Big\langle \lambda \in \mathrm{A} \Big\rangle \wedge \Big\langle \lambda \notin \Lambda \Big\rangle$$

It trivially follows from the simplification rule of inference that

$$\Big[\Big\langle \lambda \in \mathrm{A} \Big\rangle \wedge \Big\langle \lambda \notin \Lambda \Big\rangle \Big] \rightarrow \Big\langle \lambda \in \mathrm{A} \Big\rangle$$

$\therefore \langle \mathbf{A} - \mathbf{\Lambda} \rangle \subseteq \mathbf{A}$, by the definition for subsets. ■

Theorem (2216d). *Let* $\mathbf{A}$ *and* $\mathbf{\Lambda}$ *be sets.* $\mathbf{A} \cap \langle \mathbf{\Lambda} - \mathbf{A} \rangle = \varnothing$.

Proof. Let $\boldsymbol{\lambda}$ be an element in $\mathbf{A} \cap \langle \mathbf{\Lambda} - \mathbf{A} \rangle$. By the definitions for set difference, and set intersection, that is

$$\Big\langle \lambda \in \mathrm{A} \Big\rangle \wedge \Big[\Big\langle \lambda \in \Lambda \Big\rangle \wedge \Big\langle \lambda \notin \mathrm{A} \Big\rangle \Big]$$

Since logical conjunction is associative, the logical identity for this statement is $\bot$, by the negation law for logical conjunction, and by the domination law for logical conjunction. It is trivial that the logical identity for the statement $\boldsymbol{\lambda} \in \varnothing$ is $\bot$, since the empty set contains no members. Thus,

$$\Big\langle \lambda \in \mathrm{A} \Big\rangle \wedge \Big[\Big\langle \lambda \in \Lambda \Big\rangle \wedge \Big\langle \lambda \notin \mathrm{A} \Big\rangle \Big] \equiv \Big\langle \lambda \in \varnothing \Big\rangle$$

$\therefore$ by the definitions for set difference and intersection, $\mathbf{A} \cap \langle \mathbf{\Lambda} - \mathbf{A} \rangle = \varnothing$. ■

Theorem (2216e). *Let* $\mathbf{A}$ *and* $\mathbf{\Lambda}$ *be sets.* $\mathbf{A} \cup \langle \mathbf{\Lambda} - \mathbf{A} \rangle = \mathbf{A} \cup \mathbf{\Lambda}$.

Proof. Let $\boldsymbol{\lambda}$ be an element in $\mathbf{A} \cup \langle \mathbf{\Lambda} - \mathbf{A} \rangle$. By the definitions for set difference, and set union, that is

$$\Big\langle \lambda \in \mathrm{A} \Big\rangle \vee \Big[\Big\langle \lambda \in \Lambda \Big\rangle \wedge \Big\langle \lambda \notin \mathrm{A} \Big\rangle \Big]$$

Distributing the logical disjunction over logical conjunction yields

$$\Big[\Big\langle \lambda \in \mathrm{A} \Big\rangle \vee \Big\langle \lambda \in \Lambda \Big\rangle \Big] \wedge \Big[\Big\langle \lambda \in \mathrm{A} \Big\rangle \vee \Big\langle \lambda \notin \mathrm{A} \Big\rangle \Big]$$

By the negation law for logical disjunction, and by the identity law for logical conjunction, that is

$$\Big\langle \lambda \in \mathrm{A} \Big\rangle \vee \Big\langle \lambda \in \Lambda \Big\rangle$$

Thus,

$$\Big\langle \lambda \in \mathrm{A} \Big\rangle \vee \Big[\Big\langle \lambda \in \Lambda \Big\rangle \wedge \Big\langle \lambda \notin \mathrm{A} \Big\rangle \Big] \equiv \Big\langle \lambda \in \mathrm{A} \Big\rangle \vee \Big\langle \lambda \in \Lambda \Big\rangle$$

$\therefore$ by the definition of set union, $\mathbf{A} \cup \langle \mathbf{\Lambda} - \mathbf{A} \rangle = \mathbf{A} \cup \mathbf{\Lambda}$. ■

Theorem (2217). *Let* $\mathbf{A}$*,* $\mathbf{\Lambda}$*, and* $\mathbf{\Delta}$ *be sets.* $\overline{\mathbf{A} \cap \mathbf{\Lambda} \cap \mathbf{\Delta}} = \overline{\mathbf{A}} \cup \overline{\mathbf{\Lambda}} \cup \overline{\mathbf{\Delta}}$.

Proof. Let $\boldsymbol{\lambda}$ be an element in $\overline{\mathbf{A} \cap \mathbf{\Lambda} \cap \mathbf{\Delta}}$. By the definitions for set complementation and set membership, that is

$$\lambda \notin \mathrm{A} \cap \Lambda \cap \Delta \equiv \neg\big[\lambda \in \mathrm{A} \cap \Lambda \cap \Delta\big]$$

The following statement is equivalent, by the definition for set intersection,

$$\neg\Big[\big\langle\lambda \in \mathrm{A}\big\rangle \wedge \big\langle\lambda \in \Lambda\big\rangle \wedge \big\langle\lambda \in \Delta\big\rangle\Big]$$

By DeMorgans law (from logic), and by the definitions for set membership and set complementation, that is

$$\neg\Big\langle\lambda \in \mathrm{A}\Big\rangle \vee \neg\Big\langle\lambda \in \Lambda\Big\rangle \vee \neg\Big\langle\lambda \in \Delta\Big\rangle \equiv \Big\langle\lambda \in \overline{\mathrm{A}}\Big\rangle \vee \Big\langle\lambda \in \overline{\Lambda}\Big\rangle \vee \Big\langle\lambda \in \overline{\Delta}\Big\rangle$$

$\therefore$ $\overline{\mathbf{A} \cap \mathbf{\Lambda} \cap \mathbf{\Delta}} = \overline{\mathbf{A}} \cup \overline{\mathbf{\Lambda}} \cup \overline{\mathbf{\Delta}}$, by the definition for the union of sets. ■

Theorem (2218a). *Let* $\mathbf{A}$*,* $\mathbf{\Lambda}$*, and* $\mathbf{\Delta}$ *be sets.* $\langle\mathbf{A} \cup \mathbf{\Lambda}\rangle \subseteq \langle\mathbf{A} \cup \mathbf{\Lambda} \cup \mathbf{\Delta}\rangle$.

Proof. Let $\boldsymbol{\lambda}$ be an element in $\mathbf{A} \cup \mathbf{\Lambda}$. By the defintion for the union of sets, that is

$$\Big\langle\lambda \in \mathrm{A}\Big\rangle \vee \Big\langle\lambda \in \Lambda\Big\rangle$$

Let this statement be represented by the propositional variable $\boldsymbol{p}$. By the addition rule of inference, $\boldsymbol{p}$ implies $\boldsymbol{p} \vee \boldsymbol{q}$, for any propositional variable $\boldsymbol{q}$. Let $\boldsymbol{q}$ be the statement $\boldsymbol{\lambda} \in \mathbf{\Delta}$. Thus,

$$\Big\langle\lambda \in \mathrm{A}\Big\rangle \vee \Big\langle\lambda \in \Lambda\Big\rangle \rightarrow \Big\langle\lambda \in \mathrm{A}\Big\rangle \vee \Big\langle\lambda \in \Lambda\Big\rangle \vee \Big\langle\lambda \in \Delta\Big\rangle$$

$\therefore$ $\langle\mathbf{A} \cup \mathbf{\Lambda}\rangle \subseteq \langle\mathbf{A} \cup \mathbf{\Lambda} \cup \mathbf{\Delta}\rangle$, by the definitions for set union and subsets. ■

Theorem (2218b). *Let* $\mathbf{A}$*,* $\mathbf{\Lambda}$ *and* $\mathbf{\Delta}$ *be sets.* $\langle\mathbf{A} \cap \mathbf{\Lambda} \cap \mathbf{\Delta}\rangle \subseteq \langle\mathbf{A} \cap \mathbf{\Lambda}\rangle$.

Proof. Let $\boldsymbol{\lambda}$ be an element in $\mathbf{A} \cap \mathbf{\Lambda} \cap \mathbf{\Delta}$. By the definition for set intersection, that is

$$\Big\langle\lambda \in \mathrm{A}\Big\rangle \wedge \Big\langle\lambda \in \Lambda\Big\rangle \wedge \Big\langle\lambda \in \Delta\Big\rangle$$

By the simplification rule of inference,

$$\Big\langle\lambda \in \mathrm{A}\Big\rangle \wedge \Big\langle\lambda \in \Lambda\Big\rangle \wedge \Big\langle\lambda \in \Delta\Big\rangle \rightarrow \Big\langle\lambda \in \mathrm{A}\Big\rangle \wedge \Big\langle\lambda \in \Lambda\Big\rangle$$

$\therefore$ $\langle\mathbf{A} \cap \mathbf{\Lambda} \cap \mathbf{\Delta}\rangle \subseteq \langle\mathbf{A} \cap \mathbf{\Lambda}\rangle$, by the defintions for set intersection and subsets. ■

Theorem (2218c). *Let* $\mathbf{A}$*,* $\mathbf{\Lambda}$*, and* $\mathbf{\Delta}$ *be sets.* $\left\langle \mathbf{A} - \mathbf{\Lambda} \right\rangle - \mathbf{\Delta} \subseteq \left\langle \mathbf{A} - \mathbf{\Delta} \right\rangle$.

Proof. Let $\boldsymbol{\lambda}$ be an element in $\left\langle \mathbf{A} - \mathbf{\Lambda} \right\rangle - \mathbf{\Delta}$. By the definition for set difference, that is

$$\left[\left\langle \lambda \in \mathrm{A} \right\rangle \wedge \left\langle \lambda \notin \Lambda \right\rangle \right] \wedge \left\langle \lambda \notin \Delta \right\rangle$$

By the law of associativity for logical conjunction, by the law of commutativity for logical conjunction, and by the simplification rule of inference,

$$\left\{ \left[\left\langle \lambda \in \mathrm{A} \right\rangle \wedge \left\langle \lambda \notin \Lambda \right\rangle \right] \wedge \left\langle \lambda \notin \Delta \right\rangle \right\} \equiv \left\{ \left[\left\langle \lambda \in \mathrm{A} \right\rangle \wedge \left\langle \lambda \notin \Delta \right\rangle \right] \wedge \left\langle \lambda \notin \Lambda \right\rangle \right\}$$

$$\to \left\langle \lambda \in \mathrm{A} \right\rangle \wedge \left\langle \lambda \notin \Delta \right\rangle$$

$\therefore$ $\left\langle \mathbf{A} - \mathbf{\Lambda} \right\rangle - \mathbf{\Delta} \subseteq \left\langle \mathbf{A} - \mathbf{\Delta} \right\rangle$, by the definitions for set difference and subsets. ■

Theorem (2218d). *Let* $\mathbf{A}$*,* $\mathbf{\Lambda}$*, and* $\mathbf{\Delta}$ *be sets.* $\left\langle \mathbf{A} - \mathbf{\Delta} \right\rangle \cap \left\langle \mathbf{\Delta} - \mathbf{\Lambda} \right\rangle = \varnothing$.

Proof. Let $\boldsymbol{\lambda}$ be an element in $\left\langle \mathbf{A} - \mathbf{\Delta} \right\rangle \cap \left\langle \mathbf{\Delta} - \mathbf{\Lambda} \right\rangle$. By the definitions for set difference, and set intersection, that is

$$\left[\left\langle \lambda \in \mathrm{A} \right\rangle \wedge \left\langle \lambda \notin \Delta \right\rangle \right] \wedge \left[\left\langle \lambda \in \Delta \right\rangle \wedge \left\langle \lambda \notin \Lambda \right\rangle \right]$$

Since logical conjunction is associative, $\boldsymbol{\lambda}$ is in $\mathbf{\Delta}$, and $\boldsymbol{\lambda}$ is not in $\mathbf{\Delta}$. Thus, by the negation law of logic,

$$\left\langle \lambda \in \mathrm{A} \right\rangle \wedge \left\langle \perp \right\rangle \wedge \left\langle \lambda \notin \Lambda \right\rangle$$

This statement is $\perp$, by the domination law for logical conjunction. And the logical identity for $\boldsymbol{\lambda} \in \varnothing$ is trivially $\perp$, since the empty set contains no members. Hence,

$$\left[\left\langle \lambda \in \mathrm{A} \right\rangle \wedge \left\langle \lambda \notin \Delta \right\rangle \right] \wedge \left[\left\langle \lambda \in \Delta \right\rangle \wedge \left\langle \lambda \notin \Lambda \right\rangle \right] \equiv \left\langle \lambda \in \varnothing \right\rangle$$

$\therefore$ $\left\langle \mathbf{A} - \mathbf{\Delta} \right\rangle \cap \left\langle \mathbf{\Delta} - \mathbf{\Lambda} \right\rangle = \varnothing$, by the definitions for the difference of sets, and for the intersection of sets. ■

Theorem (2218e). *Let* $\mathbf{A}$*,* $\mathbf{\Lambda}$*, and* $\mathbf{\Delta}$ *be sets.* $\langle \mathbf{\Lambda} - \mathbf{A} \rangle \cup \langle \mathbf{\Delta} - \mathbf{A} \rangle = \langle \mathbf{\Lambda} \cup \mathbf{\Delta} \rangle - \mathbf{A}$.

Proof. Let $\boldsymbol{\lambda}$ be an element in $\langle \mathbf{\Lambda} - \mathbf{A} \rangle \cup \langle \mathbf{\Delta} - \mathbf{A} \rangle$. By the definitions for set difference, and the union of sets, that is

$$\Big[\Big\langle \lambda \in \Lambda \Big\rangle \wedge \Big\langle \lambda \notin \mathrm{A} \Big\rangle\Big] \vee \Big[\Big\langle \lambda \in \Delta \Big\rangle \wedge \Big\langle \lambda \notin \mathrm{A} \Big\rangle\Big]$$

Factoring $\boldsymbol{\lambda} \notin \mathbf{A}$ out, by the distributive laws for logical conjunction over disjunction,

$$\Big\{\Big[\Big\langle \lambda \in \Lambda \Big\rangle \wedge \Big\langle \lambda \notin \mathrm{A} \Big\rangle\Big] \vee \Big[\Big\langle \lambda \in \Delta \Big\rangle \wedge \Big\langle \lambda \notin \mathrm{A} \Big\rangle\Big]\Big\} \equiv$$

$$\Big\{\Big[\Big\langle \lambda \in \Lambda \Big\rangle \vee \Big\langle \lambda \in \Delta \Big\rangle\Big] \wedge \Big\langle \lambda \notin \mathrm{A} \Big\rangle\Big\}$$

$\therefore$ $\langle \mathbf{\Lambda} - \mathbf{A} \rangle \cup \langle \mathbf{\Delta} - \mathbf{A} \rangle = \langle \mathbf{\Lambda} \cup \mathbf{\Delta} \rangle - \mathbf{A}$, by the defintions for set difference, and set union. ∎

Theorem (2219). *Let* $\mathbf{A}$*, and* $\mathbf{\Lambda}$ *be sets.* $\mathbf{A} - \mathbf{\Lambda} = \mathbf{A} \cap \overline{\mathbf{\Lambda}}$.

Proof. Let $\boldsymbol{\lambda}$ be an element in $\mathbf{A} - \mathbf{\Lambda}$. By the definition for set difference,

$$\Big\langle \lambda \in \mathrm{A} \Big\rangle \wedge \Big\langle \lambda \notin \Lambda \Big\rangle$$

By the definition for set complementation,

$$\Big[\Big\langle \lambda \in \mathrm{A} \Big\rangle \wedge \Big\langle \lambda \notin \Lambda \Big\rangle\Big] \equiv \Big[\Big\langle \lambda \in \mathrm{A} \Big\rangle \wedge \Big\langle \lambda \in \overline{\Lambda} \Big\rangle\Big]$$

$\therefore$ $\mathbf{A} - \mathbf{\Lambda} = \mathbf{A} \cap \overline{\mathbf{\Lambda}}$, by the definition for the intersection of sets. ∎

Theorem (2237a). *Let* $\mathbf{\Gamma}$ *be a subset of the universal set* $\mathbf{\Omega}$*.*

$$\mathbf{\Gamma} \oplus \mathbf{\Gamma} = \varnothing$$

Proof. By Theorem 2235, $\mathbf{\Gamma} \oplus \mathbf{\Gamma} = \langle \mathbf{\Gamma} \cup \mathbf{\Gamma} \rangle - \langle \mathbf{\Gamma} \cap \mathbf{\Gamma} \rangle$. By the set idempotent laws, that is $\mathbf{\Gamma} - \mathbf{\Gamma}$, and by Theorem 2219, equivalent to $\mathbf{\Gamma} \cap \overline{\mathbf{\Gamma}}$. It follows immediately from the set complement law for the intersection of sets that $\mathbf{\Gamma} \oplus \mathbf{\Gamma} = \varnothing$. ∎

Theorem (2220). *Let* $\mathbf{A}$*, and* $\mathbf{\Lambda}$ *be sets.* $\langle \mathbf{A} \cap \mathbf{\Lambda} \rangle \cup \langle \mathbf{A} \cap \overline{\mathbf{\Lambda}} \rangle = \mathbf{A}$*.*

Proof. Let $\boldsymbol{\lambda}$ be an element in $\langle \mathbf{A} \cap \mathbf{\Lambda} \rangle \cup \langle \mathbf{A} \cap \overline{\mathbf{\Lambda}} \rangle$. By the definitions for the union of sets, and set intersection, that is,

$$\left[\left\langle \lambda \in \mathrm{A} \right\rangle \wedge \left\langle \lambda \in \Lambda \right\rangle\right] \vee \left[\left\langle \lambda \in \mathrm{A} \right\rangle \wedge \left\langle \lambda \in \overline{\Lambda} \right\rangle\right]$$

By the law of distribution for logical conjunction over disjunction, we can factor out the term $\boldsymbol{\lambda} \in \mathbf{A}$. Hence, the following statement is equivalent,

$$\left\langle \lambda \in \mathrm{A} \right\rangle \wedge \left[\left\langle \lambda \in \Lambda \right\rangle \vee \left\langle \lambda \in \overline{\Lambda} \right\rangle\right]$$

$\boldsymbol{\lambda} \in \mathbf{\Lambda} \vee \boldsymbol{\lambda} \in \overline{\mathbf{\Lambda}} \equiv \top$, by the negation laws of logic. Thus, by the identity law for logical conjunction,

$$\left\{\left[\left\langle \lambda \in \mathrm{A} \right\rangle \wedge \left\langle \lambda \in \Lambda \right\rangle\right] \vee \left[\left\langle \lambda \in \mathrm{A} \right\rangle \wedge \left\langle \lambda \in \overline{\Lambda} \right\rangle\right]\right\} \equiv \left\{\left\langle \lambda \in \mathrm{A} \right\rangle \wedge \left\langle \top \right\rangle\right\} \equiv$$

$$\left\langle \lambda \in \mathrm{A} \right\rangle$$

$\therefore$ $\langle \mathbf{A} \cap \mathbf{\Lambda} \rangle \cup \langle \mathbf{A} \cap \overline{\mathbf{\Lambda}} \rangle = \mathbf{A}$. ■

Theorem (2221). *Let* $\mathbf{A}$*,* $\mathbf{\Lambda}$*, and* $\mathbf{\Delta}$ *be sets.* $\mathbf{A} \cup \langle \mathbf{\Lambda} \cup \mathbf{\Delta} \rangle = \langle \mathbf{A} \cup \mathbf{\Lambda} \rangle \cup \mathbf{\Delta}$*, such that set union is associative.*

Proof. Let $\boldsymbol{\lambda}$ be an element in $\mathbf{A} \cup \langle \mathbf{\Lambda} \cup \mathbf{\Delta} \rangle$. By the definition for the union of sets, that is

$$\left\langle \lambda \in \mathrm{A} \right\rangle \vee \left[\left\langle \lambda \in \Lambda \right\rangle \vee \left\langle \lambda \in \Delta \right\rangle\right]$$

It trivially follows from the associative law for logical disjunction that

$$\left\langle \lambda \in \mathrm{A} \right\rangle \vee \left[\left\langle \lambda \in \Lambda \right\rangle \vee \left\langle \lambda \in \Delta \right\rangle\right] \equiv \left[\left\langle \lambda \in \mathrm{A} \right\rangle \vee \left\langle \lambda \in \Lambda \right\rangle\right] \vee \left\langle \lambda \in \Delta \right\rangle$$

$\therefore$ $\mathbf{A} \cup \langle \mathbf{\Lambda} \cup \mathbf{\Delta} \rangle = \langle \mathbf{A} \cup \mathbf{\Lambda} \rangle \cup \mathbf{\Delta}$, such that set union is associative. by the definition for the union of sets. ■

Theorem (2222). *Let* $\mathbf{A}$, $\mathbf{\Lambda}$, *and* $\mathbf{\Delta}$ *be sets.* $\mathbf{A} \cap \langle \mathbf{\Lambda} \cap \mathbf{\Delta} \rangle = \langle \mathbf{A} \cap \mathbf{\Lambda} \rangle \cap \mathbf{\Delta}$, *such that set intersection is associative.*

Proof. Let $\boldsymbol{\lambda}$ be an element in $\mathbf{A} \cap \langle \mathbf{\Lambda} \cap \mathbf{\Delta} \rangle$. By the definition for the intersection of sets, that is

$$\Big\langle \lambda \in \mathrm{A} \Big\rangle \wedge \Big[\Big\langle \lambda \in \Lambda \Big\rangle \wedge \Big\langle \lambda \in \Delta \Big\rangle \Big]$$

It trivially follows from the associative law for logical conjunction that

$$\Big\langle \lambda \in \mathrm{A} \Big\rangle \wedge \Big[\Big\langle \lambda \in \Lambda \Big\rangle \wedge \Big\langle \lambda \in \Delta \Big\rangle \Big] \equiv \Big[\Big\langle \lambda \in \mathrm{A} \Big\rangle \wedge \Big\langle \lambda \in \Lambda \Big\rangle \Big] \wedge \Big\langle \lambda \in \Delta \Big\rangle$$

$\therefore$ $\mathbf{A} \cap \langle \mathbf{\Lambda} \cap \mathbf{\Delta} \rangle = \langle \mathbf{A} \cap \mathbf{\Lambda} \rangle \cap \mathbf{\Delta}$, such that set intersection is associative, by the definition for the intersection of sets. ∎

Theorem (2223). *Let* $\mathbf{A}$, $\mathbf{\Lambda}$, *and* $\mathbf{\Delta}$ *be sets. Set union is distributive over set intersection such that*

$$\mathbf{A} \cup \Big\langle \mathbf{\Lambda} \cap \mathbf{\Delta} \Big\rangle = \Big\langle \mathbf{A} \cup \mathbf{\Lambda} \Big\rangle \cap \Big\langle \mathbf{A} \cup \mathbf{\Delta} \Big\rangle$$

Proof. Let $\boldsymbol{\lambda}$ be an element in $\mathbf{A} \cup \langle \mathbf{\Lambda} \cap \mathbf{\Delta} \rangle$. By the defintions for the union and intersection of sets, that is

$$\langle \lambda \in \mathrm{A} \rangle \vee \Big[\langle \lambda \in \Lambda \rangle \wedge \langle \lambda \in \Delta \rangle \Big]$$

By the law of distribution for logical disjunction over conjunction,

$$\langle \lambda \in \mathrm{A} \rangle \vee \Big[\langle \lambda \in \Lambda \rangle \wedge \langle \lambda \in \Delta \rangle \Big] \equiv \Big[\langle \lambda \in \mathrm{A} \rangle \vee \langle \lambda \in \Lambda \rangle \Big] \wedge \Big[\langle \lambda \in \mathrm{A} \rangle \vee \langle \lambda \in \Delta \rangle \Big]$$

$\therefore$ $\mathbf{A} \cup \langle \mathbf{\Lambda} \cap \mathbf{\Delta} \rangle = \langle \mathbf{A} \cup \mathbf{\Lambda} \rangle \cap \langle \mathbf{A} \cup \mathbf{\Delta} \rangle$, such that set union is distributive over set intersection, by the definitons for set union and set intersection. ∎

Theorem (2231). *Let* $\mathbf{A}$, *and* $\mathbf{\Lambda}$ *be subsets of a universal set* $\mathbf{\Omega}$.

$$\mathbf{A} \subseteq \mathbf{\Lambda} \textit{ if and only if } \overline{\mathbf{\Lambda}} \subseteq \overline{\mathbf{A}}$$

Proof. The proposition $\mathbf{A} \subseteq \mathbf{\Lambda}$ is defined by the universal quantification

$$\forall \lambda \Big\langle \lambda \in \mathrm{A} \rightarrow \lambda \in \Lambda \Big\rangle$$

Where $\boldsymbol{\lambda}$ is an element in the domain of discourse $\mathbf{\Omega}$. It is a tautology that the truth value for the predicate is equivalent to its contrapositive. Thus,

$$\forall \lambda \Big\langle \lambda \notin \Lambda \rightarrow \lambda \notin \mathrm{A} \Big\rangle$$

By the definition for set complementation, and subsets $\boldsymbol{\lambda} \in \overline{\mathbf{\Lambda}} \subseteq \overline{\mathbf{A}} \therefore \mathbf{A} \subseteq \mathbf{\Lambda}$ ***iff*** $\overline{\mathbf{\Lambda}} \subseteq \overline{\mathbf{A}}$. ∎

Theorem (2224). *Let* $\mathbf{A}$, $\mathbf{\Lambda}$, *and* $\mathbf{\Delta}$ *be sets.*

$$\left\langle \mathbf{A} - \mathbf{\Lambda} \right\rangle - \mathbf{\Delta} = \left\langle \mathbf{A} - \mathbf{\Delta} \right\rangle - \left\langle \mathbf{\Lambda} - \mathbf{\Delta} \right\rangle$$

Proof. Let $\boldsymbol{\lambda}$ be an element in $\langle \mathbf{A} - \mathbf{\Lambda} \rangle - \mathbf{\Delta}$. By the definition for set difference,

$$\left[\left\langle \lambda \in \mathrm{A} \right\rangle \wedge \left\langle \lambda \notin \Lambda \right\rangle\right] \wedge \left\langle \lambda \notin \Delta \right\rangle$$

Note that, by the indentity law for logical disjunction, $\boldsymbol{\lambda} \notin \mathbf{\Lambda} \equiv \boldsymbol{\lambda} \notin \mathbf{\Lambda} \vee \bot$. And since $\boldsymbol{\lambda} \in \mathbf{\Delta} \equiv \bot$, by definition, it follows that

$$\left\langle \lambda \notin \Lambda \right\rangle \equiv \left[\left\langle \lambda \notin \Lambda \right\rangle \vee \left\langle \bot \right\rangle\right] \equiv \left[\left\langle \lambda \notin \Lambda \right\rangle \vee \left\langle \lambda \in \Delta \right\rangle\right]$$

Moreover, by the double negation law of logic, and by DeMorgans laws,

$$\left\langle \lambda \notin \Lambda \right\rangle \equiv \neg\left\{\neg\left[\left\langle \lambda \notin \Lambda \right\rangle \vee \left\langle \lambda \in \Delta \right\rangle\right]\right\} \equiv \neg\left[\left\langle \lambda \in \Lambda \right\rangle \wedge \left\langle \lambda \notin \Delta \right\rangle\right]$$

Thus, the proposition $\boldsymbol{\lambda} \in \langle \mathbf{A} - \mathbf{\Lambda} \rangle - \mathbf{\Delta}$, is equivalent to

$$\left\{\left\langle \lambda \in \mathrm{A} \right\rangle \wedge \neg\left[\left\langle \lambda \in \Lambda \right\rangle \wedge \left\langle \lambda \notin \Delta \right\rangle\right]\right\} \wedge \left\langle \lambda \notin \Delta \right\rangle$$

By law of commutativity (and association) for logical conjunction, that is

$$\left[\left\langle \lambda \in \mathrm{A} \right\rangle \wedge \left\langle \lambda \notin \Delta \right\rangle\right] \wedge \neg\left[\left\langle \lambda \in \Lambda \right\rangle \wedge \left\langle \lambda \notin \Delta \right\rangle\right]$$

By the definitions for the difference of sets, set complementation, and the intersection of sets,

$$\left[\left\langle \mathrm{A} - \Lambda \right\rangle - \Delta\right] \equiv \left[\left\langle \mathrm{A} - \Delta \right\rangle \cap \overline{\left\langle \Lambda - \Delta \right\rangle}\right]$$

$\therefore$ $\langle \mathbf{A} - \mathbf{\Lambda} \rangle - \mathbf{\Delta} = \langle \mathbf{A} - \mathbf{\Delta} \rangle - \langle \mathbf{\Lambda} - \mathbf{\Delta} \rangle$, by Thereom 2.2.19

■

Theorem (2237d). *Let* $\mathbf{\Xi}$ *be a subset of a universal set* $\mathbf{\Omega}$.

$$\mathbf{\Xi} \oplus \overline{\mathbf{\Xi}} = \mathbf{\Omega}$$

Proof. By Theorem 2235, $\mathbf{\Xi} \oplus \overline{\mathbf{\Xi}} = \langle \mathbf{\Xi} \cup \overline{\mathbf{\Xi}} \rangle - \langle \mathbf{\Xi} \cap \overline{\mathbf{\Xi}} \rangle$. By the set complement laws that is $\mathbf{\Omega} - \boldsymbol{\varnothing}$. Rather, $\mathbf{\Omega} \cap \overline{\boldsymbol{\varnothing}}$, by Theorem 2219. Since $\overline{\boldsymbol{\varnothing}} = \mathbf{\Omega}$, that is $\mathbf{\Omega} \cap \mathbf{\Omega}$. Which is $\mathbf{\Omega}$, by the idempotent law for set intersection. Thus, $\mathbf{\Xi} \oplus \overline{\mathbf{\Xi}} = \mathbf{\Omega}$. ■

Theorem (2235). *Let* $\mathbf{A}$*, and* $\mathbf{\Lambda}$ *be sets.* $\mathbf{A} \oplus \mathbf{\Lambda} = \langle \mathbf{A} \cup \mathbf{\Lambda} \rangle - \langle \mathbf{A} \cap \mathbf{\Lambda} \rangle$.

