# COM 5120 Communications Theory

# Chapter 6 Information Theory

Prof. Jen-Ming Wu
Inst. of Communications Engineering
Dept. of Electrical Engineering
National Tsing Hua University
Email:jmwu@ee.nthu.edu.tw

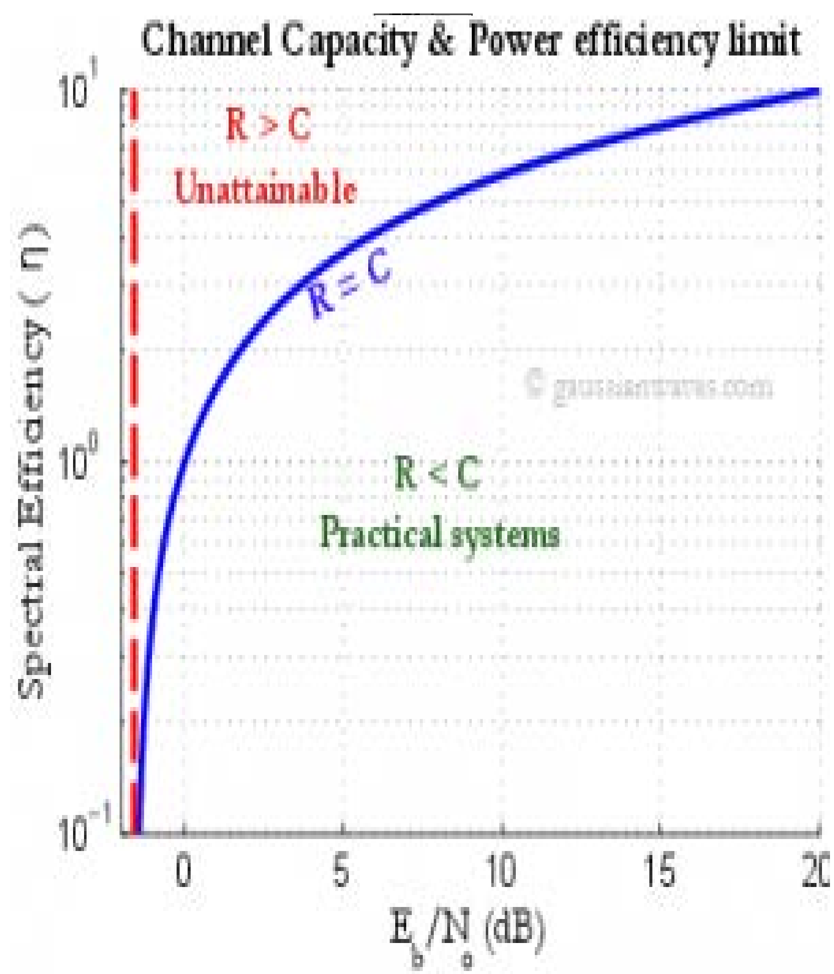
Fall, 2019

# Why Information Theory?

Information theory deals with mathematical modeling and analysis of a communication system. It tries to answer the following questions:

- What is the irreducible complexity that below which a signal source can not be further compressed?
- What is the ultimate transmission rate for reliable communication over a noisy channel?

$$R \uparrow \quad P_e \uparrow \quad \Rightarrow unreliable$$



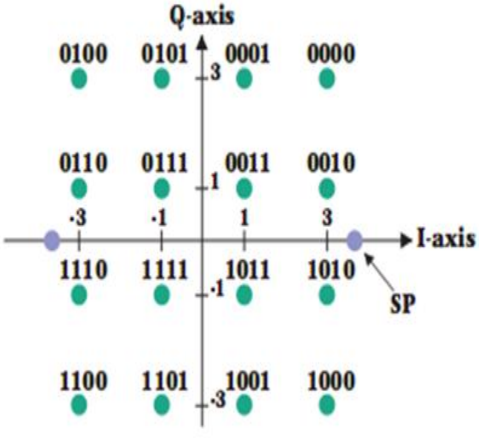
# 6.1 Mathematical Models for Information Source

- Discrete Memoryless Source (DMS)

  Assume a discrete information source $\{x_1, x_2, \ldots, x_K\}$ each source has a given probability of $p_k$, $1 \le k \le K$

  where $\sum_{k=1}^{K} p_k = 1$

$\Rightarrow$ A discrete source with statistically independent output sequence is called discrete memoryless source (DMS)



# 6.2 Measure of Information


- How to measure information?

Given a DMS $X \in \{x_1, x_2, \ldots, x_K\}$, the amount of information for $X = x$ is inverse proportional to its probability $\mathrm{P}(x)$, defined as:

$$I(x) = \log_2 \frac{1}{\mathrm{P}(x)} = -\log_2 \mathrm{P}(x) \quad \text{(bits)}$$

$$= \ln \frac{1}{\mathrm{P}(x)} \quad \text{(nats)}$$

$P(x) \uparrow \quad I(x) \downarrow$

# Properties of Information

For the DMS $\{x_1, \ldots, x_K\}$,

(1) Zero informatin event: $I(x_k) = 0$ for $p_k = 1$

$\Rightarrow$ Absolute certainty of the outcome about an event

$\Rightarrow$ No information gains for the message

(2) Non-negativity: $I(x_k) \geq 0 \quad \because \; 0 \leq p_k \leq 1$

Given message $X = x_k$, always produce some info or no info.

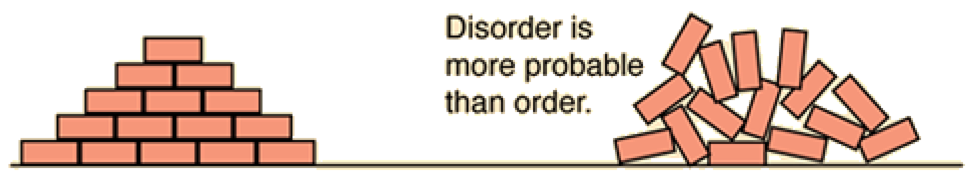
Never bring about a loss of info. (non-negative)

(3) $I(x_k) > I(x_j)$ for $p_k < p_j$

# Entropy of Information

- Definition: Entropy represents the mean value of information per source symbol

$$H(X) = \mathrm{E}[I(x_k)] = \sum_{k=1}^{K} p_k I(x_k)$$




$\Rightarrow$ Entropy is a measure of uncertainty about $X$

➢ Entropy is used to describe the degree of randomness in a system.

- Properties

$$(1) H(X) = 0 \text{ iff } \begin{cases} p_k = 1 & \text{for } k = k^* \\ p_k = 0 & \text{other} \end{cases} \quad \Rightarrow \text{no uncertainty}$$

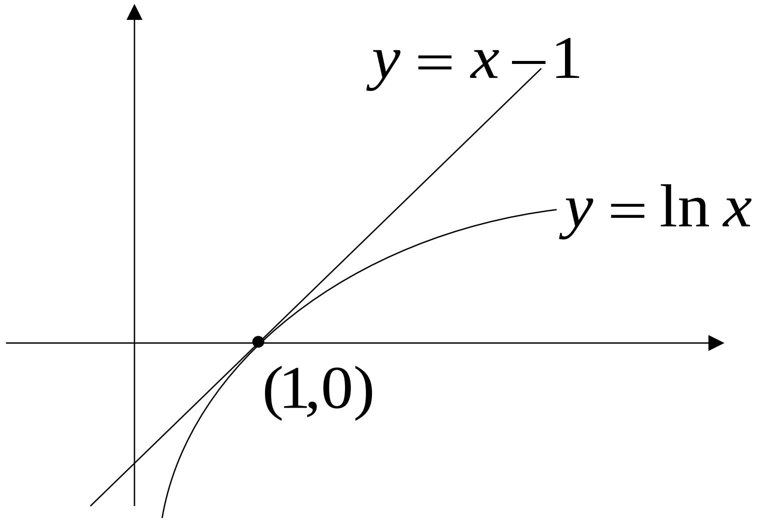
$$(2) H(X) = \log_2 K \text{ iff } p_k = \frac{1}{K} \text{ for all } K$$

