4040-849 Optimization Methods

Written Assignment 2

Christopher Wood

April 11, 2012

PROBLEM 1-a.

Solution.

Making the substitution of $f(\lambda)$ for $\frac{\tau_{zy}}{p_{max}}$, where $\lambda = \frac{z}{b}$, we get a simplified equation that can be simplified as follows.

$$\begin{aligned}
f(\lambda) &= -\frac{1}{2}\left[-\frac{1}{\sqrt{1+\lambda^2}} + \left(2 - \frac{1}{1+\lambda^2}\right)\sqrt{1+\lambda^2} - 2\lambda\right] \\
&= -\frac{1}{2}\left[-\frac{1}{\sqrt{1+\lambda^2}} + 2\sqrt{1+\lambda^2} - \frac{\sqrt{1+\lambda^2}}{1+\lambda^2} - 2\lambda\right] \\
&= \frac{0.5}{\sqrt{1+\lambda^2}} - \sqrt{1+\lambda^2} + \frac{0.5\sqrt{1+\lambda^2}}{1+\lambda^2} + \lambda \\
&= \frac{0.5}{\sqrt{1+\lambda^2}} - \sqrt{1+\lambda^2}\left(1 - \frac{0.5}{1+\lambda^2}\right) + \lambda
\end{aligned}$$

Therefore, as shown, we can reduce the problem of finding the location of the maximum shear stress for $v_1 = v_2 = 3$ reduces to maximizing the function shown below:

$$f(\lambda) = \frac{0.5}{\sqrt{1+\lambda^2}} - \sqrt{1+\lambda^2}\left(1 - \frac{0.5}{1+\lambda^2}\right) + \lambda \tag{1}$$

PROBLEM 1-b.

Solution.

In order to apply the Fibonacci method, we must be trying to minimize the objective function for a particular problem. Therefore, since we were given an objective function (1) that we must maximize, we simply negate it so that we can apply the Fibonacci method to solve it numerically. The resulting function that we seek to minimize is shown below.

$$f'(\lambda) = -f\lambda) = -\frac{0.5}{\sqrt{1+\lambda^2}} + \sqrt{1+\lambda^2}\left(1 - \frac{0.5}{1+\lambda^2}\right) - \lambda$$

Following the approach for the Fibonacci method with $n = 8$ yeilds a total of seven iteration steps we must take. The values for J, A_1, B_1, L_1, L_2^* from each of these iterations from $J = 2$ and finishing at $J = 8$ are shown below, starting with initial values of 8, 0, and 3 for n, A_1 and B_1, respectively.

Step	J	A_1	B_1	L_1	L_2^*	x_1	x_2	f_1	f_2
1	2	0	3	3	$\frac{39}{34}$	$\frac{39}{34}$	$\frac{63}{34}$	-0.2824	-0.2223
2	3	0	$\frac{63}{34}$	$\frac{63}{34}$	$\frac{39}{34}$	$\frac{27}{34}$	$\frac{39}{34}$	-0.3000	-0.2824
3	4	0	$\frac{39}{34}$	$\frac{39}{34}$	$\frac{12}{17}$	$\frac{15}{34}$	$\frac{12}{17}$	-0.2631	-0.2988
4	5	$\frac{15}{34}$	$\frac{39}{34}$	$\frac{12}{17}$	$\frac{15}{34}$	$\frac{24}{34}$	$\frac{15}{17}$	-0.2988	-0.2985
5	6	$\frac{15}{34}$	$\frac{15}{17}$	$\frac{15}{34}$	$\frac{9}{34}$	$\frac{21}{34}$	$\frac{12}{17}$	-0.2931	-0.2988
6	7	$\frac{21}{34}$	$\frac{15}{17}$	$\frac{9}{34}$	$\frac{3}{17}$	$\frac{12}{17}$	$\frac{27}{34}$	-0.2988	-0.3003
7	8	$\frac{12}{17}$	$\frac{15}{17}$	-	-	-	-	-	-

Once $J = n = 8$ we no longer have to derive the value for L_1 or L_2^*, and we can conclude that the maximum value of 1 occurs at somewhere in the interval $[A_1, B_1]$, so we compute it as $\lambda \approx \frac{A_1+B_1}{2} = \frac{12+15}{17*2} = \frac{27}{34} = 0.7941176$.

