Basins of Attraction

When Newton's method is used to solve systems with multiple solutions, it is not always certain to which solution the algorithm will converge. This application runs Newton's method on two functions in two variables; by coloring each starting position to indicate which solution is obtained, we can visually see areas in which intial positions for Newton's method will reliably produce the same result.

$$x_{n+1} = x_n - (J_F(x_n))^{-1} F(x_n)$$ *Newton's method*

At each iteration, the distance from x_n to x_{n+1} is found by applying the inverse of the Jacobian to $F(x_n)$. Ideally, when x_n is near a solution, this distance should become increasingly smaller as the process approaches the solution with increasing precision. This means that, in general, when the Jacobian is large (as measured by the absolute value of the determinant) , and its inverse is small, Newton's method will work as we would like, and yield more predictable results.

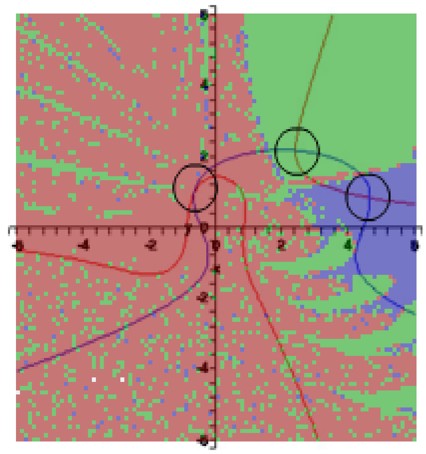
Example One

$$F(\vec{x}) = \begin{Bmatrix} f(x,y) = x^3 y - 4x^2 - y^3 + 3 \\ g(x,y) = x^2 + y^3 - 4x - 2y - 2 \end{Bmatrix}$$

$$J_F(\vec{x}) = \begin{bmatrix} 3x^2 y - 8x & x^3 - 3y^2 \\ 2x - 4 & 3y^2 - 2 \end{bmatrix}$$

$$H_f(\vec{x}) = \begin{bmatrix} 6xy - 8 & 3x^2 \\ 3x^2 & -6y \end{bmatrix}$$

$$H_g(\vec{x}) = \begin{bmatrix} 2x & 0 \\ 0 & 6y \end{bmatrix}$$



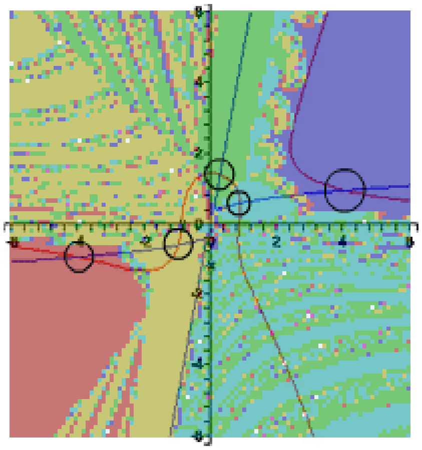
p₁ = (-0.63937220, 1.03083306)
p₂ = (2.41408884, 2.16588572)
p₃ = (4.66231111, 0.83406716)

$$\begin{aligned} det(J_F(\vec{p}_1)) &= -10.630 & det(J_F(\vec{p}_2)) &= 224.015 & det(J_F(\vec{p}_3)) &= -527.026 \\ \|H_f(\vec{p}_1)\|_{HS} &= 184.173 & \|H_f(\vec{p}_2)\|_{HS} &= 1326.465 & \|H_f(\vec{p}_3)\|_{HS} &= 8765.179 \\ \|H_g(\vec{p}_1)\|_{HS} &= 39.889 & \|H_g(\vec{p}_2)\|_{HS} &= 192.189 & \|H_g(\vec{p}_3)\|_{HS} &= 111.993 \end{aligned}$$

The radii of convergence for this system, as shown on the graph a circle around each solution, are approximately 0.3. This is not entirely accurate for p_1, as two points within the circle converge elsewhere.

