COM 5335 NETWORK SECURITY LECTURE 6 PUBLIC KEY CRYPTOGRAPHY & RSA

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Outline

- One-way Trapdoor functions
- Basic Number Theory for RSA
- RSA Digital Signatures

One-Way Trapdoor Functions

One-Way Functions

- The most basic primitive for cryptosystem is a one-way function (OWF).
  - *Informally, this is a function which is EASY to compute but HARD to invert.*

The Factorization Problem

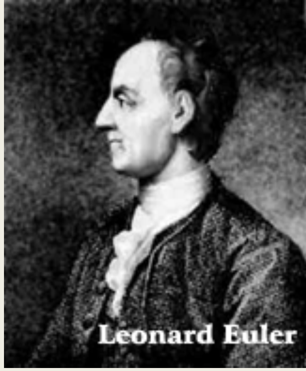
- Factorization is a well-known candidate for OWF.
  - *Randomly select two prime numbers: p and q.*
  - *It is easy to compute N=pq.*
  - *However, conversely, given N=pq, it is assumed to be HARD to obtain p or q.*

One-way Trapdoor Functions

- A one-way trapdoor function f is a one-way function with an extra property.
- There exists some secret information (called the trapdoor) that allows its possessor to EFFICIENTLY invert f.
- It is infeasible to invert f without knowledge of the trapdoor.

Basic Number Theory for RSA

Euler Totient Function



- Euler's Totient Function φ is defined by
  $$\varphi(n) = \left|\{x | 1 \leq x < n, \gcd(x, n) = 1\}\right|$$
- $\varphi(2)=|\{1\}|=1$
- $\varphi(3)=|\{1,2\}|=2$
- $\varphi(4)=\{1,3\}=2$
- $\varphi(5)=|\{1,2,3,4\}|=4$
- $\varphi(6)=|\{1,5\}|=2$

Calculation of Euler Totient Function

- Properties
- $\forall \text{ primes } p, \text{ and } \alpha \geq 1 \Rightarrow \varphi(p^\alpha) = p^{\alpha-1}(p-1)$
- $\forall m, n \in \mathbb{Z}, s.t. \gcd(m,n) = 1 \Rightarrow \varphi(mn) = \varphi(m)\varphi(n)$
- Corollary: $\varphi(pq) = (p-1)(q-1)$ for p, q primes

The Group $\mathbb{Z}_n^*$

- $\mathbb{Z}_n^* = \{k | \gcd(k, n) = 1, 1 \leq k < n\}$
- For any positive integer n, $\mathbb{Z}_n^*$ forms a group under multiplication modulo n.
- Euler's Theorem

  $\forall \alpha \in \mathbb{Z}_n^*,\ \text{ we have } \alpha^{\varphi(n)} \equiv 1 \pmod{n}$

Examples of $\mathbb{Z}_n^*$

- Z_{15}={0,1,2,3,4,5,6,7,8,9,10,11,12,13,14}
- Z_{15}*={1,2,4,7,8,11,13,14}

$(\varphi(15) = 8)\ \forall \alpha \in \mathbb{Z}_{15}^*,\ \text{we have } \alpha^8 \equiv 1 \pmod{15}$

- Z_{12}={0,1,2,3,4,5,6,7,8,9,10,11}
- Z_{12}*={1,5,7,11}

$(\varphi(12) = 4)\ \forall \alpha \in \mathbb{Z}_{12}^*,\ \text{we have } \alpha^4 \equiv 1 \pmod{12}$

RSA

- In 1977 Rivest, Shamir and Adelman proposed the first candidate trapdoor function,
  - *Now called the RSA. The story of modern cryptography followed.*
  - *The best known & widely used public-key scheme*
- It is based on exponentiation in a finite group $\mathbb{Z}_n^*$ over integers modulo a number
  - *exponentiation takes* $O(\log^3 n)$ *operations (easy)*
- It uses large integers (eg. 1024 bits)
- The security relies on difficulty of factoring large numbers
  - *factorization takes* $O\left(e^{\sqrt{\log n \log \log n}}\right)$ *operations (hard)*

