COM 5120 Communications Theory

Chapter 3 Digital Modulation

Jen-Ming Wu
Inst. of Communications Engineering
Dept. of Electrical Engineering
National Tsing Hua University

Fall, 2019

Outline

- **Signal Space Representation of Waveform**
- **Signaling Schemes with Modulation**
  1. **Modulation without Memory**
     - **M-PAM, M-PSK, M-QAM**
     - **M-ary Orthogonal Signaling (M-FSK)**
     - **Bi-orthogonal Signaling**
     - **Simplex Signaling**
  2. **Modulation with Memory**
     - **Linear Modulation with Memory**
     - **Non-linear Modulation with Memory**
- **Power Spectral Density of Digital Modulation Signal**

Signal Space Representation of Waveform

Given signal waveforms : $\{S_1(t), S_2(t), \ldots\ldots, S_M(t)\}$

Q: What is the necessary signal space to represent the waveforms?

With the basis functions :

$$\{\phi_1(t), \phi_2(t), \ldots\ldots, \phi_N(t)\} \qquad N < M$$

Then the transmission signal

$$S_i(t) = \sum_{j=1}^{N} S_{ij}\varphi_j(t), \; i = 1, 2, \ldots\ldots, M \qquad 0 \le t \le T$$

where $\{\phi_j(t)\}$ are orthornormal, i.e.,

$$\int_0^T \phi_i(t)\phi_j(t)dt = \begin{cases} 1, i = j \\ 0, i \neq j \end{cases}, \; S_{ij} = \int_0^T S_i(t)\phi_j(t)dt$$

$\Rightarrow \{\phi_j(t)\}$ spans the signal space of $S_i(t)$

Signal Space Representation of Waveform

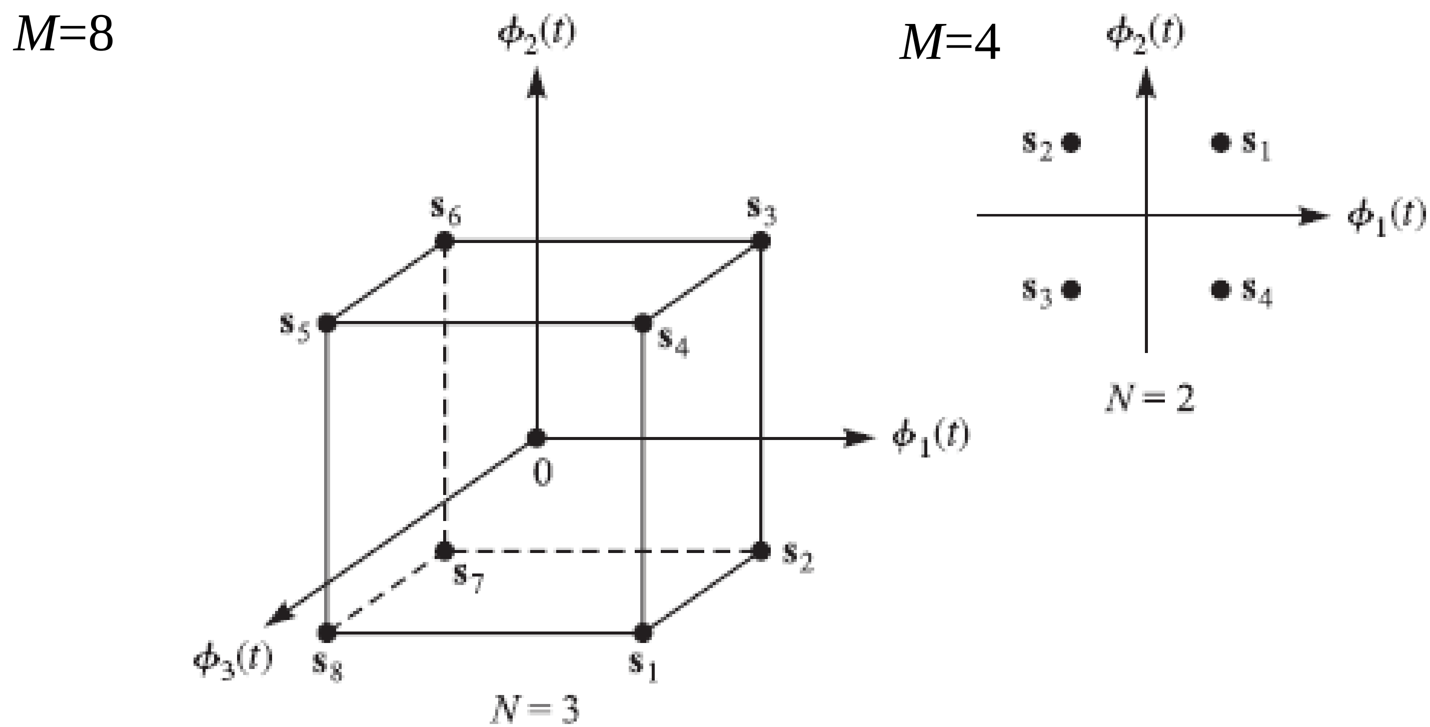
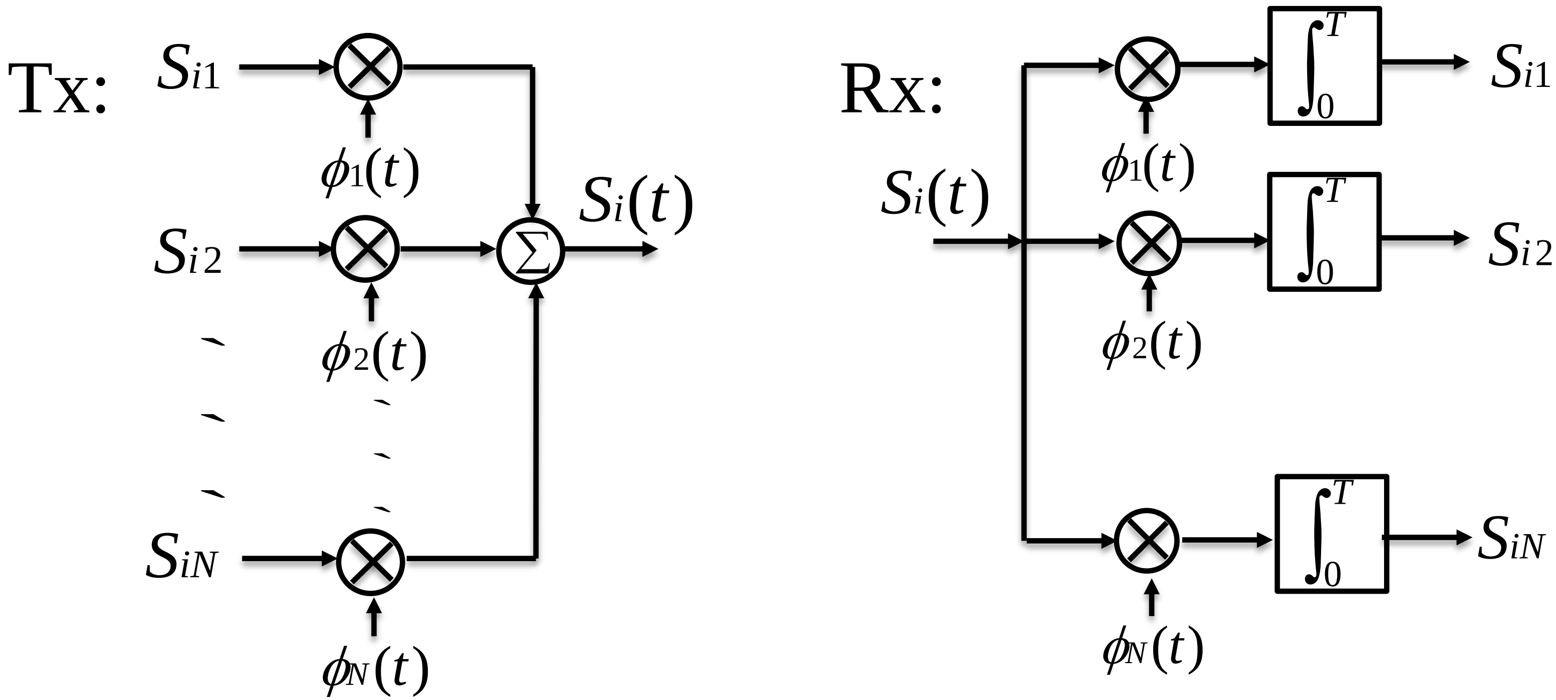


FIGURE 3.2–10
Signal space diagrams for signals generated from binary codes.

How does the signal space affect the system?

➢ **Signal space defines the Tx/Rx architecture**



The continuous $S_i(t), i = 1,..,M$ can be *fully represented* by the *signal vector*.

$$\underline{\boldsymbol{S_i}} = \left[S_{i1}, S_{i2}, \ldots, S_{iN}\right]^{\mathrm{T}}, i = 1,2,\ldots, M$$

Signal Vectorization

- Signal Energy

$$\left\|\underline{S_i}\right\|^2 = \underline{S_i}^T \underline{S_i} = \sum_{j=1}^{N} S_{ij}^{\ 2} = \int_0^T S_i(t)^2\, dt \text{ (Signal energy)}$$

- Signal Inner Product

$$\int_0^T S_i(t) S_k(t)\, dt = \int_0^T \Big(\sum_j S_{ij}\phi_j(t)\Big)\Big(\sum_{j=1}^{N} S_{kj}(t)\phi_j(t)\Big) dt = \underline{S_i}^T \underline{S_k}$$

when $k = i$,$E_i = \int_0^T S_i^{\,2}(t)\, dt = \left\|\underline{S_i}\right\|^2$

- Signal Distance

$$\int_0^T (S_i(t) - S_k(t))^2\, dt = \left\|S_i - S_k\right\|^2 = d_{ik}^{\ 2}$$

➢ The continuous time domain signal operation can be replaced by the vector operation in linear algebra.

Q: How to find the basis $\{\phi_j(t)\}$ from the waveforms $\{S_i(t)\}$?

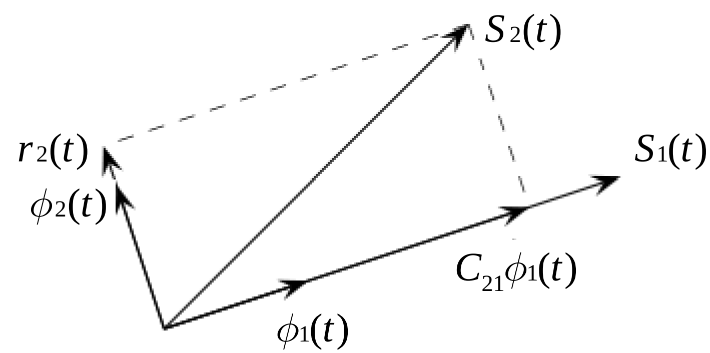
Orthogonal Expansion of Signals

✓ **Gram-Schmidt Orthogonalization Procedure :**

$$\phi_1(t) = \frac{S_1(t)}{\sqrt{E_1}} \Rightarrow S_1(t) = \sqrt{E_1}\phi_1(t)$$

$$r_2(t) = S_2(t) - C_{21}\phi_1(t), \text{ where } C_{21} = \int_0^T S_2(t)\phi_1(t)dt$$

$$\phi_2(t) = \frac{r_2(t)}{\sqrt{E_2}}, E_2 = \int_0^T r_2^{\,2}(t)dt$$



●●●

$$r_i(t) = S_i(t) - \sum_{j=1}^{i-1} C_{ij}\phi_j(t), C_{ij} = \int_0^T S_i(t)\phi_j(t)dt$$

$$\phi_i(t) = \frac{r_i(t)}{\sqrt{E_i}}, \qquad i = 1, \ldots, M$$

$$\Rightarrow \{\phi_1(t), \phi_2(t), \ldots, \phi_N(t)\} \qquad N \le M$$

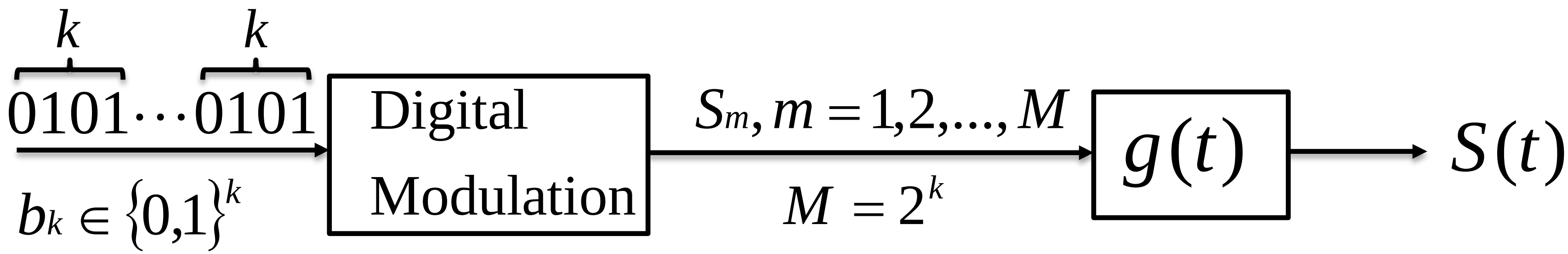
Signaling Schemes with Modulation

1. Modulation without Memory
   - M-PAM, M-PSK, M-QAM
   - M-ary Orthogonal Signaling (M-FSK)
   - Bi-orthogonal Signaling
   - Simplex Signaling

2. Modulation with Memory
   - Linear Modulation with Memory
   - Non-linear Modulation with Memory

Introduction to Digital Modulation

- The mapping of digital sequence into a set of waveforms.



$$\text{Bit rate} = \frac{1}{\mathrm{T_b}}$$

$$S(t) = \sum_i s_m g(t - iT), \quad s_m \in \{s_1, s_2, \ldots\ldots, s_M\}, \quad 0 \le t \le T$$

where $g(t)$ is the pulse shaping function.

➢ The modulation process is to embed the bit info into amplitude/phase/frequency of the symbol S_m. The signal waveform bears the information of the bit stream.

$$R_s = \text{symbol rate} = \frac{1}{kT_b} = \frac{1}{T_s}$$

The modulation schemes can be classified:

(1) By memory

- With memory :

  The mapping of $b_n \to S_m$ depends on the past symbols of S_{m-i}

  Ex: DPSK, MSK, CPM, …,etc

- Without memory(Memoryless):

  The mapping of $b_n \to S_m$ depends on the current set of b_{n} and s_m only.

  Ex: PAM, BPSK, QAM, MPSK

(2) By linearity

- Linear modulation (Ex: PAM)

  Mapping satisfy the superposition principle.

  If $a_1 \rightarrow b_1$ and $a_2 \rightarrow b_2$, then $a_1 + a_2 \rightarrow b_1 + b_2$

- Non-linear modulation (Ex: M-PSK, M-FSK)

(3)By coherence

- Coherent modulation (Ex: M-PSK, M-QAM)
- Non-coherent modulation (Ex: DPSK)

Memoryless Modulation

- Pulse-Amplitude Modulation(PAM)

The M-ary PAM signal.

$$S_m(t) = A_m g(t)\cos(2\pi f_c t) = \mathrm{Re}\left\{ A_m g(t) e^{j2\pi f_c t} \right\}, \quad \begin{array}{l} m = 1, 2, \ldots, M \\ 0 \le t \le T \end{array}$$

where $g(t)$ is the pulse shaping function :

$$A_m = (2m - 1 - M),\, m = 1,2,\ldots, M \;= \left\{ \pm 1, \pm 3, \ldots, \pm (M-1) \right\}$$

Example :

$$k = 1, M = 2, A_m \in \left\{ \pm 1 \right\} \Rightarrow 2 - PAM$$

$$k = 2, M = 4, A_m \in \left\{ \pm 1, \pm 3 \right\} \Rightarrow 4 - PAM$$

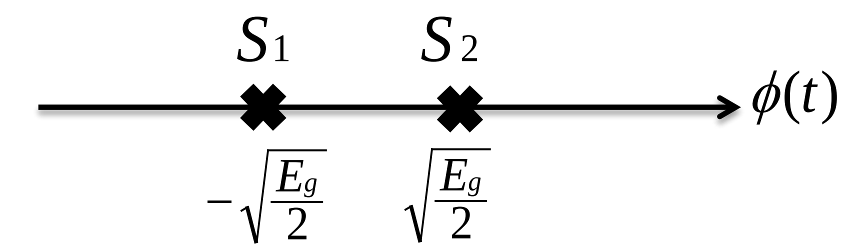
$S_m(t)$ is a one-dimension signal with

$$S_m(t) = A_m g(t)\cos(2\pi f_c t) = S_m \phi(t)$$

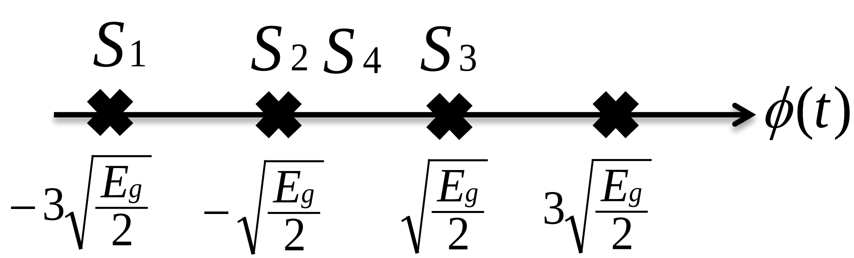
$$\text{Let } E_g = \int_0^T \left|g(t)\right|^2 dt \equiv \left\|g(t)\right\|^2,\ \phi(t) = \frac{S_m(t)}{\left|S_m(t)\right|} = \sqrt{\tfrac{2}{E_g}}\, g(t)\cos(2\pi f_c t),$$

$$\left\|\phi(t)\right\|^2 = \int_0^T \phi^2(t)dt = 1, S_m = A_m\sqrt{\frac{E_g}{2}}, m = 1,2,\ldots,M$$

