# 排隊理論 2020 期中考範圍習題練習

**1.3.** The Carry Out Curry House, a fast-food Indian restaurant, must decide on how many parallel service channels to provide. They estimate that, during the rush hours, the average number of arrivals per hour will be approximately 40. They also estimate that, on average, a server will take about 5.5 min to serve a typical customer. Using only this information, how many service channels will you recommend they install?

**1.5.** The Outfront BBQ Rib Haven does carry out only. During peak periods, two servers are on duty. The owner notices that during these periods, the servers are almost never idle. She estimates the percent idle time of each server to be 1 percent. Ideally, the percent idle time would be 10 percent to allow time for important breaks.

**(a)** If the owner decides to add a third server during these times, how much idle time would each server have then?

**(b)** Suppose that by adding the third server, the pressure on the servers is reduced, so they can work more carefully, but their service output rate is reduced by 20 percent. What now is the percent time each would be idle?

**(c)** Suppose, instead, that the owner decides to hire an aid (at a much lower salary) who servers as a gofer for the two servers, rather than hiring another full server. This allows the two servers to decrease their average service *time* by 20 percent (relative to the original service times). What now is the percent idle time of each of the two servers?

**1.10.** Suppose that an $M/G/1/K$ queue has a blocking probability of $p_k = 0.1$ with $\lambda = \mu = 1$ and $L = 5$. Find $W$, $W_q$, and $p_0$.

**1.11.** Suppose that it costs \$3 to make one dose of the small pox vaccine. Once a dose is made, its shelf life is 90 days, after which it can no longer be used. It is desired to have, on average, 300 million doses available at any given time.

**(a)** What is the yearly cost to implement this plan?

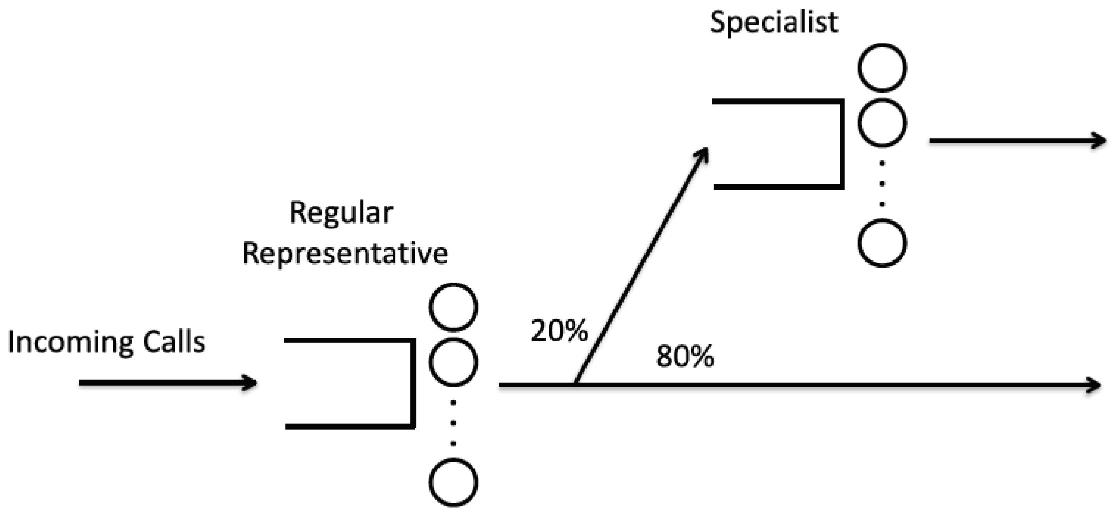
**(b)** Suppose now that the shelf life of a vaccine is randomly distributed according to an Erlang distribution with a mean of 90 days and a standard deviation of 30 days. What is the yearly cost to implement this plan?

**(c)** Suppose that a vaccine with a longer shelf life can be made, but at a greater cost. It is found that the cost to produce a vaccine with a shelf life of $x$ days is equal to $a + bx^2$, where $a = \$2.50$ and $b = \$0.00005$. What is the shelf life that minimizes the yearly cost?

