Theorem (3.2.18)**.** *Let f be the function defined by f(k, n)* $= 1^k + 2^k + \cdots + n^k$*, where k and n are positive integers. f(k, n) is* $\mathcal{O}(n^{k+1})$*.*

Proof. If f(k, n) is $\mathcal{O}(n^{k+1})$, then there exist constant witnesses C and j such that $|1^k + 2^k + \cdots + n^k| \leq C|n^{k+1}|$, for all $n > j$. If $j = 1$, then

$$f(k,n) = \left(1^k + 2^k + \cdots + n^k\right) \leq \left(n^k + n^k + \cdots + n^k\right) = n(n^k) = n^{k+1}$$

Thus, $\left(1^k + 2^k + \cdots + n^k\right) \leq n^{k+1}$, for all $n > 1$. So $|f(k,n)| \leq |n^{k+1}|$, for all $n > 1$. It follows that $f(k,n)$ is $\mathcal{O}(n^{k+1})$ with constant witnesses $C = 1$ and $j = 1$. ∎