COS4807 Assignment 1 - 791729

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Question 1

$$
\begin{aligned}
(p \to q) \leftrightarrow (p \to r) &\equiv \big((p \to q) \to (p \to r)\big) \land \big((p \to r) \to (p \to q)\big) && \text{remove biconditional} \\
&\equiv \big(\neg(p \to q) \lor (p \to r)\big) \land \big(\neg(p \to r) \lor (p \to q)\big) && \text{remove conditional} \\
&\equiv \big(\neg(\neg p \lor q) \lor (\neg p \lor r)\big) \land \big(\neg(\neg p \lor r) \lor (\neg p \lor q)\big) && \text{remove conditional} \\
&\equiv \big((\neg\neg p \land \neg q) \lor (\neg p \lor r)\big) \land \big((\neg\neg p \land \neg r) \lor (\neg p \lor q)\big) && \text{De Morgan} \\
&\equiv \big((p \land \neg q) \lor (\neg p \lor r)\big) \land \big((p \land \neg r) \lor (\neg p \lor q)\big) && \text{remove } \neg\neg \\
&\equiv \big((p \lor \neg p \lor r) \land (\neg q \lor \neg p \lor r)\big) \land \big((p \lor \neg p \lor q) \land (\neg r \lor \neg p \lor q)\big) && \text{distribute disjunction} \\
&\equiv (p \lor \neg p \lor r) \land (\neg q \lor \neg p \lor r) \land (p \lor \neg p \lor q) \land (\neg r \lor \neg p \lor q)
\end{aligned}
$$

In clausal form is: $\{\{p, \neg p, r\}, \{\neg q, \neg p, r\}, \{p, \neg p, q\}, \{\neg r, \neg p, q\}\}$

Without trivial clauses: $\{\{\neg q, \neg p, r\}, \{\neg r, \neg p, q\}\}$

Which can be abbreviated as: $\{\bar{q}\bar{p}r, \bar{r}\bar{p}q\}$

Question 2

$$
\begin{aligned}
&\big((p \to q) \to \neg(r \to q)\big) \land \big((\neg r \to p) \to (p \to q)\big) \\
&\equiv \big(\neg(p \to q) \lor \neg(r \to q)\big) \land \big(\neg(\neg r \to p) \lor (p \to q)\big) \\
&\equiv \big(\neg(\neg p \lor q) \lor \neg(\neg r \lor q)\big) \land \big(\neg(\neg\neg r \lor p) \lor (\neg p \lor q)\big) \\
&\equiv \big((\neg\neg p \land \neg q) \lor (\neg\neg r \land \neg q)\big) \land \big((\neg\neg\neg r \land \neg p) \lor (\neg p \lor q)\big) \\
&\equiv \big((p \land \neg q) \lor (r \land \neg q)\big) \land \big((\neg r \land \neg p) \lor (\neg p \lor q)\big) \\
&\equiv (p \lor r) \land (p \lor \neg q) \land (\neg q \lor r) \land (\neg q \lor \neg q) \land (\neg r \lor \neg p \lor q) \land (\neg p \lor \neg p \lor q)
\end{aligned}
$$

In abbreviated clausal form: $\{pr, p\bar{q}, \bar{q}r, \bar{q}, \bar{r}\bar{p}q, \bar{p}q\}$

1. pr
2. $p\bar{q}$
3. $\bar{q}r$
4. $\bar{q}$
5. $\bar{r}\bar{p}q$
6. $\bar{p}q$
7. rq 1,6
8. $r\bar{p}$ 3,6
9. r 3,7
10. $\bar{r}\bar{p}$ 4,5
11. $\bar{p}$ 4,6
12. $r\bar{q}$ 2,8
13. $\bar{q}\bar{r}$ 2,9
14. pr 2,7
15. $\bar{q}\bar{p}$ 3,9

Because we can resolve no new clauses and we have not attained a $\square$, we conclude that the formula is satisfiable.

Question 3

Formula	Clausal Form
$p \to (q \vee r)$	$\bar{p}qr$
$(p \to r) \to q$	$pq, q\bar{r}$
$\neg q$	$\bar{q}$

We will use resolution to attain a $\square$ to show that the set of formulas where we have negated the goal are unsatisfiable.

1. $\bar{p}qr$
2. pq
3. $q\bar{r}$
4. $\bar{q}$ negation of goal
5. qr 1,2
6. q 3,5
7. $\square$ 4,6

We have proved that the set of clauses are unsatisfiable and so shown that the formulas from which they are derived are also unstatisfiable. Therefore, we conclude that $\{p \to (q \vee r), (p \to r) \to q\} \models q$

Question 4

We wish to prove that if the set of clauses labeling the leaves of a resolution tree are satisfiable, then the clause at the root of the tree is satisfiable.

