# On Folkman Numbers

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**Abstract**

This survey contains a comprehensive overview of the results relating to Folkman numbers, a topic in general Ramsey Theory. We present data which, to the best of our knowledge, includes all known nontrivial values and bounds for definite Folkman numbers of arbitary forms. In particular, we present results, with complete citations, for general edge and vertex Folkman numbers of the form $F(a_1, \ldots, a_r; q) = \min\{|V(G)| : G \to (a_1, \ldots, a_r; q) and \omega(G) < q\}$, where the edge and vertex Folkman numbers consist of edge and vertex colorings satisfying this property.

# Contents

# Chapter 1

# Introduction

## 1.1 Scope and Notation

Folkman numbers, first introduced by Folkman in 1970 [2], are a branch of Ramsey theory concerned with the graphs in which a monochromatic copy of a particular subgraph exists for all (edge or vertex) colorings. We write $G \to (a_1, ..., a_k; p)^e$ iff for every edge coloring of an undirected simple graph $G$ not containing $K_q$, there exists a monochromatic $K_{a_i}$ in color $i$ for some $i \in \{1, ..., k\}$. The edge Folkman number is defined as $F_e(a_1, ..., a_k) = \min\{|V(G)| : G \to (a_1, ..., a_k; q)^e\}$. Similarly, the vertex Folkman number is defined as $F_v(a_1, ..., a_k) = \min\{|V(G)| : G \to (a_1, ..., a_k; q)^v\}$. In 1970 Folkman proved that for all $k > \max(s, t)$, edge- and vertex-Folkman numbers $F_e(s, t; k$ and $F_v(s, t; k)$ exist. The sets $\mathcal{F}_v(s, t; q) = \{G : G \to (s, t)^v \wedge \omega(G) < q\}$ and $\mathcal{F}_e(s, t; q) = \{G : G \to (s, t)^e \wedge \omega(G) < q\}$ are the *vertex* and *edge Folkman graphs*, respectively.

In this survey we only consider finite, simple graphs (i.e. those without loops and multiple edges). We denote the vertex and edge set of a graph $G$ using $V(G)$ and $E(G)$, respectively. $N(v)$ is the open neighborhood of vertex $v \in V(G)$, consisting of all vertices adjacent to $v$. $G[V], V \subset V(G)$ denotes the subgraph induced by the vertices $V$. Also, $\alpha(G)$ and $\omega(G)$ denote the cardinality of a maximum independent set of $G$ (i.e. a set for which all pairwise vertices are not adjacent in $G$) and the cardinality of the largest clique in $G$ (i.e. a set for which all pairwise vertices are adjacent in $G$). The chromatic number of $G$, which is the minimal number of colors required to a vertex two-coloring of $G$, is denoted as $\chi(G)$. Finally, we denote the complement of a graph $G$ as $\bar{G}$.

We denote $G - v = G[V(G) \setminus v]$. We also denote $G - e$ as the subgraph of $G$ such that $V(G - e) = V(G)$ and $E(G - e) = E(G) \setminus \{e\}$. Similarly, we denote $G + e$ as the supergraph of $G$ such that $V(G + e) = V(G)$ and $E(G + e) = E(G) \cup \{e\}$. Finally, we denote the union of two graphs $G_1$ and $G_2$ as $G_1 + G_2$, where $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{(u, v) : u \in V(G_1) \text{ and } v \in V(G_2)\}$.

Classical Ramsey numbers $R(s, t)$ are pivotal in the study of Folkman numbers. We define $R(s, t)$ as the smallest integer $n$ such that every 2-coloring of the edges of $K_n$ there exists a monochromatic $K_s$ in the first color or $K_t$ in the second color. In other words, $R(s, t) = \min\{n | K_n \to (s, t)^e\}$. We often generalize the arrowing property to graphs $G$ and $H$ such that $R(G, H) = \min\{n | K_n \to (G, H)^e\}$. Radziszowski [3] maintains a regularly updated survey containing the most recent results with known bounds and exact values for small Ramsey numbers.

We begin our exploration of finite Folkman numbers in Chapter 1.1 with a discussion of classical Folkman numbers - those edge and vertex numbers of the form $F(s, t; q)$. We then discuss multicolor Folkman numbers in Chapter 2.2.1.

