4005-800 Algorithms

Homework 7

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PROBLEM 1 - 34.2-1. *Consider the language GRAPH-ISOMORPHISM* $= \{\langle G_1, G_2\rangle : G_1$ *and* G_2 *are isomorphic graphs* $\}$. *Prove that GRAPH-ISOMORPHISM* $\in NP$ *by describing a polynomial-time algorithm to verify the language.*

Solution. By the defintion of graph isomorphism, two graphs G_1 and G_2 are isomorphic if and only if there exists a bijection $m : V(G_1) \to V(G_2)$ such that any two vertices v_i and v_j of G_1 are adjacent in G_1 if and only if $m(v_i)$ and $m(v_j)$ are adjacent in G_2. With this definition, it is enough to check the bijection m to see if it fulfills this property to verify that two graphs isomorphic. We can easily devise a polynomial-time algorithm to solve this as follows:

Algorithm 1: GRAPH-ISOMORPHISM-VERIFIER

```
function VerifyGraphIsomorphism(m)
    for all v_i ∈ V(G_1) do
        vCount = 0                                  ▷ Count number of times v_i appears in G_2
        for all v_j ∈ V(G_2) do
            if m(v_i) == v_j then
                vCount = vCount + 1
            end if
        end for
        if vCount ≠ 1 then                          ▷ v_i should only appear once in G_2
            return False
        end if
    end for

    for all v_i ∈ V(G_1) do
        for all v_j ∈ V(G_1) do
            if (v_i, v_j) ∈ E(G_1) and (m(v_i), m(v_j)) ∉ E(G_2) then
                return False
            end if
        end for
    end for

    for all v_i ∈ V(G_1) do
```

```
        for all v_j ∈ V(G_1) do
            if (m(v_i), m(v_j)) ∈ E(G_2) and (v_i, v_j) ∉ E(G_1) then
                return False
            end if
        end for
    end for

    return True
end function
```

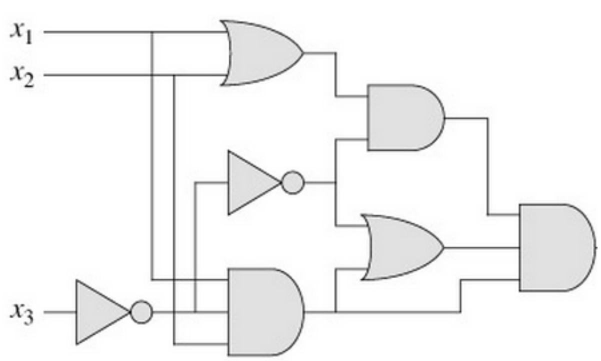
Note that m denotes the bijective mapping (the permutation) from $V(G_1)$ to $V(G_2)$. It is clear that the permutation check runs in $O(V)$ time and the edge check runs in $O(V^2)$ time. Thus, we conclude that this algorithms runs in $O(V^2)$ time and thus verifies the solution (i.e. the permutation mapping m) to the GRAPH-ISOMORPHISM problem in polynomial time.

PROBLEM 2 - 34.2-10. *Prove that if $NP \neq$ co-NP, then $P \neq NP$.*

Solution. If $NP \neq$ *co*-NP, then we know there exists a problem $Q \in NP$ such that $Q \notin$ *co*-NP. Furthermore, by definition, we know that $P \in$ *co*-$NP \cap NP$. Now, let Q be a problem in NP that is not in *co*-NP. By the definition of the set intersection, that means that $Q \notin$ *co*-$NP \cap NP$, and thus we know that $Q \in NP$ and $Q \notin P$ (becuse P is a subset of *co*-$NP \cap NP$). Therefore, since there exists a problem that is in NP but not in P, we conclude that $P \neq NP$.

PROBLEM 3 - 34.3-1. *Verify that the circuit in Figure 34.8(b) is unsatisfiable.*

Solution.



The logical equivalent expression for this circuit is as follows:

$$((x_1 \vee x_2) \wedge x_3) \wedge (x_3 \vee (x_1 \wedge x_2 \wedge \neg x_3)) \wedge (x_1 \wedge x_2 \wedge \neg x_3)$$

x_1, x_2, x_3	$x_1 \vee x_2$	$(x_1 \vee x_2) \wedge x_3$	$x_1 \wedge x_2 \wedge \neg x_3$	$x_3 \vee (x_1 \wedge x_2 \wedge \neg x_3)$	Final AND Gate
F,F,F	F	F	F	F	F
F,F,T	F	F	F	T	F
F,T,F	T	F	F	F	F
F,T,T	T	T	F	T	F
T,F,F	T	F	F	F	F
T,F,T	T	T	F	T	F
T,T,F	T	F	T	T	F
T,T,T	T	T	F	T	F

Table 1: Truth table for problem #3, where T = True and F = False.

To show that this circuit is unsatisfiable, we simply build a truth table for the boolean expression that considers all logical values for x_1, x_2, and x_3, as shown in table :

Therefore, since there is no possible combination of logical values for x_1, x_2, and x_3 such that the boolean expression is true, we conclude that it is unsatisfiable.

PROBLEM 4 - 34.4-5. *Show that the problem of determining the satisfiability of boolean formulas in disjunctive normal form is polynomial-time solvable.*

Solution. We show that the problem of determining the satisfiability of boolean formulas in disjunctive normal form is polynomial-time solvable by providing a polynomial-time algorithm that performs this task. This algorithm is realized below in Algorithm 2.

ALGORITHM 2: DNF-SOLVER

function SOLVEDNF(ψ)
    for all Logical clauses $c_i \in \psi$ **do**
        $satisfiable = True$
        $satList = makeQueue()$
        for all Literals $l_j \in c_i$ **do**
            for all Literals $l_k \in satList$ **do**
                if $l_j == \neg l_j$ **then**
                    $satisfiable =$ False ▷ Found the negation of l_j in the queue
                end if
            end for
            $PUSH(satList, l_j)$ ▷ Push this literal into the queue
        end for
        if $satisfiable == True$ **then**
            return $True$
        end if

```
    end for
    return False
end function
```

Since DNF statements are composed of disjunctions (ORs) of conjunction clauses (ANDs), it is enough to check and see if only one conjunction clause can be satisfied. Therefore, this procedure simply traverses over every clause and checks to see if there is a literal and its negation in that clause, which indicates that the clause can never be true. If this is not the case, then the clause must be satisfiable, and thus the expression is satisfiable.

The time complexity of this algorithm is $O(mn^2)$, where m is the number of clauses in ψ and n is the number of literals in the boolean expression. The reason for this is that for every clause we traverse over every literal in the clause, and for each element we perform a linear search with *satList* that can be equal to the number of literals in the clause. Therefore, since the linear search runs in $O(n)$ time, the number of literals in a clause is $O(n)$, and the number of clauses in ψ is $O(m)$, and each of these operations are nested, the resulting time complexity is $O(mn^2)$.

PROBLEM 5 - 34.5-5. *The **set-partition problem** takes as input a set S of numbers. The question is whether the numbers can be partitioned into two sets A and $A' = S - A$ such that $\sum_{x \in A} x = \sum_{x \in A'} x$. Show that the set-partition problem is NP-complete.*

Solution. In order to show that the set-partition problem, Q is NP-complete, we show that it reduces to the subset-sum problem, Q'. That is, we prove $Q \leq_p Q'$.

PROBLEM 6. *TODO*

Solution.

Algorithm 3: RECURSIVE-KNAPSACK

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TODO
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