Example : $M = 2, A_m = \pm 1$



$M = 4, A_m = \pm 1, \pm 3$

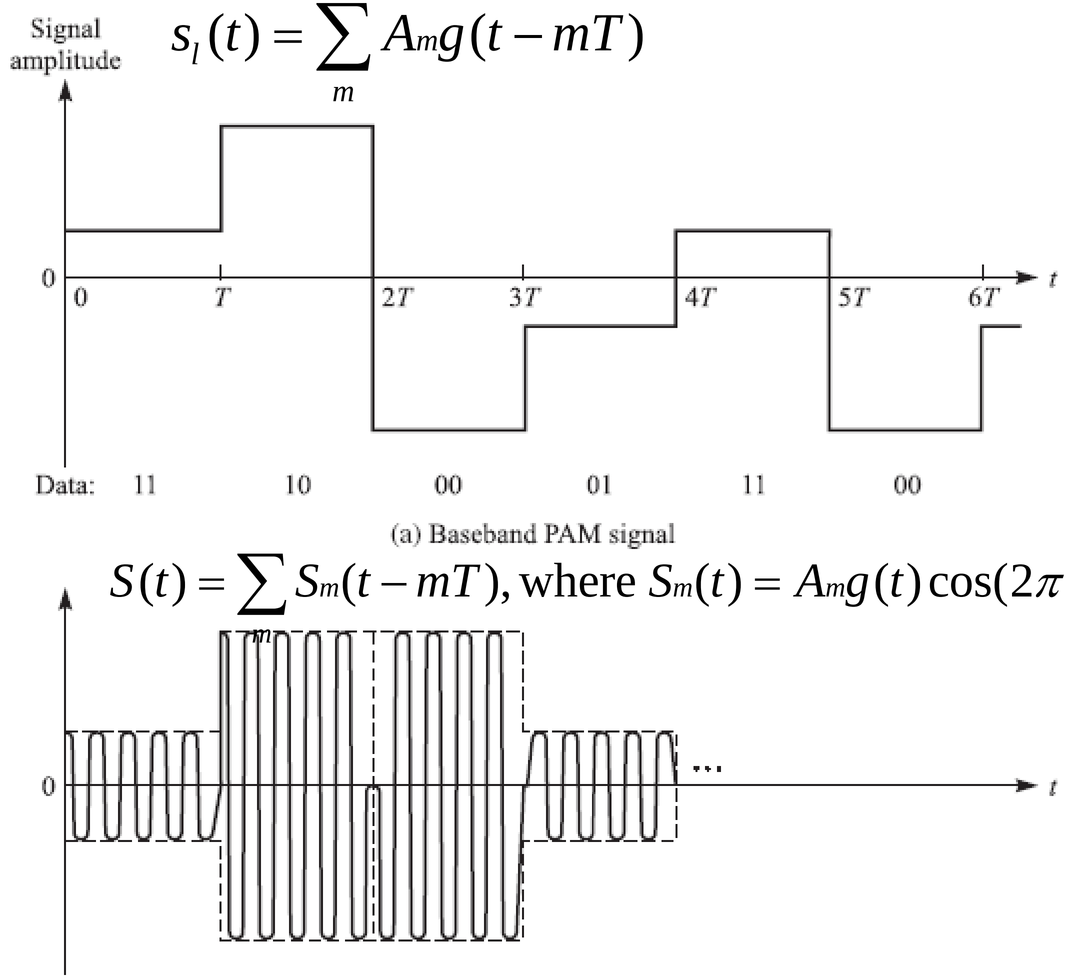


$$\Rightarrow d_{\min} = \sqrt{2E_g}$$

Signaling of M-PAM

$$s_l(t) = \sum_m A_m g(t - mT)$$

$$S(t) = \sum_m S_m(t - mT), \text{where } S_m(t) = A_m g(t)\cos(2\pi f_c t)$$



(a) Baseband PAM signal

(b) Bandpass PAM signal

FIGURE 3.2–2
Example of (a) baseband and (b) carrier-modulated PAM signals.

Mapping between bits and symbols

- Gray Encoding

• Rule : Adjacent signal symbol point differ by only 1 bit.

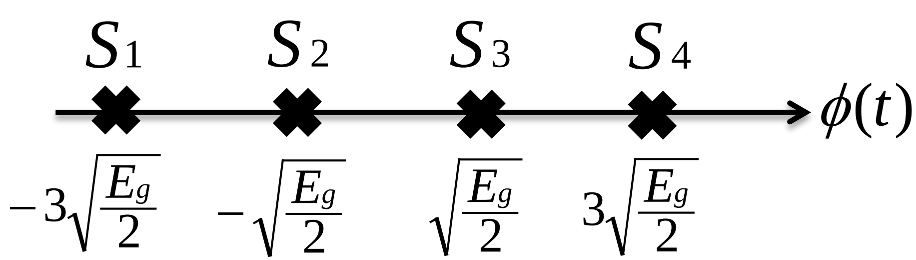
• Reason : When error occurs in a symbol , it most likely migrates to nearby points.

• With Gray Encoding we could minimize the error bits when symbol error occurs.

• The symbol error rate depends on the $d_{\min}$.

Encoding A : 00 01 10 11

Encoding B : 00 01 11 10 Which encoding is better?



Phase Modulation (or Phase Shift Keying, PSK)

The M-ary PSK signal

$$S_m(t) = g(t)\cos(2\pi f_c t + \theta_m)$$
$$= g(t)\cos(\theta_m)\cos(2\pi f_c t) - g(t)\sin(\theta_m)\sin(2\pi f_c t)$$

where $\theta_m = \frac{m-1}{M}2\pi, m = 1,2,\ldots,M$

Equal signal energy : $E_m = \int_0^T S_m{}^2(t)dt = \frac{1}{2}\int_0^T g^2(t)dt = \frac{E_g}{2}$

The orthogonal basis are :

$$\phi_1(t) = \sqrt{\tfrac{2}{E_g}}g(t)\cos(2\pi f_c t)$$

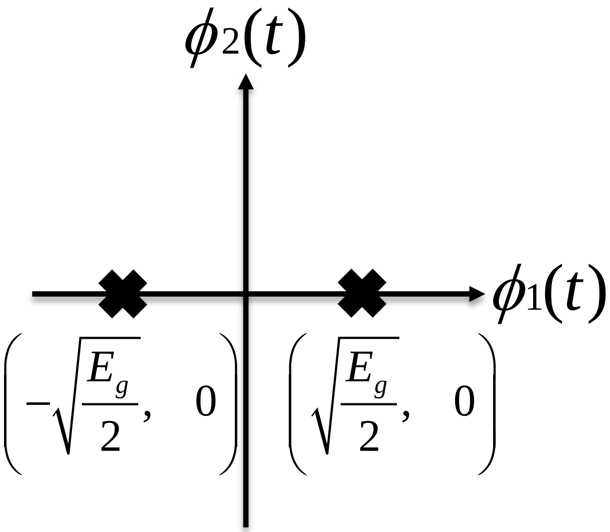
$$\phi_2(t) = -\sqrt{\tfrac{2}{E_g}}g(t)\sin(2\pi f_c t)$$

$$\Rightarrow S_m(t) = \sqrt{\tfrac{E_g}{2}}\cos(\theta_m)\phi_1(t) + \sqrt{\tfrac{E_g}{2}}\sin(\theta_m)\phi_2(t)$$

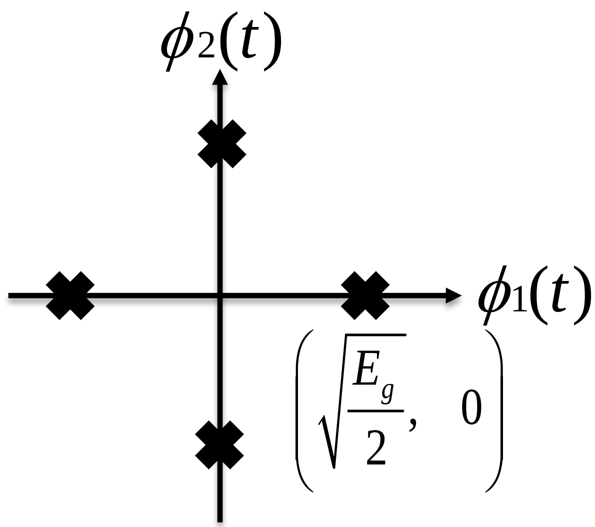
$$\underline{S}_m = \left[\sqrt{\tfrac{E_g}{2}}\cos(\theta_m), \sqrt{\tfrac{E_g}{2}}\sin(\theta_m)\right]^T, m = 1,2,\ldots,M, N = 2$$

- Example :

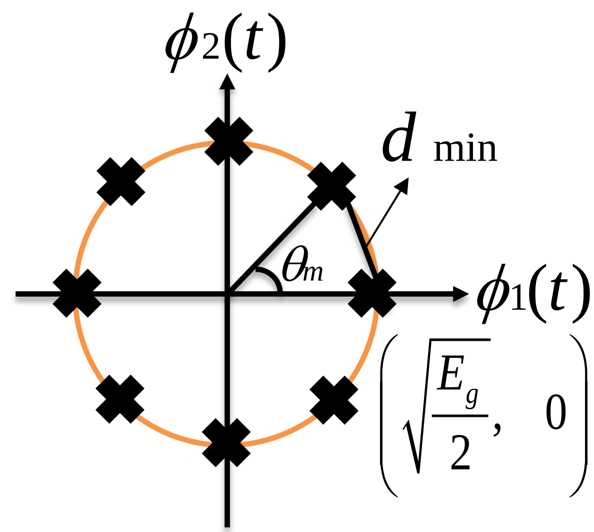
M=2,BPSK



M=4,QPSK



M=8-PSK



$$\Rightarrow d_{\min} = 2\sqrt{\frac{E_g}{2}}\sin\left(\frac{\theta_m}{2}\right) = \sqrt{2E_g\sin^2\left(\frac{\pi}{M}\right)} = \sqrt{E_g\left(1-\cos\frac{2\pi}{M}\right)}$$

- <u>M-ary Quadrature Amplitude Modulation(QAM)</u>

Consider a 2-D PAM with orthogonal basis.

$$\phi_1(t) = \sqrt{\tfrac{2}{E_g}}\, g(t)\cos(2\pi f_c t),\ 0 \le t \le T \Rightarrow \text{In-phase}$$

$$\phi_2(t) = -\sqrt{\tfrac{2}{E_g}}\, g(t)\sin(2\pi f_c t),\ 0 \le t \le T \Rightarrow \text{Quadrature-phase}$$

$$\begin{aligned} S_m(t) &= A_{mI}\, g(t)\cos(2\pi f_c t) - A_{mQ}\, g(t)\sin(2\pi f_c t) \\ &= V_m\, g(t)\cos(2\pi f_c t + \theta_m),\ \text{where } A_{mI},\ A_{mQ} \in \{\pm 1, \pm 3, \ldots, \pm(M-1)\} \end{aligned}$$

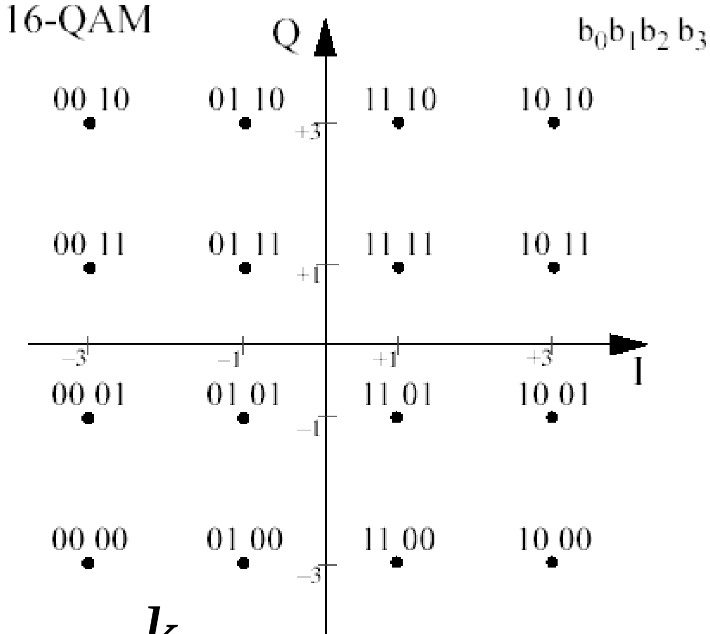
$$V_m = (A^2_{mI} + A^2_{mQ})^{\frac{1}{2}},\ \ \theta_m = \tan^{-1}\left(\frac{A_{mQ}}{A_{mI}}\right)$$

On the basis :

$$S_m(t) = S_{mI}\phi_1(t) + S_{mQ}\phi_2(t)$$

with $\underline{S}_m = \begin{bmatrix} S_{mI} & S_{mQ} \end{bmatrix}^T = \begin{bmatrix} \sqrt{\frac{E_g}{2}} A_{mI} & \sqrt{\frac{E_g}{2}} A_{mQ} \end{bmatrix}^T,\ m = 1, 2, \ldots, M$

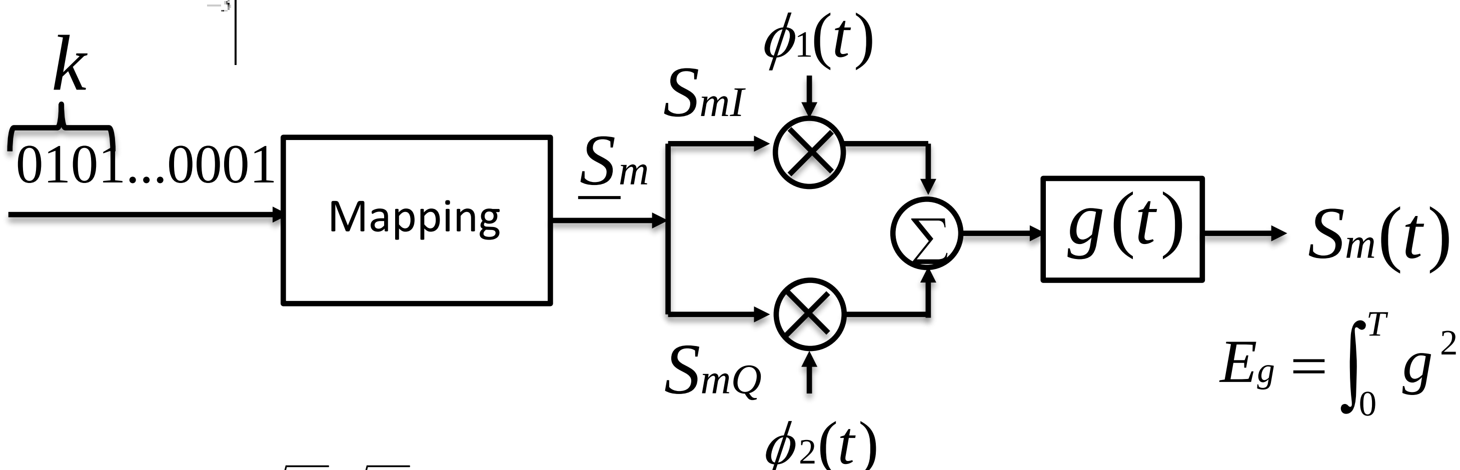
Example M=16, 16-QAM



$$\underline{S}_m = \left[\sqrt{\frac{E_g}{2}} A_{mI},\ \sqrt{\frac{E_g}{2}} A_{mQ} \right]^T \quad , m = 1, 2, \ldots, M$$

where $A_{mI}, A_{mQ} \in \{\pm 1, \pm 3, \ldots, \pm(M-1)\}$

$$d_{\min} = \sqrt{2E_g}$$



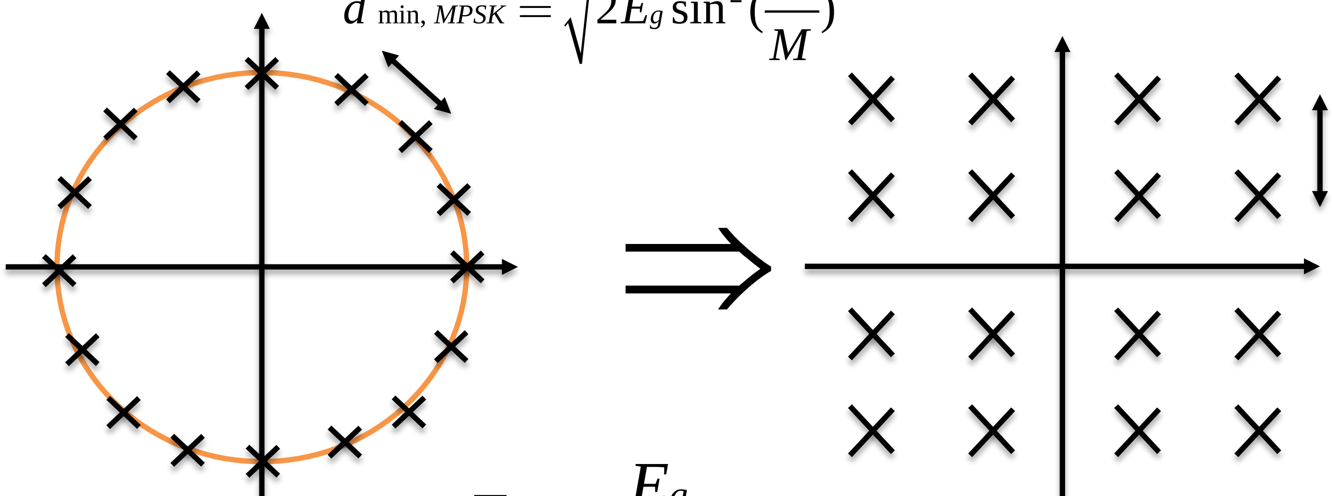
$$E_g = \int_0^T g^2(t)dt$$

$$E_{av} = \frac{1}{M} \frac{E_g}{2} \sum_{m=1}^{\sqrt{M}} \sum_{m=1}^{\sqrt{M}} (A_{mI}^2 + A_{mQ}^2) = \frac{E_g}{2M} \frac{2M(M-1)}{3} = \frac{(M-1)}{3} E_g$$

Notes on QAM v.s. PAM and MPSK

(1) QAM improves the BW efficiency of PAM.