# Chapter 2

# Classical Folkman Numbers

In this section we present results for edge and vertex Folkman numbers. In some cases there is an important connection between these two values. For example, Piwakowski et al. showed that $G \rightarrow (3,3;4)^v$ implies $G + x \rightarrow (3,3;5)^e$ for some additional vertex $x$ [4]. Such relations have been important in the computation of various Folkman numbers, as we will describe in the following sections.

## 2.1 Edge Folkman Numbers $F_e(s,t;q)$

It is clear that $G \rightarrow (s,t;q)^e$ implies that $\omega(G) \geq \max\{s,t\}$ [2], which means that $F_e(s,t;q)$ exists if and only if $q > \max\{s,t\}$. It is well known that as $q$ decreases, the difficulty in computing $F_e(s,t;q)$ increases. For this reason, exact values for $F_e(s,t;4)$ and $F_e(s,t;5)$ have been among the most well-studied problems. We present all known values of these two numbers in Tables 2.2 and 2.2.

Table 2.1: Values and Bounds for Edge Folkman numbers of the type $F_e(s,t;5)$.

| $(s,t)$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | | | | | | | | | | | | |
| 3 | | 15 [4] | | $\leq 21$ [5] | | | | | | | | |
| 4 | | | | | | | | | | | | |
| 5 | | | | | | | | | | | | |
| 6 | | | | | | | | | | | | |
| 7 | | | | | | | | | | | | |
| 8 | | | | | | | | | | | | |

Table 2.2: Values and Bounds for Edge Folkman numbers of the type $F_e(s,t;4)$.

| $(s,t)$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | | | | | | | | | | | | |
| 3 | | $\leq 786$ [6] | | | | | | | | | | |
| 4 | | | | | | | | | | | | |
| 5 | | | | | | | | | | | | |
| 6 | | | | | | | | | | | | |
| 7 | | | | | | | | | | | | |
| 8 | | | | | | | | | | | | |

Table 2.3: Folkman number bounds for small values of $k$. Note that $R(3,3) = 6$.

| $k$ | $F_e(3,3;k)$ | graphs | reference |
|---|---|---|---|
| $\leq 7$ | 6 | $K_6$ | folklore |
| 6 | 8 | $C_5 + K_3$ | [28] |
| 5 | 15 | 659 graphs | [4] |
| 4 | $\leq 786$ | $G_{786}$ | [6] |

### 2.1.1 $K_n$-Free Graphs

It is a well-known result that if $k > R(s,t) \to F_e(s,t;k) = R(s,t)$. Conversely, when $k \leq R(s,t)$, very little is known about the bounds of $F_e(s,t;k)$. Table 2.3 shows that as the value of $k$ decreases, the problem of finding $F_e(s,t;k)$ becomes increasingly difficult.

#### $F_e(3,3;4)$ **History**

The number $F_e(3,3;4)$ has a particular intriguing history, starting with the question first posed by Erdős in [1]:

*What is the order of the smallest $K_4$-free graph for which any 2-coloring of its edges must contain at least one monochromatic triangle?*

This question is equivalent to finding the smallest $K_4$-free graph that is not the union of two triangle-free graphs. Folkman first proved the existence of such graphs in 1970, showing that for all $k > \max\{s,t\}$, $F_e(s,t;k)$ must exist [2].

Erdős followed this result by offering a \$100 prize (equivalent to 300 Swiss francs at the time) for determining if $F_e(3,3;4) < 10^{10}$. In 1988, Spencer [9] presented the first proof of graphs below this bound using a probabilistic approach, which was shown to contain an error by Hovey in 1989, who adjusted the bound to $F_e(3,3;4) < 3 \times 10^9$. More than twenty years passed before Dudek and Rödl [12] proposed a construction technique for Folkman graphs based on the maximum cut of a related graph that was used to improve the bound to $F_e(3,3;4) \leq 941$. Lange et al. [6] further improved this bound to $F_e(3,3;4) \leq 786$ with the same technique using the Geomans-Williamson MAX-CUT approximation algorithm. The conjecture that $F_e(3,3;4) \leq 127$ proposed by Radziszowski et al. [10] is still an open problem, and is motivated by the intuition that $G_{127} = (\mathbb{Z}_{127}, \{(x,y)|x - y \equiv \alpha^3 (\mod 127)\})$ contains many triangles and small dense subgraphs. Following in the footsteps in Erdős, Graham offered a \$100 prize for proving that $F_e(3,3;4) \leq 100$.

