Two forms of curl in 2D

$$\nabla \wedge (\psi \hat{z}) = (\psi_y, -\psi_x, 0) \qquad \mathrm{curl}_1\, \psi = (\psi_y, -\psi_x)$$

$$\nabla \wedge (u, v, 0) = (0, 0, \partial_x v - \partial_y u) \qquad \mathrm{curl}_2\, \underline{u} = \partial_x v - \partial_y u$$

curl_1 can be interpreted as rotated grad: $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} = \begin{pmatrix} \partial_y \\ -\partial_x \end{pmatrix}$

curl_2 has alternative representation as $\frac{1}{|S|}\oint_{\partial S} \underline{u} . d\underline{x}$ for "small" S.

$\mathrm{div} \circ \mathrm{curl}_1 = 0$ and $\mathrm{curl}_2 \circ \mathrm{grad} = 0$.

Under the inner products $[\underline{a}, \underline{b}]_2 = \int \underset{\sim}{a} . \underline{b}\, dA$, $[\psi, \phi]_1 = \int \psi \phi \, dA$:

$$[\underline{u}, \mathrm{curl}_1 \psi]_2 = \int (u \psi_y - v \psi_x)\, dA = \int \psi (v_x - u_y)\, dA + \text{boundary terms}$$
$$= [\mathrm{curl}_2\, \underline{u}, \psi]_1 + \cdots$$

Likewise $[\mathrm{grad}\, \phi, \underline{u}]_2 = [\phi, -\mathrm{div}\, \underline{u}]_1 + \cdots$

Represent these relationships as "exact sequences"

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "\mathrm{grad}"] \arrow[dr, phantom, "\circlearrowleft L"] & \mathbb{R}^2 \arrow[r, "\mathrm{curl}_2"] \arrow[dr, phantom, "\circlearrowright L"] & \mathbb{R} \\
\mathbb{R} \arrow[u, "1"] & \mathbb{R}^2 \arrow[u, "2"'] \arrow[l, "-\mathrm{div}"] & \mathbb{R} \arrow[u, "1"'] \arrow[l, "\mathrm{curl}_1"]
\end{tikzcd}$$

Adjoints under inner products

which reveal two forms of scalar Laplacian:

$$L \equiv -\nabla^2 = -\mathrm{div} \circ \mathrm{grad} = \mathrm{curl}_2 \circ \mathrm{curl}_1$$

and the Helmholtz decomposition

$$\underline{u} = \mathrm{grad}\, \phi + \mathrm{curl}_1 \psi \qquad \text{for some } \phi, \psi$$

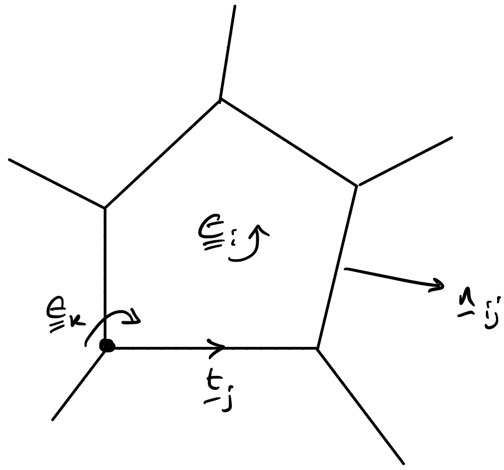
which exploits orthogonality

$$[\mathrm{grad}\, \phi, \mathrm{curl}_1 f]_2 = [\phi, -\mathrm{div} \circ \mathrm{curl}_1 f]_1 = 0$$
$$= [\mathrm{curl}_2 \circ \mathrm{grad}\, \phi, f]_1 = 0$$

with potentials determined via $L\phi = -\mathrm{div}\, \underline{u}$, $L\psi = \mathrm{curl}_2\, \underline{u}$

