4040-849 Optimization Methods

Written Assignment 2

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PROBLEM 1-a.

Solution.

Making the substitution of $f(\lambda)$ for $\frac{\tau_{zy}}{p_{max}}$, where $\lambda = \frac{z}{b}$, we get an equation that can be simplified as follows.

$$\begin{aligned}
f(\lambda) &= -\frac{1}{2}\left[-\frac{1}{\sqrt{1+\lambda^2}} + \left(2 - \frac{1}{1+\lambda^2}\right)\sqrt{1+\lambda^2} - 2\lambda\right] \\
&= -\frac{1}{2}\left[-\frac{1}{\sqrt{1+\lambda^2}} + 2\sqrt{1+\lambda^2} - \frac{\sqrt{1+\lambda^2}}{1+\lambda^2} - 2\lambda\right] \\
&= \frac{0.5}{\sqrt{1+\lambda^2}} - \sqrt{1+\lambda^2} + \frac{0.5\sqrt{1+\lambda^2}}{1+\lambda^2} + \lambda \\
&= \frac{0.5}{\sqrt{1+\lambda^2}} - \sqrt{1+\lambda^2}\left(1 - \frac{0.5}{1+\lambda^2}\right) + \lambda
\end{aligned}$$

Therefore, as shown, we can reduce the problem of finding the location of the maximum shear stress for $v_1 = v_2 = 3$ reduces to maximizing the function shown below:

$$f(\lambda) = \frac{0.5}{\sqrt{1+\lambda^2}} - \sqrt{1+\lambda^2}\left(1 - \frac{0.5}{1+\lambda^2}\right) + \lambda \tag{1}$$

PROBLEM 1-b.

Solution.

In order to apply the Fibonacci method, we must be trying to minimize the objective function for a particular problem. Therefore, since we were given an objective function (1) that we must maximize, we simply negate it so that we can apply the Fibonacci method to solve it numerically. The resulting function that we seek to minimize is shown below.

$$f^*(\lambda) = -f\lambda) = -\frac{0.5}{\sqrt{1+\lambda^2}} + \sqrt{1+\lambda^2}\left(1 - \frac{0.5}{1+\lambda^2}\right) - \lambda$$

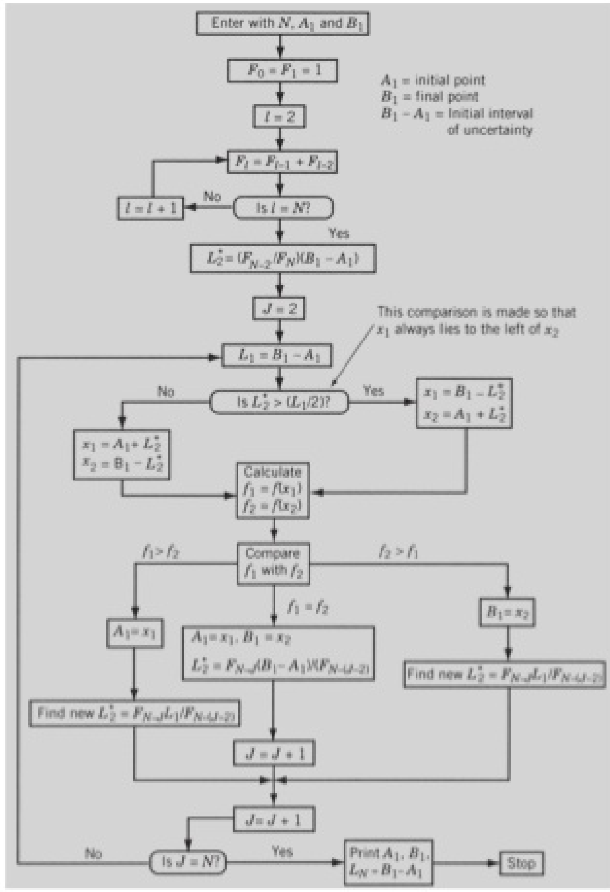

Figure 1: A flow diagram of the Fibonacci search method.