$\Rightarrow$ uniform prob. distribution leads to max uncertainty

## (3)The entropy is bounded: $0 \le H(X) \le \log_2 K$

*Proof :*

$$
\begin{aligned}
& H(X) - \log_2 K \\
&= \sum_{k=1}^{K} p_k \log_2 \frac{1}{p_k} - \log_2 K \\
&= \sum_{k=1}^{K} p_k \log_2 \frac{1}{p_k} - \sum_{k=1}^{K} p_k \log_2 K \\
&= \sum_{k=1}^{K} p_k \log_2 \frac{1}{K p_k} = \frac{1}{\ln 2} \sum_{k=1}^{K} p_k \ln \frac{1}{K p_k}
\end{aligned}
$$




By the inequality $\ln x \le x - 1$

$$
\begin{aligned}
\Rightarrow H(X) - \log_2 K &\le \frac{1}{\ln 2} \sum_{k=1}^{K} p_k \left(\frac{1}{K p_k} - 1\right) \\
&= \frac{1}{\ln 2} \sum_{k=1}^{K} \left(\frac{1}{K} - p_k\right) = 0 \quad \therefore H(X) \le \log_2 K
\end{aligned}
$$

- Joint Entropy:

The entropy of a pair of random variables $X, Y$

$$H(X,Y) = -\sum_x \sum_y \mathrm{P}(x,y) \log_2 \mathrm{P}(x,y)$$

i.e. the mean value of joint information

- Conditional Entropy:

Given $X = x$, the entropy of $Y$ is

$$H(Y \mid x) = -\sum_y \mathrm{P}(y \mid x) \log_2 \mathrm{P}(y \mid x)$$

i.e. the mean value of conditional information

- The average conditional entropy over all possible values of $X$ is

$$H(Y \mid X) = \sum_x \mathrm{P}(x) H(Y \mid x) = -\sum_x \sum_y \mathrm{P}(x,y) \log_2 \mathrm{P}(y \mid x)$$

$$Lemma\ 1: H(X,Y) = H(X) + H(Y \mid X)$$

*Proof:*

$$H(Y|X) = -\sum_x \sum_y \mathrm{P}(x,y) \log_2 \mathrm{P}(y \mid x)$$

$\mathrm{P}(y \mid x) = \mathrm{P}(x,y) / \mathrm{P}(x)$

$$= -\sum_x \sum_y \mathrm{P}(x,y) \log_2 \mathrm{P}(x,y)$$

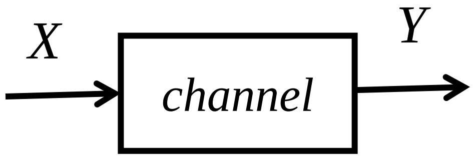
$$+ \sum_x \sum_y \mathrm{P}(x,y) \log_2 \mathrm{P}(x)$$

$$= H(X,Y) - H(X)$$

$$\therefore H(X,Y) = H(X) + H(Y|X)$$

# Mutual Information

The information provided by the occurrence of $Y = y$ about the event $X = x$ , is defined by




$$I(x, y) = \log_2 \frac{P(x|y)}{P(x)} = \log_2 \frac{P(x, y)}{P(x)P(y)}$$

called mutual information between $x$ and $y$.

Note: If X and Y are independent random variables, the occurrence of $Y = y$ provides no information about the occurrence of $X = x$. $\rightarrow I(x,y) = ?$

# Mutual Information

- $X, Y$ are independent:


$$P(x\,|\,y) = \frac{P(x)P(y)}{P(y)} = P(x) \;\Rightarrow I(x, y) = 0$$

- $X, Y$ are fully dependent:

$$P(x\,|\,y) = 1, \;\; \Rightarrow I(x, y) = I(x)$$

- The mutual information about $X, Y$ is

$$I(X, Y) = \sum_x \sum_y P(x, y) I(x, y)$$

$$= \sum_x \sum_y P(x, y) \log_2 \frac{P(x\,|\,y)}{P(x)} = \sum_x \sum_y P(x, y) \log_2 \frac{P(y\,|\,x)}{P(y)}$$

- Properties of Mutual Information

(1). $I(x,y) = I(y,x)$ $\because I(x,y) = \log_2 \frac{P(x,y)}{P(x)P(y)}$

(2). $I(x,y) \geq 0$, "=" holds when X,Y are independent

(3). $I(X,Y) \leq \min\{K_x, K_y\}$

where $K_x = size(X), K_y = size(Y)$

# Mutual Information and Entropy

$$Lemma\ 2: I(X,Y) = H(X) - H(X \mid Y) =$$ Reduction of uncertainty about $X$ after observing $Y$

Uncertainty of source (average information of $X$)

Uncertainty of X given Observation of Y

$$= \text{Amount of information about X in Y}$$


*Proof:*

$$I(X,Y) = \sum_x \sum_y \mathrm{P}(x,y) \log_2 \frac{\mathrm{P}(x \mid y)}{\mathrm{P}(x)}$$

$$= \underbrace{\sum_x \sum_y \mathrm{P}(x,y) \log_2 \mathrm{P}(x \mid y)}_{-H(X \mid Y)} - \underbrace{\sum_x \sum_y \mathrm{P}(x,y) \log_2 \mathrm{P}(x)}_{H(X)}$$

# Mutual Information and Entropy

$$Lemma\ 3:\quad I(X,Y) \le \min\{H(X), H(Y)\}$$

$$\Pr oof:\quad \because I(X,Y) = H(X) - H(X\,|\,Y)\ \ \text{(Lemma 2)}$$

$$\because\ \ I(X,Y) \le H(X)$$

$$I(X,Y) \le H(Y)$$

$$\rightarrow\ \ I(X,Y) \le \min\{H(X), H(Y)\}$$

Furthermore,

$$H(X) \le \log K_x$$

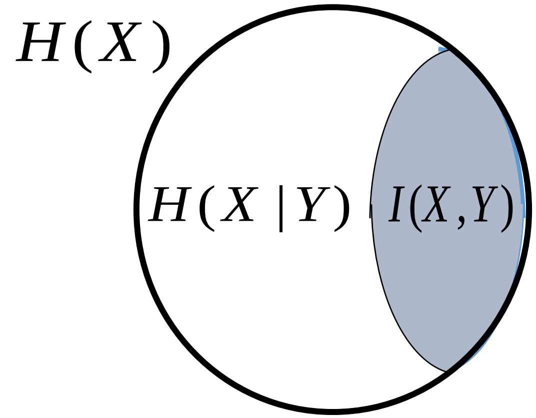
$$H(Y) \le \log K_y$$

$$\therefore I(X,Y) \le \min\{\log K_x, \log K_y\}$$

$$\rightarrow I(X,Y) \le \min\{K_x, K_y\}$$

# Remarks

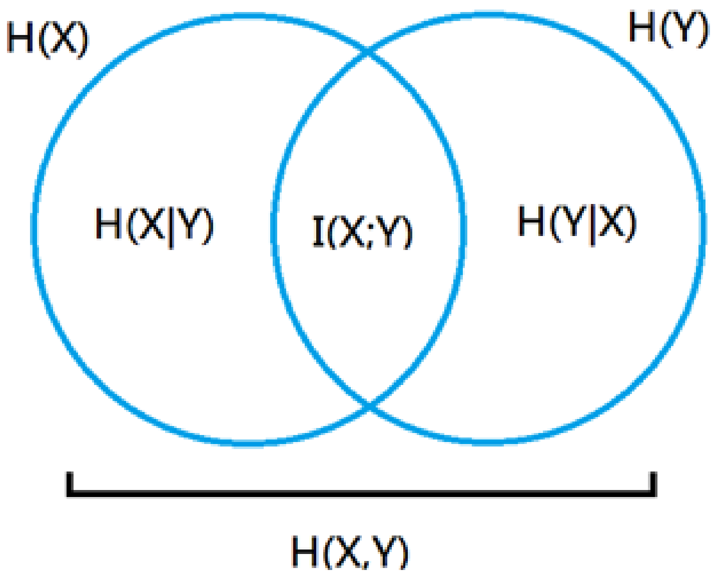
$$I(X,Y) = H(X) - H(X|Y)$$


- To decode the transmitted message correctly, the conditional entropy $H(X|Y)$ has to be minimized.
- Consequently, the mutual information is maximized if $H(X|Y)=0$, then $I(X,Y)=H(X)$.
  $\Rightarrow$ The mutual information provides all the uncertainty about $X$.

