4005-800 Algorithms

Homework 4

Christopher Wood

April 24, 2012

PROBLEM 1-a. *Prove that for any* $n \in \mathbb{N}$, $\lfloor (n+1)/2 \rfloor = \lceil n/2 \rceil$.

Solution.

Case 1: n is even ($2 \mid n$, or $n = 2m$ for some $m \in \mathbb{N}$)

$$\begin{aligned}
\left\lfloor \frac{n+1}{2} \right\rfloor &= \left\lfloor \frac{2m+1}{2} \right\rfloor \\
&= \left\lfloor \frac{2m}{2} + \frac{1}{2} \right\rfloor \\
&= m + \left\lfloor \frac{1}{2} \right\rfloor \\
&= m \\
&= \left\lceil \frac{2m}{2} \right\rceil \\
&= \left\lceil \frac{n}{2} \right\rceil
\end{aligned}$$

Case 2: n is odd ($2 \nmid n$, or $n = 2m+1$ for some $m \in \mathbb{N}$)

$$\begin{aligned}
\left\lfloor \frac{n+1}{2} \right\rfloor &= \left\lfloor \frac{(2m+1)+1}{2} \right\rfloor \\
&= \left\lfloor \frac{2(m+1)}{2} \right\rfloor \\
&= m+1 \\
&= m + \left\lceil \frac{1}{2} \right\rceil \\
&= \frac{2m}{2} + \left\lceil \frac{1}{2} \right\rceil \\
&= \left\lceil \frac{2m}{2} + \frac{1}{2} \right\rceil \\
&= \left\lceil \frac{2m+1}{2} \right\rceil \\
&= \left\lceil \frac{n}{2} \right\rceil
\end{aligned}$$

Thus, since a number $n \in \mathbb{N}$ can only be even or odd, we can conclude that for any $n \in \mathbb{N}$, $\left\lfloor \frac{n+1}{2} \right\rfloor = \left\lceil \frac{n}{2} \right\rceil$.

PROBLEM 1-b. *Prove that for any* $n \in \mathbb{N}$, $\lfloor n/2 \rfloor + 1 = \lceil (n+1)/2 \rceil$.

Solution. **Case 1:** n is even ($2 \mid n$, or $n = 2m$ for some $m \in \mathbb{N}$)

$$\begin{aligned}
\left\lfloor \frac{n}{2} \right\rfloor + 1 &= \left\lfloor \frac{2m}{2} \right\rfloor + 1 \\
&= \frac{2m}{2} + 1 \\
&= \frac{2m}{2} + \left\lceil \frac{1}{2} \right\rceil \\
&= \left\lceil \frac{2m}{2} \right\rceil + \left\lceil \frac{1}{2} \right\rceil \\
&= \left\lceil \frac{2m}{2} + \frac{1}{2} \right\rceil \\
&= \left\lceil \frac{2m+1}{2} \right\rceil \\
&= \left\lceil \frac{n+1}{2} \right\rceil
\end{aligned}$$

Case 2: n is odd ($2 \nmid n$, or $n = 2m+1$ for some $m \in \mathbb{N}$)

$$\begin{aligned}
\left\lfloor \frac{n}{2} \right\rfloor + 1 &= \left\lfloor \frac{2m+1}{2} \right\rfloor + 1 \\
&= \left\lfloor \frac{2m}{2} + \frac{1}{2} \right\rfloor + 1 \\
&= m + \left\lfloor \frac{1}{2} \right\rfloor + 1 \\
&= m + 1 \\
&= \frac{2(m+1)}{2} \\
&= \left\lceil \frac{2(m+1)}{2} \right\rceil \\
&= \left\lceil \frac{2m+2}{2} \right\rceil \\
&= \left\lceil \frac{(2m+1)+1}{2} \right\rceil \\
&= \left\lceil \frac{n+1}{2} \right\rceil
\end{aligned}$$

Thus, since a number $n \in \mathbb{N}$ can only be even or odd, we can conclude that for any $n \in \mathbb{N}$, $\left\lfloor \frac{n}{2} \right\rfloor + 1 = \left\lceil \frac{n+1}{2} \right\rceil$.

PROBLEM 1-c. *Let $D(n) = T(n+1) - T(n)$. Determine a recurrence for $D(n)$.*

Solution.