**Open Problem:** $F_e(3,3;4) \leq 100$?

The lower bound of $F_e(3,3;4) \geq 10$ was first proved by Lin in 1972 [33]. Piwakowski et al. [4] improved this bound to $F_e(3,3;4) \geq 16$ by enumerating all graphs in $\mathcal{F}_e(3,3;5)$ and proving that all of them contain $K_4$'s. In 2007, Radziszowski et al. [10] pushed this bound to $F_e(3,3;4) \geq 19$ with a construction technique that relied on the fact that $G \to (3,3;4)^v$ implies $G+x \to (3,3;5)^e$ and $F_e(3,3;5) = 15$. Table 2.5 enumerates these developments leads us to the current state of the field.

#### $F_e(3,3;5)$ **History**

This number also has a dated history, starting with the existential question posed by Erdős et al. in 1967 [1]. However, the first proof of the existence of this number predates Erdős in an unpublished manuscript by Pòsa. A viable upper bound was first proven by Schäuble [] in 1969, shortly after the proposition by Erdős et al., in which it was shown that $F_e(3,3;5) \leq 42$. Graham and Spencer [29] improved this bound to $F_e(3,3;5) \leq 23$ in 1971, and further conjectured that $F_e(3,3;5) = 23$ without supporting reasoning. This bound was subsequently reduced by Irving

Table 2.4: History of $F_e(3,3;4)$

| Year | Bounds | Who | Ref. |
|---|---|---|---|
| 1967 | any? | Erdős-Hajnal | [1] |
| 1970 | exist | Folkman | [2] |
| 1972 | $\geq 10$ | Lin | [33] |
| 1975 | $\leq 10^{10}$? | Erdős offers \$100 for proof | |
| 1986 | $\leq 8 \times 10^{11}$ | Frankl-Rödl | [8] |
| 1988 | $\leq 3 \times 10^{9}$ | Spencer | [9] |
| 1999 | $\geq 16$ | Piwakowski et al. (implicit) | [4] |
| 2007 | $\geq 19$, $\leq 127$? | Radziszowski-Xu | [10] |
| 2008 | $\leq 9697$ | Lu | [11] |
| 2008 | $\leq 941$ | Dudek-Rödl | [12] |
| 2012 | $\leq 786$ | Lange et al. | [6] |
| 2012 | $\leq 100$? | Garaham offers \$100 for proof | |

Table 2.5: History of $F_e(3,3;5)$

| Year | Bounds | Who | Ref. |
|---|---|---|---|
| 1967 | any? | Erdős-Hajnal | [1] |

[30] in 1973. At this point, work on the upper bound diverged. Hadziivanov and Nenov [13] were able to construct a 16-vertex graph in $\mathcal{F}_e(3,3;5)$, and Nenov [17] further improved this bound with a 15-vertex graph in $\mathcal{F}_e(3,3;5)$, thus proving that $F_e(3,3;5) \leq 15$. Hadziivanov and Nenov [14] found another such graph on 15 vertices in 1984. Years later, Erickson [24] found a 17-vertex graph in $\mathcal{F}_e(3,3;5)$ and subsequently conjecture that $F_e(3,3;5) = 17$. However, Bukor [31] disproved this conjecture by showing the same 16-vertex construction presented in [13]. In 1996, the upper bound of 15 was verified once more by Urbański [32] with a different construction of the 15-vertex graph in [17]. The lower bound of $F_e(3,3;5)$ has much less history. In 1972 Lin [33] proved that $F_e(3,3;5) \geq 10$, which was subsequently improved by Nenov [16] to $F_e(3,3;5) \geq 11$, and then by Hadziivanov and Nenov [15] to $F_e(3,3;5) \geq 12$. The final value of $F_e(3,3;5) = 15$, proven by Piwakowski et al. [4] in 1999, was shown by constructing all 659 15-vertex graphs in $\mathcal{F}_e(3,3;5;15)$, where each such graph has a $K_4$.

