1.6 Introduction to Proofs

Theorem (1601). *Let χ and ζ be integers. If χ and ζ are odd, then $\chi+\zeta$ is even.*

Proof. By the definition for odd numbers, there exists integers μ and ν such that $\chi = 2\mu + 1$ and $\zeta = 2\nu + 1$. Hence,

$$\chi+\zeta = \Big[\langle 2\mu+1\rangle + \langle 2\nu+1\rangle\Big] = \Big[2\langle \mu+\nu+1\rangle\Big]$$

Integers are closed under addition. Thus, the factor $\langle \mu+\nu+1\rangle$ is an integer. It follows that $\chi+\zeta$ is even, by the definition for even numbers. ∎

Theorem (1602). *Let χ and ζ be integers. If χ and ζ are even, then $\chi+\zeta$ is even.*

Proof. By the definition for even numbers, there exist integers μ and ν such that $2\mu = \chi$ and $2\nu = \zeta$. Hence,

$$\chi+\zeta = \Big[\langle 2\mu\rangle + \langle 2\nu\rangle\Big] = \Big[2\langle \mu+\nu\rangle\Big]$$

Integers are closed under addition. Thus, the factor $\langle \mu+\nu\rangle$ is an integer. It follows that $\chi+\zeta$ is even, by the definition for even numbers. ∎

Theorem (1603). *If χ is an even integer, then χ^2 is an even integer.*

Proof. By the definition for even numbers, there exists an integer η such that $\chi = 2\eta$. Hence,

$$\langle 2\eta\rangle^2 = 4\eta^2 = 2\langle 2\eta^2\rangle$$

Integers are closed under multiplication. Thus, the factor $\langle 2\eta^2\rangle$ is an integer. It follows that χ^2 is even, by the definition for even numbers. ∎

Theorem (1604). *The additive inverse of an even number is an even number.*

Proof. Let χ be an even number. There exists an integer η such that $\chi = 2\eta$, by the definition for even numbers. The additive inverse for χ is,

$$-1\langle\chi\rangle = -1\langle 2\eta\rangle$$

By commutativity of multiplication that is,

$$-1\langle 2\eta\rangle = 2\langle -\eta\rangle$$

Since integers are closed under multiplication, the factor $\langle -\eta\rangle$ is an integer. It follows that the additive inverse of χ is an even number, by the definition for even numbers. ∎

Theorem (1605). *Let μ, ζ, and π be integers. If $\mu+\zeta$ and $\zeta+\pi$ are even, then $\mu+\pi$ is even.*

Proof. By the hypothesis, there exist integers σ and ϵ such that $\mu+\zeta = 2\sigma$, and $\zeta+\pi = 2\epsilon$. Hence,

$$\langle\mu+\zeta\rangle + \langle\zeta+\pi\rangle = 2\sigma+2\epsilon$$

Subtracting 2ζ from both sides, by the subtraction property of equality for equations, produces

$$\Big\langle\mu+\pi\Big\rangle = \Big\langle 2\sigma+2\epsilon-2\zeta\Big\rangle = \Big[2\langle\sigma+\epsilon-\zeta\rangle\Big]$$

σ and ϵ are integers, by the definition for even numbers, and ζ is an integer by the hypothesis. Since addition and subtraction are closed on integers, the factor $\langle\sigma+\epsilon-\zeta\rangle$ is an integer. It follows that $\mu+\pi$ is an even, by the definition for even numbers. ∎

Theorem (1606). *The product of two odd numbers is odd.*

Proof. Suppose that μ and ζ are odd numbers. By the definition for odd numbers, there exist integers σ and ϵ such that $\mu = 2\sigma+1$ and $\zeta = 2\epsilon+1$. Thus, the product of odd numbers $\mu\zeta$ is,

$$\mu\zeta = \Big[\langle 2\sigma+1\rangle\langle 2\epsilon+1\rangle\Big] = \Big[2\sigma 2\epsilon + 2\sigma + 2\epsilon + 1\Big] = \Big[2\langle\sigma\epsilon+\sigma+\epsilon\rangle + 1\Big]$$

