2.2 Set Operations

Theorem (2.2.5). *Let A be a subset of U. $\overline{\overline{A}} = A$.*

Proof. Suppose x is an element in $\overline{\overline{A}}$. By the definition of set complementation $x \in \neg\overline{A}$, and of course by the same reasoning $x \in \neg(\neg A)$. By the logical law of double negation $x \in A$. Thus it follows directly that $\overline{\overline{A}} = A$. ■

Theorem (2.2.6a). *Let A be a set. The set identity for A is $A \cup \emptyset = A$.*

Proof. Let x be an element in $A \cup \emptyset$. By the definition of set union $(x \in A) \vee (x \in \emptyset)$. But $x \in \emptyset$ is $\perp$ because $\emptyset$ is empty. Therefore x must be in A. It follows directly that $A \cup \emptyset = A$. Thus proves the set identity law for set union. ■

Theorem (2.2.6b). *Let A be a set with universal set U. The set identity for A is $A \cap U = A$.*

Proof. Let x be an element in $A \cap U$. By the definition for set intersection, $(x \in A) \wedge (x \in U)$. We know that $x \in U$ is true because U is the universe. The logical law of identity has it that $x \in A$. Therefore it follows directly that $A \cap U = A$. Thus proves the set identity law for set intersection. ■

Theorem (2.2.7a). *Let A be a set with universal set U. U dominates set union such that $A \cup U = U$.*

Proof. Let x be an element in $A \cup U$. By the definition of set union, $(x \in A) \vee (x \in U)$. Regardless of the truth value for $x \in A$, we know $x \in U$ is always true because U is the universe. Therefore by logical domination $(x \in A) \vee (x \in U) \equiv x \in U$. It directly follows from the definitions that $A \cup U = U$. Thus proves the set domination law for the union of sets. ■

Theorem (2.2.7b). *Let A be a set. The empty set dominates set intersection such that $A \cap \emptyset = \emptyset$.*

Proof. Let x be an element in $A \cap \emptyset$. By the definition for set intersection we have $(x \in A) \wedge (x \in \emptyset)$. We know that $(x \in \emptyset) \equiv \perp$ because the empty set is empty. The logical law of domination gives us that $(x \in A) \wedge \perp \equiv \perp$. It immediately follows that $(x \in A) \wedge (x \in \emptyset) \equiv (x \in \emptyset)$, which is the definition of $A \cap \emptyset = \emptyset$. Thus proves the law of set domination for the intersection of sets. ■

Theorem (2.2.8a). *Let A be a set. A is idempotent such that $A \cup A = A$.*

Proof. Let x be an element in $A \cup A$. By the definition of set union we have, $(x \in A) \vee (x \in A)$. The logical idempotent law says
$(x \in A) \vee (x \in A) \equiv (x \in A)$ which is precisely $A \cup A = A$ by definition. Thus proves the idempotent law for the union of sets. ■

Theorem (2.2.8b). *Let A be a set. A is idempotent such that $A \cap A = A$.*

Proof. Let x be an element in $A \cap A$. By the definition for set intersection we have $(x \in A) \wedge (x \in A)$. The logical idempotent law says
$(x \in A) \wedge (x \in A) \equiv (x \in A)$. That is the definition of $A \cap A = A$. Thus proves the idempotent law for the intersection of sets. ■

Theorem (2.2.9a). *Let A be a set with universal set U. $A \cup \overline{A} = U$.*

Proof. Let x be an element in $A \cup \overline{A}$. By the definition for set union and the complement of sets we have $(x \in A) \vee (x \notin A)$. The right-hand side of this disjunction is equivalent to $x \in (U \cap \overline{A})$ according to the definition for set complementation. That is, $(x \in U) \wedge (x \notin A)$. So the original disjunction is the same as $(x \in A) \vee [(x \in U) \wedge (x \notin A)]$. We must distribute the left-hand side of this disjunction over the conjunction occurring in the right-hand side. We get $[(x \in A) \vee (x \in U)] \wedge [(x \in A) \vee (x \notin A)]$. By the logical law of negation the identity for the right-hand side of this conjunction is true. The left-hand side of this conjunction is dominated by U, according to Theorem 2.2.7a. Therefore the statement $x \in A \cup \overline{A}$ can be equivalently stated as $(x \in U) \wedge T$; the logical identity of which is $x \in U$. Thus proving the set complementation law for the union of sets, $A \cup \overline{A} = U$. ■

Theorem (2.2.9b). *Let A be a set.* $A \cap \overline{A} = \emptyset$.

Proof. Let x be an element in $A \cap \overline{A}$. By definition, $(x \in A) \wedge (x \in \overline{A})$. The right-hand side of this conjunction, according to the definitions of set complementation and logical negation, can be restated as $(x \notin A) \equiv \neg(x \in A)$. Again, by logical negation, we have $(x \in A) \wedge \neg(x \in A) \equiv \bot$. This is the very meaning of $A \cap \overline{A} = \emptyset$. Thus proves the set complementation law for the intersection of sets. ■

Theorem (2.2.10a). *Let A be a set.* $A - \emptyset = A$.

Proof. Let x be an element in $A - \emptyset \equiv A \cap \overline{\emptyset}$, the logical equivalence for which is established by Theorem 2.2.19. By definition we have, $(x \in A) \wedge (x \notin \emptyset)$. Because by supposition $\exists x(x \in (A - \emptyset))$, we know that the statement $(x \notin \emptyset)$ must be true. Therefore, by logical identity, the statement $x \in (A - \emptyset)$ is defined as $x \in A$. So $A - \emptyset = A$. ■

Theorem (2.2.10b). *Let A be a set.* $\emptyset - A = \emptyset$.