**1.12.** Customers who have purchased a Delta laptop may call a customer support center to get technical help. Initially, a call is handled by a regular service representative. If the problem cannot be handled by a regular service representative, the call is transferred to a specialist. Twenty percent of all calls are transferred to a specialist. On average, there are 40 customers being served or waiting to be served by a regular representative. On average, there are 10 customers being served or waiting to be served by a specialist. The average rate of incoming calls is 100 per hour. There are 30 regular representatives and 10 specialists.

**(a)** What is the average time spent in the system for an arbitrary customer? State any assumptions you make to answer this question.

**(b)** What is the average time spent in the system for a customer who needs to talk to a specialist?




**1.13.** Consider the following (very) simplified model of Social Security. Every year, 3 million people turn 65. A person begins to receive Social Security benefits when s/he reaches an age of 65 years. An individual (over the age of 65) has a 5% chance of dying each year, independent of all else. Social Security benefits are $40,000 per person per year.

**(a)** On average, how long does a person receive Social Security benefits?

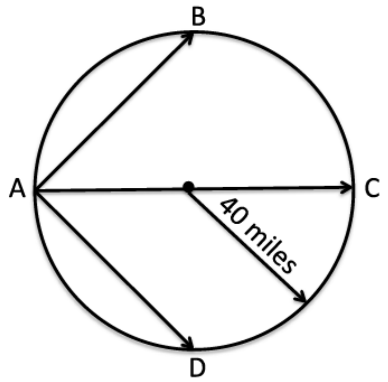
**(b)** What is the average total yearly payout in Social Security benefits?

**1.14.** Planes arrive at a circular sector of airspace according to a Poisson process with rate 20 arrivals per hour. The radius of the sector is 40 miles. Each plane travels at a speed of 400 miles per hour. There are 4 possible entrance / exit points in the sector, as shown. An aircraft is equally like to arrive and depart from any of points A, B, C, and D (but an aircraft cannot enter and exit from the same point). For example, the probability that an aircraft arrives at point A is 1/4. Given that an aircraft arrives at A, the probability that it exits at B, C, or D is 1/3 each. Assume that aircraft flights are straight paths and there are no collisions or conflict avoidance maneuvers in the sector.

**(a)** What is the average path length across the sector?

**(b)** What is the average number of aircraft in the sector?

**(c)** If we suppose that aircraft sometimes execute avoidance maneuvers to prevent conflicts/collisions, would the answer in (b) go up or down?

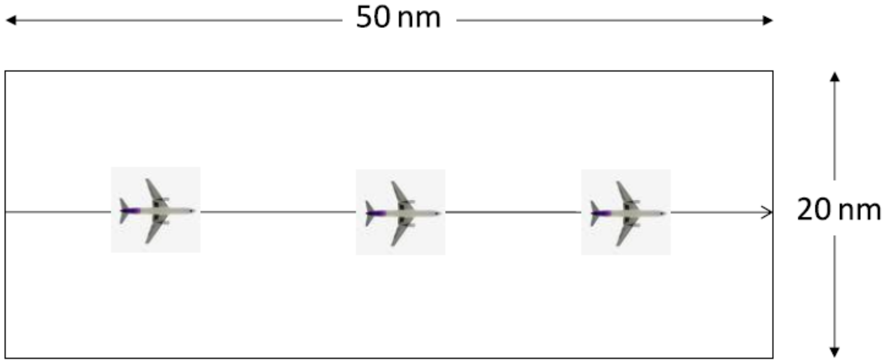



**1.15.** The length of time that a person owns a car before buying a new one has an Erlang-3 distribution with a mean of 5 years. Suppose that there are approximately 150 million cars in the United States.

**(a)** Assuming that a person's old car is destroyed when he or she buys a new car, how many cars does the auto industry expect to sell each year?

**(b)** Now assume that a person's old car is sold to somebody else when that person buys a new car. The person who buys the used car keeps it for period of time following an Erlang-3 distribution with a mean of 7 years. When that person buys another used car, his or her previous used car is assumed to be destroyed. Under the same previous assumptions, how many new cars does the auto industry expect to sell each year?

