On Finite Folkman Numbers

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March 26, 2013

Abstract

This survey contains a comprehensive overview of the results relating to Folkman numbers, a topic in general Ramsey Theory. We present data which, to the best of our knowledge, includes all known nontrivial values and bounds for definite Folkman numbers of arbitary forms. In particular, we present results, with complete citations, for general edge and vertex Folkman numbers of the form $F(a_1, \ldots, a_r; q) = \min\{|V(G)| : G \to (a_1, \ldots, a_r; q) and \omega(G) < q\}$, where the edge and vertex Folkman numbers consist of edge and vertex colorings satisfying this property.

Contents

Chapter 1

Introduction

1.1 Scope and Notation

Folkman numbers, first introduced by Folkman in 1970 [2], are concerned with the graphs in which a monochromatic copy of a particular subgraph exists for all (edge or vertex) colorings. We write $G \to (a_1, ..., a_k; p)^e$ iff for every edge coloring of an undirected simple graph G not containing K_p, there exists a monochromatic K_{a_i} in color i for some $i \in \{1, ..., k\}$. The edge Folkman number is defined as $F_e(a_1, ..., a_k) = \min\{|V(G)| : G \to (a_1, ..., a_k; p)^e\}$. Similarly, the vertex Folkman number is defined as $F_v(a_1, ..., a_k) = \min\{|V(G)| : G \to (a_1, ..., a_k; p)^v\}$. In 1970 Folkman proved that for all $k > \max(s, t)$, edge- and vertex-Folkman numbers $F_e(s, t; k$ and $F_v(s, t; k)$ exist.

In this survey we only consider finite, simple graphs (i.e. those without loops and multiple edges). We denote the vertex and edge set of a graph G using $V(G)$ and $E(G)$, respectively. $N(v)$ is the open neighborhood of vertex $v \in V(G)$, consisting of all vertices adjacent to v. $G[V], V \subset V(G)$ denotes the subgraph induced by the vertices V. Also, $\alpha(G)$ and $\omega(G)$ denote the cardinality of a maximum independent set of G (i.e. a set for which all pairwise vertices are not adjacent in G) and the cardinality of the largest clique in G (i.e. a set for which all pairwise vertices are adjacent in G). The chromatic number of G, which is the minimal number of colors required to a vertex two-coloring of G, is denoted as $\chi(G)$. Finally, we denote the complement of a graph G as $\bar{G}$.

In this survey we consider a variety of special graphs of order n, including the complete graph K_n, simple cycle C_n, and path graph P_n. Other critical graphs will be introduced as needed.

We denote $G - v = G[V(G) \setminus v]$. We also denote $G - e$ as the subgraph of G such that $V(G - e) = V(G)$ and $E(G - e) = E(G) \setminus \{e\}$. Similarly, we denote $G + e$ as the supergraph of G such that $V(G + e) = V(G)$ and $E(G + e) = E(G) \cup \{e\}$. Finally, we denote the union of two graphs G_1 and G_2 as $G_1 + G_2$, where $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{(u, v) : u \in V(G_1) \text{ and } v \in V(G_2)\}$.

We begin our exploration of finite Folkman numbers in Chapter 2 with a discussion of classical Folkman numbers - those edge and vertex numbers of the form $F(s, t; q)$. We then discuss multicolor Folkman numbers in Chapter 3.

Chapter 2

Classical Folkman Numbers

2.1 Classical Folkman Numbers

In this section we present results for edge and vertex Folkman numbers. In some cases there is an important connection between these two values. For example, Piwakowski et al. showed that $G \to (3,3;4)^v$ implies $G + x \to (3,3;5)^e$ for some additional vertex x [3]. Such relations have been important in the computation of various Folkman numbers, as we will describe in the following sections.

2.1.1 Edge Folkman Numbers $F_e(s,t;q)$

It is clear that $G \to (s,t;q)^e$ implies that $\omega(G) \geq \max\{s,t\}$ [2], which means that $F_e(s,t;q)$ exists if and only if $q > \max\{s,t\}$. It is well known that as q decreases, the difficulty in computing $F_e(s,t;q)$ increases. For this reason, exact values for $F_e(s,t;4)$ and $F_e(s,t;5)$ have been among the most well-studied problems. We present all known values of these two numbers in Tables 2.1.1 and 2.1.1.

