COM5180 Assignment #1
due May 8^{th} 23:59

Answer 7 of the following questions. You may type your answer as a pdf file or scan your hand-written solutions. Submit on iLMS by the deadline.

1. Let $\{x_n\}$ be a zero-mean, complex random WSS process. Prove that the correlation matrix $\mathbf{R}_x$ is PSD.

2. Given $\mathbf{A} \in M_{m\times n}(\mathbb{F})$. Prove that $(L_{\mathbf{A}})^\dagger = L_{\mathbf{A}^\dagger}$.

3. Define the function $f : \mathbb{R}^n \to \mathbb{R}$ as follows. $f(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x} + 2\mathbf{b}^T\mathbf{x} + c$, for some $\mathbf{A} \in M_{n\times n}(\mathbb{R})$, $\mathbf{b} \in \mathbb{R}^n$, $c \in \mathbb{R}$. Prove the followings: (1) $\nabla f(\mathbf{x}) = 2\mathbf{A}\mathbf{x} + 2\mathbf{b}$. (2) $\nabla^2 f(\mathbf{x}) = 2\mathbf{A}$. (3) f is convex if and only if $\mathbf{A}$ is PSD.

4. Prove that a matrix $\mathbf{A} \in M_{n\times n}(\mathbb{R})$ has an LU decomposition if every principal submatrix $\mathbf{A}_{\{1,\ldots,k\}}$ has nonzero determinant.

5. Let $A, M \in M_{n\times n}(\mathbb{R})$ be two symmetric matrices. Moreover, M is PD. Prove that there exists a non-singular matrix C such that $C^TMC = I_n$, and C^TAC is diagonal.

6. Prove or disprove the following statement. $T : \mathbb{C} \to \mathbb{C}$ is normal if and only if $T = T_1 + iT_2$ for some self-adjoint operators $T_1, T_2 : \mathbb{C} \to \mathbb{C}$ such that $T_1T_2 = T_2T_1$.

7. Let $\mathcal{V} = M_{n\times n}(\mathbb{F})$ be the space of $n \times n$ matrices over $\mathbb{F}$. Let $A \in \mathcal{V}$ be a fixed matrix. Let $T : \mathcal{V} \to \mathcal{V}$ be a linear operator defined by $T(B) = AB$. Prove or disprove the following statement. If A is diagonalizable, then T is diagonalizable.

8. Continued from above. Let $U : \mathcal{V} \to \mathcal{V}$ be a linear operator defined by $U(B) = AB - BA$. Prove or disprove the following statement. If A is diagonalizable, then U is diagonalizable.