Proof. Let $\boldsymbol{\lambda}$ be an element in $\mathbf{A} \oplus \mathbf{\Lambda}$. By the definition for the symmetric difference of sets,

$$\left[\left\langle \lambda \in \mathrm{A} \right\rangle \wedge \left\langle \lambda \notin \Lambda \right\rangle\right] \vee \left[\left\langle \lambda \notin \mathrm{A} \right\rangle \wedge \left\langle \lambda \in \Lambda \right\rangle\right]$$

Distributing the right-hand side over the left-hand side, by the distributive laws of logic, that is

$$\left\{\left\langle \lambda \in \mathrm{A} \right\rangle \vee \left[\left\langle \lambda \notin \mathrm{A} \right\rangle \wedge \left\langle \lambda \in \Lambda \right\rangle\right]\right\} \wedge \left\{\left\langle \lambda \notin \Lambda \right\rangle \vee \left[\left\langle \lambda \notin \mathrm{A} \right\rangle \wedge \left\langle \lambda \in \Lambda \right\rangle\right]\right\}$$

Again, by the distributive law for logical disjunction over conjunction, and by the associative law for logical conjunction, we have

$$\left[\left\langle \lambda \in \mathrm{A} \right\rangle \vee \left\langle \lambda \notin \mathrm{A} \right\rangle\right] \wedge \left[\left\langle \lambda \in \mathrm{A} \right\rangle \vee \left\langle \lambda \in \Lambda \right\rangle\right] \wedge \left[\left\langle \lambda \notin \Lambda \right\rangle \vee \left\langle \lambda \notin \mathrm{A} \right\rangle\right] \wedge$$

$$\left[\left\langle \lambda \notin \Lambda \right\rangle \vee \left\langle \lambda \in \Lambda \right\rangle\right]$$

The following identity is given by the negation laws of logic,

$$\langle \top \rangle \wedge \left[\left\langle \lambda \in \mathrm{A} \right\rangle \vee \left\langle \lambda \in \Lambda \right\rangle\right] \wedge \left[\left\langle \lambda \notin \Lambda \right\rangle \vee \left\langle \lambda \notin \mathrm{A} \right\rangle\right] \wedge \langle \top \rangle$$

By DeMorgans laws, and by the identity law for logical conjunction, that is

$$\left[\left\langle \lambda \in \mathrm{A} \right\rangle \vee \left\langle \lambda \in \Lambda \right\rangle\right] \wedge \neg\left[\left\langle \lambda \in \Lambda \right\rangle \wedge \left\langle \lambda \in \mathrm{A} \right\rangle\right]$$

Which, by the definitions for set union, set intersection, and set membership, is equivalent to

$$\left\{\lambda \in \left\langle \mathrm{A} \cup \Lambda \right\rangle \wedge \neg\left[\lambda \in \left\langle \Lambda \cap \mathrm{A} \right\rangle\right]\right\} \equiv \left\{\lambda \in \left\langle \mathrm{A} \cup \Lambda \right\rangle \wedge \lambda \notin \left\langle \Lambda \cap \mathrm{A} \right\rangle\right\}$$

$\therefore$ by the definition for set difference, $\mathbf{A} \oplus \mathbf{\Lambda} = \langle \mathbf{A} \cup \mathbf{\Lambda} \rangle - \langle \mathbf{A} \cap \mathbf{\Lambda} \rangle$. ∎

Theorem (2238a). *Let* $\mathbf{A}$*, and* $\mathbf{\Lambda}$ *be sets. The symmetric difference of sets is associative such that*

$$\langle \mathbf{A} \oplus \mathbf{\Lambda} \rangle = \langle \mathbf{\Lambda} \oplus \mathbf{A} \rangle$$

Proof. By Theorem 2235, $\mathbf{A} \oplus \mathbf{\Lambda} = \langle \mathbf{A} \cup \mathbf{\Lambda} \rangle - \langle \mathbf{A} \cap \mathbf{\Lambda} \rangle$. Because set union is associative, and because set intersection is associative, trivially $\mathbf{A} \oplus \mathbf{\Lambda} \equiv \langle \mathbf{\Lambda} \cup \mathbf{A} \rangle - \langle \mathbf{\Lambda} \cap \mathbf{A} \rangle$; by Theorem 2235, $\mathbf{\Lambda} \oplus \mathbf{A}$. ∎

Theorem (2236). *Let $\mathbf{\Gamma}$, and $\mathbf{\Xi}$ be sets.*

$$\mathbf{\Gamma} \oplus \mathbf{\Xi} = \Big\langle \mathbf{\Gamma} - \mathbf{\Xi} \Big\rangle \cup \Big\langle \mathbf{\Xi} - \mathbf{\Gamma} \Big\rangle$$

Proof. Suppose there exists an element $\boldsymbol{\zeta}$ such that $\boldsymbol{\zeta}$ is a member of $\mathbf{\Gamma} \oplus \mathbf{\Xi}$. By the definition for symmetric difference,

$$\Big[\Big\langle \zeta \in \Gamma \Big\rangle \wedge \Big\langle \zeta \notin \Xi \Big\rangle\Big] \vee \Big[\Big\langle \zeta \notin \Gamma \Big\rangle \wedge \Big\langle \zeta \in \Xi \Big\rangle\Big]$$

Because logical conjunction is associative, this statement is equivalent to

$$\Big[\Big\langle \zeta \in \Gamma \Big\rangle \wedge \Big\langle \zeta \notin \Xi \Big\rangle\Big] \vee \Big[\Big\langle \zeta \in \Xi \Big\rangle \wedge \Big\langle \zeta \notin \Gamma \Big\rangle\Big]$$

$\therefore$ $\mathbf{\Gamma} \oplus \mathbf{\Xi} = \langle \mathbf{\Gamma} - \mathbf{\Xi} \rangle \cup \langle \mathbf{\Xi} - \mathbf{\Gamma} \rangle$, by the defintions for set union and the difference of sets. ■

Lemma (2201). *Let $\mathbf{A}$, and $\mathbf{\Lambda}$ be sets.*

$$\mathbf{A} \oplus \mathbf{\Lambda} = \Big\langle \mathbf{A} \cup \mathbf{\Lambda} \Big\rangle \cap \Big\langle \overline{\mathbf{A}} \cup \overline{\mathbf{\Lambda}} \Big\rangle$$

Proof. By Theorem 2235, $\langle \mathbf{A} \oplus \mathbf{\Lambda} \rangle \equiv \langle \mathbf{A} \cup \mathbf{\Lambda} \rangle - \langle \mathbf{A} \cap \mathbf{\Lambda} \rangle$, and by Theorem 2219, that is $\langle \mathbf{A} \cup \mathbf{\Lambda} \rangle \cap \langle \overline{\mathbf{A} \cap \mathbf{\Lambda}} \rangle$. By Demorgans law for the complement of set intersection,

$$\Big\langle \mathrm{A} \cup \Lambda \Big\rangle \cap \Big\langle \overline{\mathrm{A} \cap \Lambda} \Big\rangle \equiv \Big\langle \mathrm{A} \cup \Lambda \Big\rangle \cap \Big\langle \overline{\mathrm{A}} \cup \overline{\Lambda} \Big\rangle$$

$\therefore$ $\mathbf{A} \oplus \mathbf{\Lambda} = \langle \mathbf{A} \cup \mathbf{\Lambda} \rangle \cap \langle \overline{\mathbf{A}} \cup \overline{\mathbf{\Lambda}} \rangle$ ■

Lemma (2202). *Let $\mathbf{A}$, and $\mathbf{\Lambda}$ be sets.*

$$\Big[\Big\langle \mathbf{A} \cup \mathbf{\Lambda} \Big\rangle \cap \Big\langle \overline{\mathbf{A}} \cup \overline{\mathbf{\Lambda}} \Big\rangle\Big] \cup \mathbf{\Lambda} = \mathbf{A} \cup \mathbf{\Lambda}$$

Proof. Let $\mathbf{\Omega}$ be the universe. By the law of distribution for set union over intersection, and by the associative law for set union,

$$\Big[\Big\langle \mathrm{A} \cup \Lambda \Big\rangle \cap \Big\langle \overline{\mathrm{A}} \cup \overline{\Lambda} \Big\rangle\Big] \cup \Lambda \equiv \Big\langle \mathrm{A} \cup \Lambda \cup \Lambda \Big\rangle \cap \Big\langle \overline{\mathrm{A}} \cup \overline{\Lambda} \cup \Lambda \Big\rangle$$

By the idempotent law for set union, and by the complement law for set union, that is

$$\Big\langle \mathrm{A} \cup \Lambda \Big\rangle \cap \Big\langle \overline{\mathrm{A}} \cup \Omega \Big\rangle$$

The right-hand side of this intersection is dominated by the universe, according to the domination law for set union. Thus, that is $\mathbf{A} \cup \mathbf{\Lambda}$ intersect $\mathbf{\Omega}$. $\therefore$ $[\langle \mathbf{A} \cup \mathbf{\Lambda} \rangle \cap \langle \overline{\mathbf{A}} \cup \overline{\mathbf{\Lambda}} \rangle] \cup \mathbf{\Lambda} = \mathbf{A} \cup \mathbf{\Lambda}$, by the identity law for the intersection of sets. ■

Lemma (2203). *Let* $\mathbf{A}$*, and* $\mathbf{\Lambda}$ *be sets.*

$$\left\langle \mathbf{A} \cup \mathbf{\Lambda} \right\rangle \cap \left\langle \overline{\mathbf{A}} \cup \overline{\mathbf{\Lambda}} \right\rangle \cap \mathbf{\Lambda} = \mathbf{\Lambda} \cap \overline{\mathbf{A}}$$

Proof. By the law of distribution for set intersection over set union,

$$\left\langle \mathrm{A} \cup \Lambda \right\rangle \cap \left\langle \overline{\mathrm{A}} \cup \overline{\Lambda} \right\rangle \cap \Lambda = \left\langle \mathrm{A} \cup \Lambda \right\rangle \cap \left[\left\langle \overline{\mathrm{A}} \cap \Lambda \right\rangle \cup \left\langle \overline{\Lambda} \cap \Lambda \right\rangle \right]$$

$\overline{\mathbf{\Lambda}} \cap \mathbf{\Lambda} \equiv \varnothing$, by the complement law for set intersection. Therefore, the term in the brackets is $\overline{\mathbf{A}} \cap \mathbf{\Lambda}$, by the identity law for set union. By the associative law for set intersection, what we have left is

$$\left\langle \mathrm{A} \cup \Lambda \right\rangle \cap \overline{\mathrm{A}} \cap \Lambda$$

$\mathbf{A} \cup \mathbf{\Lambda}$ is absorbed by $\mathbf{\Lambda}$, by the absorption laws for sets, since sets are commutative over intersection, by the commutative law for set intersection. $\therefore \left\langle \mathbf{A} \cup \mathbf{\Lambda} \right\rangle \cap \left\langle \overline{\mathbf{A}} \cup \overline{\mathbf{\Lambda}} \right\rangle \cap \mathbf{\Lambda} = \mathbf{\Lambda} \cap \overline{\mathbf{A}}$. ■

Theorem (2238b). *Let* $\mathbf{\Gamma}$*, and* $\mathbf{\Xi}$ *be sets.* $\left\langle \mathbf{\Gamma} \oplus \mathbf{\Xi} \right\rangle \oplus \mathbf{\Xi} = \mathbf{\Gamma}$.

Proof. By Theorem 2235,

$$\left\langle \Gamma \oplus \Xi \right\rangle \oplus \Xi = \left[\left\langle \Gamma \oplus \Xi \right\rangle \cup \Xi \right] - \left[\left\langle \Gamma \oplus \Xi \right\rangle \cap \Xi \right]$$

By Lemma 2201, that is

$$\left\{ \left[\left\langle \Gamma \cup \Xi \right\rangle \cap \left\langle \overline{\Gamma} \cup \overline{\Xi} \right\rangle \right] \cup \Xi \right\} - \left\{ \left[\left\langle \Gamma \cup \Xi \right\rangle \cap \left\langle \overline{\Gamma} \cup \overline{\Xi} \right\rangle \right] \cap \Xi \right\}$$

Since set intersection is associative, by the associative law for the intersection of sets, the identities for the terms in the difference are given immediately by Lemma 2202, and Lemma 2203. Thus,

$$\left\langle \Gamma \oplus \Xi \right\rangle \oplus \Xi = \left\langle \Gamma \cup \Xi \right\rangle - \left\langle \Xi \cap \overline{\Gamma} \right\rangle$$

By Theorem 2219, by DeMorgans law for the complement of intersections, and by the complementation law for sets, that is

$$\left[\left\langle \Gamma \cup \Xi \right\rangle - \left\langle \Xi \cap \overline{\Gamma} \right\rangle \right] \equiv \left[\left\langle \Gamma \cup \Xi \right\rangle \cap \left\langle \overline{\Xi \cap \overline{\Gamma}} \right\rangle \right] \equiv \left[\left\langle \Gamma \cup \Xi \right\rangle \cap \left\langle \overline{\Xi} \cup \Gamma \right\rangle \right]$$

$\mathbf{\Gamma}$ can be factored out, by the distribution law for set union over intersection.

$$\left\langle \Gamma \oplus \Xi \right\rangle \oplus \Xi \equiv \Gamma \cup \left\langle \Xi \cap \overline{\Xi} \right\rangle$$

$\mathbf{\Xi} \cap \overline{\mathbf{\Xi}}$ is empty, by the complement law for set intersection. And $\mathbf{\Gamma}$ union the empty set is $\mathbf{\Gamma}$, by the identity law for set union $\therefore \left\langle \mathbf{\Gamma} \oplus \mathbf{\Xi} \right\rangle \oplus \mathbf{\Xi} = \mathbf{\Gamma}$. ■

Lemma (2204). *Let $\mathbf{\Gamma}$, $\mathbf{\Pi}$, and $\mathbf{\Xi}$ be sets.*

$$\left\langle \mathbf{\Gamma} \cup \mathbf{\Pi} \right\rangle \cap \left\langle \overline{\mathbf{\Gamma}} \cup \overline{\mathbf{\Pi}} \right\rangle \cap \mathbf{\Xi} = \left\langle \mathbf{\Pi} \cap \overline{\mathbf{\Gamma}} \cap \mathbf{\Xi} \right\rangle \cup \left\langle \mathbf{\Gamma} \cap \overline{\mathbf{\Pi}} \cap \mathbf{\Xi} \right\rangle$$

Proof. Distributing the term $\mathbf{\Gamma} \cup \mathbf{\Pi}$ over the union of $\overline{\mathbf{\Gamma}}$ and $\overline{\mathbf{\Pi}}$, by the law of distribution for the intersection of sets over union, in the left-hand side of the equation expressed by the lemma is

$$\left[\left\langle \Gamma \cup \Pi \right\rangle \cap \overline{\Gamma} \right] \cup \left[\left\langle \Gamma \cup \Pi \right\rangle \cap \overline{\Pi} \right] \cap \Xi$$

Again, by the law of distribution for intersection over set union,

$$\equiv \left[\left\langle \Gamma \cap \overline{\Gamma} \right\rangle \cup \left\langle \Pi \cap \overline{\Gamma} \right\rangle \right] \cup \left[\left\langle \Gamma \cap \overline{\Pi} \right\rangle \cup \left\langle \Pi \cap \overline{\Pi} \right\rangle \right] \cap \Xi$$

$\mathbf{\Gamma} \cap \overline{\mathbf{\Gamma}}$ and $\mathbf{\Pi} \cap \overline{\mathbf{\Pi}}$ are both empty, by the domination law for set intersection. And any set, union the empty set, is itself, by the identity law for set union. Thus, by association, what is left is

$$\left[\left\langle \Pi \cap \overline{\Gamma} \right\rangle \cup \left\langle \Gamma \cap \overline{\Pi} \right\rangle \right] \cap \Xi$$

$\therefore \langle \mathbf{\Gamma} \cup \mathbf{\Pi} \rangle \cap \langle \overline{\mathbf{\Gamma}} \cup \overline{\mathbf{\Pi}} \rangle \cap \mathbf{\Xi} = \langle \mathbf{\Pi} \cap \overline{\mathbf{\Gamma}} \cap \mathbf{\Xi} \rangle \cup \langle \mathbf{\Gamma} \cap \overline{\mathbf{\Pi}} \cap \mathbf{\Xi} \rangle$, by the law of distribution for set intersection over set union, and by association for the intersection of sets. ∎

Lemma (2205). *Let $\mathbf{\Gamma}$, $\mathbf{\Pi}$, and $\mathbf{\Xi}$ be sets.*

$$\overline{\left\langle \mathbf{\Gamma} \cup \mathbf{\Pi} \right\rangle \cap \left\langle \overline{\mathbf{\Gamma}} \cup \overline{\mathbf{\Pi}} \right\rangle} \cap \mathbf{\Xi} = \left\langle \overline{\mathbf{\Gamma}} \cap \overline{\mathbf{\Pi}} \cap \mathbf{\Xi} \right\rangle \cup \left\langle \mathbf{\Gamma} \cap \mathbf{\Pi} \cap \mathbf{\Xi} \right\rangle$$

Proof. By DeMorgans Law for sets, the expression occurring in the left-hand side of the equation in the lemma is

$$\left[\left\langle \overline{\Gamma \cup \Pi} \right\rangle \cup \left\langle \overline{\overline{\Gamma} \cup \overline{\Pi}} \right\rangle \right] \cap \Xi$$

Which, again by DeMorgans law for sets, and by the complementation law for sets, is equivalent to

$$\left[\left\langle \overline{\Gamma} \cap \overline{\Pi} \right\rangle \cup \left\langle \Gamma \cap \Pi \right\rangle \right] \cap \Xi$$

By the distributive law for set intersection over set union, and by associative law for the intersection of sets, that is

$$\left\langle \overline{\Gamma} \cap \overline{\Pi} \cap \Xi \right\rangle \cup \left\langle \Gamma \cap \Pi \cap \Xi \right\rangle$$

$\therefore \overline{\langle \mathbf{\Gamma} \cup \mathbf{\Pi} \rangle \cap \langle \overline{\mathbf{\Gamma}} \cup \overline{\mathbf{\Pi}} \rangle} \cap \mathbf{\Xi} = \langle \overline{\mathbf{\Gamma}} \cap \overline{\mathbf{\Pi}} \cap \mathbf{\Xi} \rangle \cup \langle \mathbf{\Gamma} \cap \mathbf{\Pi} \cap \mathbf{\Xi} \rangle$. ∎

Theorem (2237b). *Let $\mathbf{\Gamma}$ be a subset of the universal set $\mathbf{\Omega}$.*

$$\mathbf{\Gamma} \oplus \varnothing = \mathbf{\Gamma}$$

Proof. By Theorem 2235, $\mathbf{\Gamma} \oplus \varnothing = \langle \mathbf{\Gamma} \cup \varnothing \rangle - \langle \mathbf{\Gamma} \cap \varnothing \rangle$. By the identity law for set union, and by the set domination law for intersection, that is $\mathbf{\Gamma} - \varnothing$, which by Theorem 2219 means $\mathbf{\Gamma} \cap \overline{\varnothing}$. Because $\overline{\varnothing} = \mathbf{\Omega}$, $\mathbf{\Gamma} \cap \overline{\varnothing} \equiv \mathbf{\Gamma} \cap \mathbf{\Omega}$. Thus, by the set identity law for intersection, $\mathbf{\Gamma} \oplus \varnothing = \mathbf{\Gamma}$. ■

Theorem (2237c). *Let $\mathbf{\Gamma}$ be a subset of the universal set $\mathbf{\Omega}$.*

$$\mathbf{\Gamma} \oplus \mathbf{\Omega} = \overline{\mathbf{\Gamma}}$$

Proof. By Theorem 2235, $\mathbf{\Gamma} \oplus \mathbf{\Omega} = \langle \mathbf{\Gamma} \cup \mathbf{\Omega} \rangle - \langle \mathbf{\Gamma} \cap \mathbf{\Omega} \rangle$. By the set domination law for set union, and by the set identity law for set intersection, that is $\mathbf{\Omega} - \mathbf{\Gamma}$. Hence, by Theorem 2219, $\mathbf{\Omega} \cap \overline{\mathbf{\Gamma}}$. By the identity law for set intersection, $\mathbf{\Gamma} \oplus \mathbf{\Omega} = \overline{\mathbf{\Gamma}}$. ■

Theorem (2241). *Let $\mathbf{\Gamma}$, $\mathbf{\Pi}$, and $\mathbf{\Xi}$ be sets.*

$$\textit{If } \mathbf{\Gamma} \oplus \mathbf{\Xi} = \mathbf{\Pi} \oplus \mathbf{\Xi}, \textit{ then } \mathbf{\Gamma} = \mathbf{\Pi}$$

Proof. By contraposition. Note that the statement $\mathbf{\Gamma} \oplus \mathbf{\Xi} = \mathbf{\Pi} \oplus \mathbf{\Xi}$ is by definition

$$\left\langle \Gamma \cap \overline{\Xi} \right\rangle \cup \left\langle \overline{\Gamma} \cap \Xi \right\rangle \equiv \left\langle \Pi \cap \overline{\Xi} \right\rangle \cup \left\langle \overline{\Pi} \cap \Xi \right\rangle$$

Assume there exists an element $\boldsymbol{\zeta}$ such that $\boldsymbol{\zeta} \in \mathbf{\Gamma}$ and $\boldsymbol{\zeta} \notin \mathbf{\Pi}$. Thus, $\mathbf{\Gamma} \not\subseteq \mathbf{\Pi}$, the negation of the consequent, by the definiton of subsets. By that hypothesis, $\boldsymbol{\zeta}$ has to be in $\mathbf{\Gamma} \cap \overline{\mathbf{\Xi}}$ and cannot be in $\overline{\mathbf{\Gamma}} \cap \mathbf{\Xi}$. This means that $\boldsymbol{\zeta}$ is not in $\mathbf{\Xi}$. Neither can $\boldsymbol{\zeta}$ be in $\mathbf{\Pi} \cap \overline{\mathbf{\Xi}}$. And since $\boldsymbol{\zeta} \notin \mathbf{\Xi}$, $\boldsymbol{\zeta}$ cannot be in $\overline{\mathbf{\Pi}} \cap \mathbf{\Xi}$. So $\boldsymbol{\zeta}$ is in $\mathbf{\Gamma} \oplus \mathbf{\Xi}$ but not $\mathbf{\Pi} \oplus \mathbf{\Xi}$. Therefore, $\mathbf{\Gamma} \oplus \mathbf{\Xi} \not\subseteq \mathbf{\Pi} \oplus \mathbf{\Xi}$, by the definition of subsets. The implication,

$$\left\langle \Pi \not\subseteq \Gamma \right\rangle \to \left[\left\langle \Pi \oplus \Xi \right\rangle \not\subseteq \left\langle \Gamma \oplus \Xi \right\rangle \right]$$

follows without loss of generality $\therefore$ $\mathbf{\Gamma} \neq \mathbf{\Pi}$ implies $\mathbf{\Gamma} \oplus \mathbf{\Xi} \neq \mathbf{\Pi} \oplus \mathbf{\Xi}$. ■

Theorem (2240). *Let* $\boldsymbol{\Gamma}$, $\boldsymbol{\Pi}$, *and* $\boldsymbol{\Xi}$ *be sets. The symmetric difference for sets is associative such that*

$$\left\langle \boldsymbol{\Gamma} \oplus \boldsymbol{\Pi} \right\rangle \oplus \boldsymbol{\Xi} = \boldsymbol{\Gamma} \oplus \left\langle \boldsymbol{\Pi} \oplus \boldsymbol{\Xi} \right\rangle$$

Proof. Let $\boldsymbol{\zeta}$ be an element in $\langle \boldsymbol{\Gamma} \oplus \boldsymbol{\Pi} \rangle \oplus \boldsymbol{\Xi}$. By the definition for the symmetric difference of sets, $\boldsymbol{\zeta}$ is in

$$\left[\left\langle \Gamma \oplus \Pi \right\rangle \cap \overline{\Xi} \right] \cup \left[\left\langle \overline{\Gamma \oplus \Pi} \right\rangle \cap \Xi \right]$$

By Lemma 2201, $\boldsymbol{\zeta}$ is an element of

$$\left\{ \left[\left\langle \Gamma \cup \Pi \right\rangle \cap \left\langle \overline{\Gamma} \cup \overline{\Pi} \right\rangle \right] \cap \overline{\Xi} \right\} \cup \left\{ \left[\overline{\left\langle \Gamma \cup \Pi \right\rangle \cap \left\langle \overline{\Gamma} \cup \overline{\Pi} \right\rangle} \right] \cap \Xi \right\}$$

Each superset on either side of this union is described either by Lemma 2204, or by Lemma 2205. Thus, by Lemma 2204, and 2205, $\boldsymbol{\zeta}$ is in

$$\left\langle \Pi \cap \overline{\Gamma} \cap \overline{\Xi} \right\rangle \cup \left\langle \Gamma \cap \overline{\Pi} \cap \overline{\Xi} \right\rangle \cup \left\langle \overline{\Gamma} \cap \overline{\Pi} \cap \Xi \right\rangle \cup \left\langle \Gamma \cap \Pi \cap \Xi \right\rangle \equiv \Delta$$

Now, suppose it were the case that $\boldsymbol{\zeta}$ were an element in $\boldsymbol{\Gamma} \oplus \langle \boldsymbol{\Pi} \oplus \boldsymbol{\Xi} \rangle$. Because set intersection and set union are commutative, from the definition for the symmetric difference of sets, $\boldsymbol{\zeta}$ would have to be in

$$\left[\left\langle \Pi \oplus \Xi \right\rangle \cap \overline{\Gamma} \right] \cup \left[\left\langle \overline{\Pi \oplus \Xi} \right\rangle \cap \Gamma \right]$$

By Lemma 2201, $\boldsymbol{\zeta}$ is an element of

$$\left\{ \left[\left\langle \Pi \cup \Xi \right\rangle \cap \left\langle \overline{\Pi} \cup \overline{\Xi} \right\rangle \right] \cap \overline{\Gamma} \right\} \cup \left\{ \left[\overline{\left\langle \Pi \cup \Xi \right\rangle \cap \left\langle \overline{\Pi} \cup \overline{\Xi} \right\rangle} \right] \cap \Gamma \right\}$$

And by Lemma 2204, and Lemma 2205, $\boldsymbol{\zeta}$ is an element in

$$\left\langle \Xi \cap \overline{\Pi} \cap \overline{\Gamma} \right\rangle \cup \left\langle \Pi \cap \overline{\Xi} \cap \overline{\Gamma} \right\rangle \cup \left\langle \overline{\Pi} \cap \overline{\Xi} \cap \Gamma \right\rangle \cup \left\langle \Pi \cap \Xi \cap \Gamma \right\rangle \equiv \Delta$$

Because $\boldsymbol{\zeta}$ is in $\boldsymbol{\Delta}$ whenever $\boldsymbol{\zeta}$ is in $\langle \boldsymbol{\Gamma} \oplus \boldsymbol{\Pi} \rangle \oplus \boldsymbol{\Xi}$, and because $\boldsymbol{\zeta}$ is in $\boldsymbol{\Delta}$ whenever $\boldsymbol{\zeta}$ is in $\boldsymbol{\Gamma} \oplus \langle \boldsymbol{\Pi} \oplus \boldsymbol{\Xi} \rangle$, it follows immediately that the symmetric difference for sets is associative such that $\langle \boldsymbol{\Gamma} \oplus \boldsymbol{\Pi} \rangle \oplus \boldsymbol{\Xi} = \boldsymbol{\Gamma} \oplus \langle \boldsymbol{\Pi} \oplus \boldsymbol{\Xi} \rangle$. ■

Chapter 3

Functions

Theorem (2320). *Let $\boldsymbol{\gamma}$ be the function $\boldsymbol{\gamma} : \mathbb{R} \to \mathbb{R}$, such that*

$$\forall\alpha\left\langle \gamma[\alpha] > 0 \right\rangle$$

Let $\boldsymbol{\delta}$ be the function $\boldsymbol{\delta} : \mathbb{R} \to \mathbb{R}$ defined by $\boldsymbol{\delta}[\boldsymbol{\alpha}] = \frac{\mathbf{1}}{\boldsymbol{\gamma}[\boldsymbol{\alpha}]}$. $\boldsymbol{\gamma}[\boldsymbol{\alpha}]$ is strictly increasing if and only if $\boldsymbol{\delta}[\boldsymbol{\alpha}]$ is strictly decreasing.