Table 2.1: Values and Bounds for Edge Folkman numbers of the type $F_e(s,t;5)$.

(s,t)	2	3	4	5	6	7	8	9	10	11	12	13
2												
3		15 [3]		≤ 21 [4]								
4												
5												
6												
7												
8												

Table 2.2: Values and Bounds for Edge Folkman numbers of the type $F_e(s,t;4)$.

(s,t)	2	3	4	5	6	7	8	9	10	11	12	13
2												
3		≤ 786 [5]										
4												
5												
6												
7												
8												

Table 2.3: History of $F_e(e, e; 4)$

Year	Bounds	Who	Ref.
1967	any?	Erdős-Hajnal	[1]
1970	exist	Folkman	[2]
1972	≥ 10	Lin	[6]
1975	$\leq 10^{10}$?	Erdős offers $100 for proof	
1986	$\leq 8 \times 10^{11}$	Frankl-Rödl	[7]
1988	$\leq 3 \times 10^{9}$	Spencer	[8]
1999	≥ 16	Piwakowski et al. (implicit)	[3]
2007	≥ 19	Radziszowski-Xu	[9]
2008	≤ 9697	Lu	[10]
2008	≤ 941	Dudek-Rödl	[11]
2012	≤ 786	Lange et al.	[5]
2012	≤ 100?	Garaham offers $100 for proof	

The number $F_e(3, 3; 4)$ has a particular intriguing history, starting with the question first posed by Erdős in [1]:

What is the order of the smallest K_4-free graph for which any 2-coloring of its edges must contain at least one monochromatic triangle?

This question is equivalent to finding the smallest K_4-free graph that is not the union of two triangle-free graphs. Table 2.3 enumerates main developments of the previous work on this problem and leads us to the current state of the field.

2.1.2 Vertex Folkman Numbers $F_v(s, t; q)$

TODO

Chapter 3

Multicolor Folkman Numbers

3.1 Multicolor Folkman Numbers

Multicolor Folkman numbers are unconstrained in the number of r colorings used in their specification. Variations of the multicolor Folkman numbers vary either the number of colors r or the colors (a_i) themselves. Many interesting results have been derived for these types of problems, as we will show in the following sections.

3.1.1 Exact Values

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Vertex Folkman Numbers

To date, few exact values for $F_v(a_1, \ldots, a_r; m-1)$, where $m = \sum_{i=1}^{r}(a_i - 1) + 1$ are known. For this reason, we enumerate those known values below.

- $F_v(2, 3, 3; 5) = 12$
- $F_v(3, 3; 4) = 14$

3.1.2 Generalized Results

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Vertex Folkman Numbers

In [13], Nenov proved a very interesting result for Folkman numbers of the form $F_v(3, \ldots, 3) = \min\{|V(G)| : G \to \underbrace{(3, \ldots, 3)}_{r}$ and $\omega(G) < 2r\}$, showing that $F_v(3, \ldots, 3) = 2r + 7, r \geq 3$. This was an improvement on the bounds of $2r + 5 \leq F_v(3, \ldots, 3) \leq 2r + 10, r \geq 4$, proven by Luczak et al. [14]. A similar bound of $2r + 6 \leq F_v(3, \ldots, 3) \leq 2r + 8, r \geq 3$, was proven by Nenov et al. in [12].

3.1.3 Connections with $\chi(G)$

A graph G is called an edge-critical k-chromatic graph if $\chi(G) = k$ and $\chi(G') < k$ for each proper subgraph $G' of G$ (i.e. $G' = G - v$ or $G' = G - e$). All edge critical k-chromatic graphs must be connected, $\chi G = k$, and $\chi(G - e) < k$ for all $e \in E(G)$. Similarly, a graph G is a vertex-critical k-chromatic if $\chi(G) = k$ and $\chi(G - v) < k$ for all $v \in V(G)$. It has been proven that if $G \to (a_1, \ldots, a_r)$ then $\chi(G) \leq m$, where $m = \sum_{i=1}^{r}(a_i - 1) + 1$.

The value $f(G) = \chi(G) - \omega(G)$ has been shown to have very deep connections with Folkman numbers.

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