Theorem (2.4.42)**.** *The cardinality of* $\mathbb{Z}^+ \times \mathbb{Z}^+$ *is aleph null.*

Proof. $\mathbb{Z}^+ \times \mathbb{Z}^+$ is defined as $\{\langle x, y\rangle | (x \in \mathbb{Z}^+) \wedge (y \in \mathbb{Z}^+)\}$. Since x and y are positive integers, for every ordered pair $\langle x, y\rangle$ in $\mathbb{Z}^+ \times \mathbb{Z}^+$, $\langle x, y\rangle$ exists if and only if the rational number $\frac{x}{y}$ exists. Thus, $\frac{x}{y}$ exists, and all elements in $\mathbb{Z}^+ \times \mathbb{Z}^+$ can be represented by the two dimensional list

$$\begin{array}{l} \langle 1,1\rangle \iff \frac{1}{1}, \langle 1,2\rangle \iff \frac{1}{2}, \langle 1,3\rangle \iff \frac{1}{3}, \ldots \\ \langle 2,1\rangle \iff \frac{2}{1}, \langle 2,2\rangle \iff \frac{2}{2}, \langle 2,3\rangle \iff \frac{2}{3}, \ldots \\ \langle 3,1\rangle \iff \frac{3}{1}, \langle 3,2\rangle \iff \frac{3}{2}, \langle 3,3\rangle \iff \frac{3}{3}, \ldots \\ \quad\vdots \end{array}$$

The hypotheses in the biconditional converse statements for each list entry are the list elements in the proof for the countability of rational numbers. That means $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable if and only if the rational numbers are countable. We know the rational numbers are countable. Therefore the cardinality of $\mathbb{Z}^+ \times \mathbb{Z}^+$ is $\aleph_0$. ■