COM 5120
Communications Theory

Chapter 4
Optimal Receiver for AWGN

Prof. Jen-Ming Wu
Inst. Of Communications Engineering
Dept. of Electrical Engineering
National Tsing Hua University
Hsinchu, Taiwan

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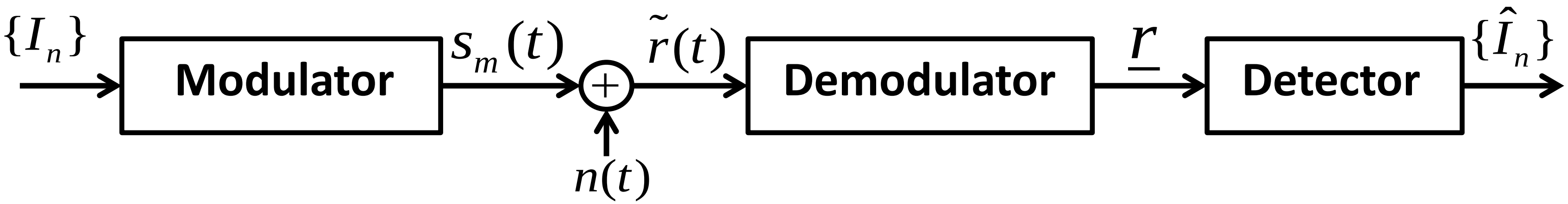
4.1 **Optimal Detection for a General Vector Channel**



Let $\{s_m(t),\ m=1,2,\ldots,M\}$ be the M transmitted signal waveforms.

The received signal is

$$\tilde{r}(t) = s_m(t) + n(t) \quad 0 \le t \le T$$

$$n(t) \sim \mathcal{N}(0, \frac{N_0}{2})$$

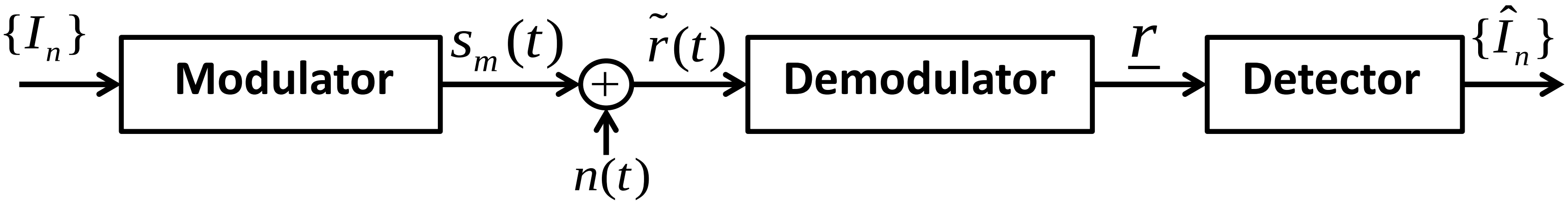
Suppose the basis set $\{\phi_1(\mathrm{t}),\ldots,\phi_N(\mathrm{t})\}$ is the minimun set for spanning $\{s_m(t)\}$

$$s_m(t) = \sum_{k=1}^{N} s_{mk}\phi_k(\mathrm{t}) \qquad k = 1,\ldots,N$$

$$s_{mk} = < s_m(t), \phi_k(t) > = \int_0^T s_m(t)\phi_k(t)dt$$

$$\underline{s}_m = [s_{m1}, s_{m2}, \ldots, s_{mN}]^T$$

4.1 **Optimal Detection for a General Vector Channel**



Note that, in general,

$$\tilde{r}(t) = s_m(t) + n(t) \notin \text{span}\{\phi_1(t),\ldots,\phi_N(t)\} \quad \text{Why?}$$

$$\because n(t) \notin \text{span}\{\phi_1(t),\ldots,\phi_N(t)\}$$

Suppose that

$$\tilde{r}(t) \in \text{span}\{\phi_1(t),\ldots,\phi_K(t)\} \quad K \geq N$$

$$\tilde{r}(t) = \sum_{k=1}^{K} r_k \phi_k(t) \quad \underline{\tilde{r}} = [r_1, r_2, \ldots, r_K]^T$$

4.1 **Optimal Detection for a General Vector Channel**

Objective : Based on $\tilde{r}(t)$ or equivalently $\underline{\tilde{r}}$,
the receiver wants to find the optimally estimated $\underline{\hat{s}}_m$.

The detector is a function that maps $\underline{\tilde{r}}$ to one of the signals $\{\underline{s}_1,\ldots,\underline{s}_M\}$.

Definition of Optimality:
The optimal detector is the detector that minimize $\Pr(\text{error})$ i.e

$$\mathrm{D}(\underline{\tilde{r}}) = \underline{s}_m \quad \text{if} \quad P(\underline{s}_m \mid \underline{\tilde{r}}) \geq P(\underline{s}_n \mid \underline{\tilde{r}}) \ \ \forall\, n \neq m$$

$$\mathrm{D}(\underline{\tilde{r}}) = \arg\max_{\underline{s}\in\{\underline{s}_1,\ldots,\underline{s}_M\}} P(\underline{s} \mid \underline{\tilde{r}})$$

This is called *Maximum A Posteriori* (MAP) detector

$$\underline{\hat{s}} = \arg\max_{\underline{s}\in\{\underline{s}_1,\ldots,\underline{s}_M\}} P(\underline{s} \mid \underline{\tilde{r}})$$

N-dimension *K*-dimension

Theorem of Irrelevance

For MAP detector, to represent the received signal, it is sufficient to have

$$r(t)=\sum_{k=1}^{N} r_k \phi_k(t) \quad \text{where} \quad r_k = < \tilde{r}(t), \phi_k(t) >$$

Given the received signal $\tilde{r}(t)=\sum_{k=1}^{K} r_k \phi_k(t) = s_m(t)+n(t)$

$$\text{where} \quad r_k \;=\; < \tilde{r}(t), \phi_k(t) > = < s_m(t), \phi_k(t) > + < n(t), \phi_k(t) >$$

$$= \begin{cases} s_{mk}+n_k & 1 \le k \le N \\ n_k & k > N \end{cases}$$

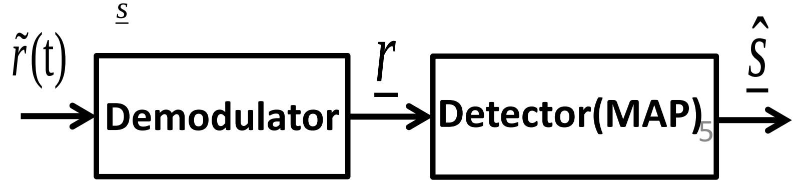
The noise vector $\underline{\tilde{n}} = [n_1, n_2, \ldots, n_K]^T$ and $\underline{\tilde{n}} \sim \mathcal{N}(0, \frac{N_0}{2}\mathbf{I_K})$, i.e. $\mathrm{E}[n_k]=0$, and $\mathrm{E}[n_k n_m]=\frac{N_0}{2}\delta_{km}$

where $\{n_k\}_{k=N+1}^{K}$ are mutually independent and independent of $\underline{s}_m$

Therefore $\underline{\hat{s}} = \arg\max_{\underline{s}} P(\underline{s} \mid \underline{\tilde{r}}) = \arg\max_{\underline{s}} \dfrac{P(\underline{s}, \underline{\tilde{r}})}{P(\underline{\tilde{r}})} = \arg\max_{\underline{s}} \dfrac{P(\underline{s})P(\underline{\tilde{r}} \mid \underline{s})}{P(\underline{\tilde{r}})}$

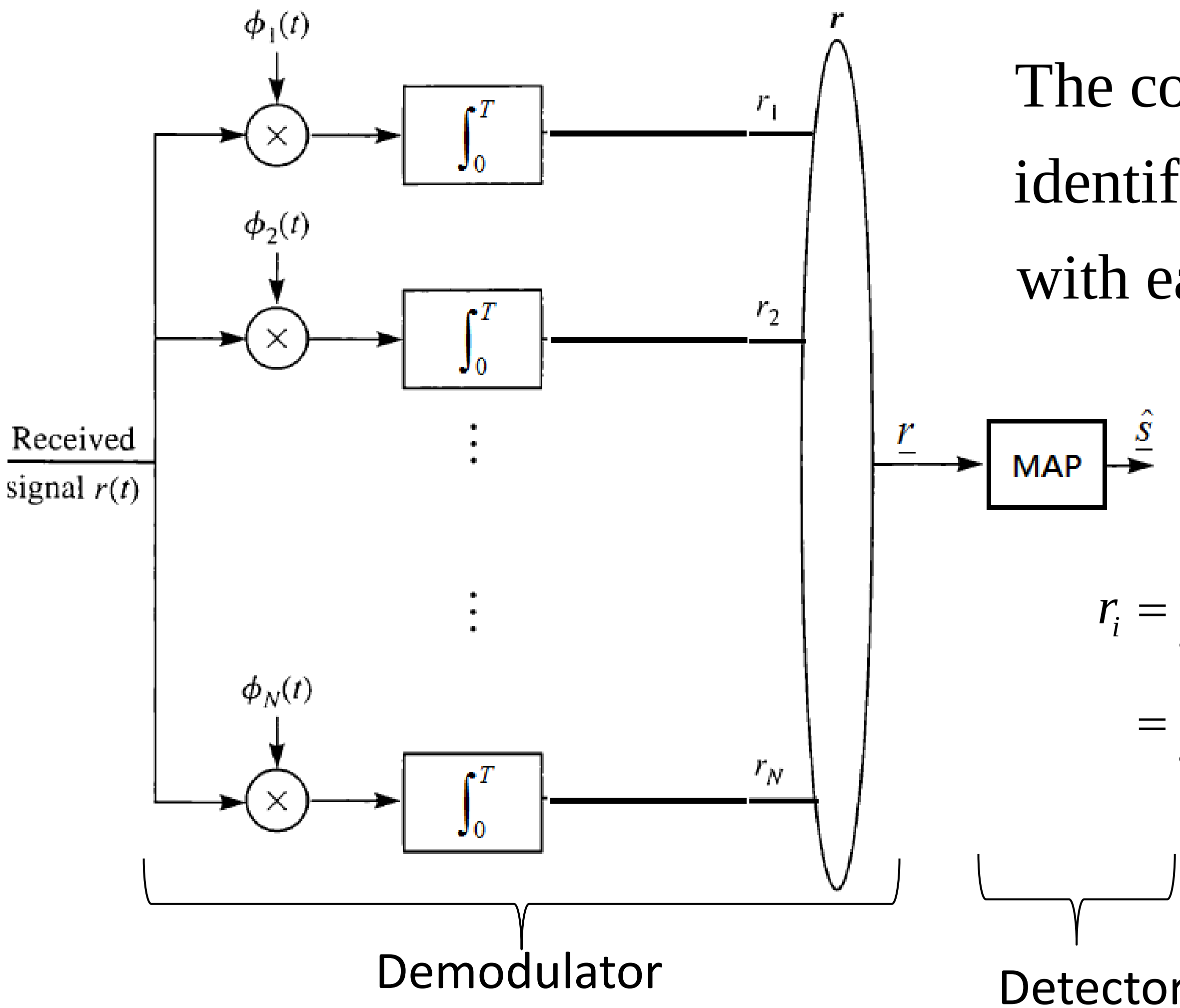
$$= \arg\max_{\underline{s}} \frac{P(\underline{s})P(r_1,\ldots,r_N, r_{N+1},\ldots,r_K \mid \underline{s})}{\sum_{\underline{s}} P(\underline{s})P(r_1,\ldots,r_N, r_{N+1},\ldots,r_K \mid \underline{s})} = \arg\max_{\underline{s}} \frac{P(\underline{s})P(r_1,\ldots,r_N \mid \underline{s})P(r_{N+1},\ldots,r_K)}{\sum_{\underline{s}} P(\underline{s})P(r_1,\ldots,r_N \mid \underline{s})P(r_{N+1},\ldots,r_K)}$$

$$= \arg\max_{\underline{s}} \frac{P(\underline{s})P(\underline{r} \mid \underline{s})}{P(\underline{r})} = \arg\max_{\underline{s}} P(\underline{s} \mid \underline{r})$$



4.2 Waveform and Vector AWGN Channel

- **Correlation demodulator**



The correlation demodulator identifies the correlation of $r(t)$ with each basis $\phi_k(t)$, $k = 1, .., N$.

$$r_i = \int_0^T r(t)\,\phi_i(t)\,\mathrm{d}t$$
$$= \int_0^T s(t)\,\phi_i(t)\,\mathrm{d}t + \int_0^T n(t)\,\phi_i(t)dt$$
$$i = 1, \ldots, N$$

✓ The receiver architecture generally contains two parts: demodulator, and detector.

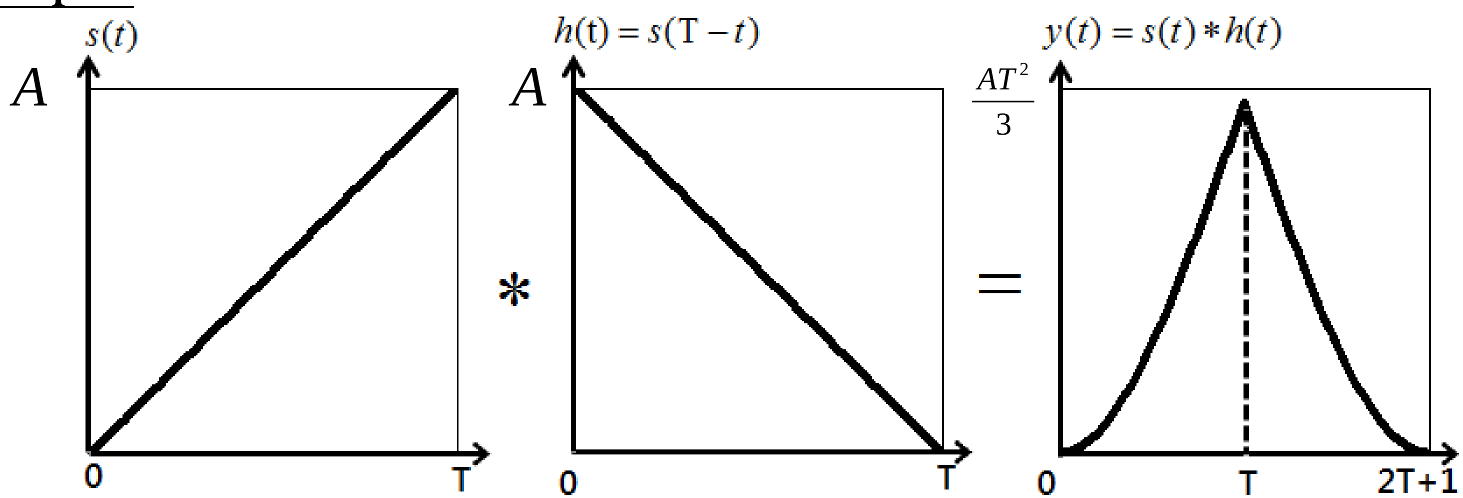
Matched Filter Demodulator

- **Matched Filter**

Definition

For a signal $s(t)$ which is zero outside $[0, \mathrm{T}]$, a filter with impulse response $h(t) = s(T - t)$ is a matched filter for $s(t)$.

Example

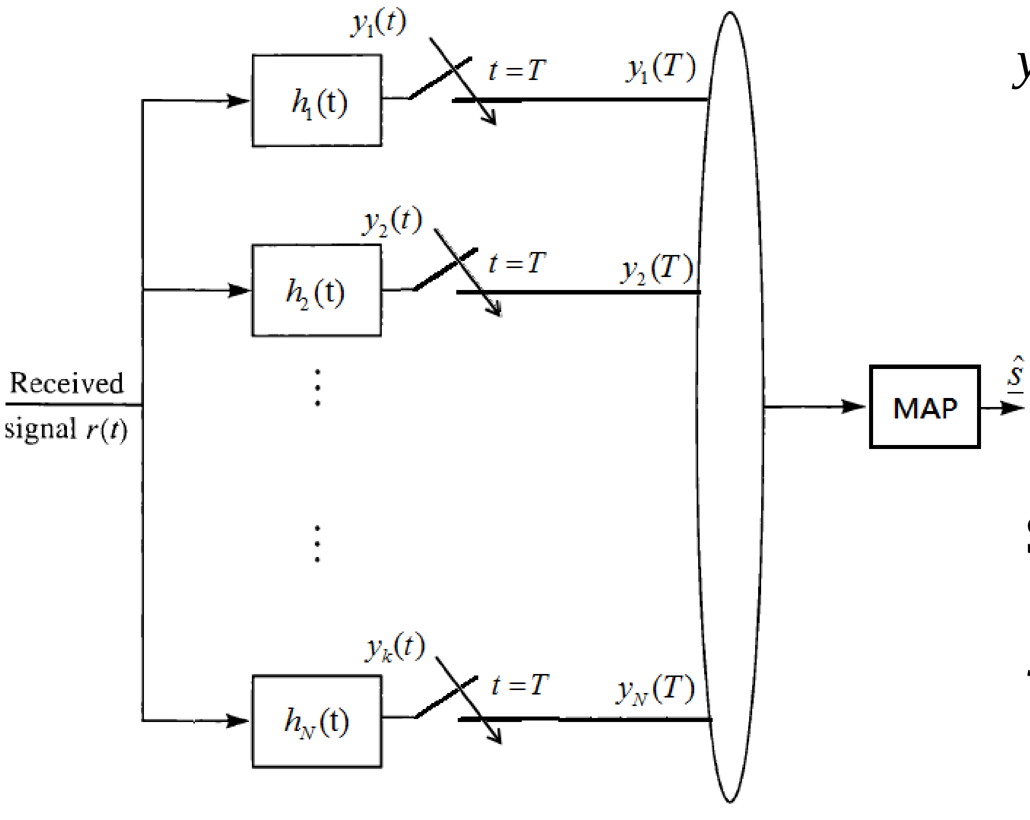


$$y(\mathrm{T}) = \int_0^T |s(\mathrm{t})|^2 dt = \frac{AT^2}{3}$$

The maximun occurs at t=T

Matched Filter Demodulator

For the matched filter demodulator, the N filters $h_k(t)$ are matched to the basis $\{\phi_k(t)\}$, *i.e.* $h_k(t) = \phi_k(\mathrm{T}\text{-}t)$.



$$y_k(t) = \int_0^t r(\tau)\, h_k(t-\tau)\, \mathrm{d}\tau$$

$$= \int_0^t r(\tau)\ \phi_k(\mathrm{T}\text{-}t+\tau)\, \mathrm{d}\tau$$

Q: Is the MF demodulator equivalent to correlation demodulator?

