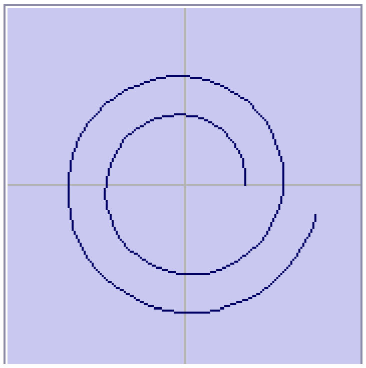
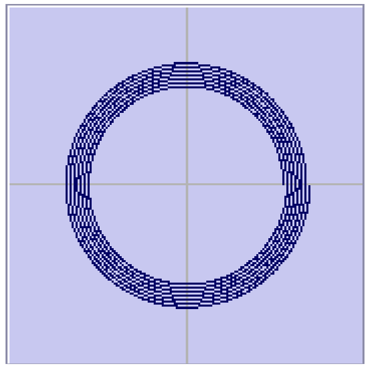
Following Level Curves

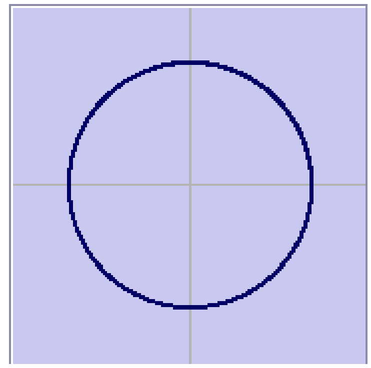
The algorithm used in this application produces discrete points on a level curve by choosing a starting point, moving a small distance along a line tangent to the curve, and iterating the process.



$f(x,y) = x^2 + y^2$
$\boldsymbol{\varepsilon = 10^{-1}}$
$x_0 = (0.5, 0.0)$
$N = 10^2$



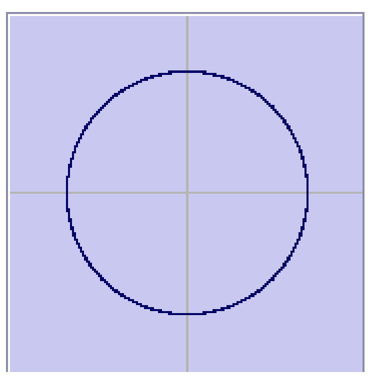
$f(x,y) = x^2 + y^2$
$\boldsymbol{\varepsilon = 10^{-2}}$
$x_0 = (0.8, 0.0)$
$N = 4*10^3$



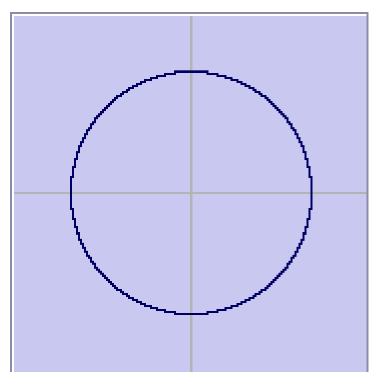
$f(x,y) = x^2 + y^2$
$\boldsymbol{\varepsilon = 10^{-3}}$
$x_0 = (1.0, 0.0)$
$N = 4*10^4$

One flaw of using this methods is that it tends to produce systematic error; with each step, x moves onto a level curve that is farther from the original curve. In the case of $f(x,y) = x^2 + y^2$, wherein each level curve is a circle, values of x move away from the origin in a spiral pattern; this effect particularly visible when the value of epsilon is too large.

The graph changes very little when reducing the step length from 10^{-4} to 10^{-5}, so these may be considered sufficiently small values for epsilon.



$f(x,y) = x^2 + y^2$
$\boldsymbol{\varepsilon = 10^{-4}}$
$x_0 = (1.0, 0.0)$
$N = 4*10^5$



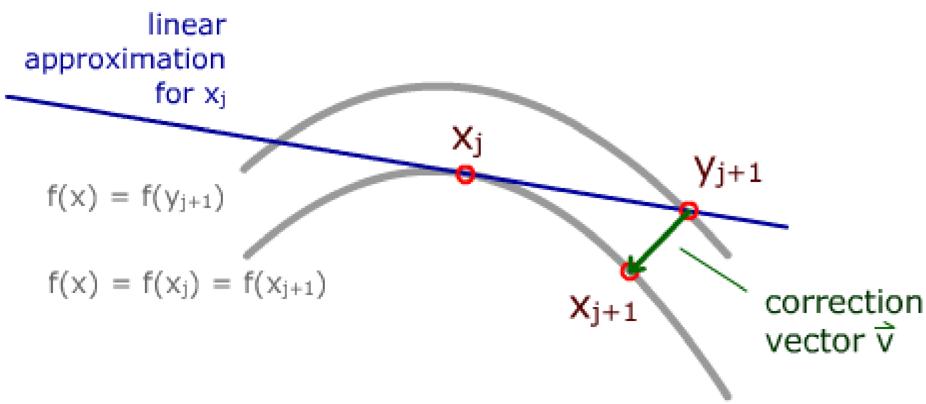
$f(x,y) = x^2 + y^2$
$\boldsymbol{\varepsilon = 10^{-5}}$
$x_0 = (1.0, 0.0)$
$N = 4*10^6$

Altitude Corrections

To ensure that the algorithm does not produce a path which strays away from the curve, at each step we can force the point back onto the right level curve.