Proof. Let x be an element in $\emptyset - A \equiv \emptyset \cap \overline{A}$, the logical equivalence for which is established by Theorem 2.2.19. Because this expression is defined as $(x \in \emptyset) \wedge (x \notin A)$ the supposition $\exists x(x \in (\emptyset - A))$ immediately contradicts $x \in \emptyset$. Meaning that no such x could possibly exist. It follows that $(\emptyset - A)$ must be empty. Hence, $\emptyset - A = \emptyset$ ■

Theorem (2.2.11a). *Let A and B be sets. The union of A and B is commutative.*

Proof. Let x be an element in $A \cup B$. This is defined as $(x \in A) \vee (x \in B)$. Because logical disjunction is commutative $(x \in A) \vee (x \in B) \equiv (x \in B) \vee (x \in A)$. This of course means $x \in (B \cup A)$. Therefore $A \cup B = B \cup A$, and the union of two sets is indeed commutative. ■

Theorem (2.2.11b). *Let A and B be sets. The intersection of A and B is commutative.*

Proof. Let x be an element in $A \cap B$. By the definition of intersection, $(x \in A) \wedge (x \in B)$. Because logical conjunction is commutative, the definition is equivalently stated as $(x \in B) \wedge (x \in A)$. Meaning that $x \in (B \cap A)$. So $A \cap B = B \cap A$, and indeed the intersection of two sets is commutative. ∎

Theorem (2.2.12). *Let A and B be sets. $A \cup (A \cap B) = A$.*

Proof. Let x be an element in $A \cup (A \cap B)$. By the definitions for set union and set intersection we have $(x \in A) \vee [(x \in A) \wedge (x \in B)]$. Treating each parenthesized object as a discrete object we can immediately apply the logical law of absorption to this definition. The result is $x \in A$. Thus it follows directly from the definition that $A \cup (A \cap B) = A$; the consequence of which proves the absorption law for the union of a set with the intersection of itself and another set. ∎

Theorem (2.2.13). *Let A and B be sets. $A \cap (A \cup B) = A$.*

Proof. Let x be an element in $A \cup (A \cap B)$. By the definitions for set union and set intersection we have $(x \in A) \wedge [(x \in A) \vee (x \in B)]$. Treating each parenthesized object as a discrete object we can immediately apply the logical law of absorption to this definition. The result is $x \in A$. Thus it follows directly from the definition that $A \cap (A \cup B) = A$; the consequence of which proves the absorption law for the intersection of a set with the union of itself and another set. ∎

Theorem (2.2.15). *Let A and B be sets.* $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

Proof. Let x be an element in $\overline{A \cup B}$. By the definition of set complementation we have $\neg[x \in (A \cup B)]$. By the definition of set union, $\neg[(x \in A) \vee (x \in B)]$. Applying DeMorgans law (from logic) to the logical operations we get $\neg(x \in A) \wedge \neg(x \in B) \equiv (x \in \overline{A}) \wedge (x \in \overline{B})$. This is the definition of $x \in (\overline{A} \cap \overline{B})$. Therefore $\overline{A \cup B} \subseteq (\overline{A} \cap \overline{B})$.

Now suppose x were an element in $\overline{A} \cap \overline{B}$. Then by definition $(x \in \overline{A}) \wedge (x \in \overline{B}) \equiv \neg(x \in A) \wedge \neg(x \in B)$. By DeMorgans law (from logic) we have $\neg[(x \in A) \vee (x \in B)]$. Since this is the definition for set union it follows that $\neg[x \in (A \cup B)]$. Finally, applying the definition of set complementation we arrive at $x \in \overline{A \cup B}$. Therefore $(\overline{A} \cap \overline{B}) \subseteq \overline{A \cup B}$.

Because $\overline{A \cup B} \subseteq (\overline{A} \cap \overline{B})$ and $(\overline{A} \cap \overline{B}) \subseteq \overline{A \cup B}$ the sets are equivalent by definition. That is, $\overline{A \cup B} = (\overline{A} \cap \overline{B})$. Thereby proving DeMorgans law for sets, that the complement of the union of two sets is equivalent to the intersection of those set complements. ∎

Theorem (2.2.16a). *Let A and B be sets.* $(A \cap B) \subseteq A$.

Proof. Let x be an element in $(A \cap B)$. Then by definition, $(x \in A) \wedge (x \in B)$. It trivially follows from the definition of logical conjunction that $x \in A$ whenever $x \in (A \cap B)$. So $(A \cap B) \subseteq A$. ∎

Theorem (2.2.16b). *Let A and B bet sets.* $A \subseteq (A \cup B)$

Proof. All of the elements in A are a subset of $A \cup B$ by the definition of set union. Therefore it trivially follows that $A \subseteq (A \cup B)$. ∎

Theorem (2.2.16c). *Let A and B be sets.* $(A - B) \subseteq A$.

Proof. Let x be an element in $A - B$. By Theorem 2.2.19 $A - B$ is equivalent to the statement $A \cap \overline{B}$, and thus by definition we have $(x \in A) \wedge (x \notin B)$. We can infer by the simplification rule that $x \in A$. It therefore follows immediately from the definition that $(A - B) \subseteq A$. ∎

Theorem (2.2.16d). *Let A and B be sets.* $A \cap (B - A) = \emptyset$.

Proof. Let x be an element in $A \cap (B - A)$. By Theorem 2.2.19 $(B - A)$ is equivalentt to $(B \cap \overline{A})$. Because set intersection is associative we can drop the parentheses, giving us $A \cap B \cap \overline{A}$. This is logically defined as $(x \in A) \wedge (x \in B) \wedge (x \notin A) \equiv \perp$. Because this statement is false $\forall x$ in the domain, $A \cap (B - A)$ is empty. ■

Theorem (2.2.16e). *Let A and B be sets.* $A \cup (B - A) = A \cup B$.

