Math 422 HMWK 2 Selected Solutions

Savage

Exercise 1

Part A

First observe that after 23 jumps of 5 units or 115 units in the clockwise direction, the grasshopper will land on 1. This is because after 2 laps of 57 units (114 units), you end back on zero. Let $v \in \{0, 1, ..., 56\}$. Then if the grasshopper jumps $23 \cdot v$ times of 5 units in the clockwise direction, it will land on number v. This means the entire dial is the set of solutions where the grasshopper can land.

Part B

If a grass hopper jumps in units of 3 starting at zero, it it will hit 0 after 19 jumps in the clockwise direction. We can write the total number jumps, N, as $N = 3k + 3 \cdot 19l$ where $0 \leq k < 19$. Observe that regardless of what l is, $3k$ and $3k + 3 \cdot 19l$ will land on the same tile. This means whatever tile the grasshopper lands on, it must be divisible by 3. Next, the grasshopper can actually achieve every multiple of 3 in the set $\{0, 1, ..., 56\}$.

Exercise 3

Silverman 2.5, Part A

We define

1. $b = 4T_5 = 4 \cdot 15 = 60$
2. $b = 4T_6 = 4 \cdot 21 = 84$
3. $b = 4T_7 = 4 \cdot 8 = 112$

Recall by Theorem 2.1 that

$$b = \frac{s^2 - t^2}{2}.$$

We do the first case, the rest are similar. In (1), this would mean we are looking for s and t such that

$$s = \sqrt{120 + t^2}.$$

Choosing $t = 1$ gives us a perfect square and gives us $s = 11$. So our PPT is $(11, 60, 61)$.

Silverman 2.5, Part B

Theorem 0.1. *Let* $n \in \mathbb{Z}^+$. *There exists a PPT of the form* $(a, 4T_n, c)$.

Proof. Let $t = 1$ and $s = 2n + 1$. Note that these are both odd integers with $t \geq 1$, so by Theorem 2.1, if we define a, b, and c as in the theorem, we really do get a PPT. What remains to show is that b is of the required form. From Theorem 2.1 we have

$$b \overset{\text{def}}{=} \frac{s^2 - t^2}{2} = \frac{(2n+1)^2 - 1}{2} = \frac{(4n^2 + 4n + 1) - 1}{2}.$$

Cancelling the 1's allows us to write

$$= 4 \cdot \frac{n^2 + n}{2} = 4 \cdot \frac{n(n+1)}{2} = 4T_n.$$

□

Exercise 5

We illustrate just part (a)

Silverman 5.1, Part A

The $\gcd(67890, 12345)$.

$$\begin{aligned} 67890 &= 12345 \cdot (5) + 6165 \\ 12345 &= 6165 \cdot (2) + 15 \\ 6165 &= 15 \cdot (411) + 0. \end{aligned}$$

So $\gcd(67890, 12345) = 15$.