(2) Given the same E_g ,QAM gives better utilization (larger $d_{\min}$) in the constellation space than MPSK.



$$d_{\min,\,MPSK} = \sqrt{2E_g \sin^2\left(\frac{\pi}{M}\right)}$$

$$d_{\min,\,MQAM} = \sqrt{2E_g}$$

$$E_{av} = \frac{E_g}{2}$$

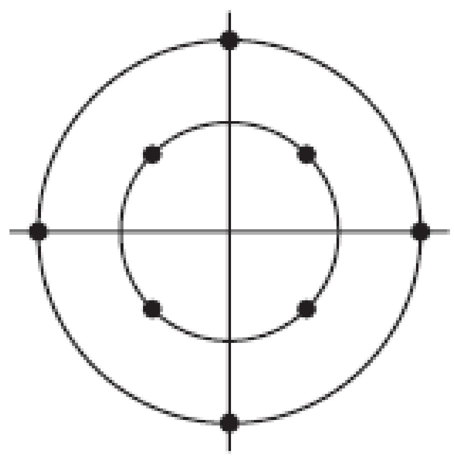
MPSK

$$E_{av} = \frac{(M-1)}{3}E_g$$

M - QAM

$\Rightarrow$ QAM has larger $d_{\min}$ and hence better error performance.

QAM can be viewed as combined PAM-PSK Modulation



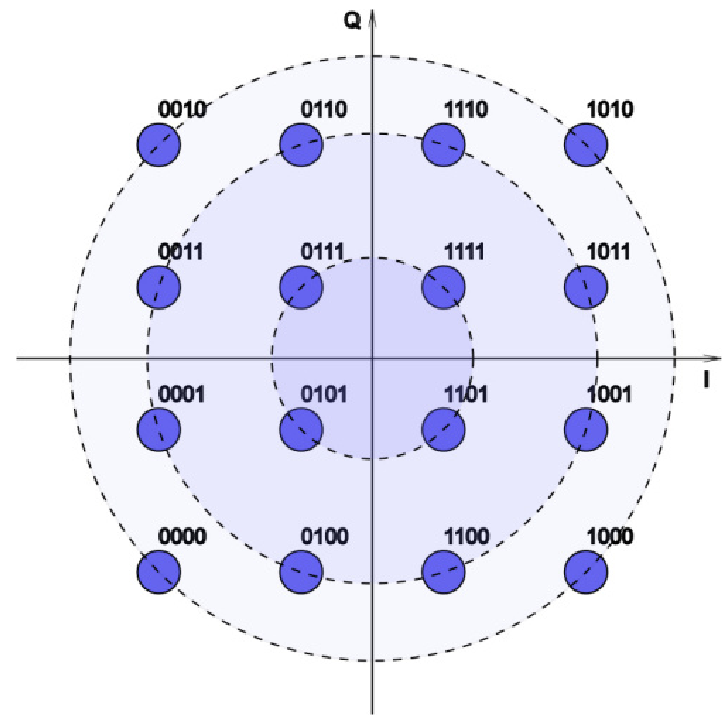


FIGURE 3.2–4
Examples of combined PAM-PSK constellations.

Multi-dimensional Signals

- Orthogonal multidimensional basis

Example :M-ary Frequency Shift Keying(MFSK)

$$S_m(t) = \sqrt{\frac{2E}{T}} \cos(2\pi f_c t + 2\pi f_m t) \;, \quad \begin{array}{l} m = 1,2,\ldots,M \\ 0 \le t \le T \end{array}$$

The orthogonal basis :

$$\phi_m(t) = \frac{S_m(t)}{\sqrt{E}} = \sqrt{\frac{2}{T}} \cos(2\pi(f_c + f_m)t) \;, \quad \begin{array}{l} m = 1,2,\ldots,M \\ 0 \le t \le T \end{array}$$

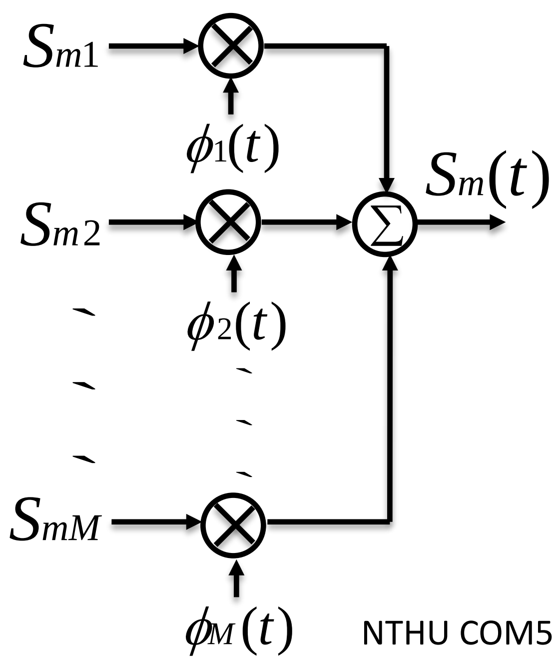
$$\begin{aligned}
\int_0^T \phi_m(\mathrm{t})\phi_n(\mathrm{t})\mathrm{dt} &= \frac{2}{\mathrm{T}} \int_0^T \cos[2\pi(f_c + f_m)t]\cos[2\pi(f_c + f_n)t]dt \\
&= \frac{1}{T}\int_0^T \cos[2\pi(2f_c + f_m + f_n)t]dt + \frac{1}{T}\int_0^T \cos[2\pi(f_m - f_n)t]dt \\
&= \frac{A_1}{T}\sin[2\pi(2f_c + f_m + f_n)T] + \frac{A_2}{T}\sin[2\pi(f_m - f_n)T] \\
&= 0 \;, \quad \text{(Has to be 0 to satisfy orthogonality)}
\end{aligned}$$

$$\Rightarrow \begin{cases} (1)\ 2f_c + f_m + f_n = \dfrac{n_c}{2T} \quad , \ n_c \in \text{ integer} \\ (2)\ \Delta f = \left| f_m - f_n \right| = \dfrac{k}{2T} \quad , \ k \in \text{integer} \ , \end{cases} \Rightarrow \boxed{\Delta f_{\min} = \frac{1}{2T}}$$

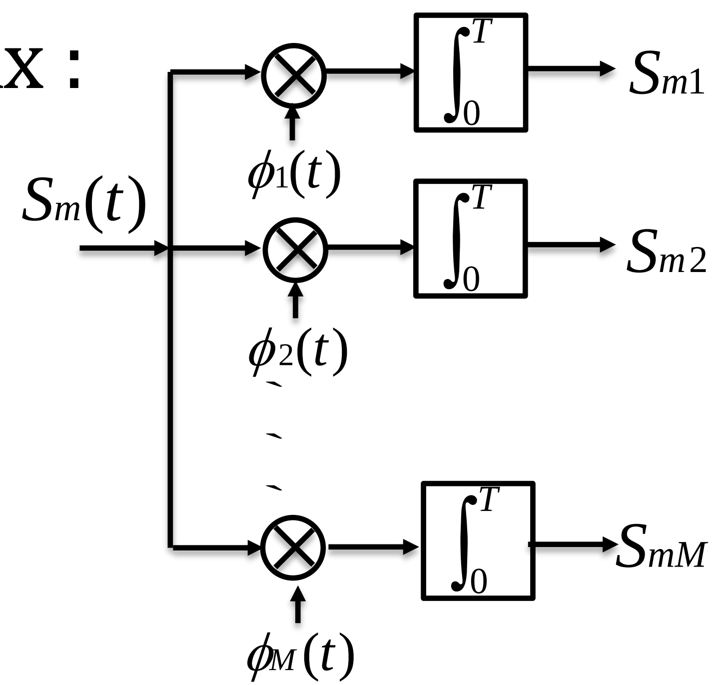
$$S_{mn} = \int_0^T S_m(t)\phi_n(t)dt \ , m = 1,2,\ldots,M$$

$$= \begin{cases} \sqrt{E} \ , m = n \\ 0 \ , m \neq n \end{cases}$$

Tx :

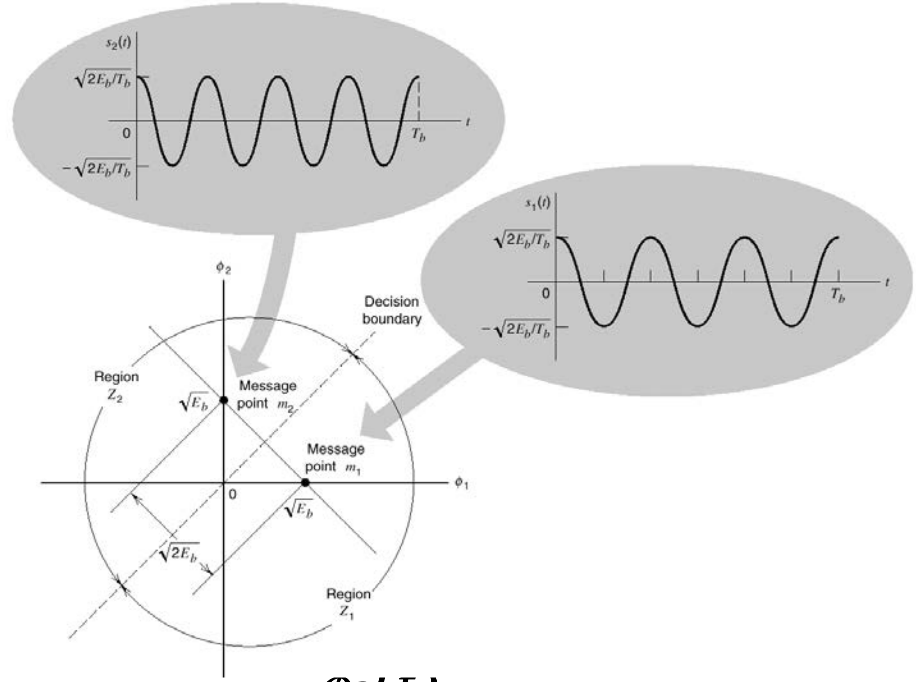


Rx :



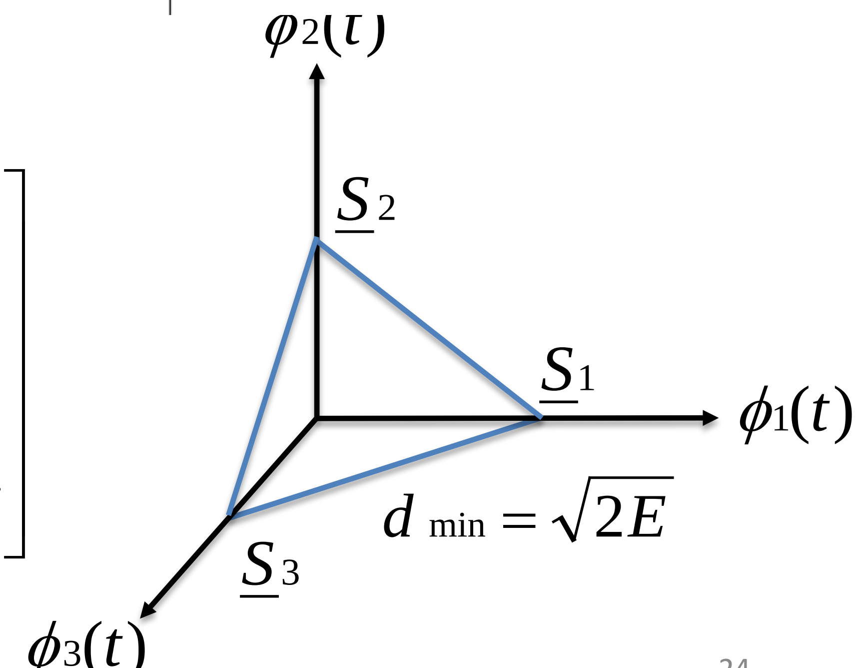
$$M = 2 \Rightarrow BFSK$$

$$S_m(t) = \begin{cases} \sqrt{\dfrac{2E}{T}} \cos[2\pi(f_c - \dfrac{\Delta f}{2})t] \,, & m = 1, \\ \sqrt{\dfrac{2E}{T}} \cos[2\pi(f_c + \dfrac{\Delta f}{2})t] \,, & m = 2 \end{cases}$$



General MFSK,

$$\underline{S}_1 = \begin{bmatrix} \sqrt{E} \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}, \quad \underline{S}_2 = \begin{bmatrix} 0 \\ \sqrt{E} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, \underline{S}_M = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ \sqrt{E} \end{bmatrix}$$



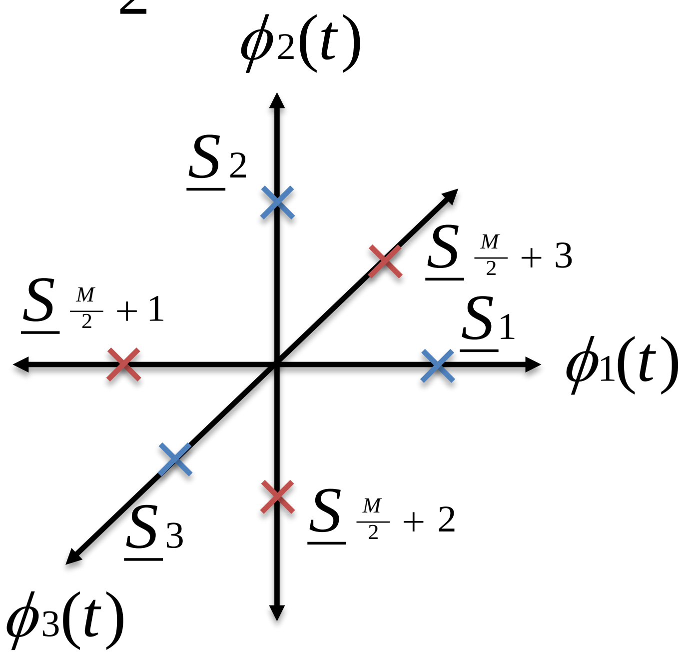
Bi-orthogonal Signaling

Construct M-biorthogonal signals from M/2 orthogonals in pairs, $M \in \text{even integer}$

$$S_m(t) = \begin{cases} \sqrt{\frac{2E}{T}} \cos[2\pi(f_c + f_m)t]\,,\ m = 1,2,\ldots,\frac{M}{2} \\ -\sqrt{\frac{2E}{T}} \cos[2\pi(f_c + f_m)t]\,,\ m = \frac{M}{2}+1, \frac{M}{2}+2,\ldots, M \end{cases}$$

with the basis functions :

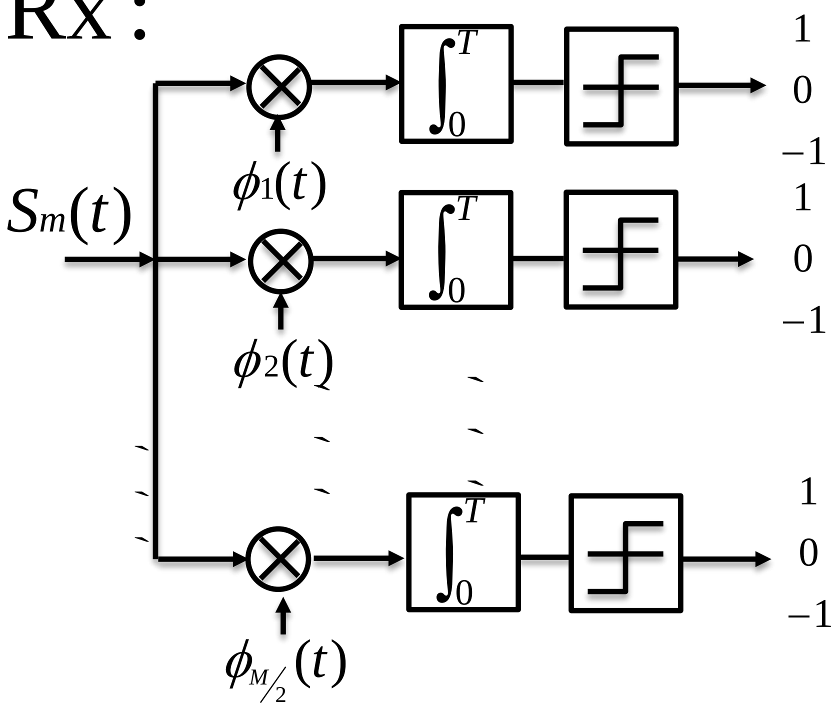
$$\phi_i(t) = \sqrt{\frac{2}{T}} \cos[2\pi(f_c + f_i)t],\ i = 1,2,\ldots,\frac{M}{2}$$



The bi-orthogonal signals has mutual correlation

$$\rho = -1, 0, +1 \quad i.e., \rho = \frac{\int_0^T S_m(t) S_n(t)\,dt}{\int_0^T S_m^{\,2}(t)\,dt} = \begin{cases} +1 & , \ n = m \\ 0 & , \ otherwise \\ -1 & , \ n = \frac{M}{2} + m \end{cases}$$

Rx :



Q: What is the benefit of the bi-orthogonal?

