2.4 Sequences and Summations

Theorem (2.4.19). *Let $\{a_n\}$ be a sequence of real numbers.*

$$\sum_{j=1}^{n}(a_j - a_{j-1}) = a_n - a_0$$

.

Proof.

$$\sum_{j=1}^{n}(a_j - a_{j-1}) = (a_n - a_{n-1}) + (a_{n-1} - a_{n-2}) + \cdots + (a_1 - a_0)$$

By associativity for addition from the field axioms for real numbers, that is

$a_n + (-a_{n-1} + a_{n-1}) + (-a_{n-2} + a_{n-2}) + (-a_{n-3} + a_{n-3}) + \cdots + (-a_1 + a_1) + -a_0$

Clearly the inner terms cancel out. Thus,

$$\sum_{j=1}^{n}(a_j - a_{j-1}) = a_n - a_0$$

■

Theorem (2.4.20).

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$$

Proof. The identity of $\left(\frac{1}{k(k+1)}\right)$ is $\left(\frac{1}{k} - \frac{1}{(k+1)}\right)$. This can be demonstrated by the equation

$$k\left(\frac{1}{k} - \frac{1}{(k+1)}\right) = \left(\frac{k+1}{k+1} - \frac{k}{k+1}\right) = \left(\frac{k+1-k}{k+1}\right) = \left(\frac{1}{k+1}\right)$$

Dividing both sides of this equation by k gives the desired identity such that

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right)$$

The sequence for which is the telescopic summation

$$\left(\frac{1}{n} - \frac{1}{n+1}\right) + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n-2} - \frac{1}{n-1}\right) + \cdots + \left(\frac{1}{1} - \frac{1}{2}\right)$$

Thus, by Theorem 2.4.19

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \left(-\frac{1}{n+1} + \frac{1}{1}\right) = \left(\frac{(-1)+(n+1)}{n+1}\right) = \frac{n}{n+1}$$

■

Theorem (2.4.21a). *The summation of odd numbers from 1 to n is n^2.*

Proof. The summation of odd numbers from 1 to n is given by,

$$\sum_{k=1}^{n} 2k - 1$$

by the definition for odd numbers. The identity of $2k - 1$ is the difference of squares $k^2 - (k-1)^2$. This identity can be demonstrated by the statement

$$k^2 - (k-1)^2 = [k + (k-1)][k - (k-1)] = (2k-1)[k + (-k+1)] = (2k-1)1$$

So the summation of odd numbers from 1 to n is the telescoping summation

$$\sum_{k=1}^{n} k^2 - (k-1)^2$$

By Theorem 2.4.19, that is $n^2 - 0^2 = n^2$. Thus,

$$\sum_{k=1}^{n} 2k - 1 = n^2$$

and indeed the summation of odd numbers from 1 to n is n^2. ∎

Theorem (2.4.21b). *The summation of natural numbers from 1 to n is*

$$\frac{n(n+1)}{2}$$

Proof. From Theorem 2.4.21a we know that

$$\sum_{k=1}^{n} 2k - 1 = n^2$$

This is the same as saying

$$n^2 = \left(-n + \sum_{k=1}^{n} 2k\right) \equiv \left(\sum_{k=1}^{n} 2k\right) = (n^2 + n) = n(n+1)$$

We can factor the coefficient 2 out of the term of summation,

$$2\sum_{k=1}^{n} k = n(n+1)$$

And of course dividing both sides by 2 gives

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

Thus, indeed, the summation of natural numbers from 1 to n is $\frac{n(n+1)}{2}$. ∎

Theorem (2.4.22). *The sum of squares from 1 to n is*

$$\frac{n(n+1)(2n+1)}{6}$$

.

