4005-800 ALGORITHMS

HOMEWORK 2

Christopher Wood

March 30, 2012

PROBLEM 1.

Solution.

The time complexity of the recurrence F_n can be defined as follows.

$$
\begin{aligned}
T_F(0) &= 1 \\
T_F(1) &= 1 \\
T_F(n) &= T_F(n-1) + T_F(n-2)
\end{aligned}
$$

The solution to $T_F(n)$ can be solved using the method of homogeneous equations, which yields the result that $T_F(n) = \Theta(\phi^n)$, where $\phi = \frac{1+\sqrt{5}}{2}$ (the golden ratio).

PROBLEM 2.

Solution.

The time complexity of the $fibIt$ routine can be found by solving the recurrence relation that defines $fibIt$. More specifically, we can such a recurrence relation for $fibIt$ by analyzing the number of additions performed, which corresponds to the following equation.

$$
\begin{aligned}
T_f(0) &= 0 \\
T_f(1) &= 0 \\
T_f(n) &= T_f(n-1) + 1
\end{aligned}
$$

This is because there is only one addition made in each recursive call from $f(n; a, b)$ to $f(n-1; b, a+b)$, and there are no additions made in the two cases where $n = 0$ and $n = 1$.

In order to solve this recurrence relation we can expand out the expression and attempt to identify the pattern. This process is shown below.

$$\begin{aligned}
T_f(n) &= T_f(n-1) + 1 \\
&= (T_f(n-2) + 1) + 1 = T_f(n-2) + 2 \\
&= (T_f(n-3) + 1) + 2 = T_f(n-3) + 3 \\
&= \dots \\
&= (T_f(n-k) + 1) + k = T_f(n-k) + k
\end{aligned}$$

Based on this pattern, we can reach the first base case of this recurrence relation $(T_f(1))$ when $(n-k) = 1$, meaning that $k = (n-1)$. Thus, we have the following.

$$\begin{aligned}
T_f(n) &= T_f(n - (n-1)) + (n-1) \\
&= T_f(1) + (n-1) \\
&= 0 + (n-1) \\
&= n - 1
\end{aligned}$$

Based on this observation we can see that $T_f(n) \in \Theta(n)$, or simply $T_f(n) = \Theta(n)$.

PROBLEM 3.

Solution.

   Base $(n = 0)$

When $n = 0$, we have the following equality.

$$\begin{aligned}
L_0(a, b) &= (f(0; a, b), f(1; a, b)) \\
&= (a, b)
\end{aligned}$$

   Induction $(n > 0)$

First, we assume that $L^n(a, b) = (f(n; a, b), f(n+1; a, b))$. Now we show that $L^{n+1}(a, b) = (f(n+1; a, b), f(n+2; a, b))$.

$$\begin{aligned}
L^{n+1}(a, b) &= L(L^n(a, b)) \text{ (by multiplication powers)} \\
&= L(f(n; a, b), f(n+1; a, b)) \text{ (by induction)} \\
&= (f(n+1; a, b), f(n; a, b) + f(n+1; a, b)) \text{ (by definition of } L\text{)} \\
&= (f(n+1; a, b), f(n+2; a, b)) \text{ (by Theorem 1)}
\end{aligned}$$

Thus, $L^{n+1}(a, b) = (f(n+1; a, b), f(n+2; a, b))$, as desired. Therefore, we know that $f(n; a, b) = (L^n(a, b))_1$.

PROBLEM 4.

Solution. TODO

PROBLEM 5.

Solution. TODO