**1.16.** Aircraft enter a sector as shown in the following figure. The sector length is 50 nautical miles (nm). The spacing between aircraft as they enter the sector is 5 nm plus an exponentially distributed random variable with a mean of 1 nm. Suppose that aircraft travel at 400 knots (nautical miles per hour). What is the average number of aircraft in a sector?




**2.2.** Derive (2.9) of Section 2.2 by the sequential use of (2.8); then employ mathematical induction to prove (2.4).

**2.9.** Verify the forward and backward Kolmogorov equations (2.25a) and (2.25b) by using (2.24) in (2.23). [*Hint*: To obtain (2.25a), let $s = t + \Delta t$. To obtain (2.25b), let $u = t - \Delta t$.]

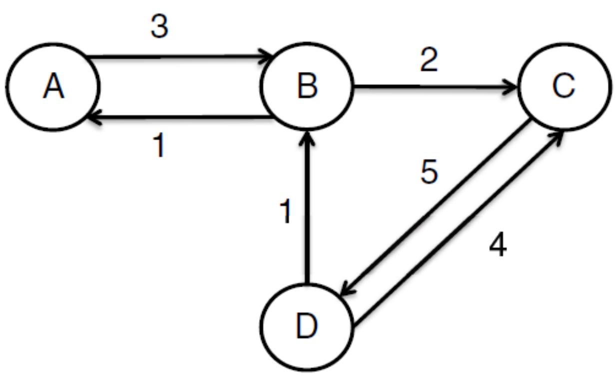
**2.12.** A certain software company has a technical support line. Requests for technical support arrive according to a Poisson process with rate $\lambda = 20$ per hour. What is the probability that:

**(a)** No calls arrive during 1 hour?

**(b)** Exactly 5 calls arrive during 1 hour?

**(c)** 5 or more calls arrive during 1 hour?

**2.13.** The following diagram represents a continuous-time Markov chain (where the numbers represent transition rates $q_{ij}$). Find the fraction of time the chain spends in each state.



**2.14.** Potential customers arrive at a one-pump gas station according to a Poisson process with rate 20 cars per hour. The amount of time required to service a car is exponentially distributed with a mean of five minutes. If there are three cars in the station (i.e., one at the pump and two in line), then arriving customers do not join the queue. Model this as a continuous-time Markov chain.

**(a)** Find the fraction of time spent in each state.
**(b)** What fraction of time is the pump being used?
**(c)** What fraction of potential customers are lost?

**2.15.** Customers arrive at a shuttle stop according to a Poisson process with rate 3 per hour. Shuttles arrive at the stop according to a Poisson process with rate 1.5 per hour. Suppose that each shuttle can hold at most 2 customers. Suppose that at most 4 people wait for the shuttle (subsequently arriving customers are turned away).

**(a)** Model this process as a continuous-time Markov chain. Give the rate transition matrix $\boldsymbol{Q}$.
**(b)** Give the probability transition matrix $\mathbf{P}$ of the embedded discrete-time Markov chain.
**(c)** Solve for the stationary probabilities $p_i$ (of the CTMC) and $\pi_i$ (of the embedded DTMC).
**(d)** What is the average number of customers who enter a shuttle?

**3.4.** What effect does simultaneously doubling $\lambda$ and $\mu$ have on $L$, $L_q$, $W$, and $W_q$ in an $M/M/1$ model?

**3.5.** For an $M/M/1$ model, derive the variance of the number of customers in the system in steady state. Use the generating function $P(z)$ in (3.15).

**3.8.** Parts arrive at a painting machine according to a Poisson process with rate $\lambda$/h. The machine can paint one part at a time. The painting process appears to be exponential with an average service time of $1/\mu$ h. It costs the company about $\$C_1$ per part per hour spent in the system (i.e., being painted or waiting to be painted). The cost of owning and operating the painting machine is strictly a function of its speed. In particular, a machine that works at an average rate of $\mu$ costs $\$\mu C_2$/h, whether or not it is always in operation. Determine the value of $\mu$ that minimizes the cost of the painting operation.

**3.13.** For the $M/M/1$ and $M/M/c$ queues, find $\mathrm{E}[T_q | T_q > 0]$, that is, the expected time one must wait in the queue, given that one must wait at all.

