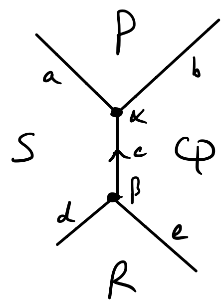
18-12-20

T1 transitions via incidence matrices.



$$\underset{\sim}{e}_c^T A = \underset{\sim}{e}_\alpha - \underset{\sim}{e}_\beta$$
$$\underset{\sim}{e}_c^T \bar{A} = \underset{\sim}{e}_\alpha + \underset{\sim}{e}_\beta$$

Assume we know the short edge c that will flip.
It is represented as the unit vector

$$\underset{\sim}{e}_{c_i} = (0, \dots, 0, 1, 0, \dots 0)^T \in \mathbb{R}^J$$

where J is the total number of edges.
Vertices α and β (where K is the total number of vertices) are given by

$$(1) \qquad \underset{\sim}{e}_\alpha = \tfrac{1}{2}\, \underset{\sim}{e}_{c_i}^T (A + \bar{A}) \qquad \underset{\sim}{e}_\beta = \tfrac{1}{2}\, \underset{\sim}{e}_{c_i}^T (\bar{A} - A) \qquad (\in \mathbb{R}^K)$$

Each cell will have a prescribed orientation $\epsilon_i = \varepsilon$ or $-\varepsilon$
(ε is a $\pi/2$ clockwise rotation, $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$)

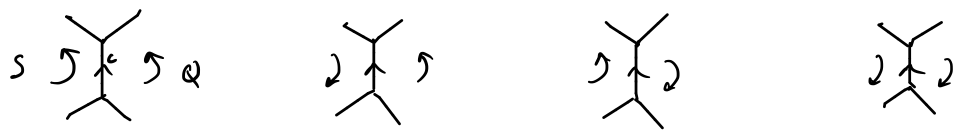
Construct $\quad \underline{n} = \frac{1}{2}\left(\mathrm{tr}(\varepsilon \epsilon_1), \dots, \mathrm{tr}(\varepsilon \epsilon_I)\right) \quad$ (I is the total number of cells)

Now $\quad \bar{B} \underset{\sim}{e}_{c_i} = \underset{\sim}{e}_S + \underset{\sim}{e}_Q = (0, \dots 0, \underset{\uparrow S}{1}, 0, \dots 0, \underset{\uparrow Q}{1}, 0, \dots 0)^T \qquad (\in \mathbb{R}^I)$

and $\quad \underline{n}^T \circ B \underset{\sim}{e}_{c_i} = (0, \dots, 0, +1, 0, \dots 0, -1, 0, \dots, 0)^T = \underset{\sim}{e}_S - \underset{\sim}{e}_Q$

[$\circ$ Elementwise multiplication]

because:



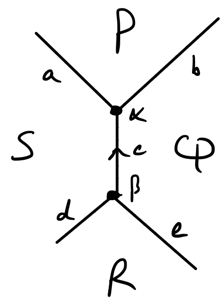
$\underline{n}^T \underset{\sim}{e}_S$	+1		−1		+1		−1		(n_S)
$\underline{n}^T \underset{\sim}{e}_Q$		+1		+1		−1		−1	(n_Q)
$\underset{\sim}{e}_S^T B \underset{\sim}{e}_c$	1		−1		1		−1		(B_{Sc})
$\underset{\sim}{e}_Q^T B \underset{\sim}{e}_c$	↑	−1	↑	−1	↑	1	↑	1	(B_{Qc})
Product	+1	↑	+1	↑	+1		+1	↑	
		−1		−1		−1		−1	

$$(2) \quad \text{So} \quad \underset{\sim}{e}_{Q_j} = \tfrac{1}{2}\left(\bar{B} \underset{\sim}{e}_{c_i} - \underline{n}^T \circ B \underset{\sim}{e}_{c_i}\right), \qquad \underset{\sim}{e}_{S_k} = \tfrac{1}{2}\left(\bar{B} \underset{\sim}{e}_{c_i} + \underline{n}^T \circ B \underset{\sim}{e}_{c_i}\right)$$

$$(3) \quad \text{Thus} \quad \underset{\sim}{e}_P = C \underset{\sim}{e}_\alpha - \underset{\sim}{e}_{Q_j} - \underset{\sim}{e}_{S_k} \quad \text{and} \quad \underset{\sim}{e}_R = C \underset{\sim}{e}_\beta - \underset{\sim}{e}_{Q_j} - \underset{\sim}{e}_{S_k} \qquad \left(C = \tfrac{1}{2} \bar{B} \bar{A}\right)$$