Let $D(n) = T(n+1) - T(n)$. If we let $n = 1$ be the base case for the recurrence as in $T(n)$, we obtain the following:

$$\begin{aligned}
D(1) &= T(2) - T(1)\\
&= T\left(\left\lceil\frac{2}{2}\right\rceil\right) + T\left(\left\lfloor\frac{2}{2}\right\rfloor\right) + 2 - 0\\
&= T(1) + T(1) + 2\\
&= 2
\end{aligned}$$

Thus, we can see that $D(1) = 2$. We now seek the general case for $D(n)$ by expanding the its representation using the definition for $T(n)$, as shown below.

$$\begin{aligned}
D(n) &= T(n+1) - T(n)\\
&= \left(T\left(\left\lceil\frac{n+1}{2}\right\rceil\right) + T\left(\left\lfloor\frac{n+1}{2}\right\rfloor\right) + (n+1)\right) - \left(T\left(\left\lceil\frac{n}{2}\right\rceil\right) + T\left(\left\lfloor\frac{n}{2}\right\rfloor\right) + n\right)\\
&= T\left(\left\lceil\frac{n+1}{2}\right\rceil\right) + T\left(\left\lfloor\frac{n+1}{2}\right\rfloor\right) - T\left(\left\lceil\frac{n}{2}\right\rceil\right) - T\left(\left\lfloor\frac{n}{2}\right\rfloor\right) + 1\\
&= T\left(\left\lceil\frac{n+1}{2}\right\rceil\right) + T\left(\left\lfloor\frac{n}{2}\right\rfloor\right) + 1\\
&= T\left(\left\lfloor\frac{n}{2}\right\rfloor + 1\right) + T\left(\left\lfloor\frac{n}{2}\right\rfloor\right) + 1
\end{aligned}$$

Now, by observing that this expression takes the same form as $D(n)$, we obtain the following:

$$\begin{aligned}
D(n) &= T\left(\left\lfloor\frac{n}{2}\right\rfloor + 1\right) + T\left(\left\lfloor\frac{n}{2}\right\rfloor\right) + 1\\
&= D(\left\lfloor\frac{n}{2}\right\rfloor) + 1
\end{aligned}$$

Now, putting these results together, we obtain the following recurrence for $D(n)$:

$$\begin{aligned}
D(1) &= 2\\
D(n) &= D(\left\lfloor\frac{n}{2}\right\rfloor) + 1
\end{aligned}$$

PROBLEM 1-d. *Prove using the strong form of induction that for any $n \in \mathbb{N}$, if $n \geq 1$ then $D(n) = \lfloor lg(n) \rfloor + 2$.*

Solution. **Base ($n = 1$)**

By the definition of $D(n)$, we know the following:

$$\begin{aligned}
D(1) &= 2\\
&= 0 + 2\\
&= \lg(1) + 2
\end{aligned}$$

$$= \left\lfloor \lg(1) \right\rfloor + 2$$

Induction $(n > 1)$
Assume that $D(k) = \left\lfloor \lg(k) \right\rfloor + 2$ for some $k \in \mathbb{N}$ such that $2 \leq k < n$. We now show that $D(n) = \left\lfloor \lg(n) \right\rfloor + 2$.

$$\begin{aligned}
D(n) &= D(\left\lfloor \frac{n}{2} \right\rfloor) + 1 \\
&= \Big(\left\lfloor \lg(\left\lfloor \frac{n}{2} \right\rfloor) \right\rfloor + 2\Big) + 1 \\
&= \Big(\left\lfloor \lg(\left\lfloor \frac{n}{2} \right\rfloor) \right\rfloor + 1\Big) + 2 \\
&= \Big(\left\lfloor \lg(\left\lfloor \frac{n}{2} \right\rfloor) \right\rfloor + \lg(2)\Big) + 2 \\
&= \Big(\left\lfloor \lg(\left\lfloor \frac{n}{2} \right\rfloor) + \lg(2) \right\rfloor\Big) + 2 \\
&= \Big(\left\lfloor \lg(2\left\lfloor \frac{n}{2} \right\rfloor) \right\rfloor\Big) + 2 \\
&= \Big(\left\lfloor \lg(n) \right\rfloor\Big) + 2 \\
&= \left\lfloor \lg(n) \right\rfloor + 2
\end{aligned}$$

Thus, $D(n) = \left\lfloor \lg(n) \right\rfloor + 2$, as desired.

