Theorem (1.6.8). *If n is a perfect square, then $n + 2$ is not a perfect square.*

Proof. Let n be a perfect square. Assume $n + 2$ is a perfect square for the purpose of contradiction. By the definition of perfect square, $\sqrt{n}$ has to be an integer, and by our assumption there exists an integer m such that $m^2 = n + 2$. So the equivalence $m^2 - (\sqrt{n})^2 = 2$ must be the difference of squares $(m + \sqrt{n})(m - \sqrt{n}) = 2$. Since the sum or difference of integers is an integer it follows that the factors of 2, $(m + \sqrt{n})$ and $(m - \sqrt{n})$, have to be integers. Because 2 is prime those integer factors can only be elements in $\{-2, -1, 1, 2\}$. Thus, there are exactly two possibilities:

(i) $m^2 - (\sqrt{n})^2 = (2)(1)$,
or (ii) $m^2 - (\sqrt{n})^2 = (-1)(-2)$.

In case (i), without loss of generality, we have a system of linear equations in two variables m and $\sqrt{n}$:

$$\begin{aligned} m + \sqrt{n} &= 2 \\ m - \sqrt{n} &= 1 \end{aligned}$$

The matrix of coefficients $A = \left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$, the inverse for which is $A^{-1} = \left[\begin{smallmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{smallmatrix}\right]$. The product of A^{-1} and the matrix of solutions yields $m = 1.5$, which is not in $\mathbb{Z}$; contradicting the assumption that m^2 was a perfect square.

In case (ii), we are presented with a similar system of linear equations. The only difference in this system compared to (i) is the matrix of solutions $S = \left[\begin{smallmatrix} -1 \\ -2 \end{smallmatrix}\right]$. $A^{-1}S$ yields $m = -1.5$, which is not in $\mathbb{Z}$, a contradiction. The assumption that m^2 was a perfect square must be false in this case, as well.

Since the assumption proves false in all possible cases, it is not possbile that both $m^2 = n + 2$, and n are perfect squares. ∎