Note that the Fibonacci numbers were generated using the equations $F(0) = 1$, $F(1) = 1$, and $F(n) = F(n-1) + F(n-2)$, as given in the slides.

PROBLEM 1-c.

Solution.

In order to apply Newton's method to find a critical point of the given objective function, we must derive the first and second derviatives of $f(\lambda)$ with respect to λ. The original function and its derivatives are shown below.

$$f(\lambda) = \frac{0.5}{\sqrt{1+\lambda^2}} - \sqrt{1+\lambda^2}\left(1 - \frac{0.5}{1+\lambda^2}\right) + \lambda$$

$$f'(\lambda) = \frac{\lambda\left(-\lambda^2 - 2\right)}{\left(\lambda^2+1\right)^{3/2}} + 1$$

$$f''(\lambda) = \frac{\lambda^2 - 2}{\sqrt{\lambda^2+1}\left(\lambda^2+1\right)^2}$$

Using these equations, we can calculate λ_{i+1} as follows:

$$\lambda_{i+1} = \lambda_i - \frac{f'(\lambda)}{f''(\lambda)}$$

Iteratively applying Newton's method with $\lambda_1 = 0.6$ as the initial point to converge towards the critical point yields the following results:

Step	λ_n	$f'(\lambda)$	$f''(\lambda)$
1	0.6	0.107199	-0.76032
2	0.740991	0.0203098	-0.485815
3	0.782797	0.00140041	-0.419968

At this point we can see that the value of the first derivative for the given value of λ_3 is less than $\epsilon = 0.01$ (i.e. $f'(\lambda_3) = 0.00140036 < 0.01$), so we conclude that the maximum of equation 1 occurs at $\lambda \approx \lambda_3 = 0.782797$.

PROBLEM 1-d.

Solution.

In order to apply the Quasi-Newton method with a fixed step size of 0.001, we must utilize the following functions:

$$f(\lambda) = \frac{0.5}{\sqrt{1+\lambda^2}} - \sqrt{1+\lambda^2}\left(1 - \frac{0.5}{1+\lambda^2}\right) + \lambda \tag{2}$$

$$f^+(\lambda) = f(\lambda + \Delta\lambda), f^-(\lambda) = f(\lambda - \Delta\lambda)$$

$$f'(\lambda_i) = \frac{f^+(\lambda_i) - f^-(\lambda_i)}{2\Delta\lambda}$$

$$f''(\lambda_i) = \frac{f^+(\lambda_i) - 2f(\lambda_i) + f^-(\lambda_i)}{\Delta\lambda^2}$$

Using these equations, we can calculate λ_{i+1} as follows:

$$\lambda_{i+1} = \lambda_i - \frac{\Delta\lambda(f^+(\lambda_i) - f^-(\lambda_i))}{2(f^+(\lambda_i) - 2f(\lambda_i) + f^-(\lambda_i))}$$

Iteratively applying the Quasi-Newton method with $\lambda_1 = 0.6$ as the initial point to converge towards the critical point yields the following results:

Step	λ_n	$f(\lambda)$	$f^+(\lambda)$	$f^-(\lambda)$	$f'(\lambda)$
1	0.6	0.291303	0.291409	0.291195	0.107199
2	0.740992	0.299837	0.299857	0.299816	0.0203097
3	0.782797	0.300281	0.300282	0.300279	0.0014006

At this point we can see that the value of the approximated first derivative for λ_3 is less than or equal to the $\epsilon = 0.01$ (i.e. $f'(\lambda_3) = 0.0014006 < 0.01$), so we conclude that the maximum of equation 1 occurs at $\lambda \approx \lambda_3 = 0.782797$.