RSA Key Setup

- Each user generates a public/private key pair by:
  - *Selecting two large primes at random: p, q*
  - *Computing their system modulus N=pq*
    - note $\varphi(N) = (p-1)(q-1)$
  - *Selecting at random the encryption key e*
    - where $1 < e < \varphi(N),\ \gcd\big(e, \varphi(N)\big) = 1$
  - *Solve following equation to find decryption key d*
    - $ed \equiv 1 \ \big(\mathrm{mod}\ \varphi(N)\big),\ \text{and } 0 \leq d \leq N$
    - Fast to do it using Euclid's Algorithm.
  - *publish their public encryption key:* $P_u = \{e,N\}$
  - *keep secret private decryption key:* $S_u = \{d,p,q\}$

RSA Encryption/Decryption

- Encrypt a message M by the sender:
  - *obtains public key of recipient $P_u=\{e,N\}$*
  - *computes: $C=M^e \bmod N$, where $0 \le M < N$*
- Decrypt the ciphertext C by the owner u:
  - *use its private key $S_u=\{d,p,q\}$*
  - *compute: $M=C^d \bmod N$*
- note that the message M must be smaller than the modulus N (block if needed)

Why RSA Works

- By Euler's Theorem:
  - $\alpha^{\varphi(N)} \equiv 1 \pmod{N}$
  - *where* $\gcd(\alpha, N) = 1$
- In RSA, we have:
  - *N=pq*
  - *φ(N)=(p-1)(q-1)*
  - *carefully chosen e & d to be inverses mod φ(N)*
  - *hence ed=1+kφ(N) for some k*
- Hence (if M is relatively prime to N):

$$C^d = (M^e)^d = M^{1+k\varphi(N)} = M(M^{\varphi(N)})^k = M^1(1)^k \equiv M \pmod{N}$$

Corollary of Euler's theorem

- Given two prime numbers p and q, and integers $n = pq$ and m, with $0<m<n$, the following relationship holds:
- $$m^{\varphi(n)+1} \equiv m \pmod{n}$$
- Proof: When $\gcd(m,n) \neq 1$, and m is a multiple of p
- → $m = cp$, $\gcd(m,q) = 1$ since $m < pq$
- → $m^{\varphi(q)} \equiv 1 \pmod{q}$
- → $[m^{\varphi(q)}]^{\varphi(p)} \equiv 1 \pmod{q}$
- → $m^{\varphi(n)} \equiv 1 \pmod{q}$ implies that $m^{\varphi(n)} = 1 + kq$
- → $m^{\varphi(n)+1} = m + kcpq = m + kcn$ (multiply $m = cp$ on both sides)
- → $m^{\varphi(n)+1} = m \pmod{n}$

Modular Exponentiation

- A useful operation for PKC:
  - *Given a, n, m, where* $a \in Z_n$ *and m is an integer,*
  - *computes* a^m *mod n.*
- By repeated squaring, a^m mod n can be computed in O(log m) multiplications in mod n, hence $O(\log^3 n)$ time, if $m<n$.

RSA Example

- Select primes: $p=17$ & $q=11$
- Compute $n = pq = 17 \times 11 = 187$
- Compute $\varphi(n)=(p-1)(q-1)=16\times 10=160$
- Select e : $\gcd(e,160)=1$; choose $e=7$
- Determine d: $de=1 \bmod 160$ and $d < 160$. $d=23$ since $23\times 7=161= 10\times 160+1$
- Publish public key $P=\{7,187\}$
- Keep secret private key $S=\{23,17,11\}$

RSA Example (cont.)

- sample RSA encryption/decryption is:
- given message M = 88
- Encryption (using public key):
  - $C = 88^7 \bmod 187 = 11$
- Decryption (using private key):
  - $M = 11^{23} \bmod 187 = 88$

Modular Exponentiation

- Use the Square and Multiply Algorithm
  - *a fast, efficient algorithm for exponentiation*
- Concept is based on repeatedly squaring base
- and multiplying in the ones that are needed to compute the result
- look at binary representation of exponent
- only takes $O(\log^2 n)$ multiples for number n
  - *eg.* $7^5 = 7^4 * 7^1 = 3*7 = 10 \bmod 11$
  - *eg.* $3^{129} = 3^{128} * 3^1 = 5*3 = 4 \bmod 11$

Modular Exponentiation

Compute a^b mod n. (b_i is the bit string of b)

$$
\begin{aligned}
&\textbf{for } i \leftarrow k \textbf{ downto } 0\\
&\quad \textbf{do } c \leftarrow 2 \times c\\
&\qquad d \leftarrow (d \times d) \bmod n\\
&\qquad \textbf{if } b_i = 1\\
&\qquad\quad \textbf{then } c \leftarrow c + 1\\
&\qquad\qquad\quad d \leftarrow (d \times a) \bmod n
\end{aligned}
$$

Equivalently, the algorithm looks at binary expansion of m. What we did is collect all the powers of two corresponding to the ones and multiply them.