# Summary

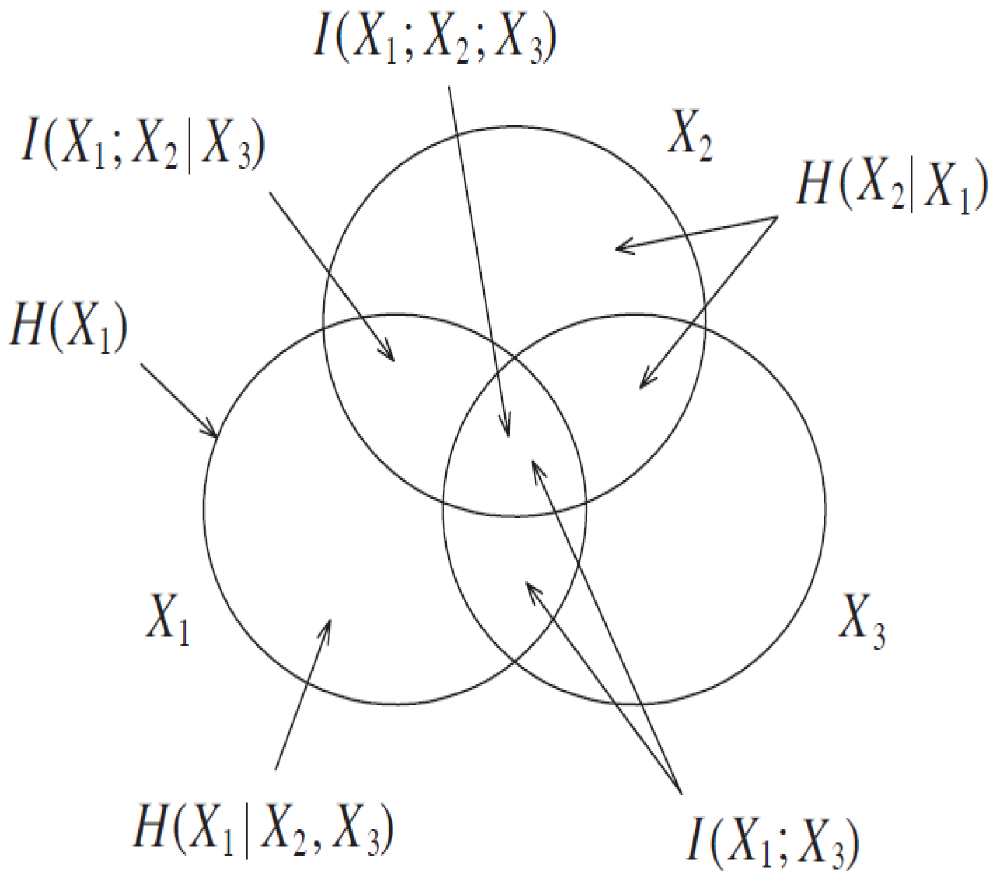
- Given r.v.'s X and Y



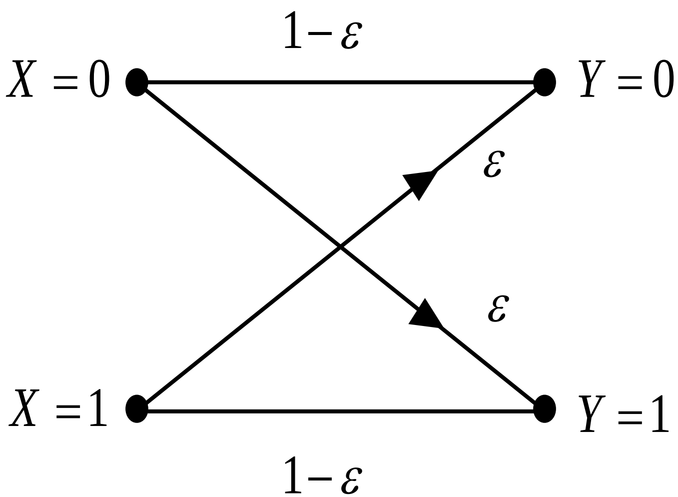

$$I(X,Y) = H(X) - H(X \mid Y)$$
$$= H(Y) - H(Y \mid X)$$

$$H(X,Y) = H(X) + H(Y \mid X)$$

- Given r.v.'s $X_1$ , $X_2$ and $X_3$



# Binary Symmetric Channel (BSC)




$$X \in \{0,1\}$$

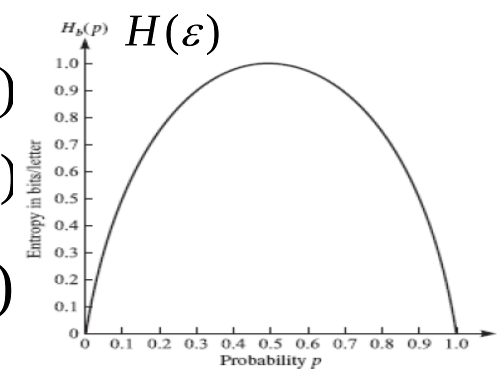
$$\mathrm{P}(Y=0 \mid X=1) = \mathrm{P}(Y=1 \mid X=0) = \varepsilon$$

$$\mathrm{P}(Y=1 \mid X=1) = \mathrm{P}(Y=0 \mid X=0) = 1-\varepsilon$$

(1) $H(X)$ is maximized when the source prior probability

$$\mathrm{P}(x) = \tfrac{1}{2} \quad \forall x = 0,1 \qquad \Rightarrow H(X) = -\frac{1}{2}\log_2\frac{1}{2} - \frac{1}{2}\log_2\frac{1}{2} = 1$$

$$\begin{aligned}(2)\, H(X|Y) &= -\sum_x \sum_y \mathrm{P}(x,y)\log_2 \mathrm{P}(x \mid y) \\ &= -p(0,0)\log_2 p(0|0) - p(1,1)\log_2 p(1|1) \\ &\quad - p(1,0)\log_2 p(1|0) - p(0,1)\log_2 p(0|1) \\ &= -(1-\varepsilon)\log_2(1-\varepsilon) - \varepsilon\log_2\varepsilon = H(\varepsilon)\end{aligned}$$

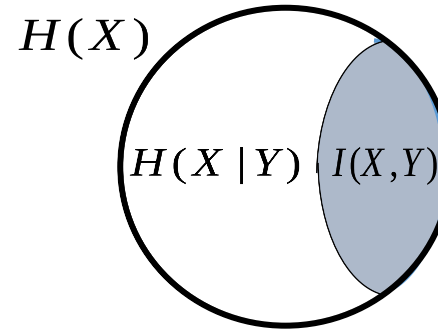


# Capacity of Binary Symmetric Channel (BSC)

- Channel Capacity is defined as the max mutual information being transmitted over the channel.


$$C = \max\{I(X,Y)\}$$

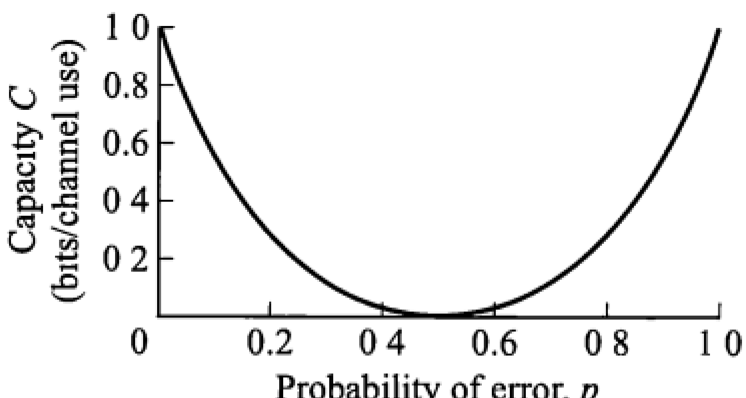
$$i.e.C = \max\{H(X) - H(X \mid Y)\}$$




From (1) and (2)