Proof. Let $\{a_n\}$ be the sequence of integers from 1 to n. The formula for the summation of squares from 1 to n can be derived from the cube of n. By theorem 2.4.19,

$$n^3 = \sum_{k=1}^{n} k^3 - (k-1)^3$$

This summation is telescopic, and thus collapses to $n^3 - (1-1)^3 = n^3$. The expansion for $(k-1)^3$ in that term of summation is $k^3 - 3k^2 + 3k - 1$, by the Binomial Theorem. Thus, yielding the algebraic identity

$$k^3 - (k-1)^3 = 3k^2 - 3k + 1$$

Hence, $n^3 = \sum_{k=1}^{n} 3k^2 - 3k + 1$, and by the field axioms,

$$n^3 = \left(3\sum_{k=1}^{n} k^2\right) - \left(3\sum_{k=1}^{n} k\right) + \left(\sum_{k=1}^{n} 1\right)$$

Note that $(\sum_{k=1}^{n} 1) = n(1)$, and by Theorem 2.4.21b, $(3\sum_{k=1}^{n} k) = 3(\frac{n(n+1)}{2})$. Thus,

$$n^3 + 3\frac{n(n+1)}{2} - n = 3\sum_{k=1}^{n} k^2$$

Eliminating the coefficient 3 from the right-hand side by division gives us the sum of squares in terms of an equation,

$$\frac{1}{3}\left(n^3 + 3\frac{n(n+1)}{2} - n\right) = \sum_{k=1}^{n} k^2$$

All that is left to do is to simplify the left-hand side $\frac{1}{3}[n^3 + 3\frac{n(n+1)}{2} - n] = \frac{2n^3+3n^2+3n-2n}{6}$. Factoring $\frac{1}{6}n$ gives $\frac{1}{6}n(2n^2+3n+3-2) = \frac{1}{6}n(2n^2+2n+n+1)$. Factoring $2n$ out of the first two terms in the sum, $\frac{1}{6}n[2n(n+1)+(n+1)]$. The simplification process is complete by factoring $(n+1)$ out of the sum, $\frac{1}{6}n(n+1)(2n+1)$. Thus, the sum of squares

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

■

Theorem (2.4.25). *Let m be a positive integer. The closed form formula for $\sum_{k=0}^{m}\lfloor\sqrt{k}\rfloor$ is $\lfloor\sqrt{m}\rfloor[\frac{1}{6}(\lfloor\sqrt{m}\rfloor-1)(4\lfloor\sqrt{m}\rfloor+1)+(m-\lfloor\sqrt{m}\rfloor^2+1)]$.*

Proof. By the properties for floor functions, there exists an integer $n=\lfloor\sqrt{k}\rfloor$ if and only if $n^2\le k<n^2+2n+1$. Thus, each integer value $n<\lfloor\sqrt{m}\rfloor$ occurs exactly $2n+1$ times, in the terms of summation. The value $n=\lfloor\sqrt{m}\rfloor$ occurs exactly $(m-\lfloor\sqrt{m}\rfloor^2+1)$ times. Subtracting those terms $n=\lfloor\sqrt{m}\rfloor$ from $\sum_{k=0}^{m}\lfloor\sqrt{k}\rfloor$ produces the sequence

$$\lfloor\sqrt{0}\rfloor(2\lfloor\sqrt{0}\rfloor+1)+\cdots+(\lfloor\sqrt{m}\rfloor-1)[2(\lfloor\sqrt{m}\rfloor-1)+1]$$

Summarily expressed as

$$\sum_{n=0}^{\lfloor\sqrt{m}\rfloor-1} n(2n+1)$$

Thus,

$$\sum_{k=0}^{m}\lfloor\sqrt{k}\rfloor=\left(\sum_{n=0}^{\lfloor\sqrt{m}\rfloor-1} n(2n+1)\right)+\lfloor\sqrt{m}\rfloor(m-\lfloor\sqrt{m}\rfloor^2+1)$$

That is, the summation of squares, and integers

$$\sum_{k=0}^{m}\lfloor\sqrt{k}\rfloor=\left(2\sum_{n=0}^{\lfloor\sqrt{m}\rfloor-1} n^2\right)+\left(\sum_{n=0}^{\lfloor\sqrt{m}\rfloor-1} n\right)+[\lfloor\sqrt{m}\rfloor(m-\lfloor\sqrt{m}\rfloor^2+1)]$$

