Prove that $\lceil x\rceil + \lceil y\rceil - \lceil x+y\rceil = (0 \vee 1)$, whenever x and y are real numbers.

Proof. There are three cases under consideration: $(x \oplus y) \in \mathbb{Z}$, $(x \wedge y) \in \mathbb{Z}$, and $(x \wedge y) \in \mathbb{R} - \mathbb{Z}$.

Suppose the case $(x \oplus y) \in \mathbb{Z}$, and $x \in \mathbb{Z}$. Then $y \in \mathbb{R} - \mathbb{Z}$, and $\lceil y\rceil = \lceil n + \epsilon\rceil$ where $n \in \mathbb{Z}$ such that $n < y < n+1$, and $\epsilon \in \mathbb{R}$ such that $0 < \epsilon < 1$. It follows that $\lceil x\rceil + \lceil y\rceil - \lceil x+y\rceil = (x+n+1) - (x+n+1)$ $= 0$. Without loss of generality, this holds whenever $x, y = y, x$.

Suppose the case that $x, y \in \mathbb{Z}$. Then $\lceil y\rceil = \lceil n + \epsilon\rceil$ where $n, \epsilon \in \mathbb{Z}$ such that $y = n$, and $\epsilon = 0$. It follows that $\lceil x\rceil + \lceil y\rceil - \lceil x+y\rceil = (x+n+0) - (x+n+0) = 0$.

Now consider the case where $(x \wedge y) \in \mathbb{R} - \mathbb{Z}$. This means that $\lceil x\rceil + \lceil y\rceil - \lceil x+y\rceil =$ $\lceil m+\epsilon\rceil + \lceil n+\sigma\rceil - \lceil m+n+\epsilon+\sigma\rceil$, where $m, n \in \mathbb{Z}$ and $\epsilon, \sigma \in \mathbb{R}$, such that $m < x < m+1,\ n < y < n+1,$ $0 < \epsilon < 1,$ and $0 < \sigma < 1$. Within this case, there are two cases to consider: $\epsilon + \sigma > 1$, and $\epsilon + \sigma \leq 1$.

If $\epsilon + \sigma > 1,$ then $\lceil m+\epsilon\rceil + \lceil n+\sigma\rceil - \lceil m+n+\epsilon+\sigma\rceil = (m+1) + (n+1) - (m+n+2) = 0$. Because $\lceil \epsilon+\sigma\rceil = 2$, whenever $\epsilon + \sigma > 1$.

If $\epsilon + \sigma \leq 1,$ then $\lceil m+\epsilon\rceil + \lceil n+\sigma\rceil - \lceil m+n+\epsilon+\sigma\rceil = (m+1) + (n+1) - (m+n+1) = 1$. Because $\lceil \epsilon+\sigma\rceil = 1$, whenever $\epsilon + \sigma \leq 1$.

$\therefore\ \lceil x\rceil + \lceil y\rceil - \lceil x+y\rceil = (0 \vee 1)$, whenever x and y are real numbers. ∎