排隊理論 HW5

○ **P5.1**

By mathmatical induction.

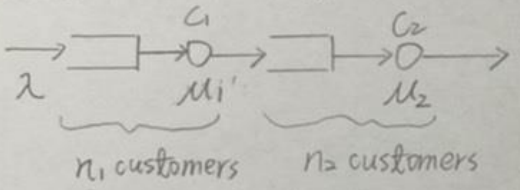
(i) We know that $F_n(0) \equiv P_r\{N(0)=n \text{ and } T>0\} = P_r\{N(0)=n\} = P_n$

(ii) When $t = x-1$, assume that $F_n(x-1) = P_n e^{-\lambda(x-1)}$ is true

(iii) When $t = x$, $F_n(x) = F_n(x-1)e^{-\lambda} = P_n e^{-\lambda x}$; Proved

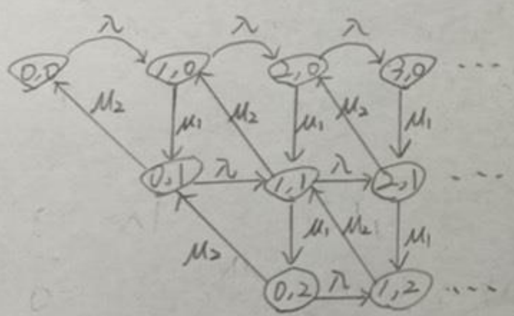
$\therefore F_n(t) = P_n e^{-\lambda t}$ #

○ **P5.4** (Series Queue)



證明 steady-state prob. 為

$P_{n_1,n_2} = P_{n_1}P_{n_2} = \rho_1^{n_1}\rho_2^{n_2}(1-\rho_1)(1-\rho_2)$



Global balance equation

$$
\begin{cases}
0 = -(\lambda+\mu_1+\mu_2)P_{n_1,n_2} + \lambda P_{n_1-1,n_2} + \mu_1 P_{n_1+1,n_2-1} + \mu_2 P_{n_1,n_2+1} \ ; \ n_1>0,\ n_2>0 \\
0 = -(\lambda+\mu_1)P_{n_1,0} + \lambda P_{n_1-1,0} + \mu_2 P_{n_1,1} \ ; \ n_1>0 \\
0 = -(\lambda+\mu_2)P_{0,n_2} + \mu_1 P_{1,n_2-1} + \mu_2 P_{0,n_2+1} \ ; \ n_2>0 \\
0 = -\lambda P_{0,0} + \mu_2 P_{0,1}
\end{cases}
$$

Then substitute $P_{n_1,n_2} = \rho_1^{n_1}\rho_2^{n_2}P_{0,0}$ into global balance equation,

$\Rightarrow 0 = -(\lambda+\mu_1+\mu_2)\rho_1^{n_1}\rho_2^{n_2}\cancel{P_{0,0}} + \lambda\rho_1^{n_1-1}\rho_2^{n_2}\cancel{P_{0,0}} + \mu_1\rho_1^{n_1+1}\rho_2^{n_2-1}\cancel{P_{0,0}} + \mu_2\rho_1^{n_1}\rho_2^{n_2+1}\cancel{P_{0,0}}$

$\Rightarrow 0 = -(\lambda+\mu_1+\mu_2) + \lambda\rho_1^{-1} + \mu_1\rho_1\rho_2^{-1} + \mu_2\rho_2$; 成立，得證 #

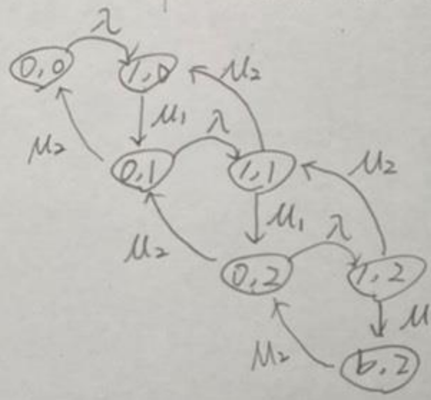
$\sum_{n_1}\sum_{n_2} P_{n_1,n_2} = 1 \Rightarrow P_{0,0}\sum_{n_1}\rho_1^{n_1}\sum_{n_2}\rho_2^{n_2} = \frac{P_{0,0}}{(1-\rho_1)(1-\rho_2)}$ (Using boundary condition)