Sampling at $t = T$

$$y_k(T) = \int_0^T r(\tau)\ \phi_k(\tau)\, \mathrm{d}\tau$$

- The MF demodulator is equivalent to correlation demodulator.

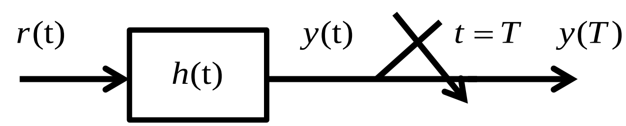
Matched Filter Demodulator

Theorem: The MF demodulator maximizes the SNR for demodulator output, i.e.

For $r(t) = s(t) + n(t)$ where $n(t) \sim \mathcal{N}(0, \frac{N_0}{2})$ and $s(t)$ is confined within $[0\ T]$,

then the SNR of $y(T)$ is maximized if $h(t)$ matched to $s(t)$.

<u>Proof</u>



$$y(\mathrm{t}) = \int_0^t r(\tau)\, h(\mathrm{t} - \tau)\, \mathrm{d}\tau$$

$$= \int_0^t s(\tau)\, h(\mathrm{t} - \tau)\, \mathrm{d}\tau + \int_0^t n(\tau)\, h(\mathrm{t} - \tau)\, \mathrm{d}\tau$$

Sampling at $t = T$

$$y(\mathrm{T}) = \underbrace{\int_0^T s(\tau)\, h(T - \tau)\, \mathrm{d}\tau}_{y_s(\mathrm{T})} + \underbrace{\int_0^T n(\tau)\, h(T - \tau)\, \mathrm{d}\tau}_{y_n(\mathrm{T})}$$

Proof (cont.)

$$\Rightarrow SNR = \frac{y_s^{\ 2}(\mathrm{T})}{E[y_n^{\ 2}(\mathrm{T})]}$$

$$E[y_n(\mathrm{T})]^2 = \int_0^T \int_0^T \underbrace{E[n(\tau_1)n(\tau_2)]}_{=\frac{N_0}{2}\delta(\tau_1-\tau_2)} h(T-\tau_1)h(T-\tau_2)\,\mathrm{d}\,\tau_1\,\mathrm{d}\,\tau_2$$

$$= \frac{N_0}{2}\int_0^T h^2(T-\tau)\,\mathrm{d}\,\tau$$

$$SNR = \frac{\left(\int_0^T s(\tau)\,h(T-\tau)\,\mathrm{d}\,\tau\right)^2}{\frac{N_0}{2}\int_0^T h^2(T-\tau)\,\mathrm{d}\,\tau} \le \frac{\left(\int_0^T s^2(\tau)\,\mathrm{d}\,\tau\right)\left(\int_0^T h^2(T-\tau)\,\mathrm{d}\,\tau\right)}{\frac{N_0}{2}\int_0^T h^2(T-\tau)\,\mathrm{d}\,\tau} = \frac{2E}{N_0}$$

Recall Cauch-Schwaz Inequality

$$\left(\int_0^T g_1(t)\,g_2(t)\,\mathrm{d}\,t\right)^2 \le \left(\int_0^T g_1^2(t)\,\mathrm{d}\,t\right)\left(\int_0^T g_2^2(t)\,\mathrm{d}\,t\right)$$

$$"=" \text{ holds when } g_1(t) = cg_2(t)$$

The "=" holds when

$$h(T-\tau) = cs(\tau) \Rightarrow h(t) = cs(T-t)$$

MAP and Maximum Likelihood (ML) Detector

The MAP $$\hat{\underline{s}} = \arg\max_{\underline{s} \in \{\underline{s}_1, \underline{s}_2, \ldots, \underline{s}_M\}} P(\underline{s} \mid \underline{r})$$

$$= \arg\max_{\underline{s}} \frac{P(\underline{s})P(\underline{r} \mid \underline{s})}{P(\underline{r})}$$

$$= \arg\max_{\underline{s}} P(\underline{s})P(\underline{r} \mid \underline{s})$$

If $P(\underline{s}_m) = \dfrac{1}{M} \quad \forall m$ (*i.e.* equal probability)

then $\hat{\underline{s}} = \arg\max_{\underline{s}} P(\underline{r} \mid \underline{s})$

This is called maximum likelihood detector (ML Detector)

$P(\underline{r} \mid \underline{s})$ is called the likelihood function.

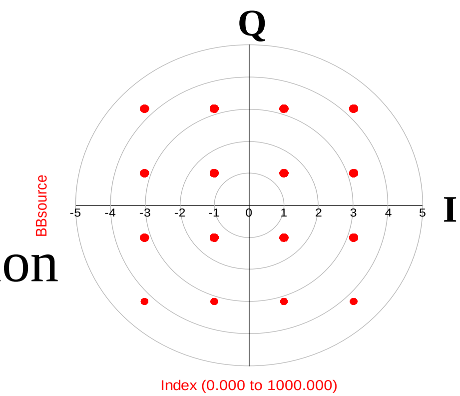
- In most cases, usually $P(\underline{s} \mid \underline{r})$ is not available, but $P(\underline{r} \mid \underline{s})$ is available.

MAP and Maximum Likelihood Detector

Example (AWGN channel case)

For $\underline{r} = \underline{s} + \underline{n}$ where $\underline{n} \sim \mathcal{N}(0, \frac{N_0}{2}\mathbf{I}_N)$ assume $\underline{s}_m$ is equally probable.

The MAP detector becomes ML detector.



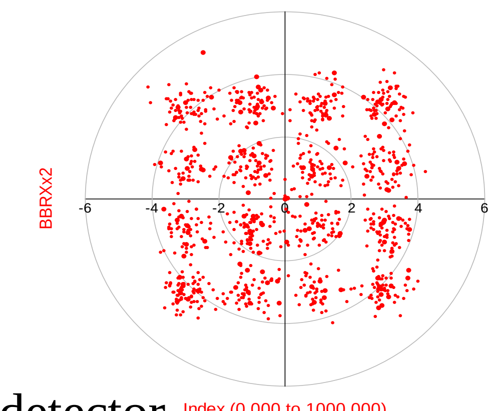
$$\hat{\underline{s}} = \underset{\underline{s}}{\operatorname{argmax}}\ p(\underline{r} \mid \underline{s})$$

$$= \underset{\underline{s}}{\operatorname{argmax}} (\frac{1}{\pi N_0})^{\frac{N}{2}} \exp(-\frac{1}{N_0}\|\underline{r} - \underline{s}\|^2) \rightarrow \text{likelihood function}$$

$$= \underset{\underline{s}}{\operatorname{argmax}}\ \ln(f(\underline{r} \mid \underline{s})) \rightarrow \text{log-likelihood function}$$

$$= \underset{\underline{s}}{\operatorname{argmax}} -\frac{1}{N_0}\|\underline{r} - \underline{s}\|^2$$

$$= \underset{\underline{s}}{\operatorname{argmin}} \|\underline{r} - \underline{s}\|^2 \quad \Rightarrow \text{Minimum distance(MD) detector}$$



If $\|\underline{s}_m\|^2 = E_s \;\; \forall m$ (e.g. MPSK)

$$\hat{\underline{s}} = \underset{\underline{s}}{\operatorname{argmin}} \|\underline{r}\|^2 - 2\underline{r} \cdot \underline{s} + \|\underline{s}\|^2 = \underset{\underline{s}}{\arg\max}\ \underline{r} \cdot \underline{s} \rightarrow \text{max correlation detector}$$

✓ With different conditions, the detector can be reformulated.

4.3 Optimal detection and error probability for band limited and power limited signals

- Binary PAM(Binary Antipodal Signaling)

$$s_1(t) = Ag(t)\cos(2\pi f_c t) \qquad 0 \le t \le T_b$$

$$s_2(t) = -Ag(t)\cos(2\pi f_c t) \qquad 0 \le t \le T_b$$

1-D signal with basis with $\phi(t) = \sqrt{\frac{2}{E_g}} g(t)\cos(2\pi f_c t)$

$$s_1 = \int_0^{T_b} s_1(t)\phi(t)dt = A\sqrt{\frac{E_g}{2}} = \sqrt{E_b} \qquad E_b = \|s_1(\mathrm{t})\|^2 = \|s_2(\mathrm{t})\|^2$$

$$s_2 = \int_0^{T_b} s_2(t)\phi(t)dt = -A\sqrt{\frac{E_g}{2}} = -\sqrt{E_b}$$

The received signal $r(t) = s(t) + n(t)$

After the decorrelator $r = s + n$

$$where \qquad s = \begin{cases} \sqrt{E_b} & if\ s_1 \text{ transmitted} \\ -\sqrt{E_b} & if\ s_2 \text{ transmitted} \end{cases} \qquad n \sim \mathcal{N}(0, \frac{N_0}{2})$$

- MAP detector

$$\underline{\hat{s}} = \underset{s\in\{s_1,..,s_M\}}{\operatorname{argmax}}\, p(s \mid r)$$

Q: How to find $p(s/r)$?

$$\underline{\hat{s}} = \underset{s}{\operatorname{argmax}}\, p(\mathrm{s})\, p(\mathrm{r} \mid \mathrm{s})$$

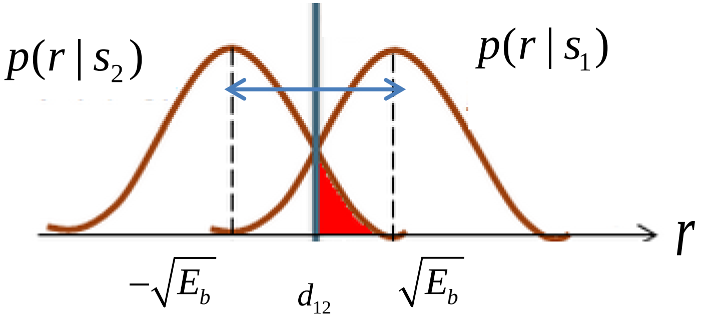
$\rightarrow$ Given $p(s)$ and probability model of noise, $p(r \mid s)$ can be obtained based on received r.

Assume equal probability of s,

$$\underline{\hat{s}} = \underset{s}{\operatorname{argmax}}\, p(\mathrm{r} \mid \mathrm{s})$$

$$= \underset{s}{\operatorname{argmin}} \left| r - s \right|^2$$

$$= \underset{s}{\operatorname{argmax}}\ \ r \cdot s$$



$$D_1 = \{r : r\sqrt{E_b} > -r\sqrt{E_b}\}$$

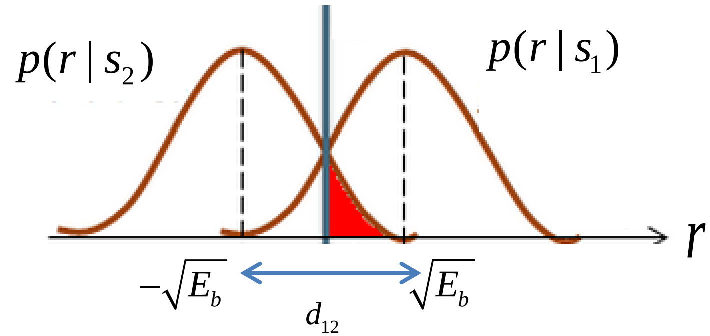
$$D_2 = \{r : -r\sqrt{E_b} > r\sqrt{E_b}\}$$

$$p(r \mid s_i) = \frac{1}{\sqrt{\pi N_0}} \exp\!\left(-\frac{1}{N_0}(r - s_i)^2\right), i = 1, 2$$

$\rightarrow$ Hence $p(s/r)$ can be obtained.

Error probability:

$$P(\text{error} \mid s_2) = P(\hat{s} = s_1 \mid s_2) = P(r \in D_1 \mid s_2)$$



$$= \int_0^{\infty} \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{(r+\sqrt{E_b})^2}{N_0}\right) dr$$

$$= \int_{\sqrt{E_b}}^{\infty} \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{x^2}{N_0}\right) dx$$

$$p(r \mid s) = \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{1}{N_0}(r-s)^2\right)$$

$$= Q\left(\frac{\sqrt{E_b}}{\sqrt{N_0/2}}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

where $Q\left(\frac{x}{\sigma_x}\right) = \int_x^{\infty} \frac{1}{\sqrt{2\pi\sigma_x}} \exp\left(\frac{-t^2}{2\sigma_x^2}\right) dt$

- Overall P_e and Point to point distance d_{12}

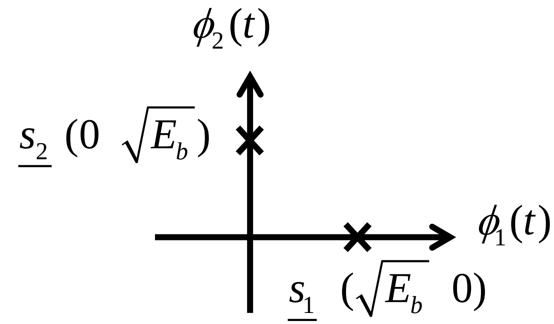
$$P_e = P(s_1)P(error \mid s_1) + P(s_2)P(error \mid s_2)$$

$$= \frac{1}{2} Q\left(\sqrt{\frac{2E_b}{N_0}}\right) + \frac{1}{2} Q\left(\sqrt{\frac{2E_b}{N_0}}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) = Q\left(\frac{d_{12}}{\sqrt{2N_0}}\right)$$

$$= Q\left(\frac{d_{12}/2}{\sqrt{N_0/2}}\right) \qquad = Q\left(\frac{d_{12}/2}{\sigma_n}\right)$$

As $d_{12} \uparrow$, $P_e \downarrow$

- **Binary Orthogonal signal**



Two dimension signal

$$\underline{s}_1 = [\sqrt{E_b}\ \ 0]^T \quad \underline{s}_2 = [0\ \ \sqrt{E_b}\,]^T$$

with the basis $\phi_1(t)$ and $\phi_2(t)$

$$\underline{r} = \underline{s} + \underline{n} = \begin{bmatrix} \sqrt{E_b} + n_1 \\ n_2 \end{bmatrix} \text{or} \begin{bmatrix} n_1 \\ \sqrt{E_b} + n_2 \end{bmatrix}$$

MAP detector $\hat{\underline{s}} = \underset{\underline{s}}{\text{argmax}}\ p(\underline{s} \mid \underline{r})$

$$= \underset{\underline{s}}{\text{argmax}}\ p(\underline{r} \mid \underline{s}) \quad \left(\because \text{ Equallly probable source symbols}\right)$$

$$= \underset{\underline{s}}{\text{argmin}} \left|\underline{r} - \underline{s}\right|^2 \quad \left(\because \text{ AWGN channel}\right)$$

$$= \underset{\underline{s}}{\text{argmax}}\ \ \underline{r} \cdot \underline{s} \quad \left(\because \text{ Equallly energy source symbols}\right)$$

$$D_1 = \{\underline{r} : \underline{r} \cdot \underline{s}_1 > \underline{r} \cdot \underline{s}_2\}$$

$$D_2 = \{\underline{r} : \underline{r} \cdot \underline{s}_1 < \underline{r} \cdot \underline{s}_2\}$$

Error probability: For the case that $\underline{s}_2$ was sent,

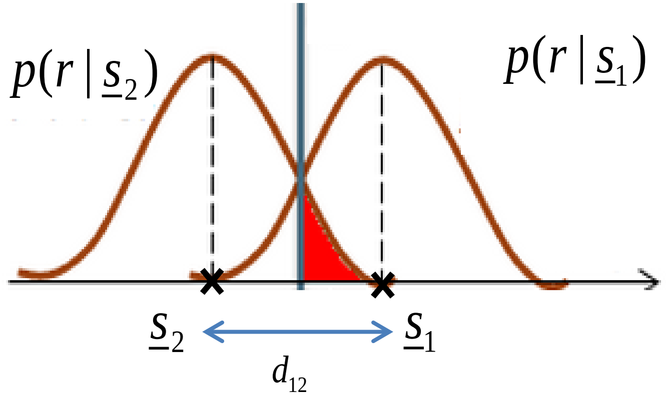
$$P(\text{error} \mid \underline{s}_2) = P(\underline{r} \mid \underline{s}_2) < P(\underline{r} \mid \underline{s}_1)$$

$$= P(\underline{r} \cdot \underline{s}_2 < \underline{r} \cdot \underline{s}_1 \mid \underline{s}_2)$$

$$= P(E_b + \sqrt{E_b} n_2 < \sqrt{E_b} n_1)$$

$$= P(n_1 - n_2 > \sqrt{E_b})$$

$$= Q(\sqrt{\frac{E_b}{\mathrm{N}_0}}) \qquad (\because \mathrm{n}_1 - \mathrm{n}_2 \sim \mathcal{N}(0, N_0))$$