The algorithm for the Fibonacci method outlined in Figure 1 was followed with 8, 0, and 3 as the initial values for N, A_1, and B_1, respectively, in order to manually compute the minimum of $f^*(\lambda)$ (and thus the maximum value of $f(\lambda)$). The values for J, A_1, B_1, L_1, L_2^* from each of the iterations of the algorithm from $J = 2$ and finishing at $J = 8$ are shown below.

Step	J	A_1	B_1	L_1	L_2^*	x_1	x_2	f_1	f_2
1	2	0	3	3	$\frac{39}{34}$	$\frac{39}{34}$	$\frac{63}{34}$	-0.2824	-0.2223
2	3	0	$\frac{63}{34}$	$\frac{63}{34}$	$\frac{39}{34}$	$\frac{27}{34}$	$\frac{39}{34}$	-0.3000	-0.2824
3	4	0	$\frac{39}{34}$	$\frac{39}{34}$	$\frac{12}{17}$	$\frac{15}{34}$	$\frac{12}{17}$	-0.2631	-0.2988
4	5	$\frac{15}{34}$	$\frac{39}{34}$	$\frac{12}{17}$	$\frac{15}{34}$	$\frac{24}{34}$	$\frac{15}{17}$	-0.2988	-0.2985
5	6	$\frac{15}{34}$	$\frac{15}{17}$	$\frac{15}{34}$	$\frac{9}{34}$	$\frac{21}{34}$	$\frac{12}{17}$	-0.2931	-0.2988
6	7	$\frac{21}{34}$	$\frac{15}{17}$	$\frac{9}{34}$	$\frac{3}{17}$	$\frac{12}{17}$	$\frac{27}{34}$	-0.2988	-0.3003
7	8	$\frac{12}{17}$	$\frac{15}{17}$	-	-	-	-	-	-

The data in this table is formatted as follows:

1. A_1 and B_1 - Values for A_1 and B_1 going into step (or round) n of the algorithm.
2. L_1 and L_2^* - Values for L_1 and L_2^* going into step (or round) n of the algorithm.
3. x_1 and x_2 - Values for x_1 and x_2 that are computed during the nth round of the algorithm.
4. $f1$ and $f2$ - Values for f_1 and f_2 computed from x_1 and x_2 during the nth round of the algorithm.

Once $J = N = 8$ the Fibonacci algorithm is terminated and the values for $A_1 = \frac{12}{17}$ and $B_1 = \frac{15}{17}$ are displayed. Also, the value $L_n = B_1 - A_1 = \frac{3}{17} \approx 0.17647$ is returned. Note that the additional values for L_1, L_2^*, x_1, x_2, f_1, and f_2 during step 7 were not computed because they were no longer needed in the algorithm.

Based on the resulting interval of interest $[A_1, B_1] = [\frac{12}{17}, \frac{15}{17}]$ that is returned from this algorithm, we can compute an approximate value for the maximum of $f(\lambda)$ to be 0.30027, using $\lambda \approx \frac{A_1+B_1}{2} = \frac{12+15}{17*2} = \frac{27}{34} = 0.7941176$ as the point where the maximum occurs, which was verified to be near the maximum value for $f(\lambda)$ when computed with Mathematica.

PROBLEM 1-c.

Solution.