**3.18.** Show the following:

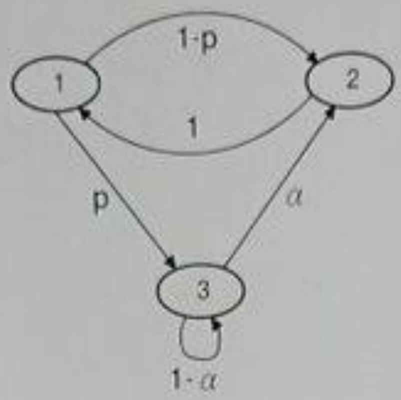
**(a)** An $M/M/1$ is always better with respect to $L$ than an $M/M/2$ with the same $\rho$.

**(b)** An $M/M/2$ is always better than two independent $M/M/1$ queues with the same service rate but each getting half of the arrivals.

**3.19.** For Problem 3.18(a), show that the opposite is true when considering $L_q$. In other words, faced with a choice between two $M/M$ systems with identical arrival rates, one with two servers and one with a single server that can work twice as fast as each of the two servers, which is the preferable system?

# Queueing Theory, Midterm Examination 1, Spring 2019

**1)** Consider the homogeneous discrete-parameter Markov chain whose state diagram is




a) (5%) Find **P**, the probability transition matrix.
b) (5%) Under what conditions (if any) will the chain be irreducible and aperiodic?
c) (5%) Solve for the stationary probability vector $\pi$.
d) (5%) What is the mean recurrence time for state 2?

**2)** (10%) Suppose $\{N(t),\ t \geq 0\}$ is a time homogeneous continuous-parameter Markov chain having infinitesimal generator $Q$ define as follows:

$$\begin{aligned} q_{ij} &= -\lambda, \quad \text{if } j = i \\ q_{ij} &= \lambda, \quad \text{if } j = i+1 \end{aligned}$$

Let $N(0) = 0$. Find the probability $\Pr\{N(t) = n\}$. Hint: The differential equation of the form

$$\frac{dy(x)}{dx} + \phi(x)y(x) = \psi(x)$$

has the solution

$$y(x) = ce^{-\int \phi(x)dx} + e^{-\int \phi(x)dx} \int e^{\int \phi(x)dx} \psi(x)dx$$

**3)** (10%) Show that Poisson arrivals implies exponential interarrival times.

**4)** (10%) Prove that if $X$ is exponentially distributed, then $X$ is memoryless.

**5)** Consider an M/M/c queue with parameters $\lambda$ and $\mu$.
(a) (5%) What is the average number of customers in the service facility?
(b) (5%) What is the probability that a server is busy?

**6)** Consider an M/M/1 queue with parameters $\lambda$ and $\mu$.
(a) (5%) Write down the global balance equations for $p_n$, $n \geq 0$.
(b) (10%) Derive the probability generating function $P(z)$.
(c) (5%) Find $p_0$ using $P(z)$.
(d) (5%) Find $p_n$ using $P(z)$.

**7)** (15%) Derive the cumulative probability distribution of the waiting time for the M/M/1 queue.
(Hint: $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$, probability density function of the Erlang-n distribution $f(x; n, \mu) = \frac{\mu(\mu x)^{n-1}}{(n-1)!} e^{-\mu x}$.)

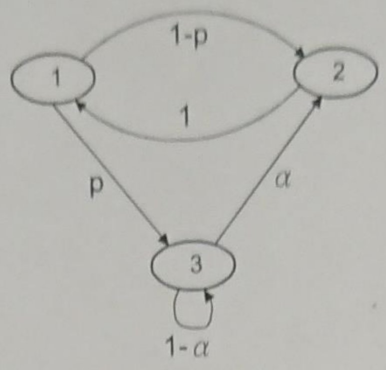
# Queueing Theory, Midterm Examination 1, Fall 2013

1) (5%) Show that Poisson arrivals implies exponential interarrival times.
2) (5%) Prove that if $X$ is exponentially distributed, then $X$ is memoryless.
3) Consider the homogeneous discrete-parameter Markov chain whose state diagram is

