Folkman Numbers

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Abstract

This survey contains a comprehensive overview of the results relating to Folkman numbers, a topic in general Ramsey Theory. We present data which, to the best of our knowledge, includes all known nontrivial values and bounds for definite Folkman numbers of arbitary forms. In particular, we present results, with complete citations, for general edge and vertex Folkman numbers of the form $F(a_1, \ldots, a_r; q) = \min\{|V(G)| : G \to (a_1, \ldots, a_r; q)$ and $\omega(G) < q\}$, where the edge and vertex Folkman numbers consist of edge and vertex colorings satisfying this property.

Contents

1 Introduction

Folkman numbers, first introduced by Folkman in 1970 [14], are a branch of Ramsey theory concerned with the graphs in which a monochromatic copy of a particular subgraph exists for all (edge or vertex) colorings. We write $G \to (a_1, ..., a_r; p)^e$ if and only if for every edge coloring of an undirected simple graph G not containing K_q, there exists a monochromatic K_{a_i} in color i for some $i \in \{1, ..., r\}$. The edge Folkman number is defined as $F_e(a_1, ..., a_k) = \min\{|V(G)| : G \to (a_1, ..., a_k; q)^e\}$. Similarly, the vertex Folkman number is defined as $F_v(a_1, ..., a_r) = \min\{|V(G)| : G \to (a_1, ..., a_r; q)^v\}$. In 1970 Jon Folkman proved that for all $q > \max(s, t)$, edge- and vertex-Folkman numbers $F_e(s, t; q)$ and $F_v(s, t; q)$ exist. The sets $\mathcal{F}_v(s, t; q) = \{G : G \to (s, t)^v \wedge \omega(G) < q\}$ and $\mathcal{F}_e(s, t; q) = \{G : G \to (s, t)^e \wedge \omega(G) < q\}$ are the *vertex* and *edge Folkman graphs*, respectively. We also define $\mathcal{H}(s, t) = \{G : G \to (s, t)$ and $\omega(G) = \max\{s, t\}\}$, which means that $F(s, t; q = \max\{s, t\}) = \min\{|V(G)| : G \in \mathcal{H}(s, t)\}$. Here, $\mathcal{H}$ is a special type of Folkman number whose avoidance clique is of the minimum size. Thus, it should be clear that $F(s, t; \max\{s, t\}) \leq F(s, t; q)$. For convenience, we denote $F(s, t; \max\{s, t\})$ as $F(s, t)$ for both vertex and edge Folkman numbers. Some related problems often consider the more general question of determining $G \to (H_1, \ldots, H_r; Q)$, which is true if and only if for every edge (or vertex) coloring there exists a monochromatic copy of H_i for some $1 \leq i \leq r$ and G is Q-free.

2 Scope and Notation

In this survey we only consider finite, simple graphs (i.e. those without loops and multiple edges). We denote the vertex and edge set of a graph G using $V(G)$ and $E(G)$, respectively. $N(v)$ is the open neighborhood of vertex $v \in V(G)$, consisting of all vertices adjacent to v. $G[V], V \subset V(G)$ denotes the subgraph induced by the vertices V. Also, $\alpha(G)$ and $\omega(G)$ denote the cardinality of a maximum independent set of G (i.e. a set for which all pairwise vertices are not adjacent in G) and the cardinality of the largest clique in G (i.e. a set for which all pairwise vertices are adjacent in G). The chromatic number of G, which is the minimal number of colors required to a vertex two-coloring of G, is denoted as $\chi(G)$. The girth of a graph G, denoted as $g(G)$, is the length of the shortest cycle in G. Finally, we denote the complement of a graph G as G^c.

We denote $G - v = G[V(G) \setminus v]$. We also denote $G - e$ as the subgraph of G such that $V(G - e) = V(G)$ and $E(G - e) = E(G) \setminus \{e\}$. Similarly, we denote $G + e$ as the supergraph of G such that $V(G + e) = V(G)$ and $E(G + e) = E(G) \cup \{e\}$. Finally, we denote the union of two graphs G_1 and G_2 as $G_1 + G_2$, where $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{(u, v) : u \in V(G_1)$ and $v \in V(G_2)\}$. From this definition, it is clear that $\omega(G_1 + G_2) = \omega(G_1) + \omega(G_2)$. As a special case, we denote $K_n - C_m$, $m \leq n$, as the induced subgraph of K_n obtained by removing all edges on some cycle C_m. The kth power of a graph G, denoted as G^k, is the graph $(V(G), E(G) \bigcup \{(u, v) : d(u, v) < k\})$.

A *hypergraph* G is the pair $(V(G), E(G)$ where $V(G)$ is the set of vertices and $E(G) = 2^{V(G)}$ is hte set of hyperedges. Traditional definitions for graph order and clique sizes apply to hypergraphs. A hypergraph is *k-uniform* if every edge $e \in E(G)$ appears exactly k times in the set (i.e. it has cardinality k).

Classical Ramsey numbers $R(s, t)$ are pivotal in the study of Folkman numbers. We define $R(s, t)$ as the smallest integer n such that every 2-coloring of the edges of K_n there exists a monochromatic K_s in the first color or K_t in the second color. In other words, $R(s, t) = \min\{n | K_n \to (s, t)^e\}$. We often generalize the arrowing property to graphs G and H such that $R(G, H) = \min\{n | K_n \to (G, H)^e\}$. Radziszowski [66] maintains a regularly updated survey containing the most recent results with known bounds and exact values for small Ramsey numbers.

We begin our exploration of finite Folkman numbers in Chapter 3.6 with a discussion of classical Folkman numbers - those edge and vertex numbers of the form $F(s, t; q)$. We then discuss multicolor Folkman numbers in Chapter **??**. Asymptotic results are then presented in Chapter 6.1, followed by a discussion of the complexity theoretical results regarding Folkman numbers and their computations in Chapter **??**

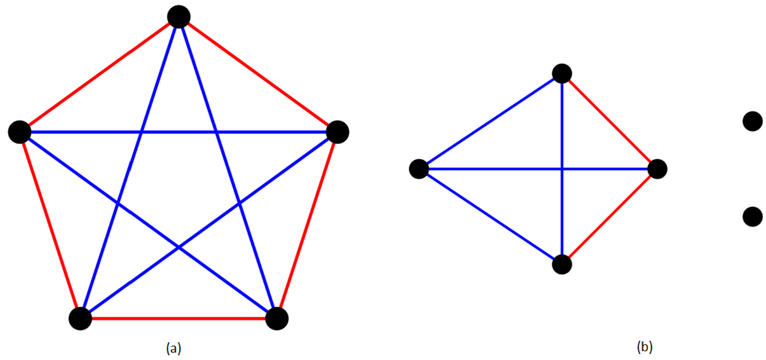

Figure 1: (a) A coloring of K_5 that does not contain a monochromatic triangle, and (b) a part of a 2-edge coloring of $G = K_3 + C_5$ that shows a triangle cannot be avoided.

3 Classical Folkman Numbers

In this section we present results for edge and vertex Folkman numbers. In some cases there is an important connection between these two values. For example, Piwakowski et al. showed that $G \rightarrow (3,3;4)^v$ implies $G + x \rightarrow (3,3;5)^e$ for some additional vertex x [64]. Such relations have been important in the computation of various Folkman numbers, as we will describe in the following sections. For convenience, we let $F(a_1, \ldots, a_r; q) = F(a_r; q)$ if $a_1 = a_2 = \cdots = a_r$.

We first begin with a fundamental result proven by Folkman in 1970.

Theorem 1. [14] $F_e(s,t;q)$ exists if and only if $q > \max\{s,t\}$.

Remark 2. This can be generalized to state that $F(a_1, \ldots, a_r; q)$ exists if and only if $q > \max\{a_1, \ldots, a_r\}$, as was affirmatively shown by Nesetril and Rödl in [63]. Furthermore, from this result, it should be clear that $G \rightarrow (s,t;q)^e$ implies that $\omega(G) \geq \max\{s,t\}$.

3.1 Edge Folkman Numbers $F_e(s,t;q)$

It is well known that as q decreases, the difficulty in computing $F_e(s,t;q)$ increases. For this reason, exact values for $F_e(s,t;4)$ and $F_e(s,t;5)$ have been among the most well-studied problems. We present all known values of these two numbers in Tables **??** and **??**.

Theorem 3. [16] $F_e(3,3;6) = 8$.