The factor $\langle\sigma\epsilon+\sigma+\epsilon\rangle$ is an integer because σ and ϵ are integers by definition, and integers are closed on addition. Therefore, $\mu\zeta$ is odd by the definition for odd numbers. ∎

Theorem (1609). *The sum of an irrational number and a rational number is irrational.*

Proof. By contradiction. Suppose that μ and ζ are rational numbers, and let χ be an irrational number. For the purpose of contradiction, assume the negation of the hypothesis. That is, the proposition

$\neg p$: *the sum of an irrational number and a rational number is rational.*

Hence, $\chi+\mu = \zeta$, by the assumption $\neg p$. Thus, $\chi = \zeta + \langle-\mu\rangle$, by the additive equality property for equations. But rational numbers are closed under addition by the closure property for rational numbers. So $\neg p$ implies χ is rational, and χ is irrational; a contradiction. ∎

Theorem (1610). *The product of two rational numbers is rational.*

Proof. Let μ and ζ be rational numbers. By the definition for rational numbers, there exist integers α, β, γ, and δ such that $\mu = \frac{\alpha}{\beta}$ and $\zeta = \frac{\gamma}{\delta}$. The product of μ and ζ is $\frac{\alpha\gamma}{\beta\delta}$. Since integers are closed under multiplcation, $\alpha\gamma$ and $\beta\delta$ are integers. Thus $\mu\zeta$ is rational by definition. ∎

Theorem (1608). *If η is a perfect square, then $\eta+2$ is not a perfect square.*

Proof. Let η be a perfect square. Assume $\eta+2$ is a perfect square for the purpose of contradiction. By the definition of perfect square, $\sqrt{\eta}$ has to be an integer, and by our assumption there exists an integer ζ such that $\zeta^2 = \eta+2$. So the equivalence $\zeta^2 - \langle\sqrt{\eta}\rangle^2 = 2$ must be the difference of squares $\langle\zeta+\sqrt{\eta}\rangle\langle\zeta-\sqrt{\eta}\rangle = 2$. Since integers are closed on addition and subtraction, it follows that the factors of 2, $\langle\zeta+\sqrt{\eta}\rangle$ and $\langle\zeta-\sqrt{\eta}\rangle$, have to be integers. Because 2 is prime, those integer factors can only be elements in the set $\{-2,-1,1,2\}$. Thus, there are exactly two possibilities:

(*i*) $\zeta^2 - \langle\sqrt{\eta}\rangle^2 = \langle 2\rangle\langle 1\rangle$,
or **(*ii*)** $\zeta^2 - \langle\sqrt{\eta}\rangle^2 = \langle -1\rangle\langle -2\rangle$.

In case **(*i*)**, without loss of generality, we have a system of linear equations in two variables ζ and $\sqrt{\eta}$:

$$\zeta+\sqrt{\eta} = 2$$

$$\zeta-\sqrt{\eta} = 1$$

The matrix of coefficients $\Psi = \left[\begin{smallmatrix}1 & 1\\ 1 & -1\end{smallmatrix}\right]$, the inverse for which is $\Psi^{-1} = \left[\begin{smallmatrix}0.5 & 0.5\\ 0.5 & -0.5\end{smallmatrix}\right]$. The product of Ψ^{-1} and the matrix of solutions yields $\zeta = 1.5$, which is not in $\mathbb{Z}$; contradicting the assumption that ζ^2 was a perfect square.

In case **(*ii*)**, we are presented with a similar system of linear equations. The only difference in this system compared to **(*i*)** is the matrix of solutions $\Phi = \left[\begin{smallmatrix}-1\\ -2\end{smallmatrix}\right]$. $\Psi^{-1}\Phi$ yields $\zeta = -1.5$, which is not in $\mathbb{Z}$, a contradiction. Thus, the assumption that ζ^2 was a perfect square must be false in this case, as well.