By Theorem 2.4.22, and by Theorem 2.4.21b

$$\sum_{k=0}^{m}\lfloor\sqrt{k}\rfloor=\left\{\frac{2}{6}\lfloor\sqrt{m}\rfloor(\lfloor\sqrt{m}\rfloor-1)[2(\lfloor\sqrt{m}\rfloor-1)+1]\right\}+$$

$$\left\{\frac{3}{6}\lfloor\sqrt{m}\rfloor(\lfloor\sqrt{m}\rfloor-1)\right\}+\lfloor\sqrt{m}\rfloor(m-\lfloor\sqrt{m}\rfloor^2+1)$$

Factoring $\frac{1}{6}\lfloor\sqrt{m}\rfloor(\lfloor\sqrt{m}\rfloor-1)$ out of the first two terms yields

$$\frac{1}{6}\lfloor\sqrt{m}\rfloor(\lfloor\sqrt{m}\rfloor-1)\{2[2(\lfloor\sqrt{m}\rfloor-1)+1]+3\}+\lfloor\sqrt{m}\rfloor(m-\lfloor\sqrt{m}\rfloor^2+1)$$

And by arithmetic simplification that is

$$\frac{1}{6}\lfloor\sqrt{m}\rfloor(\lfloor\sqrt{m}\rfloor-1)(4\lfloor\sqrt{m}\rfloor+1)+\lfloor\sqrt{m}\rfloor(m-\lfloor\sqrt{m}\rfloor^2+1)$$

Factoring $\lfloor\sqrt{m}\rfloor$ from the outer sum completes the derivation

$$\sum_{k=0}^{m}\lfloor\sqrt{k}\rfloor=\lfloor\sqrt{m}\rfloor\left[\frac{1}{6}(\lfloor\sqrt{m}\rfloor-1)(4\lfloor\sqrt{m}\rfloor+1)+(m-\lfloor\sqrt{m}\rfloor^2+1)\right]$$

. ■

Theorem (2.4.26). *Let m be a positive integer. The closed form formula for* $\sum_{k=0}^{m} \lfloor \sqrt[3]{k} \rfloor$ *is* $\lfloor \sqrt[3]{m} \rfloor [\frac{1}{4}(\lfloor \sqrt[3]{m} \rfloor^2 - \lfloor \sqrt[3]{m} \rfloor)(3\lfloor \sqrt[3]{m} \rfloor + 1) + (m - \lfloor \sqrt[3]{m} \rfloor^3 + 1)]$.

Proof. By the properties for floor functions there exists an integer $n_k = \lfloor \sqrt[3]{k} \rfloor$ such that $n_k^3 \leq k < n_k^3 + 3n_k^2 + 3n_k + 1$. This means that each value less than $\lfloor \sqrt[3]{m} \rfloor$ in the terms of summation occurs exactly $3\lfloor \sqrt[3]{k} \rfloor^2 + 3\lfloor \sqrt[3]{k} \rfloor + 1$ times. The maximum value in the terms of summation occurs $(m - \lfloor \sqrt[3]{m} \rfloor^3 + 1)$ times. Subtracting those terms consisting of the maximum value produces the sequence

$$n_0(3n_0^2 + 3n_0 + 1) + n_1(3n_1^2 + 3n_1 + 1) + \cdots + n_{\lfloor \sqrt[3]{m} \rfloor - 1}(3n^2_{\lfloor \sqrt[3]{m} \rfloor - 1} + 3_{\lfloor \sqrt[3]{m} \rfloor - 1} + 1)$$

Summarily expressed as

$$\sum_{n=0}^{\lfloor \sqrt[3]{m} \rfloor - 1} n(3n^2 + 3n + 1)$$

Thus,

$$\sum_{k=0}^{m} \lfloor \sqrt[3]{k} \rfloor = \left(\sum_{n=0}^{\lfloor \sqrt[3]{m} \rfloor - 1} n(3n^2 + 3n + 1) \right) + \lfloor \sqrt[3]{m} \rfloor (m - \lfloor \sqrt[3]{m} \rfloor^3 + 1)$$

That is, the summation of cubes, squares, and integers

$$\sum_{k=0}^{m} \lfloor \sqrt[3]{k} \rfloor = \left(3 \sum_{n=0}^{\lfloor \sqrt[3]{m} \rfloor - 1} n^3 \right) + \left(3 \sum_{n=0}^{\lfloor \sqrt[3]{m} \rfloor - 1} n^2 \right) + \left(\sum_{n=0}^{\lfloor \sqrt[3]{m} \rfloor - 1} n \right) +$$
$$\lfloor \sqrt[3]{m} \rfloor (m - \lfloor \sqrt[3]{m} \rfloor^3 + 1)$$