Remark 4. This is often touted as the one of the simpler and more elegant results for a Folkman number that comes from the graph $G = K_8 - C_5 = K_3 + C_5$. It is clear that $\omega(G) = 5$, so G is K_6-free, and by the pigeonhold principle it can be shown that coloring the edges of C_5 without a triangle is not possible. This is shown in Figure **??**.

Theorem 5. [16] $F_e(3,3;7) = 6$.

Theorem 6. [64] $F_e(3,3;5) = 15$.

Remark 7. Piwakowski found that there exists exactly one bicritical graph $G \in \mathcal{F}_v(3,3;4;15)$, such that $G + e$ contains a K_5 and $G - e$ strips the Ramsey property. This was found by enumerating all nonisomorphic graphs of order 12, filtering all graphs H that do not satsify $K_5 \not\subseteq H$, $\chi(H) \geq 5$, and for every edge $\{u,v\} \in E(\bar{H})$ there are vertices $a, b \in V(H)$ such that $H[\{x,y,a,b\}]$ is isomorphic to $K_4 - e$. Then, with additional constraints, all of the $(+e, K_5)$ critical graphs from this set were found and then subsequently checked to see if $H - e \rightarrow (3,3)^e$. With this process, it was shown that $\delta(G) = 14$, and by removing this vertex v the result is a graph $G - v \in \mathcal{F}_v(3,3;4;14)$, as shown in Figure 2.

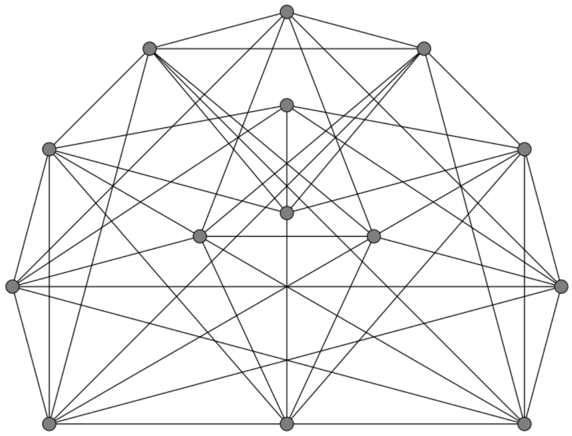

Figure 2: The unique bicritical graph $G \in \mathcal{F}_v(3,3;4;14)$ [64].

Table 1: Folkman number bounds for small values of k. Note that $R(3,3) = 6$.

k	$F_e(3,3;k)$	graphs	reference
≤ 7	6	K_6	folklore
6	8	$C_5 + K_3$	[16]
5	15	659 graphs	[64]
4	≤ 786	G_{786}	[35]

3.2 Triangle-Free Graphs

It is a well-known result that if $k > R(s,t) \rightarrow F_e(s,t;k) = R(s,t)$. Conversely, when $k \leq R(s,t)$, very little is known about the bounds of $F_e(s,t;k)$. Table 1 shows that as the value of k decreases, the problem of finding $F_e(s,t;k)$ becomes increasingly difficult.

3.2.1 $F_e(3,3;4)$ History

The number $F_e(3,3;4)$ has a particular intriguing history, starting with the question first posed by Erdős in [12]:

What is the order of the smallest K_4-free graph for which any 2-coloring of its edges must contain at least one monochromatic triangle?

This question is equivalent to finding the smallest K_4-free graph that is not the union of two triangle-free graphs. Folkman first proved the existence of such graphs in 1970, showing that for all $k > \max\{s,t\}$, $F_e(s,t;k)$ must exist [14].

Erdős followed this result by offering a \$100 prize (equivalent to 300 Swiss francs at the time) for determining if $F_e(3,3;4) < 10^{10}$. In 1988, Spencer [68] presented the first proof of graphs below this bound using a probabilistic approach, which was shown to contain an error by Hovey in 1989, who adjusted the bound to $F_e(3,3;4) < 3 \times 10^9$. More than twenty years passed before Dudek and Rödl [7] proposed a construction technique for Folkman graphs based on the maximum cut of a related graph that was used to improve the bound to $F_e(3,3;4) \leq 941$. Lange et al. [35] further improved this bound to $F_e(3,3;4) \leq 786$ with the same technique using the Geomans-Williamson MAX-CUT approximation algorithm. The conjecture that $F_e(3,3;4) \leq 127$ proposed by Radziszowski et al. [65] is still an open problem, and is motivated by the intuition that $G_{127} = (\mathbb{Z}_{127}, \{(x,y) | x - y \equiv \alpha^3 (\mod 127)\})$ contains many triangles and small dense subgraphs. Following in the footsteps in Erdős, Graham offered a \$100 prize for proving that $F_e(3,3;4) \leq 100$.

Table 2: History of $F_e(3,3;4)$

Year	Bounds	Who	Ref.
1967	any?	Erdős-Hajnal	[12]
1970	exist	Folkman	[14]
1972	≥ 10	Lin	[36]
1975	$\leq 10^{10}$?	Erdős offers \$100 for proof	
1986	$\leq 8 \times 10^{11}$	Frankl-Rödl	[15]
1988	$\leq 3 \times 10^{9}$	Spencer	[68]
1999	≥ 16	Piwakowski et al. (implicit)	[64]
2007	≥ 19, ≤ 127?	Radziszowski-Xu	[65]
2008	≤ 9697	Lu	[37]
2008	≤ 941	Dudek-Rödl	[7]
2012	≤ 786	Lange et al.	[35]
2012	≤ 100?	Garaham offers \$100 for proof	

Problem 8. $F_e(3,3;4) \leq 100$?

The lower bound of $F_e(3,3;4) \geq 10$ was first proved by Lin in 1972 [36]. Piwakowski et al. [64] improved this bound to $F_e(3,3;4) \geq 16$ by enumerating all graphs in $\mathcal{F}_e(3,3;5)$ and proving that all of them contain K_4's. In 2007, Radziszowski et al. [65] pushed this bound to $F_e(3,3;4) \geq 19$ with a construction technique that relied on the fact that $G \to (3,3;4)^v$ implies $G + x \to (3,3;5)^e$ and $F_e(3,3;5) = 15$. Table 3 enumerates these developments leads us to the current state of the field.

3.2.2 $F_e(3,3;5)$ History

This number also has a dated history, starting with the existential question posed by Erdős et al. in 1967 [12]. However, the first proof of the existence of this number predates Erdős in an unpublished manuscript by Pòsa. A viable upper bound was first proven by Schǎuble [67] in 1969, shortly after the proposition by Erdős et al., in which it was shown that $F_e(3,3;5) \leq 42$. Graham and Spencer [17] improved this bound to $F_e(3,3;5) \leq 23$ in 1971, and further conjectured that $F_e(3,3;5) = 23$ without supporting reasoning. This bound was subsequently reduced to 18 by Irving [23] in 1973.

At this point, work on the upper bound diverged. Hadziivanov and Nenov [20] were able to construct a 16-vertex graph in $\mathcal{F}_e(3,3;5)$, and Nenov [44] further improved this bound with a 15-vertex graph in $\mathcal{F}_e(3,3;5)$, thus proving that $F_e(3,3;5) \leq 15$. Hadziivanov and Nenov [21] found another such graph on 15 vertices in 1984. Years later, Erickson [13] found a 17-vertex graph in $\mathcal{F}_e(3,3;5)$ and subsequently conjecture that $F_e(3,3;5) = 17$. However, Bukor [2] disproved this conjecture by showing the same 16-vertex construction presented in [20]. In 1996, the upper bound of 15 was verified once more by Urbaǹski [69] with a different construction of the 15-vertex graph in [44].

The lower bound of $F_e(3,3;5)$ has much less history. In 1972 Lin [36] proved that $F_e(3,3;5) \geq 10$, which was subsequently improved by Nenov [43] to $F_e(3,3;5) \geq 11$, and then by Hadziivanov and Nenov [22] to $F_e(3,3;5) \geq 12$. The final value of $F_e(3,3;5) = 15$, proven by Piwakowski et al. [64] in 1999, was shown by constructing all 659 15-vertex graphs in $\mathcal{F}_e(3,3;5;15)$, where each such graph has a K_4.

3.3 $F_e(a_1, \ldots, a_r; R(a_1, \ldots, a_r))$

Folkman numbers of the form $F_e(a_1, \ldots, a_r; R(a_1, \ldots, a_r))$ are very difficult to find. To date, only a few results are known, as shown below.

Theorem 9. $F_e(3,3;6) = 8$ [16].

