COS4807 Assignment 1 - 791729

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Question 1

$$(p \to q) \leftrightarrow (p \to r) \equiv \big((p \to q) \to (p \to r)\big) \wedge \big((p \to r) \to (p \to q)\big) \qquad \text{remove biconditional}$$

$$\equiv \big(\neg(p \to q) \vee (p \to r)\big) \wedge \big(\neg(p \to r) \vee (p \to q)\big) \qquad \text{remove conditional}$$

$$\equiv \big(\neg(\neg p \vee q) \vee (\neg p \vee r)\big) \wedge \big(\neg(\neg p \vee r) \vee (\neg p \vee q)\big) \qquad \text{remove conditional}$$

$$\equiv \big((\neg\neg p \wedge \neg q) \vee (\neg p \vee r)\big) \wedge \big((\neg\neg p \wedge \neg r) \vee (\neg p \vee q)\big) \qquad \text{De Morgan}$$

$$\equiv \big((p \wedge \neg q) \vee (\neg p \vee r)\big) \wedge \big((p \wedge \neg r) \vee (\neg p \vee q)\big) \qquad \text{remove } \neg\neg$$

$$\equiv \big((p \vee \neg p \vee r) \wedge (\neg q \vee \neg p \vee r)\big) \wedge \big((p \vee \neg p \vee q) \wedge (\neg r \vee \neg p \vee q)\big) \qquad \text{distribute disjunction}$$

$$\equiv (p \vee \neg p \vee r) \wedge (\neg q \vee \neg p \vee r) \wedge (p \vee \neg p \vee q) \wedge (\neg r \vee \neg p \vee q) \quad \checkmark$$

In clausal form is: $\{\{p, \neg p, r\}, \{\neg q, \neg p, r\}, \{p, \neg p, q\}, \{\neg r, \neg p, q\}\}$ $\quad\checkmark$

Without trivial clauses: $\{\{\neg q, \neg p, r\}, \{\neg r, \neg p, q\}\}$ $\quad\checkmark$

Which can be abbreviated as: $\{\bar{q}\bar{p}r, \bar{r}\bar{p}q\}$ $\quad\checkmark$

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Question 2

$$\big((p \to q) \to \neg(r \to q)\big) \wedge \big((\neg r \to p) \to (p \to q)\big)$$

$$\equiv \big(\neg(p \to q) \vee \neg(r \to q)\big) \wedge \big(\neg(\neg r \to p) \vee (p \to q)\big)$$

$$\equiv \big(\neg(\neg p \vee q) \vee \neg(\neg r \vee q)\big) \wedge \big(\neg(\neg\neg r \vee p) \vee (\neg p \vee q)\big)$$

$$\equiv \big((\neg\neg p \wedge \neg q) \vee (\neg\neg r \wedge \neg q)\big) \wedge \big((\neg\neg\neg r \wedge \neg p) \vee (\neg p \vee q)\big)$$

$$\equiv \big((p \wedge \neg q) \vee (r \wedge \neg q)\big) \wedge \big((\neg r \wedge \neg p) \vee (\neg p \vee q)\big)$$

$$\equiv (p \vee r) \wedge (p \vee \neg q) \wedge (\neg q \vee r) \wedge (\neg q \vee \neg q) \wedge (\neg r \vee \neg p \vee q) \wedge (\neg p \vee \neg p \vee q) \quad \checkmark$$

In abbreviated clausal form: $\{pr, p\bar{q}, \bar{q}r, \bar{q}, \bar{r}\bar{p}q, \bar{p}q\}$ $\quad\checkmark$

1. pr
2. $p\bar{q}$
3. $\bar{q}r$
4. $\bar{q}$
5. $\bar{r}\bar{p}q$
6. $\bar{p}q$
7. rq 1,6 $\quad\checkmark$
8. $r\bar{p}$ 3,6 $\quad\checkmark$
9. r 3,7 $\quad\checkmark$
10. $\bar{r}\bar{p}$ 4,5 $\quad\checkmark$
11. $\bar{p}$ 4,6 $\quad\checkmark$
12. $r\bar{q}$ 2,8
13. $\bar{q}\bar{r}$ 2,9 $\quad\times$
14. pr 2,7
15. $\bar{q}\bar{p}$ 3,9 $\quad\times$

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Because we can resolve no new clauses and we have not attained a □, we conclude that the formula is satisfiable. ✓

Question 3

Formula	Clausal Form
$p \rightarrow (q \vee r)$	$\bar{p}qr$
$(p \rightarrow r) \rightarrow q$	$pq, q\bar{r}$
$\neg q$	$\bar{q}$

✓ 💬

We will use resolution to attain a □ to show that the set of formulas where we have negated the goal are unsatisfiable.