✓ The system complexity saves by 50%

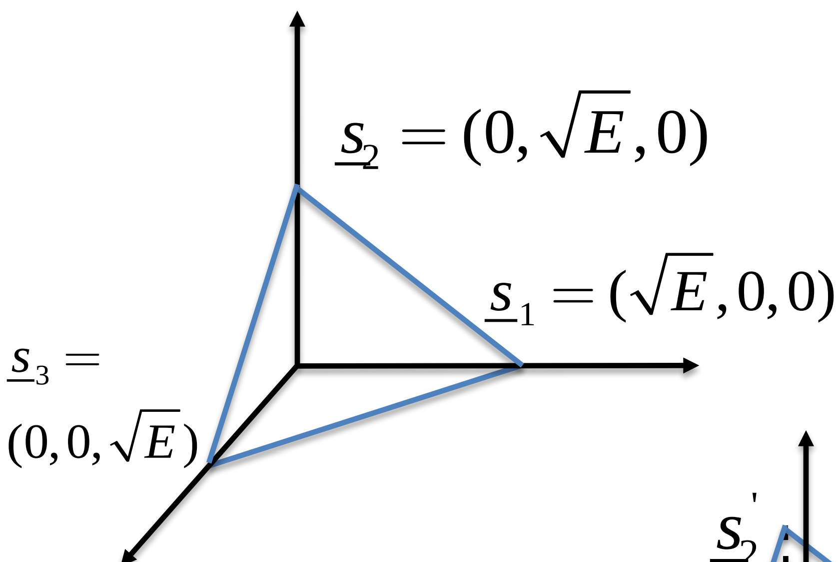
What's the advantages in bi-orthogonal signal?

(1) Complexity saving

(2) Better spectral efficiency than MFSK or M-ary orthogonal signals.

Simplex Signaling

Translate a set of M-ary orthogonal signals from the origin to the centroid of the constellation.

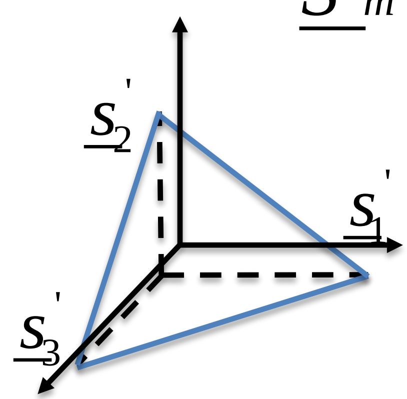


The centroid locates at :

$$\underline{S} = \frac{1}{M}\sum_{m=1}^{M} \underline{S}_m \, , \ \text{(mean value)}$$

The simplex signals are then

$$\underline{S}'_m = \underline{S}_m - \underline{S}$$



Characteristics of Simplex Signal

(1)Reduced Energy :

$$\left\|\underline{S'}_m\right\|^2 = \left\|\underline{S}_m - \underline{S}\right\|^2 = \left\|\underline{S}_m\right\|^2 - 2\underline{S}^T{}_m\underline{S} + \left\|\underline{S}\right\|^2$$

$$= E - \frac{2}{M}E + \frac{1}{M}E = (1 - \frac{1}{M})E$$

(2)Equal cross-correlation :

$$\rho_{mk} = \frac{\underline{S'}_m{}^T\,\underline{S'}_k}{\left\|\underline{S'}_m\right\|\,\left\|\underline{S'}_k\right\|} = \begin{cases} 1 & ,\ m = k \\ \dfrac{-1}{M-1} & ,\ m \neq k \end{cases}$$

where $\underline{S'}_m{}^T\,\underline{S'}_k = (\underline{S}_m - \underline{S})^T(\underline{S}_k - \underline{S}) = \underline{S}_m{}^T\,\underline{S}_k - \underline{S}_k{}^T\,\underline{S} - \underline{S}_m{}^T\,\underline{S} + \left\|\underline{S}\right\|^2$

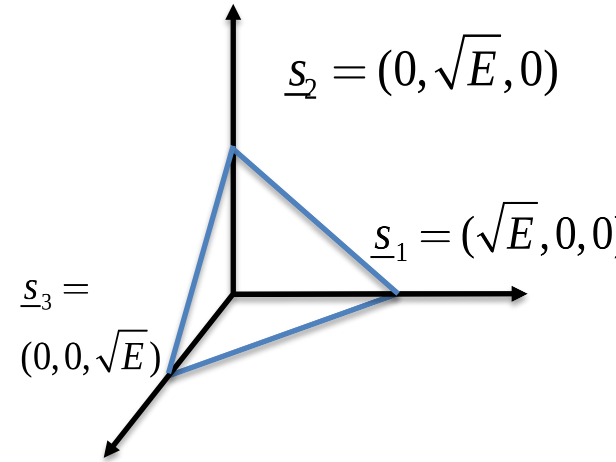
$$= -\frac{E}{M} - \frac{E}{M} + \frac{E}{M} = -\frac{1}{M}E$$

$$\rho_{mk} = \frac{-\frac{1}{M}}{1 - \frac{1}{M}} = \frac{-1}{M-1}$$

Example of Simplex Signal

- M-orthogonal signal

$$[\underline{S}_1 \;\; \underline{S}_2 \ldots \underline{S}_M] = \begin{bmatrix} \sqrt{E} & 0 & & 0 \\ 0 & \sqrt{E} & & \vdots \\ \vdots & 0 & \cdots & \vdots \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & \sqrt{E} \end{bmatrix}$$



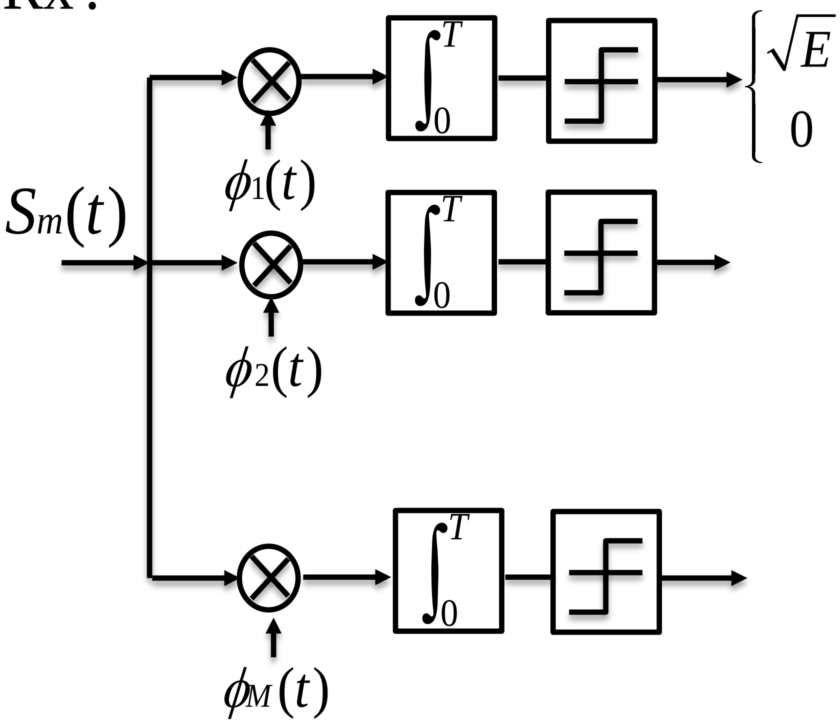
- Simplex Signal

$$[\underline{S}'_1 \;\; \underline{S}'_2 \ldots \underline{S}'_M] = \begin{bmatrix} \left(1-\frac{1}{M}\right)\sqrt{E} & -\frac{1}{M}\sqrt{E} & \cdots & -\frac{1}{M}\sqrt{E} \\ -\frac{1}{M}\sqrt{E} & \left(1-\frac{1}{M}\right)\sqrt{E} & \cdots & -\frac{1}{M}\sqrt{E} \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & & \vdots \\ -\frac{1}{M}\sqrt{E} & -\frac{1}{M}\sqrt{E} & \cdots & \left(1-\frac{1}{M}\right)\sqrt{E} \end{bmatrix}$$

- For MFSK :

$$\int S_m(t)\phi_k(t)dt = \underline{S_m}^T \bullet \frac{\underline{S_k}}{\left\|\underline{S_k}\right\|} = \begin{cases} \sqrt{E} & , \ m = k \\ 0 & , \ m \neq k \end{cases}$$

Rx :

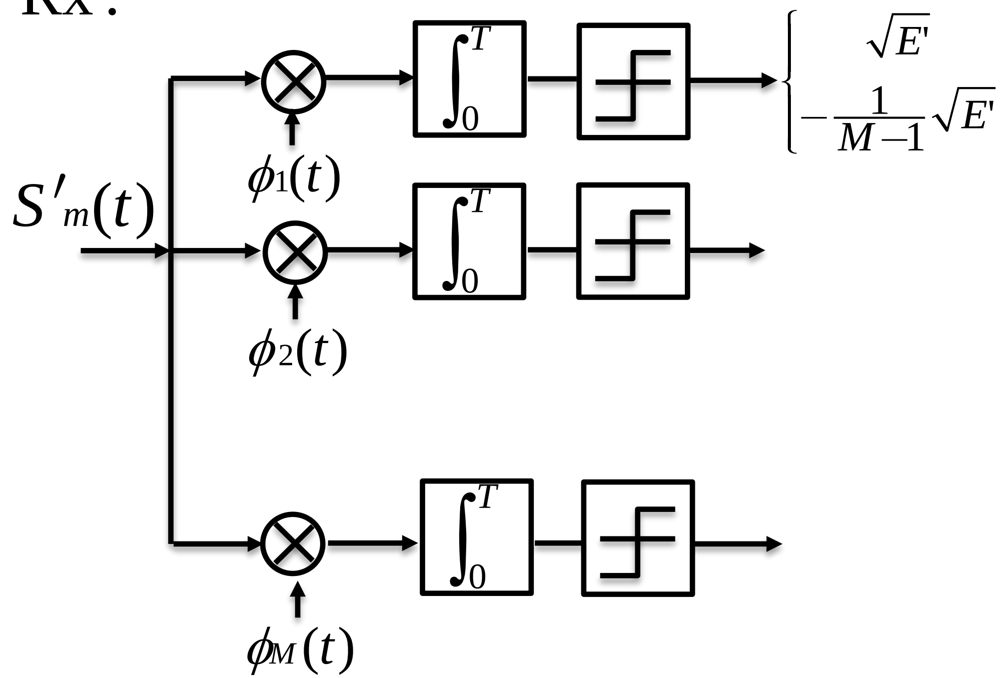


- For MFSK with simplex signaling

$$\int S'_m(t)\phi_k(t)dt = \underline{S'_m}^T \bullet \frac{\underline{S_k}}{\left\|\underline{S_k}\right\|} = \begin{cases} \sqrt{E'} & , \ m = k \\ -\dfrac{\sqrt{E'}}{M-1} & , \ m \neq k \end{cases}$$

where $E' = \left(1 - \dfrac{1}{M}\right)E$

Rx :



Test Your Understanding

1. **What is digital modulation?**

2. **How to find the signal space that represents a modulation scheme?**

Modulation with Memory

1. Linear Modulation with Memory
2. Non-linear Modulation with Memory

1. Linear Modulation with Memory (Proakis 3.3)

- Linear Correlated Encoding :

Linear modulation satisfies the superposition principle:

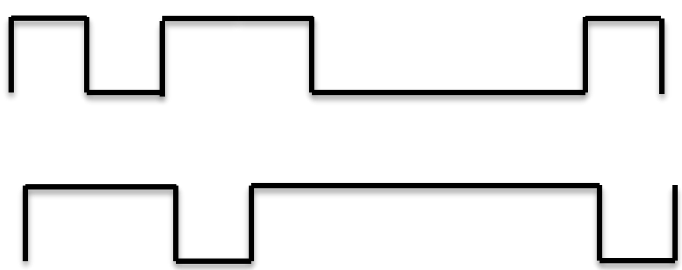
If $a_1 \to b_1$ and $a_2 \to b_2$, then $a_1 + a_2 \to b_1 + b_2$

Ex: Differential encoding

Transmit signal makes a transition when a message 1 is to transmit.

a_k 1 0 1 1 0 0 0 1

Data $a_k(t)$



Encoded $b_k(t)$

Q: Can you identify the relation of a_k and b_k from the waveforms?

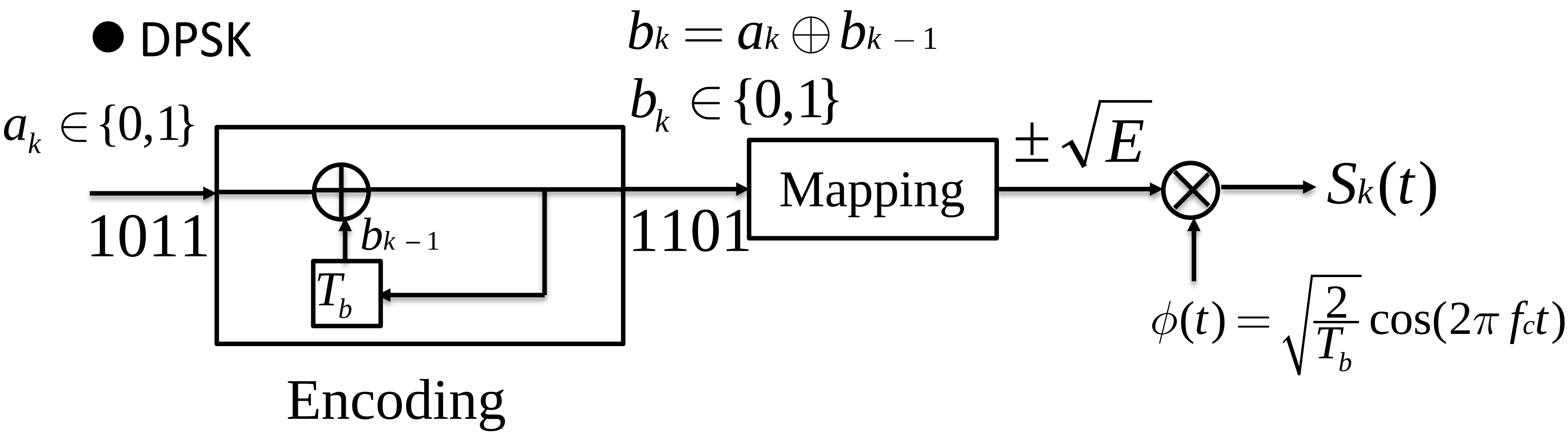
$$\Rightarrow b_k = a_k \oplus b_{k-1}$$

Q: Is this a linear modulation? Superposition principle?

If $b_k = a_k \oplus b_{k-1}$ and $b'_k = a'_k \oplus b_{k-1}$, then $(a_k + a'_k) \oplus b_{k-1} = b_k + b'_k$

Differential Encoding Tx

- DPSK



- The encoder output depends on the current input and the pervious symbol(s)

$\Rightarrow$ Encoding with memory.