Proof. Let x be an element in $A \cup (B - A)$. By Theorem 2.2.19 $(B - A)$ is equivalent to $(B \cap \overline{A})$. So by definition we have $(x \in A) \vee [(x \in B) \wedge (x \notin A)]$. Distributing logical disjunction over logical conjunction yields $[(x \in A) \vee (x \in B)] \wedge [(x \in A) \vee (x \notin A)]$. Which by logical negation and by logical identity reduces to $(x \in A) \vee (x \in B)$, that is the very definition for $A \cup B$.

Suppose the converse case in which x is an element of $A \cup B$. That is, of course as already stated, defined as $(x \in A) \vee (x \in B)$. Note the fact that the conjunction of this proposition with another true proposition is true. Let p be that proposition, $(x \in A)$. Then $p \vee \neg p \equiv (x \in A) \vee (x \notin A) \equiv T$, and thus we can make the following statement $[(x \in A) \vee (x \in B)] \wedge [(x \in A) \vee (x \notin A)]$, which holds. Factoring the term $(x \in A)$ out on the logical operators gives the form $(x \in A) \vee [(x \in B) \wedge (x \notin A)]$. This statement is the definition for $A \cup (B - A)$.

Since $A \cup (B - A) \subseteq A \cup B$ and $A \cup B \subseteq A \cup (B - A)$, $A \cup (B - A) = A \cup B$ by definition. ■

Theorem (2.2.17). *Let A, B, and C be sets.* $\overline{A \cap B \cap C} = \overline{A} \cup \overline{B} \cup \overline{C}$.

Proof. Let x be an element in $\overline{A \cap B \cap C}$. By the definitions for set complementation and logical negation we have
$x \notin (A \cap B \cap C) \equiv \neg(x \in A \cap B \cap C)$. By the definition of set intersection that is $\neg[(x \in A) \wedge (x \in B) \wedge (x \in C)]$. Using DeMorgans law (from logic) we can distribute the logical negation across the conjunctions giving us the expression $\neg(x \in A) \vee \neg(x \in B) \vee \neg(x \in C)$. Carrying out those logical negations on each respective term, and again by the definition for set complementation we have $(x \in \overline{A}) \vee (x \in \overline{B}) \vee (x \in \overline{C})$. This is the definition of $\overline{A} \cup \overline{B} \cup \overline{C}$. Hence, $\overline{A \cap B \cap C} \subseteq \overline{A} \cup \overline{B} \cup \overline{C}$.

Now suppose the converse case, where x is an element of $\overline{A} \cup \overline{B} \cup \overline{C}$. As already stated in the previous paragraph the definition for this expression is $(x \in \overline{A}) \vee (x \in \overline{B}) \vee (x \in \overline{C})$. By the definitions for complementation and logical negation that is $\neg(x \in A) \vee \neg(x \in B) \vee \neg(x \in C)$. Using DeMorgans law (from logic) we can factor out the logical negations such that $\neg[(x \in A) \wedge (x \in B) \wedge (x \in C)]$. Which, by the arguments given in the first paragraph we know is equivalent to $\overline{A \cap B \cap C}$. Thus, $\overline{A} \cup \overline{B} \cup \overline{C} \subseteq \overline{A \cap B \cap C}$.

It immediately follows from the definition of set equivalence that
$\overline{A \cap B \cap C} = \overline{A} \cup \overline{B} \cup \overline{C}$ ∎

Theorem (2.2.18a). *Let A, B, and C be sets.* $(A \cup B) \subseteq (A \cup B \cup C)$.

Proof. Let x be an element in $(A \cup B)$. We have $(x \in A) \vee (x \in B)$, by definition. Let this definition statement be represented by p. Trivially, the fact of p would be unaffected were p disjunct any proposition q. Supposing such a q existed, we would have $p \vee q \equiv T$ by logical domination (because the hypothetical supposition assumes $p \equiv T$.) Let q be the proposition $(x \in C)$. Then $p \vee q \equiv (x \in A) \vee (x \in B) \vee (x \in C)$ is the well formed statement defining the superset under interrogation. Since we know that x is in the union of A and B by the hypothesis, and because $p \vee q \equiv T$ means that x is in the union of A, B, and C, it immediately follows that
$(A \cup B) \subseteq (A \cup B \cup C)$. ∎

Theorem (2.2.18b). *Let A, B and C be sets.* $(A \cap B \cap C) \subseteq (A \cap B)$.

Proof. Let x be an element in $(A \cap B \cap C)$. The definition of this expression is $(x \in A) \wedge (x \in B) \wedge (x \in C)$. From that, obviously $(x \in A) \wedge (x \in B)$, being the definition for $(A \cap B)$. It necessarily follows that $(A \cap B \cap C) \subseteq (A \cap B)$. ■

Theorem (2.2.18c). *Let A, B, and C be sets.* $(A - B) - C \subseteq (A - C)$.

Proof. Let x be an element in $(A - B) - C$. By Theorem 2.2.19 $A - B \equiv A \cap \overline{B}$ and $(A \cap \overline{B}) - C \equiv (A \cap \overline{B}) \cap \overline{C}$. By the associative laws and the commutative laws for the intersection of sets we have, $x \in (A \cap \overline{C}) \cap \overline{B}$. By definition that is $[(x \in A) \wedge (x \in \overline{C})] \wedge (x \notin B)$. Or rather, $x \in (A - C) \wedge (x \notin B)$. By logical identity $x \in (A - C)$. Since $x \in [(A - B) - C] \implies x \in (A - C)$, $(A - B) - C \subseteq (A - C)$. ■

Theorem (2.2.18d). *Let A, B, and C be sets.* $(A - C) \cap (C - B) = \emptyset$.