The most predictable results occur around p_2 and p_3, where $|det(J_F)|$ is highest. These are also the two points at which at least one function has a large $\|H_f\|_{HS}$ value, possibly indicating that Newton's method is more well-behaved when a function has a high amount of curvature at a solution.

Example Two



p_1 = (-3.98811438 , -0.94248647)
p_2 = (-0.97183225 , -0.60553770)
p_3 = (0.28475362 , 1.39382376)
p_4 = (0.89727820 , 0.59125841)
p_5 = (4.07789941 , 0.94927179)
p_6 = (25.03179351 , 125.15864794)

$$F(\vec{x})=\begin{Bmatrix} f(x,y)=x^3y-4x^2-y^3+3 \\ g(x,y)=x^2+y^4-5xy^3 \end{Bmatrix}$$

$$J_F(\vec{x})=\begin{bmatrix} 3x^2y-8x & x^3-3y^2 \\ 2x-5y^3 & 4y^3-15xy^2 \end{bmatrix}$$

$$H_f(\vec{x})=\begin{bmatrix} 6xy-8 & 3x^2 \\ 3x^2 & -6y \end{bmatrix}$$

$$H_g(\vec{x})=\begin{bmatrix} 2 & -15y^2 \\ -15y^2 & 12y^2-30xy \end{bmatrix}$$

$$\begin{aligned} det(J_F(\vec{p}_1)) &= -901.073 \\ \|H_f(\vec{p}_1)\|_{HS} &= 4797.227 \\ \|H_g(\vec{p}_1)\|_{HS} &= 10784.081 \end{aligned}$$

$$\begin{aligned} det(J_F(\vec{p}_2)) &= 25.323 \\ \|H_f(\vec{p}_2)\|_{HS} &= 49.229 \\ \|H_g(\vec{p}_2)\|_{HS} &= 240.180 \end{aligned}$$

$$\begin{aligned} det(J_F(\vec{p}_3)) &= -80.203 \\ \|H_f(\vec{p}_3)\|_{HS} &= 101.626 \\ \|H_g(\vec{p}_3)\|_{HS} &= 1832.511 \end{aligned}$$

$$\begin{aligned} det(J_F(\vec{p}_4)) &= 22.550 \\ \|H_f(\vec{p}_4)\|_{HS} &= 47.455 \\ \|H_g(\vec{p}_4)\|_{HS} &= 196.369 \end{aligned}$$

$$\begin{aligned} det(J_F(\vec{p}_5)) &= -1014.261 \\ \|H_f(\vec{p}_5)\|_{HS} &= 5241.861 \\ \|H_g(\vec{p}_5)\|_{HS} &= 11461.211 \end{aligned}$$

$$\begin{aligned} det(J_F(\vec{p}_6)) &= 1.539(10^{11}) \\ \|H_f(\vec{p}_6)\|_{HS} &= 3.607(10^{8}) \\ \|H_g(\vec{p}_6)\|_{HS} &= 1.193(10^{11}) \end{aligned}$$

(since p_6 is not located on the graph, it is not discussed.)

Again, large Jacobians and Hessians are found at wide basins of attraction with fairly well-defined borders. P_5 has the greatest Jacobian and the greatest radius of convergence, around 0.3. Similarly, p_4 has the smallest Jacobian, with a small convergence radius of 0.15.

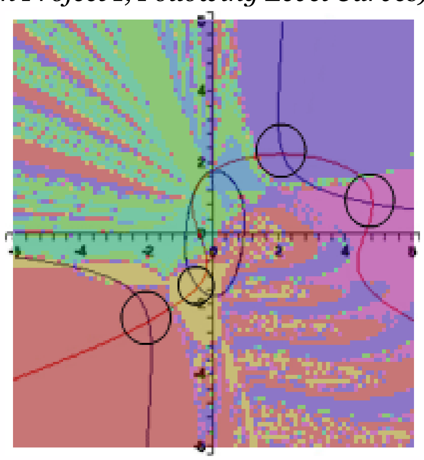
The boundaries between basins of attraction are related to, but not consistently predictable from, the graph of the functions. Some boundaries occur on or near the function curves, while others run between them.

