Homework 5

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A. Silverman 8.4. Prove that following divisibility tests work.

(a) The number a is divisible by 4 if and only if its last two digits are divisible by 4.

Proof. Let $a \in \mathbb{Z}$. Write $a = \sum_{i=0}^{n-1} a_i 10^i$. Since $100 \equiv 0 \pmod 4$, we know $10^i \equiv 0 \pmod 4$ for $i \geqslant 2$. So

$$\begin{aligned}\sum_{i=0}^{n-1} a_i 10^i &\equiv (a_0 + a_1 10) \pmod 4 + \sum_{i=2}^{n-1} a_i(0) \pmod 4 \\ &\equiv (a_0 + a_1 10) \pmod 4 + 0 \pmod 4. \\ &\equiv (a_0 + a_1 10) \pmod 4.\end{aligned}$$

□

(b) The number a is divisible by 8 if and only if its last three digits are divisible by 5.

Proof. Let $a \in \mathbb{Z}$. Write $a = \sum_{i=0}^{n-1} a_i 10^i$. Since $1000 \equiv 0 \pmod 8$, we know $10^i \equiv 0 \pmod 8$ for $i \geqslant 3$. So

$$\begin{aligned}\sum_{i=0}^{n-1} a_i 10^i \pmod 8 &\equiv \sum_{i=0}^{2} a_i 10^i \pmod 8 + \sum_{i=3}^{n-1} a_i(0) \pmod 8 \\ &\equiv \sum_{i=0}^{2} a_i 10^i \pmod 8 + 0 \pmod 8. \\ &\equiv \sum_{i=0}^{2} a_i 10^i \pmod 8.\end{aligned}$$

□

(c) The number a is divisible by 3 if and only if the sum of its digits is divisible by 3.

Proof. Let $a \in \mathbb{Z}$. Write $a = \sum_{i=0}^{n-1} a_i 10^i$. Since $10 \equiv 1 \pmod 3$, we know $10^i \equiv 1 \pmod 3$ for $i \geqslant 1$. So

$$\begin{aligned}\sum_{i=0}^{n-1} a_i 10^i \pmod 3 &\equiv \sum_{i=0}^{n-1} a_i(1) \pmod 3 \\ &\equiv \sum_{i=0}^{n-1} a_i \pmod 3.\end{aligned}$$

□

(d) The number a is divisible by 9 if and only if the sum of its digits is divisible by 9.

Proof. Let $a \in \mathbb{Z}$. Write $a = \sum_{i=0}^{n-1} a_i 10^i$. Since $10 \equiv 1 \pmod 9$, we know $10^i \equiv 1 \pmod 9$ for $i \geqslant 1$. So

$$\begin{aligned}\sum_{i=0}^{n-1} a_i 10^i \pmod 9 &\equiv \sum_{i=0}^{n-1} a_i(1) \pmod 9 \\ &\equiv \sum_{i=0}^{n-1} a_i \pmod 9.\end{aligned}$$

□

(e) The number a is divisible by 11 if and only if the alternating sum of the digits of a is divisible by 11.

Proof. Let $a \in \mathbb{Z}$. Write $a = \sum_{i=0}^{n-1} a_i 10^i$. Since $10^i \equiv 1 \pmod{11}$ for all $i \equiv 0 \pmod 2$ and $10^i \equiv -1 \pmod{11}$ for all $i \equiv 1 \pmod 2$, we have

$$\begin{aligned}\sum_{i=0}^{n-1} a_i 10^i &\equiv \sum_{i \equiv_2 0}^{n-1} a_i(1) + \sum_{i \equiv_2 1}^{n-1} a_i(-1) \pmod{11} \\ &\equiv \sum_{i=0}^{n-1} a_i(-1)^i \pmod{11}.\end{aligned}$$

□

B. Let $n \geqslant 0$.

1. Prove that

$$x^{n+1} - 1 = (x-1)(x^n + x^{n-1} + \cdots + 1).$$

Proof. We show by induction on n. Since

$$x^1 - 1 = x - 1 = (x-1)(1) = (x-1)(x^0),$$

the equality holds for $n = 0$. Suppose there exists $n \in \mathbb{N}_{>0}$ such that

$$x^{n+1} - 1 = (x-1)\sum_{i=0}^{n} x^i.$$

Then by multiplying both sides by x and applying the distributive law, we obtain

$$x^{n+2} - x = \sum_{i=0}^{n+1} x^i.$$

But

$$x^{n+2} - x = x^{n+1} - 1 - (x-1),$$

so

$$x^{n+1} - 1 = \sum_{i=1}^{n+1} x^i + (x-1) = (x-1)\sum_{i=0}^{n+1} x^i.$$

□

2. Prove that

$$x^{n+1} - y^{n+1} = (x-y)\sum_{i=0}^{n} x^i y^{n-i}.$$

Proof. By applying the result from part 1 to $\frac{x}{y}$, we get

$$\left(\frac{x}{y}\right)^{n+1} - 1 = \left(\frac{x}{y} - 1\right) \sum_{i=0}^{n} \left(\frac{x}{y}\right)^{n}.$$

Thus

$$y^{n+1}\left(\left(\frac{x}{y}\right)^{n+1} - 1\right) = y^{n+1}\left(\frac{x}{y} - 1\right) \sum_{i=0}^{n} \left(\frac{x}{y}\right)^{n}$$

But

$$y^{n+1}\left(\left(\frac{x}{y}\right)^{n+1} - 1\right) = x^{n+1} - y^{n+1},$$

and

$$\begin{aligned} y^{n+1}\left(\frac{x}{y} - 1\right) \sum_{i=0}^{n} \left(\frac{x}{y}\right)^{n} &= y\left(\frac{x}{y} - 1\right) y^{n} \sum_{i=0}^{n} \left(\frac{x}{y}\right)^{n} \\ &= (x-y) \sum_{i=0}^{n} x^{i} y^{n-i}. \end{aligned}$$

□

C. 1. What day of the week is February 21, 2030?

Thursday

2. Write a program which takes as input a year n with $2001 \leqslant n \leqslant 2099$ and outputs the day of the week of February 21 in year n.

```
days = ['Sun', 'Mon', 'Tue', 'Wed', 'Thu', 'Fri', 'Sat']


def feb21(n):
    """Return day of week of Feb 21 for year 2000 < n <
        2100"""
    n = n % 100  # store last two digits of year
    q, r = n // 12, n % 12
    return days[(q + r + (r // 4) + 2) % 7]
```