Proof. Let x be an element in $(A - C) \cap (C - B)$. By Theorem 2.2.19 this is equivalently stated as $x \in [(A \cap \overline{C}) \cap (C \cap \overline{B})]$. Since set intersection is associative the inner parentheses can be eliminated, $x \in (A \cap \overline{C} \cap C \cap \overline{B})$. An expression the definition for which is $(x \in A) \wedge (x \in \overline{C}) \wedge (x \in C) \wedge (x \in \overline{B})$. But by the logical law of negation that is $(x \in A) \wedge \perp \wedge (x \in \overline{B}) \equiv \perp$. Meaning $\neg \exists x (x \in (A - C) \cap (C - B))$. In other words, the intersection is indeed empty. ■

Theorem (2.2.18e). *Let A, B, and C be sets.* $(B - A) \cup (C - A) = (B \cup C) - A$.

Proof. Let x be an element in $(B - A) \cup (C - A)$. By Theorem 2.2.19 this is the same as $x \in [(B \cap \overline{A}) \cup (C \cap \overline{A})]$. By definition, that is, $[(x \in B) \wedge (x \in \overline{A})] \vee [(x \in C) \wedge (x \in \overline{A})]$. By the associative property for logical conjunction, and factoring out the term $(x \in \overline{A})$, we get $[(x \in B) \vee (x \in C)] \wedge (x \in \overline{A})$. This statement is the definition for $(B \cup C) - A$.

Proving the converse case, suppose that x is an element in $(B \cup C) - A$. Note that the expression is equivalent to $(B \cup C) \cap \overline{A}$. Thus we have the following definition, $[(x \in B) \vee (x \in C)] \wedge (x \in \overline{A})$. By logical distribution for conjunction over disjunction $[(x \in B) \wedge (x \in \overline{A})] \vee [(x \in C) \wedge (x \in \overline{A})]$. This statement defines the expression $x \in [(B \cap \overline{A}) \cup (C \cap \overline{A})]$. Which, as argued in the first paragraph, is logically equivalent to $x \in [(B - A) \cup (C - A)]$.

Since $[(B - A) \cup (C - A)] \subseteq [(B \cup C) - A]$ and $[(B \cup C) - A] \subseteq [(B - A) \cup (C - A)]$. It immediately follows from the definition of set equivalence that $(B - A) \cup (C - A) = (B \cup C) - A$. ■

Theorem (2.2.19). *Let A, and B be sets.* $A - B = A \cap \overline{B}$.

Proof. Let x be an element in $A - B$. By the definition for set difference, $(x \in A) \wedge (x \notin B)$. By the definition for set complementation this is the same as $(x \in A) \wedge (x \in \overline{B})$. Which is exactly the definition for $x \in (A \cap \overline{B})$.

Proving the converse trivially follows by reversing our steps in the direct form. Suppose there exists an element x such that $x \in (A \cap \overline{B})$. By the definition for set intersection we have $(x \in A) \wedge (x \in \overline{B})$. By the definition for set complementation we arrive at the definition for set difference $(x \in A) \wedge (x \notin B)$. Therefore $x \in (A - B)$.

Since $(A - B) \subseteq (A \cap \overline{B})$ and $(A \cap \overline{B}) \subseteq (A - B)$, by the definition for set equality we have $(A - B) = (A \cap \overline{B})$. ■

Theorem (2.2.20). *Let A, and B be sets.* $(A \cap B) \cup (A \cap \overline{B}) = A$.

Proof. Let x be an element in $(A \cap B) \cup (A \cap \overline{B})$. By definition then, we have $[(x \in A) \wedge (x \in B)] \vee [(x \in A) \wedge (x \notin B)]$. By the logical law of distribution for conjunction over disjunction we can factor the term $(x \in A)$ out on the conjunction. Thus, $(x \in A) \wedge [(x \in B) \vee (x \notin B)]$. By the logical law of negation we have $(x \in A) \wedge T$ which is equivalent to $x \in A$ by the identity law of logic.

Proving the converse, let x be an element in A. That is, $(x \in A)$. By the logical law of identity, $(x \in A) \wedge T \equiv (x \in A)$. Since $(x \in B) \vee (x \notin B)$ is true by the logical law of negation, we can, by logical equivalence, construct the following statement while retaining the logical identity for the statement $(x \in A)$; that is $(x \in A) \wedge [(x \in B) \vee (x \notin B)]$. By the logical law of distribution for conjunction over disjunction we have the definition of our original expression, $[(x \in A) \wedge (x \in B)] \vee [(x \in A) \wedge (x \notin B)]$. Hence, $x \in [(A \cap B) \cup (A \cap \overline{B})]$.

Because $(A \cap B) \cup (A \cap \overline{B}) \subseteq A$ and $A \subseteq (A \cap B) \cup (A \cap \overline{B})$ it follows from the definition of set equivalence that $(A \cap B) \cup (A \cap \overline{B}) = A$. ■

Theorem (2.2.21). *Let A, B, and C be sets.* $A \cup (B \cup C) = (A \cup B) \cup C$, *such that set union is associative.*

Proof. Let x be an element in $A \cup (B \cup C)$. The logical definition being $(x \in A) \vee [(x \in B) \vee (x \in C)]$. It trivially follows from the associative law for logical disjunction that $[(x \in A) \vee (x \in B)] \vee (x \in C)$. Hence, x is an element of $(A \cup B) \cup C$.