$\therefore P_{0,0} = (1-\rho_1)(1-\rho_2)$ ；得證 #

○ P5.7

* Possible system state

(n_1, n_2)	Description
(0, 0)	System empty
(1, 0)	Customer in process at 1 only
(0, 1)	Customer in process at 2 only
(1, 1)	Customer in process at 1 and 2
(0, 2)	2 customers in process at 2 only
(1, 2)	1 customer at 1, 2 customers at 2
(b, 2)	2 customers at 2 and 1 customer finished at 1 (blocked)



$\boxed{\text{其中 } \mu_1 = \mu_2 = \mu}$

$$
\begin{cases}
0 = -\lambda P_{0,0} + \mu_2 P_{0,1} \\
0 = -\mu_1 P_{1,0} + \lambda P_{0,0} + \mu_2 P_{1,1} \\
0 = -(\lambda+\mu_2) P_{0,1} + \mu_1 P_{1,0} + \mu_2 P_{0,2} \\
0 = -(\mu_1+\mu_2) P_{1,1} + \lambda P_{0,1} + \mu_2 P_{1,2} \\
0 = -(\lambda+\mu_2) P_{0,2} + \mu_1 P_{1,1} + \mu_2 P_{b,2} \\
0 = -(\mu_1+\mu_2) P_{1,2} + \lambda P_{0,2} \\
0 = -\mu_2 P_{b,2} + \mu_1 P_{1,2}
\end{cases}
$$

$$
\begin{array}{cccccc|c}
(1,0) & (0,1) & (1,1) & (0,2) & (1,2) & (b,2) & (0,0)
\end{array}
$$

$$
\left[\begin{array}{cccccc|c}
0 & \mu & 0 & 0 & 0 & 0 & \lambda \\
-\mu & 0 & \mu & 0 & 0 & 0 & -\lambda \\
\mu & -(\lambda+\mu) & 0 & \mu & 0 & 0 & 0 \\
0 & \lambda & -2\mu & 0 & \mu & 0 & 0 \\
0 & 0 & \mu & -(\lambda+\mu) & 0 & \mu & 0 \\
0 & 0 & 0 & \lambda & -2\mu & 0 & 0 \\
0 & 0 & 0 & 0 & \mu & -\mu & 0
\end{array}\right]
$$

將各個 state 解出來並用 $P_{0,0}$ 表示 (懶得解)

最後用 Boundary condition 解出 $P_{0,0}$

○ P5.8

* The possible system states

Let system state be (n_1, n_2, n_3) for station 1, 2, 3, respectively.

$$\begin{cases} 1\ (n,0,0) \\ 2\ (n,1,0) \\ 3\ (n,0,1) \\ 4\ (n,1,1) \\ 5\ (n,b,1) \\ 6\ (b,1,0) \\ 7\ (b,1,1) \\ 8\ (b,b,1) \end{cases}$$

is the eight system states, where $n \geq 0$

○ P5.13

* The flow balance equations

(a)
$$\begin{cases} \lambda_A = 30 + 0.1\lambda_A \\ \lambda_B = 0.6\lambda_A + 0.1\lambda_C \\ \lambda_C = 0.2\lambda_A + 0.1\lambda_B \\ \lambda_D = 0.8\lambda_B + 0.8\lambda_C \end{cases} \xrightarrow{\text{Solve}} \begin{cases} \lambda_A \approx 33.3333 \\ \lambda_B \approx 20.8754 \\ \lambda_C \approx 8.7542 \\ \lambda_D \approx 23.7037 \end{cases}$$

$\gamma_A = \frac{\lambda_A}{\mu_A} \approx 0.2778$, $\gamma_B = \frac{\lambda_B}{\mu_B} \approx 0.6958$

$\gamma_C = \frac{\lambda_C}{\mu_C} \approx 0.8754$, $\gamma_D = \frac{\lambda_D}{\mu_D} \approx 0.7901$