Proof. Suppose there exist real numbers $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ such that $\boldsymbol{\alpha} < \boldsymbol{\beta}$, and suppose that $\boldsymbol{\gamma}[\boldsymbol{\alpha}] < \boldsymbol{\gamma}[\boldsymbol{\beta}]$. $\boldsymbol{\gamma}$ is a strictly increasing real-valued function by definition. By the multiplicative compatibility law from the order axioms, multiplying both sides of the latter inequality by $\frac{\mathbf{1}}{\boldsymbol{\gamma}[\boldsymbol{\alpha}]\cdot\boldsymbol{\gamma}[\boldsymbol{\beta}]}$

$$\left\{ \left(\frac{\gamma[\alpha]}{\gamma[\alpha] \cdot \gamma[\beta]} \right) < \left(\frac{\gamma[\beta]}{\gamma[\alpha] \cdot \gamma[\beta]} \right) \right\} \equiv \left\{ \left(\frac{1}{\gamma[\beta]} \right) < \left(\frac{1}{\gamma[\alpha]} \right) \right\} \equiv$$
$$\left\{ \left(\delta[\beta] \right) < \left(\delta[\alpha] \right) \right\}$$

Thus, $\boldsymbol{\delta}$ is a strictly decreasing real-valued function, by definition. The converse can be proven by multiplying that inequality $\boldsymbol{\delta}[\boldsymbol{\alpha}] > \boldsymbol{\delta}[\boldsymbol{\beta}]$ by $\boldsymbol{\gamma}[\boldsymbol{\alpha}]\boldsymbol{\gamma}[\boldsymbol{\beta}]$, by the multiplicative compatibility law from the order axioms,

$$\left\{ \left(\frac{1 \cdot \gamma[\alpha] \cdot \gamma[\beta]}{\gamma[\alpha]} \right) > \left(\frac{1 \cdot \gamma[\alpha] \cdot \gamma[\beta]}{\gamma[\beta]} \right) \right\} \equiv \left\{ \left(\gamma[\beta] \right) > \left(\gamma[\alpha] \right) \right\}$$

Thus, $\boldsymbol{\gamma}$ is a strictly increasing real-valued function, by definition $\therefore$ $\boldsymbol{\gamma}[\boldsymbol{\alpha}]$ is strictly increasing if and only if $\boldsymbol{\delta}[\boldsymbol{\alpha}]$ is strictly decreasing. ∎

Theorem (2321). *Let $\boldsymbol{\gamma}$ be the function $\boldsymbol{\gamma} : \mathbb{R} \to \mathbb{R}$, such that*

$$\forall\alpha\Big\langle\gamma[\alpha] > 0\Big\rangle$$

Let $\boldsymbol{\delta}$ be the function $\boldsymbol{\delta} : \mathbb{R} \to \mathbb{R}$ defined by $\boldsymbol{\delta}[\boldsymbol{\alpha}] = \frac{\mathbf{1}}{\boldsymbol{\gamma}[\boldsymbol{\alpha}]}$. $\boldsymbol{\gamma}[\boldsymbol{\alpha}]$ is strictly decreasing if and only if $\boldsymbol{\delta}[\boldsymbol{\alpha}]$ is strictly increasing.

Proof. Suppose there exist real numbers $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ such that $\boldsymbol{\alpha} < \boldsymbol{\beta}$, and suppose that $\boldsymbol{\gamma}[\boldsymbol{\alpha}] > \boldsymbol{\gamma}[\boldsymbol{\beta}]$. $\boldsymbol{\gamma}$ is a strictly decreasing real-valued function by definition. By the multiplicative compatibility law from the order axioms, Multiplying both sides of the latter inequality by $\frac{\mathbf{1}}{\boldsymbol{\gamma}[\boldsymbol{\alpha}]\cdot\boldsymbol{\gamma}[\boldsymbol{\beta}]}$

$$\left\{\left(\frac{\gamma[\alpha]}{\gamma[\alpha]\cdot\gamma[\beta]}\right) > \left(\frac{\gamma[\beta]}{\gamma[\alpha]\cdot\gamma[\beta]}\right)\right\} \equiv \left\{\left(\frac{1}{\gamma[\beta]}\right) > \left(\frac{1}{\gamma[\alpha]}\right)\right\} \equiv \left\{\Big(\delta[\beta]\Big) > \Big(\delta[\alpha]\Big)\right\}$$

Thus, $\boldsymbol{\delta}$ is a strictly increasing real-valued function, by defintion. The converse can be proven by multiplying that inequality $\boldsymbol{\delta}[\boldsymbol{\alpha}] < \boldsymbol{\delta}[\boldsymbol{\beta}]$ by $\boldsymbol{\gamma}[\boldsymbol{\alpha}]\boldsymbol{\gamma}[\boldsymbol{\beta}]$, by the multiplicative compatibility law from the order axioms,

$$\left\{\left(\frac{1\cdot\gamma[\alpha]\cdot\gamma[\beta]}{\gamma[\alpha]} < \frac{1\cdot\gamma[\alpha]\cdot\gamma[\beta]}{\gamma[\beta]}\right)\right\} \equiv \left\{\Big(\gamma[\beta]\Big) < \Big(\gamma[\alpha]\Big)\right\}$$

Thus, $\boldsymbol{\gamma}$ is a strictly decreasing real-valued function, by defintion $\therefore$ $\boldsymbol{\gamma}[\boldsymbol{\alpha}]$ is strictly decreasing if and only if $\boldsymbol{\delta}[\boldsymbol{\alpha}]$ is strictly increasing. ∎

Theorem (2324). *Let $\boldsymbol{\alpha}$ be the function $\boldsymbol{\alpha} : \mathbb{R} \to \mathbb{R}$ defined by $\boldsymbol{\alpha}[\boldsymbol{\lambda}] = \boldsymbol{\epsilon}^{\boldsymbol{\lambda}}$. $\boldsymbol{\alpha}[\boldsymbol{\lambda}]$ is not invertible.*

Proof.

$$\Big\langle\alpha[\lambda] = \epsilon^{\lambda}\Big\rangle \to \left[\Big\langle\alpha^{-1}[\lambda]\Big\rangle = \Big\langle\log_{\epsilon}\lambda\Big\rangle\right]$$

The domain for $\boldsymbol{\alpha}^{-\mathbf{1}}[\boldsymbol{\lambda}]$ is $\mathbb{R}$, by definition. But logarithmic functions are undefined for negative-valued domains. Thus, $\boldsymbol{\alpha}[\boldsymbol{\lambda}]$ cannot be bijective, so $\boldsymbol{\alpha}[\boldsymbol{\lambda}]$ is not invertible. ∎

Theorem (2325). *Let $\boldsymbol{\alpha}$ be a function $\boldsymbol{\alpha} : \mathbb{R} \to \mathbb{R}$ defined by $\boldsymbol{\alpha}[\boldsymbol{\lambda}] = |\boldsymbol{\lambda}|$. $\boldsymbol{\alpha}[\boldsymbol{\lambda}]$ is not invertible.*

Proof. Let $\boldsymbol{\lambda}$ be a postive real number. By the definition for $\boldsymbol{\alpha}$,

$$\Big\langle\alpha[\lambda] = \lambda\Big\rangle \wedge \Big\langle\alpha[-1\cdot\lambda] = \lambda\Big\rangle$$

So if $\boldsymbol{\alpha}$ had an inverse, then

$$\Big\langle\alpha^{-1}[\lambda] = \lambda\Big\rangle \vee \Big\langle\alpha^{-1}[\lambda] = -1\cdot\lambda\Big\rangle$$

Thus, $\boldsymbol{\alpha}^{-\mathbf{1}}$ is not a function by definition. ∎

Theorem (2329a). *Let $\boldsymbol{\lambda}$ be a function $\boldsymbol{\lambda : \Delta \to A}$, and let $\boldsymbol{\epsilon}$ be a function $\boldsymbol{\epsilon : \Lambda \to \Delta}$. If both $\boldsymbol{\lambda}$ and $\boldsymbol{\epsilon}$ are injective, then $\langle \boldsymbol{\lambda \circ \epsilon} \rangle$ is injective.*

Proof. By the contrapositive. Suppose it were not the case that $\langle \boldsymbol{\lambda \circ \epsilon} \rangle$ were injective. By the definition for injective functions, with domain of discourse $\langle \boldsymbol{\iota, \zeta \in \Lambda} \rangle$, that is

$$\neg \forall \iota \forall \zeta \Big\langle \Big[\langle \lambda \circ \epsilon \rangle [\iota] = \langle \lambda \circ \epsilon \rangle [\zeta] \Big] \to \langle \iota = \zeta \rangle \Big\rangle$$

The composition of functions $\langle \boldsymbol{\lambda \circ \epsilon} \rangle [\boldsymbol{\iota}]$ is defined as $\boldsymbol{\lambda} \big[\boldsymbol{\epsilon(\iota)} \big]$. Thus, we have the equivalent universal quantification

$$\neg \forall \iota \forall \zeta \Big\langle \Big[\lambda \big[\epsilon(\iota) \big] = \lambda \big[\epsilon(\zeta) \big] \Big] \to \langle \iota = \zeta \rangle \Big\rangle$$

In other words, it follows from the negation of the direct consequent that it is not the case that $\boldsymbol{\lambda}$ is injective, by the definition for injective functions. This is sufficient to prove the logical negation of the direct hypothesis. Thus, if both $\boldsymbol{\lambda}$ and $\boldsymbol{\epsilon}$ are injective, then $\langle \boldsymbol{\lambda \circ \epsilon} \rangle$ is injective. ∎

Theorem (2329b). *Let $\boldsymbol{\delta}$ be a function $\boldsymbol{\delta : \Delta \to A}$, and let $\boldsymbol{\gamma}$ be a function $\boldsymbol{\gamma : \Lambda \to \Delta}$. If both $\boldsymbol{\delta}$ and $\boldsymbol{\gamma}$ are surjective, then $\langle \boldsymbol{\delta \circ \gamma} \rangle$ is surjective.*

Proof. By the contrapositive. Suppose it were not the case that $\langle \boldsymbol{\delta \circ \gamma} \rangle$ were surjective. By the definition for surjective functions, with domain of discourse $\boldsymbol{\lambda \in \Lambda}$ and $\boldsymbol{\alpha \in A}$, that is

$$\neg \forall \alpha \exists \lambda \Big\langle \langle \delta \circ \gamma \rangle [\lambda] = \alpha \Big\rangle$$

The composition of functions $\langle \boldsymbol{\delta \circ \gamma} \rangle [\boldsymbol{\lambda}]$ is defined as $\boldsymbol{\delta} \big[\boldsymbol{\gamma(\lambda)} \big]$. Thus, we have the equivalent universal quantification

$$\neg \forall \alpha \exists \lambda \Big\langle \delta \big[\gamma(\lambda) \big] = \alpha \Big\rangle$$

In other words, it follows from the negation of the direct consequent that it is not the case that $\boldsymbol{\delta}$ is surjective, by the definition for surjective functions. This is sufficient to prove the logical negation of the direct hypothesis. Thus, if both $\boldsymbol{\delta}$ and $\boldsymbol{\gamma}$ are surjective, then $\langle \boldsymbol{\delta \circ \gamma} \rangle$ is surjective. ∎

Theorem (2330). *Let $\boldsymbol{\delta}$ and $\langle\boldsymbol{\delta} \circ \boldsymbol{\gamma}\rangle$ be injective functions. $\boldsymbol{\gamma}$ is injective.*

Proof. By the contrapositive. Suppose that $\boldsymbol{\gamma}$ were not injective. Then by the definition for injective functions we have the following universally quantified statement, with the domain of discourse being the domain of $\boldsymbol{\gamma}$,

$$\neg\forall\mu\forall\nu\Big\langle\big\langle\gamma[\mu] = \gamma[\nu]\big\rangle \to \big\langle\mu = \nu\big\rangle\Big\rangle$$

By the equality properties for equations, and by the defintion for the composition of functions,

$$\Big\langle\gamma[\mu] = \gamma[\nu]\Big\rangle \to \Big\{\Big\langle\delta\big[\gamma(\mu)\big] = \delta\big[\gamma(\nu)\big]\Big\rangle \equiv \Big[\big\langle\delta \circ \gamma\big\rangle[\mu] = \big\langle\delta \circ \gamma\big\rangle[\nu]\Big]\Big\}$$

Thus, the universal quantification above implies

$$\neg\forall\mu\forall\nu\Big\langle\Big\{\big\langle\delta \circ \gamma\big\rangle[\mu] = \big\langle\delta \circ \gamma\big\rangle[\nu]\Big\} \to \big\langle\mu = \nu\big\rangle\Big\rangle$$

That is, it is not the case that $\langle\boldsymbol{\delta} \circ \boldsymbol{\gamma}\rangle$ is injective, by the definition for injective functions. Since it follows directly from the negation of the consequent that $\langle\boldsymbol{\delta} \circ \boldsymbol{\gamma}\rangle$ is not injective: if $\boldsymbol{\delta}$ and $\langle\boldsymbol{\delta} \circ \boldsymbol{\gamma}\rangle$ are injective functions, then $\boldsymbol{\gamma}$ is indeed injective. ■

Theorem (2336a). *Let $\boldsymbol{\lambda}$ be the function $\boldsymbol{\lambda} : \boldsymbol{\Gamma} \to \boldsymbol{\Phi}$. Let $\mathbf{A}$, and $\boldsymbol{\Lambda}$ be subsets of $\boldsymbol{\Gamma}$.*

$$\boldsymbol{\lambda}\big[\mathbf{A} \cup \boldsymbol{\Lambda}\big] = \boldsymbol{\lambda}\big[\mathbf{A}\big] \cup \boldsymbol{\lambda}\big[\boldsymbol{\Lambda}\big]$$

Proof. Suppose there exists an element $\boldsymbol{\phi}$ such that $\boldsymbol{\phi} \in \boldsymbol{\lambda}\big[\mathbf{A} \cup \boldsymbol{\Lambda}\big]$. By the definition for the image of a set under the function $\boldsymbol{\lambda}$, $\boldsymbol{\phi} = \boldsymbol{\lambda}\big[\boldsymbol{\iota}\big]$ where $\boldsymbol{\iota}$ is an element in $\mathbf{A} \cup \boldsymbol{\Lambda}$. That is, by the definition for the union of sets, $\boldsymbol{\iota} \in \mathbf{A} \vee \boldsymbol{\iota} \in \boldsymbol{\Lambda}$. Thus, $\boldsymbol{\phi} \in \boldsymbol{\lambda}\big[\mathbf{A}\big] \vee \boldsymbol{\phi} \in \boldsymbol{\lambda}\big[\boldsymbol{\Lambda}\big]$. And by the definition for set union, $\boldsymbol{\phi} \in \boldsymbol{\lambda}\big[\mathbf{A}\big] \cup \boldsymbol{\lambda}\big[\boldsymbol{\Lambda}\big]$.

Suppose there exists an element $\boldsymbol{\phi}$ such that $\boldsymbol{\phi} \in \boldsymbol{\lambda}\big[\mathbf{A}\big] \cup \boldsymbol{\lambda}\big[\boldsymbol{\Lambda}\big]$. By the defintion for set union, $\boldsymbol{\phi} \in \boldsymbol{\lambda}\big[\mathbf{A}\big] \vee \boldsymbol{\phi} \in \boldsymbol{\lambda}\big[\boldsymbol{\Lambda}\big]$. By the definition for the image of a set under the function $\boldsymbol{\lambda}$, there exists an element $\boldsymbol{\iota} \in \mathbf{A} \vee \boldsymbol{\iota} \in \boldsymbol{\Lambda}$ such that $\boldsymbol{\lambda}\big[\boldsymbol{\iota}\big] = \boldsymbol{\phi}$. By the definition for set union, $\boldsymbol{\iota} \in \mathbf{A} \cup \boldsymbol{\Lambda}$. Hence, $\boldsymbol{\phi} \in \boldsymbol{\lambda}\big[\mathbf{A} \cup \boldsymbol{\Lambda}\big]$ ■

Theorem (2368). *Let $\boldsymbol{\lambda}$ be a function $\boldsymbol{\lambda} : \mathbf{A} \to \boldsymbol{\Lambda}$, where $\mathbf{A}$ and $\boldsymbol{\Lambda}$ are finite sets, and $|\mathbf{A}| = |\boldsymbol{\Lambda}|$. $\boldsymbol{\lambda}$ is injective if and only if $\boldsymbol{\lambda}$ is surjective.*

Proof. Direct form by the contrapositive. Suppose $\boldsymbol{\lambda}$ is not surjective. This can be true only if $|\mathbf{A}| < |\boldsymbol{\Lambda}|$ (which is impossible,) or when $\boldsymbol{\lambda}$ is not injective. Thus, if $\boldsymbol{\lambda}$ is injective, then $\boldsymbol{\lambda}$ is surjective.

Converse form by the contrapositive. Suppose $\boldsymbol{\lambda}$ is not injective. This can be true only if $|\mathbf{A}| > |\boldsymbol{\Lambda}|$ (which is impossible,) or when $\boldsymbol{\lambda}$ is not surjective. Thus, if $\boldsymbol{\lambda}$ is surjective, then $\boldsymbol{\lambda}$ is injective. ■

Theorem (2.2.36b). *Let $\boldsymbol{\lambda}$ be the function $\boldsymbol{\lambda} : \boldsymbol{\Gamma} \to \boldsymbol{\Delta}$. Let* $\mathbf{A}$*, and* $\boldsymbol{\Lambda}$ *be subsets of* $\boldsymbol{\Gamma}$.

$$\Big\langle \boldsymbol{\lambda}\big[\mathbf{A} \cap \boldsymbol{\Lambda}\big] \Big\rangle \subseteq \Big\langle \boldsymbol{\lambda}\big[\mathbf{A}\big] \cap \boldsymbol{\lambda}\big[\boldsymbol{\Lambda}\big] \Big\rangle$$

Proof. Let $\boldsymbol{\iota}$ be an element in $\boldsymbol{\Gamma}$ such that $\boldsymbol{\lambda}\big[\boldsymbol{\iota}\big]$ is a member of $\boldsymbol{\lambda}\big[\mathbf{A} \cap \boldsymbol{\Lambda}\big]$. By the definition for the image of $\langle \mathbf{A} \cap \boldsymbol{\Lambda} \rangle$ under the function $\boldsymbol{\lambda}$, $\boldsymbol{\iota}$ is an element in $\langle \mathbf{A} \cap \boldsymbol{\Lambda} \rangle$. By the definition for the intersection of sets

$$\Big\langle \iota \in \mathrm{A} \Big\rangle \wedge \Big\langle \iota \in \Lambda \Big\rangle$$

Thus,

$$\Big\langle \lambda\big[\iota\big] \in \lambda\big[\mathrm{A}\big] \Big\rangle \wedge \Big\langle \lambda\big[\iota\big] \in \lambda\big[\Lambda\big] \Big\rangle$$

By the definition for set intersection, $\boldsymbol{\lambda}\big[\boldsymbol{\iota}\big]$ is a member of $\langle \boldsymbol{\lambda}\big[\mathbf{A}\big] \cap \boldsymbol{\lambda}\big[\boldsymbol{\Lambda}\big] \rangle$, $\therefore$ $\boldsymbol{\lambda}\big[\mathbf{A} \cap \boldsymbol{\Lambda}\big] \subseteq \boldsymbol{\lambda}\big[\mathbf{A}\big] \cap \boldsymbol{\lambda}\big[\boldsymbol{\Lambda}\big]$. ■

Theorem (2340a). *Let $\boldsymbol{\lambda}$ be the function $\boldsymbol{\lambda} : \boldsymbol{\Gamma} \to \mathbf{A}$. Let* $\boldsymbol{\Lambda}$*, and* $\boldsymbol{\Delta}$ *be subsets of* $\mathbf{A}$.

$$\boldsymbol{\lambda}^{-1}\big[\boldsymbol{\Lambda} \cup \boldsymbol{\Delta}\big] = \boldsymbol{\lambda}^{-1}\big[\boldsymbol{\Lambda}\big] \cup \boldsymbol{\lambda}^{-1}\big[\boldsymbol{\Delta}\big]$$

Proof. Let $\boldsymbol{\iota}$ be an element in $\boldsymbol{\lambda}^{-1}\big[\boldsymbol{\Lambda} \cup \boldsymbol{\Delta}\big]$. By the inverse function for $\boldsymbol{\lambda}^{-1}$, and by the definition for the union of sets, that is

$$\Big\langle \lambda\big[\iota\big] \in \lambda\Big[\lambda^{-1}\big[\Lambda \cup \Delta\big]\Big] \Big\rangle \equiv \Big[\Big\langle \lambda\big[\iota\big] \in \Lambda \Big\rangle \vee \Big\langle \lambda\big[\iota\big] \in \Delta \Big\rangle\Big]$$

By $\boldsymbol{\lambda}$ inverse, and by the definition for set union, that is

$$\Big[\Big\langle \iota \in \lambda^{-1}\big[\Lambda\big] \Big\rangle \vee \Big\langle \iota \in \lambda^{-1}\big[\Delta\big] \Big\rangle\Big] \equiv \Big[\iota \in \Big\langle \lambda^{-1}\big[\Lambda\big] \cup \lambda^{-1}\big[\Delta\big] \Big\rangle\Big]$$

$\therefore$ $\boldsymbol{\lambda}^{-1}\big[\boldsymbol{\Lambda} \cup \boldsymbol{\Delta}\big] = \boldsymbol{\lambda}^{-1}\big[\boldsymbol{\Lambda}\big] \cup \boldsymbol{\lambda}^{-1}\big[\boldsymbol{\Delta}\big]$. ■

Theorem (2340b). *Let $\boldsymbol{\lambda}$ be the function $\boldsymbol{\lambda} : \boldsymbol{\Gamma} \to \mathbf{A}$. Let* $\boldsymbol{\Lambda}$*, and* $\boldsymbol{\Delta}$ *be subsets of* $\mathbf{A}$.

$$\boldsymbol{\lambda}^{-1}\big[\boldsymbol{\Lambda} \cap \boldsymbol{\Delta}\big] = \boldsymbol{\lambda}^{-1}\big[\boldsymbol{\Lambda}\big] \cap \boldsymbol{\lambda}^{-1}\big[\boldsymbol{\Delta}\big]$$

Proof. Let $\boldsymbol{\iota}$ be an element in $\boldsymbol{\lambda}^{-1}\big[\boldsymbol{\Lambda} \cap \boldsymbol{\Delta}\big]$. By the inverse function for $\boldsymbol{\lambda}^{-1}$, and by the definition for the intersection of sets, that is

$$\Big\langle \lambda\big[\iota\big] \in \lambda\Big[\lambda^{-1}\big[\Lambda \cap \Delta\big]\Big] \Big\rangle \equiv \Big[\Big\langle \lambda\big[\iota\big] \in \Lambda \Big\rangle \wedge \Big\langle \lambda\big[\iota\big] \in \Delta \Big\rangle\Big]$$

By $\boldsymbol{\lambda}$ inverse, and by the definition for set intersection, that is

$$\Big[\Big\langle \iota \in \lambda^{-1}\big[\Lambda\big] \Big\rangle \wedge \Big\langle \iota \in \lambda^{-1}\big[\Delta\big] \Big\rangle\Big] \equiv \Big[\iota \in \Big\langle \lambda^{-1}\big[\Lambda\big] \cap \lambda^{-1}\big[\Delta\big] \Big\rangle\Big]$$

$\therefore$ $\boldsymbol{\lambda}^{-1}\big[\boldsymbol{\Lambda} \cap \boldsymbol{\Delta}\big] = \boldsymbol{\lambda}^{-1}\big[\boldsymbol{\Lambda}\big] \cap \boldsymbol{\lambda}^{-1}\big[\boldsymbol{\Delta}\big]$. ■

Theorem (2341). *Let $\boldsymbol{\lambda}$ be the function $\boldsymbol{\lambda : \Delta \to A}$. Let $\boldsymbol{\Lambda}$ be a subset of $\mathbf{A}$.*

$$\boldsymbol{\lambda^{-1}\left[\overline{\Lambda}\right] = \overline{\lambda^{-1}\left[\Lambda\right]}}$$

Proof. Let $\boldsymbol{\alpha}$ be an element in $\boldsymbol{\lambda^{-1}\left[\overline{\Lambda}\right]}$. By $\boldsymbol{\lambda^{-1}}$ inverse, by the definition of set complement, by $\boldsymbol{\lambda}$ inverse, and again by the defintion of set complement,

$$\left\langle \alpha \in \lambda^{-1}\left[\overline{\Lambda}\right] \right\rangle \equiv \left\langle \lambda[\alpha] \in \overline{\Lambda} \right\rangle \equiv \left\langle \lambda[\alpha] \notin \Lambda \right\rangle \equiv$$

$$\left\langle \alpha \notin \lambda^{-1}\left[\Lambda\right] \right\rangle \equiv \left\langle \alpha \in \overline{\lambda^{-1}\left[\Lambda\right]} \right\rangle$$

$\therefore$ $\boldsymbol{\lambda^{-1}\left[\overline{\Lambda}\right] = \overline{\lambda^{-1}\left[\Lambda\right]}}$. ■

Theorem (2342). *Let $\boldsymbol{\zeta}$ be a real number. $\boldsymbol{\lfloor \zeta + \frac{1}{2} \rfloor}$ is the closest integer to $\boldsymbol{\zeta}$, except when $\boldsymbol{\zeta}$ is midway between two integers, when it is the larger of these two integers.*

Proof. By cases. By the properties of floor functions, there exists an integer $\boldsymbol{\lambda}$ such that

$$\left\langle \lambda \right\rangle \leq \left\langle \zeta \right\rangle < \left\langle \lambda + 1 \right\rangle$$

and $\boldsymbol{\zeta - \lfloor \zeta \rfloor = \epsilon}$.