In the converse case, let x be an element of $(A \cup B) \cup C$. The logical definition being $[(x \in A) \vee (x \in B)] \vee (x \in C)$. It trivially follows from the associative law for logical disjunction that $(x \in A) \vee [(x \in B) \vee (x \in C)]$. Hence, x is an element of $A \cup (B \cup C)$.

Since $A \cup (B \cup C) \subseteq (A \cup B) \cup C$ and $(A \cup B) \cup C \subseteq A \cup (B \cup C)$ it follows immediately from the definition for set equality that $A \cup (B \cup C) = (A \cup B) \cup C$. Thus, the union of three sets is indeed associative. ■

Theorem (2.2.22). *Let A, B, and C be sets.* $A \cap (B \cap C) = (A \cap B) \cap C$, *such that set intersection is associative.*

Proof. Let x be an element in $A \cap (B \cap C)$. The logical definition being $(x \in A) \wedge [(x \in B) \wedge (x \in C)]$. It trivially follows from the logical law of association for conjunction that $[(x \in A) \wedge (x \in B)] \wedge (x \in C)$. Therefore by definition x is an element of $(A \cap B) \cap C$.

Proving the converse, let x be an element in $(A \cap B) \cap C$. The logical definition being $[(x \in A) \wedge (x \in B)] \wedge (x \in C)$. It trivially follows from the logical law of association for conjunction that $(x \in A) \wedge [(x \in B) \wedge (x \in C)]$. Therefore by definition x is an element of $A \cap (B \cap C)$.

Since $A \cap (B \cap C) \subseteq (A \cap B) \cap C$ and $(A \cap B) \cap C \subseteq A \cap (B \cap C)$ it immediately follows from the definition for set equality that $A \cap (B \cap C) = (A \cap B) \cap C$. Thus, the intersection of three sets is associative. ■

Theorem (2.2.23). *Let A, B, and C be sets.* $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, *such that set union is distributive over set intersection.*

Proof. Let x be an element in $A \cup (B \cap C)$. The logical definition being $(x \in A) \vee [(x \in B) \wedge (x \in C)]$. By the logical law for distribution of disjunction over conjunction we have $[(x \in A) \vee (x \in B)] \wedge [(x \in A) \vee (x \in C)]$. By definition, x is an element in $(A \cup B) \cap (A \cup C)$.

Proving the converse, let x be an element in $(A \cup B) \cap (A \cup C)$. The logical definition being $[(x \in A) \vee (x \in B)] \wedge [(x \in A) \vee (x \in C)]$. By the logical law for distribution we can factor out the term $x \in A$ over the conjunction. Thus, $(x \in A) \vee [(x \in B) \wedge (x \in C)]$, and x is an element in $A \cup (B \cap C)$ by definition.

Since $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$ and $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$, it follows immediately from the definition of set equality that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. Therefore, set union is indeed distributive over set intersection. ■

Theorem (2.2.24). *Let A, B, and C be sets.* $(A-B)-C=(A-C)-(B-C)$.

Proof. Let x be an element in $(A-B)-C$. By the definition for set difference we have $[(x \in A) \wedge (x \notin B)] \wedge (x \notin C)$. Now, the logical identity for the proposition $(x \notin B)$ is $(x \notin B) \vee \perp \equiv (x \notin B)$, and since
$(x \in C) \equiv \perp$ by reason of the hypothesis, it necessarily follows from logical identity that $(x \notin B) \equiv [(x \notin B) \vee (x \in C)]$. Thus, by these facts, the law of logical commutativity, and the law of logical association we make the logically equivalent statement with respect to the definition given by the first expression, $[(x \in A) \wedge (x \notin C)] \wedge [(x \notin B) \vee (x \in C)]$. Since the right-hand side of the conjunction is true and in its proper logical identity, if it were double negated, it would remain intact by the law of double negation. Thus applying first a negation by DeMorgans law, and second a negation directly on the propositional statement, we have $[(x \in A) \wedge (x \notin C)] \wedge \neg[(x \in B) \wedge (x \notin C)]$. By the definition for set difference and set complementation x is an element in $(A-C) \cap \overline{(B-C)}$. Which is equivalent to the expression $(A-C)-(B-C)$, by Theorem 2.2.19.

To prove the converse case, let x be an element in $(A-C)-(B-C)$. By Theorem 2.2.19 $(A-C) \cap \overline{(B-C)}$ is an equivalent expression. This expression containing the element x is defined by
$[(x \in A) \wedge (x \notin C)] \wedge \neg[(x \in B) \wedge (x \notin C)]$. Applying DeMorgans law to the statement on the right-hand side of the conjunction we get
$[(x \in A) \wedge (x \notin C)] \wedge [(x \notin B) \vee (x \in C)]$. Note that as demonstrated in the first paragraph we have the following identity $[(x \notin B) \vee (x \in C)] \equiv (x \notin B)$. Thus, by that fact, the law of logical commutativity, and by the law of logical association it follows that the identity of our statement is
$[(x \in A) \wedge (x \notin B)] \wedge (x \notin C)$. This is the definition for $x \in [(A-B)-C]$.

Since $(A-B)-C \subseteq (A-C)-(B-C)$ and
$(A-C)-(B-C) \subseteq (A-B)-C$, it follows immediately from the definition for set equality that $(A-B)-C=(A-C)-(B-C)$ ■

Theorem (2.2.31). *Let A, and B be subsets of a universal set U.*
$A \subseteq B \iff \overline{B} \subseteq \overline{A}$.