$$C = 1 + \varepsilon \log \varepsilon + (1-\varepsilon)\log(1-\varepsilon)$$

$$= 1 - H(\varepsilon) \qquad 0 \le \varepsilon \le 1$$




$C$ is maximized when

$$\varepsilon = 0 \text{ or } 1$$

$C$ is minimized when

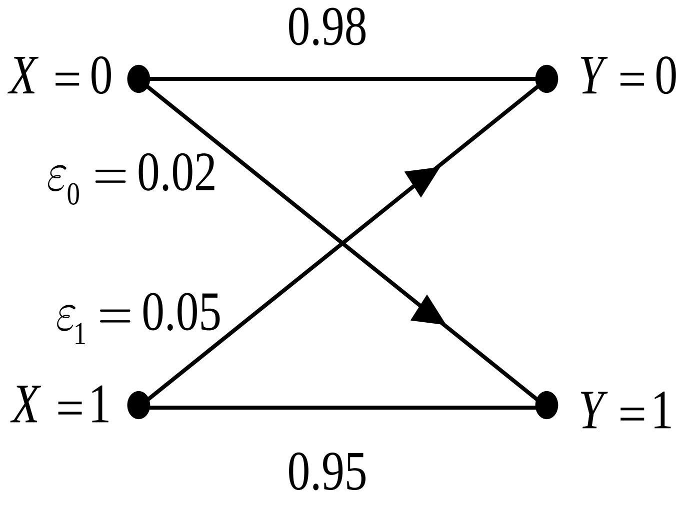
$$\varepsilon = \frac{1}{2}$$

# Mutual information of Binary Channel

**Example**: Let **X** be a binary source which has equal probable symbol $\{0, 1\}$. Let **Y** be a binary output$\{0, 1\}$. The channel has transition probability matrix

$$P_{ch} = \begin{bmatrix} 0.98 & 0.02 \\ 0.05 & 0.95 \end{bmatrix}$$

Calculate the mutual information of this channel. (Known $H(Y) = 0.9994$ and $H(X) = 1$)




**Solution:**

$$I(X, Y) = \sum_{i,j} P(x_i, y_j) log_2 \left[ \frac{P(x_i/y_j)}{P(x_i)} \right] = 0.7854$$

- Entropy

  Given discrete random variable $X$,

$$H(X) = \sum_{k=1}^{K} p_k \log_2 \frac{1}{p_k}.$$

- Differential Entropy

  Given continuous random variable $X$,

$$H(X) = \int_{-\infty}^{\infty} f(x) \log_2 \frac{1}{f(x)} dx,$$

# Capacity of DMC over AWGN

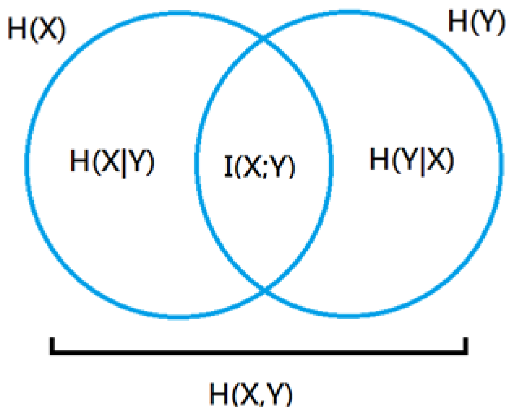
- Consider transmission of random variable $X_k, k=1,\ldots,K$ over AWGN channel with zero mean and power spectral density $(\mathrm{PSD}) = \frac{N_0}{2}$. The received r.v. is $Y_k = X_k + N_k$,

- The transmitted power is limited to $E[X_k^2] = P$, and the noise power is $\frac{N_0}{2} \times 2W = N_0 W$ with transmission bandwidth 2W.

- The Capacity of DMC over AWGN

$$C = \max I(X_k, Y_k)$$
$$= \max\{\boxed{H(X_k)} - H(X_k \mid Y_k)\}$$

Unknown for receiver




By the duality, $I(X_k, Y_k) = I(Y_k, X_k)$

$$\therefore C = \max\{H(Y_k) - H(Y_k \mid X_k)\}$$

For Rx, only $H(Y_k)$ is available, not $H(X_k)$

Also, $Y_k = X_k + N_k$

$$\Rightarrow H(Y_k \mid X_k) = H(X_k + N_k \mid X_k)$$
$$= H(X_k \mid X_k) + H(N_k \mid X_k) = H(N_k)$$

$$\Rightarrow C = I(X_k, Y_k) = H(Y_k) - H(N_k)$$

- The Capacity of DMC over AWGN

$$H(Y_k) = E[I(Y_k)] = -\int_{-\infty}^{\infty} f_{Y_k}(y_k)\log_2 f_{Y_k}(y_k)dy_k$$

$$\text{where } f_{Y_k}(y_k) = \frac{1}{\sqrt{2\pi\sigma_y^2}}\exp\left(\frac{-y_k^2}{2\sigma_y^2}\right)$$

$$\sigma_Y^2 = \mathrm{E}[X_k^2] + \sigma_N^2 = P + \frac{N_0}{2} \quad \text{Why?} \quad \because Y_k = X_k + N_k, \therefore \sigma_Y^2 = \sigma_X^2 + \sigma_N^2$$

$$\Rightarrow H(Y_k) = -\int_{-\infty}^{\infty} f_{Y_k}(y_k)[-\ln(\sqrt{2\pi\sigma_y^2}) - \frac{y_k^2}{2\sigma_y^2}](\log_2 e)dy_k$$

$$\text{Note } \int_{-\infty}^{\infty} f_{Y_k}(y_k)dy_k = 1$$

$$\int_{-\infty}^{\infty} {y_k}^2 f_{Y_k}(y_k)dy_k = \sigma_Y^2 = P + \frac{N_0}{2}$$

- The Capacity of DMC over AWGN

$$\Rightarrow H(Y_k) = -\int_{-\infty}^{\infty} f_{Y_k}(y_k)[-\ln(\sqrt{2\pi\sigma_y^2}) - \frac{y_k^2}{2\sigma_y^2}](\log_2 e)dy_k$$

$$= (\log_2 e)\left\{\ln(\sqrt{2\pi\sigma_y^2})\int_{-\infty}^{\infty} f_{Y_k}(y_k)dy_k + \frac{1}{2\sigma_y^2}\int_{-\infty}^{\infty} y_k^2 f_{Y_k}(y_k)dy_k\right\}$$

$$= \log_2 e[\ln(\sqrt{2\pi\sigma_y^2}) + \frac{1}{2}] = \log_2 e[\frac{1}{2}\ln(2\pi\sigma_y^2) + \frac{1}{2}\ln e]$$

$$= \log_2 e[\frac{1}{2}\ln(2\pi e\sigma_y^2)] = \frac{1}{2}\log_2(2\pi e\sigma_y^2)$$

Similarly,

$$H(N_k) = -\int_{-\infty}^{\infty} f_{N_k}(n_k)\log_2 f_{N_k}(n_k)dn_k = \frac{1}{2}\log_2(2\pi e\sigma_n^2)$$

$$C = H(Y_k) - H(N_k)$$

$$= \frac{1}{2}\log_2 2\pi e(P+\sigma_n^2) - \frac{1}{2}\log_2 2\pi e\sigma_n^2$$

$$= \frac{1}{2}\log_2(1+\frac{P}{\sigma_n^2}) \qquad \Rightarrow C = \frac{1}{2}\log_2(1+SNR) \quad \square$$

# Geometric Interpretation of Channel Capacity

- The capacity of discrete-time AWGN channel with input power constraint:

Let $Y_i = X_i + N_i, \quad i = 1, \ldots, n,$ where $X_i$ is the transmission symbol at basis $\phi_i(t)$.