Point to point distance d_{12} and P_e

$$P_e = P(s_1)P(error \mid s_1) + P(s_2)P(error \mid s_2)$$

$$= \frac{1}{2} Q(\sqrt{\frac{E_b}{N_0}}) + \frac{1}{2} Q(\sqrt{\frac{E_b}{N_0}})$$

$$= Q(\sqrt{\frac{E_b}{N_0}}) = Q(\frac{d_{12}}{\sqrt{2\mathrm{N}_0}}) \boxed{= Q(\frac{d_{12}/2}{\sqrt{N_0/2}})}$$

As $d_{12} \uparrow$, $P_e \downarrow$

Probability Error of M-ary Orthogonal Signals

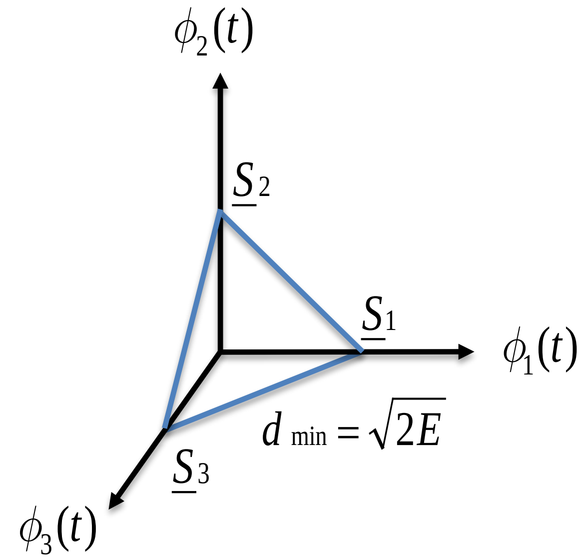
Transmitted symbols

$$\underline{s}_1 = [\sqrt{E_s} \quad 0 \quad \dots \quad 0 \quad 0]^T$$

$$\underline{s}_2 = [0 \quad \sqrt{E_s} \quad 0 \quad \dots \quad 0]^T$$

$$\vdots \qquad \qquad \ddots$$

$$\underline{s}_M = [0 \quad 0 \quad \dots \quad 0 \quad \sqrt{E_s}\,]^T$$



Suppose $\underline{s}_1$ was transmitted

$$\underline{r} = \underline{s}_1 + \underline{n} = \begin{bmatrix} \sqrt{E_s} + n_1 \\ n_2 \\ \vdots \\ n_M \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_M \end{bmatrix}$$

$$p(r_1 \mid \underline{s}_1) = \frac{1}{\sqrt{\pi N_0}} \exp(-\frac{1}{N_0}(r_1 - \sqrt{E_s}\,)^2)$$

$$p(r_k \mid \underline{s}_1) = \frac{1}{\sqrt{\pi N_0}} \exp(-\frac{r_k^{\ 2}}{N_0}),$$

$$k = 2, \dots, M$$

MAP detector

$$\hat{\underline{s}} = \underset{\underline{s}_k}{\operatorname{argmax}} \;\; \underline{r} \cdot \underline{s}_k \qquad \Rightarrow \underline{r} \cdot \underline{s}_1 = r_1 \cdot \sqrt{E_s} = E_s + \sqrt{E_s}\, n_1$$

$$\underline{r} \cdot \underline{s}_k = r_k \cdot \sqrt{E_s} = \sqrt{E_s}\, n_k, \quad k = 2, \ldots, M$$

Error probability for $\underline{s}_1$

$$P(\text{error} \mid \underline{s}_1) = 1 - P(\text{correct} \mid \underline{s}_1)$$

$$P(\text{correct} \mid \underline{s}_1) = P(\underline{r} \cdot \underline{s}_1 > \underline{r} \cdot \underline{s}_k \quad \forall\, \mathrm{k} \neq 1)$$

$$= P(r_1 > n_2, r_1 > n_3, \ldots, r_1 > n_M)$$

$$= \int_{-\infty}^{\infty} P(r_1 > n_2, r_1 > n_3, \ldots, r_1 > n_M \mid r_1)\, p(r_1)\, \mathrm{d}\, r_1$$

Given r_1

$$P(r_1 > n_2, r_1 > n_3, \ldots, r_1 > n_M \mid r_1) = \prod_{k=2}^{M} P(r_1 > n_k \mid r_1)$$

where

$$P(r_1 > n_k \mid r_1) = \int_{-\infty}^{r_1} \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{{n_k}^2}{N_0}\right) dn_k = 1 - Q\left(\frac{r_1}{\sqrt{N_0 / 2}}\right)$$

Therefore, the correct probability

$$P(\text{correct} \mid \underline{s}_1) = \int_{-\infty}^{\infty} P(\, r_1 > n_2, r_1 > n_3 \,, \dots , r_1 > n_M \mid r_1) p(r_1)\, \mathrm{d}\, r_1$$

$$= \int_{-\infty}^{\infty} [1 - Q(\frac{r_1}{\sqrt{N_0 / 2}})]^{M-1} \frac{1}{\sqrt{\pi N_0}} \exp(-\frac{1}{N_0}(r_1 - \sqrt{E_s})^2)\, \mathrm{d}\, r_1$$

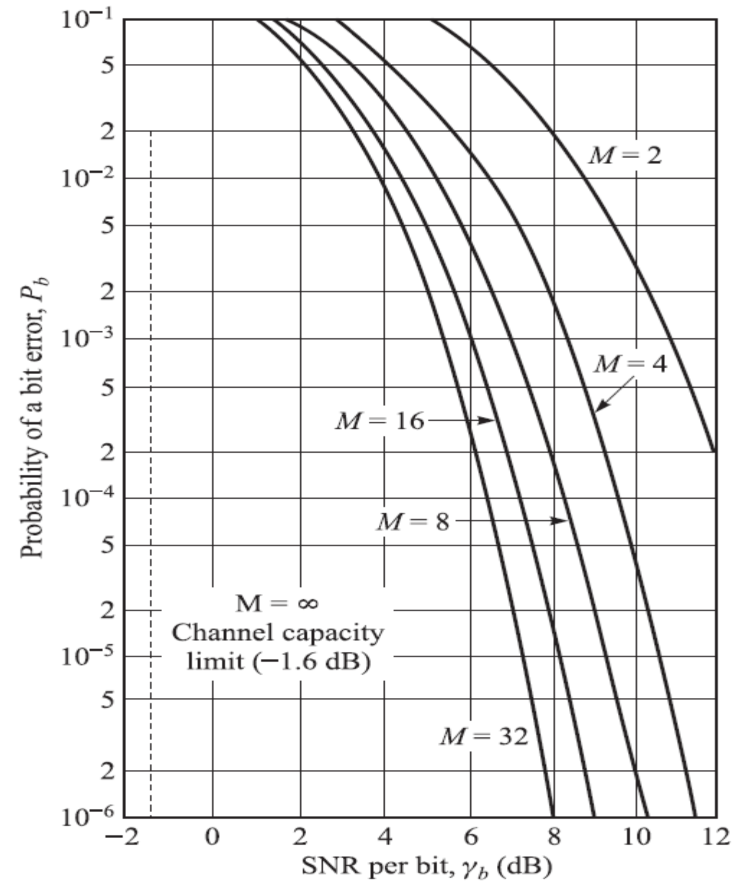
The error probability,

$$P(\text{error} \mid \underline{s}_1) = 1 - \int_{-\infty}^{\infty} [1 - Q(\frac{r_1}{\sqrt{N_0 / 2}})]^{M-1} p(r_1)\, \mathrm{d}\, r_1$$

The overall error probability,

$$P_e = \sum_{i=1}^{M} P(\underline{s}_i) P(\text{error} \mid \underline{s}_i) = P(\text{error} \mid \underline{s}_1)$$

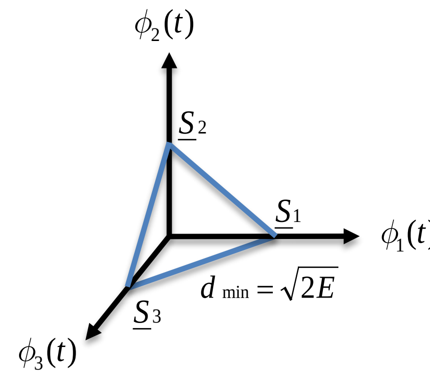
$M \uparrow \quad R_b \uparrow \quad P_e \uparrow \quad P_b \downarrow$



Q: What has to pay? A: More basis or BW.

Bit error rate of M-ary orthogonal Signals

- To compare different modulation schemes with different M values , it is more fair to convert symbol error P_e to bit error rate(BER).
- In M-ary orthogonal signals, a symbol $\underline{\mathbf{s}}_m$ has equal probability of having error to any other symbol, $\underline{\mathbf{s}}_l$.



$$P(\hat{\underline{s}} = \underline{s}_\ell \mid \underline{s} = \underline{s}_m) = \begin{cases} 1 - P_e & if\ \ell = m \\ \dfrac{P_e}{M-1} & f\ \ell \neq m \end{cases}$$

Suppose that symbol $\underline{s}_m$ represents K bits message $\mathrm{b}_{m1}\ \mathrm{b}_{m2} \ldots \mathrm{b}_{mK}$

and symbol $\underline{s}_l$ also represents K bits message $\mathrm{b}_{l1}\ \mathrm{b}_{l2} \ldots \mathrm{b}_{lK}$, $K = \log_2^M$.

➢ For a single bit error, there are 2^{K-1} combination of cases for the same symbol error.

Bit error rate (P_b) of M-ary orthogonal Signals

- Example (M=8 K=3)

Assume $\underline{s}_1$ *is sent*

$$\underline{s}_1 = 0 \quad 0 \quad 0 \quad \} \; P_c = 1 - P_e(M)$$

$$\left.\begin{aligned} \underline{s}_2 &= 0 \quad 0 \quad 1 \\ \underline{s}_3 &= 0 \quad 1 \quad 0 \\ \underline{s}_4 &= 0 \quad 1 \quad 1 \\ \underline{s}_5 &= 1 \quad 0 \quad 0 \\ \underline{s}_6 &= 1 \quad 0 \quad 1 \\ \underline{s}_7 &= 1 \quad 1 \quad 0 \\ \underline{s}_8 &= 1 \quad 1 \quad 1 \end{aligned}\right\} P_e = P(\hat{\underline{s}} \neq \underline{s}_1)$$

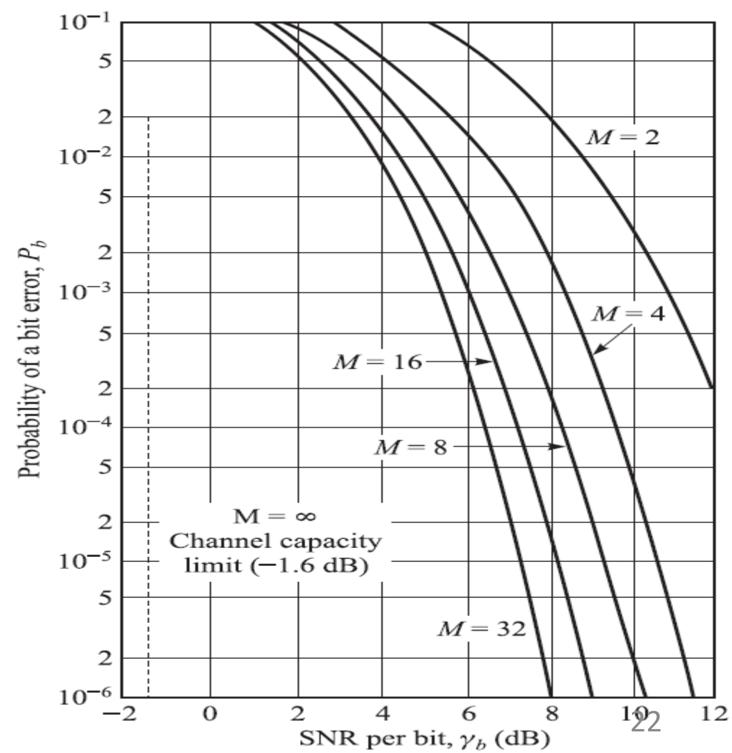
$$\Rightarrow P_b = \frac{4}{7} P_e$$

M	BER (P_b)
2	$P_b = P_e$ (M=2)
4	$P_b = \frac{2}{3} P_e$ (M=4)
8	$P_b = \frac{4}{7} P_e$ (M=8)
16	$P_b = \frac{8}{15} P_e$ (M=16)

- In general, given M, $k = \log_2 M$)

$$P_b = \frac{\frac{M}{2}}{M-1} P_e = \frac{2^{k-1}}{2^k - 1} P_e$$

$$M \uparrow \quad P_b \downarrow \qquad as\ M \to \infty \quad P_b \to \frac{1}{2} P_e$$



Union Bound of Probability of Error for M-ary Orthogonal Signals

- Union rule: Suppose we have event $E_1, E_2, \ldots, E_M$, then $P(\bigcup_{i=1}^{M} E_i) \le \sum_{i=1}^{M} P(E_i)$.
- Applying the union rule to P_e

$$P_e = \sum_{i=1}^{M} P(\underline{s}_m) P(\bigcup_{\ell \ne m} \{\underline{r} \cdot \underline{s}_m < \underline{r} \cdot \underline{s}_\ell \mid \underline{s}_m\})$$

$$= P(\bigcup_{\ell \ne 1} \{\underline{r} \cdot \underline{s}_1 < \underline{r} \cdot \underline{s}_\ell \mid \underline{s}_1\})$$

$$\le \sum_{\ell=2}^{M} P(\underline{r} \cdot \underline{s}_1 < \underline{r} \cdot \underline{s}_\ell \mid \underline{s}_1)$$

$$= (\mathrm{M}-1) P(\underline{r} \cdot \underline{s}_1 < \underline{r} \cdot \underline{s}_2 \mid \underline{s}_1)$$

$$= (\mathrm{M}-1) P(E_s + \sqrt{E_s}\,\mathrm{n}_1 < \sqrt{E_s}\,\mathrm{n}_2)$$

$$= (\mathrm{M}-1) P(\mathrm{n}_2 - \mathrm{n}_1 > \sqrt{E_s})$$

$$= (\mathrm{M}-1) Q(\frac{\sqrt{E_s}}{\sqrt{N_0}})$$

As M $\uparrow$, $P_e \uparrow$ *or* $\downarrow$?

$\rightarrow$ Depends on whether E_s is fixed.

$$\le \frac{(\mathrm{M}-1)}{2} \exp(\frac{-E_s}{2N_0})$$

$$= \frac{1}{2}(2^K - 1) \exp(\frac{-KE_b}{2N_0})$$

$$= \frac{1}{2}(\mathrm{e}^{K \ln 2} - 1) \exp(\frac{-KE_b}{2N_0}) \le \boxed{\frac{1}{2} e^{K(\ln 2 - \frac{E_b}{2N_0})}}$$

What does it suggest?

For $\frac{E_b}{2N_0} > \ln(2)$, i.e $\frac{E_b}{N_0} > 2\ln(2) = 1.42$

as $K \to \infty$, $P_e \to 0$

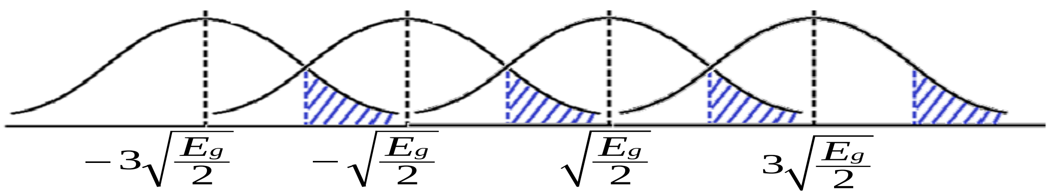
$\Rightarrow$ Reliable communication requirement.