Proof. The proposition $A \subseteq B$ is equivalent to the universally quantified statement $\forall x(x \in A \implies x \in B)$. It is tautological that the propositional function in this statement is logically equivalent to its contrapositive form (satisfying the biconditional requirement.) That is, $\forall x(x \notin B \implies x \notin A)$ is the logically equivalent statement. By the definition for set complementation that is $\forall x(x \in \overline{B} \implies x \in \overline{A})$. By universal generalization
$A \subseteq B \iff \overline{B} \subseteq \overline{A}$. ■

Theorem (2.2.35). *Let A, and B be sets. $A \oplus B = (A \cup B) - (A \cap B)$.*

Proof. Let x be an element in $A \oplus B$. This is logically defined as $[(x \in A) \wedge (x \notin B)] \vee [(x \notin A) \wedge (x \in B)]$. By the definition for set difference x is an element in $(A - B) \cup (B - A)$ which by Theorem 2.2.19 can be expressed as $(A \cap \overline{B}) \cup (B \cap \overline{A})$. By Theorem 2.2.23, which proves that set unions are distributive over set intersections, the following expression is equivalent $[A \cup (B \cap \overline{A})] \cap [\overline{B} \cup (B \cap \overline{A})]$. Again, by Theorem 2.2.23, we have $[(A \cup B) \cap (A \cup \overline{A})] \cap [(\overline{B} \cup B) \cap (\overline{B} \cup \overline{A})]$. Let the logical definition for this expression be represented in two propositional variables $p \wedge q$ such that:

$p \equiv \{[(x \in A) \vee (x \in B)] \wedge [(x \in A) \vee (x \notin A)]\}$

$q \equiv \{[(x \notin B) \vee (x \in B)] \wedge [(x \notin B) \vee (x \notin A)]\}$.

By the logical law of identity $p \equiv (x \in A) \vee (x \in B)$, and $q \equiv (x \notin B) \vee (x \notin A)$. So we have $[(x \in A) \vee (x \in B)] \wedge [(x \notin B) \vee (x \notin A)]$. Applying DeMorgans law to the right-hand side of the conjunction we get $[(x \in A) \vee (x \in B)] \wedge \neg[(x \in B) \wedge (x \in A)]$. Then by definition, x is an element in $(A \cup B) \cap \overline{(A \cap B)}$, and according to Theorem 2.2.19 x is an element in $(A \cup B) - (A \cap B)$.

Proving the converse case is trivial. Let x be an element in $(A \cup B) - (A \cap B)$. By Theorem 2.2.19 x is an element in $(A \cup B) \cap \overline{(A \cap B)}$. By definition we have, $[(x \in A) \vee (x \in B)] \wedge \neg[(x \in B) \wedge (x \in A)]$. By DeMorgans law, $[(x \in A) \vee (x \in B)] \wedge [(x \notin B) \vee (x \notin A)]$. Let this logical formula be represented in two propositional variables $p \wedge q$ such that $p \equiv (x \in A) \vee (x \in B)$ and $q \equiv (x \notin B) \vee (x \notin A)$. By the logical law of identity $p \wedge T \equiv T$, and $q \wedge T \equiv T$. By the negation law of logic, $(x \notin B) \vee (x \in B) \equiv T$, and $(x \in A) \vee (x \notin A) \equiv T$. Therefore,

$p \equiv \{[(x \in A) \vee (x \in B)] \wedge [(x \in A) \vee (x \notin A)]\}$

$q \equiv \{[(x \notin B) \vee (x \in B)] \wedge [(x \notin B) \vee (x \notin A)]\}$.

By definition, x is an element in $[(A \cup B) \cap (A \cup \overline{A})] \cap [(\overline{B} \cup B) \cap (\overline{B} \cup \overline{A})]$. By Theorem 2.2.23, factoring A out of the left-hand side of the intersection, and factoring $\overline{B}$ out of the right-hand side of the intersection, the following expression is equivalent $[A \cup (B \cap \overline{A})] \cap [\overline{B} \cup (B \cap \overline{A})]$. Again, by Theorem 2.2.23, factoring $(B \cap \overline{A})$ out of the intersection we have the following equivalent expression $(A \cap \overline{B}) \cup (B \cap \overline{A})$. Which, by Theorem 2.2.19 is equivalently stated as $(A - B) \cup (B - A)$, defined by $[(x \in A) \wedge (x \notin B)] \vee [(x \notin A) \wedge (x \in B)]$. But this is the formal definition for the symmetric difference of sets, so x must be an element in $A \oplus B$.

Since $A \oplus B \subseteq (A \cup B) - (A \cap B)$ and $(A \cup B) - (A \cap B) \subseteq A \oplus B$, it follows immediately that $A \oplus B = (A \cup B) - (A \cap B)$. ∎

Theorem (2.2.36). *Let A, and B be sets.* $A \oplus B = (A - B) \cup (B - A)$

Proof. Let x be an element in $A \oplus B$. Then by the definition for symmetric difference $[(x \in A) \wedge (x \notin B)] \vee [(x \notin A) \wedge (x \in B)]$. Because logical conjunction is associative, the statement is equivalent to $[(x \in A) \wedge (x \notin B)] \vee [(x \in B) \wedge (x \notin A)]$. According to the definition for set difference, and by the definition for set union, it follows that x is an element in $(A - B) \cup (B - A)$.

Proving the converse, suppose x were an element in $(A - B) \cup (B - A)$. The logical definition being $[(x \in A) \wedge (x \notin B)] \vee [(x \in B) \wedge (x \notin A)]$. By the associative law for logical conjunction the following statement is equivalent $[(x \in A) \wedge (x \notin B)] \vee [(x \notin A) \wedge (x \in B)]$. Since this is the definition for symmetric difference, x is an element in $A \oplus B$.