We show this by structural induction on the height of the tree.

Base Case: Let C be an arbitrary clause which is satisfiable. Then, if the resolution tree is labeled by C and has a height $n = 0$, C is also a leaf node and so is, by definition, satisifiable.

Inductive Step: With $n > 0$ we have C being formed by the resolution of two subtrees, C_1 and C_2 each with height $k < n$. Assume C_1 and C_2 are satisfiable under some interpretation $\mathscr{I}$. Since, for some complementary set of literals l, l^c, we have that either $\mathscr{I}(l) = T$ or $\mathscr{I}(l^c) = T$.

Suppose $\mathscr{I}(l) = T$, then $\mathscr{I}(l^c) = F$. If $l \in C_1$ then we have that $l^c \in C_2$ and so C_2 can be satisfied only by some other literal $l' \in C_2$ with $\mathscr{I}(l') = T$ (and $l' \neq l^c$). The resoultion rule defines $C = (C_1 - \{l\}) \cup (C_2 - \{l^c\})$ so $l' \in C$ which means that C is also satisfiable. A symmetric argument holds when $\mathscr{I}(l^c) = T$

Question 5

5(i)

Given the following sudoku board with cells numbered $1 - 9$, we define the set of clauses required to encode the problem as follows:

1	2	3
4	5	6
7	8	9

We encode the fact that each cell must contain at least one number as follows:

$$
\begin{gathered}
11 \vee 12 \vee 13, \\
21 \vee 22 \vee 23, \\
31 \vee 32 \vee 33, \\
41 \vee 42 \vee 43, \\
51 \vee 52 \vee 53, \\
61 \vee 62 \vee 63, \\
71 \vee 72 \vee 73, \\
81 \vee 82 \vee 83, \\
91 \vee 92 \vee 93
\end{gathered}
$$

We add the constraint that a cell can contain at most one value:

$$
\begin{gathered}
\overline{11} \vee \overline{12}, \overline{11} \vee \overline{13}, \overline{12} \vee \overline{13}, \\
\overline{21} \vee \overline{22}, \overline{21} \vee \overline{23}, \overline{22} \vee \overline{23}, \\
\overline{31} \vee \overline{32}, \overline{31} \vee \overline{33}, \overline{32} \vee \overline{33}, \\
\overline{41} \vee \overline{42}, \overline{41} \vee \overline{43}, \overline{42} \vee \overline{43}, \\
\overline{51} \vee \overline{52}, \overline{51} \vee \overline{53}, \overline{52} \vee \overline{53}, \\
\overline{61} \vee \overline{62}, \overline{61} \vee \overline{63}, \overline{62} \vee \overline{63}, \\
\overline{71} \vee \overline{72}, \overline{71} \vee \overline{73}, \overline{72} \vee \overline{73}, \\
\overline{81} \vee \overline{82}, \overline{81} \vee \overline{83}, \overline{82} \vee \overline{83}, \\
\overline{91} \vee \overline{92}, \overline{91} \vee \overline{93}, \overline{92} \vee \overline{93}
\end{gathered}
$$

Add the constraint that no two cells in a row can contain the same number:

$$
\begin{gathered}
\overline{11} \vee \overline{21}, \overline{12} \vee \overline{22}, \overline{13} \vee \overline{23}, \\
\overline{11} \vee \overline{31}, \overline{12} \vee \overline{32}, \overline{13} \vee \overline{33}, \\
\overline{21} \vee \overline{31}, \overline{22} \vee \overline{32}, \overline{23} \vee \overline{33}, \\
\overline{41} \vee \overline{51}, \overline{42} \vee \overline{52}, \overline{43} \vee \overline{53}, \\
\overline{41} \vee \overline{61}, \overline{42} \vee \overline{62}, \overline{43} \vee \overline{63}, \\
\overline{51} \vee \overline{61}, \overline{52} \vee \overline{62}, \overline{53} \vee \overline{63}, \\
\overline{71} \vee \overline{81}, \overline{72} \vee \overline{82}, \overline{73} \vee \overline{83}, \\
\overline{71} \vee \overline{91}, \overline{72} \vee \overline{92}, \overline{73} \vee \overline{93}, \\
\overline{81} \vee \overline{91}, \overline{82} \vee \overline{92}, \overline{83} \vee \overline{93}
\end{gathered}
$$