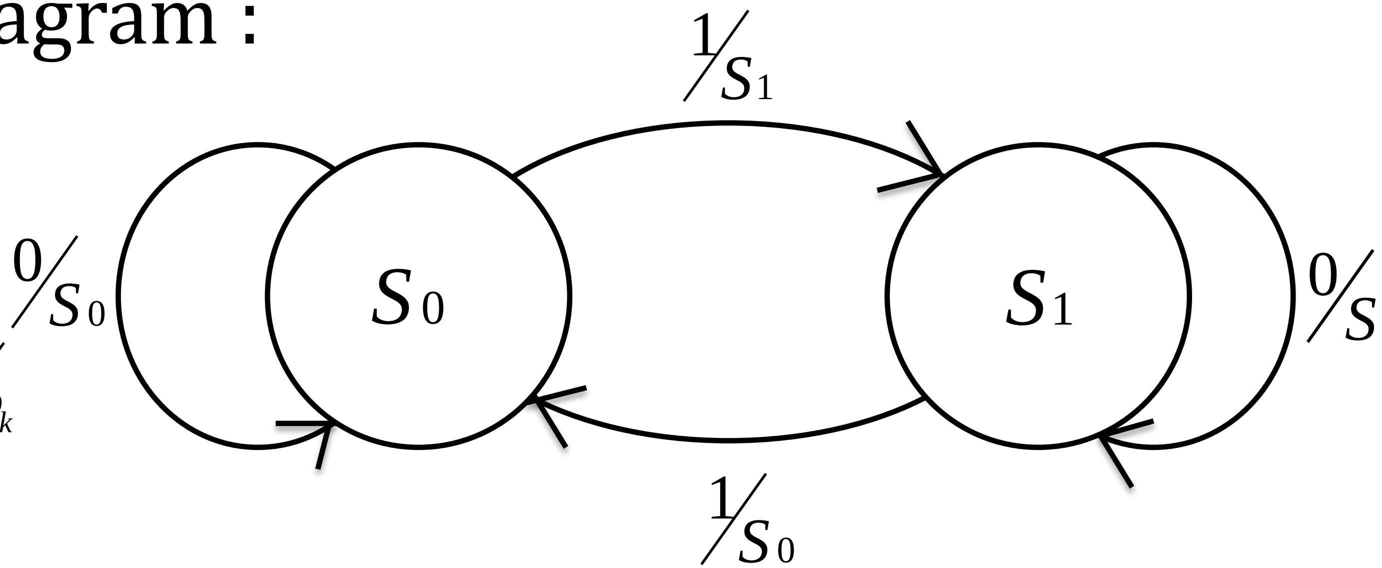
The transition can be expressed by:

- State diagram :

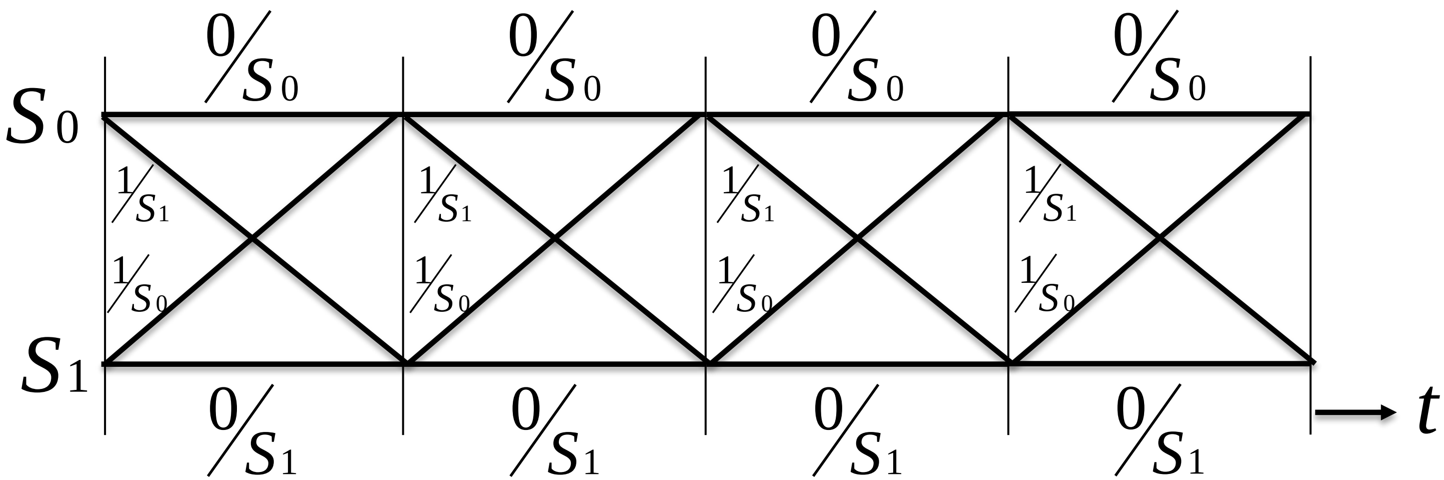
$S_0 : b_k = 0$

$S_1 : b_k = 1$

$\text{input}/\text{output} = a_k / b_k$



- Trellis diagram :



Linear modulation with memory can characterized by the Markov chain.

- ✓ The output b_k can be considered as the state of Markov chain with transition probability from s_i to s_j as

$$\prod_{ij} \quad i, j \in \{0,1\} \quad \prod_{ij} = \mathrm{P}(s_j \mid s_i)$$

Let the Markov transition matrix be: $\underline{\prod} = \begin{bmatrix} \prod_{00} & \prod_{10} \\ \prod_{01} & \prod_{11} \end{bmatrix}$

Let $\underline{\mathrm{P}}(n) = \begin{bmatrix} \mathrm{P}_0(n) \\ \mathrm{P}_1(n) \end{bmatrix}$ =Probability distribution of $b_k = 0$ and $b_k = 1$ at $\mathrm{t} = n\mathrm{T}$

$$\Rightarrow \underline{\mathrm{P}}(1) = \underline{\prod} \cdot \underline{\mathrm{P}}(0) = \begin{bmatrix} \prod_{00} & \prod_{10} \\ \prod_{01} & \prod_{11} \end{bmatrix} \begin{bmatrix} \mathrm{P}_0(0) \\ \mathrm{P}_1(0) \end{bmatrix}$$

$$\underline{\mathrm{P}}(1) = \underline{\prod} \cdot \underline{\mathrm{P}}(0)$$
$$\underline{\mathrm{P}}(2) = \underline{\prod} \cdot \underline{\mathrm{P}}(1)$$
$$\vdots$$
$$\underline{\mathrm{P}}(n) = \underline{\prod} \cdot \underline{\mathrm{P}}(n-1)$$

Remarks

If $\lim_{n\to\infty} \underline{\prod}^n$ exists, then the Markov chain stabilizes to a stationary state, and the probability distribution $\underline{\mathrm{P}}$ becomes a steady state.

$\Rightarrow$ In the steady state , the probability distribution remains unchanged, $i.e. \underline{\mathrm{P}}(n) = \underline{\mathrm{P}}(n-1) = \underline{\mathrm{P}}$.

$\Rightarrow$ By the Markov chain rule, $\underline{\mathrm{P}}(n) = \underline{\prod} \cdot \underline{\mathrm{P}}(n\text{-}1)$, $i.e. \underline{\mathrm{P}} = \underline{\prod} \cdot \underline{\mathrm{P}}$

i.e. , $\underline{\mathrm{P}}$ is the eigenvector of $\underline{\prod}$ with the eigenvalue = 1.

Ex: If the source is equiprobable and all transition probabilities = 1/2

$$\underline{\prod} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \text{ then } \lim_{n\to\infty} \underline{\prod}^n = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \qquad \Rightarrow \lim_{n\to\infty} \underline{\mathrm{P}}(n) = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

Modulation with Memory

1. Linear Modulation with Memory

2. Non-linear Modulation with Memory
   - ✓ CP-BFSK: Minimum Shift Keying(MSK)
   - ✓ CP-MFSK

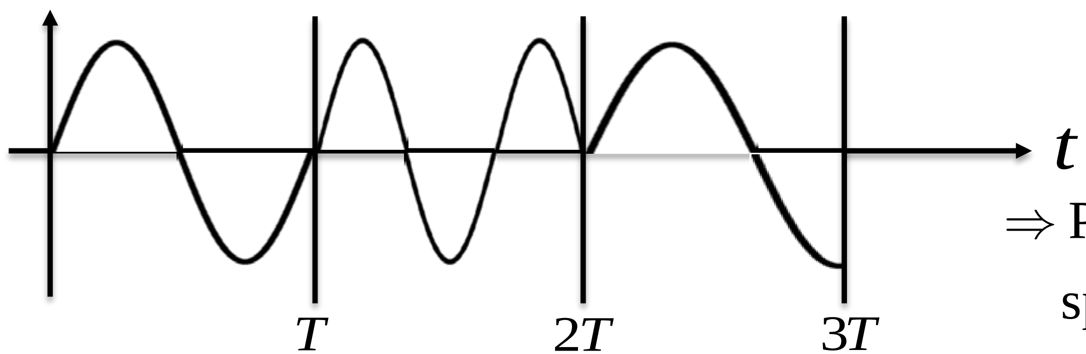
2. **Non-linear Modulation with Memory**

- Continuous Phase Modulation with FSK(CPFSK)

<u>Motivation /Observation:</u>

For memoryless FSK, the phase is dis-continuous.

$$S_m(t)=\sqrt{\tfrac{2E}{T}}\cos(2\pi f_c t+2\pi f_m t)\,,\quad \begin{matrix} m=1,2,\ldots,M \\ 0\le t\le T \end{matrix}$$



$\Rightarrow$ Produce high frequency spectral components.

- What would happen?

(1)Abrupt transition of phases leads to large spectral side-lobes.

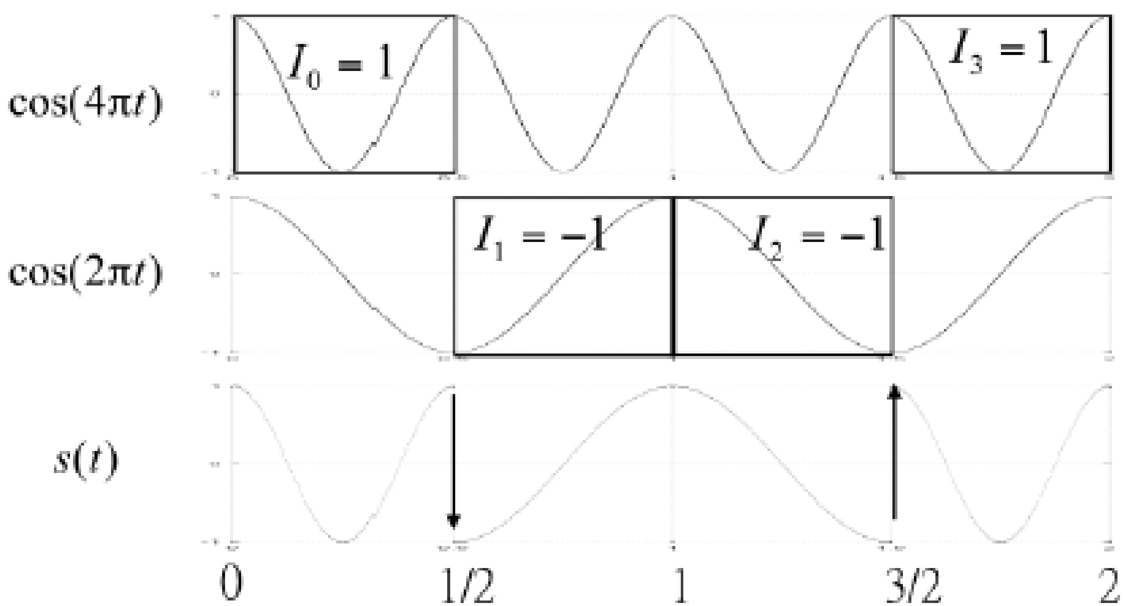
(2)Passing through a BPF to save the spectral BW, but leads to distorted signal.

✓ The continuous phase design helps to shape the spectrum of Tx signals for band-limited transmission.

Example of Memoryless BFSK

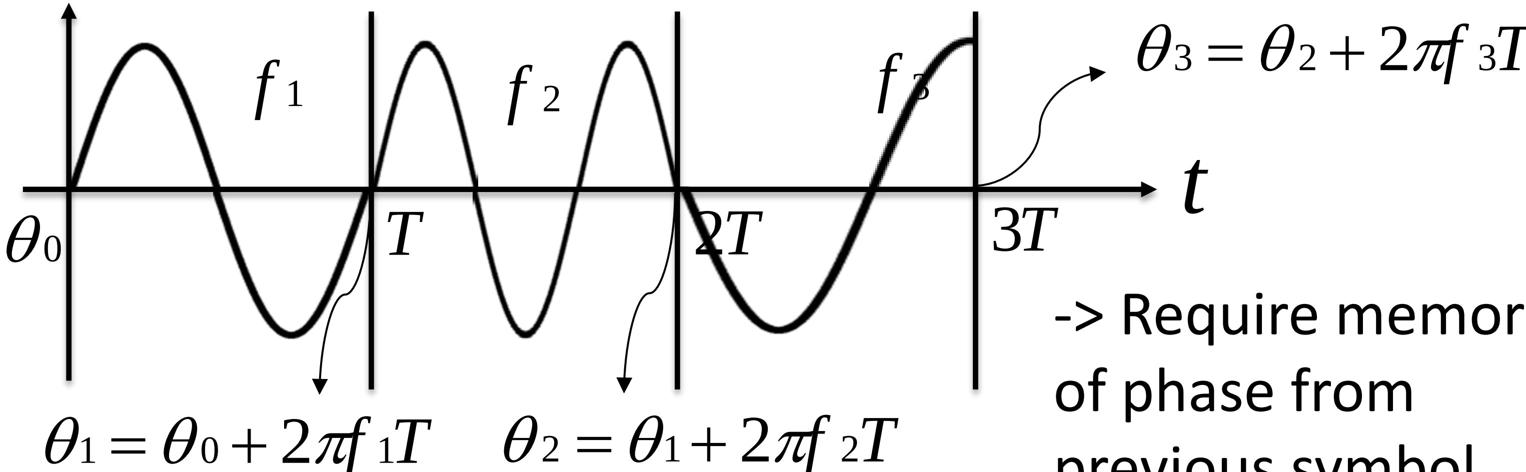
For the n^{th} symbol in time, $nT \le t \le (n+1)T$

$$S_n(t) = \begin{cases} \sqrt{\dfrac{2E}{T}} \cos[2\pi f_1 t] \text{ for symbol } 1 \\ \sqrt{\dfrac{2E}{T}} \cos[2\pi f_2 t] \text{ for symbol } 0 \end{cases},$$



41

Continuous-Phase FSK



-> Require memory of phase from previous symbol.

Minimum Shift Keying(MSK)

✓ MSK is a special form of CP-BFSK

For the n^{th} symbol in time, $nT \le t \le (n+1)T$

$$S_n(t) = \begin{cases} \sqrt{\frac{2E}{T}} \cos[2\pi f_1(t - nT) + \theta(nT)] \text{ for symbol } 1 \\ \sqrt{\frac{2E}{T}} \cos[2\pi f_2(t - nT) + \theta(nT)] \text{ for symbol } 0 \end{cases},$$

where $\theta(nT)$ denotes the phase value at end of the previous symbol.

$$\text{where}\begin{cases} f_1 = f_c + \dfrac{h}{2T} = f_c + h\cdot\Delta f_{\min}, \text{ symbol 1} \\ f_2 = f_c - \dfrac{h}{2T} = f_c - h\cdot\Delta f_{\min}, \text{ symbol 0} \end{cases} \text{and } \Delta f_{\min} = \frac{1}{2T}$$

$$\boxed{\begin{aligned} & h = T\cdot\Delta f = \text{modulation index} \\ & \Rightarrow h_{\min} = \frac{1}{2}(= T\cdot\Delta f_{\min}) \text{ (Minimum Shift)} \end{aligned}}$$

The MSK signal $S_n(t) = \sqrt{\dfrac{2E}{T}}\cos[2\pi f_c(t-nT) \underbrace{\pm 2\pi h\Delta f_{\min}(t-nT) + \theta(nT)]}_{\theta(t)}$

$$= \sqrt{\frac{2E}{T}}\cos[2\pi f_c(t-nT) + \theta(t)]$$

where $\theta(t) = \theta(nT) \pm \dfrac{\pi h}{T}(t-nT)\ ,\ nT \le t \le (n+1)T$

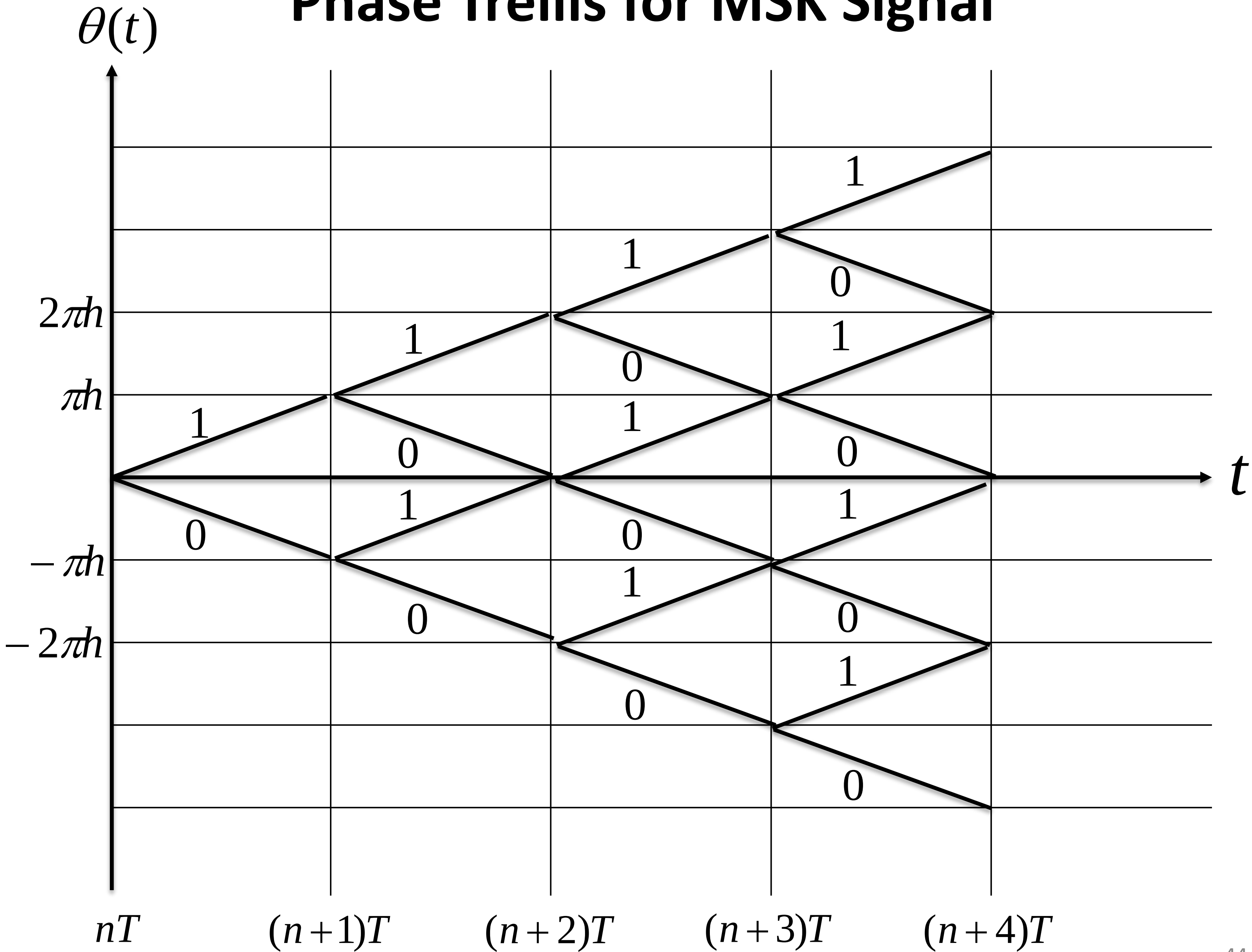
for the nth symbol and $\begin{cases} +\text{:symbol } 1 \\ -\text{:symbol } 0 \end{cases}$

The phase transition becomes continuous with phase trellis.

$$\theta[(n+1)T] - \theta[(nT)] = \begin{cases} \pi h\text{ , for symbol 1} \\ -\pi h\text{ , for symbol 0} \end{cases}$$

✓ Symbol detection is changed from frequencies to phase transitions.