$N_i \sim \mathcal{N}(0, \sigma_n^2)$ and the input power of $X_i$ is constrained, $\mathrm{E}[X_i^2] \le P$

- Transmission of $\underline{x}$ with N-dimension: $\underline{x} = [X_1, \ldots, X_{\mathrm{N}}]^T$

  The received signal vector is $\underline{y} = \underline{x} + \underline{n} = [Y_1, \ldots, Y_{\mathrm{N}}]^T$

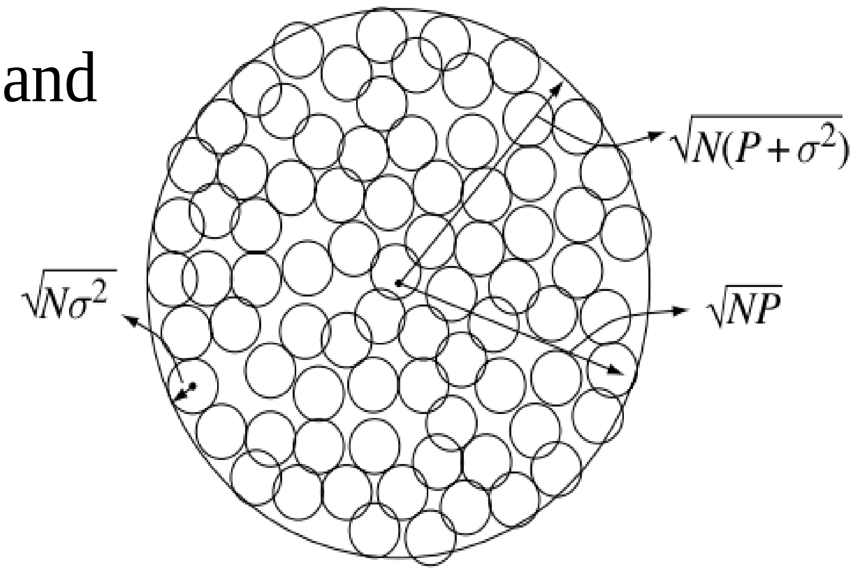
  The average received power on each dimension:

$$\frac{1}{\mathrm{N}} \left\| \underline{y} \right\|^2 = \mathrm{E}[X_i^2] + \mathrm{E}[N_i^2] \le P + \sigma_n^2$$

# Geometric Interpretation of Channel Capacity

- $\underline{y}$ is inside the N-dimensional sphere of radius $\sqrt{\mathrm{N}(P+\sigma_n^2)}$

- If $\underline{x}$ is transmitted, $\underline{y}$ will be in an N-dimension sphere of radius $\sqrt{\mathrm{N}\sigma_n^2}$ and centered at $\underline{x}$ with high probability.




- The max number of spheres with radius $\sqrt{\mathrm{N}\sigma_n^2}$ that can be packed in a sphere of radius $\sqrt{\mathrm{N}(P+\sigma_n^2)}$ is the ratio of the volumes of the sphere.

# Geometric Interpretation of Channel Capacity

The volume of a $N$-dimension sphere with radius $\gamma$

$$V_{\mathrm{N}} = B_{\mathrm{N}}\gamma^{\mathrm{N}} \propto \gamma^{\mathrm{N}}$$

$$Ex:\ N = 2, V_2 = \pi\gamma^2 \quad N = 3, V_3 = \frac{4}{3}\pi\gamma^3$$

⇒ The maximum number of different messages (or symbols) that can be resolvable at the receiver is

$$M = \frac{B_N\left(\sqrt{N(P+\sigma_n^2)}\right)^N}{B_N\left(\sqrt{N\sigma_n^2}\right)^N} = (1+\frac{P}{\sigma_n^2})^{\frac{N}{2}}$$

- The resulting transmission bit rate at each dimension:

$$R = \frac{1}{N}\log_2 M = \frac{1}{2}\log_2(1+\frac{P}{\sigma_n^2}) \ \text{ bits/transmission (symbol)}$$

# The Capacity of Band-limited AWGN Channel (Continuous Time)

Given channel bandwidth $2W$, input power constraint $P$ and noise power spectral density $\frac{N_0}{2}$

- The capacity of discrete-time channel in bits/transmission

$$C = \frac{1}{2}\log_2(1+\frac{P}{N_0 W})$$

bits/transmission
(or bits/symbol)

- The capacity of discrete-time channel in bits/sec

$$C = 2W\frac{1}{2}\log_2(1+\frac{P}{N_0 W}) = W\log_2(1+\frac{P}{N_0 W}) \quad \text{bits/sec}$$

# How to Improve the Channel Capacity?

- Primary Communication Resources
  - ✓Transmitted Power (Signal-to-Noise Ratio, SNR)
  - ✓Channel Bandwidth (sampling rate, and noise power)

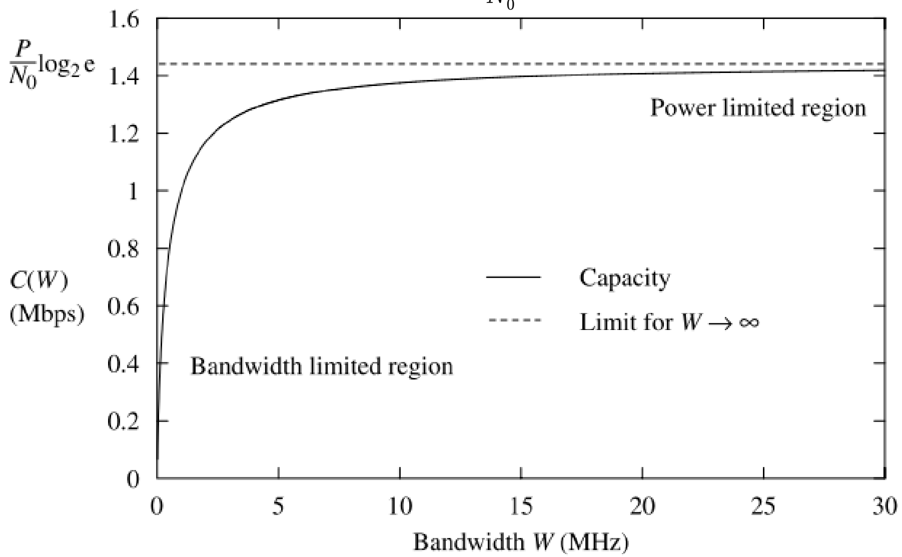
$$C = W\log_2(1+\frac{P}{N_0 W})$$

$$\ln(1+x) \cong x, \quad x << 1$$
$$\log_2(1+x) \cong (\log_2 e)x$$

(1) Fixed *P* and increase *W*

$$\text{As } W\uparrow \ \Rightarrow \mathrm{C}_\infty = \lim_{W\to\infty} W\log_2(1+\frac{P}{N_0 W}) = \frac{P}{N_0}\log_2 e$$

$$\text{Ex}: \frac{P}{N_0} = 10^6$$




➢ With infinite bandwidth, the channel capacity can not increase definitely!

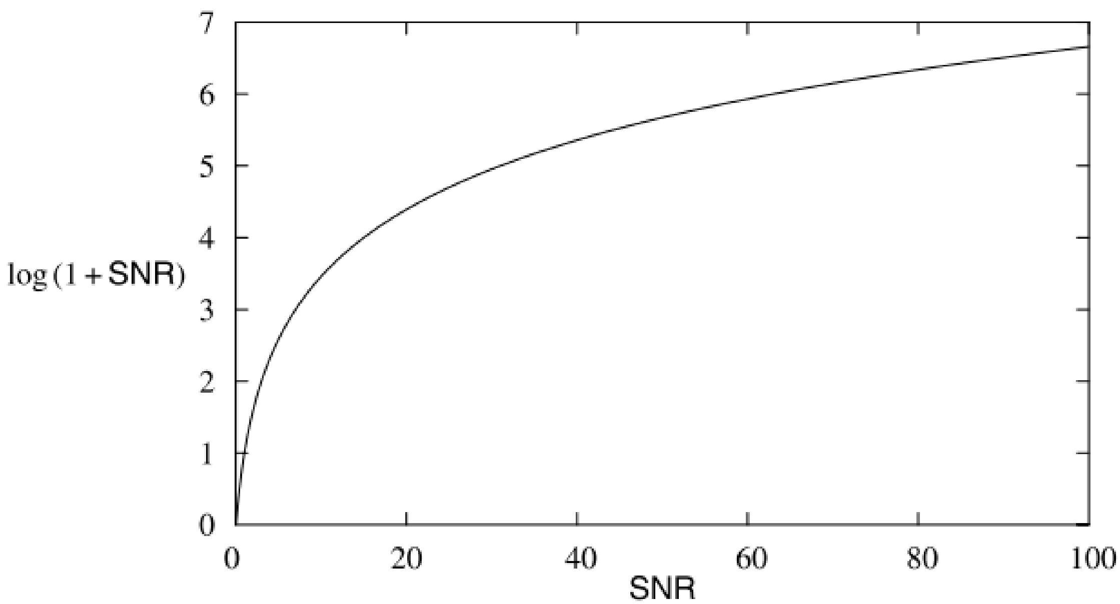
# How to Improve the Channel Capacity?

