Theorem (2.3.70c)**.** *Let x be a real number.* $\lceil \lceil \frac{x}{2} \rceil \div 2 \rceil = \lceil \frac{x}{4} \rceil$.

Proof. Let n be an integer satisfying the properties for ceiling functions with respect to x such that $\lceil \frac{x}{4} \rceil = n$. Thus establishes the fact, $4n - 4 < x \leq 4n$. We shall proceed by analyzing the statement $\lceil \lceil \frac{x}{2} \rceil \div 2 \rceil = n$. If $\lceil \lceil \frac{x}{2} \rceil \div 2 \rceil = \lceil \frac{x}{4} \rceil$ is true, then $\lceil \lceil \frac{x}{2} \rceil \div 2 \rceil = n$ will be defined, by the properties of ceiling functions, as $4n - 4 < x \leq 4n$; since this is the case for $\lceil \frac{x}{4} \rceil = n$.

First $\lceil \lceil \frac{x}{2} \rceil \div 2 \rceil = n$ says that $2n - 2 < \lceil \frac{x}{2} \rceil \leq 2n$. By the properties for ceiling functions, this is equivalently stated as (i) $\lceil \frac{x}{2} \rceil = 2n - 1$, or (logical) (ii) $\lceil \frac{x}{2} \rceil = 2n$.

> (i) $\lceil \frac{x}{2} \rceil = 2n - 1$, by the properties for ceiling functions, states that $4n - 4 < x \leq 4n - 2$.
>
> (ii) $\lceil \frac{x}{2} \rceil = 2n$, by the properties for ceiling functions, states that $4n - 2 < x \leq 4n$.

The statement $4n-4 < x \leq 4n-2$ or (logical) $4n-2 < x \leq 4n$ is the same as $4n-4 < x \leq 4n$. Thus, $\lceil \lceil \frac{x}{2} \rceil \div 2 \rceil = n$, is indeed defined by $4n-4 < x \leq 4n$. Because both sides of the equation have the same definition, the statement $\lceil \lceil \frac{x}{2} \rceil \div 2 \rceil = \lceil \frac{x}{4} \rceil$, is true. ■