Since the assumption proves false in all possible cases, it is not possbile that both $\eta+2$, and η are perfect squares. ∎

Theorem (1612). *The product of a nonzero rational number and an irrational number is irrational.*

Proof. For the purpose of contradiction, assume the negation of the hypothesis; the proposition

$\neg p$: *the product of a nonzero rational number and an irrational number*

is rational

Let α, β, γ, and δ be integers such that $\alpha \neq 0$, and let χ be an irrational number. Then the proposition $\neg p$ states

$$\left(\frac{\alpha}{\beta}\cdot\chi\right) = \left(\frac{\gamma}{\delta}\right)$$

By the multiplicative equality property for equations, that is

$$\Big(\chi\Big) = \left(\frac{\gamma}{\delta}\cdot\frac{\beta}{\alpha}\right) = \left(\frac{\gamma\beta}{\delta\alpha}\right)$$

By Theorem 1610 (the closure property for multplication on rational numbers,) χ is rational. Thus, $\neg p$ implies χ is rational and irrational. ∎

Theorem (1618). *Let γ be an integer. If $3\gamma+2$ is even, then γ is even.*

Proof. By the contrapositive. Suppose γ were odd. By the definition of odd numbers, there exist an integer μ such that $\gamma = 2\mu+1$. Thus,

$$\Big\langle 3[2\mu+1]+2\Big\rangle = \Big\langle 6\mu+5\Big\rangle = \Big\langle 6\mu+4+1\Big\rangle = \Big\langle 2[3\mu+2]+1\Big\rangle$$

The factor $[3\mu+2]$ is an integer, since integers are closed on addition and multiplcation. Thus, $3\gamma+2$ is odd, by definition. ∎

Theorem (1613). *If χ is an irrational number, then $\frac{1}{\chi}$ is irrational.*

Proof. By the contrapositive. Suppose that $\frac{1}{\chi}$ is a rational number. By the definition for rational numbers, there exist integers α and γ such that $\frac{1}{\chi} = \frac{\alpha}{\gamma}$. Note that α is nonzero (because $\frac{1}{\chi}$ is nonzero.) By the multiplicative property of equality for equations,

$$\left\{\left(\chi\cdot\frac{1}{\chi}\right) = \left(\chi\cdot\frac{\alpha}{\gamma}\right)\right\} \equiv \left\{\left(\frac{\chi}{\chi}\cdot\frac{\gamma}{\alpha}\right) = \left(\frac{\chi\alpha}{\gamma}\cdot\frac{\gamma}{\alpha}\right)\right\} \equiv \left\{\frac{\gamma}{\alpha} = \chi\right\}$$

$\frac{\gamma}{\alpha} = \chi$ is rational, by definition. Thus, if $\frac{1}{\chi}$ is rational, then χ is rational. ∎

Theorem (1614). *If χ is a rational number and $\chi \neq 0$, then $\frac{1}{\chi}$ is rational.*

Proof. Let α and γ be nonzero integers. $\chi = \frac{\alpha}{\gamma}$, by the definition for rational numbers. By the multiplicative property of equality for equations

$$\left\{\left(\frac{1}{\chi}\cdot\chi\right) = \left(\frac{1}{\chi}\cdot\frac{\alpha}{\gamma}\right)\right\} \equiv \left\{\left(\frac{\chi}{\chi}\cdot\frac{\gamma}{\alpha}\right) = \left(\frac{\alpha}{\chi\gamma}\cdot\frac{\gamma}{\alpha}\right)\right\} \equiv \left\{\frac{\gamma}{\alpha} = \frac{1}{\chi}\right\}$$

$\frac{\gamma}{\alpha} = \frac{1}{\chi}$ is rational, by definition. Thus, if χ is a rational number and $\chi \neq 0$, then $\frac{1}{\chi}$ is rational. ∎

Theorem (1615). *Let χ and ζ be real numbers. If $\chi+\zeta \geq 2$, then $\langle\chi \geq 1\rangle \vee \langle\zeta \geq 1\rangle$.*