Remark 10. This was shown with the graph $G = K_8 - C_5 = K_3 + C_5 \to (3,3)^e$.

Table 3: History of $F_e(3,3;5)$

Year	Bounds	Who	Ref.
1967	any?	Erdős-Hajnal	[12]
1969	≤ 42	Schäuble	[67]
1971	≤ 23	Graham and Spencer	[17]
1971	$= 23?$	Graham and Spencer	[17]
1972	≥ 10	Lin	[36]
1973	≤ 18	Irving	[23]
1979	≤ 16	Hadziivanov and Nenov	[20]
1980	≥ 11	Nenov	[43]
1981/84	≤ 15	Nenov, Hadziivanov and Nenov	[44], [21]
1985	≥ 12	Hadziivanov and Nenov	[22]
1993	≤ 17	Erickson	[13]
1993	$= 17?$	Erickson	[13]
1994	≤ 16	Bukor	[2]
1996	≤ 15	Urbański	[69]
1999	$\geq 15, = 15$	Piwakowski, Radziszowski, Urbański	[64]

Theorem 11. $F_e(3,4;9) = 14$ [49].

Theorem 12. $F_e(3,5;14) = 16$ [36].

Theorem 13. $F_e(4,4;18) = 20$ [36].

Theorem 14. $F_e(3,3,3;17) = 19$ [36].

3.4 Special Cases

Other special cases of edge Folkman numbers exist which do not fall under the typical two-color form. We present these results in Table 3.4. Remarks for such results are given below.

(a) $F_e(3,4,\leq 10)$ is trivially witnessed by K_9 since $R(3,4) = 9$.

(b) $F_e(3,4,9)$ was witnessed with the critical $(3,4)$-Ramsey graph $K_4 + C_5 + C_5 + C_5$.

(c) $F_e(3,5;8)$ was shown with the witnessing graph $G = K_8 + Q$ (where Q is the graph shown in Figure 6), which improved the upper bound of $F_e(3,5;8) \leq 24$ proven by Kolev and Nenov in 2008 [31]. Since $\omega(K_8 + Q) = \omega(K_8) + \omega(Q)$ and $\omega(Q) = 4$ [18], we have that $\omega(G) = 12$. Thus, since $|V(G)| = 21$ and $G \to (3,5)^e$, $F_e(3,5;13) \leq 21$ [31]. The best known lower bound for this number, $F_e(3,5;13) \geq 18$, was shown by Lin [36] in 1972. Nenov [46] showed that the equality $F_e(3,5;13) = 18$ is only possible by proving the arrowing $K_8 + C_5 + C_5 \to (3,5)^e$, but no one has been able to check this yet.

(d) $F_e(4,4;17)$ was the first result on the upper bound of this number, as previously the only known fact was that the number existed, as proved by Folkman [14].

Some special cases of interest for the reader include $F_e(3,4;7)$ and $F_e(3,5;13)$.

Problem 15. $F_e(3,4;7) =?$

Problem 16. $F_e(3,5;13) = 18?$

3.5 Vertex Folkman Numbers $F_v(s,t;q)$

Vertex Folkman numbers are of equal importance in this particular research topic.

Table 4: Table of special case exact values and bounds for edge Folkman numbers $F_e(s,t;q)$.

s	t	q	$F_e(s,t;q)$	Ref.
3	4	10	≤ 9	[66]
3	4	9	14	[49]
3	4	8	16	[26]
3	5	14	16	*
3	5	8	≤ 21	[31]
4	4	18	20	*
4	4	17	≤ 25	[34]
3	7	22	≥ 27	*

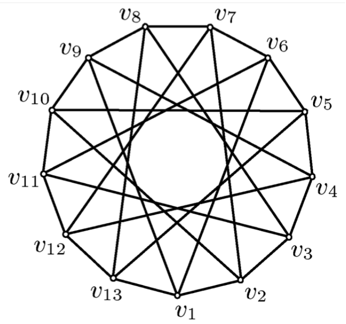

Figure 3: The complement of the critical Greenwood and Gleason [18] graph.

1. $F_v(4,4;6) \leq 14$ is an improvement on the previous upper bound of $F_v(4,4;6) \leq 35$, which was shown by Łuczak, Rucinski, and Urbański, and was obtained with the witnessing graph $G = K_1 + Q$ (since $Q \rightarrow (3,4)^v$, see Figure 3).

Theorem 17. [59, 33] $16 \leq F_v(4,4;5) \leq 25$.

Theorem 18. [52] $F_v(2,2;4) = 13$.

Theorem 19. [53] $F_v(3,4) = 13$.

Remark 20. The proof of this theorem hinged on a supporting theorem proven by Nenov, which states that for an n-vertex graph $G \in \mathcal{H}(3,4)$ it is true that $\alpha(G) \leq n-9$, and that equality in this bound implies that $n \leq 18$. Furthermore, the inequality $F_v(3,4) \leq 13$ is witnessed by the graph shown in Figure 3. The lower bound $F_v(3,4) \geq 13$ was proven by showing that no 12-vertex graphs $G \in \mathcal{H}(3,4)$ exist. By assuming that a 12-vertex graph $G \in \mathcal{H}(3,4)$, it is true that $\alpha(G) = 2$ (see the prior result), which means that such a graph G was a subgraph of P (where $\bar{P}$ is shown in Figure 4). The authors then showed that $P \in \mathcal{H}(3,4)$, thus proving that no such graphs G exist on 12 vertices.

Theorem 21. [64] $F_v(3,3;4) = 14$.

Remark 22. The authors also showed that $F_e(3,3;5) \leq F_v(3,3;4) + 1$ and $F_v(3,3;4) = 14$.

3.6 Special Cases

Theorem 23. [33] $F_v(2,2;3) = 5$.

Remark 24. This was proved using the graph C_5, by showing that $C_5 \rightarrow (2,2)$.

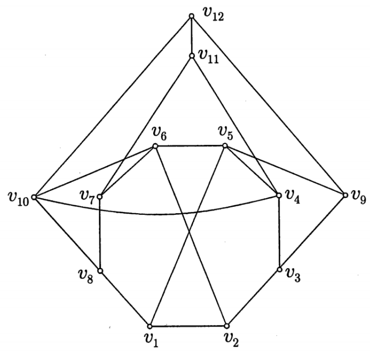

Figure 4: The complement of the graph P, which was shown by Nenov [53] that $P \notin \mathcal{H}(3,4)$.

4 General Folkman Numbers

General Folkman numbers are typically unconstrained in the number of r colorings used in their specification. Variations of these general, or multicolor, Folkman numbers vary either the number of colors r or the colors (a_i) themselves. Many interesting results have been derived for these types of problems. Research for general Folkman numbers is targeted at finding bounds for a variety of Folkman number classes, often times leading to explit values for such classes. We will discuss all related results in the following sections. First, however, we define some important terms and notation that will be used in this discussion.

For Folkman numbers $F(a_1, \ldots, a_r; q)$ we define the following useful quantities [38].

$$m = \sum_{i=1}^{k}(a_i - 1) + 1, p = \max\{a_1, \ldots, a_r\} \tag{1}$$

It is clear that $K_m \to (a_1, \ldots, a_r)^v$ and $K_{m-1} \not\to (a_1, \ldots, a_r)^v$, so if $q \geq m{+}1$ then $F_v(a_1, \ldots, a_r; q) = m$. An $(a_1, \ldots, a_r)$-vertex minimal graph G is one such that $G \in \mathcal{F}_v(a_1, \ldots, a_r; q)$ and $G - v \notin \mathcal{F}_v(a_1, \ldots, a_r; q)$ for all $v \in V(G)$. Similarly, an $(a_1, \ldots, a_r)$-edge minimal graph G is one such that $G \in F_e(a_1, \ldots, a_r; q)$ and $G - e \notin F_e(a_1, \ldots, a_r; q)$ for all $e \in E(G)$. Finally, since it is well known that if $F_v(a_1, a_2, \ldots, a_r; q) = F_v(a_2, \ldots, a_r; q)$ if $a_1 = 1$, we assume $a_i \geq 2$ for all $1 \leq i \leq r$ unless otherwise stated.

