4005-800 Algorithms

Homework 4

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PROBLEM 1-a. *TODO*

Solution.

Case 1: n is even ($2 \mid n$, or $n = 2m$ for some $m \in \mathbb{N}$)

$$\begin{aligned}
\lfloor \frac{n+1}{2} \rfloor &= \lfloor \frac{2m+1}{2} \rfloor \\
&= \lfloor \frac{2m}{2} + \frac{1}{2} \rfloor \\
&= m + \lfloor \frac{1}{2} \rfloor \\
&= m \\
&= \lceil \frac{2m}{2} \rceil \\
&= \lceil \frac{n}{2} \rceil
\end{aligned}$$

Case 2: n is odd ($2 \nmid n$, or $n = 2m+1$ for some $m \in \mathbb{N}$)

$$\begin{aligned}
\lfloor \frac{n+1}{2} \rfloor &= \lfloor \frac{(2m+1)+1}{2} \rfloor \\
&= \lfloor \frac{2(m+1)}{2} \rfloor \\
&= m+1 \\
&= m + \lceil \frac{1}{2} \rceil \\
&= \frac{2m}{2} + \lceil \frac{1}{2} \rceil \\
&= \lceil \frac{2m}{2} + \frac{1}{2} \rceil \\
&= \lceil \frac{2m+1}{2} \rceil \\
&= \lceil \frac{n}{2} \rceil
\end{aligned}$$

Thus, since a number $n \in \mathbb{N}$ can only be even or odd, we can conclude that for any $n \in \mathbb{N}$, $\lfloor \frac{n+1}{2} \rfloor = \lceil \frac{n}{2} \rceil$.

PROBLEM 1-b. *TODO*

Solution. **Case 1:** n is even ($2 \mid n$, or $n = 2m$ for some $m \in \mathbb{N}$)

$$\begin{aligned}
\lfloor \frac{n}{2} \rfloor + 1 &= \lfloor \frac{2m}{2} \rfloor + 1 \\
&= \frac{2m}{2} + 1 \\
&= \frac{2m}{2} + \lceil \frac{1}{2} \rceil \\
&= \lceil \frac{2m}{2} \rceil + \lceil \frac{1}{2} \rceil \\
&= \lceil \frac{2m}{2} + \frac{1}{2} \rceil \\
&= \lceil \frac{2m+1}{2} \rceil \\
&= \lceil \frac{n+1}{2} \rceil
\end{aligned}$$

Case 2: n is odd ($2 \not\mid n$, or $n = 2m+1$ for some $m \in \mathbb{N}$)

$$\begin{aligned}
\lfloor \frac{n}{2} \rfloor + 1 &= \lfloor \frac{2m+1}{2} \rfloor + 1 \\
&= \lfloor \frac{2m}{2} + \frac{1}{2} \rfloor + 1 \\
&= m + \lfloor \frac{1}{2} \rfloor + 1 \\
&= m + 1 \\
&= \frac{2(m+1)}{2} \\
&= \lceil \frac{2(m+1)}{2} \rceil \\
&= \lceil \frac{2m+2}{2} \rceil \\
&= \lceil \frac{(2m+1)+1}{2} \rceil \\
&= \lceil \frac{n+1}{2} \rceil
\end{aligned}$$

Thus, since a number $n \in \mathbb{N}$ can only be even or odd, we can conclude that for any $n \in \mathbb{N}$, $\lfloor \frac{n}{2} \rfloor + 1 = \lceil \frac{n+1}{2} \rceil$.

PROBLEM 1-c.

Solution.

Let $D(n) = T(n+1) - T(n)$. If we let $n = 1$ be the base case for the recurrence as in $T(n)$, we obtain the following:

$$\begin{aligned} D(1) &= T(2) - T(1) \\ &= T\left(\lceil \frac{2}{2} \rceil\right) + T\left(\lfloor \frac{2}{2} \rfloor\right) + 2 - 0 \\ &= T(1) + T(1) + 2 \\ &= 2 \end{aligned}$$

Thus, we can see that $D(1) = 2$. We now seek the general case for $D(n)$ by expanding the its representation using the definition for $T(n)$, as shown below.

$$\begin{aligned} D(n) &= T(n+1) - T(n) \\ &= \left(T\left(\lceil \frac{n+1}{2} \rceil\right) + T\left(\lfloor \frac{n+1}{2} \rfloor\right) + (n+1)\right) - \left(T\left(\lceil \frac{n}{2} \rceil\right) + T\left(\lfloor \frac{n}{2} \rfloor\right) + n\right) \\ &= T\left(\lceil \frac{n+1}{2} \rceil\right) + T\left(\lfloor \frac{n+1}{2} \rfloor\right) - T\left(\lceil \frac{n}{2} \rceil\right) - T\left(\lfloor \frac{n}{2} \rfloor\right) + 1 \\ &= T\left(\lceil \frac{n+1}{2} \rceil\right) + T\left(\lfloor \frac{n}{2} \rfloor\right) + 1 \\ &= T\left(\lfloor \frac{n}{2} \rfloor + 1\right) + T\left(\lfloor \frac{n}{2} \rfloor\right) + 1 \end{aligned}$$