Phase Trellis for MSK Signal



Signal space diagram of MSK

$$S(t) = \sqrt{\frac{2E}{T}} \cos[2\pi f_c t + \theta(t)]$$
$$= \sqrt{\frac{2E}{T}} \cos[\theta(t)] \cos(2\pi f_c t) - \sqrt{\frac{2E}{T}} \sin[\theta(t)] \sin(2\pi f_c t)$$

Let the basis function of MSK be :

$$\phi_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t) \quad , \; nT \le t \le (n+1)T$$
$$\phi_2(t) = -\sqrt{\frac{2}{T}} \sin(2\pi f_c t) \quad , \; nT \le t \le (n+1)T$$

Then the projection on $\phi_1(t)$, $\phi_2(t)$ are

$$\underline{S} = \left[\sqrt{E}\cos[\theta(t)] \,,\; \sqrt{E}\sin[\theta(t)]\right]^T = \left[S_I(t) \,,\; S_Q(t)\right]^T$$

$$\theta(t) = \theta(nT) \pm \frac{\pi}{2T}(t - nT), \quad nT \le t \le (n+1)T$$

for MSK with $\Delta f_{\min} = \dfrac{1}{2T}, \; h = \dfrac{1}{2}$

Without losing generality, assume the staring phase $\theta(nT)=0, \text{ or } \pi$

$$S_I(t)=\sqrt{E}\cos\big(\theta(t)\big), \quad S_Q(t)=\sqrt{E}\sin\big(\theta(t)\big)$$

$$\theta(t)=\theta(nT)\pm\frac{\pi}{2T}(t-nT) \quad for \quad nT\le t\le(n+1)T$$

At $t=nT \Rightarrow$ $\boxed{S_I(t)=S_I(nT)=\sqrt{E}\cos[\theta(nT)]}$

$$S_Q(t)=S_Q(nT)=\sqrt{E}\sin[\theta(nT)]=0$$

At $t=(n+1)T \Rightarrow S_I(t)=S_I[(n+1)T]=\sqrt{E}\cos[\theta(nT)\mp\frac{\pi}{2}]=0$

$$\boxed{S_Q(t)=S_Q[(n+1)T]=\sqrt{E}\sin[\theta(nT)\pm\frac{\pi}{2}]=\pm\sqrt{E}\cos[\theta(nT)]}$$

where "+"=symbol 1, "$-$" = symbol 0

The phasor representation and the constellation point of the signal are defined as :

Phasor : $\tilde{S}(t)=S_I(t)+jS_Q(t+T), \ or$

Vector : $\underline{S}=[S_I(nT) \ , \ S_Q(n+1)T] \quad \text{where} \begin{cases} S_I(nT)=\sqrt{E}\cos[\theta(nT)] \\ S_Q[(n+1)T]=\pm\sqrt{E}\cos[\theta(nT)] \end{cases}$

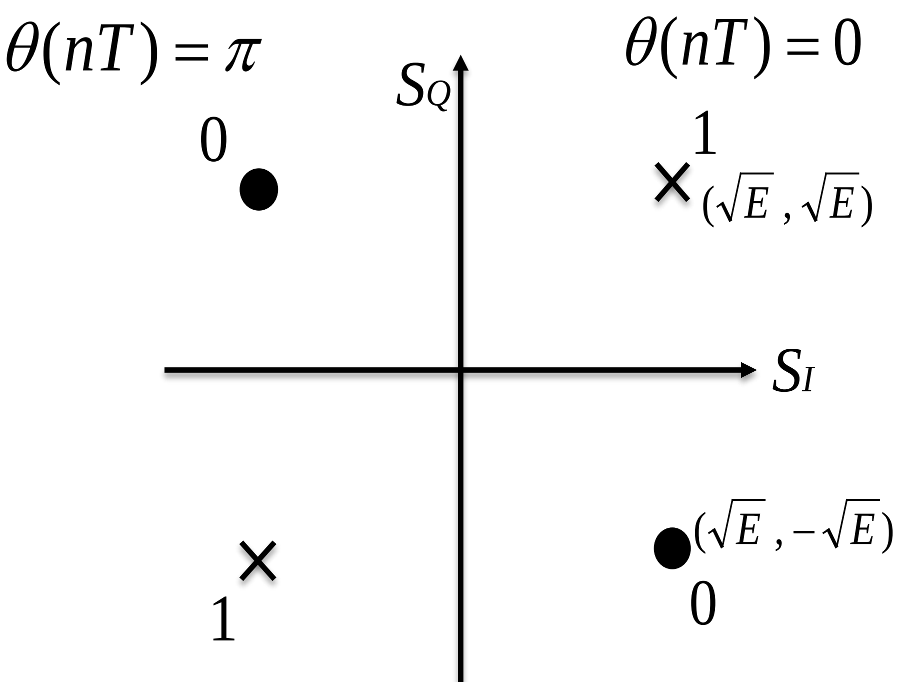
$$\stackrel{\Delta}{=} [S_I \ , \ S_Q]$$

Example : $\underline{S} = [S_I(nT)\ ,\ S_Q(n+1)T] \overset{\Delta}{=} [S_I\ ,\ S_Q]$

$$\theta(nT) = 0 \Rightarrow (S_I, S_Q) = \begin{cases} (\sqrt{E}\ ,\ \sqrt{E}) : & \text{symbol } 1, \\ (\sqrt{E}\ ,\ -\sqrt{E}) : & \text{symbol } 0, \end{cases}$$

Note that the reference initial phase $\theta(nT)$ is not necessarily 0, e.g.

$$\theta(nT) = \pi \Rightarrow (S_I, S_Q) = \begin{cases} (-\sqrt{E}\ , -\sqrt{E}) : & \text{symbol } 1 \\ (-\sqrt{E}\ , \sqrt{E}) : & \text{symbol } 0 \end{cases}$$



$$\begin{cases} S_I(nT) = \sqrt{E}\cos[\theta(nT)] \\ S_Q[(n+1)T] = \pm\sqrt{E}\cos[\theta(nT)] \end{cases}$$

$\mathit{Symbol}\ 1 : S_I, S_Q$ same sign

$\mathit{Symbol}\ 0 : S_I, S_Q$ opposite sign

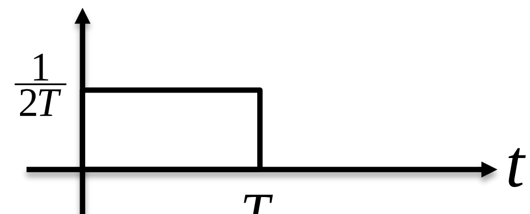
Non-linear Modulation with Memory (M-ary CPFSK)

- Constructing M-ary CPFSK from M-ary PAM

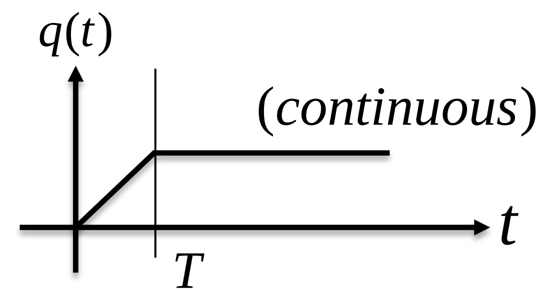
Given M-PAM $d(t)=\sum_n I_n g(t-nT),$

where $\{I_n\}\in\{\pm1,\pm3,\ldots,\pm(M-1)\}$ are M-PAM symbols.

Let $g(t)=\frac{1}{2T}rect(t-\frac{T}{2})$



Define $q(t)=\int_0^t g(\tau)d\tau=\begin{cases}0 & , t\le 0\\ \frac{t}{2T} & , 0\le t\le T\\ \frac{1}{2} & , t>T\end{cases}$



Each M-ary PAM symbol I_n is mapped into frequencies $\{\pm f_d,\pm3 f_d,\ldots,\pm(M-1) f_d\}$

The CPFSK signal is :

$$S(t)=\sqrt{\frac{2E}{T}}\cos[2\pi f_c t+\theta(t,d(t))],\quad nT\le t\le(n+1)T$$

where $\theta(t,d(t))=4\pi T f_d\int_{-\infty}^{t}d(\tau)d\tau$

$$=4\pi f_d T\int_{-\infty}^{t}\left(\sum_n I_n g(\tau-nT)\right)d\tau$$

$$=4\pi f_d T\sum_{k=-\infty}^{n-1}I_k\underbrace{\int_{-\infty}^{nT}g(\tau-kT)d\tau}_{=1/2}+4\pi f_d T I_n\int_{nT}^{t}g(\tau-nT)d\tau$$

$$=2\pi f_d T\sum_{k=-\infty}^{n-1}I_k+4\pi f_d T I_n q(t-nT)$$

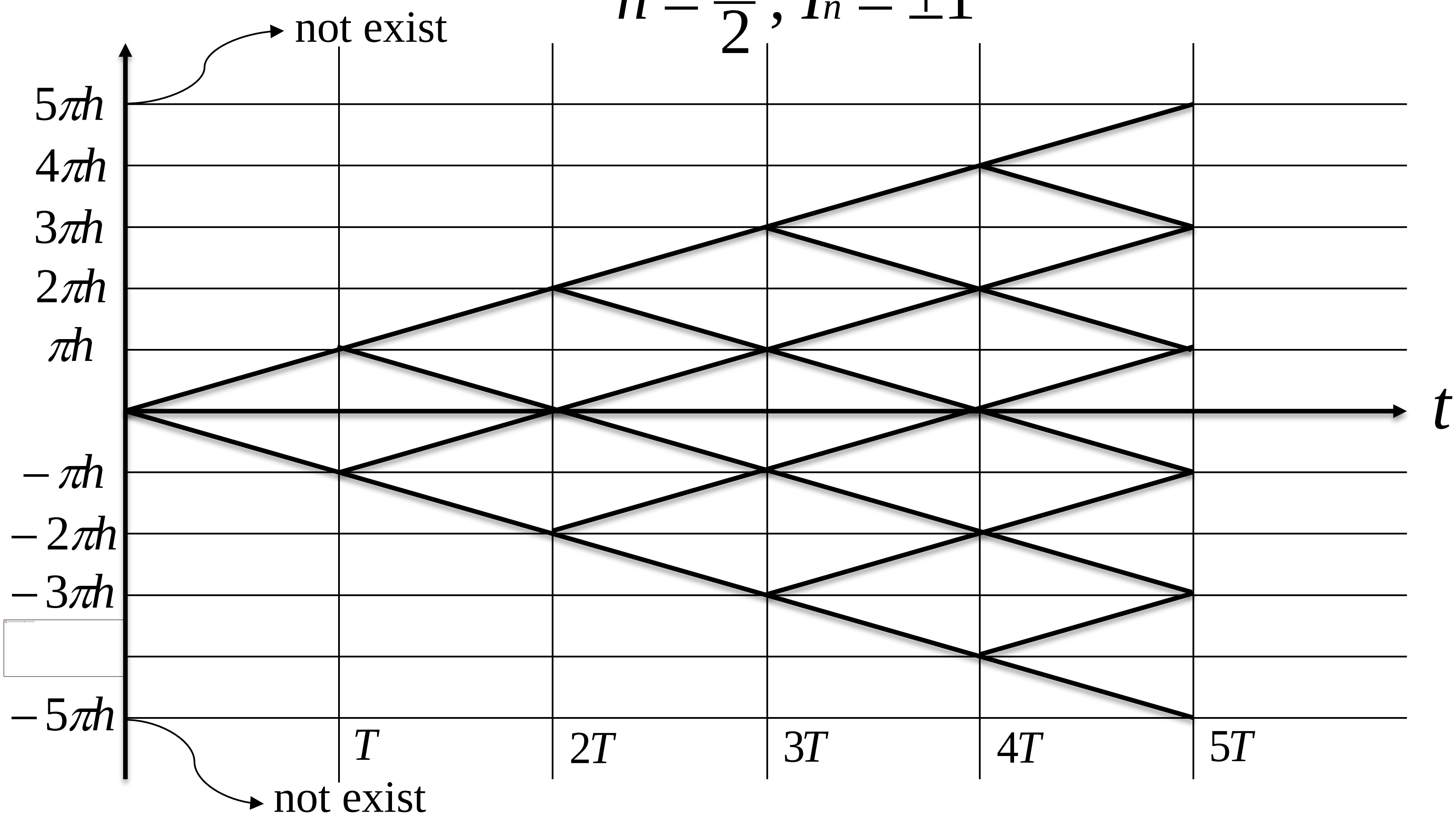
$$=\theta(nT)+2\pi h I_n q(t-nT)\ ,\ nT\le t\le(n+1)T$$

Note : $h=T\cdot\Delta f=2f_d T$

$$\begin{cases}t=nT\ ,\ \theta(t)=\theta(nT)\\ t=(n+1)T\ ,\ \theta(t)=\theta(nT)+\pi h I_n=\theta[(n+1)T]\Rightarrow \text{phase trellis}\end{cases}$$

Phase Trellis for M-ary CPFSK Signal, M=2

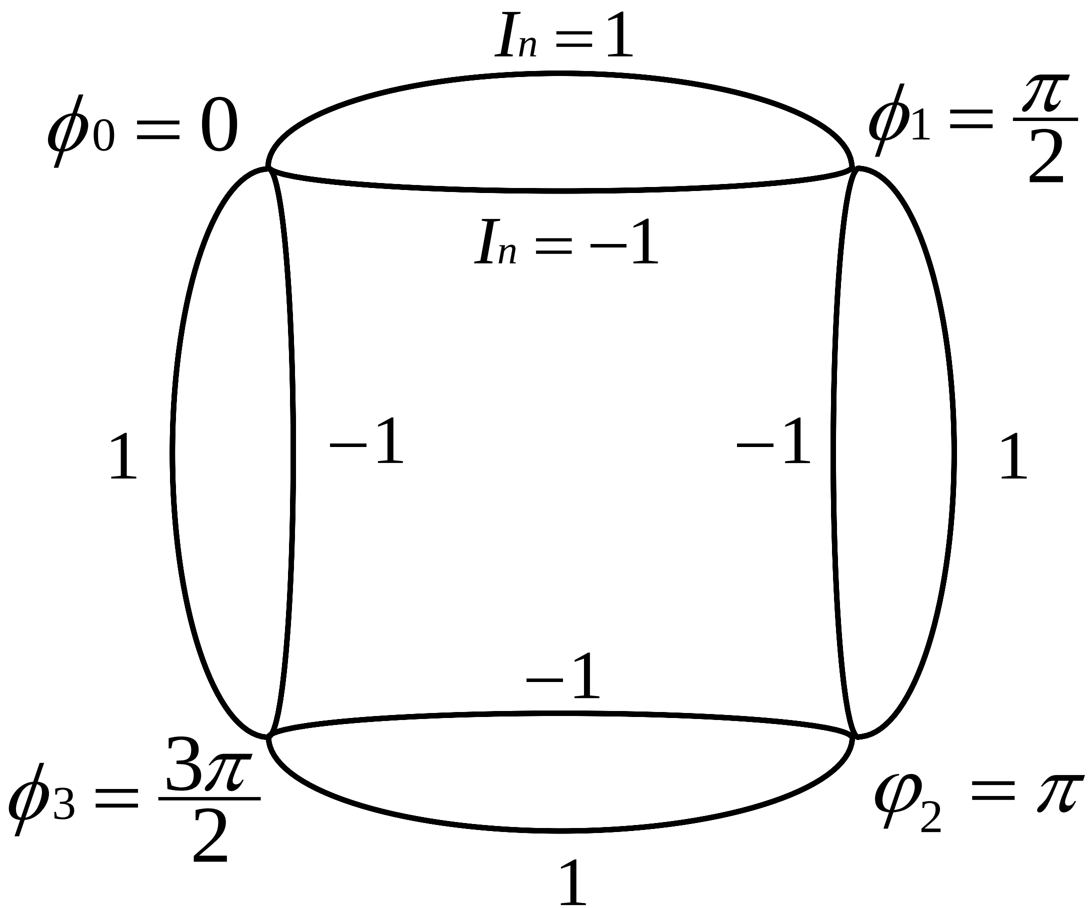
$$h = \frac{1}{2}\,,\ I_n = \pm 1$$



In fact , in this example, there are only four states.