TODO: fill in the table with the paragraph contents

### 2.1.2 Special Cases

- $F_e(3,4;\leq 10) = 9$ ($K_9$ since $R(3,4) = 9$ [3])
- $F_e(3,4;9) = 14$ [35]
- $F_e(3,4;8) = 16$ [36]
- $F_e(3,4;7) =?$
- $F_e(3,5;14) = 16$ []
- $F_e(4,4;18) = 20$ []
- $F_e(3,7;22) \geq 27$ []

**Open Problem:** $F_e(3,4;7) =?$

## 2.2 Vertex Folkman Numbers $F_v(s,t;q)$

TODO: header

Table 2.6: Values and Bounds for Vertex Folkman numbers of the type $F_v(s,t;5)$.

| $(s,t)$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | | | | | | | | | | | | |
| 3 | | | | | | | | | | | | |
| 4 | | | | | | | | | | | | |
| 5 | | | | | | | | | | | | |
| 6 | | | | | | | | | | | | |
| 7 | | | | | | | | | | | | |
| 8 | | | | | | | | | | | | |

Table 2.7: Values and Bounds for Vertex Folkman numbers of the type $F_v(s,t;4)$.

| $(s,t)$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | | | | | | | | | | | | |
| 3 | | 14 [4] | | | | | | | | | | |
| 4 | | | | | | | | | | | | |
| 5 | | | | | | | | | | | | |
| 6 | | | | | | | | | | | | |
| 7 | | | | | | | | | | | | |
| 8 | | | | | | | | | | | | |

### 2.2.1 Special Cases

TODO

# Chapter 3

# Multicolor Folkman Numbers

Multicolor Folkman numbers are unconstrained in the number of $r$ colorings used in their specification. Variations of the multicolor Folkman numbers vary either the number of colors $r$ or the colors $(a_i)$ themselves. Many interesting results have been derived for these types of problems, as we will show in the following sections.

For Folkman numbers $F(a_1, \ldots, a_n; q)$ we define $m = \sum_{i=1}^{k}(a_i - 1) + 1$ and $a = \max\{a_1, \ldots, a_k\}$. Many results make use of these values. A vertex minimal graph $G$ is one such that $G \in \mathcal{F}_v(a_1, \ldots, a_r; q)$ and $G - v \notin \mathcal{F}_v(a_1, \ldots, a_r; q)$ for all $v \in V(G)$.

## 3.1 Exact Values

TODO: header

### 3.1.1 Multicolor Edge Folkman Numbers

Nenov [22] showed that for ineteger colors $a_1, \ldots, a_r$, where and $a_i \geq$ for all $1, \ldots, r, rgeq2$, then $F_e(a_1, \ldots, a_4; R(a_1, \ldots, a_r) - 2) \leq R(a_1, \ldots, a_r) + 6$. Since $a_i \geq 3$ and $r \geq 2$, $R(a_1, \ldots, a_r) > 2 + \max\{a_1, \ldots, a_r\}$, which implies that such Folkman numbers exist. The exact value of $F_e(a_1, \ldots, a_4; R(a_1, \ldots, a_r) - 2) = R(a_1, \ldots, a_r) + 6$ what shown to be true if an only if $K_{R-7} + \bar{Q} \rightarrow (a_1, \ldots, a_r)^e$ or $K_{R-9} + C_5 + C_5 + C_5 \rightarrow (a_1, \ldots, a_r)^e$, where $R = R(a_1, \ldots, a_r)$. In this case, $\bar{Q}$ is the graph shown in Figure 3.1.

TODO: find citations for these results

- $F_e(3, 3, 3; 17) = 19$ []
- $F_e(3, 3, 3; 16) = 21$ []

### 3.1.2 Vertex Multicolor Folkman Numbers

Folkman [2] showed that $F_v(a_1, \ldots, a_k; q) = m$ for $q > m$ and $F_v(a_1, \ldots, a_k; q) = a + m$ for $q = m$. To date, few exact values for $F_v(a_1, \ldots, a_r; m - 1)$, are known. For this reason, we enumerate those known values below.