$p_{12} = e^{-\lambda}\sum_{n=0}^{2}\binom{1}{n}p^n q^{1-n}\frac{\lambda^{2-n}}{(2-n)!}$

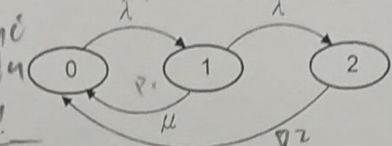
$= e^{-\lambda}\,1\cdot q\,\frac{\lambda^2}{2!} + e^{-\lambda}\,p\cdot\frac{\lambda}{1}$

$p_{12} = e^{-\lambda}(\lambda p + q\lambda^2)$




a) (5%) Find **P**, the probability transition matrix.
b) (5%) Under what conditions (if any) will the chain be irreducible and aperiodic?
c) (5%) Solve for the stationary probability vector $\pi$.
d) (5%) What is the mean recurrence time for state 2?

4) Consider the state transition diagram of a continuous-parameter Markovian queueing system shown below.




a) (5%) Solve for $p_k$, $k = 0,\ 1,\ 2$.
b) (5%) Find the average number in the system.

5) Consider a Markov chain with states $E_0, E_1, E_2, \cdots$ and with transition probabilities

$$p_{ij} = e^{-\lambda}\sum_{n=0}^{j}\binom{i}{n}p^n q^{i-n}\frac{\lambda^{j-n}}{(j-n)!}$$

where $p + q = 1\ (0 < p < 1)$.
a) (5%) Is this chain irreducible? periodic? Explain.
b) (5%) We wish to find $\pi_i$ = steady-state probability of $E_i$. Write $\pi_i$ in terms of $p_{ij}$ and $\pi_j$ for $j = 0,\ 1,\ 2,\ \cdots$.

$\binom{i}{n}p^n q^{i-n}$

6) Consider an M/M/c queue with parameters $\lambda$ and $\mu$.
(a) (5%) What is the average number of customers in the service facility?
(b) (5%) What is the probability that a server is busy?

7) Consider an M/M/1 queue with parameters $\lambda$ and $\mu$.
(a) (5%) Write down the global balance equations for $p_n$.
(b) (10%) Derive the probability generating function $P(z)$.
(c) (5%) Find $p_0$ using $P(z)$.
(d) (5%) Find $p_n$ using $P(z)$.

8) (5%) Derive the probability distribution of the total time, $W(t)$, that a customer spends in an M/M/1 system with parameters $\lambda$ and $\mu$.

9) Consider an M/M/2 queueing system where the average arrival rate is $\lambda$ customer per second and the average service time is $1/\mu$ sec, where, $\lambda < 2\mu$.
a) (5%) Find the differential-difference equations that govern the time-dependent probabilities $p_k(t)$.
b) (5%) Find the stationary probabilities, $p_k$.

# CS5531 Queueing Theory, Midterm Examination 1 , Fall 2012

1) Consider an M/M/c queue with parameters $\lambda$ and $\mu$.

(a) (5%) What is the average number of customers in the service facility? $\frac{\lambda}{\mu}$ $\frac{\lambda}{\mu}$

(b) (5%) What is the probability that a server is busy? $\frac{\lambda}{c\mu}$ - server 가 바쁜 $\frac{\lambda}{3c\mu} = P_{busy} \times 1 + 0 \times P_{i}$

server is busy $c=3$

2) The Outffront BBQ Haven does carry out only. During peak periods, three servers are on duty. The owner notices that during these periods, the servers are almost never idle. She estimates the percentage idle time of each server to be 1 percent. Ideally, the percentage idle time would be 10 percent.

(a) (5%) If the owner decides to add a fourth server during these times, what would be the percentage idle time of each server?

(b) (5%) Suppose that by adding the fourth server, the pressure on the servers is reduced, so that they can work more carefully, but their $\boxed{\text{service output}}$ rate is reduced by 10 percent. What now is the percent time each would be idle? $\mu$ $\mu$ $\mu \to 0.9\mu$

(c) (5%) Suppose, instead, the owner decides to hire an aid (at a much lower salary) who serves as a gofer for the three servers, rather than hiring another full server. This allows the three servers to decrease their average service time by 10 percent (relative to the original service rate). What now is the percent idle time of each of the three servers? $\to \frac{1}{\mu}$

3) (10%) Show that Poisson arrivals implies exponential interarrival times.