(i) Suppose the case in which $\boldsymbol{\zeta}$ is midway between two integers, or is closest to the larger of two integers $\boldsymbol{\lambda}$ and $\boldsymbol{\langle \lambda + 1 \rangle}$. Then, the inequality $\boldsymbol{\frac{1}{2} \leq \epsilon < 1}$ must be true. Thus,

$$\left[\left\langle \lambda + \frac{1}{2} \right\rangle \leq \left\langle \lambda + \epsilon \right\rangle < \left\langle \lambda + 1 \right\rangle\right] \equiv \left[\left\langle \lambda + \frac{1}{2} \right\rangle \leq \left\langle \zeta \right\rangle < \left\langle \lambda + 1 \right\rangle\right] \equiv$$

$$\left[\left\langle \lambda + 1 \right\rangle \leq \left\langle \zeta + \frac{1}{2} \right\rangle < \left\langle \lambda + 1 + \frac{1}{2} \right\rangle\right]$$

Since $\boldsymbol{\langle \lambda + 1 + \frac{1}{2} \rangle < \langle \lambda + 1 + 1 \rangle}$, the integer $\boldsymbol{\lfloor \zeta + \frac{1}{2} \rfloor}$ is $\boldsymbol{\langle \lambda + 1 \rangle}$, by the properties for floor functions, and by the law of transitivity from the order axioms.

(ii) Suppose the case in which $\boldsymbol{\zeta}$ is closest to the integer $\boldsymbol{\lambda}$. Then the inequality $\boldsymbol{0 \leq \epsilon < \frac{1}{2}}$, must be true. Thus,

$$\left[\left\langle \lambda + 0 \right\rangle \leq \left\langle \lambda + \epsilon \right\rangle < \left\langle \lambda + \frac{1}{2} \right\rangle\right] \equiv$$

$$\left[\left\langle \lambda \right\rangle \leq \left\langle \zeta \right\rangle < \left\langle \lambda + \frac{1}{2} \right\rangle\right] \equiv \left[\left\langle \lambda + \frac{1}{2} \right\rangle \leq \left\langle \zeta + \frac{1}{2} \right\rangle < \left\langle \lambda + 1 \right\rangle\right]$$

Since $\boldsymbol{\langle \lambda \rangle \leq \langle \lambda + \frac{1}{2} \rangle}$, the integer $\boldsymbol{\lfloor \zeta + \frac{1}{2} \rfloor}$ is $\boldsymbol{\langle \lambda \rangle}$, by the properties for floor functions, and by the law of transitivity from the order axioms. $\therefore$ $\boldsymbol{\lfloor \zeta + \frac{1}{2} \rfloor}$ is the closest integer to $\boldsymbol{\zeta}$, except when $\boldsymbol{\zeta}$ is midway between two integers, when it is the larger of these two integers. ■

Theorem (2343). *Let $\boldsymbol{\zeta}$ be a real number. $\lceil \boldsymbol{\zeta} - \frac{1}{2} \rceil$ is the closest integer to $\boldsymbol{\zeta}$, except when $\boldsymbol{\zeta}$ is midway between two integers, when it is the smaller of these two integers.*

Proof. By cases. By the properties of ceiling functions, there exists an integer $\boldsymbol{\lambda}$ such that

$$\left\langle \lambda - 1 \right\rangle < \left\langle \zeta \right\rangle \leq \left\langle \lambda \right\rangle$$

and $\lceil \boldsymbol{\zeta} \rceil - \boldsymbol{\zeta} = \boldsymbol{\epsilon}$.

(i) Suppose the case in which $\boldsymbol{\zeta}$ is midway between two integers, or is closest to the smaller of two integers $\langle \boldsymbol{\lambda} - \mathbf{1} \rangle$ and $\boldsymbol{\lambda}$. Then, the inequality, $\mathbf{1} > \boldsymbol{\epsilon} \geq \frac{\mathbf{1}}{\mathbf{2}}$ must be true. Thus,

$$\left[\left\langle \lambda - 1 \right\rangle < \left\langle \lambda - \epsilon \right\rangle \leq \left\langle \lambda - \frac{1}{2} \right\rangle\right] \equiv \left[\left\langle \lambda - 1 \right\rangle < \left\langle \zeta \right\rangle \leq \left\langle \lambda - \frac{1}{2} \right\rangle\right] \equiv$$

$$\left[\left\langle \lambda - 1 - \frac{1}{2} \right\rangle < \left\langle \zeta - \frac{1}{2} \right\rangle \leq \left\langle \lambda - 1 \right\rangle\right]$$

Since $\langle \boldsymbol{\lambda} - \mathbf{1} - \mathbf{1} \rangle < \langle \boldsymbol{\lambda} - \mathbf{1} - \frac{\mathbf{1}}{\mathbf{2}} \rangle$, the integer $\lceil \boldsymbol{\zeta} - \frac{\mathbf{1}}{\mathbf{2}} \rceil$ is $\langle \boldsymbol{\lambda} - \mathbf{1} \rangle$, by the properties for ceiling functions, and by the law of transitivity from the order axioms.

(ii) Suppose the case in which $\boldsymbol{\zeta}$ is closest to the integer $\boldsymbol{\lambda}$. Then, the inequality $\frac{\mathbf{1}}{\mathbf{2}} > \boldsymbol{\epsilon} \geq \mathbf{0}$ must be true. Thus,

$$\left[\left\langle \lambda - \frac{1}{2} \right\rangle < \left\langle \lambda - \epsilon \right\rangle \leq \left\langle \lambda - 0 \right\rangle\right] \equiv$$

$$\left[\left\langle \lambda - \frac{1}{2} \right\rangle < \left\langle \zeta \right\rangle \leq \left\langle \lambda \right\rangle\right] \equiv \left[\left\langle \lambda - 1 \right\rangle < \left\langle \zeta - \frac{1}{2} \right\rangle \leq \left\langle \lambda - \frac{1}{2} \right\rangle\right]$$

Since $\langle \boldsymbol{\lambda} - \frac{\mathbf{1}}{\mathbf{2}} \rangle \leq \langle \boldsymbol{\lambda} \rangle$, the integer $\lceil \boldsymbol{\zeta} - \frac{\mathbf{1}}{\mathbf{2}} \rceil$ is $\langle \boldsymbol{\lambda} \rangle$, by the properties for ceiling functions, and by the law of transitivity from the order axioms $\therefore$ $\lceil \boldsymbol{\zeta} - \frac{\mathbf{1}}{\mathbf{2}} \rceil$ is the closest integer to $\boldsymbol{\zeta}$, except when $\boldsymbol{\zeta}$ is midway between two integers, when it is the smaller of these two integers. ∎

Theorem (2346). *Let $\boldsymbol{\lambda}$ be a real number, and let $\boldsymbol{\zeta}$ be an integer.*

$$\lceil \boldsymbol{\lambda} + \boldsymbol{\zeta} \rceil = \lceil \boldsymbol{\lambda} \rceil + \boldsymbol{\zeta}$$

Proof. Given $\lceil \boldsymbol{\lambda} \rceil$, by the properties for ceiling functions we have

$$\left\langle \lceil \lambda \rceil - 1 \right\rangle < \left\langle \lambda \right\rangle \leq \left\langle \lceil \lambda \rceil \right\rangle$$

By the additive compatibility law from the order axioms, that is

$$\left\langle [\lceil \lambda \rceil + \zeta] - 1 \right\rangle < \left\langle \lambda + \zeta \right\rangle \leq \left\langle \lceil \lambda \rceil + \zeta \right\rangle$$

$\therefore$ $\lceil \boldsymbol{\lambda} + \boldsymbol{\zeta} \rceil = \lceil \boldsymbol{\lambda} \rceil + \boldsymbol{\zeta}$, by the properties for ceiling functions. ∎

Theorem (2344). *Let $\boldsymbol{\lambda}$ be a real number.*

$$\lceil\boldsymbol{\lambda}\rceil - \lfloor\boldsymbol{\lambda}\rfloor = \mathbf{1},\ \textit{if}\ \boldsymbol{\lambda} \notin \mathbb{Z}.\ \lceil\boldsymbol{\lambda}\rceil - \lfloor\boldsymbol{\lambda}\rfloor = \mathbf{0},\ \textit{if}\ \boldsymbol{\lambda} \in \mathbb{Z}$$

Proof. Let $\boldsymbol{\lambda} - \lfloor\boldsymbol{\lambda}\rfloor = \boldsymbol{\sigma}$. By the properties for ceiling functions, there exists an integer $\boldsymbol{\zeta}$ such that $\lceil\boldsymbol{\lambda}\rceil = \boldsymbol{\zeta}$, if and only if

$$\left\langle \zeta - 1 \right\rangle < \left\langle \lambda \right\rangle \leq \left\langle \zeta \right\rangle$$

By the identity of $\boldsymbol{\zeta}$, and by the additive compatibility law from the order axioms (subtracting $\lfloor\boldsymbol{\lambda}\rfloor$ from every side,) that is,

$$\left\langle \lceil\lambda\rceil - 1 \right\rangle < \left\langle \lambda \right\rangle \leq \left\langle \lceil\lambda\rceil \right\rangle \equiv$$

$$\left\langle \lceil\lambda\rceil - \lfloor\lambda\rfloor - 1 \right\rangle < \left\langle \lambda - \lfloor\lambda\rfloor \right\rangle \leq \left\langle \lceil\lambda\rceil - \lfloor\lambda\rfloor \right\rangle \equiv$$

$$\left\langle \lceil\lambda\rceil - \lfloor\lambda\rfloor - 1 \right\rangle < \left\langle \sigma \right\rangle \leq \left\langle \lceil\lambda\rceil - \lfloor\lambda\rfloor \right\rangle$$

Hence, $\lceil\boldsymbol{\sigma}\rceil = \lceil\boldsymbol{\lambda}\rceil - \lfloor\boldsymbol{\lambda}\rfloor$, by the properties for ceiling functions. There are two cases under consideration: (i) $\boldsymbol{\lambda}$ is an integer, and (ii) $\boldsymbol{\lambda}$ is a real number not in integers.

(i) If $\boldsymbol{\lambda}$ is an integer, then by the definition for the floor function, the largest integer less than or equal to $\boldsymbol{\lambda}$ is $\boldsymbol{\lambda}$. Thus,

$$\left\langle \lambda - \lfloor\lambda\rfloor \right\rangle = \left\langle \lambda - \lambda \right\rangle = \left\langle \sigma \right\rangle$$

Clearly, $\boldsymbol{\sigma} = \mathbf{0}$ in this case. So, by the definition for ceiling functions, since the smallest integer greater than or equal to $\mathbf{0}$ is $\mathbf{0}$, $\lceil\boldsymbol{\sigma}\rceil = \mathbf{0}$. Thus, $\lceil\boldsymbol{\lambda}\rceil - \lfloor\boldsymbol{\lambda}\rfloor = \mathbf{0}$, whenever $\boldsymbol{\lambda}$ is an integer.

(ii) If $\boldsymbol{\lambda}$ is a real number, but not an integer, then $\boldsymbol{\sigma}$ has to be greater than zero, but less than one. That is,

$$\left\{ \left\langle 0 \right\rangle < \left\langle \sigma \right\rangle < \left\langle 1 \right\rangle \right\} \rightarrow \left\{ \left\langle 1 - 1 \right\rangle < \left\langle \sigma \right\rangle \leq \left\langle 1 \right\rangle \right\}$$

By the properties for ceiling functions, $\lceil\boldsymbol{\lambda}\rceil - \lfloor\boldsymbol{\lambda}\rfloor = \lceil\boldsymbol{\sigma}\rceil = \mathbf{1}$, whenever $\boldsymbol{\lambda}$ is a real number, but not an integer.

$\therefore$ **$\lceil\boldsymbol{\lambda}\rceil - \lfloor\boldsymbol{\lambda}\rfloor = \mathbf{1}$, if $\boldsymbol{\lambda} \notin \mathbb{Z}$. $\lceil\boldsymbol{\lambda}\rceil - \lfloor\boldsymbol{\lambda}\rfloor = \mathbf{0}$, if $\boldsymbol{\lambda} \in \mathbb{Z}$.**

∎

Theorem (2345). *Let* $\boldsymbol{\lambda}$ *be a real number.*

$$\langle \boldsymbol{\lambda - 1} \rangle < \langle \lfloor \boldsymbol{\lambda} \rfloor \rangle \leq \langle \boldsymbol{\lambda} \rangle \leq \langle \lceil \boldsymbol{\lambda} \rceil \rangle < \langle \boldsymbol{\lambda + 1} \rangle$$

Proof. Let $\boldsymbol{\lambda} - \lfloor \boldsymbol{\lambda} \rfloor = \boldsymbol{\sigma}$, and let $\lceil \boldsymbol{\lambda} \rceil - \boldsymbol{\lambda} = \boldsymbol{\epsilon}$. By the additive and multiplicative compatibility laws from the order axioms, and by the properties for floor functions, there exists an integer $\lfloor \boldsymbol{\lambda} \rfloor = \boldsymbol{\zeta}$ such that

$$\Big\langle \zeta \Big\rangle \leq \Big\langle \lambda \Big\rangle < \Big\langle \zeta + 1 \Big\rangle \equiv$$
$$\Big\langle \lfloor \lambda \rfloor \Big\rangle \leq \Big\langle \lfloor \lambda \rfloor + \sigma \Big\rangle < \Big\langle \lfloor \lambda \rfloor + 1 \Big\rangle \equiv$$
$$\Big[\Big\langle 0 \Big\rangle \leq \Big\langle \sigma \Big\rangle < \Big\langle 1 \Big\rangle \Big] \equiv \Big[\Big\langle -1 \Big\rangle < \Big\langle -\sigma \Big\rangle \leq \Big\langle 0 \Big\rangle \Big]$$

Also, by the properties for ceiling functions, there exists an integer $\lceil \boldsymbol{\lambda} \rceil = \boldsymbol{\xi}$. From which, by similar reasoning as to that of above, we can derive

$$\Big\langle 0 \Big\rangle \leq \Big\langle \epsilon \Big\rangle < \Big\langle 1 \Big\rangle$$

Thus, combining both results by transitivity from the order axioms

$$\Big\langle -1 \Big\rangle < \Big\langle -\sigma \Big\rangle \leq \Big\langle 0 \Big\rangle \leq \Big\langle \epsilon \Big\rangle < \Big\langle 1 \Big\rangle$$

Again, by the additive compatibility law from the order axioms, and by the identities for $\lfloor \boldsymbol{\lambda} \rfloor$ and $\lceil \boldsymbol{\lambda} \rceil$, that is

$$\Big\langle \lambda - 1 \Big\rangle < \Big\langle \lambda - \sigma \Big\rangle \leq \Big\langle \lambda + 0 \Big\rangle \leq \Big\langle \lambda + \epsilon \Big\rangle < \Big\langle \lambda + 1 \Big\rangle \equiv$$
$$\Big\langle \lambda - 1 \Big\rangle < \Big\langle \lfloor \lambda \rfloor \Big\rangle \leq \Big\langle \lambda \Big\rangle \leq \Big\langle \lceil \lambda \rceil \Big\rangle < \Big\langle \lambda + 1 \Big\rangle$$

■

Theorem (2347a). *Let* $\boldsymbol{\lambda}$ *be a real number, and let* $\boldsymbol{\zeta}$ *be an integer.*

$$\langle \boldsymbol{\lambda} < \boldsymbol{\zeta} \rangle \textit{ if and only if } \langle \lfloor \boldsymbol{\lambda} \rfloor < \boldsymbol{\zeta} \rangle$$

Proof. Assume $\boldsymbol{\lambda} < \boldsymbol{\zeta}$. By the properties of the floor function, $\lfloor \boldsymbol{\lambda} \rfloor \leq \boldsymbol{\lambda}$. Thus,

$$\Big\langle \lfloor \lambda \rfloor \Big\rangle \leq \Big\langle \lambda \Big\rangle < \Big\langle \zeta \Big\rangle$$

It immediately follows from transitivity that $\lfloor \boldsymbol{\lambda} \rfloor < \boldsymbol{\zeta}$.

In the converse case, assume $\lfloor \boldsymbol{\lambda} \rfloor < \boldsymbol{\zeta}$. Since $\lfloor \boldsymbol{\lambda} \rfloor$ and $\boldsymbol{\zeta}$ are integers, we can infer from additive compatibility that $\lfloor \boldsymbol{\lambda} \rfloor + \mathbf{1} \leq \boldsymbol{\zeta}$. And by the properties of the floor function, we have

$$\Big\langle \lfloor \lambda \rfloor \Big\rangle \leq \Big\langle \lambda \Big\rangle < \Big\langle \lfloor \lambda \rfloor + 1 \Big\rangle$$

Thus,

$$\Big\langle \lfloor \lambda \rfloor \Big\rangle \leq \Big\langle \lambda \Big\rangle < \Big\langle \lfloor \lambda \rfloor + 1 \Big\rangle \leq \Big\langle \zeta \Big\rangle$$

This statement says that $\boldsymbol{\lambda} < \boldsymbol{\zeta}$.

$\therefore$ $\langle \boldsymbol{\lambda} < \boldsymbol{\zeta} \rangle$ ***if and only if*** $\langle \lfloor \boldsymbol{\lambda} \rfloor < \boldsymbol{\zeta} \rangle$. ■

Theorem (2347b). *Let $\boldsymbol{\lambda}$ be a real number, and let $\boldsymbol{\zeta}$ be an integer.*

$$\langle \boldsymbol{\zeta} < \boldsymbol{\lambda} \rangle \textit{ if and only if } \langle \boldsymbol{\zeta} < \lceil \boldsymbol{\lambda} \rceil \rangle$$

Proof. Assume $\boldsymbol{\zeta} < \boldsymbol{\lambda}$. By the properties of the ceiling function, $\boldsymbol{\lambda} \leq \lceil \boldsymbol{\lambda} \rceil$. Thus,

$$\left\langle \zeta \right\rangle < \left\langle \lambda \right\rangle \leq \left\langle \lceil \lambda \rceil \right\rangle$$

It immediately follows that $\boldsymbol{\zeta} < \lceil \boldsymbol{\lambda} \rceil$.

In the converse case, assume $\boldsymbol{\zeta} < \lceil \boldsymbol{\lambda} \rceil$. Since $\lceil \boldsymbol{\lambda} \rceil$ and $\boldsymbol{\zeta}$ are integers, by the additive compatibility law from the order axioms, we can infer $\boldsymbol{\zeta} \leq \lceil \boldsymbol{\lambda} \rceil - \mathbf{1}$. And by the properties of the ceiling function, we have

$$\left\langle \lceil \lambda \rceil - 1 \right\rangle < \left\langle \lambda \right\rangle \leq \left\langle \lceil \lambda \rceil \right\rangle$$

Thus,

$$\left\langle \zeta \right\rangle \leq \left\langle \lceil \lambda \rceil - 1 \right\rangle < \left\langle \lambda \right\rangle \leq \left\langle \lceil \lambda \rceil \right\rangle$$

This statement says that $\boldsymbol{\zeta} < \boldsymbol{\lambda}$.

$\therefore$ $\langle \boldsymbol{\zeta} < \boldsymbol{\lambda} \rangle$ ***if and only if*** $\langle \boldsymbol{\zeta} < \lceil \boldsymbol{\lambda} \rceil \rangle$.

Theorem (2348a). *Let $\boldsymbol{\lambda}$ be a real number, and let $\boldsymbol{\zeta}$ be an integer.*

$$\langle \boldsymbol{\lambda} \leq \boldsymbol{\zeta} \rangle \textit{ if and only if } \langle \lceil \boldsymbol{\lambda} \rceil \leq \boldsymbol{\zeta} \rangle$$

Proof. Direct form by the contrapositive. Assume the negation of the direct consequent, such that $\lceil \boldsymbol{\lambda} \rceil > \boldsymbol{\zeta}$. By Theorem 2347b, $\boldsymbol{\lambda} > \boldsymbol{\zeta}$. Thus,

$$\left[\left\langle \lambda \right\rangle \leq \left\langle \zeta \right\rangle \right] \rightarrow \left[\left\langle \lceil \lambda \rceil \right\rangle \leq \left\langle \zeta \right\rangle \right]$$

Converse form by the contrapositive. Assume the negation of the direct hypothesis, such that $\boldsymbol{\lambda} > \boldsymbol{\zeta}$. By Theorem 2347b, $\lceil \boldsymbol{\lambda} \rceil > \boldsymbol{\zeta}$. Thus,

$$\left[\left\langle \lceil \lambda \rceil \right\rangle \leq \left\langle \zeta \right\rangle \right] \rightarrow \left[\left\langle \lambda \right\rangle \leq \left\langle \zeta \right\rangle \right]$$

$\therefore$ $\langle \boldsymbol{\lambda} \leq \boldsymbol{\zeta} \rangle$ ***if and only if*** $\langle \lceil \boldsymbol{\lambda} \rceil \leq \boldsymbol{\zeta} \rangle$. ∎

Theorem (2348b). *Let $\boldsymbol{\lambda}$ be a real number, and let $\boldsymbol{\zeta}$ be an integer.*

$$\left\langle \boldsymbol{\zeta} \leq \boldsymbol{\lambda} \right\rangle \textit{ if and only if } \left\langle \boldsymbol{\zeta} \leq \lfloor \boldsymbol{\lambda} \rfloor \right\rangle$$

Proof. Direct form by the contrapositive. Assume the negation of the direct consequent, such that $\boldsymbol{\zeta} > \lfloor \boldsymbol{\lambda} \rfloor$. By Theorem 2347a, $\boldsymbol{\zeta} > \boldsymbol{\lambda}$. Thus,

$$\left[\left\langle \zeta \right\rangle \leq \left\langle \lambda \right\rangle\right] \rightarrow \left[\left\langle \zeta \right\rangle \leq \left\langle \lfloor \lambda \rfloor \right\rangle\right]$$

Converse form by the contrapositive. Assume the negation of the direct hypothesis, such that $\boldsymbol{\zeta} > \boldsymbol{\lambda}$. By Theorem 2347a, $\boldsymbol{\zeta} > \lfloor \boldsymbol{\lambda} \rfloor$. Thus,

$$\left[\left\langle \zeta \right\rangle \leq \left\langle \lfloor \lambda \rfloor \right\rangle\right] \rightarrow \left[\left\langle \zeta \right\rangle \leq \left\langle \lambda \right\rangle\right]$$

$\therefore$ $\left\langle \boldsymbol{\zeta} \leq \boldsymbol{\lambda} \right\rangle$ ***if and only if*** $\left\langle \boldsymbol{\zeta} \leq \lfloor \boldsymbol{\lambda} \rfloor \right\rangle$. ■

Theorem (2366). *Let $\boldsymbol{\sigma}$ be the invertible function $\boldsymbol{\sigma} : \boldsymbol{\Theta} \rightarrow \boldsymbol{\Omega}$, and let $\boldsymbol{\phi}$ be the invertible function $\boldsymbol{\phi} : \boldsymbol{\Phi} \rightarrow \boldsymbol{\Theta}$.*

The inverse of the composition $\boldsymbol{\sigma} \circ \boldsymbol{\phi}$ is given by

$$\left\langle \boldsymbol{\sigma} \circ \boldsymbol{\phi} \right\rangle^{-1} = \boldsymbol{\phi}^{-1} \circ \boldsymbol{\sigma}^{-1}$$

Proof. By Theorem 2329a and Theorem 2329b, and by the definition for bijective functions, $\boldsymbol{\sigma} \circ \boldsymbol{\phi}$ is invertible. Thus,

$$\left\langle \sigma \circ \phi \right\rangle^{-1} \circ \left\langle \sigma \circ \phi \right\rangle = \iota_\alpha$$

What remains to be determined is whether $\left\langle \boldsymbol{\phi}^{-1} \circ \boldsymbol{\sigma}^{-1} \right\rangle \circ \left\langle \boldsymbol{\sigma} \circ \boldsymbol{\phi} \right\rangle = \boldsymbol{\iota}_{\boldsymbol{\alpha}}$. Let $\boldsymbol{\alpha}$ be an element in the domain $\boldsymbol{\Phi}$ such that

$$\left[\left\langle \phi^{-1} \circ \sigma^{-1} \right\rangle \circ \left\langle \sigma \circ \phi \right\rangle\right]\left[\alpha\right] = \alpha$$

By the definition for the composition of functions, that is

$$\phi^{-1}\left[\sigma^{-1}\left[\sigma\left[\phi\left[\alpha\right]\right]\right]\right] = \alpha$$

Clearly, $\left\langle \boldsymbol{\phi}^{-1} \circ \boldsymbol{\sigma}^{-1} \right\rangle \circ \left\langle \boldsymbol{\sigma} \circ \boldsymbol{\phi} \right\rangle = \boldsymbol{\iota}_{\boldsymbol{\alpha}}$

$\therefore$ *the inverse of the composition $\boldsymbol{\sigma} \circ \boldsymbol{\phi}$ is given by $\left\langle \boldsymbol{\sigma} \circ \boldsymbol{\phi} \right\rangle^{-1} = \boldsymbol{\phi}^{-1} \circ \boldsymbol{\sigma}^{-1}$.* ■

Theorem (2349). *Let $\boldsymbol{\zeta}$ be an integer.*

$$\textit{If } \zeta \textit{ is even, then } \left\lfloor \frac{\zeta}{2} \right\rfloor = \frac{\zeta}{2}; \textit{ if } \zeta \textit{ is odd, then } \left\lfloor \frac{\zeta}{2} \right\rfloor = \frac{\langle \zeta - 1 \rangle}{2}$$

Proof. By cases.