And the constraint that no two cells in a column can contain the same number:

$$
\begin{gathered}
\overline{11} \vee \overline{41}, \overline{12} \vee \overline{42}, \overline{13} \vee \overline{43}, \\
\overline{11} \vee \overline{71}, \overline{12} \vee \overline{72}, \overline{13} \vee \overline{73}, \\
\overline{41} \vee \overline{71}, \overline{42} \vee \overline{72}, \overline{43} \vee \overline{73}, \\
\overline{21} \vee \overline{51}, \overline{22} \vee \overline{52}, \overline{23} \vee \overline{53}, \\
\overline{21} \vee \overline{81}, \overline{22} \vee \overline{82}, \overline{23} \vee \overline{83}, \\
\overline{51} \vee \overline{81}, \overline{52} \vee \overline{82}, \overline{53} \vee \overline{83}, \\
\overline{31} \vee \overline{61}, \overline{32} \vee \overline{62}, \overline{33} \vee \overline{63}, \\
\overline{31} \vee \overline{91}, \overline{32} \vee \overline{92}, \overline{33} \vee \overline{93}, \\
\overline{61} \vee \overline{91}, \overline{62} \vee \overline{92}, \overline{63} \vee \overline{93}
\end{gathered}
$$

Finally we encode the given starting positions as unit clauses:

$12, 61$

5(ii)

1. After applying unit propogation with $\mathscr{I}(12) = T$ we arrive at the following set of clauses:

$$
\begin{aligned}
S = \{&21 \vee 23, \\
&31 \vee 33, \\
&41 \vee 43, \\
&51 \vee 52 \vee 53, \\
&61 \vee 62 \vee 63, \\
&71 \vee 73, \\
&81 \vee 82 \vee 83, \\
&91 \vee 92 \vee 93, \\
&\overline{21} \vee \overline{23}, \\
&\overline{31} \vee \overline{33}, \\
&\overline{41} \vee \overline{43}, \\
&\overline{51} \vee \overline{52}, \overline{51} \vee \overline{53}, \overline{52} \vee \overline{53}, \\
&\overline{61} \vee \overline{62}, \overline{61} \vee \overline{63}, \overline{62} \vee \overline{63}, \\
&\overline{71} \vee \overline{73}, \\
&\overline{81} \vee \overline{82}, \overline{81} \vee \overline{83}, \overline{82} \vee \overline{83}, \\
&\overline{91} \vee \overline{92}, \overline{91} \vee \overline{93}, \overline{92} \vee \overline{93}, \\
&\overline{21} \vee \overline{31}, \overline{23} \vee \overline{33}, \\
&\overline{41} \vee \overline{51}, \overline{43} \vee \overline{53}, \\
&\overline{41} \vee \overline{61}, \overline{43} \vee \overline{63}, \\
&\overline{51} \vee \overline{61}, \overline{52} \vee \overline{62}, \overline{53} \vee \overline{63}, \\
&\overline{71} \vee \overline{81}, \overline{73} \vee \overline{83}, \\
&\overline{71} \vee \overline{91}, \overline{73} \vee \overline{93}, \\
&\overline{81} \vee \overline{91}, \overline{82} \vee \overline{92}, \overline{83} \vee \overline{93}, \\
&\overline{41} \vee \overline{71}, \overline{43} \vee \overline{73}, \\
&\overline{21} \vee \overline{51}, \overline{23} \vee \overline{53}, \\
&\overline{21} \vee \overline{81}, \overline{23} \vee \overline{83}, \\
&\overline{51} \vee \overline{81}, \overline{52} \vee \overline{82}, \overline{53} \vee \overline{83}, \\
&\overline{31} \vee \overline{61}, \overline{33} \vee \overline{63}, \\
&\overline{31} \vee \overline{91}, \overline{33} \vee \overline{93}, \\
&\overline{61} \vee \overline{91}, \overline{62} \vee \overline{92}, \overline{63} \vee \overline{93}\}
\end{aligned}
$$

2. Setting $\mathscr{I}(61) = T$, and applying unit propogation to the resultant set of clauses S, we are able to resolve all the remaining clauses and are left with $S = \emptyset$.

   We conclude that the initial set of clauses is satisfiable and take the solution to be valuations $\mathscr{I}(12) = T, \mathscr{I}(11) = F, \mathscr{I}(13) = F, \mathscr{I}(22) = F, \mathscr{I}(32) = F, \mathscr{I}(42) = F, \mathscr{I}(72) = F, \mathscr{I}(61) = T, \mathscr{I}(62) = F, \mathscr{I}(63) = F, \mathscr{I}(41) = F, \mathscr{I}(51) = F, \mathscr{I}(43) = T, \mathscr{I}(53) = F, \mathscr{I}(73) = F, \mathscr{I}(52) = T, \mathscr{I}(71) = T, \mathscr{I}(31) = F, \mathscr{I}(91) = F, \mathscr{I}(33) = T, \mathscr{I}(23) = F, \mathscr{I}(93) = F, \mathscr{I}(21) = T, \mathscr{I}(92) = T, \mathscr{I}(81) = F, \mathscr{I}(82) = F, \mathscr{I}(83) = T$

Which is the following solution

2	1	3
3	2	1
1	3	2