By the closed form formula for each individual summation we have,

$$\sum_{k=0}^{m} \lfloor \sqrt[3]{k} \rfloor = \left\{ \frac{3}{4} \lfloor \sqrt[3]{m} \rfloor^2 (\lfloor \sqrt[3]{m} \rfloor - 1)^2 \right\} +$$
$$\left\{ \frac{2}{4} \lfloor \sqrt[3]{m} \rfloor (\lfloor \sqrt[3]{m} \rfloor - 1)[2(\lfloor \sqrt[3]{m} \rfloor - 1) + 1] \right\} + \left\{ \frac{2}{4} \lfloor \sqrt[3]{m} \rfloor (\lfloor \sqrt[3]{m} \rfloor - 1) \right\} +$$
$$\lfloor \sqrt[3]{m} \rfloor (m - \lfloor \sqrt[3]{m} \rfloor^3 + 1)$$

Algebraic simplification completes the derivation

$$\sum_{k=0}^{m} \lfloor \sqrt[3]{k} \rfloor = \lfloor \sqrt[3]{m} \rfloor \left[\frac{1}{4} (\lfloor \sqrt[3]{m} \rfloor^2 - \lfloor \sqrt[3]{m} \rfloor)(3\lfloor \sqrt[3]{m} \rfloor + 1) + (m - \lfloor \sqrt[3]{m} \rfloor^3 + 1) \right]$$

■

Theorem (2.4.36). *A subset of a countable set is countable.*

Proof. Let A and B be sets such that $A \subseteq B$. B is countable by the hypothesis, and so by the definition for countability $|B| \leq \aleph_0$. By the definition of subset, $|A| \leq |B| \therefore |A| \leq \aleph_0$ and it follows that the subset of a countable set is countable. ∎

Theorem (2.4.37). *Let A, and B be sets such that $A \subseteq B$. If A is uncountable, then B is uncountable.*

Proof. By the hypothesis, $|A| > \aleph_0$ by the definition for countability since A is uncountable. By the definition for subset, the cardinality of B is at least the cardinality of A. Therefore the least cardinality for B is $|B| > \aleph_0$, and it follows that B is uncountable. ∎

Theorem (2.4.38). *Let A, and B be sets with equal cardinality.*
$|P(A)| = |P(B)|$.

Proof. The cardinality of a power set is 2 to the power of the set cardinality. By the hypothesis, $|A| = |B| = n$. Therefore, $|P(A)| = 2^n$, and
$|P(B)| = 2^n$. ∎

Theorem (2.4.40). *The union of two countable sets is countable.*

Proof. By cases. Let A, and B be countable sets. There are three cases that must be considered. (i) A and B are finite, (ii) exclusively A or B is finite and the other is countably infinite, (iii) A and B are both countably infinite.

(i) Suppose A and B are finite. There exist natural numbers m, and n such that $|A| = m$ and $|B| = n$. The maximum cardinality for $A \cup B$ occurs when A and B are disjoint, where the cardinality is $m + n$. $m + n$ is a natural number less than $\aleph_0$. Thus, $A \cup B$ is finite and countable by definition.

(ii) Without loss of generality suppose A is finite with cardinality n, and B is countably infinite. It must be that a sequence exists $\{a_i\} = \{a_0, a_1, \ldots a_n\}$ containing all elements in A. Since a bijection exists between B and $\mathbb{N}$ by the definition for countability, a sequence exists $\{b_i\} = \{b_0, b_1, b_2, \ldots\}$ containing all elements in B. Clearly, for the union of A and B a sequence exists $\{c_i\} = \{a_0, a_1, \ldots, a_n, b_0, b_1, b_2, \ldots\}$. Infinite sequences are countable by definition, so $A \cup B$ is a countably infinite set.