(i) Assume $\boldsymbol{\zeta}$ is even. By the definition for even numbers, there exists an integer $\boldsymbol{\iota}$ such that $\boldsymbol{\zeta = 2\iota}$. Thus, by that identity for $\boldsymbol{\zeta}$, and by the multiplicative inverse from the field axioms,

$$\left(\frac{\zeta}{2}\right) = \left(\frac{2\iota}{2}\right) = \left\langle \iota \right\rangle$$

By that identity $\boldsymbol{\iota}$, and by the definition for the floor function,

$$\left\lfloor \frac{\zeta}{2} \right\rfloor = \left\lfloor \iota \right\rfloor = \left\langle \iota \right\rangle$$

$\therefore$ **if $\boldsymbol{\zeta}$ is even, then** $\left\lfloor \frac{\zeta}{2} \right\rfloor = \frac{\zeta}{2}$, by the identity $\boldsymbol{\iota}$.

(ii) Assume $\boldsymbol{\zeta}$ is odd. By the definition for odd numbers, there exists an integer $\boldsymbol{\iota}$ such that $\boldsymbol{\zeta = 2\iota + 1}$. Thus, by that identity for $\boldsymbol{\zeta}$, by the multiplicative inverse from the field axioms, and by the definition for floor functions,

$$\left\lfloor \frac{\zeta}{2} \right\rfloor = \left\lfloor \frac{2\iota + 1}{2} \right\rfloor = \left\lfloor \iota + \frac{1}{2} \right\rfloor = \left\langle \iota \right\rangle$$

Also, by that identity for $\boldsymbol{\zeta}$, and by the additive and multiplicative inverse from the field axioms,

$$\left(\frac{\langle \zeta - 1 \rangle}{2}\right) = \left(\frac{[\langle 2\iota + 1 \rangle - 1]}{2}\right) = \left(\frac{2\iota}{2}\right) = \left\langle \iota \right\rangle$$

$\therefore$ **if $\boldsymbol{\zeta}$ is odd, then** $\left\lfloor \frac{\zeta}{2} \right\rfloor = \frac{\langle \zeta - 1 \rangle}{2}$, by the identity $\boldsymbol{\iota}$. ∎

Theorem (2369a). *Let $\boldsymbol{\lambda}$ be a real number.*

$$\left\lceil \lfloor \lambda \rfloor \right\rceil = \lfloor \lambda \rfloor$$

Proof. By the properties of floor functions, there exists an integer $\boldsymbol{\iota}$ such that $\lfloor \boldsymbol{\lambda} \rfloor = \boldsymbol{\iota}$, and by the identity $\boldsymbol{\iota}$,

$$\left\langle \lfloor \lambda \rfloor = \iota \right\rangle \equiv \left\langle \left\lceil \lfloor \lambda \rfloor \right\rceil = \lceil \iota \rceil \right\rangle$$

$\boldsymbol{\iota}$ is the smallest integer that is greater than or equal to $\boldsymbol{\iota}$. Therefore, by the definition for the ceiling function $\lceil \boldsymbol{\iota} \rceil = \boldsymbol{\iota}$. Hence,

$$\left\langle \lfloor \lambda \rfloor = \iota \right\rangle \wedge \left\langle \left\lceil \lfloor \lambda \rfloor \right\rceil = \lceil \iota \rceil = \iota \right\rangle$$

$\therefore$ $\left\lceil \lfloor \boldsymbol{\lambda} \rfloor \right\rceil = \lfloor \boldsymbol{\lambda} \rfloor$, by the identity $\boldsymbol{\iota}$. ∎

Theorem (2350). *Let $\boldsymbol{\zeta}$ be a real number.*

$$\lfloor -\boldsymbol{\zeta} \rfloor = -\lceil \boldsymbol{\zeta} \rceil, \textbf{\textit{ and }} \lceil -\boldsymbol{\zeta} \rceil = -\lfloor \boldsymbol{\zeta} \rfloor.$$

Proof. By cases.

(i) By the multiplicative compatibility laws from the order axioms, and by the properties for ceiling functions, there exists an integer $\lceil \boldsymbol{\zeta} \rceil = \boldsymbol{\lambda}$ such that

$$\left\{ \left\langle \lambda - 1 \right\rangle < \left\langle \zeta \right\rangle \leq \left\langle \lambda \right\rangle \right\} \equiv \left\{ \left\langle -\lambda + 1 \right\rangle > \left\langle -\zeta \right\rangle \geq \left\langle -\lambda \right\rangle \right\}$$

By the properties for floor functions, $\lfloor -\boldsymbol{\zeta} \rfloor = -\boldsymbol{\lambda}$. And $-\mathbf{1} \times \lceil \boldsymbol{\zeta} \rceil = -\boldsymbol{\lambda}$, by the multiplicative equality property for equations. Thus, $\lfloor -\boldsymbol{\zeta} \rfloor = -\lceil \boldsymbol{\zeta} \rceil$, by the identity $-\boldsymbol{\lambda}$.

(ii) By the multiplicative compatibility laws from the order axioms, and by the properties of floor functions, there exists an integer $\lfloor \boldsymbol{\zeta} \rfloor = \boldsymbol{\lambda}$ such that

$$\left\{ \left\langle \lambda \right\rangle \leq \left\langle \zeta \right\rangle < \left\langle \lambda + 1 \right\rangle \right\} \equiv \left\{ \left\langle -\lambda \right\rangle \geq \left\langle -\zeta \right\rangle > \left\langle -\lambda - 1 \right\rangle \right\}$$

By the properties of ceiling functions, $\lceil -\boldsymbol{\zeta} \rceil = -\boldsymbol{\lambda}$. And $-\mathbf{1} \times \lfloor \boldsymbol{\zeta} \rfloor = -\boldsymbol{\lambda}$, by the multiplicative property for equations. Thus, $\lceil -\boldsymbol{\zeta} \rceil = -\lfloor \boldsymbol{\zeta} \rfloor$, by the identity $-\boldsymbol{\lambda}$.

$\therefore$ $\lfloor -\boldsymbol{\zeta} \rfloor = -\lceil \boldsymbol{\zeta} \rceil$, **and** $\lceil -\boldsymbol{\zeta} \rceil = -\lfloor \boldsymbol{\zeta} \rfloor$. ■

Theorem (2370a). *Let $\boldsymbol{\lambda}$ be a real number.*

$$\Big\lfloor \lceil \boldsymbol{\lambda} \rceil \Big\rfloor = \lceil \boldsymbol{\lambda} \rceil$$

Proof. By the properties of ceiling functions, there exists an integer $\boldsymbol{\iota}$ such that $\lceil \boldsymbol{\lambda} \rceil = \boldsymbol{\iota}$, and by the identity $\boldsymbol{\iota}$,

$$\left\langle \lceil \lambda \rceil = \iota \right\rangle \equiv \left\langle \Big\lfloor \lceil \lambda \rceil \Big\rfloor = \lfloor \iota \rfloor \right\rangle$$

$\boldsymbol{\iota}$ is the largest integer that is less than or equal to $\boldsymbol{\iota}$. Therefore, by the definition for the floor function $\lfloor \boldsymbol{\iota} \rfloor = \boldsymbol{\iota}$. Hence,

$$\left\langle \lceil \lambda \rceil = \iota \right\rangle \wedge \left\langle \Big\lfloor \lceil \lambda \rceil \Big\rfloor = \lfloor \iota \rfloor = \iota \right\rangle$$

$\therefore$ $\Big\lfloor \lceil \boldsymbol{\lambda} \rceil \Big\rfloor = \lceil \boldsymbol{\lambda} \rceil$, by the identity $\boldsymbol{\iota}$. ■

Theorem (2367a). *Suppose that* $\mathbf{A}$*, and* $\mathbf{\Lambda}$ *are sets with universal set* $\mathbf{\Omega}$*. Let* $\boldsymbol{\lambda}_{\mathbf{A}\cap\mathbf{\Lambda}}$ *be the characteristic function* $\boldsymbol{\lambda}_{\mathbf{A}\cap\mathbf{\Lambda}} : \mathbf{\Omega} \rightarrow \{\mathbf{0}, \mathbf{1}\}$*, let* $\boldsymbol{\lambda}_{\mathbf{A}}$ *be the characteristic function* $\boldsymbol{\lambda}_{\mathbf{A}} : \mathbf{\Omega} \rightarrow \{\mathbf{0}, \mathbf{1}\}$*, and let* $\boldsymbol{\lambda}_{\mathbf{\Lambda}}$ *be the characteristic function* $\boldsymbol{\lambda}_{\mathbf{\Lambda}} : \mathbf{\Omega} \rightarrow \{\mathbf{0}, \mathbf{1}\}$*.*

$$\boldsymbol{\lambda}_{\mathbf{A}\cap\mathbf{\Lambda}}\left[\boldsymbol{\iota}\right] = \boldsymbol{\lambda}_{\mathbf{A}}\left[\boldsymbol{\iota}\right] \times \boldsymbol{\lambda}_{\mathbf{\Lambda}}\left[\boldsymbol{\iota}\right]$$

Proof. By cases. There are two cases under consideration. (*i*) $\boldsymbol{\iota}$ is a member of $\mathbf{A} \cap \mathbf{\Lambda}$, or (*ii*) it is not the case that $\boldsymbol{\iota}$ is a member of $\mathbf{A} \cap \mathbf{\Lambda}$.

(i) Suppose $\boldsymbol{\iota}$ were an element in $\mathbf{A} \cap \mathbf{\Lambda}$. Note that, $\boldsymbol{\lambda}_{\mathbf{A}\cap\mathbf{\Lambda}}\left[\boldsymbol{\iota}\right] = \mathbf{1}$, by the definition for characteristic functions. Now, by the definition for the intersection of sets, and by the definition for characteristic functions,

$$\left[\left\langle \iota \in \mathrm{A}\right\rangle \wedge \left\langle \iota \in \Lambda\right\rangle\right] \equiv \left[\left\langle \lambda_{\mathrm{A}}\left[\iota\right] = 1\right\rangle \wedge \left\langle \lambda_{\Lambda}\left[\iota\right] = 1\right\rangle\right]$$

It follows immediately from logical identity, and by multiplicative identity from the field axioms that $\boldsymbol{\lambda}_{\mathbf{A}\cap\mathbf{\Lambda}}\left[\boldsymbol{\iota}\right] = \boldsymbol{\lambda}_{\mathbf{A}}\left[\boldsymbol{\iota}\right] \times \boldsymbol{\lambda}_{\mathbf{\Lambda}}\left[\boldsymbol{\iota}\right]$, in this case.

(ii) Suppose it were not the case that $\boldsymbol{\iota}$ were an element in $\mathbf{A} \cap \mathbf{\Lambda}$. Note that, $\boldsymbol{\lambda}_{\mathbf{A}\cap\mathbf{\Lambda}}\left[\boldsymbol{\iota}\right] = \mathbf{0}$, by the definition for characteristic functions. Now. by the definitions for set intersection and set membership, and by DeMorgans law for set intersection,

$$\left[\iota \notin \left\langle \mathrm{A} \cap \Lambda\right\rangle\right] \equiv \neg\left[\left\langle \iota \in \mathrm{A}\right\rangle \wedge \left\langle \iota \in \Lambda\right\rangle\right] \equiv \left[\left\langle \iota \notin \mathrm{A}\right\rangle \vee \left\langle \iota \notin \Lambda\right\rangle\right]$$

And, by the definition for characteristic functions,

$$\left[\left\langle \iota \notin \mathrm{A}\right\rangle \vee \left\langle \iota \notin \Lambda\right\rangle\right] \equiv \left[\left\langle \lambda_{\mathrm{A}}\left[\iota\right] = 0\right\rangle \vee \left\langle \lambda_{\Lambda}\left[\iota\right] = 0\right\rangle\right]$$

Without loss of generality we can suppose $\boldsymbol{\lambda}_{\mathbf{A}}\left[\boldsymbol{\iota}\right] = \mathbf{0}$. It follows immediately from logical identity, and from the multiplicative property for zero that $\boldsymbol{\lambda}_{\mathbf{A}\cap\mathbf{\Lambda}}\left[\boldsymbol{\iota}\right] = \mathbf{0} \times \boldsymbol{\lambda}_{\mathbf{A}}\left[\boldsymbol{\iota}\right] = \mathbf{0}$. Thus, $\boldsymbol{\lambda}_{\mathbf{A}\cap\mathbf{\Lambda}}\left[\boldsymbol{\iota}\right] = \boldsymbol{\lambda}_{\mathbf{A}}\left[\boldsymbol{\iota}\right] \times \boldsymbol{\lambda}_{\mathbf{\Lambda}}\left[\boldsymbol{\iota}\right]$, in this case. ∎

Lemma (2301). *Let* $\boldsymbol{\lambda}$ *be a real number, and let* $\boldsymbol{\zeta}$ *be an integer.*

$$\lfloor \boldsymbol{\lambda} + \boldsymbol{\zeta} \rfloor = \lfloor \boldsymbol{\lambda} \rfloor + \boldsymbol{\zeta}$$

Proof. Given $\lfloor \boldsymbol{\lambda} \rfloor$, by the properties for floor functions we have

$$\left\langle \lfloor \lambda \rfloor \right\rangle \leq \left\langle \lambda \right\rangle < \left\langle \lfloor \lambda \rfloor + 1 \right\rangle$$

By the additive compatibility law from the order axioms, that is

$$\left\langle \lfloor \lambda \rfloor + \zeta \right\rangle \leq \left\langle \lambda + \zeta \right\rangle < \left\langle \left[\lfloor \lambda \rfloor + \zeta\right] + 1 \right\rangle$$

$\therefore$ $\lfloor \boldsymbol{\lambda} + \boldsymbol{\zeta} \rfloor = \lfloor \boldsymbol{\lambda} \rfloor + \boldsymbol{\zeta}$, by the properties for floor functions.

∎

Theorem (**2367b**). *Suppose that* $\mathbf{A}$, *and* $\mathbf{\Lambda}$ *are sets with universal set* $\mathbf{\Omega}$. *Let* $\boldsymbol{\lambda}_{\mathbf{A}\cup\mathbf{\Lambda}}$ *be the characteristic function* $\boldsymbol{\lambda}_{\mathbf{A}\cup\mathbf{\Lambda}} : \mathbf{\Omega} \rightarrow \{\mathbf{0},\mathbf{1}\}$, *let* $\boldsymbol{\lambda}_{\mathbf{A}}$ *be the characteristic function* $\boldsymbol{\lambda}_{\mathbf{A}} : \mathbf{\Omega} \rightarrow \{\mathbf{0},\mathbf{1}\}$, *and let* $\boldsymbol{\lambda}_{\mathbf{\Lambda}}$ *be the characteristic function* $\boldsymbol{\lambda}_{\mathbf{\Lambda}} : \mathbf{\Omega} \rightarrow \{\mathbf{0},\mathbf{1}\}$.

$$\boldsymbol{\lambda}_{\mathbf{A}\cup\mathbf{\Lambda}}\left[\boldsymbol{\iota}\right] = \boldsymbol{\lambda}_{\mathbf{A}}\left[\boldsymbol{\iota}\right] + \boldsymbol{\lambda}_{\mathbf{\Lambda}}\left[\boldsymbol{\iota}\right] - \boldsymbol{\lambda}_{\mathbf{A}}\left[\boldsymbol{\iota}\right] \times \boldsymbol{\lambda}_{\mathbf{\Lambda}}\left[\boldsymbol{\iota}\right]$$

Proof. By cases. There are two major cases under consideration.

(i) Assume $\boldsymbol{\iota}$ is not an element in $\mathbf{A} \cup \mathbf{\Lambda}$. Note that, by the definition for characteristic functions, $\boldsymbol{\lambda}_{\mathbf{A}\cup\mathbf{\Lambda}}\left[\boldsymbol{\iota}\right] = \mathbf{0}$. Now, by the definition for set union $\boldsymbol{\iota}$ is in neither $\mathbf{A}$ nor $\mathbf{\Lambda}$. So $\boldsymbol{\lambda}_{\mathbf{A}}\left[\boldsymbol{\iota}\right] = \mathbf{0}$, and $\boldsymbol{\lambda}_{\mathbf{\Lambda}}\left[\boldsymbol{\iota}\right] = \mathbf{0}$. Hence,

$$\left\langle \lambda_{\mathrm{A}}\left[\iota\right] + \lambda_{\Lambda}\left[\iota\right] - \lambda_{\mathrm{A}}\left[\iota\right] \times \lambda_{\Lambda}\left[\iota\right] \right\rangle = \left\langle 0 + 0 - 0 \times 0 \right\rangle = \left\langle 0 \right\rangle$$

$\therefore\ \boldsymbol{\lambda}_{\mathbf{A}\cup\mathbf{\Lambda}}\left[\boldsymbol{\iota}\right] = \boldsymbol{\lambda}_{\mathbf{A}}\left[\boldsymbol{\iota}\right] + \boldsymbol{\lambda}_{\mathbf{\Lambda}}\left[\boldsymbol{\iota}\right] - \boldsymbol{\lambda}_{\mathbf{A}}\left[\boldsymbol{\iota}\right] \times \boldsymbol{\lambda}_{\mathbf{\Lambda}}\left[\boldsymbol{\iota}\right]$.

(ii) Assume $\boldsymbol{\iota}$ is an element in $\mathbf{A} \cup \mathbf{\Lambda}$. By the definition for characteristic functions $\boldsymbol{\lambda}_{\mathbf{A}\cup\mathbf{\Lambda}}\left[\boldsymbol{\iota}\right] = \mathbf{1}$. Also, by the definition for set union

$$\left\langle \iota \in \mathrm{A} \right\rangle \vee \left\langle \iota \in \Lambda \right\rangle$$

There are three subcases.

(a) Suppose $\boldsymbol{\iota}$ is an element in $\mathbf{A}$, but not in $\mathbf{\Lambda}$. By the definition for characteristic functions $\boldsymbol{\lambda}_{\mathbf{A}}\left[\boldsymbol{\iota}\right] = \mathbf{1}$, and $\boldsymbol{\lambda}_{\mathbf{\Lambda}}\left[\boldsymbol{\iota}\right] = \mathbf{0}$. Thus,

$$\left\langle \lambda_{\mathrm{A}}\left[\iota\right] + \lambda_{\Lambda}\left[\iota\right] - \lambda_{\mathrm{A}}\left[\iota\right] \times \lambda_{\Lambda}\left[\iota\right] \right\rangle = \left\langle 1 + 0 - 1 \times 0 \right\rangle = \left\langle 1 \right\rangle$$

(b) Suppose $\boldsymbol{\iota}$ is not an element in $\mathbf{A}$, but is an element in $\mathbf{\Lambda}$. Without loss of generality this case has the same result as case (a).

(c) Suppose $\boldsymbol{\iota}$ is in the intersection of $\mathbf{A}$ and $\mathbf{\Lambda}$.

$$\left\langle \lambda_{\mathrm{A}}\left[\iota\right] + \lambda_{\Lambda}\left[\iota\right] - \lambda_{\mathrm{A}}\left[\iota\right] \times \lambda_{\Lambda}\left[\iota\right] \right\rangle = \left\langle 1 + 1 - 1 \times 1 \right\rangle = \left\langle 1 \right\rangle$$

$\therefore\ \boldsymbol{\lambda}_{\mathbf{A}\cup\mathbf{\Lambda}}\left[\boldsymbol{\iota}\right] = \boldsymbol{\lambda}_{\mathbf{A}}\left[\boldsymbol{\iota}\right] + \boldsymbol{\lambda}_{\mathbf{\Lambda}}\left[\boldsymbol{\iota}\right] - \boldsymbol{\lambda}_{\mathbf{A}}\left[\boldsymbol{\iota}\right] \times \boldsymbol{\lambda}_{\mathbf{\Lambda}}\left[\boldsymbol{\iota}\right]$. ∎

Theorem (**2367c**). *Let* $\boldsymbol{\Lambda}$ *be a set with universal set* $\boldsymbol{\Omega}$. *Let* $\boldsymbol{\lambda}_{\overline{\boldsymbol{\Lambda}}}$ *be the characteristic function* $\boldsymbol{\lambda}_{\overline{\boldsymbol{\Lambda}}} : \boldsymbol{\Omega} \to \{\mathbf{0}, \mathbf{1}\}$.

$$\boldsymbol{\lambda}_{\overline{\boldsymbol{\Lambda}}}\big[\boldsymbol{\iota}\big] = \mathbf{1} - \boldsymbol{\lambda}_{\boldsymbol{\Lambda}}\big[\boldsymbol{\iota}\big]$$

Proof. By cases. There are two cases. Either (*i*) $\boldsymbol{\iota}$ is in $\boldsymbol{\Lambda}$, xor (*ii*) $\boldsymbol{\iota}$ is in $\overline{\boldsymbol{\Lambda}}$.

(*i*) Let $\boldsymbol{\iota}$ be an element in $\boldsymbol{\Lambda}$. By the double negation law, by the definition for set membership, by the definition for set complement, and again by the defintion for set membership, that is

$$\neg\Big[\neg\Big\langle \iota \in \Lambda \Big\rangle\Big] \equiv \neg\Big\langle \iota \notin \Lambda \Big\rangle \equiv \neg\Big\langle \iota \in \overline{\Lambda} \Big\rangle \equiv \Big\langle \iota \notin \overline{\Lambda} \Big\rangle$$

Thus, by the definition for characteristic functions

$$\Big\langle \lambda_{\overline{\Lambda}}\big[\iota\big] = 0 \Big\rangle \wedge \Big\langle \lambda_{\Lambda}\big[\iota\big] = 1 \Big\rangle$$

$\therefore\ \boldsymbol{\lambda}_{\overline{\boldsymbol{\Lambda}}}\big[\boldsymbol{\iota}\big] = \mathbf{1} - \boldsymbol{\lambda}_{\boldsymbol{\Lambda}}\big[\boldsymbol{\iota}\big]$.

(*ii*) Let $\boldsymbol{\iota}$ be an element in $\overline{\boldsymbol{\Lambda}}$. By the double negation law, by the definition for set membership, by the definition for set complement, and again by the defintion for set membership, that is

$$\neg\Big[\neg\Big\langle \iota \in \overline{\Lambda} \Big\rangle\Big] \equiv \neg\Big\langle \iota \notin \overline{\Lambda} \Big\rangle \equiv \neg\Big\langle \iota \in \Lambda \Big\rangle \equiv \Big\langle \iota \notin \Lambda \Big\rangle$$

Thus, by the definition for characteristic functions

$$\Big\langle \lambda_{\overline{\Lambda}}\big[\iota\big] = 1 \Big\rangle \wedge \Big\langle \lambda_{\Lambda}\big[\iota\big] = 0 \Big\rangle$$

$\therefore\ \boldsymbol{\lambda}_{\overline{\boldsymbol{\Lambda}}}\big[\boldsymbol{\iota}\big] = \mathbf{1} - \boldsymbol{\lambda}_{\boldsymbol{\Lambda}}\big[\boldsymbol{\iota}\big]$. ∎

Lemma (**2302**). *Let* $\boldsymbol{\lambda}$ *be a real number, such that* $\lfloor \boldsymbol{\lambda} \rfloor + \boldsymbol{\epsilon} = \boldsymbol{\lambda}$.

$$\left\{ \left\lfloor \boldsymbol{\lambda} \right\rfloor + \left\lfloor \boldsymbol{\lambda} + \frac{\mathbf{1}}{\mathbf{3}} \right\rfloor + \left\lfloor \boldsymbol{\lambda} + \frac{\mathbf{2}}{\mathbf{3}} \right\rfloor \right\} = \left\{ \mathbf{3}\boldsymbol{\lambda} - \mathbf{3}\boldsymbol{\epsilon} + \left\lfloor \boldsymbol{\epsilon} \right\rfloor + \left\lfloor \boldsymbol{\epsilon} + \frac{\mathbf{1}}{\mathbf{3}} \right\rfloor + \left\lfloor \boldsymbol{\epsilon} + \frac{\mathbf{2}}{\mathbf{3}} \right\rfloor \right\}$$

Proof. Given the real number $\boldsymbol{\lambda}$, by the properties for floor functions there exists a real number $\boldsymbol{\epsilon}$ and an integer $\boldsymbol{\lambda} - \boldsymbol{\epsilon}$ such that $\lfloor \boldsymbol{\lambda} \rfloor = \boldsymbol{\lambda} - \boldsymbol{\epsilon}$. Thus, $\boldsymbol{\lambda} = \boldsymbol{\lambda} - \boldsymbol{\epsilon} + \boldsymbol{\epsilon}$. By the identity $\boldsymbol{\lambda}$,

$$\left\lfloor \lambda \right\rfloor + \left\lfloor \lambda + \frac{1}{3} \right\rfloor + \left\lfloor \lambda + \frac{2}{3} \right\rfloor =$$

$$\left\lfloor \lambda - \epsilon + \epsilon \right\rfloor + \left\lfloor \lambda - \epsilon + \epsilon + \frac{1}{3} \right\rfloor + \left\lfloor \lambda - \epsilon + \epsilon + \frac{2}{3} \right\rfloor$$

Since $\boldsymbol{\lambda} - \boldsymbol{\epsilon}$ is an integer, by Lemma 2301 that is

$$\Big\langle \lambda - \epsilon \Big\rangle + \left\lfloor \epsilon \right\rfloor + \Big\langle \lambda - \epsilon \Big\rangle + \left\lfloor \epsilon + \frac{1}{3} \right\rfloor + \Big\langle \lambda - \epsilon \Big\rangle + \left\lfloor \epsilon + \frac{2}{3} \right\rfloor =$$

$$\Big\langle 3\lambda - 3\epsilon \Big\rangle + \left\lfloor \epsilon \right\rfloor + \left\lfloor \epsilon + \frac{1}{3} \right\rfloor + \left\lfloor \epsilon + \frac{2}{3} \right\rfloor$$

This completes the proof. ∎

Theorem (2367d). *Suppose* $\mathbf{A}$*, and* $\mathbf{\Lambda}$ *are sets with universal set* $\mathbf{\Omega}$*. Let* $\boldsymbol{\lambda}_{\mathbf{A}\oplus\mathbf{\Lambda}}$ *be the characteristic function* $\boldsymbol{\lambda}_{\mathbf{A}\oplus\mathbf{\Lambda}} : \mathbf{\Omega} \to \{\mathbf{0}, \mathbf{1}\}$*, let* $\boldsymbol{\lambda}_{\mathbf{A}}$ *be the characteristic function* $\boldsymbol{\lambda}_{\mathbf{A}} : \mathbf{\Omega} \to \{\mathbf{0}, \mathbf{1}\}$*, and let* $\boldsymbol{\lambda}_{\mathbf{\Lambda}}$ *be the characteristic function* $\boldsymbol{\lambda}_{\mathbf{\Lambda}} : \mathbf{\Omega} \to \{\mathbf{0}, \mathbf{1}\}$*.*

$$\boldsymbol{\lambda}_{\mathbf{A}\oplus\mathbf{\Lambda}}\left[\boldsymbol{\iota}\right] = \boldsymbol{\lambda}_{\mathbf{A}}\left[\boldsymbol{\iota}\right] + \boldsymbol{\lambda}_{\mathbf{\Lambda}}\left[\boldsymbol{\iota}\right] - \mathbf{2}\Big[\boldsymbol{\lambda}_{\mathbf{A}}\left[\boldsymbol{\iota}\right]\boldsymbol{\lambda}_{\mathbf{\Lambda}}\left[\boldsymbol{\iota}\right]\Big]$$

Proof. By cases. There are two cases under consideration. Either (i) $\boldsymbol{\iota}$ is an element in $\mathbf{A} \oplus \mathbf{\Lambda}$, or (ii) $\boldsymbol{\iota}$ is not an element in $\mathbf{A} \oplus \mathbf{\Lambda}$.

