Project for Math 2605

This project is aimed at investigating the "basins of attraction" for Newton's method of solving

$$f(x, y) = 0$$
$$g(x, y) = 0$$

The goal is to experimentally investigate the following important question:

• *In using Newton's method, how close to an exact solution does our starting guess need to be?*

As you may guess, the answer depends on the gradients and Hessians of f and g. In particular, if ∇f and ∇g are nearly parallel at some solution, we may expect trouble there, since that $J_{\mathbf{F}}$, where $\mathbf{F} = \begin{bmatrix} f \\ g \end{bmatrix}$ will have small determinant, and so its inverse will have large entries.

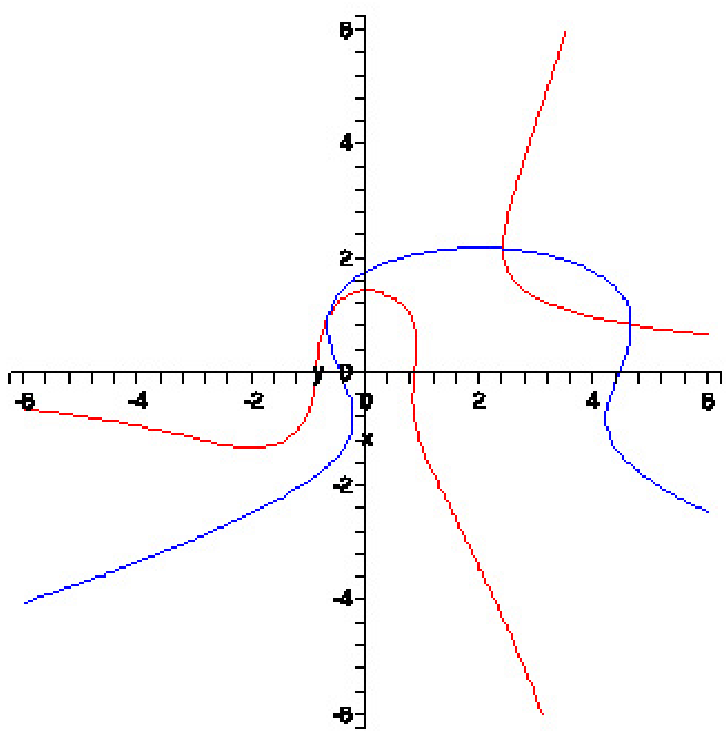
For example, consider the functions

$$f(x, y) = x^3y - 4x^2 - y^3 + 3$$

and

$$g(x, y) = x^2 + y^3 - 4x - 2y - 2\ .$$

Here is a graph showing the two curves defined implicitly by $f(x, y) = 0$ and $g(x, y) = 0$.



You see three solutions in the region shown, which is $-6 \le x, y \le 6$. Call the left most solution the *blue solution*, the right most solution the *green solution* and the middle solution the *red solution*. The goal is to write a program that will start from a dense grid of points in the graph, and for each one, determine which solution, if any, Newton's method leads to form there. Then, graphing your data, you can see which starting points end up where, and answer the question raised at the beginning.

What your program should do: Divide the square region $-6 \leq x, y \leq 6$ up into a 100 by 100 grid. Index the squares in a grid by i and j with $0 \leq i, j \leq 99$. The upper left hand corner of the i, jth square in the grid will be

$$(-6 + 12i/100, 6 - 12j/100) \ .$$

(1) Write code that runs Newton's method for the system above starting from each of these $10,000$ points. Label the starting point *blue*, *green*, *red* or *white* depending on whether you get convergence to the *blue*, *green* or *red* solutions, or none of the above.

(2) Write code, or use some graphing software, to produce a picture by coloring in the pixels of a 100 by 100 pixel square in which each pixel is colored in with the corresponding color. You are free to seek assistance with the graphics here. If you use somebody else's help to do this, just credit them. You will not lose points for that.

Questions to be answered For each solution of the system, determine from your graph the approximate radius r of the largest disk about that solution such that every starting point in the disk converges to that solution. Measure it as closely as you can from your graph.

(1) List your observe radii of convergence for each of the solutions. Also, compute $\det(J_{\mathbf{F}})$, and $\|H_f\|_{\rm HS}$ $\|H_g\|_{\rm HS}$ at each of the solutions. Do you see any relation between these numbers and the radii?

(2) What do you notice about the boundary between the different basins of attraction?

(3) We can learn only so much from a single example. Therefore, do the same for the system $f(x, y) = 0$ and $g(x, y) = 0$ where now

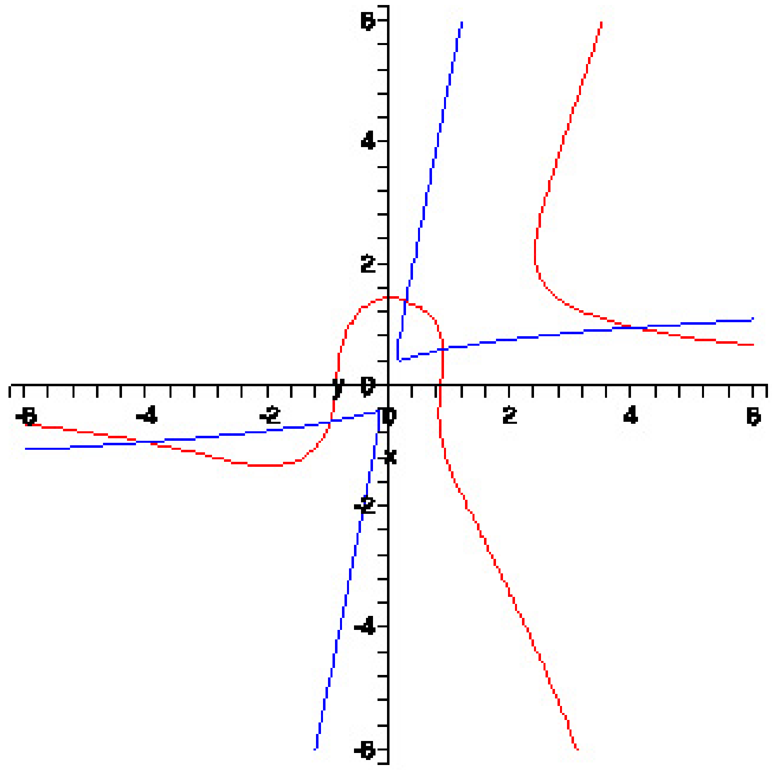
$$f(x, y) = x^3y - 4x^2 - y^3 + 3$$

and

$$g(x, y) = x^2 + y^4 - 5xy^3 \ .$$

Answer questions *(1)* and *(2)* for this system.

Only $g(x, y)$ has changed, and you will be able to use you coding with slight modifications. This time there are 5 solutions in the same region, so you will need more colors.



(4) Here is another interesting example. do the same for the system $f(x,y) = 0$ and $g(x,y) = 0$ where now

$$f(x,y) = x^3y - 4x^2 - y^2 + 3$$

and

$$g(x,y) = x^2 + y^3 - 4x - 2y - 2 \; .$$

All that has changed from the first system considered is that the y^3 term in f has changed to y^2. If you plot the two curves implicitly defined by $f(x,y) = 0$ and $g(x,y) = 0$ in the region $-2 \le x \le 2$ and $0 \le y \le 2$, you see that they are nearly tangent to one another along a stretch between two solutions, near $x_0 = -0.5697$ and $y_0 = 1.200$. Answer questions *(1)* and *(2)* for this system.

Extra Credit Implement this as a java applet in which f and g can be entered by the user, along with the region to be investigated.