Since $A \oplus B \subseteq (A - B) \cup (B - A)$ and $(A - B) \cup (B - A) \subseteq A \oplus B$ it immediately follows from the definition of set equality that $A \oplus B = (A - B) \cup (B - A)$. ■

Theorem (2.2.37a). *Let A be a subset of the universal set U.* $A \oplus A = \emptyset$.

Proof. By Theorem 2.2.35, $A \oplus A = (A \cup A) - (A \cap A)$. By the set idempotent law, that is $A - A$, and by Theorem 2.2.19, equivalent to $A \cap \overline{A}$. It follows immediately from the set complementation law that $A \oplus A = \emptyset$. ■

Theorem (2.2.37b). *Let A be a subset of the universal set U.* $A \oplus \emptyset = A$.

Proof. By Theorem 2.2.35, $A \oplus \emptyset = (A \cup \emptyset) - (A \cap \emptyset)$. By set identity, and by set domination, that is $A - \emptyset$, which by Theorem 2.2.19 means $A \cap \overline{\emptyset}$. Because $\overline{\emptyset} = U$, we have by set identity that $A \oplus \emptyset = A$. ■

Theorem (2.2.37c). *Let A be a subset of the universal set U.* $A \oplus U = \overline{A}$.

Proof. By Theorem 2.2.35, $A \oplus U = (A \cup U) - (A \cap U)$. By set domination, and by set identity, that is $U - A$. By Theorem 2.2.19, $U \cap \overline{A}$. By the identity law for sets $A \oplus U = \overline{A}$. ■

Theorem (2.2.37d). *Let A be a subset of a universal set U. $A \oplus \overline{A} = U$.*

Proof. By Theorem 2.2.35, $A \oplus \overline{A} = (A \cup \overline{A}) - (A \cap \overline{A})$. By the set complementation laws that is $U - \emptyset$. Rather, $U \cap \overline{\emptyset}$ by Theorem 2.2.19. Since $\overline{\emptyset} = U$ we have $U \cap U$, which is obviously U, by the idempotent law. Thus, $A \oplus \overline{A} = U$. ∎

Theorem (2.2.38a). *Let A, and B be sets. The symmetric difference of sets is associative such that $(A \oplus B) = (B \oplus A)$.*

Proof. By Theorem 2.2.35, $A \oplus B = (A \cup B) - (A \cap B)$. Because set union is associative, and because set intersection is associative, we have $(B \cup A) - (B \cap A)$. Again, according to Theorem 2.2.35, that is $B \oplus A$. ∎

Theorem (2.2.38b). *Let A, and B be sets. The symmetric difference of sets is subject to absorption such that $(A \oplus B) \oplus B = A$.*

Proof. By Theorem 2.2.35, $(A \oplus B) \oplus B = [(A \oplus B) \cup B] - [(A \oplus B) \cap B]$. By Theorem 2.2.19 that is $[(A \oplus B) \cup B] \cap \overline{[(A \oplus B) \cap B]}$, and by DeMorgans law $[(A \oplus B) \cup B] \cap [\overline{(A \oplus B)} \cup \overline{B}]$. Again, by Theorem 2.2.35, that is $\{[(A \cup B) - (A \cap B)] \cup B\} \cap \{\overline{[(A \cup B) - (A \cap B)]} \cup \overline{B}\}$. By Theorem 2.2.19, $\{[(A \cup B) \cap \overline{(A \cap B)}] \cup B\} \cap \{[(A \cup B) \cap \overline{(A \cap B)}] \cup \overline{B}\}$. Applying DeMorgans law, three times in succession we get $\{[(A \cup B) \cap (\overline{A} \cup \overline{B})] \cup B\} \cap \{[(\overline{A} \cap \overline{B}) \cup (A \cap B)] \cup \overline{B}\}$. By distribution and association we make the following equivalent statement, $[(A \cup B \cup B) \cap (\overline{A} \cup \overline{B} \cup B)] \cap \{[(\overline{A} \cap \overline{B}) \cup \overline{B}] \cup [(A \cap B) \cup \overline{B}]\}$. Now this expression shall be reduced. By the idempotent law, the complementation law, set absorption, and distribution we have $[(A \cup B) \cap (\overline{A} \cup U)] \cap \{\overline{B} \cup [(A \cup \overline{B}) \cap (B \cup \overline{B})]\}$. By complementation and domination we make the following equivalent statement, $[(A \cup B) \cap U] \cap \{\overline{B} \cup [(A \cup \overline{B}) \cap U]\}$. By set identity, and the idempotent law that is $(A \cup B) \cap (A \cup \overline{B})$. Factoring out A by the law of distribution, and by the set complementation laws, and by set identity we find that
$A \cup (B \cap \overline{B}) \equiv A \cup \emptyset \equiv A$
$\therefore (A \oplus B) \oplus B = A$. ∎

Theorem (2.2.40). *Let A, B, and C be sets. The symmetric difference for sets is associative such that $(A \oplus B) \oplus C = A \oplus (B \oplus C)$.*