(i) Assume $\boldsymbol{\iota}$ is an element in $\mathbf{A} \oplus \mathbf{\Lambda}$. By the defintion for the symmetric difference of sets, that is

$$\Big[\Big\langle \iota \in \mathrm{A} \Big\rangle \wedge \Big\langle \iota \notin \Lambda \Big\rangle\Big] \vee \Big[\Big\langle \iota \notin \mathrm{A} \Big\rangle \wedge \Big\langle \iota \in \Lambda \Big\rangle\Big]$$

Without loss of generality, assume $\langle \boldsymbol{\iota} \in \mathbf{A} \rangle \wedge \langle \boldsymbol{\iota} \notin \mathbf{\Lambda} \rangle$. By the definition for characteristic functions,

$$\Big[\Big\langle \lambda_{\mathrm{A}\oplus\Lambda}\left[\iota\right] = 1 \Big\rangle \wedge \Big\langle \lambda_{\mathrm{A}}\left[\iota\right] = 1 \Big\rangle \wedge \Big\langle \lambda_{\Lambda}\left[\iota\right] = 0 \Big\rangle\Big] \to$$

$$\lambda_{\mathrm{A}\oplus\Lambda}\left[\iota\right] = \lambda_{\mathrm{A}}\left[\iota\right] + \lambda_{\Lambda}\left[\iota\right] - 2\Big[\lambda_{\mathrm{A}}\left[\iota\right]\lambda_{\Lambda}\left[\iota\right]\Big] = \Big\langle 1 + 0 - 2[1][0] \Big\rangle = \Big\langle 1 \Big\rangle$$

(ii) Assume $\boldsymbol{\iota}$ is not an element in $\mathbf{A}\oplus\mathbf{\Lambda}$. There are two subcases. (a) $\boldsymbol{\iota}$ is in the intersection of $\mathbf{A}$ and $\mathbf{\Lambda}$, xor (b) $\boldsymbol{\iota}$ is in $\mathbf{\Omega}$ minus $\mathbf{A} \cup \mathbf{\Lambda}$.

(a) Assume $\boldsymbol{\iota}$ is in the intersection of $\mathbf{A}$ and $\mathbf{\Lambda}$. Thus, by the definition for characteristic functions,

$$\Big[\Big\langle \lambda_{\mathrm{A}\oplus\Lambda}\left[\iota\right] = 0 \Big\rangle \wedge \Big\langle \lambda_{\mathrm{A}}\left[\iota\right] = 1 \Big\rangle \wedge \Big\langle \lambda_{\Lambda}\left[\iota\right] = 1 \Big\rangle\Big] \to$$

$$\lambda_{\mathrm{A}\oplus\Lambda}\left[\iota\right] = \lambda_{\mathrm{A}}\left[\iota\right] + \lambda_{\Lambda}\left[\iota\right] - 2\Big[\lambda_{\mathrm{A}}\left[\iota\right]\lambda_{\Lambda}\left[\iota\right]\Big] = \Big\langle 1 + 1 - 2[1][1] \Big\rangle = \Big\langle 0 \Big\rangle$$

(b) Assume $\boldsymbol{\iota}$ is in $\mathbf{\Omega}$ minus $\mathbf{A} \cup \mathbf{\Lambda}$. By the definition for characteristic functions,

$$\Big[\Big\langle \lambda_{\mathrm{A}\oplus\Lambda}\left[\iota\right] = 0 \Big\rangle \wedge \Big\langle \lambda_{\mathrm{A}}\left[\iota\right] = 0 \Big\rangle \wedge \Big\langle \lambda_{\Lambda}\left[\iota\right] = 0 \Big\rangle\Big] \to$$

$$\lambda_{\mathrm{A}\oplus\Lambda}\left[\iota\right] = \lambda_{\mathrm{A}}\left[\iota\right] + \lambda_{\Lambda}\left[\iota\right] - 2\Big[\lambda_{\mathrm{A}}\left[\iota\right]\lambda_{\Lambda}\left[\iota\right]\Big] = \Big\langle 0 + 0 - 2[0][0] \Big\rangle = \Big\langle 0 \Big\rangle$$

This completes the proof. ∎

Theorem (2369c). *Let $\boldsymbol{\lambda}$ and $\boldsymbol{\iota}$ be real numbers.*

$$\boldsymbol{\lceil\lambda\rceil + \lceil\iota\rceil - \lceil\lambda+\iota\rceil = 0,\ \textit{or}\ 1}$$

Proof. By cases. There are two possible cases. (*i*) $\boldsymbol{\lambda}$ or $\boldsymbol{\iota}$ (or both) are integers, or (*ii*) neither $\boldsymbol{\lambda}$ nor $\boldsymbol{\iota}$ is an integer.

(i) Assume $\boldsymbol{\lambda}$ or $\boldsymbol{\iota}$ (or both) are integers. Without loss of generality $\boldsymbol{\iota}$ is an integer. By Theorem 2346,

$$\Big[\lceil\lambda\rceil + \lceil\iota\rceil - \lceil\lambda+\iota\rceil\Big] = \Big[\lceil\lambda\rceil + \lceil\iota\rceil - \lceil\lambda\rceil + \lceil\iota\rceil\Big] = 0$$

(ii) Assume neither $\boldsymbol{\lambda}$ nor $\boldsymbol{\iota}$ is an integer. There exist real numbers $\boldsymbol{\epsilon}$ and $\boldsymbol{\sigma}$ such that $\boldsymbol{\lceil\lambda\rceil - \lambda = \epsilon}$, and $\boldsymbol{\lceil\iota\rceil - \iota = \sigma}$. By Theorem 2344, $\boldsymbol{\lceil\lambda\rceil = \lfloor\lambda\rfloor + 1}$, and $\boldsymbol{\lceil\iota\rceil = \lfloor\iota\rfloor + 1}$. Hence, the identities for $\boldsymbol{\lambda}$, and $\boldsymbol{\iota}$ are

$$\Big[\Big\langle\lambda\Big\rangle = \Big\{\lfloor\lambda\rfloor + \langle 1-\epsilon\rangle\Big\}\Big] \wedge \Big[\Big\langle\iota\Big\rangle = \Big\{\lfloor\iota\rfloor + \langle 1-\sigma\rangle\Big\}\Big]$$

Thus,

$$\boldsymbol{\lceil\lambda\rceil + \lceil\iota\rceil - \lceil\lambda+\iota\rceil} \equiv$$

$$\Big\langle\lfloor\lambda\rfloor + 1\Big\rangle + \Big\langle\lfloor\iota\rfloor + 1\Big\rangle - \Bigg\lceil\Big\{\lfloor\lambda\rfloor + \langle 1-\epsilon\rangle\Big\} + \Big\{\lfloor\iota\rfloor + \langle 1-\sigma\rangle\Big\}\Bigg\rceil \equiv$$

$$\Big\langle\lfloor\lambda\rfloor + \lfloor\iota\rfloor + 2\Big\rangle - \Bigg\lceil\lfloor\lambda\rfloor + \lfloor\iota\rfloor + \Big[2 - \langle\epsilon+\sigma\rangle\Big]\Bigg\rceil$$

There are two possible subcases. Either (*a*) $\boldsymbol{\epsilon + \sigma \geq 1}$, or (*b*) $\boldsymbol{\epsilon + \sigma < 1}$.

(a) Assume $\boldsymbol{\epsilon + \sigma \geq 1}$. By the additive compatibility law from the order axioms, $\boldsymbol{1 \geq 2 - \langle\epsilon+\sigma\rangle}$. This means that $\mathbf{1}$ is the smallest integer that is greater than or equal to $\boldsymbol{2 - \langle\epsilon+\sigma\rangle}$. Thus, by the definition for ceiling functions, and by Theorem 2369a,

$$\Big[\lceil\lambda\rceil + \lceil\iota\rceil - \lceil\lambda+\iota\rceil\Big] \equiv \Big\langle\lfloor\lambda\rfloor + \lfloor\iota\rfloor + 2\Big\rangle - \Big\lceil\lfloor\lambda\rfloor + \lfloor\iota\rfloor + 1\Big\rceil = 1$$

(b) Assume $\boldsymbol{\epsilon + \sigma < 1}$. By the additive compatibility law from the order axioms, $\boldsymbol{1 < 2 - \langle\epsilon+\sigma\rangle}$. This means that 2 is the smallest integer that is greater than or equal to $\boldsymbol{2 - \langle\epsilon+\sigma\rangle}$. Thus, by the definition for ceiling functions, and by Theorem 2369a,

$$\Big[\lceil\lambda\rceil + \lceil\iota\rceil - \lceil\lambda+\iota\rceil\Big] \equiv \Big\langle\lfloor\lambda\rfloor + \lfloor\iota\rfloor + 2\Big\rangle - \Big\lceil\lfloor\lambda\rfloor + \lfloor\iota\rfloor + 2\Big\rceil = 0$$

$\therefore$ $\boldsymbol{\lceil\lambda\rceil + \lceil\iota\rceil - \lceil\lambda+\iota\rceil = 0}$**, or 1**, whenever $\boldsymbol{\lambda}$ and $\boldsymbol{\iota}$ are real numbers. ∎

Theorem (2370c). *Let $\boldsymbol{\iota}$ be a real number.*

$$\left\lceil \left\lceil \frac{\boldsymbol{\iota}}{\mathbf{2}} \right\rceil \div \mathbf{2} \right\rceil = \left\lceil \frac{\boldsymbol{\iota}}{\mathbf{4}} \right\rceil$$

Proof. By cases. By the properties for ceiling functions there exists an integer $\boldsymbol{\lambda}$ such that $\left\lceil \frac{\boldsymbol{\iota}}{\mathbf{4}} \right\rceil = \boldsymbol{\lambda}$, (and by the multiplicative compatibility laws from the order axioms,) if and only if

$$\left[\left\langle \lambda - 1 \right\rangle < \left\langle \frac{\iota}{4} \right\rangle \leq \left\langle \lambda \right\rangle\right] \equiv \left[\left\langle 2\lambda - 2 \right\rangle < \left\langle \frac{\iota}{2} \right\rangle \leq \left\langle 2\lambda \right\rangle\right]$$

There are two cases to consider. by the definition for the ceiling function, $\left\lceil \frac{\boldsymbol{\iota}}{\mathbf{2}} \right\rceil$ is the integer (*i*) $\mathbf{2}\boldsymbol{\lambda}$, or (*ii*) $\mathbf{2}\boldsymbol{\lambda} - \mathbf{1}$,

(i) If $\left\lceil \frac{\boldsymbol{\iota}}{\mathbf{2}} \right\rceil$ is $\mathbf{2}\boldsymbol{\lambda}$, then the proof is trivial. By the identity $\boldsymbol{\lambda}$, the multiplicative inverse from the field axioms, and by the definition of ceiling functions,

$$\left\lceil \left\lceil \frac{\iota}{2} \right\rceil \div 2 \right\rceil = \left\langle \lceil 2\lambda \div 2 \rceil \right\rangle = \left\langle \lceil \lambda \rceil \right\rangle = \left\langle \lambda \right\rangle$$

(ii) Suppose $\left\lceil \frac{\boldsymbol{\iota}}{\mathbf{2}} \right\rceil = \mathbf{2}\boldsymbol{\lambda} - \mathbf{1}$. By the properties for ceiling functions, and by the law of transitivity from the order axioms,

$$\left[\left\langle 2\lambda - 2 \right\rangle < \left\langle \left\lceil \frac{\iota}{2} \right\rceil \right\rangle \leq \left\langle 2\lambda - 1 \right\rangle\right] \to \left[\left\langle 2\lambda - 2 \right\rangle < \left\langle \left\lceil \frac{\iota}{2} \right\rceil \right\rangle \leq \left\langle 2\lambda \right\rangle\right]$$

Thus, by the multiplicative compatibility law from the order axioms,

$$\left[\left\langle 2\lambda - 2 \right\rangle < \left\langle \left\lceil \frac{\iota}{2} \right\rceil \right\rangle \leq \left\langle 2\lambda \right\rangle\right] \equiv \left[\left\langle \lambda - 1 \right\rangle < \left\langle \left\lceil \frac{\iota}{2} \right\rceil \div 2 \right\rangle \leq \left\langle \lambda \right\rangle\right]$$

By the properties for ceiling functions, $\left\lceil \left\lceil \frac{\iota}{2} \right\rceil \div 2 \right\rceil = \lambda$.

$\therefore \left\lceil \left\lceil \frac{\boldsymbol{\iota}}{\mathbf{2}} \right\rceil \div \mathbf{2} \right\rceil = \left\lceil \frac{\boldsymbol{\iota}}{\mathbf{4}} \right\rceil$ ∎

Lemma (2304). *Let $\boldsymbol{\lambda}$ be a real number, such that $\lfloor \boldsymbol{\lambda} \rfloor + \boldsymbol{\epsilon} = \boldsymbol{\lambda}$.*

$$\left\lfloor \mathbf{3}\boldsymbol{\lambda} \right\rfloor = \left\lfloor \boldsymbol{\lambda} \right\rfloor + \left\lfloor \boldsymbol{\lambda} + \frac{\mathbf{1}}{\mathbf{3}} \right\rfloor + \left\lfloor \boldsymbol{\lambda} + \frac{\mathbf{2}}{\mathbf{3}} \right\rfloor$$

if and only if

$$\left\lfloor \mathbf{3}\boldsymbol{\epsilon} \right\rfloor = \left\lfloor \boldsymbol{\epsilon} \right\rfloor + \left\lfloor \boldsymbol{\epsilon} + \frac{\mathbf{1}}{\mathbf{3}} \right\rfloor + \left\lfloor \boldsymbol{\epsilon} + \frac{\mathbf{2}}{\mathbf{3}} \right\rfloor$$

Proof. By Lemma 2302, and 2303, the left-hand side of the equivalence is

$$\left\{\left\langle 3\lambda - 3\epsilon \right\rangle + \left\lfloor 3\epsilon \right\rfloor\right\} = \left\{\left\langle 3\lambda - 3\epsilon \right\rangle + \left\lfloor \epsilon \right\rfloor + \left\lfloor \epsilon + \frac{1}{3} \right\rfloor + \left\lfloor \epsilon + \frac{2}{3} \right\rfloor\right\}$$

The right-hand side of the equivalence follows immediately from the inverse law of addition. ∎

Theorem (2370e). *Let $\boldsymbol{\lambda}$, and $\boldsymbol{\iota}$ be real numbers.*

$$\lfloor \boldsymbol{\lambda} \rfloor + \lfloor \boldsymbol{\iota} \rfloor + \lfloor \boldsymbol{\lambda} + \boldsymbol{\iota} \rfloor \leq \lfloor \mathbf{2}\boldsymbol{\lambda} \rfloor + \lfloor \mathbf{2}\boldsymbol{\iota} \rfloor$$

Proof. By cases. There exist real numbers $\boldsymbol{\epsilon}$ and $\boldsymbol{\sigma}$ such that $\boldsymbol{\lambda} - \lfloor \boldsymbol{\lambda} \rfloor = \boldsymbol{\epsilon}$. By the property for floor functions, $\lfloor \boldsymbol{\lambda} \rfloor = \boldsymbol{\lambda} - \boldsymbol{\epsilon}$, if and only if

$$\Big\langle \lambda - \epsilon \Big\rangle \leq \Big\langle \lambda \Big\rangle < \Big\langle \lambda - \epsilon + 1 \Big\rangle$$

Without loss of generality with respect to $\boldsymbol{\iota}$, by the additive compatibility law from the order axioms, there exists an integer $\lfloor \boldsymbol{\lambda} + \boldsymbol{\iota} \rfloor = \langle \boldsymbol{\lambda} - \boldsymbol{\epsilon} \rangle + \langle \boldsymbol{\iota} - \boldsymbol{\sigma} \rangle$, if and only if

$$\Big\langle \lambda - \epsilon + \iota - \sigma \Big\rangle \leq \Big\langle \lambda + \iota \Big\rangle < \Big\langle \lambda - \epsilon + \iota - \sigma + 1 \Big\rangle$$

Thus, by the identities for the floor of $\boldsymbol{\lambda}$, the floor of $\boldsymbol{\iota}$, and the floor of the sum of $\boldsymbol{\lambda}$ and $\boldsymbol{\iota}$, we deduce

$$\lfloor \lambda \rfloor + \lfloor \iota \rfloor + \lfloor \lambda + \iota \rfloor = \Big[\langle \lambda - \epsilon \rangle + \langle \iota - \sigma \rangle + \langle \lambda - \epsilon + \iota - \sigma \rangle \Big] =$$

$$2\langle \lambda + \iota \rangle - 2\langle \epsilon + \sigma \rangle$$

Now, by multiplicative compatibility, and transitivity from the order axioms,

$$\Big\langle \lambda - \epsilon \Big\rangle \leq \Big\langle \lambda \Big\rangle < \Big\langle \lambda - \epsilon + 1 \Big\rangle \equiv \Big\langle 2\lambda - 2\epsilon \Big\rangle \leq \Big\langle 2\lambda \Big\rangle < \Big\langle 2\lambda - 2\epsilon + 2 \Big\rangle \equiv$$

$$\Big\langle 2\big[\lambda - \epsilon\big] \Big\rangle \leq \Big\langle 2\lambda \Big\rangle \leq \Big\langle 2\big[\lambda - \epsilon\big] + 1 \Big\rangle$$

So $\lfloor \mathbf{2}\boldsymbol{\lambda} \rfloor$ is the integer (*i*) $\mathbf{2}\langle \boldsymbol{\lambda} - \boldsymbol{\epsilon} \rangle$, or (*ii*) $\mathbf{2}\langle \boldsymbol{\lambda} - \boldsymbol{\epsilon} \rangle + \mathbf{1}$, by the properties for the floor function.

(i) Let $\lfloor \mathbf{2}\boldsymbol{\lambda} \rfloor = \mathbf{2}\langle \boldsymbol{\lambda} - \boldsymbol{\epsilon} \rangle$. Without loss of generality with respect to $\boldsymbol{\iota}$,

$$\Big\{ \lfloor 2\lambda \rfloor + \lfloor 2\iota \rfloor \Big\} = \Big\{ 2\langle \lambda - \epsilon \rangle + 2\langle \iota - \sigma \rangle \Big\} = \Big\{ 2\langle \lambda + \iota \rangle - 2\langle \epsilon + \sigma \rangle \Big\} =$$

$$\Big\{ \lfloor \lambda \rfloor + \lfloor \iota \rfloor + \lfloor \lambda + \iota \rfloor \Big\}$$

(ii) Let $\lfloor \mathbf{2}\boldsymbol{\lambda} \rfloor = \mathbf{2}\langle \boldsymbol{\lambda} - \boldsymbol{\epsilon} \rangle + \mathbf{1}$. Without loss of generality with respect to $\boldsymbol{\iota}$,

$$\Big\{ \lfloor 2\lambda \rfloor + \lfloor 2\iota \rfloor \Big\} = \Big\{ 2\langle \lambda - \epsilon \rangle + 1 + 2\langle \iota - \sigma \rangle + 1 \Big\} = \Big\{ 2\langle \lambda + \iota + 1 \rangle - 2\langle \epsilon + \sigma \rangle \Big\}$$

$$> \Big\{ \lfloor \lambda \rfloor + \lfloor \iota \rfloor + \lfloor \lambda + \iota \rfloor \Big\}$$

$\therefore\ \lfloor \boldsymbol{\lambda} \rfloor + \lfloor \boldsymbol{\iota} \rfloor + \lfloor \boldsymbol{\lambda} + \boldsymbol{\iota} \rfloor \leq \lfloor \mathbf{2}\boldsymbol{\lambda} \rfloor + \lfloor \mathbf{2}\boldsymbol{\iota} \rfloor$. ∎

Theorem (2371a). *Let $\boldsymbol{\lambda}$ be a positive real number.*

$$\left\lfloor \sqrt{\lfloor \boldsymbol{\lambda} \rfloor} \right\rfloor = \left\lfloor \sqrt{\boldsymbol{\lambda}} \right\rfloor$$

Proof. By the properties for floor functions, there exists an integer $\lfloor \sqrt{\lfloor \boldsymbol{\lambda} \rfloor} \rfloor$ such that $\lfloor \sqrt{\boldsymbol{\lambda}} \rfloor = \lfloor \sqrt{\lfloor \boldsymbol{\lambda} \rfloor} \rfloor$, if and only if

$$\left\langle \left\lfloor \sqrt{\lfloor \lambda \rfloor} \right\rfloor \right\rangle \le \left\langle \sqrt{\lambda} \right\rangle < \left\langle \left\lfloor \sqrt{\lfloor \lambda \rfloor} \right\rfloor + 1 \right\rangle$$

By the multiplicative compatibility law from the order axioms, that is

$$\left\langle \left\lfloor \sqrt{\lfloor \lambda \rfloor} \right\rfloor \right\rangle^2 \le \left\langle \lambda \right\rangle < \left\langle \left\lfloor \sqrt{\lfloor \lambda \rfloor} \right\rfloor + 1 \right\rangle^2$$

$\lfloor \boldsymbol{\lambda} \rfloor$ is the largest integer that is less than or equal $\boldsymbol{\lambda}$, so by the definition of the floor function, $\lfloor \boldsymbol{\lambda} \rfloor \le \boldsymbol{\lambda}$. Also, $\lfloor \sqrt{\lfloor \boldsymbol{\lambda} \rfloor} \rfloor^{\mathbf{2}}$ is an integer by the defintion of floor functions, since integers are closed under multiplication. Hence, $\lfloor \sqrt{\lfloor \boldsymbol{\lambda} \rfloor} \rfloor^{\mathbf{2}} \le \lfloor \boldsymbol{\lambda} \rfloor$. So by the transitivity law from the order axioms,

$$\left\langle \left\lfloor \sqrt{\lfloor \lambda \rfloor} \right\rfloor \right\rangle^2 \le \left\langle \lfloor \lambda \rfloor \right\rangle \le \left\langle \lambda \right\rangle < \left\langle \left\lfloor \sqrt{\lfloor \lambda \rfloor} \right\rfloor + 1 \right\rangle^2 \equiv$$

$$\left\langle \left\lfloor \sqrt{\lfloor \lambda \rfloor} \right\rfloor \right\rangle^2 \le \left\langle \lfloor \lambda \rfloor \right\rangle < \left\langle \left\lfloor \sqrt{\lfloor \lambda \rfloor} \right\rfloor + 1 \right\rangle^2$$

By the multiplicative compatibility law from the order axioms, the following is an equivalent statement,

$$\left\langle \left\lfloor \sqrt{\lfloor \lambda \rfloor} \right\rfloor \right\rangle \le \left\langle \sqrt{\lfloor \lambda \rfloor} \right\rangle < \left\langle \left\lfloor \sqrt{\lfloor \lambda \rfloor} \right\rfloor + 1 \right\rangle$$

$\therefore$ $\left\lfloor \sqrt{\lfloor \boldsymbol{\lambda} \rfloor} \right\rfloor = \left\lfloor \sqrt{\boldsymbol{\lambda}} \right\rfloor$, by the properties of floor functions. ∎

Theorem (2371b). *Let $\boldsymbol{\lambda}$ be a positive real number.*

$$\left\lceil \sqrt{\lceil \boldsymbol{\lambda} \rceil} \right\rceil = \left\lceil \sqrt{\boldsymbol{\lambda}} \right\rceil$$

Proof. By the properties for ceiling functions, there exists an integer $\lceil \sqrt{\lceil \boldsymbol{\lambda} \rceil} \rceil$ such that $\lceil \sqrt{\boldsymbol{\lambda}} \rceil = \lceil \sqrt{\lceil \boldsymbol{\lambda} \rceil} \rceil$, if and only if

$$\left\langle \left\lceil \sqrt{\lceil \lambda \rceil} \right\rceil - 1 \right\rangle < \left\langle \sqrt{\lambda} \right\rangle \leq \left\langle \left\lceil \sqrt{\lceil \lambda \rceil} \right\rceil \right\rangle$$

By the multiplicative compatibility law from the order axioms, that is

$$\left\langle \left\lceil \sqrt{\lceil \lambda \rceil} \right\rceil - 1 \right\rangle^2 < \left\langle \lambda \right\rangle \leq \left\langle \left\lceil \sqrt{\lceil \lambda \rceil} \right\rceil \right\rangle^2$$

$\lceil \boldsymbol{\lambda} \rceil$ is the smallest integer that is greater than or equal to $\boldsymbol{\lambda}$, so by the definition of the ceiling function, $\boldsymbol{\lambda} \leq \lceil \boldsymbol{\lambda} \rceil$. Also, $\lceil \sqrt{\lceil \boldsymbol{\lambda} \rceil} \rceil^{\mathbf{2}}$ is an integer by the definition of floor functions, since integers are closed under multiplication. Hence, $\lceil \boldsymbol{\lambda} \rceil \leq \lceil \sqrt{\lceil \boldsymbol{\lambda} \rceil} \rceil^{\mathbf{2}}$. So by the transitivity law from the order axioms,

$$\left\langle \left\lceil \sqrt{\lceil \lambda \rceil} \right\rceil - 1 \right\rangle^2 < \left\langle \lambda \right\rangle \leq \left\langle \lceil \lambda \rceil \right\rangle \leq \left\langle \left\lceil \sqrt{\lceil \lambda \rceil} \right\rceil \right\rangle^2 \equiv$$

$$\left\langle \left\lceil \sqrt{\lceil \lambda \rceil} \right\rceil - 1 \right\rangle^2 < \left\langle \lceil \lambda \rceil \right\rangle \leq \left\langle \left\lceil \sqrt{\lceil \lambda \rceil} \right\rceil \right\rangle^2$$

By the multiplicative compatibility law from the order axioms, the following is an equivalent statement,

$$\left\langle \left\lceil \sqrt{\lceil \lambda \rceil} \right\rceil - 1 \right\rangle < \left\langle \sqrt{\lceil \lambda \rceil} \right\rangle \leq \left\langle \left\lceil \sqrt{\lceil \lambda \rceil} \right\rceil \right\rangle$$

$\therefore \left\lceil \sqrt{\lceil \boldsymbol{\lambda} \rceil} \right\rceil = \left\lceil \sqrt{\boldsymbol{\lambda}} \right\rceil$, by the properties of ceiling functions. ∎

Theorem (2372). *Let* $\boldsymbol{\lambda}$ *be a real number, such that* $\lfloor \boldsymbol{\lambda} \rfloor + \boldsymbol{\epsilon} = \boldsymbol{\lambda}$.

$$\left\lfloor \mathbf{3}\boldsymbol{\lambda} \right\rfloor = \left\lfloor \boldsymbol{\lambda} \right\rfloor + \left\lfloor \boldsymbol{\lambda} + \frac{\mathbf{1}}{\mathbf{3}} \right\rfloor + \left\lfloor \boldsymbol{\lambda} + \frac{\mathbf{2}}{\mathbf{3}} \right\rfloor$$

Proof. By cases. By Lemma 2304, it is sufficent to prove

$$\left\lfloor 3\epsilon \right\rfloor = \left\langle \mathrm{A} = \left\lfloor \epsilon \right\rfloor \right\rangle + \left\langle \Lambda = \left\lfloor \epsilon + \frac{1}{3} \right\rfloor \right\rangle + \left\langle \Delta = \left\lfloor \epsilon + \frac{2}{3} \right\rfloor \right\rangle$$

Let $\boldsymbol{p}$ be the propostion: $\boldsymbol{\Delta} \geq \boldsymbol{\Lambda} \geq \mathbf{A} \geq \mathbf{0}$. The proof for which is trivial. There are three cases:

(i) $\left\langle 0 \right\rangle \leq \left\langle \epsilon \right\rangle < \left\langle \frac{1}{3} \right\rangle$

(ii) $\left\langle \frac{1}{3} \right\rangle \leq \left\langle \epsilon \right\rangle < \left\langle \frac{2}{3} \right\rangle$

(iii) $\left\langle \frac{2}{3} \right\rangle \leq \left\langle \epsilon \right\rangle < \left\langle 1 \right\rangle$

(i) Since $\boldsymbol{p}$, $\boldsymbol{\Delta}$ is sufficient for inferring $\mathbf{A}$, and $\boldsymbol{\Lambda}$ in this case. By the additive compatibility law from the order axioms, and by the law of transitivity from the order axioms,

$$\left[\left\langle \frac{0}{3} + \frac{2}{3} \right\rangle \leq \left\langle \epsilon + \frac{2}{3} \right\rangle < \left\langle \frac{1}{3} + \frac{2}{3} \right\rangle \right] \equiv \left[\left\langle 0 \right\rangle \leq \left\langle \epsilon + \frac{2}{3} \right\rangle < \left\langle 1 \right\rangle \right]$$

Thus, $\boldsymbol{\Delta} = \mathbf{0}$, by the properties of floor functions. Hence, $\mathbf{A} + \boldsymbol{\Lambda} + \boldsymbol{\Delta} = \mathbf{0}$, by $\boldsymbol{p}$. Also, by the multiplicative compatibility law from the order axioms, and by the properties of floor functions,

$$\left[\left\langle 3 \cdot 0 \right\rangle \leq \left\langle 3 \cdot \epsilon \right\rangle < \left\langle 3 \cdot \frac{1}{3} \right\rangle \right] \equiv \left[\lfloor 3\epsilon \rfloor = 0 \right]$$

$\therefore$ $\lfloor \mathbf{3}\boldsymbol{\epsilon} \rfloor = \mathbf{A} + \boldsymbol{\Lambda} + \boldsymbol{\Delta}$, by the identity $\mathbf{0}$, in this case.