1. $\bar{p}qr$
2. pq
3. $q\bar{r}$
4. $\bar{q}$ negation of goal ✓
5. qr 1,2 ✓
6. q 3,5 ✓
7. □   4,6 ✓

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We have proved that the set of clauses are unsatisfiable and so shown that the formulas from which they are derived are also unstatisfiable. Therefore, we conclude that $\{p \rightarrow (q \vee r), (p \rightarrow r) \rightarrow q\} \models q$ ✓

Question 4

We wish to prove that if the set of clauses labeling the leaves of a resolution tree are satisfiable, then the clause at the root of the tree is satisfiable.

We show this by structural induction on the height of the tree. ✓

Base Case: Let C be an arbitrary clause which is satisfiable. Then, if the resolution tree is labeled by C and has a height $n = 0$, C is also a leaf node and so is, by definition, satisifiable. 💬

Inductive Step: With $n > 0$ we have C being formed by the resolution of two subtrees, C₁ and C₂ 💬 with height $k < n$. Assume C_1 and C_2 are satisfiable under some interpretation $\mathscr{I}$. Since, for some complementary set of literals l, l^c, we have that either $\mathscr{I}(l) = T$ or $\mathscr{I}(l^c) = T$.

Suppose $\mathscr{I}(l) = T$, then $\mathscr{I}(l^c) = F$. If $l \in C_1$ then we have that $l^c \in C_2$ and 💬 C_2 can be satisfied only by some other literal $l' \in C_2$ with $\mathscr{I}(l') = T$ (and $l' \neq l^c$). The resolution rule defines $C = (C_1 - \{l\}) \cup (C_2 - \{l^c\})$ so $l' \in C$ which means that C is also satisfiable. A symmetric argument holds when $\mathscr{I}(l^c) = T$

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Question 5

5(i)

Given the following sudoku board with cells numbered $1 - 9$, we define the set of clauses required to encode the problem as follows:

1	2	3
4	5	6
7	8	9

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We encode the fact that each cell must contain at least one number as follows:

$$11 \vee 12 \vee 13,$$
$$21 \vee 22 \vee 23,$$
$$31 \vee 32 \vee 33,$$
$$41 \vee 42 \vee 43,$$
$$51 \vee 52 \vee 53,$$
$$61 \vee 62 \vee 63,$$
$$71 \vee 72 \vee 73,$$
$$81 \vee 82 \vee 83,$$
$$91 \vee 92 \vee 93$$

We add the constraint that a cell can contain at most one value:

$$\overline{11} \vee \overline{12}, \overline{11} \vee \overline{13}, \overline{12} \vee \overline{13},$$
$$\overline{21} \vee \overline{22}, \overline{21} \vee \overline{23}, \overline{22} \vee \overline{23},$$
$$\overline{31} \vee \overline{32}, \overline{31} \vee \overline{33}, \overline{32} \vee \overline{33},$$
$$\overline{41} \vee \overline{42}, \overline{41} \vee \overline{43}, \overline{42} \vee \overline{43},$$
$$\overline{51} \vee \overline{52}, \overline{51} \vee \overline{53}, \overline{52} \vee \overline{53},$$
$$\overline{61} \vee \overline{62}, \overline{61} \vee \overline{63}, \overline{62} \vee \overline{63},$$
$$\overline{71} \vee \overline{72}, \overline{71} \vee \overline{73}, \overline{72} \vee \overline{73},$$
$$\overline{81} \vee \overline{82}, \overline{81} \vee \overline{83}, \overline{82} \vee \overline{83},$$
$$\overline{91} \vee \overline{92}, \overline{91} \vee \overline{93}, \overline{92} \vee \overline{93}$$

Add the constraint that no two cells in a row can contain the same number:

$$\overline{11} \vee \overline{21}, \overline{12} \vee \overline{22}, \overline{13} \vee \overline{23},$$
$$\overline{11} \vee \overline{31}, \overline{12} \vee \overline{32}, \overline{13} \vee \overline{33},$$
$$\overline{21} \vee \overline{31}, \overline{22} \vee \overline{32}, \overline{23} \vee \overline{33},$$
$$\overline{41} \vee \overline{51}, \overline{42} \vee \overline{52}, \overline{43} \vee \overline{53},$$
$$\overline{41} \vee \overline{61}, \overline{42} \vee \overline{62}, \overline{43} \vee \overline{63},$$
$$\overline{51} \vee \overline{61}, \overline{52} \vee \overline{62}, \overline{53} \vee \overline{63},$$
$$\overline{71} \vee \overline{81}, \overline{72} \vee \overline{82}, \overline{73} \vee \overline{83},$$
$$\overline{71} \vee \overline{91}, \overline{72} \vee \overline{92}, \overline{73} \vee \overline{93},$$
$$\overline{81} \vee \overline{91}, \overline{82} \vee \overline{92}, \overline{83} \vee \overline{93}$$