In order to apply Newton's method to find a critical point of the given objective function (1), we must derive the first and second derviatives of $f(\lambda)$ with respect to λ. The original function and its derivatives are shown below (the simplified versions of these equations were derived with the help of Mathematica and verified manually).

$$f(\lambda) = \frac{0.5}{\sqrt{1+\lambda^2}} - \sqrt{1+\lambda^2}\left(1 - \frac{0.5}{1+\lambda^2}\right) + \lambda$$

$$f'(\lambda) = \frac{\lambda\left(-\lambda^2 - 2\right)}{\left(\lambda^2+1\right)^{3/2}} + 1$$

$$f''(\lambda) = \frac{\lambda^2 - 2}{\sqrt{\lambda^2+1}\left(\lambda^2+1\right)^2}$$

Using these equations, we can calculate λ_{i+1} as follows:

$$\lambda_{i+1} = \lambda_i - \frac{f'(\lambda_i)}{f''(\lambda_i)}$$

Iteratively applying Newton's method with $\lambda_1 = 0.6$ as the initial point and $\epsilon = 0.01$ as the termination criteria to converge towards the critical point yields the following results:

Step	λ_n	$f'(\lambda_n)$	$f''(\lambda_n)$
1	0.6	0.107199	-0.76032
2	0.740991	0.0203098	-0.485815
3	0.782797	0.00140041	-0.419968

The data in this table is formatted as follows:

1. λ_n - Value for λ at step n in the method.
2. $f'(\lambda_n)$ - Value of the first derivative at λ_n.
3. $f''(\lambda_n)$ - Value of the second derivative at λ_n.

By step 3 we can see that the value of the first derivative for the given value of $f(\lambda)$ is less than $\epsilon = 0.01$ (i.e. $f'(\lambda_3) = 0.00140036 < 0.01$), so we conclude that the maximum of $f(\lambda)$ occurs at $\lambda \approx \lambda_3 = 0.782797$, so the maximum is approximately $f(0.782797) = 0.300281$.

PROBLEM 1-d.

Solution.

In order to apply the Quasi-Newton method with a fixed step size of 0.001, we must utilize the following functions:

$$f(\lambda) = \frac{0.5}{\sqrt{1+\lambda^2}} - \sqrt{1+\lambda^2}\left(1 - \frac{0.5}{1+\lambda^2}\right) + \lambda \tag{2}$$

$$f^+(\lambda) = f(\lambda + \Delta\lambda), f^-(\lambda) = f(\lambda - \Delta\lambda)$$

$$f'(\lambda) = \frac{f^+(\lambda) - f^-(\lambda)}{2\Delta\lambda}$$

$$f''(\lambda) = \frac{f^+(\lambda) - 2f(\lambda) + f^-(\lambda)}{\Delta\lambda^2}$$

Using these equations, we can calculate λ_{i+1} as follows:

$$\lambda_{i+1} = \lambda_i - \frac{\Delta\lambda(f^+(\lambda_i) - f^-(\lambda_i))}{2(f^+(\lambda_i) - 2f(\lambda_i) + f^-(\lambda_i))}$$

Iteratively applying the Quasi-Newton method with $\lambda_1 = 0.6$ as the initial point and $\epsilon = 0.01$ as the termination criteria to converge towards the critical point yields the following results:

Step	λ_n	$f(\lambda_n)$	$f^+(\lambda_n)$	$f^-(\lambda_n)$	$f'(\lambda_n)$
1	0.6	0.291303	0.291409	0.291195	0.107199
2	0.740992	0.299837	0.299857	0.299816	0.0203097
3	0.782797	0.300281	0.300282	0.300279	0.0014006

The data in this table is formatted as follows:

1. λ_n - Value for λ at step n in the method.
2. $f(\lambda_n)$ - Value of the original function at λ_n.
3. $f^+(\lambda_n)$ - Value of the function $f^+(\lambda)$ at λ_n.
4. $f^-(\lambda_n)$ - Value of the function $f^-(\lambda)$ at λ_n.
5. $f'(\lambda_n)$ - Value of the second derivative at λ_n.

By step 3 we can see that the value of the approximated first derivative for $f(\lambda)$ is less than or equal to the $\epsilon = 0.01$ (i.e. $f'(\lambda_3) = 0.0014006 < 0.01$), so we conclude that the maximum of $f(\lambda)$ occurs at $\lambda \approx \lambda_3 = 0.782797$, so the maximum is approximately $f(0.782797) = 0.300281$.