The uncorrected algorithm found the point on the tangent line y_{j+1}, which has a different altitude from x_j. We can find a point at the correct altitude by adding the correction, **v**, to y_{j+1}.

$$x_{j+1} = y_{j+1} + \vec{v}$$



In order to get a point close to y_{j+1}, the direction of **v** should be the direction of quickest ascent/descent, so we use the gradient.

$$\vec{v} = |\vec{v}| \frac{1}{|\nabla f(y_{j+1})|} \nabla f(y_{j+1})$$

The length of the gradient indicates the steepness of ascent or descent; it is the ratio of change in altitude to change in horizontal position. Since we can calculate the gradient and the change in altitude, this can be used to determine |v|.

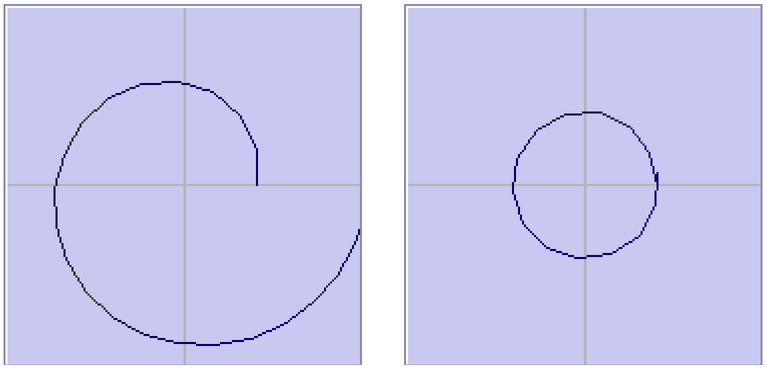
$$\nabla f(y_{j+1}) = \frac{f(x_j) - f(y_{j+1})}{|\vec{v}|}$$

$$|\vec{v}| = \frac{f(x_j) - f(y_{j+1})}{\nabla f(y_{j+1})}$$

This is all we need to write out the full formula for the new corrected point:

$$x_{j+1} = y_{j+1} + \frac{f(x_j) - f(y_{j+1})}{|\nabla f(y_{j+1})|^2} \nabla f(y_{j+1})$$

$f(x,y) = x^2+y^2$
$\varepsilon = 0.5$



Uncorrected Using correction

Using this corrective method at each step, the new algorithm can withstand drastic reductions in epsilon. Since this method ensures that the curve will close, reducing epsilon serves only to improve the smoothness of the curve. Even ε = 0.5 provides a decent, although somewhat rough graph, in a mere 14 iterations.

The quality of $\varepsilon = 10^{-4}$ and 10^{-5} graphs can be matched with $\varepsilon = 10^{-2}$ and 10^{-3} using step correction.

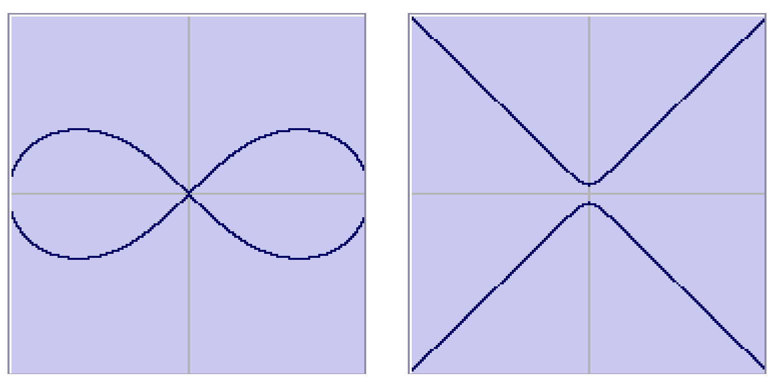
Critical Points

If the algorithm reaches a point at which $\nabla f(x_n) = 0$, the gradient has no direction. There is no way to continue to x_{n+1}, so the iteration terminates.

If the critical point is a local minimum or maximum, this behavior is accurate; the curve only consists of a single point. Saddle points, however, are problematic, and may result in incompletion of the graph.

Fortunately, it is uncommon for the algorithm to directly hit a critical point, as the gradient is calculated to 64-bit precision. The result will be precisely (0, 0) for this problem to occur. In the case of this figure-8 curve, beginning with x_0 = (1, 0) will result in the algorithm terminating at the critical point. This can be sidestepped by changing x0 to (1, 0.001), which will not significantly impact the curve.

$f(x,y) = (x^2+y^2)^2-x^2+y^2$
$\varepsilon = 10^{-4}$
$x_0 = (0.05, 0.05)$



Window range: x(-1, 1) y(-1, 1) Window range: x(-0.1, 0.1) y(-0.1, 0.1)

This level curve graph seemingly passes through a critical point. Upon closer inspection, it is actually following a curve that passes close to it.