4.1 Multicolor Edge Folkman Numbers

In 1983, Nenov [46] showed that for ineteger colors $a_1, \ldots, a_r$, where and $a_i \geq$ for all $1, \ldots, r, rgeq2$, then $F_e(a_1, \ldots, a_4; R(a_1, \ldots, a_r) - 2) \geq R(a_1, \ldots, a_r) + 6$. The Folkman number $F_e(3,3,3;15) = 23$ is one such number of this kind [45]. Since $a_i \geq 3$ and $r \geq 2$, $R(a_1, \ldots, a_r) > 2 + \max\{a_1, \ldots, a_r\}$, which implies that such Folkman numbers exist. The exact value of $F_e(a_1, \ldots, a_4; R(a_1, \ldots, a_r) - 2) = R(a_1, \ldots, a_r) + 6$ was shown to be true if an only if $K_{R-7} + Q \to (a_1, \ldots, a_r)^e$ or $K_{R-9} + C_5 + C_5 + C_5 \to (a_1, \ldots, a_r)^e$, where $R = R(a_1, \ldots, a_r)$. In this case, $\bar{Q}$ is the graph shown in Figure 6. Based on this result, Nenov showed that $F_e(3,5;12) \geq 12$ and $F_e(4,4;16) \geq 24$. In combination with the bound $F_e(3,4;7) \geq 15$ and $F_e(3,4;8) = 16$ (found in [26]), the bound $F_e(3,4;7) \geq 17$ emerges. Of further interest, the Folkman number $F_e(3,3,3;15) = 23$ proves that $K_8 + C_5 + C_5 + C_5 \to (3,3,3)^e$ [61].

General upper bounds for edge Folkman numbers have been found to be closely related to certain Ramsey numbers. In particular, Kolev [34] considered the following relationship. Let $a, \alpha \in \mathbb{Z} \cup \{0\}$, such that $R(3,a) = R(3,a-1) + a - \alpha, \alpha \geq 4$. Under these conditions, given the existence of a graph U such that $\omega(U) = a{-}1$, $U \to (a{-}1, a{-}2)^v$, and $U \to (3, \overbrace{a-3, \ldots, a-3}^{r})^v$, and $R(3,a) - 3a + \alpha + 5 \geq R(3, a-2)$, then $F_e(3, a; R - a + \alpha + 4) \leq R(3,a) - 2a + \alpha + 4 + |V(U)|$. Searching for such graphs U is very much a nontrivial problem, and to date there have been no such graphs that serve as appropriate candidates for this theorem.

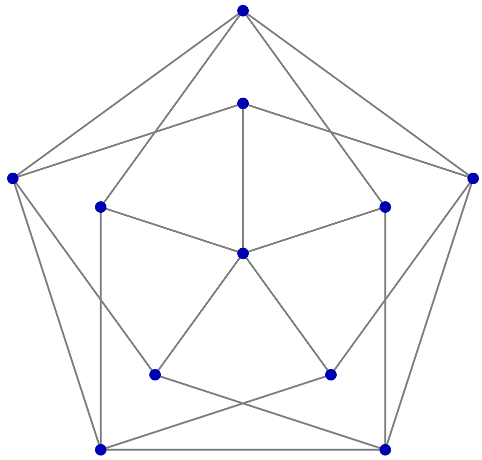

Figure 5: The Grőtzsch graph showing that $F_e(2, 2, 2; 3) = 11$.

- $F_e(3, 3, 3; 17) = 19$
- $F_e(3, 3, 3; 16) = 21$

Theorem 25. [38] For any $p \geq 2$, $F_v(2, p) = 2p + 1$.

4.2 Multicolor Vertex Folkman Numbers

Folkman [14] showed that $F_v(a_1, \ldots, a_k; q) = m$ for $q > m$ and $F_v(a_1, \ldots, a_k; q) = a + m$ for $q = m$. To date, few exact values for $F_v(a_1, \ldots, a_r; m - 1)$, are known. We state known results in the following theorems.

Theorem 26. $F_v(2, 3, 3; 5) = 12$

Theorem 27. $F_v(3, 3; 4) = 14$.

Remark 28. The inequality $F_v(3, 3; 4) \leq 14$ was proven by Nenov [44] in 1981 with a Ramsey graph $R(3, 3)$ that does not contain a clique of size 4, and the inequality $F_v(3, 3; 4) \geq 14$ was proven by Piwakowski et al. [64] in 1999 using computations to exhaustively check all possible Folkman graphs containing less than 14 vertices.

Theorem 29. $F_v(2, 2, 2; 3) = 11$.

Remark 30. This was shown by the Grotzsch graph in Figure 5. []

Theorem 31. [47] $F_v(2, 2, 2, 2; 4) = 11$.

Remark 32. Clearly, this means that the smallest 5-chromatic K_4-free graph has 11 vertices.

Theorem 33. [24] $F_v(2, 2, 2, 2; 3) = 22$.

Remark 34. Clearly, this means that the indicating the smallest 5-chromatic K_4-free graph has 22 vertices.

Theorem 35. $F_v(2, 3, 3; 4) \geq 18$

Theorem 36. $F_v(2, 2, 2, 3; 4) \leq 30$

Theorem 37. [58] $F_v(3, \ldots, 3; 2r) = 2r + 7, r \geq 3$.

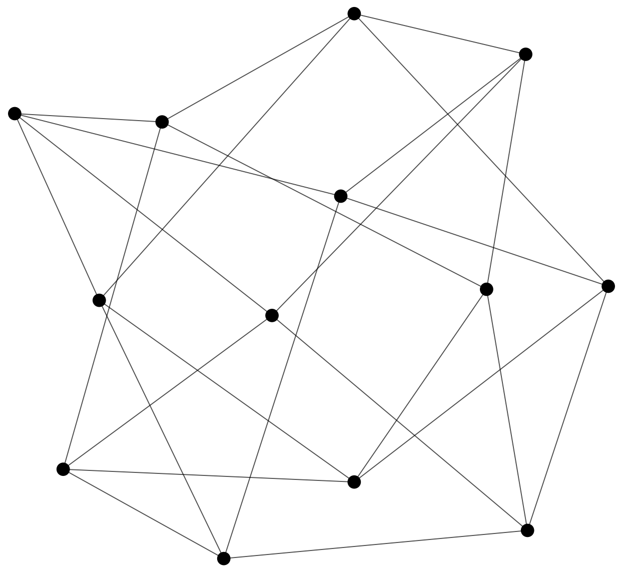

Figure 6: The Nenov graph from [61].

Remark 38. This was a very important result for $F_v(3,\ldots,3) = \min\{|V(G)| : G \to (\underbrace{3,\ldots,3}_{r})$ and $\omega(G) < 2r\}$. Furthermore, it was an improvement on the bounds of $2r+5 \leq F_v(3,\ldots,3;2r) \leq 2r+10, r \geq 4$, proven by Luczak et al. [39]. A similar bound of $2r+6 \leq F_v(3,\ldots,3) \leq 2r+8, r \geq 3$, was proven by Nenov et al. in [51].

4.2.1 $F_v(a_1,\ldots,a_r;q=m)$

In 2001, Łuczak et al. [39] proved that $F_v(a_1,\ldots,a_r;q) \geq 2m-q+1$. Computing the vertex Folkman number becomes very interesting when $q=m$, and Łuczak went on to construct very large classes, and in certain cases, infinitely many graphs that satisfy this property. Let $k \in \{a_r, a_r+1,\ldots,m-1\}$, n an integer such that $n > 2k$, $s = \gcd\{s,k\}$, and $t = k/s$. Using the graph $G = G(n,k) = C_n^{k-1} + K_{m-k-1}$, Łuczak et al. showed that $G \in \mathcal{F}_v(a_1,\ldots,a_r;q=m)$ if and only if $\sum_{i=1}^{r}\lfloor(a_i-1)/t\rfloor < s$. Recall that $\omega(G) = m-1$ because $\omega(C_n^{k-1}) + \omega(K_{m-k-1}) = m-k-1+k = m-1$, so G is K_m-free. This construction is unique in that it enables the construction of infinitely many vertex Folkman graphs. In particular, if $\gcd\{n,k\} = 1$, then $s=1$ and $t=k$, so $\sum_{i=1}^{r}\lfloor(a_i-1)/t\rfloor = 0 < s = 1$, which means that $G \in \mathcal{F}_v(a_1,\ldots,a_r;q=m)$ [39]. Since there exists infinitely many integers n,k that satsify this construction criteria, clearly an infinite number of graphs can be produced.

It is interesting to note that not all such graphs are minimal with respect to their membership in $\mathcal{F}_v(a_1,\ldots,a_r;q=m)$. In fact, G is only $(a_1,\ldots,a_r)$-vertex minimal if $k=p$, $n \geq 2k+1$, and of course, if $\gcd\{n,k\} = 1$.

