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1. Householder matrix $H = I - 2\vec{u}\vec{u}^T$ where $\vec{u}$ is an unit vector.

$\Rightarrow \langle \vec{u}, \vec{u} \rangle = \|\vec{u}\|^2 = 1$. $\forall \vec{u} \in \mathbb{R}^n$.

(1) $\because H^T = (I - 2\vec{u}\vec{u}^T)^T = I^T - 2(\vec{u}\vec{u}^T)^T = I - 2\vec{u}\vec{u}^T = H$.

$\Rightarrow H$ is symmetric.

(2)
$\because H^T = H \Rightarrow H^2 = H^T H = (I - 2\vec{u}\vec{u}^T)^T (I - 2\vec{u}\vec{u}^T) = (I - 2\vec{u}\vec{u}^T)(I - 2\vec{u}\vec{u}^T)$

$= I - 2\vec{u}\vec{u}^T - 2\vec{u}\vec{u}^T + 4\vec{u}\vec{u}^T\vec{u}\vec{u}^T$

$= I - 4\vec{u}\vec{u}^T + 4\vec{u}\langle \vec{u}, \vec{u} \rangle \vec{u}^T = I$

and $HH^T = H^2 = (I - 2\vec{u}\vec{u}^T)(I - 2\vec{u}\vec{u}^T) = I$. $\Rightarrow H^{-1} = H^T$.

$\therefore H$ is orthogonal.

(3)
$\because H^T = H$ and $H^{-1} = H^T \Rightarrow H = H^{-1} = H^T$

$\therefore H^2 = H^{-1}H = HH^{-1} = I$.

(4) Let $\langle \vec{v}, \vec{u} \rangle = 0$. $\forall \vec{u}, \vec{v}$.

$\because H\vec{u} = (I - 2\vec{u}\vec{u}^T)\vec{u} = \vec{u} - 2\vec{u}\vec{u}^T\vec{u} = \vec{u} - 2\vec{u}\langle \vec{u}, \vec{u} \rangle = \vec{u} - 2\vec{u} = -\vec{u}$.

$\Rightarrow \lambda = -1$.

$\because H\vec{v} = (I - 2\vec{u}\vec{u}^T)\vec{v} = \vec{v} - 2\vec{u}\vec{u}^T\vec{v} = \vec{v} - 2\vec{u}\langle \vec{v}, \vec{u} \rangle = \vec{v}$. $\Rightarrow \lambda = 1$.

$\therefore$ The eigenvalues of H are ± 1.

(5) $\because$ The eigenvalues of H are ± 1.

$\Rightarrow \det(H) = \lambda_1 \cdot \lambda_2 = 1 \cdot (-1) = -1$. #

2. Let v be a column vector with $\|v\|_2 = 1$.

The Householder transformation corresponding to the vector v is the orthogonal matrix $H = I_n - 2vv^T$.

① Set $k=1$ and initialize $B=A$.

② Compute $S = \sqrt{\sum_{i=k+1}^{n} b_{ik}^2}$.

If $S=0$ then set $k=k+1$ and recompute S.

③ Set $SG = \begin{cases} -1 & \text{if } b_{k+1,k} < 0 \\ +1 & \text{if } b_{k+1,k} \geq 0. \end{cases}$

④ Set $z = \frac{1}{2}(1 + SG \cdot b_{k+1}/s)$.

⑤ Set $v_i = 0$ for $i = 1, 2, \cdots, k$, Set $v_{k+1} = \sqrt{z}$ and set $v_i = \frac{SG \cdot b_k}{2v_{k+1}^{\ s}}$, $i = k+2, \cdots, n$.

⑥ Set $v = [v_1, v_2, \cdots, v_n]^T$ and let $H = I_n - 2vv^T$.

⑦ Compute $A = HBH$.

⑧ If $k = n-2$ then output A and stop.

⑨ Set $k = k+1$, $B = A$. and return ②. *

3.

$\because P = I - \frac{2vv^T}{v^Tv} = I - \frac{2vv^T}{\langle v, v\rangle}$ is Householder.

$\Rightarrow P^T = P = P^{-1}$, $\det(P) = -1$ and eigenvalues of P are ± 1.

$\because Q = I - YTY^T$.

$\Rightarrow QP = (I - YTY^T)\left(I - \frac{2vv^T}{\langle v,v\rangle}\right) = I - \frac{2vv^T}{\langle v,v\rangle} - YTY^T + \frac{2}{\langle v,v\rangle} YTY^Tvv^T.$

$= I - Y_+ T_+ Y_+^T.$

where $Y_+ \in M_{m\times(j+1)}(\mathbb{R})$ and $T_+ \in M_{(j+1)\times(j+1)}(\mathbb{R})$ is upper-triangular. #

5. $\forall x, y \in \mathbb{R}^n$.

$xy^T \in M_{n\times n}(\mathbb{R})$ has $\operatorname{rank}(xy^T) \le 1$ and $tr(xy^T) = \sum_{j=1}^{n} x_j y_j = x^T y$.

$\therefore$ $\lambda = 0$ of multiplicity $n-1$ if $\operatorname{rank}(x^T y) = 1$

$\lambda = 0$ of multiplicity n if $\operatorname{rank}(x^T y) = 0$.

If $x=0$ or $y=0$ then $\det(I + xy^T) = \det(I) = 1 = 1 + x^T y$.

If $\operatorname{rank}(xy^T) = 1$ then take any vector not in $Null(xy^T)$ and add it to a basis of $Null(xy^T)$.

$\Rightarrow \exists\, S$ is invertiable s.t. $S(xy^T)S^{-1} = \begin{bmatrix} 0 & * \\ 0 & x^T y \end{bmatrix}$.

$\Rightarrow S(I + xy^T)S^{-1} = \begin{bmatrix} I & * \\ 0 & x^T y \end{bmatrix}$.

$\therefore \det\left[S(I + xy^T)S^{-1}\right] = \det(S)\,\det(I + xy^T)\,\det(S^{-1}) = \det(I + xy^T) = 1 + x^T y$.

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