Theorem (2.2.21). *Let A, B, and C be sets. $A \cup (B \cup C) = (A \cup B) \cup C$, such that set union is associative.*

Proof. Let x be an element in $A \cup (B \cup C)$. The logical definition being $(x \in A) \vee [(x \in B) \vee (x \in C)]$. It trivially follows from the associative law for logical disjunction that $[(x \in A) \vee (x \in B)] \vee (x \in C)$. Hence, x is an element of $(A \cup B) \cup C$.

In the converse case, let x be an element of $(A \cup B) \cup C$. The logical definition being $[(x \in A) \vee (x \in B)] \vee (x \in C)$. It trivially follows from the associative law for logical disjunction that $(x \in A) \vee [(x \in B) \vee (x \in C)]$. Hence, x is an element of $A \cup (B \cup C)$.

Since $A \cup (B \cup C) \subseteq (A \cup B) \cup C$ and $(A \cup B) \cup C \subseteq A \cup (B \cup C)$ it follows immediately from the definition for set equality that $A \cup (B \cup C) = (A \cup B) \cup C$. Thus, the union of three sets is indeed associative. ■