(ii) Since $\boldsymbol{p}$, $\boldsymbol{\Lambda}$ is sufficent for inferring $\mathbf{A}$ in this case. By the additive compatibility law from the order axioms, and by the law of transitivity,

$$\left[\left\langle \frac{1}{3} + \frac{1}{3} \right\rangle \leq \left\langle \epsilon + \frac{1}{3} \right\rangle < \left\langle \frac{2}{3} + \frac{1}{3} \right\rangle \right] \equiv \left[\left\langle 0 \right\rangle \leq \left\langle \epsilon + \frac{1}{3} \right\rangle < \left\langle 1 \right\rangle \right]$$

Thus, $\boldsymbol{\Lambda} = \mathbf{0}$, by the properties of floor functions. Hence, $\mathbf{A} + \boldsymbol{\Lambda} = \mathbf{0}$, by $\boldsymbol{p}$. Now, for $\boldsymbol{\Delta}$, by the additive compatibility law from the order axioms, and by the law of transitivity from the order axioms,

$$\left[\left\langle \frac{1}{3} + \frac{2}{3} \right\rangle \leq \left\langle \epsilon + \frac{2}{3} \right\rangle < \left\langle \frac{2}{3} + \frac{2}{3} \right\rangle \right] \equiv \left[\left\langle 1 \right\rangle \leq \left\langle \epsilon + \frac{2}{3} \right\rangle < \left\langle 1 + 1 \right\rangle \right]$$

Thus, $\boldsymbol{\Delta} = \mathbf{1}$, by the properties of floor functions. Hence, $\mathbf{A} + \boldsymbol{\Lambda} + \boldsymbol{\Delta} = \mathbf{1}$. Also, by the multiplicative compatibility law from the order axioms, and by the properties of floor functions,

$$\left[\left\langle 3 \cdot \frac{1}{3} \right\rangle \leq \left\langle 3 \cdot \epsilon \right\rangle < \left\langle 3 \cdot \frac{2}{3} \right\rangle\right] \equiv$$

$$\left[\left\langle 1 \right\rangle \leq \left\langle 3\epsilon \right\rangle < \left\langle 1+1 \right\rangle\right] \equiv \left[\lfloor 3\epsilon \rfloor = 1\right]$$

$\therefore$ $\boldsymbol{\lfloor 3\epsilon \rfloor = A + \Lambda + \Delta}$, by the identity **1**, in this case.

(iii) $\mathbf{A = 0}$ can be inferred from the law of transitivity from the order axioms, and by the properties of floor functions, in this case. For $\boldsymbol{\Lambda}$, by the additive compatibility law from the order axioms, and by the law of transitivity from the order axioms,

$$\left[\left\langle \frac{2}{3} + \frac{1}{3} \right\rangle \leq \left\langle \epsilon + \frac{1}{3} \right\rangle < \left\langle \frac{3}{3} + \frac{1}{3} \right\rangle\right] \equiv \left[\left\langle 1 \right\rangle \leq \left\langle \epsilon + \frac{1}{3} \right\rangle < \left\langle 1+1 \right\rangle\right]$$

Thus, $\boldsymbol{\Lambda = 1}$, by the properties of floor functions. For $\boldsymbol{\Delta}$, by the additive compatibility law from the order axioms, and by the law of transitivity from the order axioms,

$$\left[\left\langle \frac{2}{3} + \frac{2}{3} \right\rangle \leq \left\langle \epsilon + \frac{2}{3} \right\rangle < \left\langle \frac{3}{3} + \frac{2}{3} \right\rangle\right] \equiv \left[\left\langle 1 \right\rangle \leq \left\langle \epsilon + \frac{2}{3} \right\rangle < \left\langle 1+1 \right\rangle\right]$$

Thus, $\boldsymbol{\Delta = 1}$, by the properties of floor functions. Hence, $\boldsymbol{A + \Lambda + \Delta = 2}$. Also, by the multiplicative compatibility law from the order axioms, and by the properties of floor functions,

$$\left[\left\langle 3 \cdot \frac{2}{3} \right\rangle \leq \left\langle 3 \cdot \epsilon \right\rangle < \left\langle 3 \cdot 1 \right\rangle\right] \equiv$$

$$\left[\left\langle 2 \right\rangle \leq \left\langle 3\epsilon \right\rangle < \left\langle 2+1 \right\rangle\right] \equiv \left[\lfloor 3\epsilon \rfloor = 2\right]$$

$\therefore$ $\boldsymbol{\lfloor 3\epsilon \rfloor = A + \Lambda + \Delta}$, by the identity **2**, in this case.

This completes the proof. ■

Chapter 4

Sequences and Summations

Theorem (**2419**)**.** *Let* $\{\boldsymbol{\lambda_\zeta}\}$ *be a sequence of real numbers.*

$$\boldsymbol{\sum_{\iota=1}^{\zeta} \Big\langle \lambda_\iota - \lambda_{\iota-1} \Big\rangle = \lambda_\zeta - \lambda_0}$$

Proof.

$$\sum_{\iota=1}^{\zeta} \Big\langle \lambda_\iota - \lambda_{\iota-1} \Big\rangle = \Big\langle \lambda_\zeta - \lambda_{\zeta-1} \Big\rangle + \Big\langle \lambda_{\zeta-1} - \lambda_{\zeta-2} \Big\rangle + \cdots + \Big\langle \lambda_1 - \lambda_0 \Big\rangle$$

By associativity for addition from the field axioms for real numbers, that is

$$\lambda_\zeta + \Big\langle -\lambda_{\zeta-1} + \lambda_{\zeta-1} \Big\rangle + \Big\langle -\lambda_{\zeta-2} + \lambda_{\zeta-2} \Big\rangle + \cdots + \Big\langle -\lambda_1 + \lambda_1 \Big\rangle + -\lambda_0$$

The inner terms cancel out by the inverse law for addition from the field axioms. $\therefore$

$$\boldsymbol{\sum_{\iota=1}^{\zeta} \Big\langle \lambda_\iota - \lambda_{\iota-1} \Big\rangle = \lambda_\zeta - \lambda_0}$$

■

Theorem (**2436**)**.** *A subset of a countable set is countable.*

Proof. Let $\mathbf{A}$ and $\boldsymbol{\Lambda}$ be sets such that $\mathbf{A}$ is a subset of the countable set $\boldsymbol{\Lambda}$. By the definition for countability, the cardinality of $\boldsymbol{\Lambda}$ is less than or equal to $\boldsymbol{\aleph_0}$. By the definition for subsets, the cardinality of $\mathbf{A}$ is less than or equal to $\boldsymbol{\Lambda}$. Hence, the cardinality of $\mathbf{A}$ is less than or equal to $\boldsymbol{\aleph_0}$ $\therefore$ the subset of a countable set is countable. ■

Lemma (2403). *Let $\boldsymbol{\iota}$ be a natural number such that $\lfloor\sqrt{\boldsymbol{\iota}}\rfloor = \boldsymbol{\lambda}$.*

$$\sum_{\boldsymbol{\iota}=1}^{2\boldsymbol{\lambda}+1} \lfloor\sqrt{\boldsymbol{\iota}}\rfloor = \boldsymbol{\lambda}[2\boldsymbol{\lambda}+1]$$

Proof. It is trivial that

$$\sum_{\iota=1}^{2\lambda+1} 1 = 2\lambda + 1$$

By the inverse law for multiplication from the field axioms, by the distributive law for real numbers, and by the identity $\boldsymbol{\lambda}$, that is

$$\left\{\lambda \sum_{\iota=1}^{2\lambda+1} 1 = \lambda[2\lambda+1]\right\} \equiv \left\{\sum_{\iota=1}^{2\lambda+1} \lambda = \lambda[2\lambda+1]\right\} \equiv \left\{\sum_{\iota=1}^{2\lambda+1} \lfloor\sqrt{\iota}\rfloor = \lambda[2\lambda+1]\right\}$$

■

Theorem (2420).

$$\sum_{\boldsymbol{\epsilon}=1}^{\boldsymbol{\lambda}} \left\langle \frac{1}{\boldsymbol{\epsilon}[\boldsymbol{\epsilon}+1]} \right\rangle = \frac{\boldsymbol{\lambda}}{\boldsymbol{\lambda}+1}$$

Proof. The identity $\left\langle \frac{\mathbf{1}}{\boldsymbol{\epsilon}[\boldsymbol{\epsilon}+\mathbf{1}]} \right\rangle$ is $\left\langle \frac{\mathbf{1}}{\boldsymbol{\epsilon}} - \frac{\mathbf{1}}{[\boldsymbol{\epsilon}+\mathbf{1}]} \right\rangle$. This can be demonstrated by the equation

$$\epsilon\left\langle \frac{1}{\epsilon} - \frac{1}{[\epsilon+1]} \right\rangle = \left\langle \frac{\epsilon+1}{\epsilon+1} - \frac{\epsilon}{\epsilon+1} \right\rangle = \left\langle \frac{\epsilon+1-\epsilon}{\epsilon+1} \right\rangle = \left\langle \frac{1}{\epsilon+1} \right\rangle$$

Dividing both sides of this equation by $\boldsymbol{\epsilon}$, by the inverse law for multiplication from the field axioms, gives the desired identity such that

$$\left\langle \sum_{\epsilon=1}^{\lambda} \frac{1}{\epsilon[\epsilon+1]} \right\rangle = \left\langle \sum_{\epsilon=1}^{\lambda} \frac{1}{\epsilon} - \frac{1}{\epsilon+1} \right\rangle$$

The sequence for which is the telescopic summation

$$\left\langle \frac{1}{\lambda} - \frac{1}{\lambda+1} \right\rangle + \left\langle \frac{1}{\lambda-1} - \frac{1}{\lambda} \right\rangle + \left\langle \frac{1}{\lambda-2} - \frac{1}{\lambda-1} \right\rangle + \cdots + \left\langle \frac{1}{1} - \frac{1}{2} \right\rangle$$

Thus, by Theorem 2419

$$\left\langle \sum_{\boldsymbol{\epsilon}=1}^{\boldsymbol{\lambda}} \frac{1}{\boldsymbol{\epsilon}[\boldsymbol{\epsilon}+1]} \right\rangle = \left\langle -\frac{1}{\boldsymbol{\lambda}+1} + \frac{1}{1} \right\rangle = \left\langle \frac{[-1]+[\boldsymbol{\lambda}+1]}{\boldsymbol{\lambda}+1} \right\rangle = \frac{\boldsymbol{\lambda}}{\boldsymbol{\lambda}+1}$$

■

Theorem (2421a). *The summation of odd numbers from* $\mathbf{1}$ *to* $\boldsymbol{\phi}$ *is* $\boldsymbol{\phi^2}$*.*

Proof. There exists an integer $\boldsymbol{\lambda}$, by the definition of odd numbers, such that the summation of odd numbers from $\mathbf{1}$ to $\boldsymbol{\phi}$ is given by,

$$\sum_{\lambda=1}^{\phi} 2\lambda - 1$$

The identity for $\mathbf{2\boldsymbol{\lambda} - 1}$ is the difference of squares $\boldsymbol{\lambda^2 - \langle \lambda - 1 \rangle^2}$. This identity can be demonstrated by the statement

$$\left\langle \lambda^2 - [\lambda - 1]^2 \right\rangle = \left\langle \Big[\lambda + \langle \lambda - 1 \rangle\Big]\Big[\lambda - \langle \lambda - 1 \rangle\Big] \right\rangle =$$

$$\left\langle \Big[2\lambda - 1\Big]\Big[\lambda + \langle -\lambda + 1 \rangle\Big] \right\rangle = \left\langle [2\lambda - 1]1 \right\rangle$$

So the summation of odd numbers from $\mathbf{1}$ to $\boldsymbol{\phi}$ is the telescoping summation

$$\sum_{\lambda=1}^{\phi} \lambda^2 - \langle \lambda - 1 \rangle^2$$

By Theorem 2419, that is $\boldsymbol{\phi^2 - 0^2 = \phi^2}$. Thus,

$$\sum_{\lambda=1}^{\phi} 2\lambda - 1 = \phi^2$$

and indeed the summation of odd numbers from $\mathbf{1}$ to $\boldsymbol{\phi}$ is $\boldsymbol{\phi^2}$. ∎

Theorem (2437). *Let* $\mathbf{A}$*, and* $\boldsymbol{\Lambda}$ *be sets such that* $\mathbf{A}$ *is a subset of* $\boldsymbol{\Lambda}$*. If* $\mathbf{A}$ *is uncountable, then* $\boldsymbol{\Lambda}$ *is uncountable.*

Proof. Direct proof. The cardinality for $\mathbf{A}$ is greater than $\boldsymbol{\aleph_0}$, by the definition for countability. By the definition of subsets, the cardinality of $\boldsymbol{\Lambda}$ is at least the cardinality of $\mathbf{A}$. Hence, $\boldsymbol{\Lambda}$ is uncountable, by the definition for countability. ∎

Theorem (2438). *Let* $\mathbf{A}$*, and* $\boldsymbol{\Lambda}$ *be sets with equal cardinality.*

$$\Big|\boldsymbol{\mathcal{P}}\langle \mathbf{A} \rangle\Big| = \Big|\boldsymbol{\mathcal{P}}\langle \boldsymbol{\Lambda} \rangle\Big|$$

Proof. By the hypothesis, and by the defintion for set cardinality, there exists an integer $\boldsymbol{\iota}$ such that $\big|\mathbf{A}\big| = \big|\boldsymbol{\Lambda}\big| = \boldsymbol{\iota}$. The cardinality of a power set is 2 to the power of the set cardinality. Thus,

$$\big|\boldsymbol{\mathcal{P}}\langle \mathbf{A} \rangle\big| = \big|\boldsymbol{\mathcal{P}}\langle \boldsymbol{\Lambda} \rangle\big| = \mathbf{2}^{\boldsymbol{\iota}}$$

∎

Theorem (2421b). *The summation of natural numbers from* $\mathbf{1}$ *to* $\boldsymbol{\lambda}$ *is*

$$\frac{\boldsymbol{\lambda}[\boldsymbol{\lambda}+\mathbf{1}]}{\mathbf{2}}$$

Proof. It is possible to derive the closed formula for the summation of natural numbers from $\mathbf{1}$ to $\boldsymbol{\lambda}$ from the summation of odd numbers from $\mathbf{1}$ to $\boldsymbol{\lambda}$. By the definition for odd numbers, an integer $\boldsymbol{\phi}$ exists such that, by Theorem 2.4.21a

$$\sum_{\phi=1}^{\lambda} 2\phi - 1 = \lambda^2$$

By the associative law for addition from the field axioms,

$$\left\langle \sum_{\phi=1}^{\lambda} 2\phi - 1 \right\rangle \equiv \left\langle \sum_{\phi=1}^{\lambda} 2\phi + \sum_{\phi=1}^{\lambda} -1 \right\rangle \equiv \left\langle -\lambda + \sum_{\phi=1}^{\lambda} 2\phi \right\rangle$$

Thus, by that identity, and by the inverse law for addition from the field axioms,

$$\left\{ \left\langle \lambda^2 \right\rangle = \left\langle -\lambda + \sum_{\phi=1}^{\lambda} 2\phi \right\rangle \right\} \equiv \left\{ \left\langle \sum_{\phi=1}^{\lambda} 2\phi \right\rangle = \left\langle \lambda^2 + \lambda \right\rangle = \lambda \left\langle \lambda + 1 \right\rangle \right\}$$

By the distributive laws for real numbers from the field axioms, that is

$$2\sum_{\phi=1}^{\lambda} \phi = \lambda \left\langle \lambda + 1 \right\rangle$$

And by the inverse law for multiplication from the field axioms,

$$\sum_{\boldsymbol{\phi}=\mathbf{1}}^{\boldsymbol{\lambda}} \boldsymbol{\phi} = \frac{\boldsymbol{\lambda}[\boldsymbol{\lambda}+\mathbf{1}]}{\mathbf{2}}$$

∎

Lemma (2401). *Let $\boldsymbol{\lambda}$ be a positive integer.*

$$\frac{\mathbf{1}}{\mathbf{3}}\left[\boldsymbol{\lambda}^{\mathbf{3}} + \mathbf{3}\left\langle\frac{\boldsymbol{\lambda}[\boldsymbol{\lambda}+\mathbf{1}]}{\mathbf{2}}\right\rangle - \boldsymbol{\lambda}\right] = \left[\frac{\boldsymbol{\lambda}\langle\boldsymbol{\lambda}+\mathbf{1}\rangle\langle\mathbf{2}\boldsymbol{\lambda}+\mathbf{1}\rangle}{\mathbf{6}}\right]$$

Proof. By the distributive laws for real numbers, and by the associative law for multiplication,

$$3\left\langle\frac{\lambda[\lambda+1]}{2}\right\rangle = \left\langle\frac{3\lambda^2+3\lambda}{2}\right\rangle$$

Since the rational number $\frac{\mathbf{2}}{\mathbf{2}} = \mathbf{1}$ by the inverse law for multiplication, by the multiplicative identity law from the field axioms, (and by the identity established above,)

$$\left[\lambda^3 + 3\left\langle\frac{\lambda[\lambda+1]}{2}\right\rangle - \lambda\right] = \left[\frac{2\lambda^3}{2} + \frac{3\lambda^2+3\lambda}{2} - \frac{2\lambda}{2}\right]$$

By the distributive law for real numbers

$$\frac{1}{3}\left[\frac{2\lambda^3}{2} + \frac{3\lambda^2+3\lambda}{2} - \frac{2\lambda}{2}\right] = \left[\frac{2\lambda^3+3\lambda^2+3\lambda-2\lambda}{6}\right]$$

Repeated factoring, by the distributive laws for real numbers, completes the proof.

$$\left[\frac{2\lambda^3+3\lambda^2+3\lambda-2\lambda}{6}\right] = \left[\frac{\lambda\langle 2\lambda^2+2\lambda+\lambda+1\rangle}{6}\right] = \left[\frac{\lambda\langle 2\lambda[\lambda+1]+[\lambda+1]\rangle}{6}\right]$$
$$= \left[\frac{\boldsymbol{\lambda}\langle\boldsymbol{\lambda}+\mathbf{1}\rangle\langle\mathbf{2}\boldsymbol{\lambda}+\mathbf{1}\rangle}{\mathbf{6}}\right]$$

■

Lemma (2404). *Let $\boldsymbol{\iota}$ be a natural number such that $\lfloor\sqrt[3]{\boldsymbol{\iota}}\rfloor = \boldsymbol{\lambda}$.*

$$\sum_{\boldsymbol{\iota}=\mathbf{1}}^{\mathbf{3}\boldsymbol{\lambda}^{\mathbf{2}}+\mathbf{3}\boldsymbol{\lambda}+\mathbf{1}} \lfloor\sqrt[3]{\boldsymbol{\iota}}\rfloor = \boldsymbol{\lambda}\left[\mathbf{3}\boldsymbol{\lambda}^{\mathbf{2}}+\mathbf{3}\boldsymbol{\lambda}+\mathbf{1}\right]$$

Proof. Similiar to Lemma 2403, it is trivial that

$$\sum_{\iota=1}^{3\lambda^2+3\lambda+1} 1 = 3\lambda^2+3\lambda+1$$

By the inverse law for multiplication from the field axioms, by the distributive law for real numbers, and by the identity lambda, that is

$$\left\{\lambda\sum_{\iota=1}^{3\lambda^2+3\lambda+1} 1 = \lambda\left[3\lambda^2+3\lambda+1\right]\right\} \equiv \left\{\sum_{\iota=1}^{3\lambda^2+3\lambda+1} \lambda = \lambda\left[3\lambda^2+3\lambda+1\right]\right\} \equiv$$
$$\left\{\sum_{\iota=1}^{3\lambda^2+3\lambda+1} \lfloor\sqrt[3]{\iota}\rfloor = \lambda\left[3\lambda^2+3\lambda+1\right]\right\}$$

■

Theorem (2422). *The sum of squares from* $\mathbf{1}$ *to* $\boldsymbol{\lambda}$ *is*

$$\boldsymbol{\frac{\lambda\langle\lambda+1\rangle\langle 2\lambda+1\rangle}{6}}$$

Proof. The formula for the summation of squares from $\mathbf{1}$ to $\boldsymbol{\lambda}$ can be derived from the cube of $\boldsymbol{\lambda}$. It is trivial that $\boldsymbol{\lambda^3 = \lambda^3 - \langle 1-1\rangle^3}$. By this identity for $\boldsymbol{\lambda}$, the cube of $\boldsymbol{\lambda}$ is the telescopic summation given by Theorem 2419,

$$\lambda^3 = \sum_{\iota=1}^{\lambda} \iota^3 - \langle \iota - 1\rangle^3$$

The expansion for $\boldsymbol{\langle \iota - 1\rangle^3}$ is $\boldsymbol{\iota^3 - 3\iota^2 + 3\iota - 1}$, by the Binomial Theorem. Thus, by the inverse law for addition from the field axioms, yielding the algebraic identity

$$\iota^3 - \langle \iota - 1\rangle^3 = 3\iota^2 - 3\iota + 1$$

Hence, $\boldsymbol{\lambda^3 = \sum_{\iota=1}^{\lambda} 3\iota^2 - 3\iota + 1}$. By the commutative law for addition from the field axioms, and by the distributive law for real numbers, that is

$$\lambda^3 = \left(3\sum_{\iota=1}^{\lambda}\iota^2\right) - \left(3\sum_{\iota=1}^{\lambda}\iota\right) + \left(\sum_{\iota=1}^{\lambda}1\right)$$

Note that $\boldsymbol{\sum_{\iota=1}^{\lambda} 1 = \lambda\langle 1\rangle}$. And by Theorem 2421b, $\boldsymbol{\sum_{\iota=1}^{\lambda} \iota = \frac{\lambda\langle\lambda+1\rangle}{2}}$. Thus, by those identities, and by the inverse law for addition from the field axioms

$$\lambda^3 + 3\frac{\lambda\langle\lambda+1\rangle}{2} - \lambda = 3\sum_{\iota=1}^{\lambda}\iota^2$$

Eliminating the coefficient $\mathbf{3}$ from the right-hand side, by the inverse law for multiplication from the field axioms, gives us the sum of squares in terms of an equation,

$$\frac{1}{3}\left[\lambda^3 + 3\frac{\lambda\langle\lambda+1\rangle}{2} - \lambda\right] = \sum_{\iota=1}^{\lambda}\iota^2$$

By Lemma 2401, that is

$$\boldsymbol{\sum_{\iota=1}^{\lambda}\iota^2 = \frac{\lambda\langle\lambda+1\rangle\langle 2\lambda+1\rangle}{6}}$$

∎

Lemma (2402). *Let $\boldsymbol{\iota}$ be a positive integer such that* $\lfloor \sqrt{\boldsymbol{\iota}} \rfloor = \boldsymbol{\lambda}$.

$$\left(\mathbf{2} \sum_{\phi=0}^{\boldsymbol{\lambda}-\mathbf{1}} \phi^{\mathbf{2}} \right) + \left(\sum_{\phi=0}^{\boldsymbol{\lambda}-\mathbf{1}} \phi \right) \equiv \frac{\boldsymbol{\lambda}\langle \boldsymbol{\lambda} - \mathbf{1} \rangle}{\mathbf{6}} \left[\mathbf{4}\boldsymbol{\lambda} + \mathbf{1} \right]$$

Proof. Let $\boldsymbol{\Phi}$ be two times the sum of squares from zero to $\boldsymbol{\lambda}$ minus one. Let $\mathbf{X}$ be the sum of integers from zero to $\boldsymbol{\lambda}$ minus one. By theorems 2422, and 2421b, and by shifting the index of summation,

$$\left\{ \Phi + \mathrm{X} \right\} \equiv 2 \left[\frac{\lambda[\lambda - 1][2\langle \lambda - 1 \rangle + 1]}{6} \right] + \left[\frac{\lambda[\lambda - 1]}{2} \right]$$

Since the rational number $\frac{\mathbf{3}}{\mathbf{3}} = \mathbf{1}$ by the inverse law for multiplication, by the multiplicative identity law from the field axioms

$$\left\{ \mathrm{X} \right\} \equiv \frac{3\lambda[\lambda - 1]}{6}$$

Thus, by the identity $\mathbf{X}$, factoring $\frac{\mathbf{1}}{\mathbf{6}}\boldsymbol{\lambda}\langle \boldsymbol{\lambda} - \mathbf{1} \rangle$ out from the sum of $\boldsymbol{\Phi}$ and $\mathbf{X}$, by the distributive laws for real numbers,

$$\left\{ \Phi + \mathrm{X} \right\} \equiv \frac{\lambda\langle \lambda - 1 \rangle}{6} \left[2 \left\langle 2[\lambda - 1] + 1 \right\rangle + 3 \right]$$

By the distributive laws for real numbers, and by the associative law for addition from the field axioms,

$$\left\{ 2 \left\langle 2[\lambda - 1] + 1 \right\rangle + 3 \right\} = \left\{ 4[\lambda - 1] + 5 \right\} = \left\{ 4\lambda + 1 \right\}$$

$\therefore$

$$\left\{ \boldsymbol{\Phi} + \mathbf{X} \right\} \equiv \frac{\boldsymbol{\lambda}\langle \boldsymbol{\lambda} - \mathbf{1} \rangle}{\mathbf{6}} \left[\mathbf{4}\boldsymbol{\lambda} + \mathbf{1} \right]$$

■

Lemma (2405). *Let $\boldsymbol{\iota}$ be a positive integers such that* $\left\lfloor \sqrt[3]{\boldsymbol{\iota}} \right\rfloor = \boldsymbol{\lambda}$.