(assuming $\underset{\sim}{\Delta} \underset{\sim}{u} \equiv (\mathrm{curl}_1 \circ \mathrm{curl}_2 - \mathrm{grad} \circ \mathrm{div})\underline{u} = -\nabla^2 \underline{u} = 0 \Rightarrow \underline{u} = \underline{0}$, i.e. no holes)

Four forms of curl over polygons



Signed incidence matrices A_{jk}, B_{ij}

Satisfy $\quad BA = 0$

$$\underset{\text{Vertices}}{\mathcal{V}} \xrightarrow{A} \underset{\text{Edges}}{\mathcal{E}} \xrightarrow{B} \underset{\text{Faces}}{\mathcal{F}}$$

Inner products
$$[\phi, \psi]_{\mathcal{V}} = \phi^T M^{\mathcal{V}} \psi \equiv \sum_{k,k'} \phi_k M^{\mathcal{V}}_{kk'} \psi_{k'}$$
$$[\underline{u}, \underline{v}]_{\mathcal{E}} = \underline{u}^T \underline{\underline{M}}^{\mathcal{E}} \underline{v}$$
$$[f, g]_{\mathcal{F}} = f^T M^{\mathcal{F}} g$$

Define $\quad \{\text{grad}^{\mathcal{V}} \phi\}_j = \sum_k A_{jk} \dfrac{\underline{t}_j}{t_j^2} \phi_k \qquad \{\text{curl}^c \underline{u}\}_i = \dfrac{1}{A_i} \sum_j B_{ij}\, \underline{t}_j \cdot \underline{u}_j$

Then $\quad \{\text{curl}^c \circ \text{grad}^{\mathcal{V}} \phi\}_i = \dfrac{1}{A_i} \sum_{j,k} B_{ij} A_{jk} \phi_k = 0$

and $\quad \{\text{curl}^{\mathcal{V}} \phi\}_j = \underline{\epsilon}_k\, \text{grad}^{\mathcal{V}} \phi \qquad \{\text{div}^c \underline{u}\}_i = \dfrac{1}{A_i} \sum_j \underline{n}_{ij} \cdot \underline{u}_j$

$$\underline{n}_{ij} \equiv -\underline{\epsilon}_i B_{ij}\, \underline{t}_j$$

Then $\quad \{\text{div}^c \circ \text{curl}^{\mathcal{V}} \phi\} = \dfrac{1}{A_i} \sum_{jk} B_{ij} A_{jk} \phi_k = 0$

Two exact sequences, plus adjoints under inner products

$$\begin{tikzcd}
\mathcal{V} \arrow[r, "\text{grad}^{\mathcal{V}}"] \arrow[dr, phantom, "\textcolor{red}{L_{\mathcal{V}}}"] & \mathcal{E} \arrow[r, "\text{curl}^c"] \arrow[dr, phantom, "\textcolor{blue}{L_{\mathcal{F}}}"] & \mathcal{F} \\
\mathcal{V} \arrow[u, "M^{\mathcal{V}}"] & \mathcal{E} \arrow[l, "-\widetilde{\text{div}}^{\mathcal{V}}"] \arrow[u, "M^{\mathcal{E}}"] & \mathcal{F} \arrow[l, "\widetilde{\text{curl}}^c"] \arrow[u, "M^{\mathcal{F}}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{V} \arrow[r, "\text{curl}^{\mathcal{V}}"] \arrow[dr, phantom, "\textcolor{red}{L_{\mathcal{V}}}"] & \mathcal{E} \arrow[r, "\text{div}^c"] \arrow[dr, phantom, "\textcolor{blue}{L_{\mathcal{F}}}"] & \mathcal{F} \\
\mathcal{V} \arrow[u, "M^{\mathcal{V}}"] & \mathcal{E} \arrow[l, "\widetilde{\text{curl}}^{\mathcal{V}}"] \arrow[u, "M^{\mathcal{E}}"] & \mathcal{F} \arrow[l, "-\widetilde{\text{grad}}^c"] \arrow[u, "M^{\mathcal{F}}"]
\end{tikzcd}$$

