通訊數學 Assg 1　108064535 陳文遠

1、 $x = [x_1\ x_2 \cdots x_n]$; $R_x = E[\underline{x} \cdot \underline{x}^*] = \begin{bmatrix} E[|x_1|^2] & E[x_1\bar{x}_2] & \cdots & E[x_1\bar{x}_n] \\ E[x_2\bar{x}_1] & E[|x_2|^2] & \cdots & E[x_2\bar{x}_n] \\ \vdots & \vdots & \ddots & \vdots \\ E[x_n\bar{x}_1] & E[x_n\bar{x}_2] & \cdots & E[|x_n|^2] \end{bmatrix}$

$\because R_x = R_x^*$, $\therefore R_x$ is self-adjoint

pick $\forall \underline{y}$, $\underline{y}^* R_x \underline{y} = \underline{y}^* E[\underline{x}\underline{x}^*]\underline{y} = E[\underline{y}^*\underline{x}\underline{x}^*\underline{y}]$　length must bigger then 0

$= E[(\underline{y}^*\underline{x})(\underline{y}^*\underline{x})^*] = E[\underline{\|(\underline{y}^*\underline{x})\|^2}] \geq 0$

$\therefore R_x$ is positive-definite matrix #

2、We know that,

$A \in M_{m\times n}(\mathbb{F})$

$L_A : \mathbb{F}^n \longrightarrow \mathbb{F}^m$, $(L_A)^\dagger : \mathbb{F}^m \longrightarrow \mathbb{F}^n$

Let B is pseudoinverse of A , $B = A^\dagger$

$(L_A)^\dagger = L_B \Rightarrow \underline{(L_A)^\dagger = L_{A^\dagger}}$ #

3. (1) $f(x) = \sum_j \sum_k A_{jk} x_j x_k + 2\sum_j b_j x_j + C$

$f_j'(x) = 2\sum_k A_{jk} x_k + 2b_j$

$\Rightarrow \underline{\nabla f(x) = 2Ax + 2b}$ #

(2) $\nabla^2 f(x) = \nabla(\nabla f(x)) = \nabla(2Ax + 2b) = f_j''(x)$

$\boxed{\because f_j'(x) = 2\sum_k A_{jk} x_k + 2b_j}$

$\Rightarrow f_j''(x) = 2\sum_k A_{jk} \Rightarrow \underline{\nabla^2 f(x) = 2A}$ #

(3) $f(y) = f(x)^T(y-x) + (y-x)^T \nabla^2 f(x + \alpha(y-x))(y-x)$

$\because A$ is PSD $\Rightarrow (y-x)^T \nabla^2 f(x+\alpha(y-x))(y-x) \geq 0$

$\Rightarrow f(y) \geq f(x) + \nabla f(x)^T (y-x)$

$\therefore f(x)$ is convex #

4. (i) $n=1$, trivial

(ii) set $n = 2 \sim n-1$ is true, where $n-1 > 1$

$$L = \begin{bmatrix} L' & 0 \\ X^T & 1 \end{bmatrix}, \quad U = \begin{bmatrix} U' & Y \\ 0 & u \end{bmatrix}, \quad L', U' \in M_{(n-1)\times(n-1)}$$

$$M = \begin{bmatrix} M' & R \\ -S^T & m \end{bmatrix}, \quad L'U' = M', \quad L'Y = R, \quad (U')^T X = S, \quad u = m - X^T Y$$

M': the leading pricipal minors are nonzero.

the induction hypothsis guarantees the existence of the factorization, $M' = L'U'$

5. M is PD $\Rightarrow \exists$ a nonsingular matrix C

$\det(M) > 0$

$\because A^T A = A A^T \Rightarrow A$ is normal $\Rightarrow$ Diagonalizable

$$C^T A C = D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$ #

$$\because M^T M = M M^T, \quad \therefore C^T M C = D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} = I_n$$ #

6. T is normal $\Rightarrow TT^* = T^*T$

then $T = T_1 + jT_2 \Rightarrow (T_1 + jT_2)(T_1 + jT_2)^* = (T_1 + jT_2)^*(T_1 + jT_2)$

$\Rightarrow \cancel{T_1 T_1^*} + jT_1 T_2^* + jT_2 T_1^* - \cancel{T_2 T_2^*} = \cancel{T_1^* T_1} + jT_1^* T_2 + jT_2^* T_1 - \cancel{T_2^* T_2}$

$\Rightarrow -j(T_1 T_2 - T_2 T_1) = -j(T_2 T_1 - T_1 T_2)$

$\Rightarrow T_1T_2 - T_2T_1 = 0 \Rightarrow \underline{T_1T_2 = T_2T_1}$

T is normal $\Rightarrow T = T_1 + jT_2$, T_1, T_2 is self-adjoint

$\Rightarrow \underline{T_1T_2 = T_2T_1 \; ; \text{ Proved } \#}$

7. if A is diagonalizable $\Rightarrow \exists$ unitary basis $C \Rightarrow [A]_C = D$

$\Rightarrow$ <u>T is diagonalizable #</u>

8. if A is diagonalizable $\Rightarrow U$ is diagonalizable

$A = Q^{-1}DQ$, U is similar to $U(B) = DB - BD$

$\Rightarrow D = \text{diag}[d_1, \cdots, d_n]$, $U(E_{ij}) = DE_{ij} - E_{ij}D = (d_i - d_j)E_{ij}$

<u>∴ U is diagonalizable #</u>