$$\left(\mathbf{3}\sum_{\boldsymbol{\phi}=\mathbf{0}}^{\boldsymbol{\lambda}-\mathbf{1}} \boldsymbol{\phi}^{\mathbf{3}}\right) + \left(\mathbf{3}\sum_{\boldsymbol{\phi}=\mathbf{0}}^{\boldsymbol{\lambda}-\mathbf{1}} \boldsymbol{\phi}^{\mathbf{2}}\right) + \left(\sum_{\boldsymbol{\phi}=\mathbf{0}}^{\boldsymbol{\lambda}-\mathbf{1}} \boldsymbol{\phi}\right) \equiv \frac{\boldsymbol{\lambda}^{\mathbf{3}} - \boldsymbol{\lambda}^{\mathbf{2}}}{\mathbf{4}} \left[\mathbf{3}\boldsymbol{\lambda} + \mathbf{1}\right]$$

Proof. Let $\mathbf{\Phi}$ be three times the summation of cubes from zero to lambda minus one. Let $\mathbf{X}$ be three times the summation of squares from zero to lambda minus one. Let $\mathbf{\Omega}$ be the summation of integers from zero to lambda minus one. By theorems 2422 and 2421b, by the closed formula for the summation of cubes, and by shifting the index of summation,

$$\left\{\Phi + \mathrm{X} + \Omega\right\} \equiv 3\left[\frac{\lambda^2\left[\lambda-1\right]^2}{4}\right] + 3\left[\frac{\lambda\left[\lambda-1\right]\left[2\langle\lambda-1\rangle+1\right]}{6}\right] + \left[\frac{\lambda\left[\lambda-1\right]}{2}\right]$$

By the multiplicative identity law from the field axioms,

$$\left\langle\frac{3}{6}\right\rangle = \left\langle\frac{2\cdot 3}{2\cdot 6}\right\rangle = \left\langle\frac{6}{12}\right\rangle = \left\langle\frac{2}{4}\right\rangle$$

$$\left\langle\frac{1}{2}\right\rangle = \left\langle\frac{2\cdot 1}{2\cdot 2}\right\rangle = \left\langle\frac{2}{4}\right\rangle$$

Thus, by the identities for $\mathbf{X}$ and $\mathbf{\Omega}$, factoring $\frac{\mathbf{1}}{\mathbf{4}}\boldsymbol{\lambda}\langle\boldsymbol{\lambda}-\mathbf{1}\rangle$ out from the sum of $\mathbf{\Phi}$, $\mathbf{X}$, and $\mathbf{\Omega}$, by the distributive laws from real numbers,

$$\left\{\Phi + \mathrm{X} + \Omega\right\} \equiv \frac{\lambda\langle\lambda-1\rangle}{4}\left[3\lambda\left[\lambda-1\right] + 2\left\langle 2\left[\lambda-1\right]+1\right\rangle + 2\right]$$

By the distributive law for real numbers, and by the inverse law for addition from the field axioms,

$$\left\{2\left\langle 2\left[\lambda-1\right]+1\right\rangle + 2\right\} = \left\{4\left[\lambda-1\right]+2+2\right\} = \left\{4\lambda-4+4\right\} = \left\{4\lambda\right\}$$

Hence, by the identity four lambda,

$$\left\{\Phi + \mathrm{X} + \Omega\right\} \equiv \frac{\lambda\langle\lambda-1\rangle}{4}\left[3\lambda\left[\lambda-1\right] + 4\lambda\right]$$

Factoring out lambda and distributing three, by the distributive laws for real numbers,

$$\left\{\Phi + \mathrm{X} + \Omega\right\} \equiv \frac{\lambda^2\langle\lambda-1\rangle}{4}\left[3\lambda-3+4\right]$$

The proof is complete, by the distributive law for real numbers distributing the second power of lambda, and the inverse law for addition from the field axioms. ∎

Theorem (2425). *Let* $\boldsymbol{\phi}$ *be a positive integer such that* $\lfloor\sqrt{\boldsymbol{\phi}}\rfloor = \boldsymbol{\lambda}$. *The closed form formula for* $\boldsymbol{\Phi} = \sum_{\boldsymbol{\iota}=\mathbf{0}}^{\boldsymbol{\phi}} \lfloor\sqrt{\boldsymbol{\iota}}\rfloor$ *is*

$$\frac{\boldsymbol{\lambda}\langle\boldsymbol{\lambda}-\mathbf{1}\rangle}{\mathbf{6}}\Big[\mathbf{4}\boldsymbol{\lambda}+\mathbf{1}\Big]+\boldsymbol{\lambda}\Big[\boldsymbol{\phi}-\boldsymbol{\lambda}^{\mathbf{2}}+\mathbf{1}\Big]$$

Proof. By the associative law for addition from the field axioms,

$$\Theta = \sum_{\iota=0}^{\lambda-1} \lfloor\sqrt{\iota}\rfloor =$$

$$\left(\sum_{\iota=1}^{2\beta_0+1} \lfloor\sqrt{\iota}\rfloor_0\right) + \left(\sum_{\iota=1+2\beta_0+1}^{2\beta_0+1+2\beta_1+1} \lfloor\sqrt{\iota}\rfloor_1\right) + \cdots + \left(\sum_{\iota=1+\cdots+2\beta_{\langle\lambda-2\rangle}+1}^{0+\cdots+2\beta_{\langle\lambda-1\rangle}+1} \lfloor\sqrt{\iota}\rfloor_{\langle\lambda-1\rangle}\right)$$

$\lfloor\sqrt{\boldsymbol{\iota}}\rfloor_{\boldsymbol{\tau}}$ is the unique integer $\boldsymbol{\beta_\tau}$, for $\boldsymbol{\tau} = \mathbf{0}$ to $\langle\boldsymbol{\lambda}-\mathbf{1}\rangle$. Hence, by Lemma 2403 and the distributive law for real numbers from the field axioms, by the partitioning from above, and shifting the index of summation,

$$\Theta = \Big\langle 2\beta_0^2 + \beta_0\Big\rangle + \Big\langle 2\beta_1^2 + \beta_1\Big\rangle + \cdots + \Big\langle 2\beta_{[\lambda-1]}^2 + \beta_{[\lambda-1]}\Big\rangle$$

By the commutative law for addition from the field axioms, that series is two times the finite summation of squares, plus the finite summation of integers from zero to lambda minus one.

$$\Theta = \sum_{\tau=0}^{\lambda-1} 2\tau^2 + \sum_{\tau=0}^{\lambda-1} \tau$$

Lambda occurs exactly $\langle\boldsymbol{\phi}-\boldsymbol{\lambda}^{\mathbf{2}}+\mathbf{1}\rangle$ times in big phi. Thus, by Lemma 2402 and the distributive law for real numbers from the field axioms,

$$\boldsymbol{\Phi} = \frac{\boldsymbol{\lambda}\langle\boldsymbol{\lambda}-\mathbf{1}\rangle}{\mathbf{6}}\Big[\mathbf{4}\boldsymbol{\lambda}+\mathbf{1}\Big]+\boldsymbol{\lambda}\Big[\boldsymbol{\phi}-\boldsymbol{\lambda}^{\mathbf{2}}+\mathbf{1}\Big]$$

■

Theorem (**2426**)**.** *Let m be a positive integer such that* $\boldsymbol{\left\lfloor \sqrt[3]{\phi} \right\rfloor = \lambda}$*. The closed form formula for* $\boldsymbol{\Phi = \sum_{\iota=0}^{\phi} \left\lfloor \sqrt[3]{\iota} \right\rfloor}$ *is*

$$\boldsymbol{\frac{\lambda^3 - \lambda^2}{4}\left[3\lambda + 1\right] + \lambda\left[\phi - \lambda^3 + 1\right]}$$

Proof. Let $\mathbf{X}$ be the function $\mathbf{X} : \mathbb{N} \to \mathbb{N}$ such that $\boldsymbol{X[\beta] = 3\beta^2 + 3\beta + 1}$. By the associative law for addition from the field axioms,

$$\Theta = \sum_{\iota=0}^{\lambda-1} \left\lfloor \sqrt[3]{\iota} \right\rfloor =$$

$$\left(\sum_{\iota=1}^{\mathrm{X}[\beta_0]} \left\lfloor \sqrt[3]{\iota} \right\rfloor_0 \right) + \left(\sum_{\iota=1+\mathrm{X}[\beta_0]}^{\mathrm{X}[\beta_0]+\mathrm{X}[\beta_1]} \left\lfloor \sqrt[3]{\iota} \right\rfloor_1 \right) + \cdots + \left(\sum_{\iota=1+\cdots+\mathrm{X}[\beta_{\lambda-2}]}^{0+\cdots+\mathrm{X}[\beta_{\lambda-1}]} \left\lfloor \sqrt[3]{\iota} \right\rfloor_{\langle \lambda-1 \rangle} \right)$$

$\boldsymbol{\left\lfloor \sqrt[3]{\iota} \right\rfloor_\tau}$ is the unique integer $\boldsymbol{\beta_\tau}$, for $\boldsymbol{\tau = 0}$ to $\boldsymbol{\langle \lambda - 1 \rangle}$. Hence, by Lemma 2404, by the partitioning from above, and shifting the index of summation,

$$\Theta = \left\langle \beta_0 \mathrm{X}[\beta_0] \right\rangle + \left\langle \beta_1 \mathrm{X}[\beta_1] \right\rangle + \cdots + \left\langle \beta_{\langle \lambda-1 \rangle} \mathrm{X}[\beta_{\langle \lambda-1 \rangle}] \right\rangle$$

By the distributive law for real numbers from the field axioms, and the commutative law for addition, that series is three times the finite summation of cubes, plus three times the finite summation of squares, plus the finite summation of integers from zero to lambda minus one.

$$\Theta = \sum_{\tau=0}^{\lambda-1} 3\tau^3 + \sum_{\tau=0}^{\lambda-1} 3\tau^2 + \sum_{\tau=0}^{\lambda-1} \tau$$

Lambda occurs exactly $\boldsymbol{\langle \phi - \lambda^3 + 1 \rangle}$ times in big phi. Thus, by Lemma 2405 and the distributive law for real numbers from the field axioms,

$$\boldsymbol{\Phi = \frac{\lambda^3 - \lambda^2}{4}\left[3\lambda + 1\right] + \lambda\left[\phi - \lambda^3 + 1\right]}$$

■

Theorem (2440). *The union of two countable sets is countable.*

Proof. By cases. Let $\mathbf{A}$, and $\boldsymbol{\Lambda}$ be countable sets. There are three possible cases. (i) $\mathbf{A}$ and $\boldsymbol{\Lambda}$ are finite, (ii) exclusively $\mathbf{A}$ or $\boldsymbol{\Lambda}$ is finite and the other is countably infinite, (iii) $\mathbf{A}$ and $\boldsymbol{\Lambda}$ are both countably infinite.

$\boldsymbol{(i)}$ Assume $\mathbf{A}$ and $\boldsymbol{\Lambda}$ are finite. There exist natural numbers $\boldsymbol{\lambda}$, and $\boldsymbol{\iota}$ such that $\left|\mathbf{A}\right| = \boldsymbol{\lambda}$ and $\left|\boldsymbol{\Lambda}\right| = \boldsymbol{\iota}$. The maximum cardinality for $\mathbf{A} \cup \boldsymbol{\Lambda}$ occurs when the intersection of $\mathbf{A}$ and $\boldsymbol{\Lambda}$ is the empty set. The cardinality for such a union is $\boldsymbol{\lambda} + \boldsymbol{\iota}$. $\boldsymbol{\lambda} + \boldsymbol{\iota}$ is a natural number by the closure property for addition on integers. Thus, $\boldsymbol{\lambda} + \boldsymbol{\iota}$ is less than $\boldsymbol{\aleph_0}$. By the definition for countably finite sets, $\mathbf{A} \cup \boldsymbol{\Lambda}$ is countable.

$\boldsymbol{(ii)}$ Without loss of generality assume $\mathbf{A}$ is finite with cardinality $\boldsymbol{\lambda}$, and $\boldsymbol{\Lambda}$ is countably infinite. A finite sequence $\left\{\boldsymbol{\alpha_\iota}\right\}$ containing all members of $\mathbf{A}$, and an infinite sequence $\left\{\boldsymbol{\beta}_{\mathbb{N}}\right\}$ containing all members of $\boldsymbol{\Lambda}$ exists. For the union of $\mathbf{A}$ and $\boldsymbol{\Lambda}$ there exists a sequence $\left\{\boldsymbol{\delta}\right\}$ such that

$$\{\delta_{\mathbb{N}}\} = \{\alpha_0, \alpha_1, \ldots, \alpha_\lambda, \beta_{\lambda+1}, \beta_{\lambda+2}, \beta_{\lambda+3}, \ldots\}$$

Clearly this is countably infinite by the definition for countability since $\boldsymbol{\lambda} + \boldsymbol{\chi}$ is a natural number for all $\boldsymbol{\chi}$ in natural numbers, by the closure property for addition on natural numbers.

$\boldsymbol{(iii)}$ assume both $\mathbf{A}$ and $\boldsymbol{\Lambda}$ are countable infinite sets. Since each set cardinality is $\boldsymbol{\aleph_0}$, an infinite sequence $\left\{\boldsymbol{\alpha}_{\mathbb{N}}\right\}$ containing all members of $\mathbf{A}$, and an infinite sequence $\left\{\boldsymbol{\beta}_{\mathbb{N}}\right\}$ containing all members of $\boldsymbol{\Lambda}$ exist. For the union of $\mathbf{A}$ and $\boldsymbol{\Lambda}$ there exists a infinite sequence $\left\{\boldsymbol{\delta}\right\}$ such that

$$\left\{\delta_{\mathbb{N}}\right\} = \left\{\alpha_{0_0}, \beta_{0_1}, \alpha_{1_2}, \beta_{1_3}, \alpha_{2_4}, \beta_{2_5}, \ldots\right\}$$

Thus a bijection exists between $\mathbb{N}$ and the union of $\mathbf{A}$ and $\boldsymbol{\Lambda}$, and that union is countable by the definition for countability. ■

Theorem (2443). *The set of all finite bit strings is countable.*

Proof. Let $\{a_{n-1}\}$ be the sequence of bits for any finite bit string a(base-2) of length n. The unique base-2 expansion for $\{a_{n-1}\}$ is the integer

$$a(\text{base-10}) = \sum_{i=0}^{n-1} a_i 2^i$$

Also, this integer can be converted to the unique base-2 bit string for a(base-10) by

$$a(\text{base-2}) = \sum_{i=0}^{n-1} \left[a(\text{base-10}) (\text{mod } 2^{i+1})\right] 10^i$$

Since an invertible function exists between each finite bit string and some positive integer, there exists, a one-to-one correspondence between $\mathbb{Z}$ and the set of all finite bit strings. Thus, the cardinality for the set of all finite bit strings is $\aleph_0$, and the set of all finite bit strings is countable, by definition. ■

Theorem (2441). *The union of a countable number of countable sets is countable.*

Proof. Let A_i be a countable set, for integers $i = 0$ to $n \leq \infty$ such that

$$S = \bigcup_{i=0}^{n} A_i$$

The function $f : \mathbb{N} \to A_i$ is the sequence $\{a_{ij}\} = a_{i0}, a_{i1}, a_{i2}, \ldots$. Thus, by f, all elements a_{ij} in S can be listed in the second dimension

$$a_{00}, a_{01}, a_{02}, \ldots$$

$$a_{10}, a_{11}, a_{12}, \ldots$$

$$a_{20}, a_{21}, a_{22}, \ldots$$

$$\vdots$$

By tracing the diagonal path along the two dimensional listing for S we get the countable order

$$a_{00}, a_{01}, a_{10}, a_{20}, a_{11}, a_{02}, \ldots$$

$\therefore |S| \leq \aleph_0$, and indeed the union of a countable number of countable sets is countable. ■

Theorem (2442). *The cardinality of $\mathbb{Z}^+ \times \mathbb{Z}^+$ is aleph null.*

Proof. $\mathbb{Z}^+ \times \mathbb{Z}^+$ is defined as $\{\langle x, y\rangle | (x \in \mathbb{Z}^+) \wedge (y \in \mathbb{Z}^+)\}$. Since x and y are positive integers, for every ordered pair $\langle x, y\rangle$ in $\mathbb{Z}^+ \times \mathbb{Z}^+$, $\langle x, y\rangle$ exists if and only if the rational number $\frac{x}{y}$ exists. Thus, $\frac{x}{y}$ exists, and all elements in $\mathbb{Z}^+ \times \mathbb{Z}^+$ can be represented by the two dimensional list

$$\langle 1, 1\rangle \iff \frac{1}{1}, \langle 1, 2\rangle \iff \frac{1}{2}, \langle 1, 3\rangle \iff \frac{1}{3}, \ldots$$

$$\langle 2, 1\rangle \iff \frac{2}{1}, \langle 2, 2\rangle \iff \frac{2}{2}, \langle 2, 3\rangle \iff \frac{2}{3}, \ldots$$

$$\langle 3, 1\rangle \iff \frac{3}{1}, \langle 3, 2\rangle \iff \frac{3}{2}, \langle 3, 3\rangle \iff \frac{3}{3}, \ldots$$

$$\vdots$$

The hypotheses in the biconditional converse statements for each list entry are the list elements in the proof for the countability of rational numbers. That means $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable if and only if the rational numbers are countable. We know the rational numbers are countable. Therefore the cardinality of $\mathbb{Z}^+ \times \mathbb{Z}^+$ is $\aleph_0$. ■

Chapter 5

Algorithms

5.1 The sum of integers in a list.

Algorithm 3103 : Calculate the sum of integers in a list.

function SUM($\rho_1, \rho_2, \ldots, \rho_\epsilon$: list of integers)
  $\phi \leftarrow 0$
  for $\iota \leftarrow 1, \epsilon$ **do**
    $\phi \mathrel{+}= \rho_\iota$
  return ϕ

Python

```python
def sum(a):
    ''' Calculate the sum of integers in a list. '''
    total = 0
    for i in range(len(a)):
        total += a[i]
    return total
```

Java

```java
public static int sum(int[] a) {
    // Calculate the sum of integers in a list.
    int total = 0;
    for (int i = 0; i < a.length; i++) {
        total = total + a[i];
    }
    return total;
}
```

5.2 Max adjacent list entry difference.

Algorithm 3104 : The maximum adjacent list entry difference.

function MAX DIFFERENCE($\rho_1, \rho_2, \ldots, \rho_\epsilon$: list of integers)
  $\mu \leftarrow 0$
  for $\iota = 2, \epsilon$ **do**
    $\sigma = \rho_\iota - \rho_{\iota-1}$
    if $\mu < \sigma$ **then**
      $\mu = \sigma$
  return μ

Python

```
def max_difference(a):
    ''' Max adjacent list entry difference. '''
    maximum = 0
    for i in range(1, len(a)):
        difference = a[i] - a[i-1]
        if maximum < difference:
            maximum = difference
    return maximum
```

Java

```
public static int maxDifference(int[] a) {
    // Max adjacent list entry difference.
    int maximum = 0, difference;
    for (int i = 1; i < a.length; i++) {
        difference = a[i] - a[i-1];
        if (maximum < difference) {
            maximum = difference;
        }
    }
    return maximum;
}
```

5.3 Find duplicate list entries.

Algorithm 3105 : Find duplicate list entries.

function DUPLICATES($\rho_1, \rho_2, \ldots, \rho_\epsilon$: integers in nondecreasing order)
  $\phi \leftarrow \varnothing$
  for $\iota = 1, \epsilon - 1$ **do**
    if $\rho_\iota = \rho_{\iota+1}$ **then**
      $\phi \leftarrow \phi \cup \{\rho_{\iota+1}\}$
  return ϕ

Python

```python
def duplicates(a):
    ''' Find duplicate list entries. '''
    a.sort()
    positives = set()
    for i in range(len(a)-1):
        if a[i] == a[i+1]:
            positives = positives | {a[i+1]}
    return positives
```

Java

```java
public static Set duplicates(int[] a) {
    // Find duplicate list entries.
    Arrays.sort(a);
    Set<Integer> positives = new HashSet<Integer>();
    for (int i = 0; i < a.length-1; i++) {
        if (a[i] == a[i+1]) {
            positives.add(a[i+1]);
        }
    }
    return positives;
}
```

5.4 Count negative valued list entries.

Algorithm 3106 : Count negative valued list entries.

function NEGATIVES($\rho_1, \rho_2, \ldots, \rho_\epsilon$: list of integers)
  $\phi \leftarrow 0$
  for $\iota = 1, \epsilon$ **do**
    if $\rho_\iota < 0$ **then**
      $\phi \mathrel{+}= 1$
  return ϕ

Python

```python
def negatives(a):
    ''' Count negative list entries. '''
    count = 0
    for i in range(len(a)):
        if a[i] < 0:
            count += 1
    return count
```

Java

```java
public static int negatives(int[] a) {
    // Count negative list entries.
    int count = 0;
    for (int i = 0; i < a.length; i++) {
        if (a[i] < 0) {
            count++;
        }
    }
    return count;
}
```

5.5 Find the last even list entry.

Algorithm 3107 : Find the last even list entry.

function LAST EVEN($\rho_1, \rho_2, \ldots, \rho_\epsilon$: list of integers)
  $\phi \leftarrow$ null
  for $\iota = 1, \epsilon$ **do**
    if $\neg\left[\rho_\iota \langle \bmod 2 \rangle\right]$ **then**
      $\phi \leftarrow \iota$
  return ϕ ▷ returns null if every integer ρ_ι is odd

Python

```python
def last_even(a):
    ''' Find the last even list entry. '''
    index = None
    for i in range(len(a)):
        if not a[i] % 2:
            index = i
    return index
```

Java

```java
public static Integer lastEven(int[] a) {
    // Find the last even list entry.
    Integer index = null;
    for (int i = 0; i < a.length; i++) {
        if (a[i] % 2 == 0) {
            index = i;
        }
    }
    return index;
}
```

5.6 Find the largest even list entry.

Algorithm 3108 : Find the largest even list entry.

function LARGEST EVEN($\rho_1, \rho_2, \ldots, \rho_\epsilon$: list of integers)
  $\phi \leftarrow$ null
  for $\iota = 1, \epsilon$ **do**
    if $\neg\big[\rho_\iota \langle \text{mod } 2 \rangle\big]$ **then**
      if $\langle \phi = \text{null} \rangle \vee \langle \rho_\iota > \rho_\phi \rangle$ **then**
        $\phi \leftarrow \iota$
  return ϕ ▷ returns null if every integer is odd

Python

```python
def largest_even(a):
    ''' Find the largest even list entry. '''
    index = None
    for i in range(len(a)):
        if not a[i] % 2:
            if (not index) or (a[i] > a[index]):
                index = i
    return index
```

Java

```java
public static Integer largestEven(int[] a) {
    // Find the largest even list entry.
    Integer index = null;
    for (int i = 0; i < a.length; i++) {
        if (a[i] % 2 == 0) {
            if ((index == null) || (a[i] > a[index])) {
                index = i;
            }
        }
    }
    return index;
}
```

5.7 Palindrome strings of characters.

Algorithm 3109 : Determine whether a string of characters is a palindrome

function PALINDROME($\rho_1, \rho_2, \ldots, \rho_\epsilon$: string of characters)
  for $\iota = 1, \left\lfloor \frac{\epsilon}{2} \right\rfloor$ **do**
    if $\rho_\iota \neq \rho_{\langle \epsilon+1 \rangle - \iota}$ **then**
      return $\bot$
  return $\top$ ▷ all of the characters matched

Python

```
from math import floor

def palindrome(a):
    ''' Determine whether a string is a palindrome. '''
    for i in range(floor((len(a))/2)):
        if a[i] != a[(len(a)-1)-i]:
            return False
    return True
```

Java

```
public static boolean palindrome(String a) {
    // Determine whether a string in a palindrome.
    int right  = a.length() - 1;
    int limit  = Math.floor(a.length() / 2);
    for (int i = 0; i < limit; i++) {
        if (a.charAt(i) !=  a.charAt(right - i)) {
            return false;
        }
    }
    return true;
}
```

5.8 Compute β^λ

Algorithm 3110 : Compute β^λ

function POWER(λ: integer; β: real number)
  $\epsilon \leftarrow |\lambda|$
  $\rho \leftarrow 1$
  while $\epsilon > 0$ **do** ▷ multiply β by itself $|\lambda|$ times
    $\rho \leftarrow \rho \times \beta$
    $\epsilon -= 1$
  if $\lambda < 0$ **then** ▷ λ is negative so get the inverse
    $\rho \leftarrow \frac{1}{\rho}$
  return ρ

Python

```python
from fractions import Fraction

def power(n, x):
    ''' Compute x**n '''
    exponent, product = abs(n), 1
    while exponent > 0:
        product  *= x
        exponent -= 1
    if n < 0:
        product = Fraction(1, product)
    return product
```

Java

```java
public static double power(int n, int x) {
    // Compute x**n
    int exponent = Math.abs(n);
    double product = 1;
    while (exponent > 0) {
        product *= x;
        exponent--;
    }
    if (n < 0) {
        return 1 / product;
    }
    return product;
}
```

5.9 Swap $\boldsymbol{\lambda}$ and $\boldsymbol{\iota}$.

Algorithm 3111 : Swap λ and ι.

function SWAP(λ: object; ι: object)
  $\tau \leftarrow \lambda$
  $\lambda \leftarrow \iota$
  $\iota \leftarrow \tau$
  return λ, ι

Python

```
def swap(x, y):
    ''' Swap x and y. '''
    temp = x
    x = y
    y = temp
    return x, y
```

Java

```
public static Object[] swap(Object x, Object y) {
    // Swap x and y.
    Object[] swapped = new Object[2];
    Object temp = x;
    x = y;
    y = temp;
    swapped[0] = x; swapped[1] = y;
    return swapped;
}
```

5.10 Insert an integer into a list.

Algorithm 3115 : Insert an integer λ at the correct index position in a list of integers in increasing order.

function INSERT(λ: integer; $\rho_1, \ldots, \rho_\epsilon$: integers in increasing order)
  if $\rho_\epsilon \leq \lambda$ **then**
    $\rho_{\epsilon+1} \leftarrow x$
  else
    $\iota \leftarrow \epsilon$
    while $\iota \wedge \langle \lambda < \rho_\iota \rangle$ **do** ▷ make room for λ
      $\rho_{\iota+1} \leftarrow \rho_\iota$
      $\iota -= 1$
    $\rho_{\iota+1} \leftarrow \lambda$
  return $\rho_1, \rho_2, \ldots, \rho_{\epsilon+1}$

Python

```python
def insert(x, a):
    '''
    Insert an integer x at the correct index
    position in a list of integers in increasing
    order.
    '''
    a.sort()
    a.append(None)
    if a[-2] <= x:
        a[-1] = x
    else:
        i = len(a)-2
        while i+1 and (x < a[i]):
            a[i+1] = a[i]
            i -= 1
        a[i+1] = x
    return a
```

Java

```java
public static int[] insert(int x, int[] a) {
    // Insert an integer x at the correct index
    // position in a list of integers in increasing
    // order.
    Arrays.sort(a);
    a = Arrays.copyOf(a, a.length + 1);
    if (a[a.length - 2] <= x) {
        a[a.length - 1] = x;
    } else {
        int i = a.length - 2;
        while ((i + 1 > 0) && (x < a[i])) {
            a[i+1] = a[i];
            i--;
        }
        a[i+1] = x;
    }
    return a;
}
```