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Narrowing the Edge Folkman Number Bounds

Independent Study Proposal

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1 Background

Edge Folkman numbers, first introduced by Folkman in 1970 [1], are concerned with the study of graphs in which a monochromatic coloring of a particular subgraph always exists. We write $G \rightarrow (a_1, ..., a_k; p)^e$ if for ever edge coloring of an undirected simple graph G not containing K_p, there exists a monochromatic K_{a_i} in color i for some $i \in \{1, ..., k\}$. The edge Folkman number is defined as $F_e(a_1, ..., a_k) = \min\{|V(G)| : G \rightarrow (a_1, ..., a_k; p)^e\}$. In 1970 Folkman proved that for all $k > \max(s, t)$, edge- and vertex- Folkman numbers $F_e(s, t; k$ and $F_v(s, t; k)$ exist. Prior to this, Erdos and Hajnal pose the problem of finding $F_e(3, 3; 4)$, which can be informally stated as the following question [2]:

What is the order of the smallest K_4-free graph for which any 2-coloring of its edges must contain at least one monochromatic triangle?

This is equivalent to finding the smallest K_4-free graph that is not the union of two triangle-free graphs. Since the proposition of this problem, there has been a significant amount of work towards aimed at narrowing the upper and lower bounds of $F_e(3, 3; 4)$. Table 1 enumerates the work on this problem and leads us to the current state of the field.

Table 1: History of $F_e(e, e; 4)$

Year	Bounds	Who	Ref.
1967	any?	Erdős-Hajnal	[2]
1970	exist	Folkman	[1]
1972	≥ 10	Lin	[3]
1975	$\leq 10^{10}$?	Erdős offers $100 for proof	
1986	$\leq 8 \times 10^{11}$	Frankl-Rődl	[5]
1988	$\leq 3 \times 10^{9}$	Spencer	[6]
1999	≥ 16	Piwakowski et al (implicit)	[7]
2007	≥ 19	Radziszowski-Xu	[8]
2008	≤ 9697	Lu	[9]
2008	≤ 941	Dudek-Rődel	[10]
2012	≤ 786	Lange et al	[14]
2012	≤ 100?	Garaham offers $100 for proof	

2 Problems

2.1 Lower Bound

The current lower bound for $F_e(3,3;4)$ stands at 19 [8]. The proof used for this bound, which relies on the existence of independent sets of size k in a graph G on 18 vertices, where $k = 4$, cannot easily be extended to graphs on 19 vertices. It is estimated that more than 10^{19} nonisomorphic, K_4-free graphs on 19 vertices exist [4].

The proof for this bound relies on the fact that $H \to (3,3;4)^v \Rightarrow H+x \to (3,3;5)^e$ and $F_e(3,3;5) = 15$. More specifically, using all nonisomorphic graphs on 18 vertices with contained independent sets of size $k = 4$, the authors were show that, for each candidate graph G, the graph

2.2 Upper Bound

It was proven by Dudek and Rodel that $G \to (3,3)^e$ if and only if $MC(H_G) < 2t_\Delta(G)$ [10], where $MC(H_G)$ is the size of the maximum cut of graph $H = (E(G), \{(e_1, e_2) | \{e_1, e_2, e_3\}$ form a triangle in $G\}$. Since MAX-CUT is an NP-complete problem, computations using this fact rely on approximation algorithms, such as the one proposed by Geomans and Williamson (TODO: CITE). In fact, Lange et al [14] used this approximation approach, which formulates the MAX-CUT problem as a semidefinite program (SDP). In their work, the authors examined graphs of the form (TODO: general form here) Using these computations, the authors were able to show that $MC(H_{G_{786}}) \leq 857750$, where $G_{786} = G(786,3) = (\mathbb{Z}_{786}, \{(x, y) | x \neq y, x - y \equiv \alpha^r \mod 786, \alpha \in \mathbb{Z}_{786}\}$, and since $2t_\Delta(G_{786}) = 857762$, the authors conclude that $F_e(3,3;4) \leq 786$.

It has been conjectured that $F_e(3,3;4) \leq 127$ by using a witnessing graph $G_{127} = (\mathbb{Z}_{127}, E = \{(x, y) | x - y = \alpha^3 \mod 127\})$. It was first conjectured that $G_{127} \to (3,3)^e$ in [8], where the intuition for this conjecture is that G_{127} has a large number of triangles and many small dense subgraphs. We will attempt to prove or disprove this conjecture by showing that $G_{127} \not\rightarrow (3,3)^e$ using subgraph H extensions. More specifically, we will identify subgraphs H witnessing $H \not\rightarrow (3,3)^e$ and then attempt to extend H to all of G.

Unfortunately, due to the computationally intensive nature of this procedure, we will reduce $\{H | H \not\rightarrow (3,3)^e\}$ to 3-SAT and leverage the power of SAT solvers such as Minisat [11] and zChaff [12]. The reduction works by mapping the edges in $E(H)$ to variables in $\phi_G \in$ 3-SAT, and for edge adding the following clauses to ϕ_G:

$$(x + y + z) \wedge (\bar{x} + \bar{y} + \bar{z})$$

Clearly, $H \not\rightarrow (3,3)^e \Leftrightarrow \phi_G$ is satisfiable.

3 Summary of Proposed Study

The work for this project will be two-fold.

1. addressing lower bound 2. addressing upper bound

A major part of this work will be identifying candidate subgraphs H for extension that can easily be solved using the aforementioned 3-SAT solvers.

4 Project Goals and Deliverables

The following project goals have been identified for this work

- Research current methods and potential techniques for computing the upper and lower bounds of Folkman numbers (in particular, $F_e(3,3;4)$).
- Implement software to aid in the computation of Folkman number bounds.
- Begin a survey paper outlining the history of Folkman number research [13].

By achieving these goals I will generate the following deliverables:

- Publication-ready paper and more in-depth progress report for the entire project
- Report that weighs the performance of various SAT solvers for solving this arrowing problem with G127
- Report discussing the subgraph pruning techniques used for the lower bound

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