And the constraint that no two cells in a column can contain the same number:

$$\overline{11} \vee \overline{41}, \overline{12} \vee \overline{42}, \overline{13} \vee \overline{43},$$
$$\overline{11} \vee \overline{71}, \overline{12} \vee \overline{72}, \overline{13} \vee \overline{73},$$
$$\overline{41} \vee \overline{71}, \overline{42} \vee \overline{72}, \overline{43} \vee \overline{73},$$
$$\overline{21} \vee \overline{51}, \overline{22} \vee \overline{52}, \overline{23} \vee \overline{53},$$
$$\overline{21} \vee \overline{81}, \overline{22} \vee \overline{82}, \overline{23} \vee \overline{83},$$
$$\overline{51} \vee \overline{81}, \overline{52} \vee \overline{82}, \overline{53} \vee \overline{83},$$
$$\overline{31} \vee \overline{61}, \overline{32} \vee \overline{62}, \overline{33} \vee \overline{63},$$
$$\overline{31} \vee \overline{91}, \overline{32} \vee \overline{92}, \overline{33} \vee \overline{93},$$
$$\overline{61} \vee \overline{91}, \overline{62} \vee \overline{92}, \overline{63} \vee \overline{93}$$

Finally we encode the given starting positions as unit clauses:

12, 61 ✔

5(ii)

1. After applying unit propogation with $\mathscr{I}(12) = T$ we arrive at the following set of clauses:

$$S = \{21 \lor 23, \quad ✗$$
$$31 \lor 33, \quad ✗ \quad 💬$$
$$41 \lor 43, \quad ✗$$
$$51 \lor 52 \lor 53,$$
$$61 \lor 62 \lor 63,$$
$$71 \lor 73,$$
$$81 \lor 82 \lor 83,$$
$$91 \lor 92 \lor 93,$$
$$\overline{21} \lor \overline{23}, \quad ✗$$
$$\overline{31} \lor \overline{33},$$
$$\overline{41} \lor \overline{43},$$
$$\overline{51} \lor \overline{52}, \overline{51} \lor \overline{53}, \overline{52} \lor \overline{53},$$
$$\overline{61} \lor \overline{62}, \overline{61} \lor \overline{63}, \overline{62} \lor \overline{63},$$
$$\overline{71} \lor \overline{73},$$
$$\overline{81} \lor \overline{82}, \overline{81} \lor \overline{83}, \overline{82} \lor \overline{83},$$
$$\overline{91} \lor \overline{92}, \overline{91} \lor \overline{93}, \overline{92} \lor \overline{93},$$
$$\overline{21} \lor \overline{31}, \overline{23} \lor \overline{33},$$
$$\overline{41} \lor \overline{51}, \overline{43} \lor \overline{53},$$
$$\overline{41} \lor \overline{61}, \overline{43} \lor \overline{63},$$
$$\overline{51} \lor \overline{61}, \overline{52} \lor \overline{62}, \overline{53} \lor \overline{63},$$
$$\overline{71} \lor \overline{81}, \overline{73} \lor \overline{83},$$
$$\overline{71} \lor \overline{91}, \overline{73} \lor \overline{93},$$
$$\overline{81} \lor \overline{91}, \overline{82} \lor \overline{92}, \overline{83} \lor \overline{93},$$
$$\overline{41} \lor \overline{71}, \overline{43} \lor \overline{73},$$
$$\overline{21} \lor \overline{51}, \overline{23} \lor \overline{53},$$
$$\overline{21} \lor \overline{81}, \overline{23} \lor \overline{83},$$
$$\overline{51} \lor \overline{81}, \overline{52} \lor \overline{82}, \overline{53} \lor \overline{83},$$
$$\overline{31} \lor \overline{61}, \overline{33} \lor \overline{63},$$
$$\overline{31} \lor \overline{91}, \overline{33} \lor \overline{93},$$
$$\overline{61} \lor \overline{91}, \overline{62} \lor \overline{92}, \overline{63} \lor \overline{93}\}$$

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2. Setting $\mathscr{I}(61) = T$, and applying unit propogation to the resultant set of clauses S, we are able to resolve all the remaining clauses and are left with $S = \emptyset$. ✗

We conclude that the initial set of clauses is satisfiable and take the solution to be valuations $\mathscr{I}(12) = T, \mathscr{I}(11) = F, \mathscr{I}(13) = F, \mathscr{I}(22) = F, \mathscr{I}(32) = F, \mathscr{I}(42) = F, \mathscr{I}(72) = F, \mathscr{I}(61) = T, \mathscr{I}(62) = F, \mathscr{I}(63) = F, \mathscr{I}(41) = F, \mathscr{I}(51) = F, \mathscr{I}(43) = T, \mathscr{I}(53) = F, \mathscr{I}(73) = F, \mathscr{I}(52) = T, \mathscr{I}(71) = T, \mathscr{I}(31) = F, \mathscr{I}(91) = F, \mathscr{I}(33) = T, \mathscr{I}(23) = F, \mathscr{I}(93) = F, \mathscr{I}(21) = T, \mathscr{I}(92) = T, \mathscr{I}(81) = F, \mathscr{I}(82) = F, \mathscr{I}(83) = T$

Which is the following solution

2	1	3
3	2	1
1	3	2

✓