(iii) Suppose A and B are infinitely countable sets. Since the set cardinalities are $\aleph_0$, A and B are bijective with $\mathbb{N}$. Thus, the elements in A can be ordered by the sequence $\{a_i\} = \{a_0, a_1, a_2, \ldots\}$, and the elements in B can be ordered by the sequence $\{b_i\} = \{b_0, b_1, b_2, \ldots\}$. The union of A and B can be ordered by the sequence $\{c_i\} = \{a_0, b_0, a_1, b_1, a_2, b_2, \ldots\}$. Thus a bijection exists between $\mathbb{N}$ and the union of A and B, and the cardinality of that union is $\aleph_0$. ∎

Theorem (2.4.41). *The union of a countable number of countable sets is countable.*

Proof. Let A_i be a countable set, for integers $i = 0$ to $n \leq \infty$ such that

$$S = \bigcup_{i=0}^{n} A_i$$

The function $f : \mathbb{N} \to A_i$ is the sequence $\{a_{ij}\} = a_{i0}, a_{i1}, a_{i2}, \ldots$. Thus, by f, all elements a_{ij} in S can be listed in the second dimension

$$\begin{gathered} a_{00}, a_{01}, a_{02}, \ldots \\ a_{10}, a_{11}, a_{12}, \ldots \\ a_{20}, a_{21}, a_{22}, \ldots \\ \vdots \end{gathered}$$

By tracing the diagonal path along the two dimensional listing for S we get the countable order

$$a_{00}, a_{01}, a_{10}, a_{20}, a_{11}, a_{02}, \ldots$$

$\therefore |S| \leq \aleph_0$, and indeed the union of a countable number of countable sets is countable. ■

Theorem (2.4.42). *The cardinality of $\mathbb{Z}^+ \times \mathbb{Z}^+$ is aleph null.*

Proof. $\mathbb{Z}^+ \times \mathbb{Z}^+$ is defined as $\{\langle x, y\rangle | (x \in \mathbb{Z}^+) \wedge (y \in \mathbb{Z}^+)\}$. Since x and y are positive integers, for every ordered pair $\langle x, y\rangle$ in $\mathbb{Z}^+ \times \mathbb{Z}^+$, $\langle x, y\rangle$ exists if and only if the rational number $\frac{x}{y}$ exists. Thus, $\frac{x}{y}$ exists, and all elements in $\mathbb{Z}^+ \times \mathbb{Z}^+$ can be represented by the two dimensional list

$$\begin{gathered} \langle 1,1\rangle \iff \frac{1}{1}, \langle 1,2\rangle \iff \frac{1}{2}, \langle 1,3\rangle \iff \frac{1}{3}, \ldots \\ \langle 2,1\rangle \iff \frac{2}{1}, \langle 2,2\rangle \iff \frac{2}{2}, \langle 2,3\rangle \iff \frac{2}{3}, \ldots \\ \langle 3,1\rangle \iff \frac{3}{1}, \langle 3,2\rangle \iff \frac{3}{2}, \langle 3,3\rangle \iff \frac{3}{3}, \ldots \\ \vdots \end{gathered}$$

The hypotheses in the biconditional converse statements for each list entry are the list elements in the proof for the countability of rational numbers. That means $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable if and only if the rational numbers are countable. We know the rational numbers are countable. Therefore the cardinality of $\mathbb{Z}^+ \times \mathbb{Z}^+$ is $\aleph_0$. ■

Theorem (2.4.43). *The set of all finite bit strings is countable.*

Proof. Let $\{a_{n-1}\}$ be the sequence of bits for any finite bit string a(base-2) of length n. The unique base-2 expansion for $\{a_{n-1}\}$ is the integer

$$a(\text{base-10}) = \sum_{i=0}^{n-1} a_i 2^i$$

Also, this integer can be converted to the unique base-2 bit string for a(base-10) by

$$a(\text{base-2}) = \sum_{i=0}^{n-1} \left[a(\text{base-10})(\text{mod } 2^{i+1})\right] 10^i$$

Since an invertible function exists between each finite bit string and some positive integer, there exists, a one-to-one correspondence between $\mathbb{Z}$ and the set of all finite bit strings. Thus, the cardinality for the set of all finite bit strings is $\aleph_0$, and the set of all finite bit strings is countable, by definition. ■