Probability of Error for M-ary PAM

One dimension signal with

$$s_m = \sqrt{\frac{E_g}{2}} A_m \quad m = 1, \ldots, M$$

$$A_m = (2m - 1 - M), m = 1, 2, \ldots, M = \{\pm 1, \pm 3, \ldots, \pm (M-1)\}$$



$$d_{\min} = \sqrt{2E_g}$$

MAP detector

$$\underline{\hat{s}} = \underset{s_m}{\operatorname{argmax}}\ p(s_m \mid r) = \underset{s_m}{\arg\min}\ |r - s_m|$$

$$D_m = \{\underline{r} : |\mathrm{r} - s_m| < |\mathrm{r} - s_\ell| \quad \forall \ell \neq m\}$$

For inner points, m = 2,…, M-1

$$P(\text{error} \mid s_m) = P(|\mathrm{r} - s_m| > \frac{d_{\min}}{2}) = 2\,\mathrm{Q}\left(\frac{\frac{d_{\min}}{2}}{\sqrt{\frac{N_0}{2}}}\right) = 2\,\mathrm{Q}\left(\frac{d_{\min}}{\sqrt{2N_0}}\right)$$

For outer points, m = 1, M

$$P(\text{error} \mid s_1) = P(\text{error} \mid s_M) = \mathrm{Q}\left(\frac{d_{\min}}{\sqrt{2N_0}}\right)$$

- Error probability (M-PAM)

$$P_e=\sum_{m=1}^{M}P(s_m)P(error\mid s_m)=\frac{1}{M}\sum_{m=2}^{M-1}2Q(\frac{d_{\min}}{\sqrt{2\mathrm{N}_0}})+\frac{2}{M}Q(\frac{d_{\min}}{\sqrt{2\mathrm{N}_0}})=\frac{2(\mathrm{M}-1)}{M}Q(\frac{d_{\min}}{\sqrt{2\mathrm{N}_0}})$$

With average symbol energy

$$E_{av}=\frac{1}{M}\sum_{m=1}^{M}E_m=\frac{d_{\min}}{4M}\sum_{m=1}^{M}(2m-1-M)^2$$

$$\boxed{d_{\min}=\sqrt{2E_g}}$$

$$=\frac{d_{\min}{}^2}{4M}\sum_{m=1}^{M}[4m^2+(M+1)^2-4m(M+1)]=\frac{d_{\min}{}^2}{12}(M^2-1)=\frac{E_g}{6}(M^2-1)$$

$$\Rightarrow\ \text{For M-PAM, } d_{\min}=\sqrt{\frac{12E_{av}}{M^2-1}}=\sqrt{\frac{12KE_b}{M^2-1}},\text{ where } E_{av}=KE_b$$

$$\Rightarrow P_e=\frac{2(\mathrm{M}-1)}{M}Q(\frac{d_{\min}}{\sqrt{2\mathrm{N}_0}})=\frac{2(\mathrm{M}-1)}{M}Q(\sqrt{\frac{6KE_b}{(\mathrm{M}^2-1)N_0}})\approx 2Q(\sqrt{\frac{6(\log_2\mathrm{M})E_b}{(\mathrm{M}^2-1)N_0}})\quad M\uparrow\ P_e\uparrow$$

Q: If double the size, $M'=2M$, how much more E_{b} is required to maintain the same P_{e}?

$$\Rightarrow P_e'=2Q(\sqrt{\frac{6\log_2^{M'}E_b'}{((M')^2-1)N_0}})\approx 2Q(\sqrt{\frac{6\log_2^{M}E_b'}{4(\mathrm{M}^2-1)N_0}})\ \Rightarrow E_b'\approx 4E_b$$

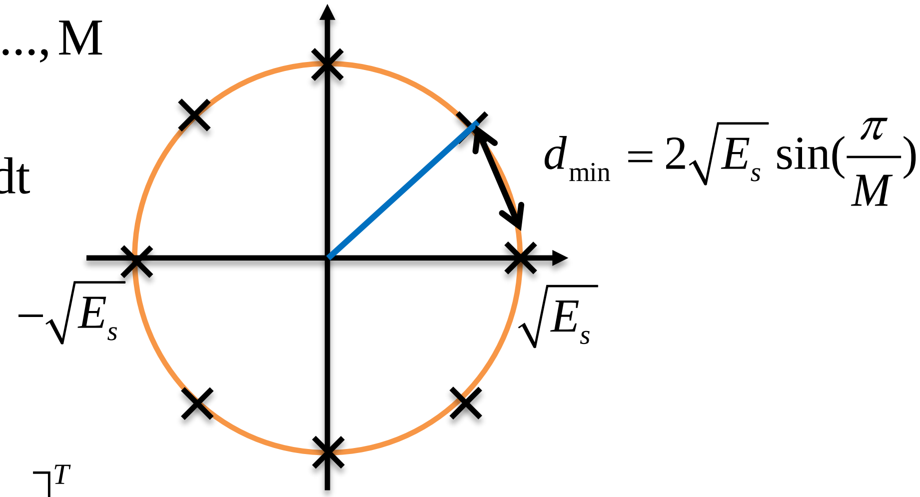
Probability of Error for M-ary PSK

$$s_m(t) = g(t)\cos(2\pi f_c t + \frac{2\pi}{M}(m-1)) \qquad m = 1,\ldots,\mathrm{M}$$

$$\mathrm{E}_s = \|s_m(\mathrm{t})\|^2 = \frac{1}{2}\mathrm{E}_g \qquad where \;\; \mathrm{E}_g = \int_0^T g^2(\mathrm{t})\,\mathrm{dt}$$

$$\phi_1(t) = \sqrt{\tfrac{2}{E_g}}\, g(t)\cos(2\pi f_c t)$$

$$\phi_2(t) = -\sqrt{\tfrac{2}{E_g}}\, g(t)\sin(2\pi f_c t)$$



$$\underline{s}_m = \left[\sqrt{\tfrac{E_g}{2}}\cos(\frac{2\pi}{M}(m-1)), \sqrt{\tfrac{E_g}{2}}\sin(\frac{2\pi}{M}(m-1))\right]^T$$

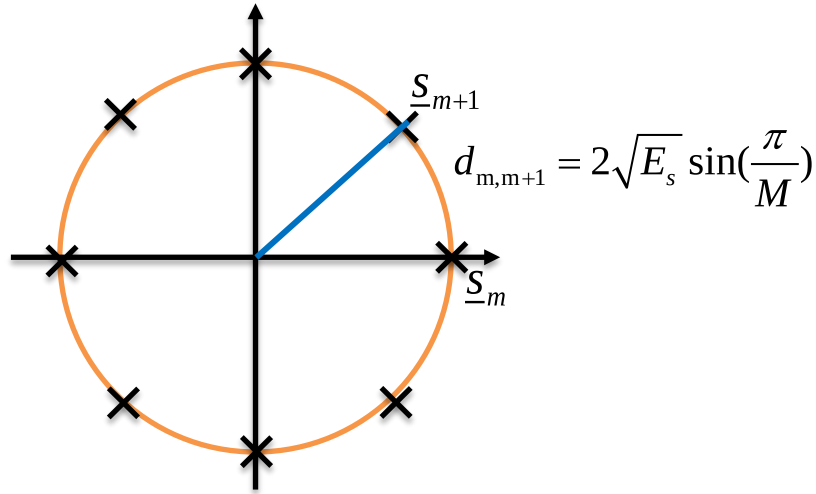
$$= \left[\sqrt{E_s}\cos(\frac{2\pi}{M}(m-1)), \sqrt{E_s}\sin(\frac{2\pi}{M}(m-1))\right]^T, m = 1, 2, \ldots, M$$

MAP detector (assume AWGN and equal prob. of symbol occurrence)

$$\hat{\underline{s}} = \underset{\underline{s}}{\operatorname{argmax}}\; p(\underline{s}\,|\,\underline{r}) = \underset{\underline{s}}{\operatorname{argmax}}\; \underline{r}\cdot\underline{s}$$

Choose $\hat{\underline{s}} = \underline{s}_m$ where $\underline{r}$ has the largest projection onto $\underline{s}_m$

$$P_e = \sum_{i=1}^{M} P(\underline{s}_m) P\Big(\bigcup_{\ell \neq m} \{\underline{r}\cdot\underline{s}_m < \underline{r}\cdot\underline{s}_\ell \mid \underline{s}_m\}\Big)$$
$$= P\Big(\bigcup_{\ell \neq m} \{\underline{r}\cdot\underline{s}_m < \underline{r}\cdot\underline{s}_\ell \mid \underline{s}_m\}\Big)$$
$$\leq \sum_{\ell=2}^{M} P(\underline{r}\cdot\underline{s}_m < \underline{r}\cdot\underline{s}_\ell \mid \underline{s}_m)$$
$$\cong P_e(\underline{s}_m, \underline{s}_{m+1}) + P_e(\underline{s}_m, \underline{s}_{m-1})$$



$P_e(\underline{s}_m, \underline{s}_\ell)$ represents the pairwise error probability when $\underline{s}_m$ is sent and $\underline{s}_\ell$ is detected.

$$P_e(\underline{s}_m, \underline{s}_{m+1}) = \mathrm{P}(\mathcal{N}(0, \frac{N_0}{2}) > \frac{d_{\mathrm{m},m+1}}{2}) = \mathrm{Q}(\frac{d_{\min}}{\sqrt{2N_0}})$$

$$P_e(\underline{s}_m, \underline{s}_{m-1}) = \mathrm{P}(\mathcal{N}(0, \frac{N_0}{2}) > \frac{d_{\mathrm{m},m-1}}{2}) = \mathrm{Q}(\frac{d_{\min}}{\sqrt{2N_0}})$$

$\because$ In *MPSK* $d_{\mathrm{m,m+1}} = 2\sqrt{E_s}\,\sin(\frac{\pi}{M})$

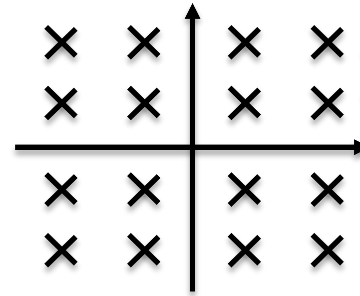
$$\therefore P_e \cong 2\,\mathrm{Q}\Big(\frac{2\sqrt{E_s}\,\sin(\frac{\pi}{M})}{\sqrt{2N_0}}\Big) = 2\,\mathrm{Q}\Big(\sqrt{\frac{2E_s}{N_0}}\sin(\frac{\pi}{M})\Big)$$

$M \uparrow \quad R_b \uparrow \quad P_e \uparrow$

Q: If double the size, $M' = 2M$, how much more $\frac{E_\mathrm{b}}{N_0}$ is required to maintain the same P_e?

For large M, $\sin(\frac{\pi}{M}) \approx \frac{\pi}{M} \Rightarrow P_e' = 2\,\mathrm{Q}\Big(\frac{2\sqrt{\log_2^{M'} E_b'}\,\frac{\pi}{M'}}{\sqrt{2N_0}}\Big) \approx 2\,\mathrm{Q}\Big(\frac{\sqrt{\log_2^{M} E_b'}\,\frac{\pi}{M}}{\sqrt{2N_0}}\Big) \Rightarrow \frac{E_b'}{N_0} = 4(\frac{E_b}{N_0})$

Detection of of M-ary QAM



$$s_m(t) = A_{mI}\, g(t)\cos(2\pi f_c t) - A_{mQ}\, g(t)\sin(2\pi f_c t)$$

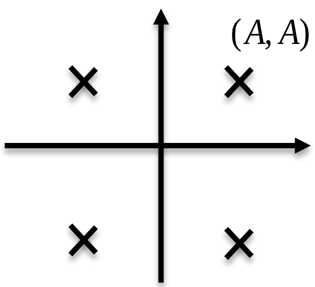
$$\underline{s}_m = \left[s_{mI} \;\; s_{mQ} \right]^T = \left[\sqrt{\frac{E_g}{2}} A_{mI} \;\; \sqrt{\frac{E_g}{2}} A_{mQ} \right]^T , \quad m = 1, 2, \ldots, M \quad A_{mI}, A_{mQ} \in \{\pm 1, \pm 3, .., \pm(M-1)\}$$

$$E_s = \frac{E_g}{3}(M-1)$$

MAP detector (assume AWGN and equal prob. of symbol occurrence)

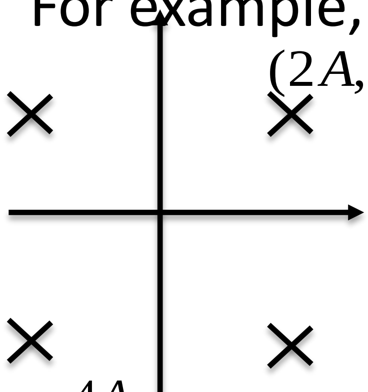
$$\hat{\underline{s}} = \arg\min_{\underline{s}} \left\| \underline{r} - \underline{s} \right\|^2$$

To determine the error performance, we first specify the constellation. The error probability is usually dominated by the minimum distance of constellation. For example, consider the following 4-QAM.



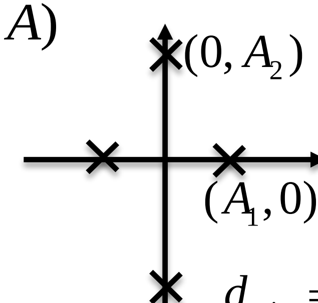
$d_{\min} = 2A$

$E_s = 2A^2$



$d_{\min} = 4A$

$E_s = 8A^2$

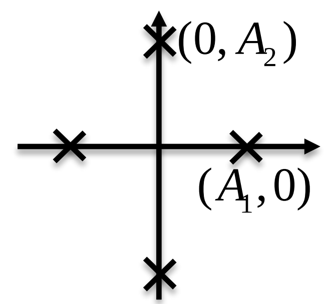


$A_1 = A$

$A_2 = \sqrt{3}A$

$d_{\min} = 2A$

$$E_s = \frac{A^2 + A^2 + 3A^2 + 3A^2}{4} = 2A^2$$



$A_1 = A_2 = A$

$d_{\min} = \sqrt{2}A$

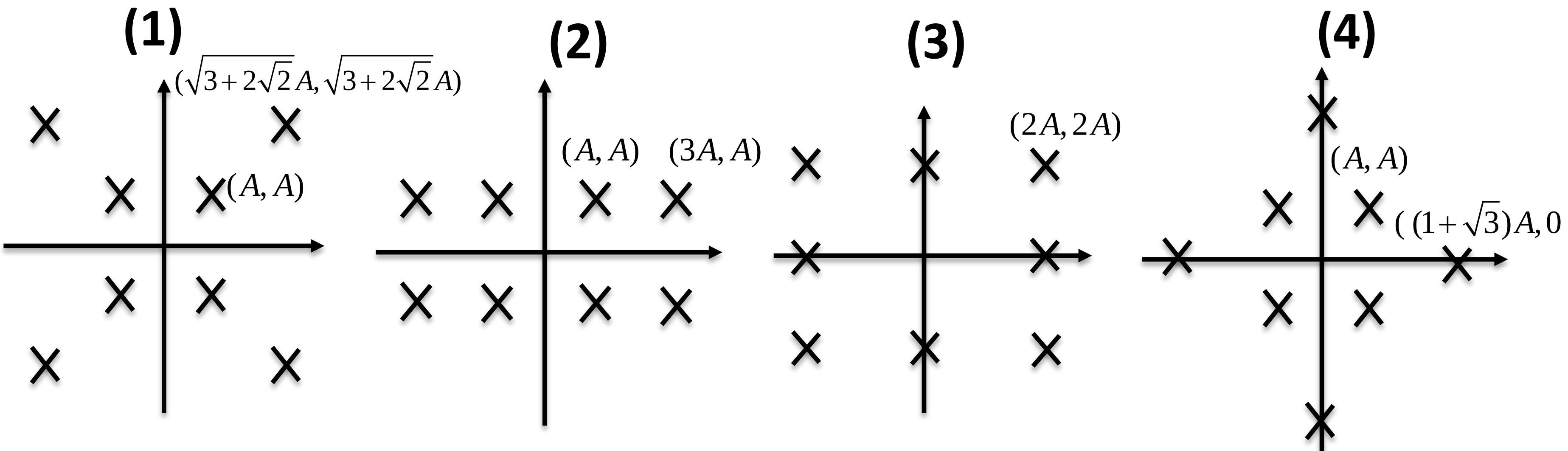
$E_s = A^2$

Q: What is the optimal distribution of the point on the signal space that yields the best performance in P_e and energy efficiency?

Constellation Figure of Merit (CFM)

$$CFM = \frac{d_{\min}^2}{E_{av}}$$ -> Yields quick evaluation among constellations.

Example (M=8 8-QAM)



(1) $d_{\min} = 2A$, $\mathrm{E}_{av1} = 6.83A^2$

(2) $d_{\min} = 2A$, $\mathrm{E}_{av2} = 6A^2$

(3) $d_{\min} = 2A$, $\mathrm{E}_{av3} = 6A^2$

(4) $d_{\min} = 2A$, $\mathrm{E}_{av4} = 4.73A^2$

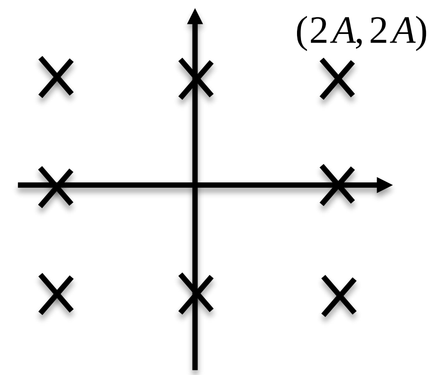
✓ The constellation (4) is known to be the best 8-QAM constellation in terms of CFM.

Q: How do you compare the 8-QAM in (2) and (3)?

✓ In general, the optimal constellation for M-QAM is difficult to proof.

