4005-800 Algorithms

Homework 6

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PROBLEM 1 - 16.1-2.

Suppose that instead of always selecting the first activity to finish, we instead select the last activity to start that is compatible with all previously selected activities. Describe how this approach is a greedy algorithm, and prove that it yields an optimal solution.

Solution.

In the context of the activity selection problem, the notion of a greedy algorithm is one that chooses an activity at each iteration such that we are left with as much resources (time) for other activities as possible. Thus, by choosing the last activity a_k to start (assuming that all activities are sorted in monotonically decreasing order by start time), we are guaranteed to leave as much time for other compatible activities as possible that be selected before the start of a_k, so this is a greedy approach. This is because choosing an activity with an earler start time would consume more resources, and therefore is not greedy.

We can also show that this approach yields an optimal solution to the original problem. Consider, for example, any non-empty subsequence of activities $S' =< a_1, a_2, ..., a_n >$, which is in monotonically decreasing order based on start times. We now prove that the largest set of compatible activities for S' is 1+(the largest set of compatible activities in $< a_2, a_3, ..., a_n >$). In other words, a_1 is included in some maximum-size subsequence of mutually compatible activities of S'.

Proof. Let A be the largest set of mutually compatble activities in S', and let a_j be the activitiy with the latest start time. If $a_j = a_1$, then the result follows immediately. If $a_j \neq a_1$, then consider $A' = (A - \{a_j\}) \cup \{a_1\}$. Note that A' is a mutually compatible set of activities since $s_1 \geq s_j$. That is, there is more time before a_1, and by picking a_j there is nothing after (since a_j has the latest start time). Now, since $|A'| = |A|$, the result follows immediately. ∎

PROBLEM 2 - 26.1-4.

*Let f be a flow in a network, and let α be a real number. The **scalar flow product**, denoted αf, is a function $V \times V \to \mathbb{R}$ defined by*

$$(\alpha f)(u, v) = \alpha \times f(u, v)$$

*Prove that the flows in a network form a **convex set**. That is, show that if f_1 and f_2 are flows, then so is $\alpha f_1 + (1 - \alpha) f_2$ for all α in the range $0 \leq \alpha \leq 1$.*

Solution.

To show that flows in a network form a **convex set**, we consider the expresion of scalar flow products $\alpha f_1 + (1-\alpha)f_2$, assuming that f_1 and f_2 are valid flows in the same network. Manipulating this expression yields the following:

$$\begin{aligned}
\alpha f_1 + (1-\alpha)f_2 &= \alpha f_1 - \alpha f_2 + f_2 && (1)\\
&= \alpha(f_1 - f_2) + f_2 && (2)
\end{aligned}$$

Now, we consider this expression on a case-by-case basis for α.

Case 1: $\alpha = 0$
When $\alpha = 0$, the expression $\alpha(f_1 - f_2) + f_2$ reduces to f_2, which is known to be a valid flow.

Case 2: $\alpha = 1$
When $\alpha = 1$, the expression $\alpha(f_1 - f_2) + f_2$ reduces to f_1, which is known to be a valid flow.

Case 3: $0 < \alpha < 1$
Since f_1 and f_2 are flows, we know they both satisfy the capacity constraint and flow conservation rules. Thus, we know that for all $u, v \in V$, $f_1(u,v) \leq c(u,v)$ and $f_2(u,v) \leq c(u,v)$. Using this relationship with the original convex criteria described above yields the following for all $u, v \in V$:

$$\begin{aligned}
\alpha f_1(u,v) + (1-\alpha)f_2(u,v) &\leq \alpha c(u,v) + (1-\alpha)c(u,v)\\
&= \alpha c(u,v) + c(u,v) - \alpha c(u,v)\\
&= c(u,v)
\end{aligned}$$

Thus, we see that convex criteria expression satisfies the capacity constraint. We now consider this expression in terms of the flow conservation. Since f_1 and f_2 are flows, we know that for each pair of vertices $u, v \in V$, $f_1(u,v) = -f_1(v,u)$ and $f_2(u,v) = -f_2(v,u)$ by skew symmetry. Further, by flow conservation, we know that for all vertices $u \in V - \{s,t\}$ we have $\sum_{v \in V} f_1(u,v) = 0$ and $\sum_{v \in V} f_2(u,v) = 0$. Now, using this fact, we examine the convex critera expression described above for all vertices $u \in V - \{s,t\}$ as follows:

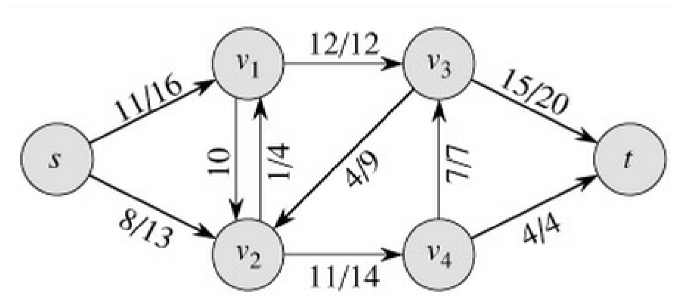
$$\begin{aligned}
\sum_{v \in V} \alpha f_1(u,v) + (1-\alpha)f_2(u,v) &= \sum_{v \in V} \alpha f_1(u,v) + \sum_{v \in V} f_2(u,v) - \sum_{v \in V} \alpha f_2(u,v)\\
&= \sum_{v \in V} \alpha f_1(u,v) - \sum_{v \in V} \alpha f_2(u,v)\\
&= \alpha \sum_{v \in V} f_1(u,v) - \alpha \sum_{v \in V} f_2(u,v)\\
&= 0
\end{aligned}$$

Thus, since the convex critera expression $\alpha f_1(u,v) + (1-\alpha)f_2(u,v)$ satisfies both the capacity constraint and the flow constraint in this case, we conclude that it is also a flow.

PROBLEM 3 - 26.2-2.

In Figure 26.1(b), *what is the flow across the cut* $(\{s, v_2, v_4\}, \{v_1, v_3, t\})$*? What is the capacity of this cut?*

Solution. The graph in Figure 26.1(b) is shown below.



By considering the cut $(\{s, v_2, v_4\}, \{v_1, v_3, t\})$, we know the edges that cross the cut boundary are $\{(u, v) : u \in S, v \in T, (u, v) \in E\} = \{(s, v_1), (v_2, v_1), (v_3, v_2), (v_4, v_3), (v_4, t)\}$. The net flow of this $s - t$-cut is then as follows:

$$\begin{aligned} f(S, T) &= f(s, v_1) + f(v_2, v_1) + f(v_3, v_2) + f(v_4, v_3) + f(v_4, t) \\ &= 11 + 1 + (-4) + 7 + 4 \\ &= 19 \end{aligned}$$

Since the capacity of the cut is defined as the sum of the edge capacities from all vertices $u \in S$ to $v \in T$, we have the following capacity:

$$\begin{aligned} c(S, T) &= c(s, v_1) + c(v_2, v_1) + c(v_4, v_3) + c(v_4, t) \\ &= 16 + 4 + 7 + 4 \\ &= 31 \end{aligned}$$