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1. $H = A + jB$ is PD.

$\because$ H is self-adjoint $\Rightarrow \bar{H}^T = \bar{A}^T - j\bar{B}^T = A^T - jB^T = H = A + jB$.

$\Rightarrow A^T = A$ and $B^T = -B$.

$\Rightarrow$ A is symmetric and B is skew-symmetric.

$\because C = \begin{bmatrix} A & -B \\ B & A \end{bmatrix} \Rightarrow \bar{C}^T = C^T = \begin{bmatrix} A^T & B^T \\ -B^T & A^T \end{bmatrix} = \begin{bmatrix} A & -B \\ B & A \end{bmatrix} = C$.

$\Rightarrow$ C is symmetric. $\Rightarrow$ C is orthogonally diagonalizable.

$\Rightarrow \exists\ Q$ is orthogonal s.t. $Q^T C Q = D$ where Q is consist of orthonormal eigenvectors of C.

$\Rightarrow Q^{-1} = Q^T$. $\Rightarrow \quad C = QDQ^T = Q\sqrt{D}\sqrt{D}Q^T = (Q\sqrt{D}Q^T)(Q\sqrt{D}Q^T)$

$= (Q\sqrt{D}Q^T)^T (Q\sqrt{D}Q^T)$.

Let $P = Q\sqrt{D}Q^T$. $\Rightarrow C = P^T P$ and P is invertiable

$\Leftrightarrow$ C is positive definite. #

2. $n(x) = \sqrt{\bar{x}^T A x}$ is a vector norm $\Leftrightarrow$ A is PD.

"$\Rightarrow$":

$\because n(x)$ is a vector norm $\Rightarrow n(x) = \sqrt{\bar{x}^T A x} \geq 0$.

$\Rightarrow \bar{x}^T A x \geq 0$. $\Rightarrow \langle Ax, x \rangle \geq 0$.

$\because \langle Ax, x \rangle = \langle x, Ax \rangle = \overline{(Ax)}^T x = \bar{x}^T \bar{A}^T x = \bar{x}^T A x \geq 0$.

$\Rightarrow \bar{A}^T = A \Rightarrow$ A is Hermitiain $\Rightarrow$ A is PD.

"$\Leftarrow$": $\because$ A is PD $\Rightarrow \exists\ x \in \mathbb{C}^n$ s.t. $\bar{x}^T A x > 0$.

Let $n(x) = \sqrt{\bar{x}^T A x}$. $\because \bar{x}^T A x > 0 \Rightarrow n(x) = \sqrt{\bar{x}^T A x} > 0$.

if $x = 0$ then $n(0) = \sqrt{\bar{0}^T A 0} = 0$.

$\therefore n(x) \geq 0,\ \forall x \in \mathbb{C}^n \Rightarrow n(x)$ is a vector norm.

Hence $n(x) = \sqrt{\bar{x}^T A x}$ is a vector norm $\Leftrightarrow$ A is PD. #

3.

Let λ_i be the eigenvalues of T and v_i be the eigenvectors corresponding to λ_i.

$\because$ T has n distinct eigenvalues. $\Rightarrow$ $\det(T-\lambda I_V)$ can splits.

$\Rightarrow$ $\exists$ β is an orthonormal basis s.t. $[T]_\beta$ is upper-triangular and diagonalizable.

Let $A=[T]_\beta$ and $f(x)$ be the minimal polynomial of A.

$\Rightarrow f(x) = (x-\lambda_1)(x-\lambda_2)\cdots\cdots(x-\lambda_n) = \prod_{i=1}^{n}(x-\lambda_i)$.

$\Rightarrow f(A) = \prod_{i=1}^{n}(A-\lambda_i I) \Rightarrow f(T) = \prod_{i=1}^{n}(T-\lambda_i I_V)$.

$\because UT=TU \Rightarrow U[T(v_i)] = U(\lambda_i v_i) = \lambda_i U(v_i) = T[U(v_i)]$.

$\Rightarrow U(v_i) = v_i$, $i=1,\dots,n$.

$\because U(v_i) = v_i = f(T)(v_i) = v_i$, $i=1,\dots,n$.

$\Rightarrow U = f(T)$ where f is the minimal polynomial. #

4. T is normal $\Rightarrow T^*T = TT^*$.

(a) $\forall x \in V$.

$\because$ T is normal $\Rightarrow \|T(x)\| = \|T^*(x)\| \Rightarrow (I-T)$ is normal.