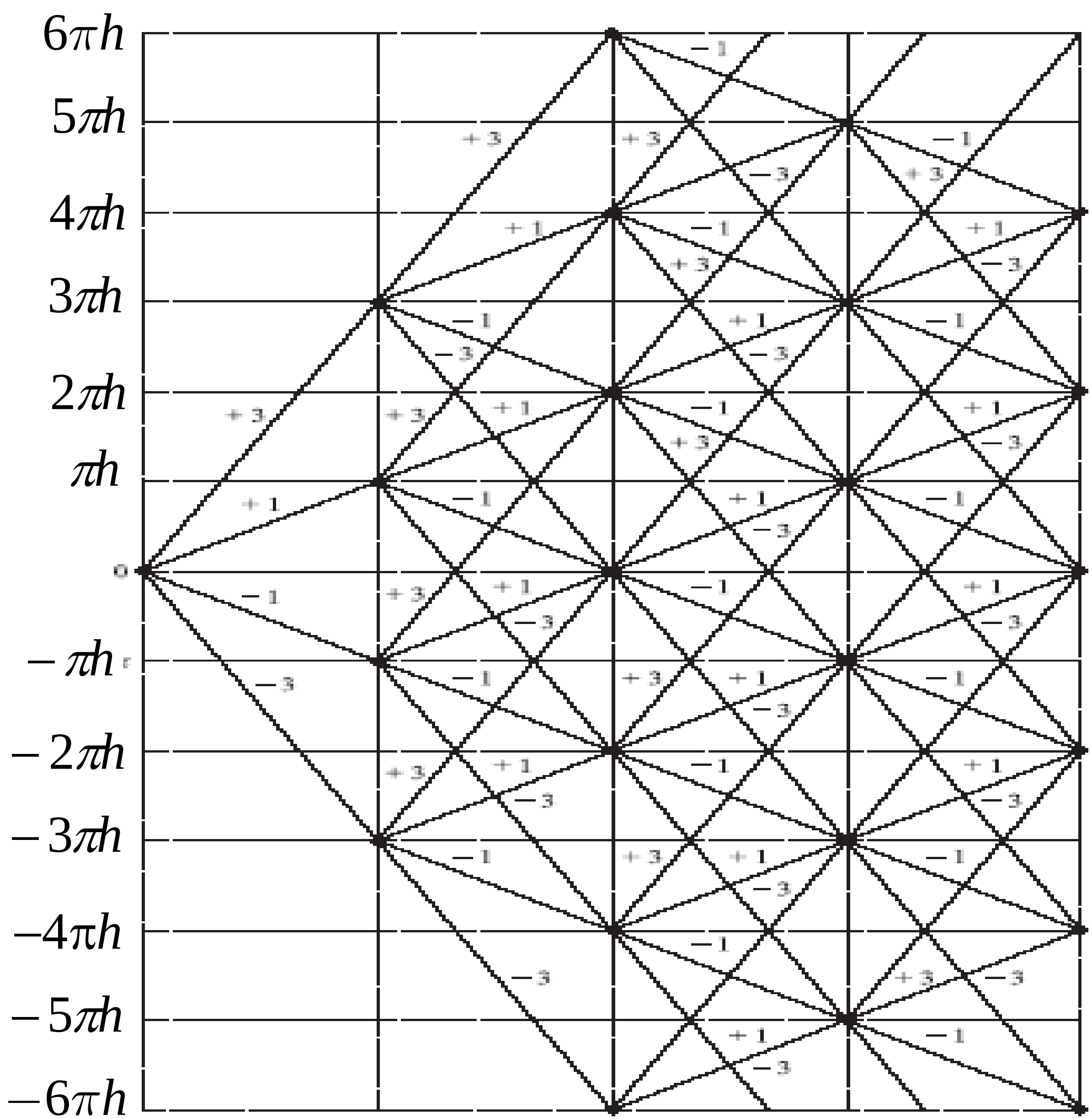
$$(\pi h I_n = \frac{\pi}{2} \Rightarrow \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi)$$

- State diagram (Binary CFSK , $h = \frac{1}{2}$, $I_n = \pm 1$)



Phase Trellis for M-ary CPFSK Signal (M=4)

$$I_n \in \{\pm 1, \pm 3\}$$



Thus CPM can be decoded by Viterbi trellis decoding.

- The number of possible phase states is determined by h.

For $h = \dfrac{m}{p}$, where m and p are mutually prime integers,

we have ：

$$\begin{cases} p \text{ states , i.e., } \left\{0, \dfrac{\pi}{\mathrm{p}} m, \ldots, \dfrac{(\mathrm{p}-1)\pi}{\mathrm{p}} m\right\} \text{ when m is even} \\ 2p \text{ states , i.e., } \left\{0, \dfrac{\pi}{\mathrm{p}} m, \ldots, \dfrac{(2\mathrm{p}-1)\pi}{\mathrm{p}} m\right\} \text{ when m is odd} \end{cases}$$

i.e. there are $\dfrac{2\pi}{\pi h} = \dfrac{2}{h}$ states.

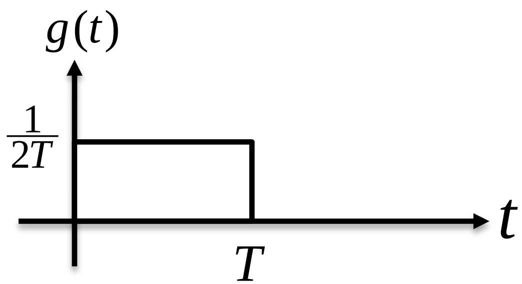
Example :

$$h = \frac{1}{3} \Rightarrow \ \frac{2}{h} = 6 \text{ states } (= 2\mathrm{p})$$

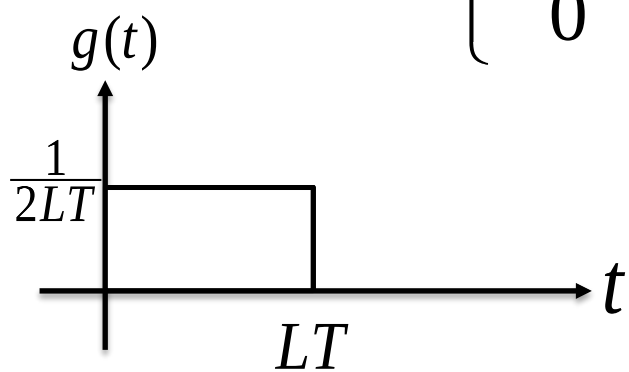
$$h = \frac{2}{3} \Rightarrow \ \frac{2}{h} = 3 \text{ states } (= \mathrm{p})$$

In general , CPM has many variations by choosing different combination of *h* , M , and *g*(*t*).

$$\text{For CPFSK}: g(t) = \frac{1}{2T} rect\left(\frac{t-\frac{T}{2}}{T}\right)$$



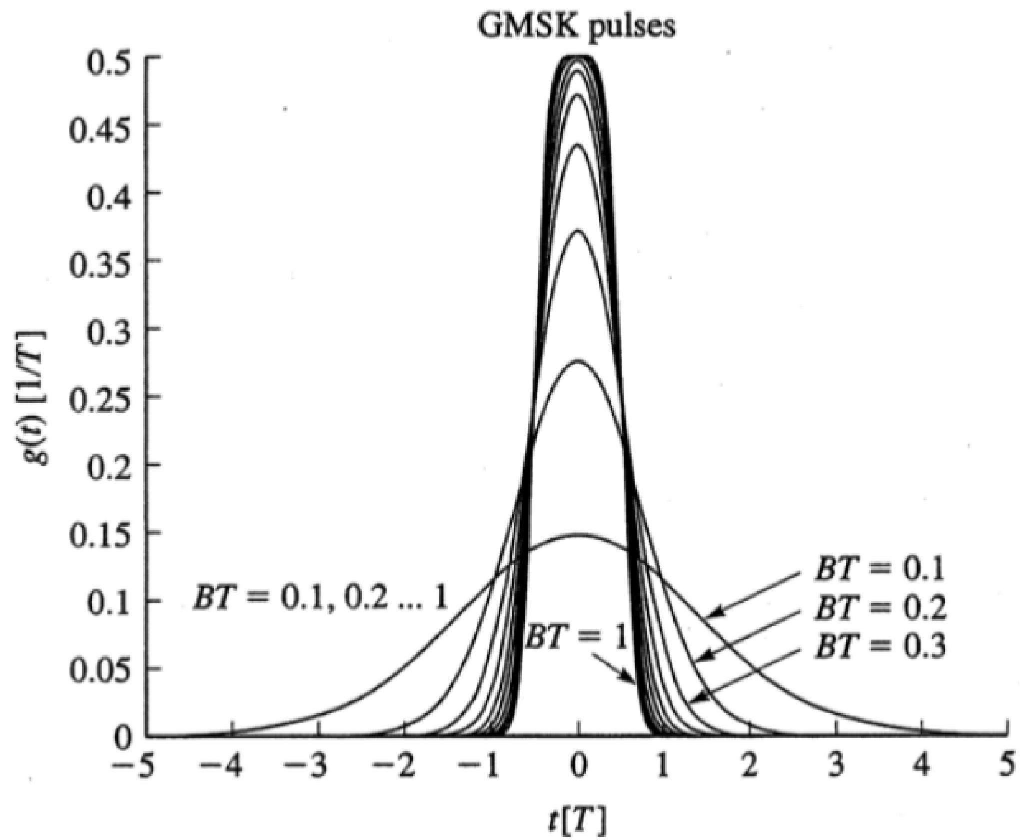
$$\text{For L - Rect CPFSK}: g(t) = \begin{cases} \frac{1}{2LT} & ,0 \le t \le LT \\ 0 & ,otherwise \end{cases}$$



- Gaussian pulse shaping function for MSK(GMSK)

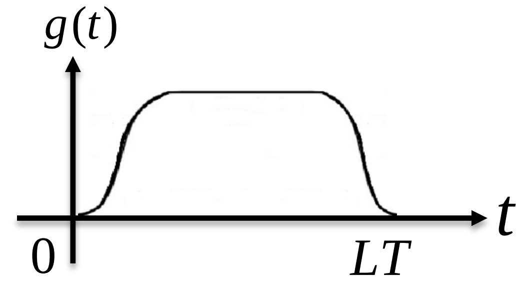
$$g(t) = \frac{1}{\sqrt{\ln 2}}\{Q[2\pi B(t-\frac{T}{2})] - Q[2\pi B(t+\frac{T}{2})]\}$$

Note:
GMSK with BT = 0:3 is used in the European digital cellular communication system, called GSM (2G).



- LRC(L - Raised Cosine)

$$g(t)=\begin{cases}\frac{1}{2LT}[1-\cos(\frac{2\pi t}{LT})] & ,0\le t\le LT\\ 0 & ,otherwise\end{cases}$$



In general, the $g(t)$ can be classified into two types:

Full - response CPM : $g(t)=0$, for $t\ge T$

Partial - response CPM : $g(t)\neq 0$, for $t\ge T$

Power Spectral Density of Digital Modulation Signal

Why studying power spectral characteristics?

- Defines the bandwidth requirement for a modulation scheme in the transmission channel.

Power Spectral Density of Digital Modulation Signal

Given a low-pass (i.e. baseband) signal $v(t)$ that is WSS random proc.,

the passband signal : $s(t) = v(t)\cos(2\pi f_c t) = \mathrm{Re}\left\{ v(t)e^{j2\pi f c t} \right\}$

It can be shown that the power spectral density :

$$S_s(f) = \frac{1}{4}[S_v(f - f_c) + S_v(f + f_c)]$$

Proof. $\mathrm{R}_s(\tau) = \mathrm{E}[v(t+\tau)\cos(2\pi f_c(t+\tau))v(t)\cos(2\pi f_c t)]$

$$= \mathrm{E}\left[v(\mathrm{t}+\tau)v(\mathrm{t})\frac{1}{2}[\cos(2\pi f_c(2t+\tau))+\cos(2\pi f_c\tau)] \right]$$

$$= \frac{1}{2}\mathrm{E}\left[v(t+\tau)v(t)\right]\cos(2\pi f_c\tau) \quad \text{(assume ergodic r.p.)}$$

$$= \frac{1}{2}\mathrm{R}_{\mathrm{vv}}(\tau)\cos(2\pi f_c\tau)$$

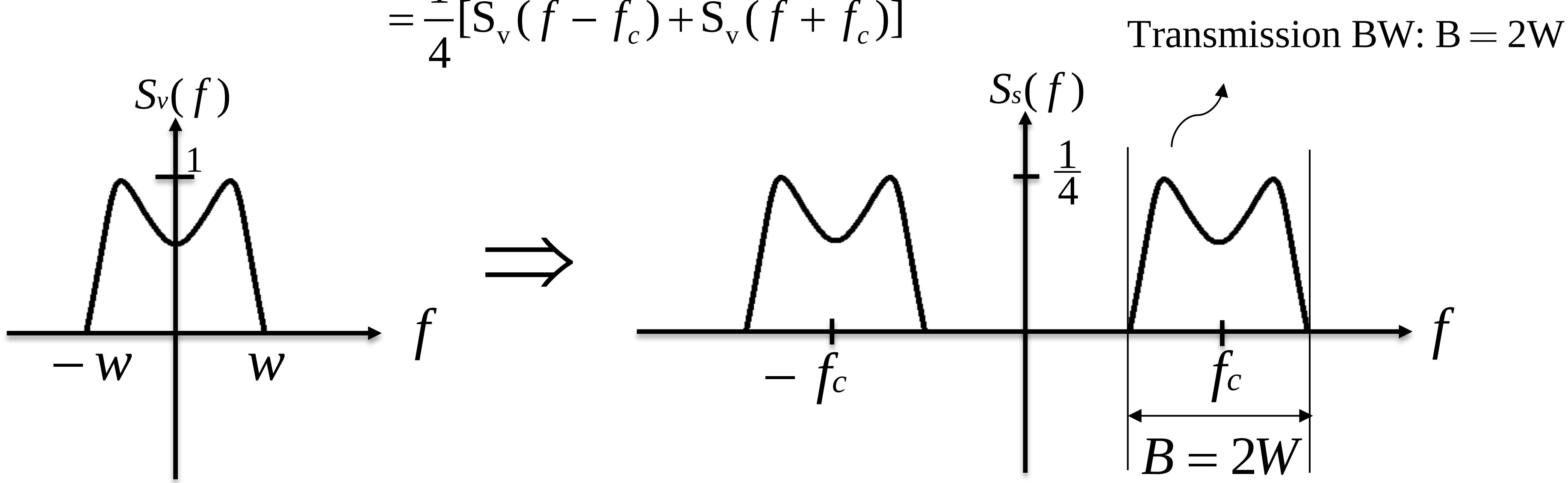
Power Spectral Density of Digital Modulation Signal

The PSD of $s(t)$ is the F.T. of $\mathrm{R}_s(\tau)$.

$$\mathrm{S}_s(f) = F\{\mathrm{R}_s(\tau)\} = F\left\{\frac{1}{2}\mathrm{R}_{vv}(\tau)\cos(2\pi f_c \tau)\right\}$$

$$= \frac{1}{2}\mathrm{S}_v(f) * \frac{1}{2}[\delta(f - f_c) + \delta(f + f_c)]$$

$$= \frac{1}{4}[\mathrm{S}_v(f - f_c) + \mathrm{S}_v(f + f_c)]$$



Therefore , we can focus on baseband $S_v(f)$ and the passband PSD $S_s(f)$ can be obtained accordingly.

Power Spectral Density of Cyclo-Stationary Signals

Let the low-pass (i.e. baseband) signal $v(t)$ be

$$v(t) = \sum_{n=-\infty}^{\infty} I_n g(t - nT),$$

where

T is the symbol period,

$\{ I_n \}$ is W.S.S. random sequence with $E[I_n] = \mu_n$

and autocorrelation $R_{II}[k] = E\left[I_n I_{n+k}^{*} \right]$.

Lemma : $v(t)=\sum_{n=-\infty}^{\infty} I_n g(t-nT)$ is cyclo-stationary random process

Proof : $E\left[v(t)\right]=\mu_I \cdot \sum_n g(t-nT) \Rightarrow$ periodic in t with priod of T.

$$R_{vv}(t,\tau)=E[v(t+\tau)v^*(t)]=\sum_n \sum_m E[I_n I^*_m]\cdot g(t+\tau-nT)\cdot g^*(t-mT)$$

Let $k=n-m,$ then $E[I_n I^*_m]=R_{II}[n-m]\equiv R_{II}[k]$ $\because \{I_n\}$ is W.S.S.

$$g(t+\tau-nT)=g(t-(m+k)T+\tau)$$

$$R_{vv}(t,\tau)\overset{n=k+m}{=}\sum_k R_{II}[k]\cdot \sum_m g(t-(m+k)T+\tau)g^*(t-mT)$$

$\Rightarrow$ Periodic in t with period T

$\Rightarrow v(t)$ is cyclo-stationary.