4) (10%) Prove that if $X$ is exponentially distributed, then $X$ is memoryless.

$W_q(t) = Pr\{T_q \geq t\}$

5) Consider an M/M/1 queue with parameters $\lambda$ and $\mu$.

(a) (5%) Write down the global balance equations for $p_n$.

(b) (10%) Derive the probability generating function $P(z)$.

(c) (5%) Find $p_0$ using $P(z)$.

(d) (5%) Find $p_n$ using $P(z)$.

$P_n = \begin{cases} 1-\rho, & n=0 \\ \rho^n(1-\rho), & n \geq 1 \end{cases}$

$e^{-\mu t(1-\rho)}$

6) (10%) Derive the probability distribution of the total time, $W(t)$, that a customer spends in an M/M/1 queue with parameters $\lambda$ and $\mu$.

7) Consider a birth-death process in which $\lambda_n = \lambda, 0 \leq n \leq k$, $\lambda_n = 2\lambda, n \geq k+1$, and $\mu_n = \mu, n \geq 1$.

a) (10%) Write down the stationary equations.

b) (10%) Find the stationary probabilities $p_n, n \geq 0$.

# Queueing Theory, Midterm Examination 1, Fall 2011

1) (10%) Show that Poisson arrivals implies exponential interarrival times.

0.99
3
2.97

2) (10%) Prove that if $X$ is exponentially distributed, then $X$ is memoryless.

3) Consider an M/M/c queue with parameters $\lambda$ and $\mu$.

(a) (5%) What is the average number of customers in the service facility?

(b) (5%) What is the probability that a server is busy?

4) The Outffront BBQ Haven does carry out only. During peak periods, three servers are on duty. The owner notices that during these periods, the servers are almost never idle. She estimates the percentage idle time of each server to be 1 percent. Ideally, the percentage idle time would be 10 percent.

(a) (5%) If the owner decides to add a fourth server during these times, what would be the percentage idle time of each server?

(b) (5%) Suppose that by adding the fourth server, the pressure on the servers is reduced, so that they can work more carefully, but their service output rate is reduced by 10 percent. What now is the percent time each would be idle?

$\mu' = 0.9\mu$

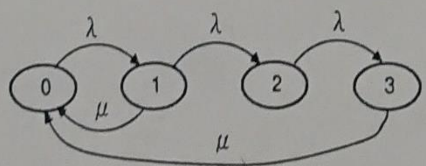
(c) (5%) Suppose, instead, the owner decides to hire an aid (at a much lower salary) who serves as a gofer for the three servers rather than hiring another full server. This allows the three servers to decrease their average service time by 10 percent (relative to the original service rate). What now is the percent idle time of each of the three servers?

5) (5%) A queueing system is being observed. We see that all $c$ ($c \geq 2$) identical exponential servers are busy, with $k$ more customers waiting, and decide to shut off the arrival stream. On the average, how long will it take until there are exactly $c - 2$ customers in the system?

$\frac{c+k-(c-1)}{c\mu} + \frac{1}{(c-1)\mu}$

0.74
400 | 2920
2800
1700

6) Consider the state transition diagram of the Markov chain shown below.




0.99
0.9
0.89

(a) (5%) Write down the stationary equations.

(b) (5%) Solve for stationary probabilities $p_k$, $k = 0, 1, 2, 3$.

(c) (5%) Find the average number in the system.

7) (10%) Draw the state-transition diagram for the imbedded discrete-parameter Markov chain induced by considering the state transition instances of a birth-death process with birth rate $\lambda$ and death rate $\mu$.

8) Consider an M/M/1 queue with parameters $\lambda$ and $\mu$.

(a) (5%) Write down the global balance equations for $p_n$, $n \geq 0$.

(b) (10%) Derive the probability generating function $P(z)$.

(c) (5%) Find $p_0$ using $P(z)$.

(d) (5%) Find $p_n$ using $P(z)$.

0.7425
400 | 2970
2800