(4) and

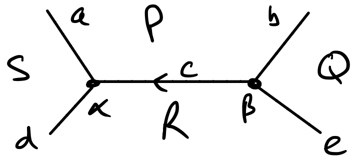
$$\underset{\sim}{e}_a = \underset{\sim}{e}_S^T \bar{B} \circ (\bar{A} \underset{\sim}{e}_\alpha - \underset{\sim}{e}_c) \qquad \underset{\sim}{e}_d = \underset{\sim}{e}_S^T \bar{B} \circ (\bar{A} \underset{\sim}{e}_\beta - \underset{\sim}{e}_c)$$
$$\underset{\sim}{e}_b = \underset{\sim}{e}_Q^T \bar{B} \circ (\bar{A} \underset{\sim}{e}_\alpha - \underset{\sim}{e}_c) \qquad \underset{\sim}{e}_e = \underset{\sim}{e}_Q^T \bar{B} \circ (\bar{A} \underset{\sim}{e}_\beta - \underset{\sim}{e}_c)$$



Here we are using $\bar{A}\,\underset{\sim}{e}_\alpha = \underset{\sim}{e}_a + \underline{e}_b + \underline{e}_c$, $\bar{A}\,\underset{\sim}{e}_\beta = \underline{e}_c + \underline{e}_d + \underline{e}_e$

and $\underline{e}_S^T \bar{B} = \underline{e}_d + \underline{e}_c + \underline{e}_a + \cdots$ $\qquad$ $\underline{e}_Q^T \bar{B} = \underline{e}_e + \underline{e}_c + \underline{e}_b + \cdots$

Having identified all vertices, edges and cells neighbouring $\underset{\sim}{c}$, now do some surgery:



This has a particular orientation (flipping c anticlockwise), hence the need to introduce Σ.

Let $\tilde{A}$ and $\tilde{B}$ be the updated incidence matrices.

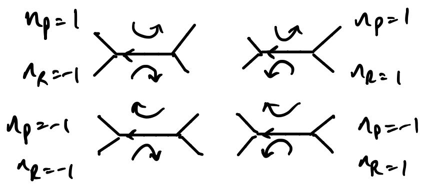
The only change to A is

A	α	β
b	$*$	0
d	0	$\square$

$\longrightarrow$

$\tilde{A}$	α	β
b	0	$*$
d	$\square$	0

The only change to B is



B	c
S	$\bullet$
P	0
Q	$\bullet$
R	0

$\longrightarrow$

$\tilde{B}$	c
S	0
P	$-\underline{n}^T\underline{e}_P$
Q	0
R	$\underline{n}^T\underline{e}_R$

So
$$\tilde{A} = A + (\underline{e}_b^T A\, \underline{e}_\alpha)\, \underline{e}_b \left[\underline{e}_\beta^T - \underline{e}_\alpha^T\right] + (\underline{e}_d^T A\, \underline{e}_\beta)\, \underline{e}_d \left[\underline{e}_\alpha^T - \underline{e}_\beta^T\right] \tag{5}$$
$$\tilde{B} = B - (\underline{e}_S^T B\, \underline{e}_c)\, \underline{e}_S \underline{e}_c^T - (\underline{e}_Q^T B\, \underline{e}_c)\, \underline{e}_Q \underline{e}_c^T + (\underline{n}^T \underline{e}_R)\, \underline{e}_R \underline{e}_c^T - (\underline{n}^T \underline{e}_P)\, \underline{e}_P \underline{e}_c^T$$

1. Starting from A, B and $\underset{\sim}{n}$, find $\bar{A}$, $\bar{B}$, C
2. Knowing $\underline{e}_c$, find $\underline{e}_\alpha, \underline{e}_\beta$ using (1) and $\underline{e}_S, \underline{e}_Q$ using (2)
3. Then find $\underline{e}_P, \underline{e}_R$ using (3) and $\underline{e}_b, \underline{e}_d$ using (4)
4. Construct $\tilde{A}$ and $\tilde{B}$ using (5).