Note : As $v(t)$ is cyclo-stationary , the averaged auto-correlation can be obtained by taking time-average over one period T.

The time averaged auto-correlation function is

$$\overline{R}_{vv}(\tau) = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} R_{vv}(t,\tau)dt$$

$$= \sum_k R_{II}[k] \sum_m \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} g(t+\tau-kT-mT)g^*(t-mT)dt$$

let $t' = t - mT$

$$= \sum_k R_{II}[k] \cdot \sum_m \frac{1}{T}\int_{-\frac{T}{2}-mT}^{\frac{T}{2}-mT} g(t'+\tau-kT)g^*(t')dt'$$

$$= \sum_k R_{II}[k] \cdot \frac{1}{T}\int_{-\infty}^{\infty} g(t'+\tau-kT)g^*(t')dt'$$

$$= \frac{1}{T}\sum_k R_{II}[k] R_{gg}\left(\tau - kT\right)$$

where $R_{gg}(\tau) \triangleq \int_{-\infty}^{\infty} g(t+\tau)g^*(t)dt = g(\tau) * g^*(-\tau)$

$$\Rightarrow R_{gg}(\tau-kT) = \int_{-\infty}^{\infty} g(t+\tau-kT)g^*(t)dt$$

The power spectral density :

$$S_{vv}(f) = \mathbb{F}\left\{\overline{R}_{vv}(\tau)\right\} = \mathbb{F}\left\{\frac{1}{T}\sum_k R_{II}[k]R_{gg}\left(\tau - kT\right)\right\}$$

Recall from Fourier transform properties, we note that

$$g(\tau) \overset{\mathbb{F}}{\leftrightarrow} G(f) \Rightarrow g^*(\tau) \leftrightarrow G^*(-f)$$

$$g^*(-\tau) \leftrightarrow G^*(f)$$

$$\begin{aligned}
\Rightarrow S_{vv}(f) &= \frac{1}{T}\sum_k R_{II}[k]\mathbb{F}\left\{R_{gg}(\tau - kT)\right\} \\
&= \frac{1}{T}\sum_k R_{II}[k]\cdot G(f)\cdot G^*(f)\cdot e^{-j2\pi fkT} \\
&= \frac{1}{T} S_{II}(f)\left|G(f)\right|^2
\end{aligned}$$

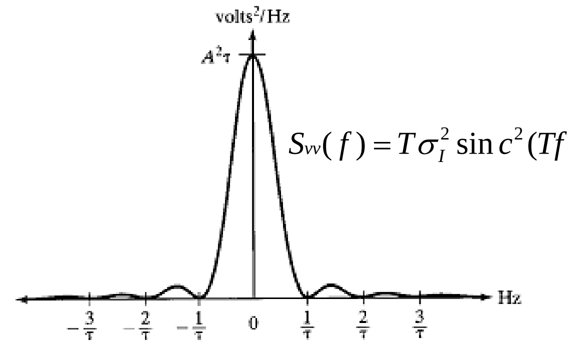
where $S_{II}(f) = \sum_{k=-\infty}^{\infty} R_{II}[k]e^{-j2\pi fkT}$

represents the PSD of the discrete random process $\{I_n\}$

$$\Rightarrow \quad \boxed{S_{vv}(f) = \frac{1}{T} S_{II}(f)\left|G(f)\right|^2}$$

- **Example : Power Spectral Density of M-PAM**

$$I_n \in \{\pm1, \pm3, \ldots, \pm(M-1)\} \Rightarrow \text{M-PAM}$$



$$\mu_I = 0\ ,\quad R_{II}[k] = \begin{cases} \sigma_I^2 & , k = 0 \\ 0 & , k \neq 0 \end{cases} = \sigma_I^2 \delta[k]$$

$$\Rightarrow S_{II}(f) = \sigma_I^2$$

$$\Rightarrow S_{vv}(f) = \frac{1}{T}\sigma_I^2 \cdot |G(f)|^2$$

Required transmission Bandwidth B=2/*T*

If $g(t) = \begin{cases} 1 & , 0 \leq t \leq T \\ 0 & , otherwise \end{cases}$, $S_{vv}(f) = \frac{1}{T} \cdot \sigma_I^2 \cdot |T \sin c(Tf)|^2 = T\sigma_I^2 \sin c^2(Tf)$

In general, for memoryless modulation with non-zero mean value,

$$R_{II}[k] = \begin{cases} \sigma_I^2 + \mu_I^2 & , k = 0 \\ \mu_I^2 & , k \neq 0 \end{cases} = \sigma_I^2 \delta[k] + \sum_{n\neq0} \mu_I^2 \delta[k-n]$$

$$\Rightarrow S_{II}(f) = \sigma_I^2 + \mu_I^2 \sum e^{-j2\pi kfT} = \sigma_I^2 + \frac{\mu_I^2}{T}\sum_k \delta(f - \frac{k}{T})$$

$$(= \sigma_I^2, \text{if } \mu_I = 0)$$

$$\Rightarrow S_{vv}(f) = \frac{1}{T} S_{II}(f) |G(f)|^2$$

- **Example : Power Spectral Density of QPSK modulation**

$$S_i(t)=\sqrt{\frac{2E}{T}}\cos[(2i-1)\frac{\pi}{4}]g(t)\cos(2\pi f_c t)-\sqrt{\frac{2E}{T}}\sin[(2i-1)\frac{\pi}{4}]g(t)\sin(2\pi f_c t)$$

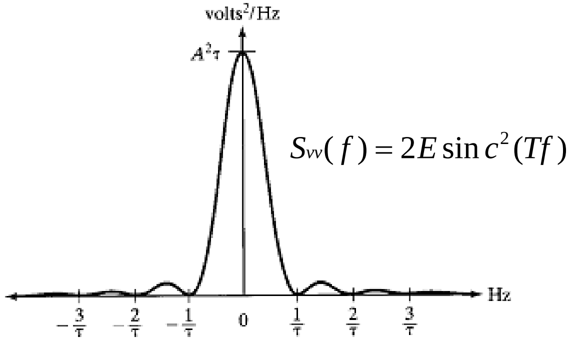
$$=\mathrm{Re}\left\{\tilde{S}_i e^{j2\pi f_c t}g(t)\right\},0\le t\le T$$

where $g(t)=\begin{cases}1 & , \ 0\le t\le T\\ 0 & , \ otherwise\end{cases}$, $\tilde{S}_i=\pm\sqrt{\frac{E}{T}}\pm j\sqrt{\frac{E}{T}}=S_I+jS_Q$

Let the baseband waveform be $v(t)=\sum_n \tilde{I}_n g(t-nT)$

where $\tilde{I}_n=S_I+jS_Q \quad S_I,S_Q\in\left\{\pm\sqrt{\frac{E}{T}}\right\}$

$$R_{II}[k]=\begin{cases}E[\left|\tilde{S}_i\right|^2]=\frac{2E}{T} & ,k=0\\ 0 & ,k\ne 0\end{cases}$$



$$\overline{R}_{vv}(\tau)=\frac{1}{T}\sum_k R_{II}[k]R_{gg}(\tau-kT)=\frac{1}{T}E[\left|\tilde{S}_i\right|^2]\cdot R_{gg}(\tau)$$

$$S_{vv}(f)=\frac{1}{T}\cdot E[\left|\tilde{S}_i\right|^2]\cdot\left|G(f)\right|^2=\frac{1}{T}\cdot\frac{2E}{T}\cdot\left|T\sin c(Tf)\right|^2=2E\sin c^2(Tf)$$

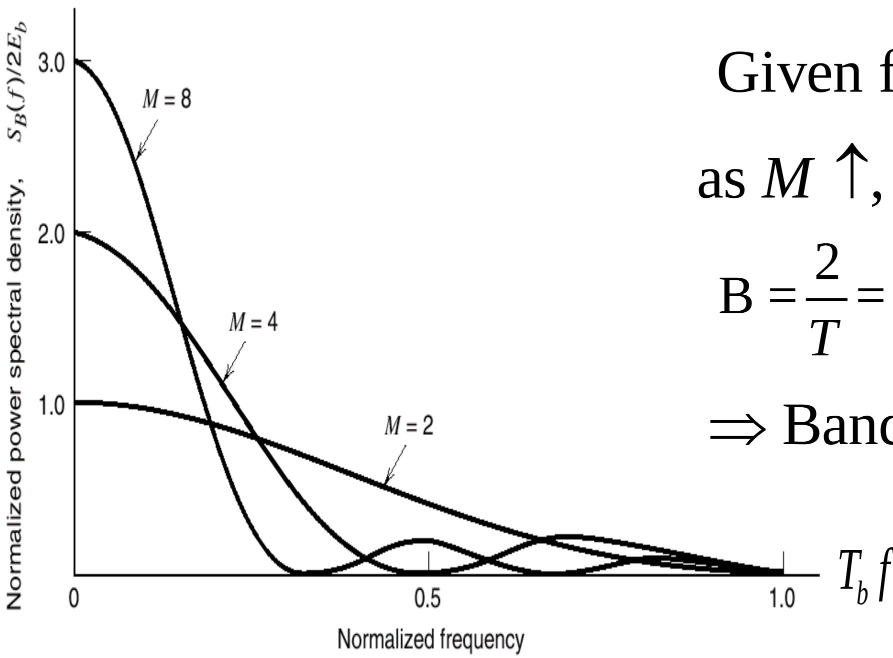
Power Spectral Density of MPSK modulation

$$S_i(t) = \sqrt{\frac{2E}{T}}\cos[(2i-1)\tfrac{\pi}{M}]\cos(2\pi f_c t) - \sqrt{\frac{2E}{T}}\sin[(2i-1)\tfrac{\pi}{M}]\sin(2\pi f_c t), 0 \le t \le T$$

$$= \mathrm{Re}\left\{\tilde{S}_i e^{j2\pi f_c t} g(t)\right\}$$

$$S_{vv}(f) = \frac{1}{T}\cdot E[\left|\tilde{S}_i\right|^2]\cdot\left|G(f)\right|^2 = \frac{1}{T}\cdot\frac{2E}{T}\cdot\left|T\mathrm{sinc}(Tf)\right|^2 = 2E\mathrm{sinc}^2(Tf)$$

$$= 2E\mathrm{sinc}^2(fT_b \log_2 M)$$



Given fixed T_b, i.e. fixed bit rate,

as $M \uparrow$, the required transmission bandwidth

$$\mathrm{B} = \frac{2}{T} = \frac{2}{T_b \log_2 M} \downarrow$$

$\Rightarrow$ Bandwidth efficiency $\rho = \dfrac{R_b}{B} \uparrow$

What has to pay for increased ρ?

$\Rightarrow \mathrm{P}_e \downarrow$

Power Spectral Density of MSK modulation

- Consider a single transmission of a MSK signal,

$$s(t)=\sqrt{\frac{2E}{T}}\cos[2\pi f_c t+\theta(t)]$$

$$=\underbrace{\sqrt{\frac{2E}{T}}\cos[\theta(t)]}_{s_I(t)}\cos(2\pi f_c t)-\underbrace{\sqrt{\frac{2E}{T}}\sin[\theta(t)]}_{s_Q(t)}\sin(2\pi f_c t),\quad 0\le t\le T$$

where $\theta(t)=\theta(0)\pm\frac{\pi}{2T}t$, $\quad$ (*i.e.* $\quad h=\frac{1}{2}$)

The baseband PSD is the combination of $s_I(t)$ and $s_Q(t)$,

$$i.e.\ \mathrm{S}_B(f)=\mathrm{S}_I(f)+\mathrm{S}_Q(f)$$

- For $s_I(t)$, assume $\theta(0)=0$ or π,

$$s_I(t)=\sqrt{\frac{2E}{T}}\cos\left(\theta(0)\pm\frac{\pi t}{2T}\right)=\sqrt{\frac{2E}{T}}\left[\cos\theta(0)\cos(\pm\frac{\pi t}{2T})-\cancel{\sin\theta(0)\sin(\pm\frac{\pi t}{2T})}\right]$$

$$=\sqrt{\frac{2E}{T}}\cos\theta(0)\cos(\pm\frac{\pi t}{2T})=\cos\theta(0)g_I(t)$$

where $g_I(t)=\begin{cases}\sqrt{\frac{2E}{T}}\cos(\frac{\pi t}{2T}), & -T\le t\le T\\ 0\quad , & other\end{cases}$

$\Rightarrow$ At $t=0$, $s_I(t)=\sqrt{\frac{2E}{T}}\cos\theta(0)$

The PSD of $s_I(t)$ is $\mathrm{S}_I(f)=\frac{1}{2T}\left|G_I(f)\right|^2=\frac{1}{2T}\frac{32ET}{\pi^2}\left[\frac{\cos(2\pi Tf)}{16T^2f^2-1}\right]^2=\frac{16E}{\pi^2}\left[\frac{\cos(2\pi Tf)}{16T^2f^2-1}\right]^2$

Power Spectral Density of MSK modulation

- For $s_Q(t)$, assume $\theta(0) = 0$ or π,

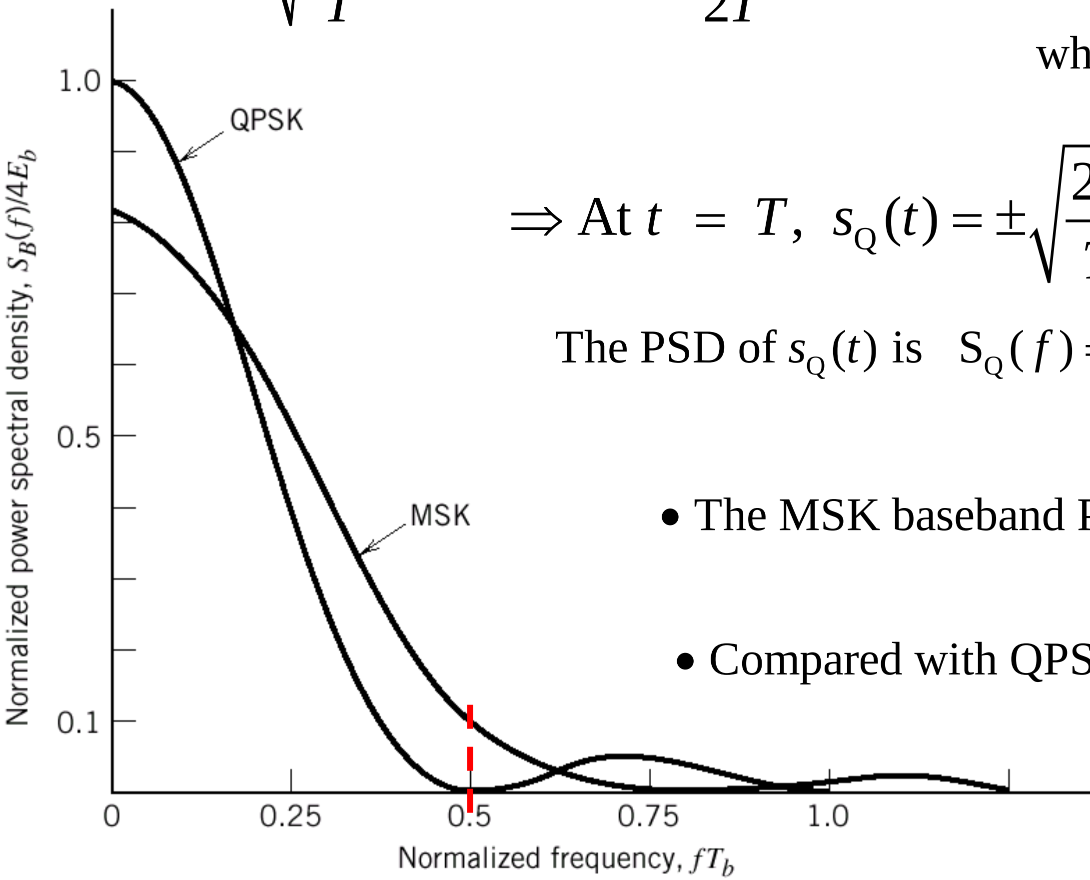
$$s_Q(t) = \sqrt{\frac{2E}{T}}\sin\left(\theta(0) \pm \frac{\pi t}{2T}\right) = \sqrt{\frac{2E}{T}}\left[\sin\theta(0)\cos(\pm\frac{\pi t}{2T}) + \cos\theta(0)\sin(\pm\frac{\pi t}{2T})\right]$$

$$= \pm\sqrt{\frac{2E}{T}}\cos\theta(0)\sin(\pm\frac{\pi t}{2T}) = \pm\cos\theta(0)\, g_Q(t)$$

where $g_Q(t) = \begin{cases} \sqrt{\frac{2E}{T}}\sin(\frac{\pi t}{2T}), & 0 \le t \le 2T \\ 0 \quad , & \textit{other} \end{cases}$

$$\Rightarrow \text{At } t = T,\ s_Q(t) = \pm\sqrt{\frac{2E}{T}}\cos\theta(0)$$

The PSD of $s_Q(t)$ is $S_Q(f) = \frac{1}{2T}\left|G_Q(f)\right|^2 = \frac{16E}{\pi^2}\left[\frac{\cos(2\pi T f)}{16T^2 f^2 - 1}\right]^2$



- The MSK baseband PSD is $S_B(f) = \frac{32E}{\pi^2}\left[\frac{\cos(2\pi T f)}{16T^2 f^2 - 1}\right]^2$

- Compared with QPSK, the MSK achieves the same BER, *but* MSK produces less interference outside the channel bandwidth.