Error Probability of M-ary QAM

- In general, the error probability for M-ary QAM is



$$P_e = \sum_{m=1}^{M} P(s_m)P(error \mid s_m)$$

$$P(error \mid s_m) = P(s_m \rightarrow s_n \mid s_m) \quad \forall n \neq m$$

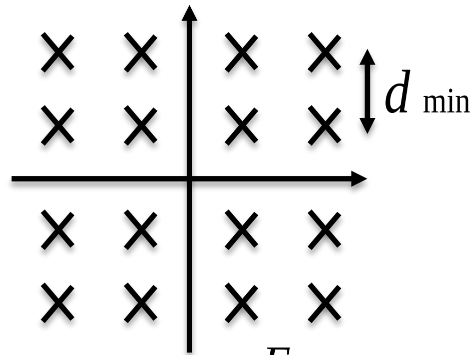
$$= \sum_{\substack{1 \le n \le M \\ n \neq m}} \mathrm{Q}\left(\frac{\frac{d_{mn}}{2}}{\sqrt{\frac{N_0}{2}}} \right) \le (\mathrm{M}-1)\mathrm{Q}\left(\frac{d_{min}}{\sqrt{2N_0}} \right)$$

$$P_e = \sum_{m=1}^{M} P(s_m)P(error \mid s_m) \le \sum_{m=1}^{M} \frac{M-1}{M} \mathrm{Q}\left(\frac{d_{min}}{\sqrt{2N_0}} \right)$$

$$\le (\mathrm{M}-1)\mathrm{Q}\left(\frac{d_{min}}{\sqrt{2N_0}} \right)$$

Error Probability of M-ary QAM

- For square M-QAM (e.g. M=4, 16, 64, ...,etc), they can be viewed as two $\sqrt{M}-\mathrm{PAM}$ in each dimension.



The correct probability $P_c = (1-P_{\sqrt{M}})^2$

The error probability for $\sqrt{M}-\mathrm{PAM}$

$$P_{\sqrt{M}} = \frac{2(\sqrt{M}-1)}{\sqrt{M}} Q\left(\sqrt{\frac{6}{M-1}\frac{E_{\sqrt{M}-PAM}}{N_0}}\right) = 2\left(1-\frac{1}{\sqrt{M}}\right)Q\left(\sqrt{\frac{3}{M-1}\frac{E_s}{N_0}}\right)$$

$$E_s = \frac{E_g}{3}(M-1)$$

$$where\ E_s = 2E_{\sqrt{M}-PAM} \qquad \text{Note}: E_s = \frac{(M-1)d_{\min}^2}{6} \qquad E_{\sqrt{M}-PAM} = \frac{(M-1)d_{\min}^2}{12}$$

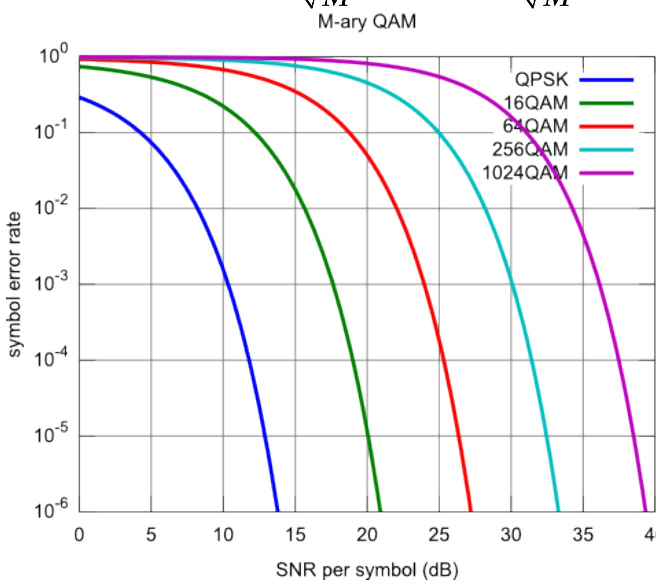
The error probability for square M-QAM $P_e = 1-P_c = 1-(1-P_{\sqrt{M}})^2 = 2P_{\sqrt{M}} - P_{\sqrt{M}}^2$

$$\rightarrow P_e \le 2P_{\sqrt{M}} = 4\left(1-\frac{1}{\sqrt{M}}\right)Q\left(\sqrt{\frac{3}{(M-1)}\frac{E_s}{N_0}}\right)$$

$$\le 4Q\left(\sqrt{\frac{3}{(M-1)}\frac{KE_b}{N_0}}\right)$$

$$\rightarrow \text{CFM}=\frac{d_{\min}^2}{E_s} = \frac{6}{M-1}$$

$$M\uparrow P_e \uparrow \ \text{CFM} \downarrow$$



Comparison of M-QAM and MPSK error probability

The error probability for M-QAM

$$P_{e,MQAM} \cong 4\,Q\left(\sqrt{\frac{3E_s}{(M-1)N_0}}\right)$$

The error probability for MPSK

$$P_{e,MPSK} \cong 2\,Q\left(\sqrt{\frac{2E_s}{N_0}}\sin\left(\frac{\pi}{M}\right)\right)$$

$$\left.\begin{matrix} \\ \\ \end{matrix}\right\} M\uparrow \quad P_e \uparrow$$

Q: For fixed P_e which modulation is more power efficient?

i.e. when M increase which modulation require more power?

- Compare the argument ratio, R_M,

$$R_M = \left(\frac{\arg P_{e,MQAM}}{\arg P_{e,MPSK}}\right)^2 = \frac{\frac{3}{M-1}}{2\sin^2\left(\frac{\pi}{M}\right)} = \frac{3}{2(M-1)\sin^2\left(\frac{\pi}{M}\right)} \approx \frac{3M^2}{2(M-1)\pi^2}$$

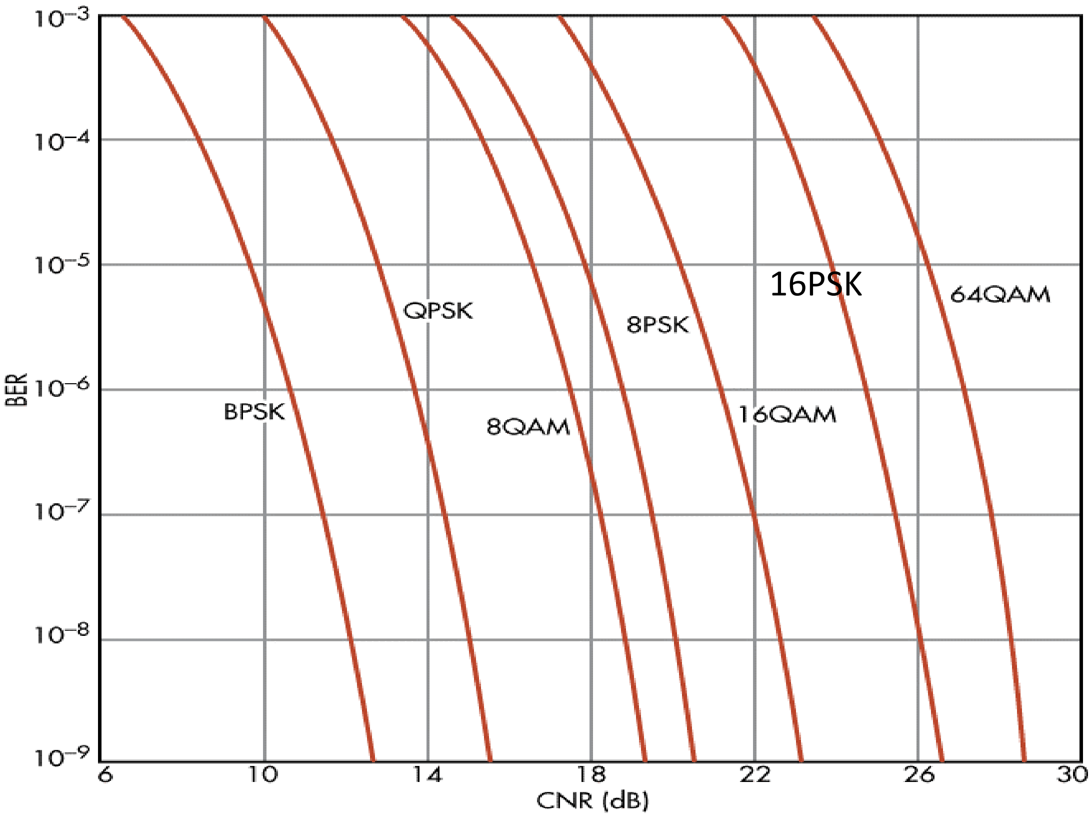
$M\uparrow R_M\uparrow \;\Rightarrow$ MPSK requires more power to keep the same P_e

Comparison of M-QAM and MPSK error probability

$$\text{MQAM}: P_e \le 4Q\left(\sqrt{\frac{3}{(M-1)}\frac{E_s}{N_0}}\right)$$

$$M \uparrow \quad R_b \uparrow \quad P_e \uparrow$$

$$\text{MPSK}: P_e \cong 2\,\text{Q}\left(\frac{2\sqrt{E_s}\sin\left(\frac{\pi}{M}\right)}{\sqrt{2N_0}}\right) = 2\,\text{Q}\left(\sqrt{\frac{2E_s}{N_0}}\sin\left(\frac{\pi}{M}\right)\right)$$



4.6 Comparison of Digital Modulation

- For multiphase signals(e.g. MPSK, MQAM) the bandwidth is governed by $g(t)$ with symbol time T. The spectral efficiency

$$\rho = \frac{R}{BW} = \frac{\log_2 M / T}{2/T} = \frac{1}{2}\log_2 M \quad \text{bits/s/Hz}$$

$$M\uparrow \ \rho\uparrow \quad P_e \quad \uparrow$$

Disadvantage: As M $\uparrow$, require increased energy to maintain fixed P_e

- For orthogonal signals(e.g. coherent MFSK $\Delta f = \frac{1}{2T}$)

$$M = 2^K \quad BW = M\Delta f = \frac{M}{2T}$$

$$\rho = \frac{R}{BW} = \frac{\log_2 M / T}{M/2T} = \frac{2\log_2 M}{M} \quad \text{bits/s/Hz}$$

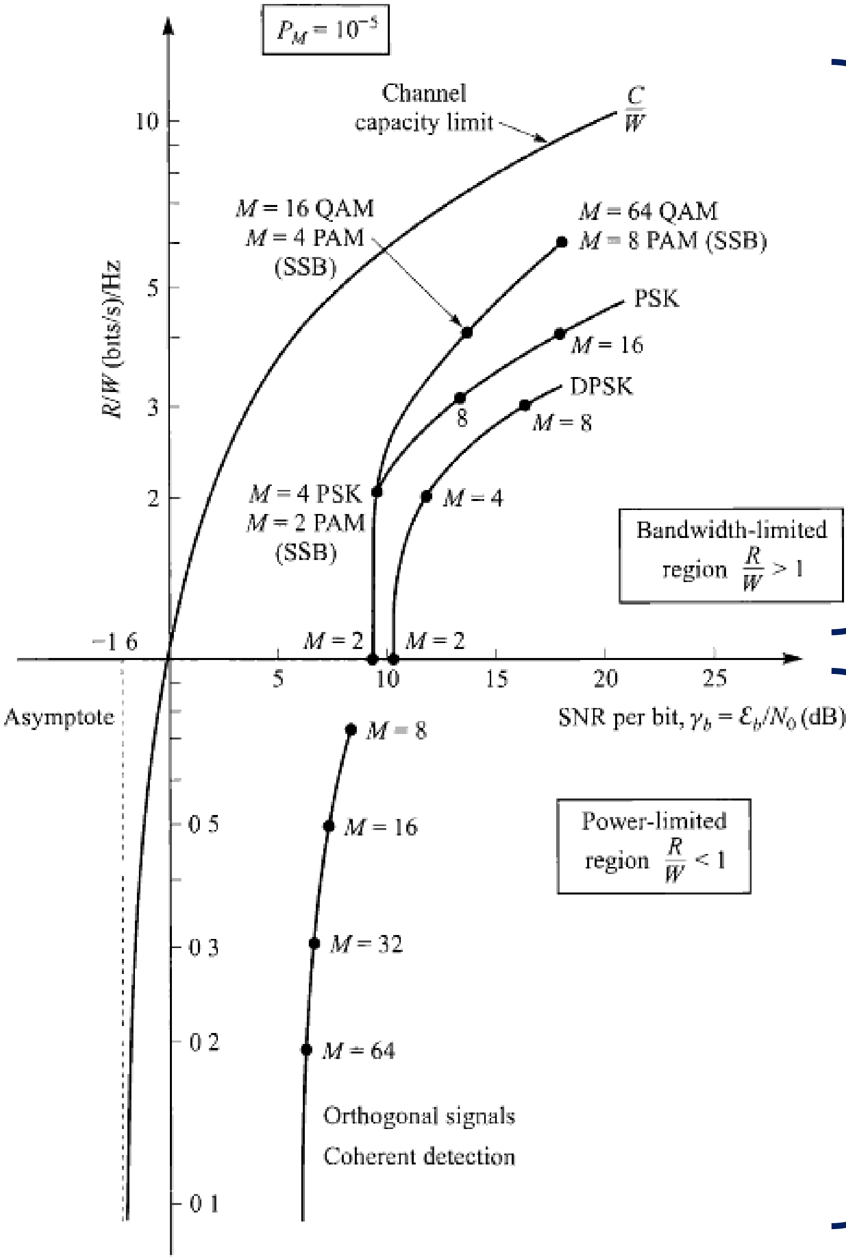
$$M\uparrow \ \mathrm{R}\uparrow \ P_e \downarrow \quad \rho\downarrow$$

Advantage: As M $\uparrow$, less energy needed to maintain fixed P_e.

In general, there is no “best” modulation depending on the system requirement.

FIGURE 4.6–1

Comparison of several modulation schemes at $P_e = 10^{-5}$ symbol error probability.



✓Bandwidth efficient modulations $\frac{R}{W} \geq 1$

- As M increases, MQAM/MPSK improves the spectral efficiency (R/W). (BW efficient)
- But MFSK decreases in R/W, as M increases. (BW inefficient)

✓Power efficient modulations $\frac{R}{W} < 1$

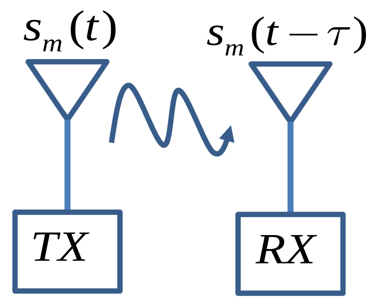
- As M increases, MQAM/MPSK requires more energy (i.e. E_b) to keep the P_e. (Power inefficient)
- But MFSK requires less E_b to keep the same P_e. So MFSK is power efficient.

4.5 Optimal Detection in Presence of Uncertainty

- **<u>Non-coherent detection of carrier modulated signals</u>**

-The Tx and Rx are not synchronized in phase.

-Time delay of transmission signal is unknown.

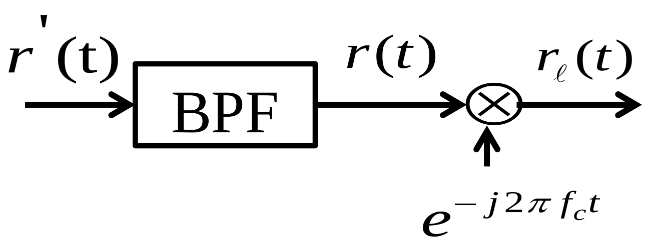


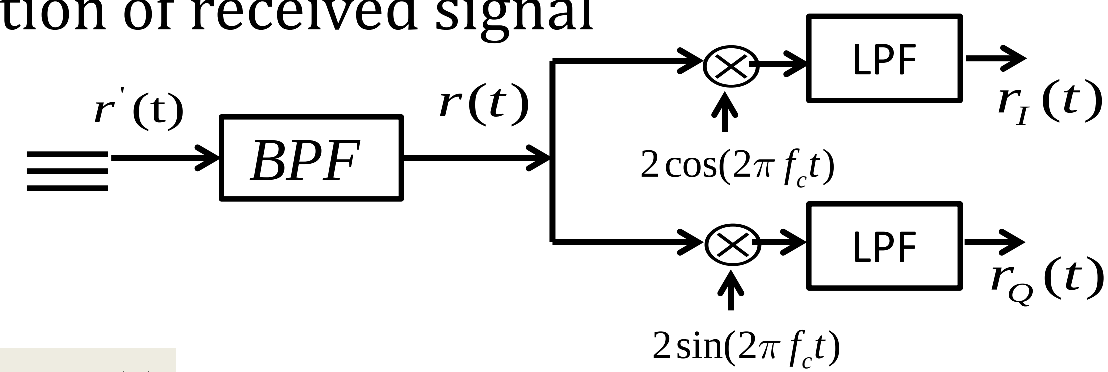
$$s_m(t) = s_{mI}(t)\cos(2\pi f_c t) - s_{mQ}(t)\sin(2\pi f_c t) = \mathrm{Re}\left\{s_{m\ell}(t)e^{j2\pi f_c t}\right\}$$

The received signals with AWGN noise

$$r(t) = s_m(t-\tau) + n(t) = \mathrm{Re}\left\{\underbrace{s_{m\ell}(t-\tau)}_{\approx\, s_{m\ell}(t)\ \text{Why?}}\underbrace{e^{-j2\pi f_c \tau}}_{=\ \text{unknown random phase}\ \triangleq\ e^{j\phi}} e^{j2\pi f_c t}\right\} + n(t)$$