$\because$ Node A is a M/M/∞ queue $\rightarrow L_A = \gamma_A$

$\because$ Node B, C, D are M/M/C queue $\rightarrow L_B = \frac{\gamma_B}{1-\gamma_B}$ ……

$$\therefore L = L_A + L_B + L_C + L_D = \gamma_A + \frac{\gamma_B}{1-\gamma_B} + \frac{\gamma_C}{1-\gamma_C} + \frac{\gamma_D}{1-\gamma_D}$$

$$= 0.2778 + 2.2873 + 7.0257 + 3.7642$$

$$= \underline{13.355}\ \#$$

(b) Using Little's formula $\Rightarrow W = \frac{L}{\lambda} = \frac{13.355}{30} = \underline{0.445\text{ (hours)}}$ #

◦ P5.15

$\lambda_E = 10 + (\lambda_A + \lambda_B + \lambda_C + \lambda_D) \times 0.1 = 10 + 0.1\lambda_E \Rightarrow \lambda_E \approx 11.1111$

$\lambda_A = 0.1\lambda_E = 1.1111 \Rightarrow \gamma_A = \frac{\lambda_A}{\mu} \approx 0.2222$ $\Rightarrow \gamma_E = \frac{\lambda_E}{3\mu} \approx 0.7407$

$\lambda_B = 0.2\lambda_E = 2.2222 \Rightarrow \gamma_B = \frac{\lambda_B}{\mu} \approx 0.4444$

$\lambda_C = 0.3\lambda_E = 3.3333 \Rightarrow \gamma_C = \frac{\lambda_C}{\mu} \approx 0.6666$

$\lambda_D = 0.4\lambda_E = 4.4444 \Rightarrow \gamma_D = \frac{\lambda_D}{\mu} \approx 0.8888$

(a) 將 node A, B, C, D 視為 M/M/1，node E 視為 M/M/3

$$\begin{cases} L_A = \frac{\gamma_A}{1-\gamma_A} \approx 0.2857, & L_D = \frac{\gamma_D}{1-\gamma_D} \approx 7.9928 \\ L_B = \frac{\gamma_B}{1-\gamma_B} \approx 0.7999, & L_E = \frac{\gamma_E}{1-\gamma_E} \approx 2.8565 \\ L_C = \frac{\gamma_C}{1-\gamma_C} \approx 1.9994 \end{cases}$$

(b) $L = L_A + L_B + L_C + L_D + L_E = 13.9343$

Using Little's Formula $\Rightarrow L = \lambda W \Rightarrow W = \frac{L}{\gamma} = \frac{13.9343}{10} \approx 1.3934$ (hours) #

◦ P5.16

$\lambda_A = 20 \Rightarrow \rho_A = \frac{\lambda_A}{30} \approx 0.6667$

$\lambda_B = 10 + 0.1\lambda_C \Rightarrow \lambda_B \approx 13.3333 \Rightarrow \rho_B = \frac{\lambda_B}{30} \approx 0.4444$

$\lambda_C = \lambda_A + \lambda_B \Rightarrow \lambda_C \approx 33.3333 \Rightarrow \rho_C = \frac{\lambda_C}{50} \approx 0.6667$

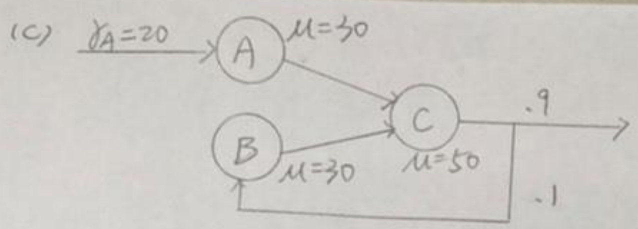
(a) 將 node A, B, C 視為 M/M/1

$L_A = \frac{\rho_A}{1-\rho_A} \approx 2.0003$ $\quad L_C = \frac{\rho_C}{1-\rho_C} \approx 0.6667$

$L_B = \frac{\rho_B}{1-\rho_B} \approx 0.7999$ $\quad L = L_A + L_B + L_C = 3.4669$

(b) $W = \frac{L}{\gamma_A + \gamma_B} = \frac{3.4669}{30} \approx 0.1156$ (hours)

(c)



$\lambda_A = 20 \Rightarrow \rho_A = \frac{\lambda_A}{30} \approx 0.6667$

$\lambda_B = 0.1\lambda_C \Rightarrow \lambda_B \approx 2.2222 \Rightarrow \rho_B = \frac{\lambda_B}{30} \approx 0.0741$

