Theorem (2.4.41)**.** *The union of a countable number of countable sets is countable.*

Proof. Let $S = \bigcup_{i\in\mathbb{N}}^{\aleph_0} A_i$ where the cardinality for A_i is at most $\aleph_0$. Then the sequence $\{a_{ij}\} = a_{i0},\ a_{i1},\ a_{i2},\ \ldots$ exists for A_i. Thus, all elements in S can be listed in two dimensions as

$$\begin{array}{l} a_{00}, a_{01}, a_{02}, \ldots \\ a_{10}, a_{11}, a_{12}, \ldots \\ a_{20}, a_{21}, a_{22}, \ldots \\ \quad\vdots \end{array}$$

The sequence $\{s_{ij}\}$ consisting of all elements of all elements in S exists. This sequence can be derived by tracing the diagonal paths along the two dimensional list for S,

$$\{s_{ij}\} = a_{00},\ a_{01},\ a_{10},\ a_{20},\ a_{11},\ a_{02},\ \ldots.$$

$\therefore |S| = \aleph_0$, and indeed the union of a countable number of countable sets is countable. ∎