The baseband representation of received signal



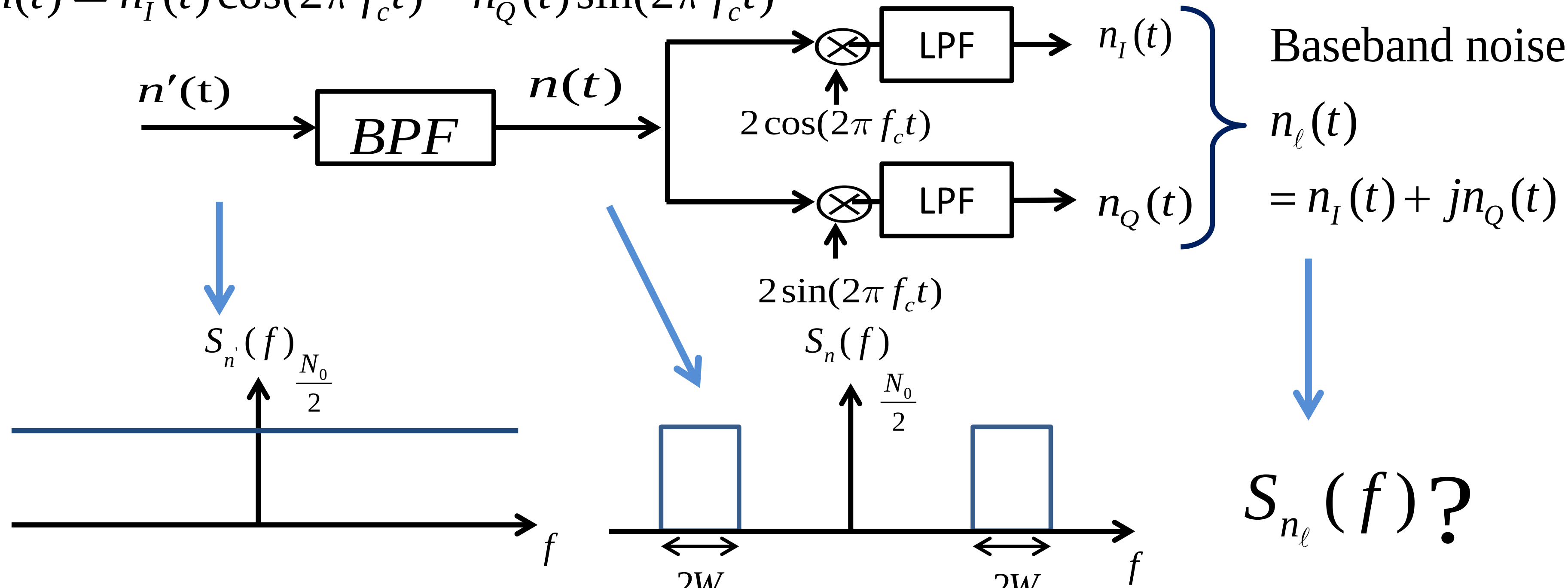


$$r_\ell(t) = s_{m\ell}(t)e^{j\phi} + n_\ell(t) = r_I(t) + jr_Q(t)$$ *where* $n_\ell(t) = n_I(t) + jn_Q(t)$

$n_I(t)$ and $n_Q(t)$ are given by $n(t) = n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t)$

Passband and Baseband Noises

$$n(t) = n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t)$$



- The autocorrelation of baseband noise

$$R_{n_\ell}(\tau) = E[n_\ell(\mathrm{t}+\tau)n_\ell^*(\mathrm{t})]$$

$$= \mathrm{E}[(n_I(\mathrm{t}+\tau) + jn_Q(\mathrm{t}+\tau))(n_I(t) - jn_Q(t))]$$

$$= \mathrm{E}[n_I(\mathrm{t}+\tau)n_I(t)] + \mathrm{E}[n_Q(\mathrm{t}+\tau)n_Q(t)] - j\,\mathrm{E}[n_I(\mathrm{t}+\tau)n_Q(t))] + j\,\mathrm{E}[n_I(t)n_Q(\mathrm{t}+\tau)]$$

$$= R_{n_I}(\tau) + R_{n_Q}(\tau)$$

Passband and Baseband Noises

$$n_I(t) = 2n(t)\cos(2\pi f_c t), \qquad |f| \le W$$

$$\rightarrow R_{n_I}(\tau) = E[n_I(\mathrm{t}+\tau)n_I(\mathrm{t})], \qquad |f| \le W$$

$$= E[4n(\mathrm{t}+\tau)n(\mathrm{t})\cos(2\pi f_c(\mathrm{t}+\tau))\cos(2\pi f_c t)], \qquad |f| \le W$$

$$= 2R_n(\tau)\cos(2\pi f_c \tau), \qquad |f| \le W$$

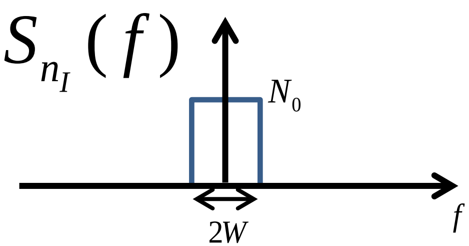
The power spectrum of baseband noise in I-channel is

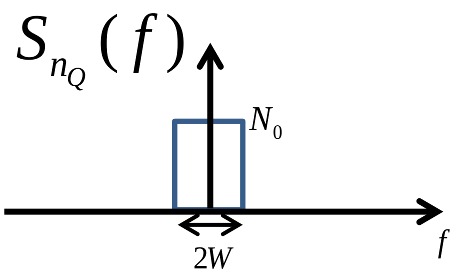
$$S_{n_I}(f) = \mathbb{F}\{R_{n_I}(\tau)\} = S_n(f+f_c) + S_n(f-f_c) = N_0, \qquad |f| \le W$$

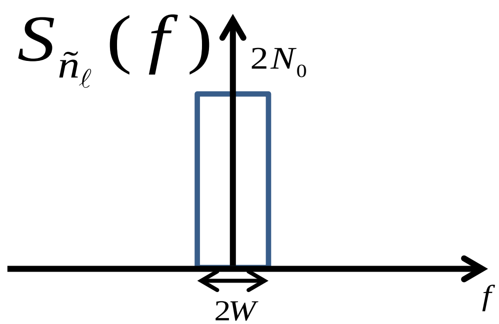
Similarly, we can also obtain baseband noise in Q-channel

$$S_{n_Q}(f) = N_0 \qquad |f| \le W$$

$$\rightarrow S_{n_\ell}(f) = S_{n_I}(f) + S_{n_Q}(f) = 2N_0, \quad |f| \le W$$







Detection of Non-coherent Carrier Modulated Signals

Assume $\underline{s}_1$ was sent,

$$\underline{r}_{\ell} = e^{j\phi}\underline{s}_{1\ell} + \underline{n}_{\ell} \quad \sim \mathcal{N}(e^{j\phi}\underline{s}_{m\ell}, 2N_0\mathbf{I})$$

where $\underline{s}_{1\ell} = [\sqrt{2E_s} \quad 0 \quad \dots \quad 0 \quad 0]^T$

$\underline{s}_{2\ell} = [0 \quad \sqrt{2E_s} \quad 0 \quad \dots \quad 0]^T$

$\underline{s}_{M\ell} = [0 \quad 0 \quad \dots \quad 0 \quad \sqrt{2E_s}]^T$

MAP detector

$$\hat{\underline{s}}_{m\ell} = \arg\max_{\underline{s}_{m\ell}} P(\underline{s}_{m\ell} \mid \underline{r}_{\ell}) = \arg\max_{\underline{s}_{m\ell}} P_m P(\underline{r}_{\ell} \mid \underline{s}_{m\ell})$$

Denote $P_m = P(\underline{s}_{m\ell})$

Consider least favorable case, taking ϕ as a uniformly distributed r.v.

$$f_{\phi}(\phi) = \frac{1}{2\pi}, \text{ where } \phi \in [0, \ 2\pi]$$

$$P(\underline{r}_{\ell} \mid \underline{s}_{m\ell}) = \int_0^{2\pi} f_{\phi}(\phi)\mathcal{N}(e^{j\phi}\underline{s}_{m\ell}, 2N_0\mathbf{I})\, d\phi$$

$$\hat{\underline{s}}_{m\ell} = \arg\max_{\underline{s}_{m\ell}} \frac{P_m}{2\pi}\left(\frac{1}{4\pi N_0}\right)^N \int_0^{2\pi} \exp\left(-\frac{\left\|\underline{r}_{\ell} - e^{j\phi}\underline{s}_{m\ell}\right\|^2}{4N_0}\right) d\phi$$

Note : $\|\underline{a} - \underline{b}\|^2$

$= (\underline{a} - \underline{b})^T (\underline{a} - \underline{b})^*$

$= \|\underline{a}\|^2 + \|\underline{b}\|^2 - (\underline{a}^T \underline{b}^* + \underline{b}^T \underline{a}^*)$

$= \|\underline{a}\|^2 + \|\underline{b}\|^2 - 2\,\mathrm{Re}\{\underline{a}^T \underline{b}^*\}$

$\|\tilde{\underline{s}}_{m\ell}\|^2 = 2E_s$

$$= \arg\max_{\underline{s}_{m\ell}} \frac{P_m}{2\pi}\left(\frac{1}{4\pi N_0}\right)^N \exp\left(-\frac{E_s}{2N_0}\right)\int_0^{2\pi} \exp\left(\frac{1}{2N_0}\mathrm{Re}\{\underbrace{\underline{r}_{\ell}^T (e^{j\phi}\underline{s}_{m\ell})^*}\}\right) d\phi \quad \text{Why?}$$

Let $\underline{r}_{\ell} \cdot \underline{s}_{m\ell} = \underline{r}_{\ell}^T \underline{s}_{m\ell}^* = \|\underline{r}_{\ell}\|\|\underline{s}_{m\ell}\| e^{j\theta}$

$$= \arg\max_{\underline{s}_{m\ell}} \frac{P_m}{2\pi}\left(\frac{1}{4\pi N_0}\right)^N \exp\left(-\frac{E_s}{2N_0}\right)\int_0^{2\pi} \exp\left(\frac{1}{2N_0}\left|\underline{r}_{\ell} \cdot \underline{s}_{m\ell}\right| \cos(\theta - \phi)\right) d\phi$$

Define $$I_0(x) = \frac{1}{2\pi}\int_0^{2\pi} \exp(x\cos\phi)d\phi$$

We have

$$\int_0^{2\pi} \exp(\frac{1}{2N_0}\left|\underline{r}_\ell \cdot \underline{s}_{m\ell}\right|\cos(\phi-\theta))d\phi = \int_0^{2\pi} \exp(\frac{1}{2N_0}\left|\underline{r}_\ell \cdot \underline{s}_{m\ell}\right|\cos(\phi'))d\phi'$$

$$= 2\pi I_0(\frac{\left|\underline{r}_\ell \cdot \underline{s}_{m\ell}\right|}{2N_0})$$

$$\Rightarrow \hat{\underline{s}}_{m\ell} = \arg\max_{\underline{s}_{m\ell}} P_m(\frac{1}{4\pi N_0})^N \exp(-\frac{E_s}{2N_0})I_0(\frac{\left|\underline{r}_\ell \cdot \underline{s}_{m\ell}\right|}{2N_0})$$

Since $I_0(x)$ is a monotonically increasing function,

and assume $P_m = \frac{1}{M}\ \forall m,\ \forall m$, the MAP detector becomes

$$\hat{\underline{s}}_{m\ell} = \arg\max_{\underline{s}_{m\ell}}\left|\underline{r}_\ell \cdot \underline{s}_{m\ell}\right| = \arg\max_{m}\sqrt{r_{mI}^2 + r_{mQ}^2}$$

where $r_{mI} = \mathrm{Re}\{\underline{r}_\ell \cdot \underline{s}_{m\ell}\}$

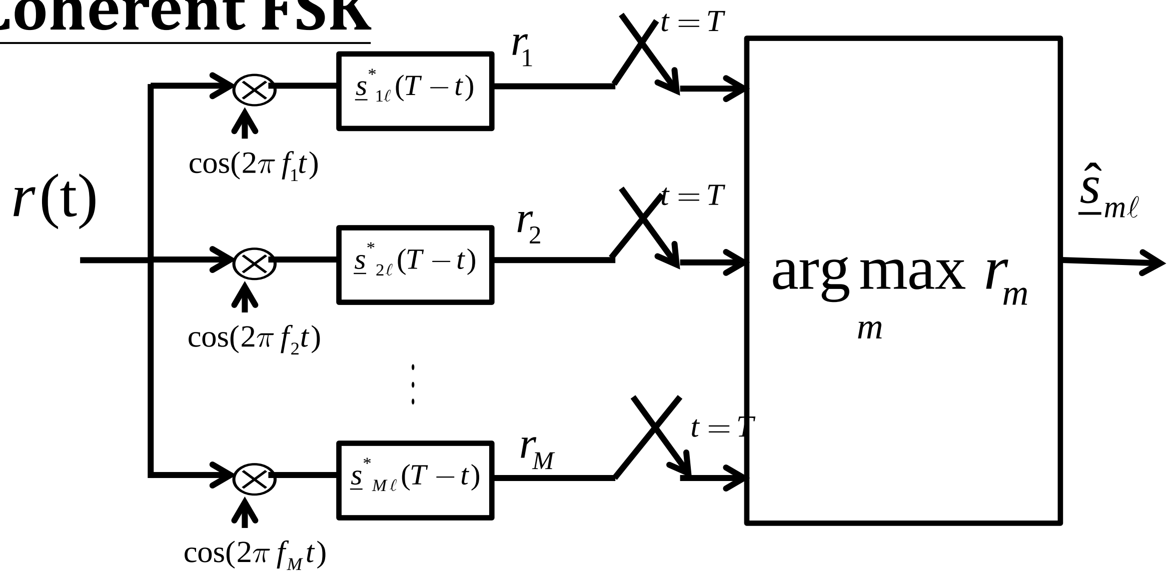
$r_{mQ} = \mathrm{Im}\{\underline{r}_\ell \cdot \underline{s}_{m\ell}\}$

$Note: \underline{r}_\ell = e^{j\phi}\underline{s}_{1\ell} + \underline{n}_\ell$

$\rightarrow \left|\underline{r}_\ell \cdot \underline{s}_{m\ell}\right| = \sqrt{r_{mI}^2 + r_{mQ}^2}$

Comparison of coherent and non-coherent MFSK architecture

Coherent FSK

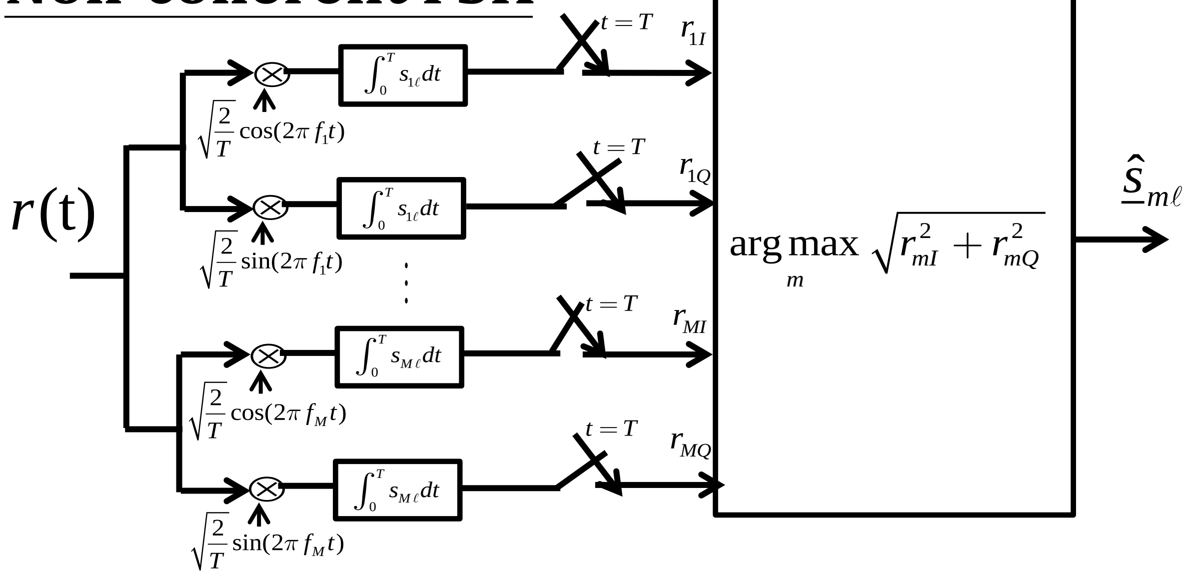


$$\hat{\underline{s}}_\ell = \underset{\underline{s}_{m\ell}}{\arg\max}\ p(\underline{s}_{m\ell} \mid \underline{r}_\ell)$$

$$= \underset{\underline{s}_{m\ell}}{\arg\max}\ \underline{r}_\ell \cdot \underline{s}_{m\ell}$$

$$= \underset{\underline{s}_{m\ell}}{\arg\max}\ r_m$$

Non-coherent FSK



$$\hat{\underline{s}}_\ell = \underset{\underline{s}_{m\ell}}{\arg\max}\left|\underline{r}_\ell \cdot \underline{s}_{m\ell}\right|$$

where $r_{mI} = \mathrm{Re}\{\underline{r}_\ell \cdot \underline{s}_{m\ell}\}$

$r_{mQ} = \mathrm{Im}\{\underline{r}_\ell \cdot \underline{s}_{m\ell}\}$

$$\rightarrow \left|\underline{r}_\ell \cdot \underline{s}_{m\ell}\right| = \sqrt{r_{mI}^2 + r_{mQ}^2}$$