Theorem 39. [38, 39] For positive integers $r \geq 2$, $F_v(a_1,\ldots,a_r;m \geq p+1) = m+p$

Remark 40. A looser bound of $F_v(a_1,\ldots,a_r;m) \leq m+p$ still holds with the exception of the unique distinguishing graph $G = K_{m-p-1} + \bar{C}_{2p+1}$ in $\mathcal{F}_v(a_1,\ldots,a_r;m)$ with vertex order $m+p$. This is the only distinguishing graph in $\mathcal{F}_v(a_1,\ldots,a_r;m)$. Furthermore it was shown that if $G \to (a_1,\ldots,a_r)$, $\omega(G) < m$, and $|V(G)| = m+p$, then $G = K_{m+p} - C_{2p+1}$.

Theorem 41. [51] For $a_1,\ldots,a_r$, $r \geq 2$, and positive integers m and p such that $m \geq p+2$, $F_v(a_1,\ldots,a_r;m-1) \geq m+p+2$.

Remark 42. This was followed up by Kolev and Nenov in 2006 with Theorem 46. Equality of $F_v(a_1,\ldots,a_r;m-1) = m+p+2$ only occurs when $p=2$ and $m \geq 5$ [19, 39, 46].

Theorem 43. [57] Let $a_1, \ldots, a_r$ be positive integers and $m \geq p+2$. If G is a graph such that $G \to (a_1, \ldots, a_r)$ and $\omega(G) < m-1$, then $|V(G)| \geq m + p + \alpha(G) - 1$ and furthermore, if $|V(G)| = m + p + \alpha(G) - 1$ then $|V(G)| \geq m + 3p$.

4.2.2 $F_v(a_1, \ldots, a_r; m-1)$

It is well known that $F_v(a_1, \ldots, a_r; m) < F_v(a_1, \ldots, a_r; m-1)$, and furthermore, that $\mathcal{F}_v(a_1, \ldots, a_r; m-1)$ is non-empty if and only if $m \geq p+2$. To date, there are few vertex Folkman numbers of the form $F_v(a_1, \ldots, a_r; m-1)$. In 1955 Mycielski [40] found an 11-vertex graph G such that $G \to (2,2,2)^v$ and $\omega(G) = 2$. This proved that $F_v(2,2,2;3) \leq 11$. The lower bound of $F_v(2,2,2;3) = 11$ was shown to be exact by Chavátal in 1974 [3].

$F_v(2,2,2,2;4) = 11$ was proven disjointly by Nenov in 1984 [47], who showed that $F_v(2,2,2,2;4) \geq 11$, and then again by Nenov in 1983 [46, 42], who showed that $F_v(2,2,2,2;4) \leq 11$ (see [50] as well).

Theorem 44. [55] For $a_1 = \cdots = a_r = 2$, it is true that

$$F_v(\underbrace{2, \ldots, 2}_{r}) = \begin{cases} 11 & : r = 3, 4, \\ r+5 & : r \geq 5 \end{cases}$$

Remark 45. The exact value of $F_v(\underbrace{2, \ldots, 2}_{r}) = r+5, r \geq 5$ was proved in [46, 42, **?**, 19]. Furthermore, it was shown by Nenov [46, 58] that $K_{r-5} + C_5 + C_5$ is the only extremal graph in $\mathcal{F}_v(\underbrace{2, \ldots, 2}_{r})$.

Few other numbers of the form $F_v(a_1, \ldots, a_r; m-1)$ are known. One particularly interesting case is the number $F_v(3,3;4) = 14$, which was proved over the span of almost two decades by Nenov [44] and Piwakowski et al. [64]. Nenov showed that $F_v(3,3;4) \leq 14$ using constructions, and Piwakowski et al. showed that $F_v(3,3;4) \geq 14$ using computational techniques. Other numbers include $F_v(3,4;5) = 13$ [53], $F_v(2,2,4;5) = 13$ [51], and $F_v(4,4;6) = 14$ [41].

In 2002, Nenov [55] showed that $F_v(2,2,2,4;6) = F_v(2,3,4;6) = 14$. To do this, he also showed that if $G \to (a_1, \ldots, a_r)^v$ and there exists one $a_i \geq 2$, then $G \to (a_1, \ldots, a_{i-1}, 2, a_i - 1, a_{i+1}, \ldots, a_r)$. Thus, if $G \to (2,3,4)^v$, for example, then by this fact we also have that $G \to (2,2,2,4)^v$.

Theorem 46. [27] $F_v(a_1, \ldots, a_r; m-1) \leq m + 3p$ for $p \geq 3$.

Remark 47. This improved the upper bound of $F_v(a_1, \ldots, a_r; m-1) \leq p^2 + m$ when $m \geq 2p+2$ shown by Łuczak et al. [**?**]. This bound is exact for $F_v(2,2,3;4) = 14$ [4] (though a looser bound of $10 \leq F_v(2,2,3;4) \leq 14$ was proved by Nenov in 2000 [51]) and $F_v(3,3;4) = 14$ [64]. The value $F_v(2,2,3;4) = 14$ is among the known $F_v(a_1, \ldots, a_r; m-1)$ such that $p \leq 4$. Coles et al. obtained this result by constructing all Folkman graphs G on 14 vertices from three-vertex extensions of smaller K_4-free graphs G' on 11 vertices, and for all such graphs G checking to see if $G - v \in \mathcal{F}_v(2,2,3;4;13)$. Large-scale computations were used to exhaustively show that no such graphs on 13 vertices exist, and thus the exact value of $F_v(2,2,3;4) = 14$ holds. Łuczak et al. also claimed that $F_v(a_1, \ldots, a_r; m-1) \leq 3p^2 + p - mp + 2m - 3$, for $p+3 \leq m \leq 2p+1$, without proof.

The boundary case for $F_v(a_1, \ldots, a_r; m-1 = p+1)$ for $p \geq 5$ was studied by Kolev and Nenov in 2006 [30]. With the constraint on m, it is clear that only two such vertex Folkman numbers exist; namely, $F_v(2,2,p;p+1)$ and $F_v(3,p;p+1)$.

Lemma 48. For a graph G, if $G \to (3,p)^v$, then $G \to (2,2,p)^v$.

Theorem 49. [30] $F_v(2,2,p;p+1) \leq F_v(3,p;p+1)$.

Remark 50. Using Theorem 83, the following inequalities immediately follow.

$$F_v(3,p;p+1) \leq 4p+2 \tag{2}$$

$$F_v(2,2,p;p+1) \leq 4p+2 \tag{3}$$

See section 4.2.5 for results that improve these bounds.

In 2001, Nenov [54] proved that $F_v(2,3,3;5) = F_v(2,2,2,3;5) = 12$.

Lemma 51. Let $a_1,\ldots,a_r$ be positive integers and m and p satisfy (1). If G is a graph such that $\omega(G) < m-1$, $G \to (a_1,\ldots,a_r)^v$ and $N(u) \subseteq N(v)$ for some $u,v \in V(G)$, then $|V(G)| \geq m+p+3$.

Lemma 52. Let $a_1,\ldots,a_r$ be positive integers and m and p satisfy (1). If G is a graph such that $\omega(G) < m-1$, $G \to (a_1,\ldots,a_r)^v$, and $\alpha(G) \neq 2$, then $|V(G)| \geq m+p+3$.

Lemma 53. Let n and p be positive integers and $p \geq 2$. Let G be a graph such that

$$\left.\begin{array}{l} b_1,\ldots,b_s \in \mathbb{Z} \\ 1 \leq b_1 \leq b_s \leq p \\ \sum_{i=1}^{s}(b_i-1)+1 = n \end{array}\right\} \Rightarrow G \to (b_1,\ldots,b_2)^v$$

In 2002, Nenov [56] presented a variety of significant results on the vertex Folkman number $F_v(a_1,\ldots,a_r;m-1)$. We begin with three of his preliminary theorems.

Theorem 54. Let $p \geq 3$ such that $F_v(2,2,p;p+1) \geq 2p+5$. Then, for each $t \geq 2$, $F_v(\underbrace{2,\ldots,2}_{t},p;t+p-1) \geq t+2p+3$.

Theorem 55. For positive integers $a_1,\ldots,a_r$, $p \geq 3$, and $m \geq p+2$, if $F_v(2,2,p;p+1) \geq 2p+5$, then $F_v(a_1,\ldots,a_r;m-1) \geq m+p+3$.