For example: compute 2^{21} mod 22.

21='10101'

4	3	2	1	0
a^{16}	a^8	a^4	a^2	a^1
1	0	1	0	1

$2^1=2 \pmod{22}$ $2^2=4 \pmod{22}$ $2^4=16 \pmod{22}$

$2^8=16*16=256=220+36=36 \pmod{22}=14 \pmod{22}$

$2^{16}=14*14=196=22*8+20=20 \pmod{22}$

Therefore,

$2^{21}=2^{16}*2^4*2^1=20*16*2=20*32=$

$=20*10 \pmod{22}=200 \pmod{22}=22*9+2=2 \pmod{22}$.

Some Remarks on RSA

The Hardness to Invert RSA

- Thus far, the best way known to invert RSA is to first factor n.
- The best running time for a fully proved algorithm is Dixon’s random squares algorithms which runs in time:
  $O\left(e^{\sqrt{\log n \log \log n}}\right)$
- But, in practice we may consider others.

- Let *l*=|*p*| where *p* is the smallest prime divisor of *n*. The Elliptic Curve algorithm takes *expected time*

$$O\left(e^{\sqrt{2\log l \log\log l}}\right)$$

- The Quadratic Sieve algorithm runs in *expected time*:

$$O\left(e^{\sqrt{\log n \log\log n}}\right)$$

- The recommended size for *n* these days is 1024 bits.

Knowledge of φ(n) is equivalent to knowledge of the factorization

$\varphi(n) \Longleftarrow$ factorization

- To compute φ(n) from p and q:
- φ(n) =(p-1)(q-1)=n+1-(p+q).

$\varphi(n) \Longrightarrow$ factorization

- To compute out p and q from φ(n).
- Since pq=n and p+q=n+1- φ(n).
- Define 2b= n+1- φ(n) since φ(n) is even.
- p and q must be the root of equation
- x2-2bx+n=0. Thus p and q equal to $b \pm \sqrt{b^2 - n}$

RSA Key Generation Remarks

- Users of RSA must:
  - *determine two primes at random p, q*
  - *select either e or d and compute the other*
- Primes p,q must not be easily derived from modulus N=p.q
  - *means must be sufficiently large*
  - *typically guess and use probabilistic test*
- Exponents e, d are inverses, so use Inverse algorithm to compute the other

RSA Security

- three approaches to attacking RSA:
  - *brute force key search (infeasible given size of numbers)*
  - *mathematical attacks (based on difficulty of computing $\varphi(N)$, by factoring modulus N)*
  - *timing attacks (on running of decryption)*

Factoring Problem

- To attack RSA, we can do either of the followings.
  - *factor N=p.q, hence find $\varphi(N)$ and then d*
  - *determine $\varphi(N)$ directly and find d*
  - *find d directly*
- If we can crack factoring => we can crack RSA, but not vice versa (i.e. if we crack RSA we may not be able to do factoring).
- Currently we believed RSA is equivalent to factoring
  - *have seen slow improvements over the years*
    - as of Aug-99 best is 130 decimal digits (512) bit with GNFS
  - *biggest improvement comes from improved algorithm*
    - cf “Quadratic Sieve” to “Generalized Number Field Sieve”
  - *barring dramatic breakthrough 1024+ bit RSA secure*
    - ensure p, q of similar size and matching other constraints

How to choose p and q

(1). The two primes should not be too close to each other (e. g. one should be a few decimal digits longer than the other).

Also, any one of p and q should not be too small due to the Elliptic Curve algorithm

Reason: $n = pq \Rightarrow n = \big((p+q)/2\big)^2 - \big((p-q)/2\big)^2 = t^2 - s^2$

Since p and q are close together we get: s is small and t is an integer only slightly larger than $\sqrt{n}$. If you test the successive integers $l > \sqrt{n}$ you will soon find one such that n= t^2-s^2, at which point you have p=t+s and q=t-s.

(2). p-1 and q-1 should have a fairly small gcd and both have at least one large prime factor.

(3). Of course, if someone discovers a factorization method that works quickly under certain other conditions on p and q, then further users of RSA would have to take care to avoid those conditions as well.

Summary

- We have covered:
- The principles of public-key cryptography
- RSA algorithm, implementation, security