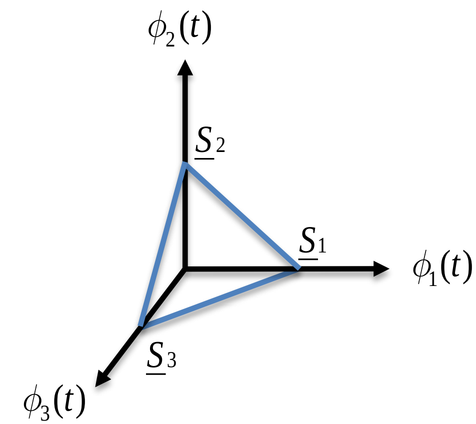
Error probability of Orthogonal Signaling with Noncoherent Detection

Assume M equiprobable and equal energy carrier modulated signals, the signal constellation are

$$\underline{s}_{1\ell} = [\sqrt{2E_s} \quad 0 \quad \dots \quad 0 \quad 0]^T$$

$$\underline{s}_{2\ell} = [0 \quad \sqrt{2E_s} \quad 0 \quad \dots \quad 0]^T$$

$$\underline{s}_{M\ell} = [0 \quad 0 \quad \dots \quad 0 \quad \sqrt{2E_s}\,]^T$$



Assume $\underline{s}_{1\ell}$ was sent, the received signal is

$$\underline{r}_{\ell} = e^{j\phi}\underline{s}_{1\ell} + \underline{n}_{\ell}$$

where $\underline{n}_{\ell} = [n_{I1} + jn_{Q1}, \dots, n_{IM} + jn_{QM}]^T$

MAP detector

$$\hat{\underline{s}}_{m\ell} = \arg\max_{\underline{s}_{m\ell}} \left| \underline{r}_{\ell} \cdot \underline{s}_{m\ell} \right|$$

$$= \arg\max_{\underline{s}_{m\ell}} \{\sqrt{\mathrm{Re}(\underline{r}_{\ell} \cdot \underline{s}_{m\ell})^2 + \mathrm{Im}(\underline{r}_{\ell} \cdot \underline{s}_{m\ell})^2}\}$$

$$= \arg\max_{\underline{s}_{m\ell}} R_m$$

$$\begin{cases} R_1 = \left|\underline{r}_\ell \cdot \underline{s}_{1\ell}\right| = \left|2E_s e^{j\phi} + \underline{n}_\ell \cdot \underline{s}_{1\ell}\right| & m=1 \\ R_m = \left|\underline{r}_\ell \cdot \underline{s}_{m\ell}\right| = \left|\underline{n}_\ell \cdot \underline{s}_{m\ell}\right| & m = 2 \sim M \end{cases}$$

$$\begin{cases} \mathrm{Re}\{\underline{r}_\ell \cdot \underline{s}_{1\ell}\} \sim \mathcal{N}(2E_s \cos\phi, 2E_s N_0) \\ \mathrm{Im}\{\underline{r}_\ell \cdot \underline{s}_{1\ell}\} \sim \mathcal{N}(2E_s \sin\phi, 2E_s N_0) \end{cases} \quad m=1$$

$$\begin{cases} \mathrm{Re}\{\underline{r}_\ell \cdot \underline{s}_{1\ell}\} \sim \mathcal{N}(0, 2E_s N_0) \\ \mathrm{Im}\{\underline{r}_\ell \cdot \underline{s}_{1\ell}\} \sim \mathcal{N}(0, 2E_s N_0) \end{cases} \quad \mathrm{m} = 2, \ldots, \mathrm{M}$$

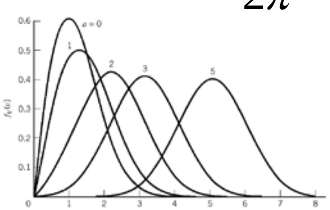
R_1 is Rician distributed with mean $s = 2E_s \quad \sigma^2 = 2E_s N_0$

R_m is *Rayleigh* distributed with mean $s = 0 \quad \sigma^2 = 2E_s N_0$

$$f_{R_1}(r_1) = \begin{cases} \dfrac{r_1}{\sigma^2} I_0\left(\dfrac{s r_1}{\sigma^2}\right) \exp\left(\dfrac{-r_1^2 + s^2}{2\sigma^2}\right) & r_1 > 0 \\ 0 & o.w \end{cases}$$

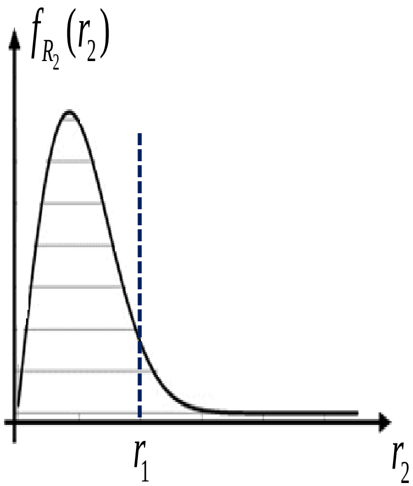
$$I_0(x) = \frac{1}{2\pi}\int_0^{2\pi} \exp(x\cos\phi)\,d\phi$$

$$f_{R_m}(r_m) = \begin{cases} \dfrac{r_m}{\sigma^2} \exp\left(\dfrac{-r_m^2}{2\sigma^2}\right) & r_m > 0 \\ 0 & o.w \end{cases}$$



✓ With the system model and probability model, we can start to find P_e of MAP detector.

$$P_c = P\{R_2 < R_1, R_3 < R_1, \ldots, R_M < R_1\}$$
$$= \int_0^\infty P(R_2 < r_1, R_3 < r, \ldots, R_M < r_1 \mid r_1) f_{R_1}(r_1)\,\mathrm{d}\,r_1$$
$$= \int_0^\infty [P(R_2 < r_1)]^{M-1} f_{R_1}(r_1)\,\mathrm{d}\,r_1$$



$$P(R_2 < r_1) = \int_0^{r_1} f_{R_2}(r_2)dr_2 = \int_0^{r_1} \frac{r_2^{\ 2}}{\sigma^2} e^{\frac{-x^2}{2\sigma^2}} dr = -e^{\frac{-x^2}{2\sigma^2}} \Big|_0^{r_1} = 1 - e^{\frac{-r_1^2}{2\sigma^2}}$$

With binomial expansion,

$$(1-x)^{M-1} = \sum_{n=0}^{M-1} (-1)^n \binom{M-1}{n} x^n = 1 - \binom{M-1}{1} x + \binom{M-1}{2} x^2 - \ldots.. \rightarrow \text{Let } x = e^{\frac{-r_1^2}{2\sigma^2}}$$

$$\therefore P_c = \int_0^\infty \sum_{n=0}^{M-1} (-1)^n \binom{M-1}{n} \exp(-\frac{nr_1^{\ 2}}{2\sigma^2}) \cdot \left\{ \frac{r_1}{\sigma^2} I_0(\frac{sr_1}{\sigma^2}) \exp(-\frac{r_1^{\ 2}+s^2}{2\sigma^2}) \right\} dr_1$$

$$= \sum_{n=0}^{M-1} (-1)^n \binom{M-1}{n} \int_0^\infty \frac{r_1}{\sigma^2} I_0(\frac{sr_1}{\sigma^2}) \exp(-\frac{(n+1)r_1^{\ 2}+s^2}{2\sigma^2}) dr_1$$

$$= \sum_{n=0}^{M-1} (-1)^n \binom{M-1}{n} \exp(\frac{-ns^2}{2(\mathrm{n}+1)\sigma^2}) \underbrace{\int_0^\infty \frac{r_1}{\sigma^2} I_0(\frac{sr_1}{\sigma^2}) \exp(-\frac{(n+1)r_1^2 + \frac{s^2}{n+1}}{2\sigma^2}) dr_1}_{A}$$

$$\text{where } A = \frac{1}{n+1} \underbrace{\int_0^\infty \frac{r'^2}{\sigma^2} I_0(\frac{s'r'}{\sigma^2}) \exp(-\frac{r'^2+s'^2}{2\sigma^2}) dr'}_{=1} = \frac{1}{n+1} \quad \textit{with } s' = \frac{s}{\sqrt{n+1}}, \quad r' = \sqrt{n+1}\, r_1$$

- Correct probability of non-coherent MFSK

$$P_c = \sum_{n=0}^{M-1} \frac{(-1)^n}{n+1} \binom{M-1}{n} \exp\left(\frac{-n}{n+1}\frac{E_s}{N_0}\right) \quad where\ s = 2E_s\ \sigma^2 = 2E_s N_0$$

Error probability

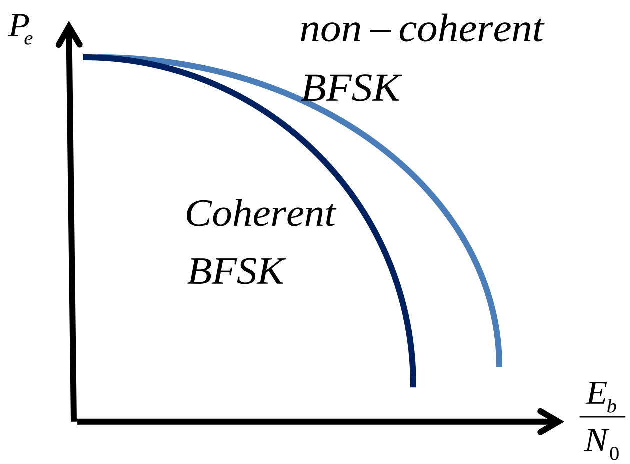
$$P_e = 1 - P_c = \sum_{n=1}^{M-1} \frac{(-1)^{n+1}}{n+1} \binom{M-1}{n} \exp\left(\frac{-n}{n+1}\frac{E_s}{N_0}\right)$$

- For noncoherent BFSK (i.e. $M = 2 \quad E_s = E_b$)

$$P_{e-noncoherent} = \frac{1}{2}\exp\left(\frac{-E_b}{2N_0}\right)$$

- Comparing with coherent BFSK

$$P_{e-coherent} = Q\left(\sqrt{\frac{E_b}{N_0}}\right) \le \frac{1}{2}\exp\left(\frac{-E_b}{2N_0}\right)$$



4.8 Detection of Signaling Scheme with Memory

Maximum Likelihood Sequence Detector(MLSD)

-The modulation with memory can be represented by a trellis.

-The transmitted sequence corresponds to a path through the trellis.

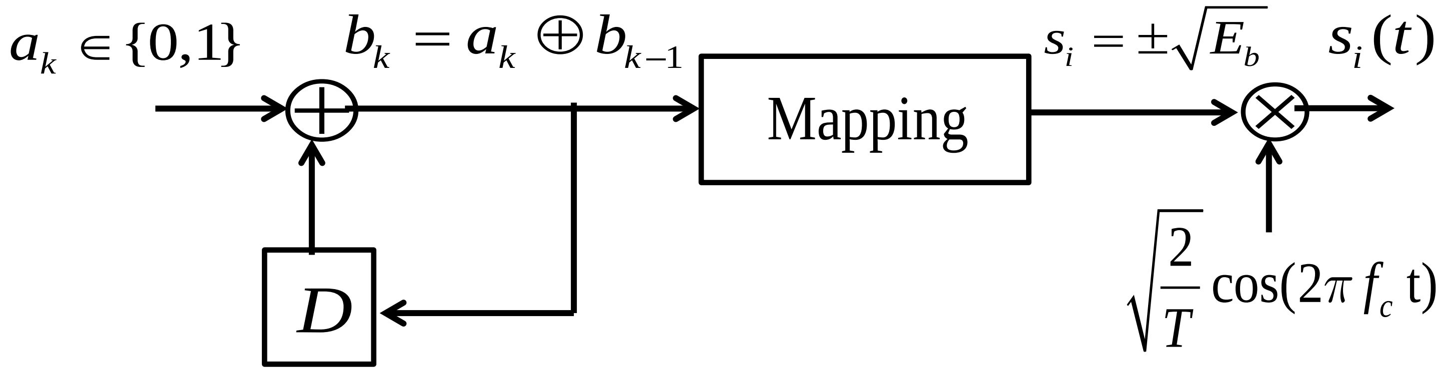
Assume transmitted sequence of K-symbols, the MLSD becomes

$$(\hat{\underline{s}}^{(1)}, \hat{\underline{s}}^{(2)}, \ldots, \hat{\underline{s}}^{(K)}) = \underset{(\underline{s}^{(1)}, \underline{s}^{(2)}, \ldots, \underline{s}^{(K)})}{\arg\min} \sum_{k=1}^{K} \left\| \underline{r}^{(k)} - \underline{s}^{(K)} \right\|^2$$

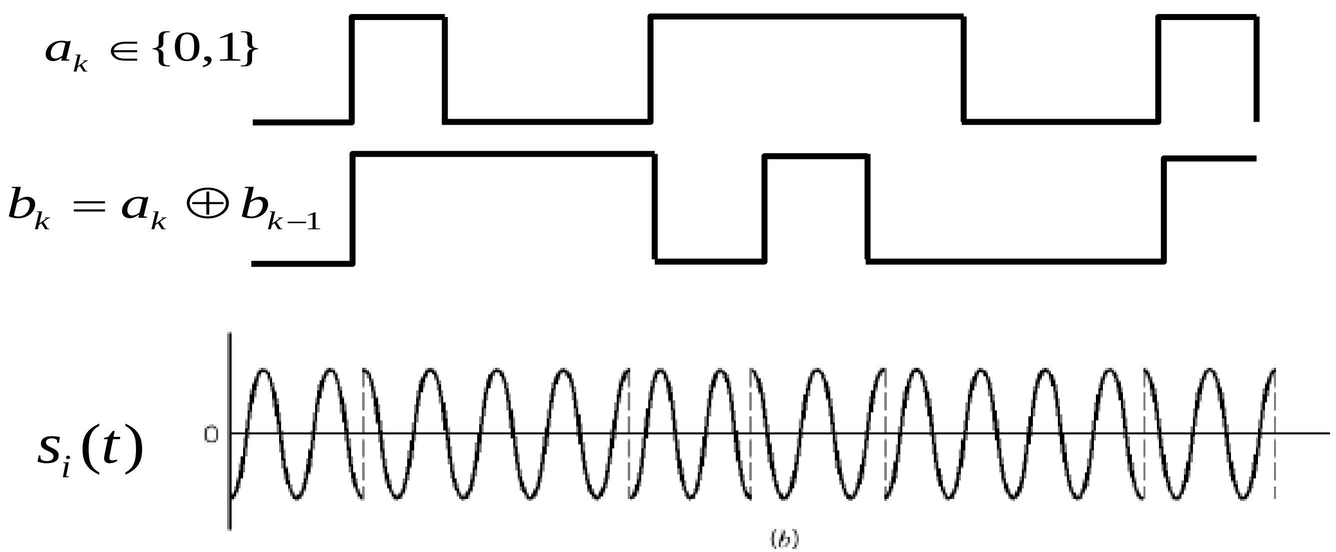
$$= \underset{(\underline{s}^{(1)}, \underline{s}^{(2)}, \ldots, \underline{s}^{(K)})}{\arg\min} \sum_{k=1}^{K} D(\underline{r}^{(k)}, \underline{s}^{(K)})$$

The computation is complicated, to the order of $O(M^K)$.

Example(NRZI differential encoding DPSK)

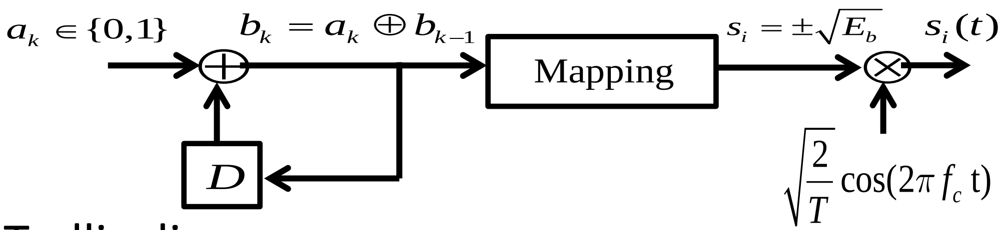


Timing diagram



b_k toggles when $a_k = 1$

Example(NRZI differential encoding DPSK)

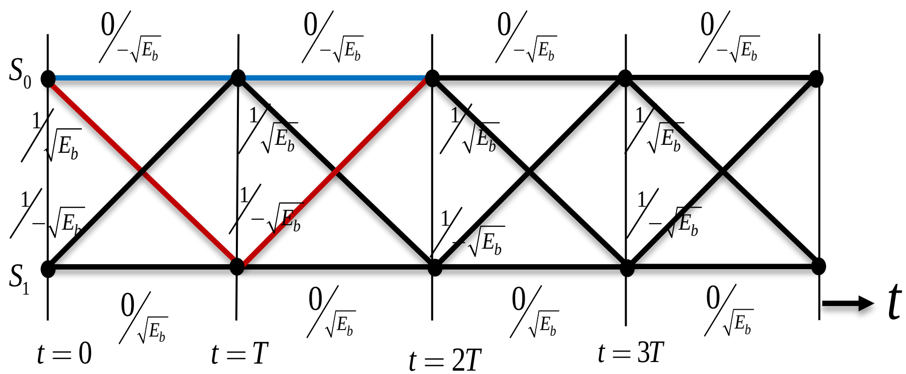


$S_0 : b_k = 0, s_k = -\sqrt{E_b}$

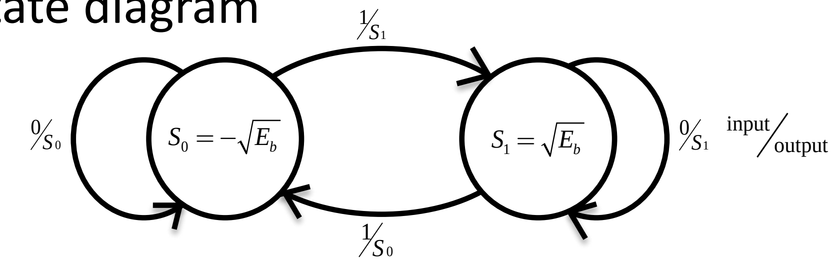
$S_1 : b_k = 1, s_k = +\sqrt{E_b}$

$\text{input}/\text{output} = a_k / b_k$

Trellis diagram



State diagram



Define the distance metric

$D_{s_i}(a_{k-2}, a_{k-1})$

Current state **Information bits at k-2 and k-1**

Assume initial state at t=0 is S_0.