$\lambda_C = \lambda_A + \lambda_B \Rightarrow \lambda_C \approx 22.2222 \Rightarrow \rho_C = \frac{\lambda_C}{50} \approx 0.4444$

$L_A = \frac{\rho_A}{1-\rho_A} \approx 2.0003$, $L_B = \frac{\rho_B}{1-\rho_B} \approx 0.08$, $L_C = \frac{\rho_C}{1-\rho_C} \approx 0.7999$

$L = L_A + L_B + L_C = 2.8802$

$W = \frac{L}{20} = \frac{2.8802}{20} \approx 0.144$ (hours) #

05.17

(a) $L = \lambda W \Rightarrow 5 = 40W \Rightarrow W = \frac{1}{8}$ (hours) = 7.5 (minutes) #

(b) $W = 7.5 + 5 = 12.5$ (minutes) #

05.18

$\lambda_A = 10 + 0.1\lambda_A + 0.1\lambda_B + 0.1\lambda_C \Rightarrow \lambda_A \approx 13.7174 \Rightarrow \rho_A = \frac{\lambda_A}{20} \approx 0.6859$

$\lambda_B = 0.9\lambda_A \Rightarrow \lambda_B \approx 12.3457 \Rightarrow \rho_B = \frac{\lambda_B}{20} \approx 0.6173$

$\lambda_C = 0.9\lambda_B \Rightarrow \lambda_C \approx 11.1111 \Rightarrow \rho_C = \frac{\lambda_C}{20} \approx 0.5556$

$\lambda_D = 0.9\lambda_C \Rightarrow \lambda_D \approx 10 \Rightarrow \rho_D = \frac{\lambda_D}{20} = 0.5$

$L = \frac{\rho_A}{1-\rho_A} + \frac{\rho_B}{1-\rho_B} + \frac{\rho_C}{1-\rho_C} + \frac{\rho_D}{1-\rho_D}$

$\approx 2.1837 + 1.613 + 1.2502 + 1 = 6.0469$

(a) $W = \frac{L}{10} = 0.60469$

(b) $\lambda_A = \gamma + 0.1\lambda_A + 0.1\lambda_B + 0.1\lambda_C$; $\lambda_C = 0.81\lambda_A$; $\lambda_D = 0.729\lambda_A$

$\Rightarrow \lambda_A = \gamma + 0.1\lambda_A + 0.1(0.9\lambda_A) + 0.1(0.81\lambda_A)$

$\Rightarrow \lambda_A = \gamma + 0.1\lambda_A + 0.09\lambda_A + 0.081\lambda_A \Rightarrow \lambda_A = \dfrac{\gamma}{0.729}$

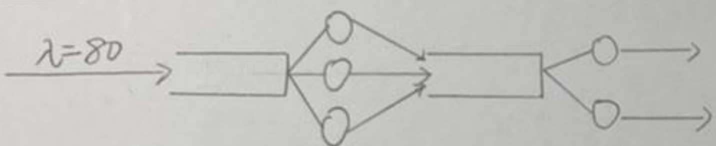
$\rho_A \le 1 \Rightarrow \dfrac{\lambda_A}{20} \le 1 \Rightarrow \lambda_A \le 20 \Rightarrow \dfrac{\gamma}{0.729} \le 20$

$\Rightarrow \gamma \le 14.58$

The maximum $\gamma = 14$ #

(c) True #

5.19



$\mu_1 = \frac{60}{2} = 30$ $\mu_2 = \frac{60}{1} = 60$

$\rho_1 = \dfrac{\lambda}{3\times\mu_1} = \dfrac{80}{3\times 30} = \dfrac{80}{90} \approx 0.8889$

$\rho_2 = \dfrac{\mu_1}{2\times\mu_2} = \dfrac{30}{2\times 60} = \dfrac{30}{120} \approx 0.25$

(a) $L_1 = \dfrac{\rho_1}{1-\rho_1} \approx 8.001$ #, $L_2 = \dfrac{\rho_2}{1-\rho_2} = 0.25$ #

(b) $L = L_1 + L_2 = 8.251$

$W = \dfrac{L}{\lambda} = \dfrac{8.251}{80} \approx 0.1031$ (hours) #

(c) True #