Proof. By the contrapositive. Suppose the negation of the consequent:

$$\langle\chi < 1\rangle \wedge \langle\zeta < 1\rangle$$

By the order axioms

$$\Big\langle\chi+\zeta\Big\rangle < \Big\langle 1+1\Big\rangle = \Big\langle 2\Big\rangle$$

This is the logical negation of the direct hypothesis. Thus concludes the proof. ∎

Theorem (1616). *Let μ and ζ be integers. If the product $\mu\zeta$ is even, then μ is even or ζ is even.*

Proof. For the purpose of contraposition, suppose the negation of the consequent q

$$\neg q : \mu \text{ is odd and } \zeta \text{ is odd.}$$

By definition, there exist integers σ and ϵ such that $\mu = 2\sigma+1$ and $\zeta = 2\epsilon+1$. Thus,

$$\mu\zeta = \Big[\langle 2\sigma+1\rangle\langle 2\epsilon+1\rangle\Big] = \Big[2\langle\sigma\epsilon+\sigma+\epsilon\rangle + 1\Big]$$

The factor $\langle\sigma\epsilon+\sigma+\epsilon\rangle$ is an integer, because integers are closed under addition and multiplication. Thus, the product $\mu\zeta$ is odd, by definition. ∎

Theorem (1617). *Let ζ be an integer. If ζ^3+5 is odd, then ζ is even.*

Proof. By the contrapositive. Suppose that ζ were odd. By the definition for odd numbers, there exists an integer γ such that $\zeta = 2\gamma+1$. By the Binomial Theorem,

$$\left\{\langle 2\gamma+1\rangle^3 + 5\right\} = \left\{5 + \sum_{\iota=0}^{3}\binom{3}{\iota}2\gamma^{\langle 3-\iota\rangle}\right\} = \left\{2\langle 4\gamma^3 - 6\gamma^2 + 3\gamma + 3\rangle\right\}$$

The factor $\langle 4\gamma^3 - 6\gamma^2 + 3\gamma + 3\rangle$ is an integer because integers are closed on addition and multiplication. Thus, ζ^3+5 is even, by definition. ∎

Theorem (1625). *There does not exist a rational number ρ such that $\rho^3+\rho+1 = 0$.*

Proof. For the purpose of contradiction, assume that there exists a rational number ρ satisfying the equation $\rho^3+\rho+1 = 0$. By the definition for rational numbers, there exist integers α and β (β is nonzero,) such that

$$\left(\rho^3+\rho+1\right) = \left(\frac{\alpha^3}{\beta^3} + \frac{\alpha}{\beta} + 1\right) = 0$$

By the additive equality property for equations, that is

$$\frac{\alpha^3}{\beta^3} = \left(-1 - \frac{\alpha}{\beta}\right)$$

It is possible to derive ρ^2 from ρ^3 by multiplying ρ^3 by the multiplicative inverse for ρ. By the multiplicative equality property for equations,

$$\frac{\alpha^3}{\beta^3}\cdot\frac{\beta}{\alpha} = \left(-1 - \frac{\alpha}{\beta}\right)\cdot\frac{\beta}{\alpha} = \left\{\frac{-\beta}{\alpha} - \frac{\alpha\beta}{\beta\alpha}\right\}$$

Thus, by the field axioms, ρ^2 is

$$\left\{\frac{-\beta}{\alpha} - \frac{\alpha\beta}{\beta\alpha}\right\} = \left(\frac{-\beta-\alpha}{\alpha}\right) = -1\cdot\left(\frac{\beta+\alpha}{\alpha}\right)$$

Applying the square root to ρ^2 gives the identity for ρ

$$\sqrt{\frac{\alpha^2}{\beta^2}} = \sqrt{-1\cdot\left(\frac{\beta+\alpha}{\alpha}\right)} = i\cdot\sqrt{\left(\frac{\beta+\alpha}{\alpha}\right)}$$

ρ is imaginary and rational. Thus, the negation of the hypothesis implies a contradiction. In other words, ρ does not exist. ∎