Remark 56. Nenov [58] also proved the special case of this theorem with $a_1 = \cdots = a_r = 3$, $r \geq 3$, showing that $F_v(3,\ldots,3;m-1) = 2r+7$ (see Theorem 37).

Theorem 57. [56] Let $m \geq 6$. Then, the following hold:

$$F_v(a_1,\ldots,a_r;m-1) = \begin{cases} m+6 & : p=3, \\ m+7 & : p=4 \end{cases}$$

Remark 58. For $p=4$, equality was shown by proving the upper and lower bounds of $F_v(a_1,\ldots,a_r;m-1)$. The upper bound of $F_v(a_1,\ldots,a_r;m-1 \geq m+7$ is a direct result from $F_v(2,2,4;5) = 13$ [51] and Theorem 55. The lower bound was proved with the critical Greenwood and Gleason graph shown in Figure 3.

4.2.3 $F_v(2_r;q$

The vertex Folkman number $F_v(\underbrace{2,\ldots,2}_{r};q=r-1)$ has received considerable attention in recent years [6, 60]. This problem is interesting because it is directly related to the chromatic number of a graph. In particular, it is well known that $G \to (2_r)^v$ if and only if $\chi(G) \geq r+1$. If less than $r+1$ colors could be used to color the vertices of G, then clearly there exists a color i that is not contained in an r-coloring of the vertices. We begin with some foundational results for this problem.

Theorem 59. [6]For a graph G with $\chi(G) \geq r+1$ and $\omega(G) \leq r$ that $|V(G)| \geq r+3$ and, furthermore, that $G = K_{r-3}+C_5$ is the only graph that witnesses equality in this bound.

Remark 60. This means that for positive integers $r \geq 2$ that $F_v(2_r;r+1) = r+3$ and $K_{r-3}+C_5$ is the only graph in $\mathcal{F}_v(2_r;r+1)$.

From Theorem 1 it follows that $F_v(2_r;r-1)$ exists if and only if $r \geq 4$. Similarly, $F_v(2_r;r-2)$ exists if and only if $r \geq 5$. Nenov extended these results in the following theorems.

Theorem 61. [46] $F_v(2_r;r-1) = r+7$ if $r \geq 8$.

These results were for $r \geq 4$ by Nenov [60], whose results are shown in the following theorems.

Theorem 62. [60] Let r be a positive integer such that $r \geq 4$. Then, the following hold:

1. $F_v(2_r; r-1) \geq r+7$;
2. $F_v(2_r; r-1) = r+7$ if $r \geq 6$;
3. $F_v(2_5; 4) \leq 16$.

Remark 63. $F_v(2_5; 4)$ is the only vertex Folkman number of this form whose value is not explicitly known.

Theorem 64. [60] Let $r \geq 5$ be a nonnegative integer. Then, the following hold:

1. $F_v(2_r; r-2) \geq r+9$;
2. $F_v(2_r; r-2) = r+9$ if $r \geq 8$

Remark 65. To date, the numbers $F_v(2_r; r-2)$ for $5 \leq r \leq 7$ are unknown.

Jensen and Royle [?] proved a similar result, shown in Theorem 66

Theorem 66. [24] $F_v(2_4; 3) = 22$.

Nenov [60] proves many interesting results for the upper bound of $F_v(2_r; q)$. Using a modified construction of the graph P, whose complement is shown in Figure 3, he was able to show new constructions that place a tighter upper bound on this vertex Folkman number. The result of this construction is summarized in Theorem 67

Theorem 67. Let r and s be non-negative integers and $r \geq 3s+6$, then $F_v(2_r; r-s-1) \leq r+2s+7$.

Remark 68. This is an immedidate result from the construction of the graph $\tilde{\text{P}}$= $K_{r-3s-6} + P + \underbrace{C_5 + \cdots + C_5}_{s}$. In particular, since $\chi(P\ ) = r+1$, $\omega(P\ ) = r-s-2$, $|V(P\ )| = r+2s+7$, $P \in \mathcal{F}_v(2_r; r-s-1)$ (where P is a subgraph of P), the Theorem immediately follows. Also, since $r \geq 3s+6$, it follows that $r-s-1 > 2$, which therefore means that $F_v(2_r; r-s-1)$ exists.

Nenov provided a similar construction to improve the upper bound of $F_v(2_r; r-s-2)$ for nonnegative integer s, captured in Theorem 69

Theorem 69. [60] Let r and s be nonnegative integers such that $r \geq 4s+8$, then $F_v(2_r; r-s-8) \leq r+2s+9$.

Remark 70. Similar to the proof of Theorem 67, a graph $Q\ = K_{r-3s-8} + Q + \underbrace{C_5 + \cdots + C_5}_{s}$, where Q is an $R(6,3)$ Ramsey graph such that $|V(Q)| = 17$, $\alpha(Q) = 2$ and $\omega(Q) = 5$ [25].

It is clear that $K_{r+1} \rightarrow (2_r)^v$ and $K_r \rightarrow (2_r)^v$, which therefore means that $F_v(2_r; q) = r+1$ if $q \geq r+2$. In 2012, Nenov [62] studied such numbers where $k \geq 1$, since $k \leq -2$ will not fall within this bound (i.e. $q < r+2$). As an immediate result from Folkman [14], it is clear that $F_v(2_r; q)$ exists if an only if $q \geq 3$. Therefore, for the following results on $F_v(2_r; r-k+1)$, it is required that $r \geq k+2$.

Theorem 71. [62] Let r and k be integers such that $-1 \leq k \leq 5$ and $r \geq k+2$. Then,

$$F_v(2_r; r-k-1) \geq r+2k+3,$$

and

$F_v(2_r; r-k-1) = r+2k+3$ if $k \in \{0,2,3,4,5\}$ and $r \geq 2k+2$ or $k \in \{-1,1\}$ and $r \geq 2k+3$.

Theorem 72. [62] Let $r \geq 8$ be a natural number. Then,

1. $F_v(2_r; r-5) \geq r+14$ and $F_v(2_r; r-5) = r+14$ if and only if $r \geq 13$;
2. $F_v(2_r; r-6) \geq r+16$ if $r \geq 9$ and $F_v(2_r; r-6) = r+16$ if $r \geq 15$;
3. $F_v(2_r; r-7) \geq r+17$, $r \geq 10$ and $F_v(2_r; r-7) = r+17$ if and only if $r \geq 16$;
4. $F_v(2_r; r-8) \geq r+18$, $r \geq 11$ and $F_v(2_r; r-8) = r+18$ if and only if $r \geq 17$;
5. $F_v(2_r; r-9) \geq r+20$, $f \geq 12$ and $F_v(2_r; r-9) = r+20$ if $r \geq 19$.

Theorem 73. [62] Let $r \geq 13$ be a natural number. Then,

1. $F_v(2_r; r-10) \geq r+21$ and $F_v(2_r; r-10) = r+21$ if $R(10,3) > 41$ and $r \geq 20$;
2. If $R(10,3) \leq 41$ then $F_v(2_r; r-10) \geq r+22$ and $F_v(2_r; r-10) = r+22$ if $r \geq 21$.

Remark 74. To date, the exact value of $R(10,3)$ is unknown. For more information, see the dynamic survey on small Ramsey numbers maintained by Radziszowski [66].

Theorem 75. [62] Let r and k be natural numbers such that $r \geq k+2$ and $k \geq 12$. Then,

1. $F_v(2_r; r-k+1) \geq r+k+11$;
2. If $k = 12$ and $r \geq 22$ then $F_v(2_r; r-11) = r+23$.

When $R < 2k+2$ and $1 \leq k \leq 5$, only the following vertex Folkman numbers are known.

1. $F_v(2,2,2;3) = 11$ [3, 40]
2. $F_v(2,2,2,2;4) = 22$ [24]
3. $F_v(2_r;4) = 11$ [47, 50]
4. $12 \leq F_v(2,2,2,2,2;4) \leq 16$ [60]

It is also interesting to note that if $k \geq 2$ then $|\mathcal{F}_v(2_r; r-1)| \geq 1$. Nenov [62] gives the example when $r \geq 8$, showing that $G = K_{r-8} + C_5 + C_5 + C_5 \in \mathcal{F}_v(2_r; r-1)$ and G is minimal.

