DEPARTMENT OF COMPUTER SCIENCE, ROCHESTER INSTITUTE OF TECHNOLOGY

Study of the Edge Folkman Number Bounds

Independent Study Proposal

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1 Background

Edge Folkman numbers, first introduced by Folkman in 1970 [1], are concerned with the graphs in which a monochromatic copy of a particular subgraph exists for all edge colorings. We write $G \rightarrow (a_1,...,a_k;p)^e$ iff for every edge coloring of an undirected simple graph G not containing K_p, there exists a monochromatic K_{a_i} in color i for some $i \in \{1,...,k\}$. The edge Folkman number is defined as $F_e(a_1,...,a_k) = \min\{|V(G)| : G \rightarrow (a_1,...,a_k;p)^e\}$. Similarly, the vertex Folkman number is defined as $F_v(a_1,...,a_k) = \min\{|V(G)| : G \rightarrow (a_1,...,a_k;p)^v\}$. In 1970 Folkman proved that for all $k > \max(s,t)$, edge- and vertex- Folkman numbers $F_e(s,t;k$ and $F_v(s,t;k)$ exist. Prior to this, Erdős and Hajnal posed the problem of finding $F_e(3,3;4)$, which can be equivalently stated as the following question [2]:

What is the order of the smallest K_4-free graph for which any 2-coloring of its edges must contain at least one monochromatic triangle?

This is equivalent to finding the smallest K_4-free graph that is not the union of two triangle-free graphs. Since the proposition of this problem, there has been a significant amount of work aimed at narrowing the upper and lower bounds of $F_e(3,3;4)$. Table 1 enumerates main developments of the previous work on this problem and leads us to the current state of the field.

Table 1: History of $F_e(e,e;4)$

Year	Bounds	Who	Ref.
1967	any?	Erdős-Hajnal	[2]
1970	exist	Folkman	[1]
1972	≥ 10	Lin	[3]
1975	$\leq 10^{10}$?	Erdős offers \$100 for proof	
1986	$\leq 8\times 10^{11}$	Frankl-Rödl	[5]
1988	$\leq 3\times 10^{9}$	Spencer	[6]
1999	≥ 16	Piwakowski et al. (implicit)	[7]
2007	≥ 19	Radziszowski-Xu	[8]
2008	≤ 9697	Lu	[9]
2008	≤ 941	Dudek-Rödl	[10]
2012	≤ 786	Lange et al.	[16]
2012	≤ 100?	Garaham offers \$100 for proof	

2 Two Problems

2.1 Lower Bound

The current lower bound for $F_e(3,3;4)$ stands at 19 [8]. The proof technique for this bound, which relies on the fact that $H \to (3,3;4)^v \Rightarrow H + x \to (3,3;5)^e$, constructed all nonisomorphic graphs G on 18 vertices that contained independent sets I of cardinality 4. Then, for each graph G, the authors used the fact that all induced subgraphs $H \in \mathscr{F}_v(3,3;4)$, where H is composed of the vertex set $V(G) \setminus I$. Using computations, the authors then constructed all possible candidate graphs G from these subgraphs H by taking the union of the vertices in I with every vertex set $A \subset M$, where M is the set of all subsets $A \subset V(H)$ such that the graph induced by A is a maximal triangle-free subgraph, and testing to see whether the resultant graph G arrows $(3,3)^e$. The fact that $\chi(G) \geq 6$ if $G \in \mathscr{F}_e(3,3;4)$ was used to further restrict the candidate graphs G that were checked.

2.2 Upper Bound

Dudek and Rödl proved that $G \to (3,3)^e$ if and only if $MC(H_G) < 2t_\Delta(G)$ [10], where $MC(H_G)$ is the size of the maximum cut of graph $H = (E(G), \{(e_1,e_2)|\{e_1,e_2,e_3\}$ form a triangle in $G\}$. Since MAX-CUT is an NP-complete problem, computations using this fact rely on approximation algorithms, such as the one proposed by Geomans and Williamson [14]. In fact, Lange et al. [16] used this approximation approach, which formulates the MAX-CUT problem as a semidefinite program (SDP). In their work, the authors examined graphs of the form $G(n,r) = (\mathbb{Z}_n, \{(x,y)|x \neq y, x-y \equiv \alpha^r \mod n, \alpha \in \mathbb{Z}_n\}$. Using these computations, the authors were able to show that $MC(H_{G_{786}}) \leq 857750$, where $G_{786} = G(786,3)$, and since $2t_\Delta(G_{786}) = 857762$, it is clear that $F_e(3,3;4) \leq 786$.

