Theorem (2.2.35)**.** *Let A, and B be sets. $A \oplus B = (A \cup B) - (A \cap B)$.*

Proof. Let x be an element in $A \oplus B$. This is logically defined as $[(x \in A) \wedge (x \notin B)] \vee [(x \notin A) \wedge (x \in B)]$. By the definition for set difference x is an element in $(A - B) \cup (B - A)$ which by Theorem 2.2.19 can be expressed as $(A \cap \overline{B}) \cup (B \cap \overline{A})$. By Theorem 2.2.23, which proves that set unions are distributive over set intersections, the following expression is equivalent $[A \cup (B \cap \overline{A})] \cap [\overline{B} \cup (B \cap \overline{A})]$. Again, by Theorem 2.2.23, we have $[(A \cup B) \cap (A \cup \overline{A})] \cap [(\overline{B} \cup B) \cap (\overline{B} \cup \overline{A})]$. Let the logical definition for this expression be represented in two propositional variables $p \wedge q$ such that:

$p \equiv \{[(x \in A) \vee (x \in B)] \wedge [(x \in A) \vee (x \notin A)]\}$
$q \equiv \{[(x \notin B) \vee (x \in B)] \wedge [(x \notin B) \vee (x \notin A)]\}$.

By the logical law of identity $p \equiv (x \in A) \vee (x \in B)$, and $q \equiv (x \notin B) \vee (x \notin A)$. So we have $[(x \in A) \vee (x \in B)] \wedge [(x \notin B) \vee (x \notin A)]$. Applying DeMorgans law to the right-hand side of the conjunction we get $[(x \in A) \vee (x \in B)] \wedge \neg[(x \in B) \wedge (x \in A)]$. Then by definition, x is an element in $(A \cup B) \cap \overline{(A \cap B)}$, and according to Theorem 2.2.19 x is an element in $(A \cup B) - (A \cap B)$.

Proving the converse case is trivial. Let x be an element in $(A \cup B) - (A \cap B)$. By Theorem 2.2.19 x is an element in $(A \cup B) \cap \overline{(A \cap B)}$. By definition we have, $[(x \in A) \vee (x \in B)] \wedge \neg[(x \in B) \wedge (x \in A)]$. By DeMorgans law, $[(x \in A) \vee (x \in B)] \wedge [(x \notin B) \vee (x \notin A)]$. Let this logical formula be represented in two propositional variables $p \wedge q$ such that $p \equiv (x \in A) \vee (x \in B)$ and $q \equiv (x \notin B) \vee (x \notin A)$. By the logical law of identity $p \wedge T \equiv T$, and $q \wedge T \equiv T$. By the negation law of logic, $(x \notin B) \vee (x \in B) \equiv T$, and $(x \in A) \vee (x \notin A) \equiv T$. Therefore,

$p \equiv \{[(x \in A) \vee (x \in B)] \wedge [(x \in A) \vee (x \notin A)]\}$
$q \equiv \{[(x \notin B) \vee (x \in B)] \wedge [(x \notin B) \vee (x \notin A)]\}$.

By definition, x is an element in $[(A \cup B) \cap (A \cup \overline{A})] \cap [(\overline{B} \cup B) \cap (\overline{B} \cup \overline{A})]$. By Theorem 2.2.23, factoring A out of the left-hand side of the intersection, and factoring $\overline{B}$ out of the right-hand side of the intersection, the following expression is equivalent $[A \cup (B \cap \overline{A})] \cap [\overline{B} \cup (B \cap \overline{A})]$. Again, by Theorem 2.2.23, factoring $(B \cap \overline{A})$ out of the intersection we have the following equivalent expression $(A \cap \overline{B}) \cup (B \cap \overline{A})$. Which, by Theorem 2.2.19 is equivalently stated as $(A - B) \cup (B - A)$, defined by $[(x \in A) \wedge (x \notin B)] \vee [(x \notin A) \wedge (x \in B)]$. But this is the formal definition for the symmetric difference of sets, so x must be an element in $A \oplus B$.

Since $A \oplus B \subseteq (A \cup B) - (A \cap B)$ and $(A \cup B) - (A \cap B) \subseteq A \oplus B$, it follows immediately that $A \oplus B = (A \cup B) - (A \cap B)$. ∎