Now, by observing that this expression takes the same form as $D(n)$, we obtain the following:

$$\begin{aligned} D(n) &= T\left(\lfloor \frac{n}{2} \rfloor + 1\right) + T\left(\lfloor \frac{n}{2} \rfloor\right) + 1 \\ &= D(\lfloor \frac{n}{2} \rfloor) + 1 \end{aligned}$$

Now, putting these results together, we obtain the following recurrence for $D(n)$:

$$\begin{aligned} D(1) &= 2 \\ D(n) &= D(\lfloor \frac{n}{2} \rfloor) + 1 \end{aligned}$$

PROBLEM 1-d. *TODO*

Solution. **Base ($n = 1$)**
By the definition of $D(n)$, we know the following:

$$\begin{aligned} D(1) &= 2 \\ &= 0 + 2 \\ &= \lg(1) + 2 \\ &= \lfloor \lg(1) \rfloor + 2 \end{aligned}$$

Induction $(n > 1)$

Assume that $D(k) = \lfloor \lg(k) \rfloor + 2$ for some $k \in \mathbb{N}$ such that $2 \le k < n$. We now show that $D(n) = \lfloor \lg(n) \rfloor + 2$.

$$
\begin{aligned}
D(n) &= D(\lfloor \frac{n}{2} \rfloor) + 1 \\
&= \Big(\lfloor \lg(\frac{n}{2}) \rfloor + 2 \Big) + 1 \\
&= \Big(\lfloor \lg(n) - \lg(2) \rfloor + 2 \Big) + 1 \\
&= \Big(\lfloor \lg(n) - 1 \rfloor + 2 \Big) + 1 \\
&= \Big((\lfloor \lg(n) \rfloor - 1) + 2 \Big) + 1 \\
&= \Big(\lfloor \lg(n) \rfloor + 1 \Big) + 1 \\
&= \lfloor \lg(n) \rfloor + 2
\end{aligned}
$$

Thus, $D(n) = \lfloor \lg(n) \rfloor + 2$, as desired.

PROBLEM 1-e. *TODO*

Solution. By the definition of $D(n)$, we observe the following:

$$
\begin{aligned}
\sum_{k=1}^{n-1} D(k) &= \sum_{k=1}^{n-1} \Big(T(k+1) - T(k) \Big) \\
&= \big(T(2) - T(1)\big) + \big(T(3) - T(2)\big) + \big(T(4) - T(3)\big) + ... + \big(T(n) - T(n-1)\big) \\
&= T(n) - T(1)
\end{aligned}
$$

Therefore, since $\sum_{k=1}^{n-1} D(k)$ turns into a telescoping sum, we see that it collapses to $T(n) - T(1)$, and since $D(n) = \lfloor \lg(n) \rfloor + 2$, we also know the following:

$$
\begin{aligned}
T(n) - T(1) &= \sum_{k=1}^{n-1} D(k) \\
&= \sum_{k=1}^{n-1} \big(\lfloor \lg(k) \rfloor + 2 \big)
\end{aligned}
$$

Thus, we see that $T(n) - T(1) = \sum_{k=1}^{n-1} \big(\lfloor \lg(k) \rfloor + 2 \big)$, and since $T(1) = 0$, we conclude that $T(n) = \sum_{k=1}^{n-1} \big(\lfloor \lg(k) \rfloor + 2 \big)$.

PROBLEM 1-f. *TODO*

Solution. Using the fact that $T(n) = \sum_{k=1}^{n-1} \big(\lfloor \lg(k) \rfloor + 2 \big)$. We now evaluate this summation as follows:

$$
\begin{aligned}
T(n) &= \sum_{k=1}^{n-1} \big(\lfloor\ \lg(n)\rfloor + 2\big) \\
&= \sum_{k=1}^{n-1} \lfloor\ \lg(k)\rfloor + \sum_{k=1}^{n-1} 2 \\
&< \sum_{k=1}^{n-1} \lfloor\ \lg(n)\rfloor + \sum_{k=1}^{n-1} 2 \\
&= (n-1)\ \lg(n) + 2(n-1) \\
&= n\ \lg(n) -\ \lg(n) + 2n - 1 \\
&= O(n\ \lg(n)) \\
&= O(n\ \log(n))
\end{aligned}
$$