$$C = W \log_2 (1 + \frac{P}{N_0 W})$$

(2) Increase $P$ with fixed $W$.

$$\text{As } P \uparrow \;\; \Rightarrow \frac{P}{N_0 W} >> 1 \;\Rightarrow C \cong W \log_2 \frac{P}{N_0 W}$$

- $C$ increases at a logarithmic rate

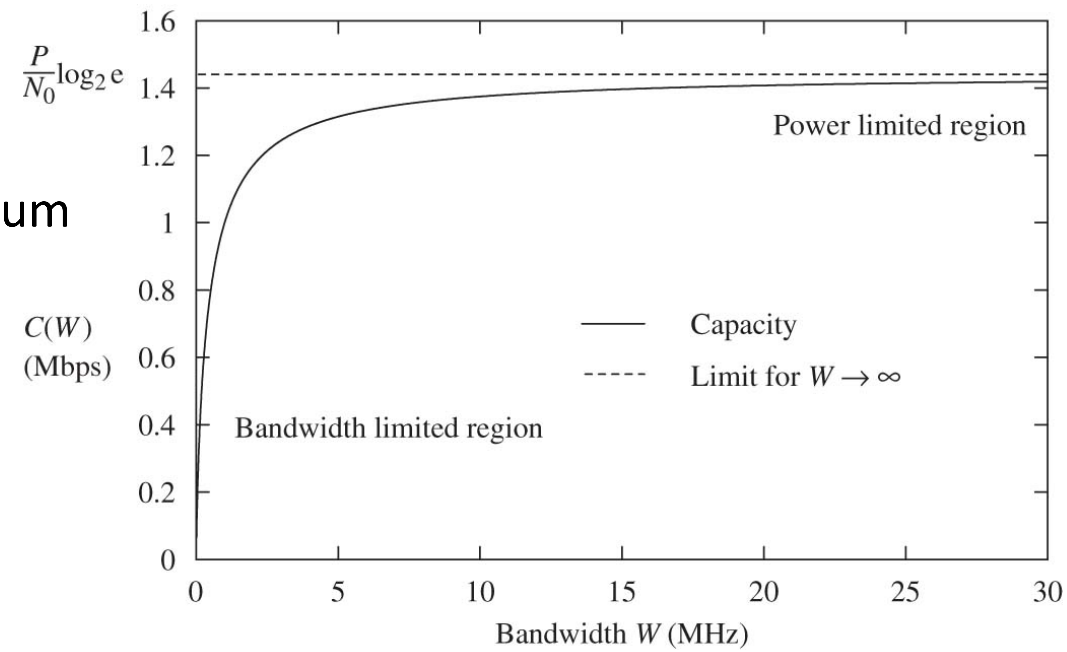



- Power inefficient at high SNR region !

# How to Improve the Channel Capacity?

- Types of Communication Channels
  - ✓ Power Limited
    - Noise
    - Finite Power
    - Propagation Loss
  - ✓ Band Limited
    - Freq response of medium
    - Multiple users share the medium

$$C = W \log_2 (1 + \frac{\overline{P}}{N_0 W})$$



# Fundamental Limit and Relation of Power Efficiency and Bandwidth Efficiency

## (1) Power efficiency

Let the power efficiency be defined as $\gamma = \dfrac{P}{C}$ (J/bit)

$$\min \frac{P}{C} = \lim_{W \to \infty} \frac{P}{C} = \lim_{W \to \infty} \frac{P}{W \log_2 (1 + \frac{P}{N_0 W})}$$

$$= \frac{P}{\frac{P}{N_0} \log_2 e} = N_0 \ln 2 \quad \to \text{(A)}$$

$$E_b = \frac{E_s}{\log_2 M} = \frac{P T_s}{\log_2 M} = \frac{P}{R} \quad \Rightarrow R = \frac{P}{E_b} \quad \to \text{(B)}$$

$\dfrac{T_s}{\log_2 M} = T_b$, $R \leftarrow$ bit rate

For reliable communications $\Rightarrow R \leq C$

$$\Rightarrow \frac{P}{E_b} \leq C \quad \Rightarrow \frac{P}{C} \leq E_b \overset{(A)(B)}{\Rightarrow} N_0 \ln 2 \leq E_b \Rightarrow \underbrace{\frac{E_b}{N_0}}_{\text{SNR per bit}} \geq \ln 2 \ (\cong 0.693 = \text{-}1.6\text{dB})$$

# What's the minimum energy to transmit one bit at room temperature?

- Shannon-von Neumann-Landauer Limit

$$N_0 \ln 2 \leq E_b$$

The noise power $N_0 = kT$ where
$k$ = Boltzmann constant = $1.38 \times 10^{-23}$ (J/K)

- At $T$ = 300K (= 27°C), the min energy required to transmit <u>one bit</u> is

$$E_{b,\text{mm}} = kT \ln 2 = 1.38 \times 10^{-23} \times 300 \times \ln 2$$

$$= 2.9 \times 10^{-21} \text{(J/bit)}$$

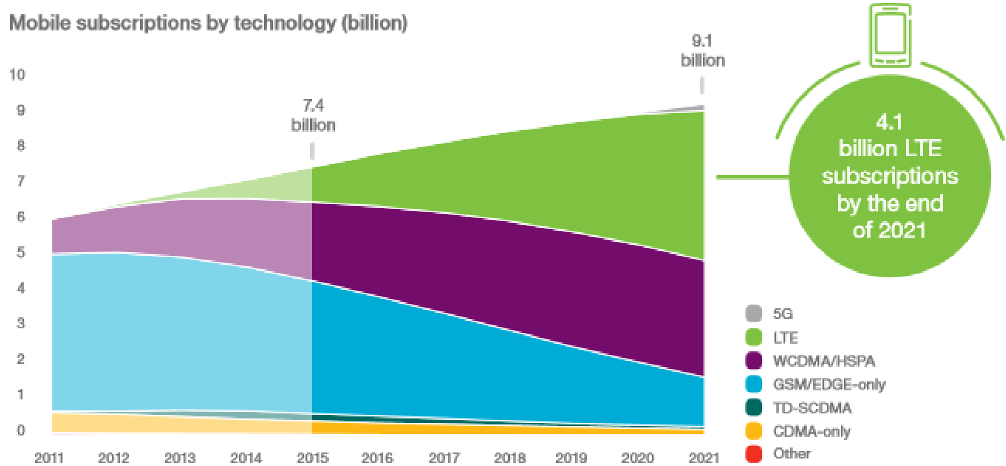
# Communication Energy Efficiency

- Wireless communication performance under energy constraints
  - Shannon-Von Neumann-Landauer Bound:

    Minimum energy/bit = $kTln2$ = 2.9 x$10^{-21}$ J/bit at 27°C
  - Complexity-energy-performance trade-off

| | Rate (Mb/s) | P_Rx (mW) | Rx (nJ/bit) | P_Tx (mW) | Tx (nJ/bit) |
|---|---|---|---|---|---|
| 802.11g | 22 | 140 | 6.4 | 450 | 20.4 |
| 802.11n | 200 | 1000 | 5 | 1800 | 9.0 |
| BT 2.0 | 0.7 | 45 | 64.3 | 62 | 88.6 |
| BT EDR | 2.2 | 48 | 21.8 | 65 | 29.5 |

Source: Wireless Net Designline




- Worldwide mobile subscribers
  - Over 7-Billions (as of 2016)
  - Increasing 2M/day

Source: Dazeinfo

- Energy efficiency under energy constraints
  - Portable device/handset operations are always limited by the battery.
  - Every mW saving in wireless transceiver represents MW's of saving for greener communications.