- $F_v(2, 3, 3; 5) = 12$ []
- $F_v(3, 3; 4) = 14$ []
- $F_v(2, 2, 3; 4) = 14$ [34]. A looser bound of $10 \leq F_v(2, 2, 3; 4) \leq 14$ was proved by Nenov in 2000 [19].
- $F_v(2, 2, 2; 3) = 11$ []
- $F_v(2, 2, 2, 2; 4) = 11$ [18]
- $F_v(2, 2, 2, 2; 3) = 22$ [23]

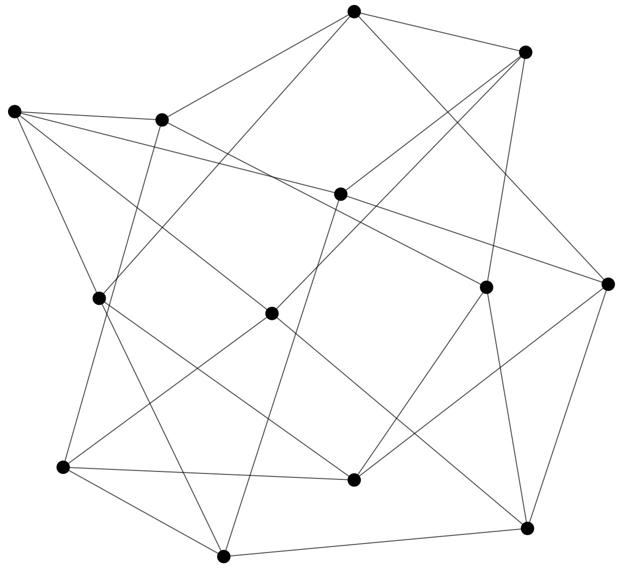

Figure 3.1: The Nenov graph from [22].

## 3.2 Generalized Results

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### 3.2.1 Vertex Folkman Numbers

In [21], Nenov proved a very interesting result for Folkman numbers of the form $F_v(3,\ldots,3) = \min\{|V(G)| : G \to \underbrace{(3,\ldots,3)}_{r} \text{ and } \omega(G) < 2r\}$, showing that $F_v(3,\ldots,3) = 2r+7, r \geq 3$. This was an improvement on the bounds of $2r+5 \leq F_v(3,\ldots,3) \leq 2r+10, r \geq 4$, proven by Luczak et al. [25]. A similar bound of $2r+6 \leq F_v(3,\ldots,3) \leq 2r+8, r \geq 3$, was proven by Nenov et al. in [19].

In 2001, Łuczak et al. [25] proved that $F_v(a_1,\ldots,a_r;q) \geq 2m-q+1$. Computing the vertex Folkman number becomes very interesting when $q = m$, and Łuczak went on to construct very large classes, and in certain cases, infinitely many graphs that satisfy this property. Let $k \in \{a_r, a_r+1, \ldots, m-1\}$, $n$ an integer such that $n > 2k$, $s = \gcd\{s,k\}$, and $t = k/s$. Using the graph $G = G(n,k) = C_n^{k-1} + K_{m-k-1}$, Łuczak et al. showed that $G \in \mathcal{F}_v(a_1,\ldots,a_r; q = m)$ if and only if $\sum_{i=1}^{r} \lfloor (a_i-1)/t \rfloor < s$. As a corollary, one can see that if $k$ and $n$ are relatively prime then $s = 1$ and $t = k$, so $\sum_{i=1}^{r} \lfloor (a_i-1)/t \rfloor = 0 < s = 1$, which means that $G \in \mathcal{F}_v(a_1,\ldots,a_r; q = m)$ [25]. This construction can produce infintely many graphs $G \in \mathcal{F}_v(a_1,\ldots,a_r;q)$, where $G = G(n,k) = C_n^{k-1} + K_{m-k-1}$ for $n \geq 2k+1$ and $\gcd\{k,n\} = 1$, since there exists infinitely many integers $n,k$ that satisfy this property.

## 3.3 Connections with $\chi(G)$

A graph $G$ is called an edge-critical $k$-chromatic graph if $\chi(G) = k$ and $\chi(G') < k$ for each proper subgraph $G' of G$ (i.e. $G' = G - v$ or $G' = G - e$). All edge critical $k$-chromatic graphs must be connected, $\chi G = k$, and $\chi(G - e) < k$ for all $e \in E(G)$. Similarly, a graph $G$ is a vertex-critical $k$-chromatic if $\chi(G) = k$ and $\chi(G - v) < k$ for all $v \in V(G)$. It has been proven that if $G \to (a_1,\ldots,a_r)$ then $\chi(G) \leq m$, where $m = \sum_{i=1}^{r}(a_i - 1) + 1$.

# Chapter 4

# Complexity Connections

TODO

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