It has been conjectured that $F_e(3,3;4) \leq 127$, which is motivated by the graph $G_{127} = (\mathbb{Z}_{127}, E = \{(x,y) | x - y = \alpha^3 \mod 127\})$ [8]. The intuition for this conjecture is that G_{127} has a large number of triangles and many small dense subgraphs. Proving or disproving the conjecture that $G_{127} \rightarrow (3,3)^e$ would be a significant result for the upper bound of $F_e(3,3;4)$.

3 Summary of Proposed Computational Work

A major thread of this project is to attack both the upper and lower bounds of $F_e(3,3;4)$. Unfortunately, it will not be feasible to leverage the same technique used by Radziszowski et al. to prove the current bound of 19. The reason for this is that the estimated number of independent sets of size 5 in nonisomorphic graphs on 19 vertices is expected to be more than 10^{19} [4]. Clearly, the number of candidate graphs G to check for $G \rightarrow (3,3)^e$ needs to be smaller, so we will work towards devising constraints similar to the chromatic number $\chi(G) \geq 6$. In doing so, we will leverage *nauty* [15] to generate all nonisomorphic graphs based on these constraints. We will also investigate other facts about graph structure imposed by membership in the set $\mathscr{F}_e(3,3;4)$, which will hopefully enable us to more efficiently enumerate all graphs on 19 vertices to check for membership in this set.

Attacking the upper bound will be an entirely different form of computation. In particular, we will need to find a graph G on n vertices where $G \rightarrow (3,3)^e$. As previously discussed, we will be focusing on the graph $G_{127} = G(127,3)$ in an attempt to prove the conjecture that $G_{127} \rightarrow (3,3)^e$. To determine this we will decompose the problem of arrowing into problems on subgraphs H that witness $H \not\rightarrow (3,3)^e$, and then carefully extend H to encompass all of G.

Due to the computationally intensive nature of this procedure, we will reduce $\{H | H \not\rightarrow (3,3)^e\}$ to 3-SAT and leverage the power of SAT solvers such as Minisat [11] and zChaff [12]. The reduction works by mapping the edges in $E(H)$ to variables in $\phi_G \in$ 3-SAT, and for edge adding the following clauses to ϕ_G:

$$(x + y + z) \wedge (\bar{x} + \bar{y} + \bar{z})$$

Clearly, $H \not\rightarrow (3,3)^e \Leftrightarrow \phi_G$ is satisfiable. Therefore, if ϕ_G is not satisfiable, then $G \rightarrow (3,3)^e$, and so $F_e(3,3;4)$.

A major part of this task will be identifying candidate subgraphs H for extension that can easily be solved using the aforementioned 3-SAT solvers. We will also take this opportunity to compare the performance of popular k-SAT solvers published in the literature [12] [11]. It may also be interesting to experiment with NAE-SAT wrappers for open-source 3-SAT solvers, in which an NAE-SAT (not-all-equivalent SAT) 3-CNF formula is satisfiable if and only if there is both a true and false literal in each clause. Such a wrapper may enable existing SAT solvers to decide if the $\phi_{G_{127}}$ formula is satisfiable or not by placing an additional constraint on the size of the state space (i.e. the backtracking tree).

4 Project Goals and Deliverables

The following project goals have been identified for this work:

- Research current methods and potential techniques for computing the upper and lower bounds of Folkman numbers (in particular, $F_e(3,3;4)$).
- Experiment with SAT solvers in an attempt to solve the $G_{127} \rightarrow (3,3)^e$ conjecture.
- Implement software to aid in the computation of Folkman number bounds.
- Begin a survey paper outlining the history of Folkman number research [13].

By achieving these goals I will generate the following deliverables:

- An online weekly log of the project progress.
- Publication-ready paper and more in-depth progress report for the entire project.
- Report that weighs the performance of various SAT solvers for solving this arrowing problem with G_{127}.
- Report discussing the subgraph pruning techniques used for the lower bound.

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