
Narrowing the Edge Folkman Number Bounds

Independent Study Proposal

Christopher Wood

Advisor: Professor Stanisław Radziszowski

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1 Background

Edge Folkman numbers, first introduced by Folkman in 1970 [1], are concerned with the study of graphs in which a monochromatic coloring of a particular subgraph always exists. We write $G \rightarrow (a_1, \dots, a_k; p)^e$ if for every edge coloring of an undirected simple graph G not containing K_p , there exists a monochromatic K_{a_i} in color i for some $i \in \{1, \dots, k\}$. The edge Folkman number is defined as $F_e(a_1, \dots, a_k) = \min\{|V(G)| : G \rightarrow (a_1, \dots, a_k; p)^e\}$. In 1970 Folkman proved that for all $k > \max(s, t)$, edge- and vertex- Folkman numbers $F_e(s, t; k)$ and $F_v(s, t; k)$ exist. Prior to this, Erdos and Hajnal pose the problem of finding $F_e(3, 3; 4)$, which can be informally stated as the following question [2]:

What is the order of the smallest K_4 -free graph for which any 2-coloring of its edges must contain at least one monochromatic triangle?

This is equivalent to finding the smallest K_4 -free graph that is not the union of two triangle-free graphs. Since the proposition of this problem, there has been a significant amount of work towards aimed at narrowing the upper and lower bounds of $F_e(3, 3; 4)$. Table 1 enumerates the work on this problem and leads us to the current state of the field.

Table 1: History of $F_e(e, e; 4)$

Year	Bounds	Who	Ref.
1967	any?	Erdős-Hajnal	[2]
1970	exist	Folkman	[1]
1972	≥ 10	Lin	[3]
1975	$\leq 10^{10}?$	Erdős offers \$100 for proof	
1986	$\leq 8 \times 10^{11}$	Frankl-Rödl	[5]
1988	$\leq 3 \times 10^9$	Spencer	[6]
1999	≥ 16	Piwakowski et al (implicit)	[7]
2007	≥ 19	Radziszowski-Xu	[8]
2008	≤ 9697	Lu	[9]
2008	≤ 941	Dudek-Rödel	[10]
2012	≤ 786	Lange et al	[14]
2012	$\leq 100?$	Graham offers \$100 for proof	

2 Problems

2.1 Lower Bound

The current lower bound for $F_e(3, 3; 4)$ stands at 19 [8]. The proof used for this bound, which relies on the existence of independent sets of size k in a graph G on 18 vertices, where $k = 4$, cannot easily be extended to graphs on 19 vertices. It is estimated that more than 10^{19} nonisomorphic, K_4 -free graphs on 19 vertices exist [4].

The proof for this bound relies on the fact that $H \rightarrow (3, 3; 4)^v \Rightarrow H + x \rightarrow (3, 3; 5)^e$ and $F_e(3, 3; 5) = 15$. More specifically, using all nonisomorphic graphs on 18 vertices with contained independent sets of size $k = 4$, the authors were show that, for each candidate graph G , the graph

2.2 Upper Bound

Dudek and Rodel proved that $G \rightarrow (3, 3)^e$ if and only if $MC(H_G) < 2t_\Delta(G)$ [10], where $MC(H_G)$ is the size of the maximum cut of graph $H = (E(G), \{(e_1, e_2) | \{e_1, e_2, e_3\} \text{ form a triangle in } G\})$. Since MAX-CUT is an NP-complete problem, computations using this fact rely on approximation algorithms, such as the one proposed by Geomans and Williamson (TODO: CITE). In fact, Lange et al [14] used this approximation approach, which formulates the MAX-CUT problem as a semidefinite program (SDP). In their work, the authors examined graphs of the form $G(n, r) = (\mathbb{Z}_n, \{(x, y) | x \neq y, x - y \equiv \alpha^r \pmod n, \alpha \in \mathbb{Z}_n\})$. Using these computations, the authors were able to show that $MC(H_{G_{786}}) \leq 857750$, where $G_{786} = G(786, 3)$, and since $2t_\Delta(G_{786}) = 857762$, it is clear that $F_e(3, 3; 4) \leq 786$.

It has been conjectured that $F_e(3, 3; 4) \leq 127$ by using a witnessing graph $G_{127} = (\mathbb{Z}_{127}, E = \{(x, y) | x - y = \alpha^3 \pmod{127}\})$. It was first conjectured that $G_{127} \rightarrow (3, 3)^e$ in [8], where the in-

tuition for this conjecture is that G_{127} has a large number of triangles and many small dense subgraphs. We will attempt to prove or disprove this conjecture by showing that $G_{127} \not\rightarrow (3,3)^e$ using subgraph H extensions. More specifically, we will identify subgraphs H witnessing $H \not\rightarrow (3,3)^e$ and then attempt to extend H to all of G .

3 Summary of Proposed Study

The ultimate goal of this work is to attack both the upper and lower bounds of $F_e(3,3;4)$. Unfortunately, it will not be feasible to leverage the same technique used by Radziszowski et al to prove the current bound of 19. The reason for this is that the estimate number of independent sets in non isomorphic graphs on 19 vertices is expected to be more than 10^{15} (TODO: CITE). Clearly, the number of candidate graphs G to check if $G \rightarrow (3,3)^e$ needs to be smaller, so we will work towards devising constraints similar to the chromatic number $\chi(G) \geq 6$.

Attacking the upper bound will be an entirely different form of computation. In particular, we will need to find a graph G on n vertices where $G \rightarrow (3,3)^e$. As previously discussed, we will be focusing on the graph $G_{127} = G(127,3)$ in an attempt to prove the conjecture that $G_{127} \rightarrow (3,3)^e$. To determine prove this we will decompose the problem arrowing problems on subgraphs H that witness $H \not\rightarrow (3,3)^e$, and then carefully extend H to encompass all of G .

Unfortunately, due to the computationally intensive nature of this procedure, we will reduce $\{H | H \not\rightarrow (3,3)^e\}$ to 3-SAT and leverage the power of SAT solvers such as Minisat [11] and zChaff [12]. The reduction works by mapping the edges in $E(H)$ to variables in $\phi_G \in 3\text{-SAT}$, and for edge adding the following clauses to ϕ_G :

$$(x + y + z) \wedge (\bar{x} + \bar{y} + \bar{z})$$

Clearly, $H \not\rightarrow (3,3)^e \Leftrightarrow \phi_G$ is satisfiable. Therefore, if ϕ_G is not satisfiable, then $G \rightarrow (3,3)^e$, and so $F_e(3,3;4)$.

A major part of this work will be identifying candidate subgraphs H for extension that can easily be solved using the aforementioned 3-SAT solvers. We will also take this opportunity to compare the performance of popular k-SAT solvers published in the literature [12] [11].

4 Project Goals and Deliverables

The following project goals have been identified for this work

- Research current methods and potential techniques for computing the upper and lower bounds of Folkman numbers (in particular, $F_e(3,3;4)$).
- Implement software to aid in the computation of Folkman number bounds.

- Begin a survey paper outlining the history of Folkman number research [13].

By achieving these goals I will generate the following deliverables:

- Publication-ready paper and more in-depth progress report for the entire project
- Report that weighs the performance of various SAT solvers for solving this arrowing problem with G127
- Report discussing the subgraph pruning techniques used for the lower bound

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