Example Three

Notice that the solutions p_3, p_4, and p_5 happen to be very near each other. As a result, this region is highly volatile, with no distinct areas of attraction. For these three points, $|det(J_F)|<10$, further demonstrating that a large Jacobian is necessary to produce distinct basins of attraction.

$$F(\vec{x})=\begin{Bmatrix} f(x,y)=x^3y-4x^2-y^2+3 \\ g(x,y)=x^2+y^3-4x-2y-2 \end{Bmatrix}$$

(Graph produced by the application created in Project 1, Following Level Curves)



p_1 = (-1.95578114, -2.44008955)
p_2 = (-0.50239172, -1.47564407)
p_3 = (-0.51925322, 1.31794823)
p_4 = (-0.62867814, 1.07346392)
p_5 = (-0.07818591, 1.72473868)
p_6 = (2.02997559, 2.17990776)
p_7 = (4.66229150, 0.83522863)

$$J_F(\vec{x})=\begin{bmatrix} 3x^2y-8x & x^3-2y \\ 2x-4 & 3y^2-2 \end{bmatrix}$$

$$H_f(\vec{x})=\begin{bmatrix} 6xy-8 & 3x^2 \\ 3x^2 & -2 \end{bmatrix} \qquad H_g(\vec{x})=\begin{bmatrix} 2 & 0 \\ 0 & 6y \end{bmatrix}$$

$$\begin{aligned} det(J_F(\vec{p}_1)) &= -216.543 \\ \|H_f(\vec{p}_1)\|_{HS} &= 693.111 \\ \|H_g(\vec{p}_1)\|_{HS} &= 218.345 \end{aligned}$$

$$\begin{aligned} det(J_F(\vec{p}_2)) &= 27.289 \\ \|H_f(\vec{p}_2)\|_{HS} &= 17.763 \\ \|H_g(\vec{p}_2)\|_{HS} &= 82.391 \end{aligned}$$

$$\begin{aligned} det(J_F(\vec{p}_3)) &= 2.775 \\ \|H_f(\vec{p}_3)\|_{HS} &= 151.866 \\ \|H_g(\vec{p}_3)\|_{HS} &= 66.532 \end{aligned}$$

$$\begin{aligned} det(J_F(\vec{p}_4)) &= -3.411 \\ \|H_f(\vec{p}_4)\|_{HS} &= 151.995 \\ \|H_g(\vec{p}_4)\|_{HS} &= 45.484 \end{aligned}$$

$$\begin{aligned} det(J_F(\vec{p}_5)) &= -9.789 \\ \|H_f(\vec{p}_5)\|_{HS} &= 81.601 \\ \|H_g(\vec{p}_5)\|_{HS} &= 111.090 \end{aligned}$$

$$\begin{aligned} det(J_F(\vec{p}_6)) &= 131.010 \\ \|H_f(\vec{p}_6)\|_{HS} &= 653.796 \\ \|H_g(\vec{p}_6)\|_{HS} &= 175.072 \end{aligned}$$

$$\begin{aligned} det(J_F(\vec{p}_7)) &= -529.127 \\ \|H_f(\vec{p}_7)\|_{HS} &= 8744.986 \\ \|H_g(\vec{p}_7)\|_{HS} &= 29.114 \end{aligned}$$

Step-count coloration

This implementation also includes the ability to change the color brightness based upon the number of iterations required to reach the solution. Darker pixels indicate a greater number of steps, whereas areas of quick convergence are brighter. In addition to producing a more interesting-looking picture, this version gives more information regarding which starting points are optimal.