Two scalar Laplacians: $\quad \textcolor{red}{L_{\mathcal{V}} = -\widetilde{\text{div}}^{\mathcal{V}}\, \text{grad}^{\mathcal{V}} = \widetilde{\text{curl}}^{\mathcal{V}}\, \text{curl}^{\mathcal{V}}}$

(for suitable $M^{\mathcal{E}}$) $\quad \textcolor{blue}{L_{\mathcal{F}} = -\text{div}^c\, \widetilde{\text{grad}}^c = \text{curl}^c\, \widetilde{\text{curl}}^c}$

Helmholtz: $\quad \underline{u} = \underbrace{\left(\text{grad}^{\mathcal{V}} \phi + \widetilde{\text{curl}}^c f\right)}_{\text{Parallel to edges}} + \underbrace{\left(\text{curl}^{\mathcal{V}} \psi - \widetilde{\text{grad}}^c g\right)}_{\text{Normal to edges}}$

with $\quad L_{\mathcal{V}} \phi = -\widetilde{\text{div}}^{\mathcal{V}} \underline{u}, \qquad L_{\mathcal{F}} f = \text{curl}^c \underline{u} \qquad$ (again assuming no holes)

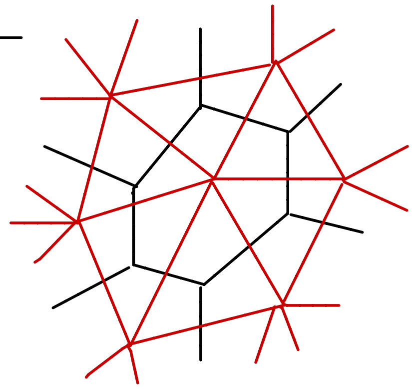
$\qquad\quad L_{\mathcal{V}} \psi = \widetilde{\text{curl}}^{\mathcal{V}} \underline{u} \qquad\quad L_{\mathcal{F}}\, g = \text{div}^c \underline{u}$

Eight forms of curl over primal and dual networks

$$T \xleftarrow{A} L \xleftarrow{B} C$$

Triangles   Links   Cell centres

$$A^T B^T = 0$$



$$\begin{tikzcd}
T & L \arrow[l, "CURL^v"'] & C \arrow[l, "grad^c"'] \\
T \arrow[u, "M^v"] \arrow[r, "\widetilde{CURL}^v"'] \arrow[ur, phantom, "L_T"] & L \arrow[u, "M^E"] \arrow[r, "-\widetilde{div}^c"'] \arrow[ur, phantom, "L_C"] & C \arrow[u, "M^F"']
\end{tikzcd}
\qquad
\begin{tikzcd}
T & L \arrow[l, "div^v"'] & C \arrow[l, "CURL^c"'] \\
T \arrow[u, "M^v"] \arrow[r, "-\widetilde{grad}^v"'] \arrow[ur, phantom, "L_T"] & L \arrow[u, "M^E"] \arrow[r, "\widetilde{CURL}^c"'] \arrow[ur, phantom, "L_C"] & C \arrow[u, "M^F"']
\end{tikzcd}$$

Two scalar Laplacians:
$$L_T = -div^v \circ \widetilde{grad}^v = CURL^v \circ \widetilde{CURL}^v$$
$$L_C = -\widetilde{div}^c \circ grad^c = \widetilde{CURL}^c \circ CURL^c$$

So for a general triangulation / tesselation, there are

16 operators, including 8 curls, 8 potentials, 4 scalar Laplacians.

When links and edges are orthogonal

8 operators, 4 curls, 4 potentials, 2 Laplacians

$$L_V = L_T, \quad L_P = L_C.$$

$$\widetilde{curl}^v = CURL^v$$
$$-\widetilde{div}^v = div^v$$
$$-\widetilde{grad}^v = grad^v$$
$$\widetilde{CURL}^v = curl^v$$

etc.