

**Theorem (2.3.68).** *Let  $f$  be a function  $f : A \implies B$ , where  $A$  and  $B$  are finite sets, and  $|A| = |B|$ .  $f$  is injective if and only if  $f$  is surjective.*

*Proof.* Direct form by the contrapositive. Suppose the negation of the statement given by the definition for surjective functions,  $\exists y \forall x (f(x) \neq y)$ . This statement can only be true if either  $|A| < |B|$  (contradicting the hypothesis,) or  $\exists x \exists y ((f(x) = f(y)) \wedge (x \neq y))$ . Since contradiction is  $\perp$  by the law for logical negation, by the identity law for logical disjunction,  $f$  is defined as not injective. Thus, if  $f$  is injective, then  $f$  is surjective.

Converse form by the contrapositive. Suppose the negation of the statement given by the definition for injective functions,  $\exists x \exists y ((f(x) = f(y)) \wedge (x \neq y))$ . This statement can only be true if either  $|A| > |B|$  (contradicting the hypothesis,) or  $\exists y \forall x (f(x) \neq y)$ . Since contradiction is  $\perp$  by the law for logical negation, by the identity law for logical disjunction,  $f$  is defined as not surjective. Thus, if  $f$  is surjective, then  $f$  is injective.

$$\therefore \forall x \forall y ((f(x) = f(y)) \implies (x = y)) \iff \forall y \exists x (f(x) = y). \quad \blacksquare$$