Prove that if x is a positive real number, then $\left\lfloor\sqrt{\lfloor x\rfloor}\right\rfloor = \lfloor\sqrt{x}\rfloor$.

Proof. $\lfloor\sqrt{x}\rfloor = n$ if and only if $n \leq \sqrt{x} < n+1$. That is, $n^2 \leq x < (n+1)^2 \;\equiv\; n^2 \leq x < n^2+2n+1$. Then there are two case under consideration. Case $(i):$ $\lfloor x\rfloor = n^2$, or case $(ii):$ $\lfloor x\rfloor = n^2+2n$.

$(i): \lfloor x\rfloor = n^2$ if and only if $n^2 \leq x < n^2+1$. Consequently, $\left\lfloor\sqrt{\lfloor x\rfloor}\right\rfloor = \left\lfloor\sqrt{n^2}\right\rfloor = \lfloor n\rfloor = n$. Since it has already been established that $\lfloor\sqrt{x}\rfloor = n$, then in this case it follows that $\left\lfloor\sqrt{\lfloor x\rfloor}\right\rfloor = \lfloor\sqrt{x}\rfloor$.

$(ii): \lfloor x\rfloor = n^2+2n$ if and only if $n^2+2n \leq x < n^2+2n+1$. It follows that $\left\lfloor\sqrt{\lfloor x\rfloor}\right\rfloor = \lfloor\sqrt{n^2+2n}\rfloor = \lfloor\sqrt{n^2+2n}+\sqrt{1}-\sqrt{1}\rfloor = \lfloor\sqrt{n^2+2n+1}-1\rfloor = \left\lfloor\sqrt{(n+1)^2}-1\right\rfloor = \lfloor(n+1)-1\rfloor = n$. Since $\lfloor\sqrt{x}\rfloor = n$, and $\left\lfloor\sqrt{\lfloor x\rfloor}\right\rfloor = n$, in this case it follows that $\lfloor\sqrt{x}\rfloor = \left\lfloor\sqrt{\lfloor x\rfloor}\right\rfloor$.

$\therefore$ if x is a positive real number, then $\left\lfloor\sqrt{\lfloor x\rfloor}\right\rfloor = \lfloor\sqrt{x}\rfloor$. ■