Homework on §3
Due: Thursday, February 14

1. Silverman 5.4.
2. Silverman 6.1.
3. Silverman 6.3. To prove the program works, show by induction that b divides both $g_n - ax_n$ and $w_n - av_n$. (Incidentally, in Python you don't need either of the variables s or t.)
4. Silverman 6.4.
5. Silverman 7.1.
6. Silverman 7.2.
7. Two players play the following game: The numbers 25 and 36 are written on a board. On each player's turn, they select any two numbers currently up and write on the board the positive difference of those two numbers, provided the difference is not already written on the board. The game continues until a player cannot write anything down. The last player to write down a number wins. Does the game always end? If so, assuming perfect play, who wins?

Solution: The answer is that the second player wins. We prove this in a series of steps. First, let B be the set of numbers on the board at the conclusion of the game. Let d be the least element of B; since $36 \in B$, B is nonempty, and since $B \subset \mathbb{N}$, by Well-Ordering, such a d exists.

Lemma. $\forall b \in B, d \mid b$.

Proof. Let $b \in B$. By the rules of the game, all positive elements of the sequence $b - d, b - 2d, \cdots$ lie in B. Via the Division Algorithm, $\exists q, r \in \mathbb{Z}$ such that $b = qd + r$ and $0 \leq r < d$. Since $d \leq b$, we have $q \geq 0$. If $r > 0$, then $r \in B$. But $r < d$, so this contradicts the minimality of d. Therefore $r = 0$, and so $d \mid b$. □

Lemma. $d = 1$

Proof. We know $25, 36 \in B$. By the previous lemma, $d \mid 25$ and $d \mid 36$. But $\gcd(25, 36) = 1$, so $d = 1$. □

Lemma. $B = \{1, 2, \ldots, 36\}$.

I leave this as an exercise.

From there, one sees that there must be 34 turns. Since 34 is even, the second player goes last.