Math 112C Homework 2

Due: Wednesday, April 17

Ffor problems requiring code, you can just include the *output* of your code, and if you want, include the code at the end of your submission. If the plot looks right, I will not look at your code. If it looks wrong, I will *try* to see if I can identify any glaring issues.

1. In class, we derived the **Crank Nicolson** scheme for the diffusion equation $\partial_t u = D\partial_{xx}u$,

$$\frac{U_j^{n+1} - U_j^n}{k} = \frac{D}{2}\left[\frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{h^2} + \frac{U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}}{h^2}\right] \tag{1}$$

   Derive the **order** of this method. Hint: move everything to one side and then Taylor series. Your leading terms (after cancellations) are your answer. (This is basically the trapezoidal rule you did on the last homework)

2. Numerically solve the diffusion equation $\partial_t u = D\partial_{xx}u$. with *forward* first order in time, *central* (second-order) in space, as we discussed in class, so

$$\frac{U_j^{n+1} - U_j^n}{k} = D\frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{h^2} \tag{2}$$

   Take **periodic boundary** conditions, $u(-1,t) = u(1,t)$, and initial condition $u(x,0) = 5 \cdot \mathbb{1}_{x\in[-0.1,0.1]}$ and $D = 1$. Solve it for $0 \le t \le 10$. Does your solution look right? It would be great to plot snapshots along the way. Try experimenting with different stepsizes.

   (Optional but much harder) use Crank-Nicolson instead. Can you take bigger timesteps?

3. It still makes perfect sense to discuss **stability** of PDE methods, but it's a bit more involved. This is called *Von Neumann* stability analysis. This might require some hints from the TA or Chris.

   Take the scheme for the diffusion equation

$$\frac{U_j^{n+1} - U_j^n}{k} = D\frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{h^2} \tag{3}$$

   And we will "guess" a form of the solution

$$\hat{u}(x,t) = T(t)e^{i\omega x} \tag{4}$$

   If you plug this in, you should get something like

$$T(t+k) = AT(t) \qquad \Longrightarrow T^{n+1} = AT^n \tag{5}$$

   which we know converges when $|A| < 1$.

   1. What is A?
   2. Using the condition $|A| < 1$, find the stability condition. Your answer should be some conditions on h, k and how they relate to D.

This is a bit tough, but your answer to part 1 should have a trig function in it. Use the *worst* case (maximum) of this trig function in your analysis. And if you want, go ahead to Google this to help figure it out.

4. **Reflection** (optional). What are your thoughts+feelings on this assignment?

- [ ] Was anything easier than expected?
- [ ] Harder than expected?
- [ ] Did it convince you that you've learned stuff in the class so far?
- [ ] Do you have outstanding confusions?