$\because (I-T)[T(x)]=0$, $(I-T^*)[T(x)]=0$ and $T^2=T \Rightarrow T^*T(x) = T^2(x) = T(x)$.

$\because T[(I-T)(x)]=0$, $T^*[(I-T)(x)]=0$ and $(T^*)^2=(T^2)^* \Rightarrow T^*T(x) = T^*(x)$.

$\therefore T(x) = T^*(x) \Rightarrow T$ is self-adjoint.

(b) $\forall x \in V$.

Let $U = T^{k-1} \Rightarrow U^2 = T^{2k-2} = T_0$ and U is normal.

$\because \|U^*U(x)\|^2 = \langle U^*U(x), U^*U(x) \rangle = \langle x, U^*UU^*U(x) \rangle$

$= \langle x, U^*U^*UU(x) \rangle = \langle x, 0 \rangle = 0 \Rightarrow U^*U(x)=0$.

$\therefore \|U(x)\|^2 = \langle U(x), U(x) \rangle = \langle x, U^*U(x) \rangle = \langle x, 0 \rangle = 0$. $\Rightarrow U(x)=0$.

$\Rightarrow U(x) = T^{k-1}(x) = 0$.

Similarily , $T(x)=0$.

(c) $\forall x \in V$.

$\because$ T is normal $\Rightarrow \|T(x)\| = \|T^*(x)\| \Rightarrow (I-T)$ is normal.

$\because T^3=T^2$ and $(I-T)[T(x)]=0 \Rightarrow T(x)=T^2(x)$.

$\because T^3=T^2$ and $T[(I-T)(x)]=0 \Rightarrow T^2(x)=T(x)$.

$\therefore T^2=T \Rightarrow T$ is idempotent. #

5. $\langle\, ,\rangle$ and $\langle\, ,\rangle' : V \to \mathbb{R}$.

(a) $\forall\, x_1, x_2, y \in V$ and $\forall\, a \in \mathbb{F}$.

$\because\ a\langle x_1, y\rangle' + \langle x_2, y\rangle' = \langle ax_1, y\rangle' + \langle x_2, y\rangle' = \langle ax_1 + x_2, y\rangle' = \langle T(ax_1+x_2), y\rangle$

and $a\langle x_1, y\rangle' + \langle x_2, y\rangle' = a\langle T(x_1), y\rangle + \langle T(x_2), y\rangle = \langle aT(x_1) + T(x_2), y\rangle$

$\Rightarrow T(ax_1 + x_2) = aT(x_1) + T(x_2) \Rightarrow T$ is linear.

Let U be another linear operator on V.

$a\langle x_1, y\rangle' + \langle x_2, y\rangle' = \langle ax_1 + x_2, y\rangle' = \langle U(ax_1+x_2), y\rangle$ and

$a\langle x_1, y\rangle' + \langle x_2, y\rangle' = a\langle U(x_1), y\rangle + \langle U(x_2), y\rangle = \langle aU(x_1) + U(x_2), y\rangle$.

$\because\ a\langle x_1, y\rangle' + \langle x_2, y\rangle' = \langle aT(x_1) + T(x_2), y\rangle = \langle aU(x_1) + U(x_2), y\rangle$

and $\langle ax_1 + x_2, y\rangle' = \langle T(ax_1+x_2), y\rangle = \langle U(ax_1+x_2), y\rangle$

$\therefore T = U \Rightarrow T$ is unique.

(b) T is PD $\Leftrightarrow T^* = T$ and $\langle T(x), x\rangle > 0$.

① $\forall\, x, y \in V$. Let $U = T^*$.

$\because\ T$ is unique on $V \Rightarrow U = T$ and $U = T^* \Rightarrow T = T^*$

$\therefore T$ is self-adjoint.

② $\because\ T$ is self-adjoint $\Rightarrow T^*T = TT^* \Rightarrow T$ is normal.

$\because\ \langle\cdot,\cdot\rangle'$ is an inner product $\Rightarrow \langle x, x\rangle' = \langle T(x), x\rangle > 0$, $\forall\, x \neq 0$.

$\therefore \langle T(x), x\rangle > 0$.

$\because\ T$ is self-adjoint and $\langle T(x), x\rangle > 0$, $\forall\, x \neq 0 \Leftrightarrow T$ is PD.

Hence T is PD. #