Theorem (2.4.22)**.** *The sum of squares from 1 to n is $\frac{n(n+1)(2n+1)}{6}$.*

Proof. Let n and k be integers. By the Binomial Theorem, $(k-1)^3 = k^3 - 3k^2 + 3k - 1$. This means that $k^3 - (k-1)^3 = 3k^2 - 3k + 1$. By Theorem 2.4.19, $n^3 = \sum_{k=1}^{n} k^3 - (k-1)^3$. Thus, $n^3 = \sum_{k=1}^{n} 3k^2 - 3k + 1$. This statement is equivalent to $n^3 = (3\sum_{k=1}^{n} k^2) - (3\sum_{k=1}^{n} k) + (\sum_{k=1}^{n} 1)$. By Theorem 2.4.21a, and by Theorem 2.4.21b, that is $n^3 = (3\sum_{k=1}^{n} k^2) - 3\frac{n(n+1)}{2} + n$; or rather $n^3 + 3\frac{n(n+1)}{2} - n = 3\sum_{k=1}^{n} k^2$. Eliminating the coefficient 3 from the right-hand side, all that is left to do is to simplify the left-hand side $\frac{1}{3}[n^3 + 3\frac{n(n+1)}{2} - n] = \sum_{k=1}^{n} k^2$. On the left-hand side we have $\frac{2n^3+3n^2+3n-2n}{6} = \frac{2n^3+3n^2+n}{6}$. Factoring $\frac{1}{6}n$ out of this expression gives $\frac{1}{6}n(2n^2+3n+1) = \frac{1}{6}n(2n^2+2n+n+1)$. Factoring the first two terms in the sum, $\frac{1}{6}n[2n(n+1)+(n+1)]$. The simplification process is completed by factoring $(n+1)$ out of the sum, $\frac{1}{6}n(n+1)(2n+1)$. That is, $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$. ■