PROBLEM 1-e. *Then prove that* $T(n) - T(1) = \sum_{k=1}^{n-1} D(k)$, *and show that an immediate consequence is that* $T(n) = \sum_{k=1}^{n-1} \lfloor lg(k) \rfloor + 2$.

Solution. By the definition of $D(n)$, we observe the following:

$$\begin{aligned}
\sum_{k=1}^{n-1} D(k) &= \sum_{k=1}^{n-1} \Big(T(k+1) - T(k)\Big) \\
&= \big(T(2) - T(1)\big) + \big(T(3) - T(2)\big) + \big(T(4) - T(3)\big) + ... + \big(T(n) - T(n-1)\big) \\
&= T(n) - T(1)
\end{aligned}$$

Therefore, since $\sum_{k=1}^{n-1} D(k)$ turns into a telescoping sum, we see that it collapses to $T(n) - T(1)$, and since $D(n) = \left\lfloor \lg(n) \right\rfloor + 2$, we also know the following:

$$\begin{aligned}
T(n) - T(1) &= \sum_{k=1}^{n-1} D(k) \\
&= \sum_{k=1}^{n-1} \big(\left\lfloor \lg(k) \right\rfloor + 2\big)
\end{aligned}$$

Thus, we see that $T(n) - T(1) = \sum_{k=1}^{n-1} \big(\lfloor \lg(k) \rfloor + 2\big)$, and since $T(1) = 0$, we conclude that $T(n) = \sum_{k=1}^{n-1} \big(\lfloor \lg(k) \rfloor + 2\big)$.

PROBLEM 1-f. *Now show that $T(n) = \sum_{k=1}^{n-1} \lfloor lg(k) \rfloor + 2$ implies that $T(n) = O(nlog(n))$.*

Solution. Using the fact that $T(n) = \sum_{k=1}^{n-1} \left(\Big\lfloor \lg(k) \Big\rfloor + 2\right)$. We now evaluate this summation as follows:

$$
\begin{aligned}
T(n) &= \sum_{k=1}^{n-1} \big(\lfloor \lg(n) \rfloor + 2\big) \\
&= \sum_{k=1}^{n-1} \lfloor \lg(k) \rfloor + \sum_{k=1}^{n-1} 2 \\
&< \sum_{k=1}^{n-1} \lfloor \lg(n) \rfloor + \sum_{k=1}^{n-1} 2 \\
&= (n-1)\lg(n) + 2(n-1) \\
&= n\lg(n) - \lg(n) + 2n - 1 \\
&= O(n\lg(n)) \\
&= O(n\log(n))
\end{aligned}
$$

PROBLEM Extra credit. *Prove that for any $n \in \mathbb{N}$, $\lfloor lg(2\lfloor n/2 \rfloor) \rfloor = \lfloor lg(n) \rfloor$.*

Solution.

Case 1: n is even ($2 \mid n$, or $n = 2m$ for some $m \in \mathbb{N}$)

$$
\begin{aligned}
\lfloor \lg(2\lfloor n/2 \rfloor) \rfloor &= \lfloor \lg(2\lfloor 2m/2 \rfloor) \rfloor \\
&= \lfloor \lg(2\lfloor m \rfloor) \rfloor \\
&= \lfloor \lg(2m) \rfloor \\
&= \lfloor \lg(n) \rfloor
\end{aligned}
$$

Case 2: n is odd ($2 \nmid n$, or $n = 2m+1$ for some $m \in \mathbb{N}$)

$$
\begin{aligned}
\lfloor \lg(2\lfloor n/2 \rfloor) \rfloor &= \lfloor \lg(2\lfloor (2m+1)/2 \rfloor) \rfloor \\
&= \lfloor \lg(2\lfloor (2m/2) + (1/2) \rfloor) \rfloor \\
&= \lfloor \lg(2\lfloor m + (1/2) \rfloor) \rfloor \\
&= \lfloor \lg(2(m + \lfloor 1/2 \rfloor) \rfloor \\
&= \lfloor \lg(2m + 2\lfloor 1/2 \rfloor) \rfloor
\end{aligned}
$$

$$
\begin{aligned}
&= \lfloor \lg(2m+1) \rfloor \\
&= \lfloor \lg(n) \rfloor
\end{aligned}
$$