# Fundamental Limit and Relation of Power Efficiency and Bandwidth Efficiency

(2) Bandwidth efficiency ($\eta = C/W$)

$$\frac{R}{W} \le \frac{\text{C}}{W} \Rightarrow \log_2\left(1 + \frac{E_b / T_b}{N_0 W}\right) \le \log_2\left(1 + \frac{E_b}{N_0} \cdot \frac{C}{W}\right) \quad ①$$

$$\because P = E_b R \le E_b C$$

$$\text{Define } \cancel{C_\infty} = \lim_{W \to \infty} C = \lim_{W \to \infty} W \log_2\left(1 + \frac{P}{N_0 W}\right)$$

$$\le \lim_{W \to \infty} W \log_2\left(1 + \frac{E_b C_\infty}{N_0 W}\right) \cong (\log_2 e) \frac{E_b \cancel{C_\infty}}{N_0} \quad \text{for } R \le C$$

$$\left(\text{Note that, for } x << 1, \quad \ln(1 + x) \cong x\right)$$

$$\Rightarrow \frac{E_b}{N_0} \ge \ln 2 = \text{-1.6dB}$$

$$\text{As } W \to \infty \Rightarrow C_\infty \to \frac{P}{N_0} \log_2 e$$

$$\Rightarrow \eta \to 0$$

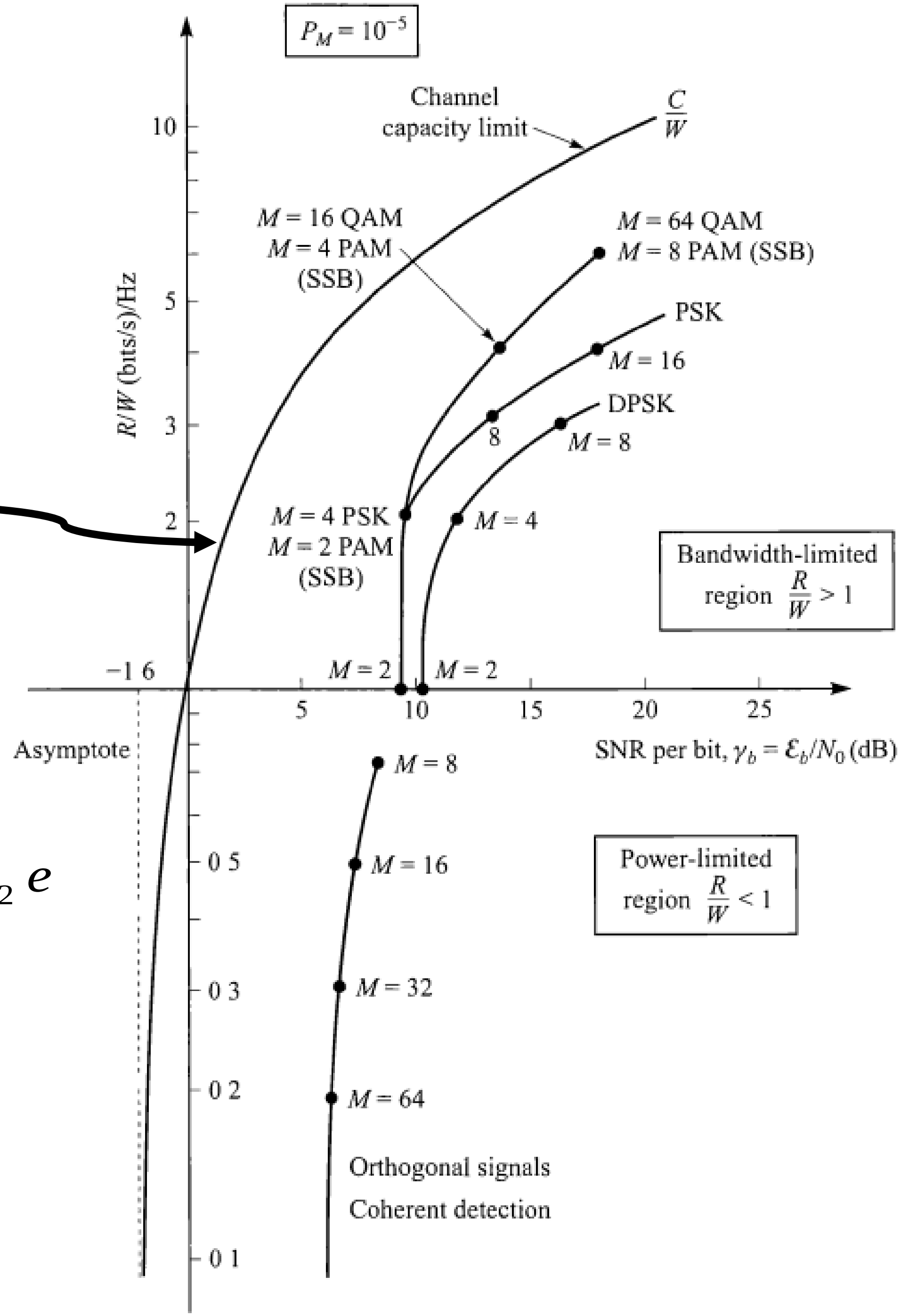
Also from ① $$\frac{E_b}{N_0} = \frac{2^{C/W} - 1}{C/W}$$

$$\frac{E_b}{N_0} \to -1.6dB$$

$$C = W \log_2(1 + \frac{P}{N_0 W})$$

$$\eta = \frac{C}{\mathrm{W}} = \log_2(1 + \frac{P}{N_0 W})$$

$$= \log_2(1 + \frac{\mathrm{E}_b C}{N_0 W})$$




$$\frac{E_b}{N_0} = \frac{2^{C/W} - 1}{C/W}$$

$$\text{As } W \to \infty \quad \Rightarrow C_\infty \to \frac{P}{N_0} \log_2 e$$

$$\Rightarrow \eta \to 0$$

$$\frac{E_b}{N_0} \to -1.6dB$$

$$\Rightarrow \frac{E_b}{N_0} \geq \ln 2 \ (\cong 0.693 = \text{-1.6dB})$$

# Achieving Channel Capacity w/ Orthogonal Signals

- From Ch4, for M-ary orthogonal signals, the error performance is

$$P_c = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (1 - Q(x))^{M-1} e^{-\frac{1}{2}\left(x - \sqrt{\frac{2E_s}{N_0}}\right)^2} \, \mathrm{d}\, x$$

$$P_e = 1 - P_c = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [1 - (1 - Q(x))^{M-1}] e^{-\frac{1}{2}\left(x - \sqrt{\frac{2E_s}{N_0}}\right)^2} \, \mathrm{d}\, x$$

- With the inequality $(1-x)^n \geq 1 - nx$, *for* $0 \leq x \leq 1$

$$1 - [1 - Q(x)]^{M-1} \leq (M-1)Q(x) < MQ(x) < Me^{-\frac{x^2}{2}}$$

- When $x$ is small, *i.e.* $x < x_0$ for some small $x_0$, the above union bound is loose. Use the tighter bound $1 - [1 - Q(x)]^{M-1} \leq 1$

➢ Let the SNR be $\gamma = \frac{E_s}{N_0}$

$$P_e \leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x_0} e^{-\frac{1}{2}(x - \sqrt{2\gamma})^2} dx + \frac{M}{\sqrt{2\pi}} \int_{x_0}^{\infty} e^{-\frac{1}{2}x^2} e^{-\frac{1}{2}(x - \sqrt{2\gamma})^2} dx \equiv P_e(x_0)$$

- Note that the upper bound of $P_e(x_0)$ is minimized when $\frac{\partial P_e\left(x_0\right)}{\partial x_0}=0$

$$\frac{\partial P_e\left(x_0\right)}{\partial x_0}=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(x-\sqrt{2\gamma})^2}-\frac{M}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}e^{-\frac{1}{2}(x-\sqrt{2\gamma})^2}=0$$

$$\Rightarrow e^{\frac{x_0^2}{2}}=M, \quad \text{i.e.} \quad x_0=\sqrt{2\ln M}=\sqrt{2\ln 2\log_2 M}=\sqrt{2k\ln 2}$$

- The first term in $P_e(x_0)$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x_0}e^{-\frac{1}{2}(x-\sqrt{2\gamma})^2}dx \overset{Let\ u=\left(x-\sqrt{2\gamma}\right)/\sqrt{2}}{=} \frac{1}{\sqrt{\pi}}\int_{-\infty}^{-\left(\sqrt{2\gamma}-x_0\right)/\sqrt{2}}e^{-u^2}du,$$

$$=Q\left(\sqrt{2\gamma}-x_0\right)<e^{-\frac{1}{2}(\sqrt{2\gamma}-x_0)^2}, \quad x_0\le\sqrt{2\gamma}$$

