Theorem (3.2.10)**.** *Let f be the function defined by $f(x) = x^3$. Let g be the function defined by $g(x) = x^4$. $f(x)$ is $\mathcal{O}(g(x))$, but $g(x)$ is not $\mathcal{O}(f(x))$.*

Proof. $f(x)$ is $\mathcal{O}(g(x))$ with constant witnesses $C = 1$, and $k = 1$, by the definition of big-O. That is, $|f(x)| \leq C|g(x)|$, for all $x > 1$.

If $g(x)$ were $\mathcal{O}(f(x))$, then there would exist constant witnesses C and k such that $|g(x)| \leq C|f(x)|$. For $x > 1$, we would have $x^4 \leq C \cdot x^3$, which implies that $x \leq C$. But no constant C exists satisfying the unbounded domain for x. Therefore it is not the case that $g(x)$ is $\mathcal{O}(f(x))$. ■