Proof. Let x be an element in $(A \oplus B) \oplus C$. By the definition for the symmetric difference of sets, $x \in [(A \oplus B) \cap \overline{C}] \cup [\overline{(A \oplus B)} \cap C]$. By Theorem 2.2.36, x is an element in $\{[(A - B) \cup (B - A)] \cap \overline{C}\} \cup \{\overline{[(A - B) \cup (B - A)]} \cap C\}$. And by Theorem 2.2.19, $\{[(A \cap \overline{B}) \cup (B \cap \overline{A})] \cap \overline{C}\} \cup \{\overline{[(A \cap \overline{B}) \cup (B \cap \overline{A})]} \cap C]\}$. Applying DeMorgans law for sets to the subset that is the right-hand side of the union in this superset, twice, produces the following logical superset equivalence, $x \in \{[(A \cap \overline{B}) \cup (B \cap \overline{A})] \cap \overline{C}\} \cup \{[(\overline{A} \cup B) \cap (\overline{B} \cup A)] \cap C]\}$. Then, distributing the terms in the subset that is the right-hand side of the union of this superset gives the logical subset equivalence, $[(\overline{A} \cup B) \cap (\overline{B} \cup A)] \cap C \equiv \{[\overline{A} \cap (\overline{B} \cup A)] \cup [B \cap (\overline{B} \cup A)]\} \cap C$. The subset terms must be distributed further, $\{[(\overline{A} \cap \overline{B}) \cup (\overline{A} \cap A)] \cup [(B \cap \overline{B}) \cup (B \cap A)]\} \cap C$. By the negation, and identity laws, $[(\overline{A} \cup B) \cap (\overline{B} \cup A)] \cap C \equiv [(\overline{A} \cap \overline{B}) \cup (B \cap A)] \cap C$. Distributing C, $[(\overline{A} \cup B) \cap (\overline{B} \cup A)] \cap C \equiv (C \cap \overline{A} \cap \overline{B}) \cup (C \cap B \cap A)$. Finally, carrying out distribution on the subset that is the left-hand side of the union of the superset gives us the logically equivalent superset statement
$(A \cap \overline{B} \cap \overline{C}) \cup (B \cap \overline{A} \cap \overline{C}) \cup (C \cap \overline{A} \cap \overline{B}) \cup (C \cap B \cap A)$.

Now let x be an element in $A \oplus (B \oplus C)$. By the definition for the symmetric difference of sets, $x \in [A \cap \overline{(B \oplus C)}] \cup [\overline{A} \cap (B \oplus C)]$. By Theorem 2.2.36, x is an element in $\{A \cap \overline{[(B - C) \cup (C - B)]}\} \cup \{\overline{A} \cap [(B - C) \cup (C - B)]\}$. And by Theorem 2.2.19, $\{A \cap \overline{[(B \cap \overline{C}) \cup (C \cap \overline{B})]}\} \cup \{\overline{A} \cap [(B \cap \overline{C}) \cup (C \cap \overline{B})]\}$. Applying DeMorgans laws for sets to the subset that is the left-hand side of the union of this superset, twice, produces the following logical superset equivalence, $x \in \{A \cap [(\overline{B} \cup C) \cap (\overline{C} \cup B)]\} \cup \{\overline{A} \cap [(B \cap \overline{C}) \cup (C \cap \overline{B})]\}$. Then, distributing the terms in the subset that is the left-hand side of the union of this superset gives the logical subset equivalence $A \cap [(\overline{B} \cup C) \cap (\overline{C} \cup B)] \equiv A \cap \{[\overline{B} \cap (\overline{C} \cup B)] \cup [C \cap (\overline{C} \cup B)]\}$. The subset terms must be distributed further, $A \cap \{[(\overline{B} \cap \overline{C}) \cup (\overline{B} \cap B)] \cup [(C \cap \overline{C}) \cup (C \cap B)]\}$. By the negation, and identity laws, $A \cap [(\overline{B} \cup C) \cap (\overline{C} \cup B)] \equiv A \cap [(\overline{B} \cap \overline{C}) \cup (C \cap B)]$. Distributing A, $A \cap [(\overline{B} \cup C) \cap (\overline{C} \cup B)] \equiv (A \cap \overline{B} \cap \overline{C}) \cup (A \cap C \cap B)$. Finally, carrying out distribution on the subset that is the right-hand side of the union of the superset gives us the logically equivalent superset statement
$(A \cap \overline{B} \cap \overline{C}) \cup (A \cap C \cap B) \cup (\overline{A} \cap B \cap \overline{C}) \cup (\overline{A} \cap C \cap \overline{B})$.

Because $x \in [(A \cap \overline{B} \cap \overline{C}) \cup (A \cap C \cap B) \cup (\overline{A} \cap B \cap \overline{C}) \cup (\overline{A} \cap C \cap \overline{B})]$, whenever x is an element in $(A \oplus B) \oplus C$ and x is an element in $A \oplus (B \oplus C)$, it follows that $(A \oplus B) \oplus C = A \oplus (B \oplus C)$ $\therefore$ the symmetric difference for sets is indeed, associative. ∎

Theorem (2.2.41). *Let A, B, and C be sets. If $A \oplus C = B \oplus C$, then $A = B$.*

Proof. By contraposition. Note that the statement $A \oplus C = B \oplus C$ is by definition $(A \cap \overline{C}) \cup (\overline{A} \cap C) = (B \cap \overline{C}) \cup (\overline{B} \cap C)$. Assume there exists an element x such that $x \in A$ and $x \notin B$. Thus, $A \not\subseteq B$. By the hypothesis, x has to be in $(A \cap \overline{C})$ and cannot be in $(\overline{A} \cap C)$. This means that x is not in C. Neither can x be in $(B \cap \overline{C})$. And since $x \notin C$, x cannot be in $(\overline{B} \cap C)$. So x is in $A \oplus C$ but not $B \oplus C$. Therefore, $A \oplus C \not\subseteq B \oplus C$. The implication, if $B \not\subseteq A$, then $B \oplus C \not\subseteq A \oplus C$, trivially follows without loss of generality. Conclusively, $A \neq B$ implies $A \oplus C \neq B \oplus C$. ∎