$At \;\; t = 2T$, the two paths entering S_0 are,

$$D_0(0,0) = (r_1 + \sqrt{E_b})^2 + (r_2 + \sqrt{E_b})^2 \quad (blue\ line)$$

$$D_0(1,1) = (r_1 - \sqrt{E_b})^2 + (r_2 + \sqrt{E_b})^2 \quad (red\ line)$$

Select one survial path entering $S_0 = \min\{D_0(0,0), D_0(1,1)\}$.

$At \;\; t = 2T$, the two paths entering S_1 are,

$$D_1(0,1) = (r_1 + \sqrt{E_b})^2 + (r_2 - \sqrt{E_b})^2$$

$$D_1(1,0) = (r_1 - \sqrt{E_b})^2 + (r_2 - \sqrt{E_b})^2$$

Select one survial path entering $S_1 = \min\{D_1(0,1), D_1(1,0)\}$.

Example cont.

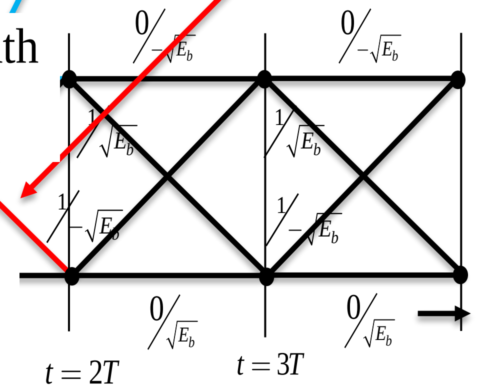
Assume the previous survivor paths at t=2T for S_0 and S_1 are $D_0(0,0)$ *and* $D_1(0,1)$.

At t=3T, upon receipt of r_3, the paths entering S_0 are

$$\left.\begin{aligned} D_0(0,0,0) &= D_0(0,0)+(r_3+\sqrt{E_b})^2 \\ D_0(0,1,1) &= D_1(0,1)+(r_3+\sqrt{E_b})^2 \end{aligned}\right\} \text{Select one survial path entering } S_0$$

At t=3T, the paths entering S_1 are

$$\left.\begin{aligned} D_1(0,0,1) &= D_0(0,0)+(r_3-\sqrt{E_b})^2 \\ D_1(0,1,0) &= D_1(0,1)+(r_3-\sqrt{E_b})^2 \end{aligned}\right\} \text{Select one survial path entering } S_1$$



- At each stage, $t=kT$, the process keep two survivor paths (for S_0 and S_1 respectively), entering the next stage.
- The next stage, $t=(k+1)T$, can leverage on the previous survivor path to find MLSD result and save the computation complexity.
- Detection with memory: the process continued as each new signal sample is received. (The Viterbi algorithm)

4.9 Optimum Receiver for CPM signals

Continuous Phase Modulation (CPM) signal (Ex: CPFSK):

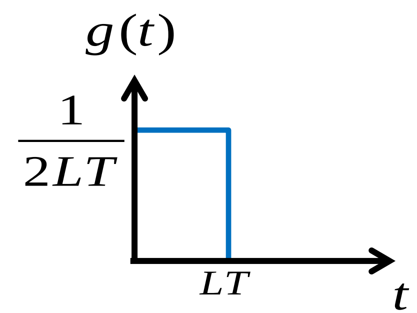
$$s(t) = \sqrt{\frac{2E}{T}} \cos[2\pi f_c t + \phi(t, \underline{I})]$$

For the carrier phase of a CPM, the phase trellis is

$$\varphi(t, \underline{I}) = 2\pi h \sum_{k=-\infty}^{n} I_k q(t - kT) \qquad \text{where } \ h = T \cdot \Delta f = 2 f_d T$$

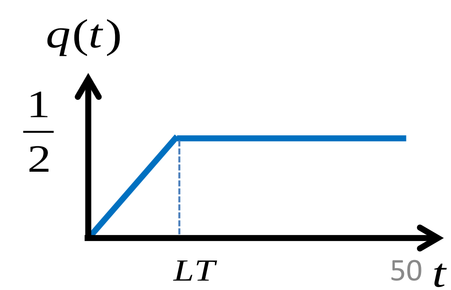
$$= \pi h \sum_{k=-\infty}^{n-L} I_k + 2\pi h \sum_{k=n-L+1}^{n} I_k q(t - kT)$$

$$= \theta_n + \theta(t, \underline{I}) \qquad nT \le t \le (n+1)T$$



$$\theta_{n+1} = \theta_n + \pi h I_n$$

$$where \ \ q(t) = \int_0^t g(\tau) d\tau \qquad g(t) = \frac{1}{2LT} rect\left(\frac{t - \frac{LT}{2}}{LT}\right)$$



The modulation index $h = T\Delta f = 2 f_d T = \frac{m}{p}$ $m, \mathrm{p} \in$ relative prime integer

$L = 1 \Rightarrow$ full response CPM: The phase trellis has $\begin{cases} p & \\ 2p & \end{cases}$ *phase states* $\begin{matrix} m \ \ even \\ m \ \ odd \end{matrix}$

$L > 1 \Rightarrow$ a partial response CPM: The phase trellis has additional states due to the partial response of $g(t)$

The phase trellis has (for L=1)

$$p \text{ phase states } \theta = \left\{ 0, \frac{\pi m}{p}, \frac{2\pi m}{p} \dots \frac{(p-1)\,\mathrm{m}\,\pi}{p} \right\} \qquad \text{for even } m$$

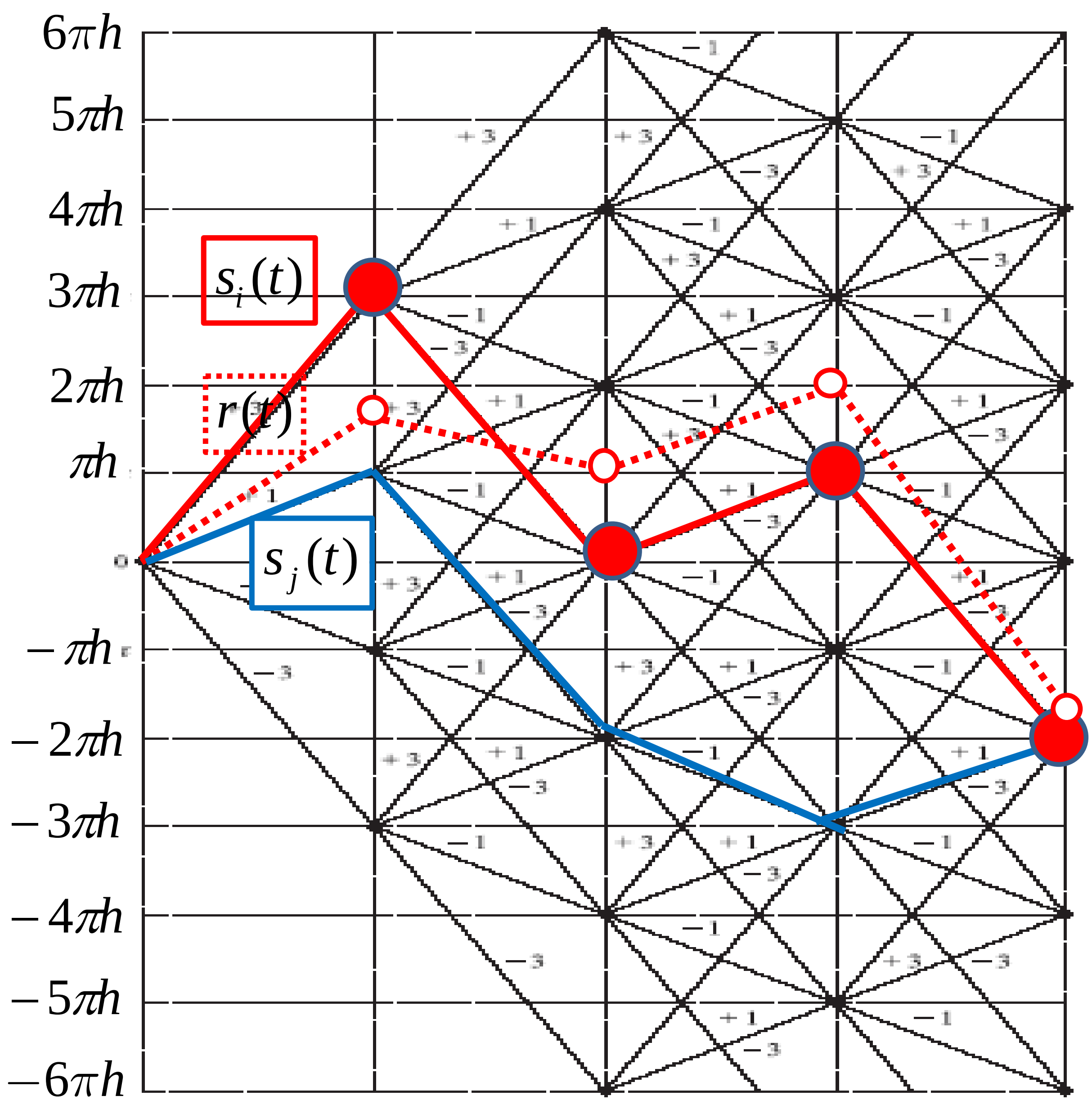
$$2p \text{ phase states } \theta = \left\{ 0, \frac{\pi m}{p}, \frac{2\pi m}{p} \dots \frac{(2p-1)\,\mathrm{m}\,\pi}{p} \right\} \qquad \text{for } odd\ m$$

<u>Example</u>

$$h = \frac{2}{3} \quad \theta \in \{\frac{2\pi}{3}, \frac{4\pi}{3}, \frac{6\pi}{3}\} \qquad h = \frac{1}{2} \quad \theta \in \{\frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\}$$

Phase Trellis for M-ary CPFSK Signal (M=4)

$I_n \in \{\pm 1, \pm 3\}$



Thus CPM can be decoded by Viterbi trellis decoding.

The state of the CPM signal at $t = nT$ may be expressed as combined phase state and correlative states as a vector.

State vector at $t = nT$:

$$\underline{s}_n = (\underset{p \text{ or } 2p}{\theta_n}, \underset{M}{I_{n-1}}, \underset{M}{I_{n-2}}, \ldots, \underset{M}{I_{n-L+1}})$$

$$N_s = \begin{cases} pM^{L-1} & m \in even \\ 2pM^{L-1} & m \in odd \end{cases} \quad (The\ number\ of\ states)$$

At t=(n+1)T state vector becomes

$$\underline{s}_{n+1} = (\theta_{n+1}, I_n, I_{n-1}, \ldots, I_{n-L+2})$$

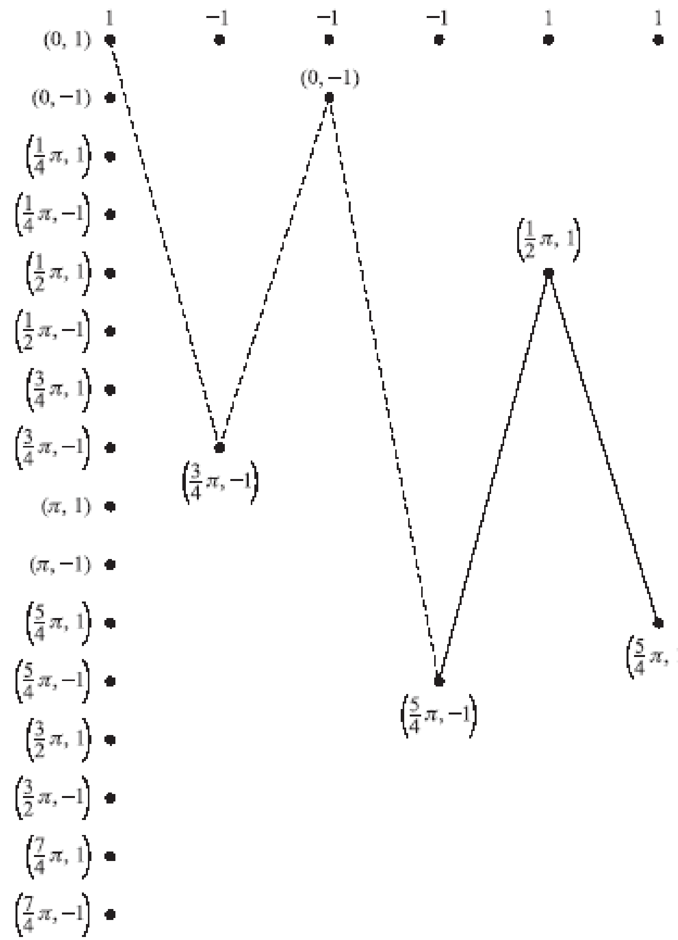


FIGURE 4.9–2
A single signal path through the trellis.

Performance of CPM

Let $s_i(t)$ and $s_j(t)$ be signalings that correspond to length *N* phase trajectories $\phi(t,\underline{I}_i)$ and $\phi(t,\underline{I}_j)$

$$s_i(t)=\sqrt{\frac{2E_s}{T}}\cos[2\pi f_c t+\phi(t,\underline{I}_i)]$$

The Euclidean distance between $s_i(t)$ and $s_j(t)$ over interval of NT is

$$\begin{aligned}
d_{ij}^2 &= \int_0^{NT}(s_i(t)-s_j(t))^2\,dt \\
&= \int_0^{NT} s_i^2(t)dt+\int_0^{NT} s_j^2(t)dt-2\int_0^{NT} s_i(t)s_j(t)dt \\
&= 2NE_s-2\frac{2E_s}{T}\int_0^{NT}\cos[2\pi f_c t+\phi(t,\underline{I}_i)]\cos[2\pi f_c t+\phi(t,\underline{I}_j)]dt \\
&= 2NE_s-\frac{2E_s}{T}\int_0^{NT}\cos[\phi(t,\underline{I}_i)-\phi(t,\underline{I}_j)]dt \\
&= \frac{2E_s}{T}\int_0^{NT}\left[1-\cos\left[\phi(t,\underline{I}_i)-\phi(t,\underline{I}_j)\right]\right]dt \quad \triangleq 2E_b\delta_{ij}{}^2
\end{aligned}$$

where $$\delta_{ij}{}^2=\frac{\log_2^M}{T}\int_0^{NT}\left[1-\cos\left[\phi(t,\underline{I}_i)-\phi(t,\underline{I}_j)\right]\right]dt=\frac{\log_2^M}{T}\int_0^{NT}\left[1-\cos\left[\phi(t,\xi)\right]\right]dt$$

$$\phi(t,\xi)=2\pi h\sum_{i=-\infty}^{n} q(\mathrm{t}-\xi T) \qquad\qquad \xi\triangleq\underline{I}_i-\underline{I}_j$$

Since $$\underline{I}_i, \underline{I}_j \in \{\pm 1, \pm 3, ..., \pm (M-1)\}$$

$$\xi = \underline{I}_i - \underline{I}_j \in \{0, \pm 2, \pm 4, ..., \pm 2(M-1)\}$$

The error rate performance of CPM can be measured by the minimum distance between any two phase trajectories of length *N*.

The pair-wise error probability is

$$P_{e,pair} = P\left(\mathcal{N}(0, N\sigma_n^2) > \frac{d_{ij}}{2}\right) = Q\left(\frac{\frac{d_{ij}}{2}}{\sqrt{N\sigma_n^2}}\right)$$

$$Let\ \sigma_n^2 = \frac{N_0}{2}$$

$$P_{e,pair} = Q\left(\frac{\sqrt{2E_b\delta_{ij}^2}}{2\sqrt{N\frac{N_0}{2}}}\right) = Q\left(\sqrt{\frac{E_b\delta_{ij}^2}{NN_0}}\right)$$

The union error probability of CPM is

$$P_e = K_{\delta_{\min}} Q\left(\sqrt{\frac{E_b\delta_{\min}^2}{NN_0}}\right) \qquad \delta_{\min}^2 = \min \delta_{i,j}^2 \quad \forall i, j$$

$K_{\delta_{\min}}$ is the number of paths having the minimum distance