4.2.4 Avoiding Smaller Cliques

Obtaining bounds on Folkman numbers becomes much more difficult as the forbidden clique size decreases. Luczak et al. [39] obtained the bound of $F_v(\underbrace{2,\ldots,2}_{r}; r+1-k) \leq r+2k+3$ when $0 \leq k \leq (r-2)/3$. This bound is exact when $k = 0, 1$.

For a graph G with girth $g(G) \geq 2p$ and $|V(G)| \geq 2m-1$, Luczak observed that $G^c \to (a_1, \ldots, a_r)^v$. Since any r-coloring of G will yield a vertex set S of $2a_i - 1$ vertices for some $i \in \{1, \ldots, r\}$. Since $g(G) \geq 2p$, it is clear that $G^c[S]$ does not contain a cycle, and thus is bipartite, meaning that $\alpha(G^c[S]) \leq \lceil |S|/s \rceil = a_i$. Therefore, if $\alpha(G^c[S]) < w$, then G does not contain a clique of size w, and therefore $G \in \mathcal{F}_v(a_1, \ldots, a_r; w)$ [39]. This observation led to the theorem by Luczak et al. [39] stating that if G is a graph such that $g(G) \geq 2p$, $\alpha(G) < l$, $w \geq l$, and $w - l + \frac{1}{2}|V(G)| \geq m$, then $G = K_{w-l} + G^c \in \mathcal{F}_v(a_1, \ldots, a_r; w)$.

Theorem 76. [39] Let $q = 2m - w \geq e^{e^{e^2}}$ and $B = 2q(\log\log q)/\log q + 2\log\log q(\log q)^{2p-1}$. If $w \geq B$ and $p \leq \log q/\log\log q$, then $F_v(a_1, \ldots, a_r; w) \leq q + B$.

Remark 77. This bound was obtained using a probabilistic construction with the Galois circulant graph $G(n,r)$.

Given that $F_v(a_1, \ldots, a_r; m) < F_v(a_1, \ldots, a_r; m-1)$, it might be intuitive to attempt to prove the existence of Folkman numbers using a construction based on recurrence relation between these two Folkman numbers. In fact, this is exactly what Łuczak et al. [39] did to study the most restrictive case of the vertex Folkman numbers (when $w = p+1$), as shown in the following Theorem.

Theorem 78. [39] For all $r \geq 2$ and $2 \leq a_1 \leq \cdots \leq a_r$, the following recurrence inequality holds:

$$F_v(a_1,\ldots,a_r;p+1) \leq 1 + (1+(r-1)(F_2-1)) \cdot F_1 + \binom{1+(r-1)(F_2-1)}{F_2} F_2,$$

where $F_1 = F_v(a_1-1,\ldots,a_r-1;p)$ and $F_2 = F_v(a_2,\ldots,a_r;p+1)$.

Remark 79. This type of construction is actually a modification of Folkman's original existential proof [14].

Corollary 80. [39] $F_v(k,l;l+1) \leq 2\sum_{i=0}^{k-1} \frac{l!}{(l-i)!} - 1$.

Remark 81. This was proved by induction on k using the fact that $F_v(a_1,\ldots,a_r;m) \leq p+m$, which implies that $F_v(2,l;l+1) \leq 2l+1 = 2\sum_{i=0}^{1} \frac{l!}{(l-i)!} - 1$ for all $l \geq 2$. When $k = l$, the upper bound of $2\sum_{i=0}^{k-1} \frac{l!}{(l-i)!} - 1$ collapses to $\lfloor 2k!(e-1) \rfloor - 1$, where e is the number of edges in the graph G. Also, this bound implies that $F_v(3,3;4) \leq 19$, which supports the result from Piwakowski et al. [64] that $F_v(3,3;4) = 14$.

4.2.5 Important Inequalities Vertex Folkman Numbers

Theorem 82. [48] For $p \geq 3$, $F(p,p) < \lfloor p!e \rfloor - 1$.

Few results in this topic are known. In 2007, Kolev [33] considered the product of two vertex Folkman numbers $F_v(a_1,\ldots,a_r;s+1)$ and $F_v(b_1,\ldots,b_r;t+1)$, for positive integers s and t. The product of these numbers was shown to bound $F_v(a_1,\ldots,a_r;st+1)$, meaning that $F_v(a_1,\ldots,a_r;st+1) \leq F_v(a_1,\ldots,a_r;s+1) \times F_v(b_1,\ldots,b_r;t+1)$. This result can be generalized to $F_v(\underbrace{kl,\ldots,kl}_{r};kl+1) \leq F_v(\underbrace{k,\ldots,k}_{r};k+1) \times F_v(\underbrace{l,\ldots,l}_{r};l+1)$ if we let $a_i = s = k$, $b_i = t = l$.

Nenov [59, 28] also showed that $F_v(p+1,p+1;p+2) \leq (p+1)F_v(p,p;p+1)$. Using the fact that $F_v(4,4;5) \leq 25$ [29], Kolev [33] showed that $F_v(p,p;p+1) \leq \frac{25}{24}p!$, where $p \geq 4$, which improved the upper bound of $F_v(p,p;p+1) \leq \lfloor 2p!(e-1) \rfloor - 1$ that was presented by Łuczak et al. in 2001 [39]. This result is obtained by induction on p.

Theorem 83. [30] Let $a_1 \leq \cdots \leq a_r, r \geq 2$ be positive integers and $a_r = b_1 + \cdots + b_s$, where b_i are positive integers too, and $b_i \geq a_{r-1}$, $i = 1,\ldots,s$, then

$$F_v(a_1,\ldots,a_r;a_r+1) \leq \sum_{i=1}^{s} F_v(a_1,\ldots,a_{r-1},b_i;b_i+1) \tag{4}$$

Many interesting results arise from this theorem. Such corollaries are listed below.

Corollary 84. [30] Let $p \geq 4$ and $p = 4k+l$, $0 \leq l \leq 3$. From Theorem 83 it is true that

$$F_v(3,p;p+1) \leq (k-1)F_v(3,4;5) + F_v(3,4+l;5+l) \tag{5}$$

$$F_v(2,2,p;p+1) \leq (k-1)F_v(2,2,4;5) + F_v(2,2,4+l;5+l) \tag{6}$$

Corollary 85. [30] Using Theorem 83, $F_v(3,p;p+1) \leq F_v(3,p-4;p-3) + F_v(3,4;5)$ and $F_v(2,2,p;p+1) \leq F_v(2,2,p-4;p-3) + F_v(2,2,4;5)$ for $p \geq 8$.

Kolev and Nenov [30] found further inequalities after examining $p = 5,6,7$ and $F_v(3,4;5) = 13$ [53].

Corollary 86. Let $p \geq 4$, then

$$F_v(3,p;p+1) \leq \frac{13p}{4} \text{ for } p \equiv 0 \mod 4$$

$$F_v(3,p;p+1) \leq \frac{13p+23}{4} \text{ for } p \equiv 1 \mod 4$$

$$F_v(3,p;p+1) \le \frac{13p+26}{4} \text{ for } p \equiv 2 \mod 4$$

$$F_v(3,p;p+1) \le \frac{13p+29}{4} \text{ for } p \equiv 3 \mod 4$$

In a similar fashion, when $p = 5$, Kolev and Nenov [30] used the equality $F_v(2,2,4;5) = 13$ [52], $F_v(2,2,6;7) \le 22$ [54], $F_v(2,2,7;8) \le 28$ [54].

Corollary 87. Let $p \ge$, then

$$F_v(2,2,p;p+1) \le \frac{13p}{4} \text{ for } p \equiv 0 \mod 4$$

$$F_v(2,2,p;p+1) \le \frac{13p+23}{4} \text{ for } p \equiv 1 \mod 4$$

$$F_v(2,2,p;p+1) \le \frac{13p+10}{4} \text{ for } p \equiv 2 \mod 4$$

$$F_v(2,2,p;p+1) \le \frac{13p+21}{4} \text{ for } p \equiv 3 \mod 4$$

Conjecture 88. Using $F_v(3,4;5) = 13$, Kolev and Nenov [30] conjectured that

$$F_v(3,p;p+1) \le \frac{13p}{4} \text{ for } p \ge 4, \tag{7}$$

Remark 89. Using $F_v(3,4;5) = 13$ (see Theorem (POINT BACK)), if 88 is true for $p = 5,6,7$ then it is also true for $p \ge 4$.

Conjecture 90.

$$F_v(2,2,p;p+1) \le \frac{13p}{4} \text{ for } p \ge 4. \tag{8}$$

Remark 91. Using $F_v(2,2,4;5) = 13$ (see Theorem (POINT BACK)), if 90 is true for $p = 5,6,7$ then it is also true for $p \ge 4$.