- The 2nd term in $P_e(x_0)$

$$\frac{M}{\sqrt{2\pi}}\int_{x_0}^{\infty}e^{-\frac{1}{2}x^2}e^{-\frac{1}{2}(x-\sqrt{2\gamma})^2}dx \overset{\text{Let } u=(x-\sqrt{\gamma/2})}{=} \frac{M}{\sqrt{2\pi}}e^{-\frac{\gamma}{2}}\int_{x_0-\sqrt{\gamma/2}}^{\infty}e^{-u^2}du$$

$$<\begin{cases} Me^{-\frac{\gamma}{2}}, & x_0\le\sqrt{\gamma/2} \\ Me^{-\frac{\gamma}{2}}e^{-(x_0-\sqrt{\gamma/2})^2}, & x_0>\sqrt{\gamma/2}\end{cases}$$

- Combining the two terms in $P_e(x_0)$ and $M = e^{x_0^2/2}$,

$$P_e < \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x_0} e^{-\frac{1}{2}(x-\sqrt{2\gamma})^2} dx + \frac{M}{\sqrt{2\pi}} \int_{x_0}^{\infty} e^{-\frac{1}{2}x^2} e^{-\frac{1}{2}(x-\sqrt{2\gamma})^2} dx$$

$$< \begin{cases} e^{-\frac{1}{2}(\sqrt{2\gamma}-x_0)^2} + e^{\frac{x_0^2}{2}} e^{-\frac{\gamma}{2}}, & 0 \le x_0 \le \sqrt{\gamma/2} \\ e^{-\frac{1}{2}(\sqrt{2\gamma}-x_0)^2} + e^{\frac{x_0^2-\gamma}{2}} e^{-(x_0-\sqrt{\gamma/2})^2}, & \sqrt{\gamma/2} \le x_0 \le \sqrt{2\gamma} \end{cases}$$

$$= \begin{cases} e^{\frac{x_0^2-\gamma}{2}} (1 + e^{-(x_0-\sqrt{\gamma/2})^2}), & 0 \le x_0 \le \sqrt{\gamma/2} \\ 2e^{-\frac{1}{2}(\sqrt{2\gamma}-x_0)^2}, & \sqrt{\gamma/2} \le x_0 \le \sqrt{2\gamma} \end{cases}$$

$$< \begin{cases} 2e^{\frac{x_0^2-\gamma}{2}}, & 0 \le x_0 \le \sqrt{\gamma/2} \\ 2e^{-\frac{1}{2}(\sqrt{2\gamma}-x_0)^2}, & \sqrt{\gamma/2} \le x_0 \le \sqrt{2\gamma} \end{cases} \qquad \text{(A)}$$

# Reliable Communication with $\frac{E_b}{N_0} \geq \ln 2$

- By reliable communication, we mean that $P_e \to 0$ is possible.

$$P_e < \begin{cases} 2e^{\frac{x_0^2-\gamma}{2}}, & 0 \leq x_0 \leq \sqrt{\gamma/2} \\ 2e^{-\frac{1}{2}(\sqrt{2\gamma}-x_0)^2}, & \sqrt{\gamma/2} \leq x_0 \leq \sqrt{2\gamma} \end{cases} \quad \text{(A)}$$

- The min $P_e$ occurs at $x_0 = \sqrt{2\ln M} = \sqrt{2k\ln 2}$. Also let

$$\gamma = k\gamma_b = k\frac{E_b}{N_0}$$

$$\Rightarrow P_e < \begin{cases} 2e^{-k\left(\gamma_b - 2\ln 2\right)/2}, & \ln M \leq \gamma/4 \\ 2e^{-k\left(\sqrt{\gamma_b}-\sqrt{\ln 2}\right)^2}, & \gamma/4 \leq \ln M \leq \gamma \end{cases}$$

- As $k \to \infty, P_e \to 0$ is possible, if $\gamma_b > \ln 2$.

# Reliable Communication with $R < C_\infty$

- Since $C_\infty = \lim_{W\to\infty} W \log_2\left(1+\frac{P}{N_0 W}\right) = \log_2 e \lim_{W\to\infty} W \ln\left(1+\frac{P}{N_0 W}\right) = (\log_2 e)\left(\frac{P}{N_0}\right)$,

$$x_0 = \sqrt{2k\ln 2} = \sqrt{2RT\ln 2}, \qquad \gamma = \frac{E_s}{N_0} = \frac{TP}{N_0} = TC_\infty \ln 2$$

From (A), $$P_e < \begin{cases} 2e^{\frac{x_0^2-\gamma}{2}}, & 0 \le x_0 \le \sqrt{\gamma/2} \\ 2e^{-\frac{1}{2}(\sqrt{2\gamma}-x_0)^2}, & \sqrt{\gamma/2} \le x_0 \le \sqrt{2\gamma} \end{cases} \qquad \text{(A)}$$

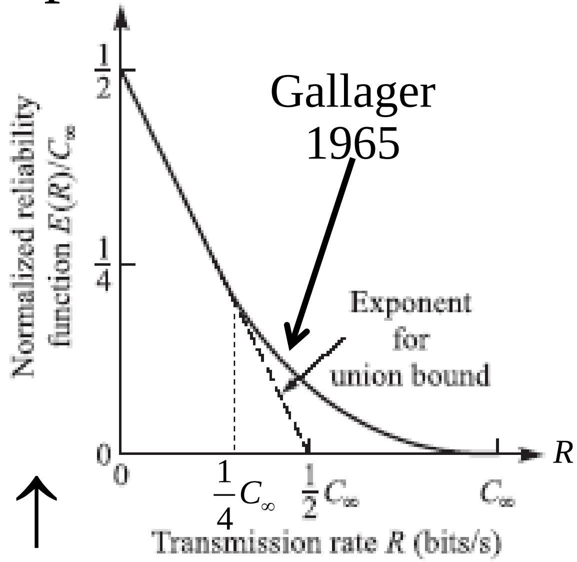
$$\Rightarrow P_e < \begin{cases} 2\times 2^{-T\left(\frac{1}{2}C_\infty - R\right)}, & 0 \le R \le \frac{1}{4}C_\infty \\ 2\times 2^{-T\left(\sqrt{C_\infty}-\sqrt{R}\right)^2}, & \frac{1}{4}C_\infty \le R \le C_\infty \end{cases}$$

$\Rightarrow$ As $T(=kT_b) \to \infty, P_e \to 0$ is possible, if

$$R < C_\infty = (\log_2 e)\left(\frac{P}{N_0}\right) \qquad (i.e. \frac{E_b}{N_0} \ge \ln 2)$$

# The Channel Reliability Function

- Channel reliability function is defined as the exponential factor (Gallager 1965)




$$E(R) = \begin{cases} \frac{1}{2}C_\infty - R, & 0 \le R \le \frac{1}{4}C_\infty \\ \left(\sqrt{C_\infty} - \sqrt{R}\right)^2, & \frac{1}{4}C_\infty \le R \le C_\infty \end{cases}$$

$$\Rightarrow P_e < 2 \times 2^{-TE(R)} \quad \text{As } R\uparrow, E(R)\downarrow \ P_e \uparrow$$

- Recall union bound on $P_e$ of orthogonaling signaling from Ch4,

$$P_e < e^{-\frac{k}{2}\left(\frac{E_b}{N_0} - 2\ln 2\right)} \quad \Rightarrow P_e < \frac{1}{2} \times 2^{-T\left(\frac{1}{2}C_\infty - R\right)}, \quad 0 \le R \le \frac{1}{2}C_\infty$$

$\Rightarrow$ The exponent $\frac{E_b}{N_0} - 2\ln 2 \ \ge 0$ is not as tight as $E(R)$, *i.e.* $\frac{E_b}{N_0} \ge \ln 2$, due to the looseness of the union bound.

# Summary

- Measure of information, entropy, and mutual information
- Channel capacity and mutual information
  - ✓Binary Symmetry Channel
  - ✓AWGN  Channel
- Band-width efficiency and power efficiency
  - ✓Minimum energy per bit
- Reliable communications and channel reliability function