The problem of determining whether or not there exists an integer p such that $F_v(2,2,p;p+1) \ne F_v(3,p;p+1)$ is still open.

4.3 Edge and Vertex Folkman Number Connections

Given the Ramsey number $R(a_1,\dots,a_r)$, it is clear that $F_e(a_1,\dots,a_r;q)$ for all $q > R(a_1,\dots,a_r)$, simply because the Ramsey number already attains this lower bound on the size of the graph n. Given this relation, the following lemma emerges.

Lemma 92. [39] Let $R_i = R(a_1,\dots,a_{i-1},a_i-1,a_{i+1},\dots,a_r)$ for all $i \in \{1,2,\dots,r\}$. If $H \in \mathcal{F}_v(R_1,R_2,\dots,R_r;w)$, then $H + v \in \mathcal{F}_v(a_1,\dots,a_r;w+1)$.

Remark 93. Clearly, this enables one to derive bounds on edge Folkman numbers using known bounds on related vertex Folkman numbers that are increased by one.

Corollary 94. [39] For $k,l \ge 3$, let $M = \max\{R(k-1,l),R(k,l-1)\}$ and $m = \min\{R(k-1,l),R(k,l-1)\}$, then it is true that $F_e(k,l;M+2) \le 2\sum_{i=0}^{m-1} \frac{M!}{(M-i)!}$.

Remark 95. This bound collapses to $F_e(k,k;M+2) \le \lfloor 2M!(e-1)\rfloor$ when $k = l$. Furthermore, this is a very loose upper bound. The bound obtained when $k = l = 3$ is $F_e(3,3;5) \le 20$ is close to the true value of 15, but other values for k and l are much farther off. However, it is still very useful for closing the gap between the original existential question and true values.

5 Relationship with the Chromatic Number

If a graph G satisfies the condititition that $\chi(G) = k$ and $\omega(G) \leq$, then G is k-chromatic and K_{r+1}-free. It is well known that if $G \rightarrow (a_1, \ldots, a_r)^v$ then $\chi(G) \geq m$. A graph G is called an edge-critical k-chromatic graph if $\chi(G) = k$ and $\chi(G') < k$ for each proper subgraph $G' of G$ (i.e. $G' = G - v$ or $G' = G - e$). All edge critical k-chromatic graphs must be connected, $\chi G = k$, and $\chi(G - e) < k$ for all $e \in E(G)$. Similarly, a graph G is a vertex-critical k-chromatic if $\chi(G) = k$ and $\chi(G - v) < k$ for all $v \in V(G)$.

It has been proven that if $G \rightarrow (a_1, \ldots, a_r)$ then $\chi(G) \leq m$, where $m = \sum_{i=1}^{r}(a_i - 1) + 1$. In 1972, Lin [36] showed that $G \rightarrow (a_1, \ldots, a_r)^3 \Rightarrow \chi(G) \geq R(a_1, \ldots, a_r)$.

The value $f(G) = \chi(G) - \omega(G)$ has been shown to have very deep connections with Folkman numbers.

Theorem 96. [60] Let G be a graph such that $f(G) \leq 2$. Then, $|V(G)| \geq \chi(G) + 2f(G)$.

Corollary 97. Nenov went on to show that for a graph G such that $f(G) \leq 4$, $|V(G)| \leq \chi(G) + 2f(G)$.

Theorem 98. [60] Let $G \in \mathcal{F}_v(2_r; q)$, $q \geq 3$ and $|V(G)| = F_v(2_r; q)$, then

1. G is a vertex-critical $(r + 1)$-chromatic graph;
2. if $q < r + 3$ then $\omega(G) = q - 1$.

6 Asymptotic Bounds for Folkman Numbers

Dudek et al. [8] asked the general question of determining $F_v(r, \mathcal{G})$, the minimal order of a graph H such that $\omega(H) = \omega(\mathcal{G})$ and for every r-coloring of the vertices of H there exists a monochromatic, induced copy of $\mathcal{G}$. This is equivalent to finding a minimal graph $H \underbrace{\rightarrow}_{\text{ind}} (G)_b^r$

Clearly, the induced subgraph requirement places a tighter constraint on the structure of H. Critical results are captured in the following theorems.

Theorem 99. [8] $F_v(r, n, q = n + 1) = F_v(n, K_n) \leq cn^2(\log n)^4$ for some constant $c = C(r)$. Therefore, $F_v(r, n, q = n + 1) = F_v(n, K_n) = \mathcal{O}(n^2(\log n)^4)$.

Remark 100. This was proven with a construction for graphs H of order $Cn^2 \log^4 n, C = C(\alpha)$, such that $\omega(H) = n$ and for all induced subgraphs $H[V]$, where $V \subset V(H)$ and $|V| = \lfloor \alpha |V(H)| \rfloor$, $H[V]$ contians a copy of K_n, and thus avoids a clique of size $n + 1$. These graphs were randomly constructed from the vertex set $\mathcal{V}$ of projected planes $PG(2, q)$, where q is prime.

Theorem 101. [8] $F_v(r, n\lceil (2 + \epsilon)n \rceil) \leq Cn$ for a $r \in \mathbb{N}$, some arbitrarily small constant $\epsilon > 0$ and constant $C = C(r, \epsilon)$.

Remark 102. This is a direct result of Theorem 99, which arises when cliques of size bigger than $q = (2 + o(1))$ are forbidden. This result is also complementary to the results obtained by Łuczak et al. [39] and Kolev et al. [27], who found that $F_v(r, n, r(n - 1)) \leq r(n - 1) + n^2 + 1$ and $F_v(r, n, r(n - 1)) \leq r(n - 1) + 3n + 1$, respectively. The bound obtained by Kolev et al. is significantly tighter than that of Łuczak et al. as n tends towards infinity. The proof was shown by probabalistic construction from random graphs $G = G(m, 1 - \frac{c}{m})$, where $c = c(r, \epsilon)$, such that $\omega(G) < \frac{2 \log c}{c} m$ and every subset of vertices $U \subset V(G), |U| = \lceil \alpha m \rceil$ induces a clique of size at least $\frac{2 \log c}{(2+\epsilon)c} c$. These criteria were then used to show that

$$F(r, c, \lceil (2 + \epsilon)n \rceil) \leq F(r, n, \lceil \frac{2 \log c}{c} m \rceil) \leq m \leq \frac{(2 + \epsilon)c}{2 \log c} n.$$

The special two-color case of this vertex Folkman number, $F_v(2, n, n+1)$ was shown to be equal to $\mathcal{O}(n!)$ by Nenov [48]. This was an improvement on the upper bound obtained by Folkman [14], which was proven using an iterated power function.

Efforts to improve Folkman's theorem using the induced subgraph constraint and without controlling the forbidden clique size were started in 1991 by Brown et al. [1], in which they proved the following theorem.

Theorem 103. For every $r \in \mathbb{N}$ there exists constants C and c such that for every graph G of order n,

$$cn^2 \leq \max_G \left\{ \min_H \left\{ |V(H)| : H \underbrace{\rightarrow}_{\text{ind}} (G)^v_r \right\} \right\} \leq Cn^2 \log^2 n.$$

Remark 104. This is equivalent to saying

$$cn^2 \leq \max\{F(r, \mathcal{G})\} \leq Cn^2 \log^2 n.$$

Also, note that all logarithms are base e unless otherwise explicitly stated.

The order of graphs satisfying this induced arrow property was further constrained by Dudek et al. [8], who adding the additional constraint that $\omega(H) = \omega(G)$. In doing so, they were only able to achieve an upper bound of $Cn^3 \log^3 + n$, for some constant C, as shown in the following theorem.

Theorem 105. For a given $r \in \mathbb{N}$, there exists a constant $C = C(r)$ such that for every graph G of order n,

$$\min \left\{ |V(H)| : H \underbrace{\rightarrow}_{\text{ind}} (G)^v_r \text{ and } \omega(H) = \omega(G) \right\} \leq Cn^3 \log^3 n.$$

Theorem 106. [9] $F(r, \mathcal{G}) \leq \frac{cn^3}{\omega(\mathcal{G})}(\log n)^5$ for some constant $c = \mathcal{O}(r)$.

Remark 107. More details warranted.

6.1 Asymptotic Results for Hypergraphs

The induced Folkman number $F(r, \mathcal{G})$ of a k-uniform hypbergraph $\mathcal{G}$ of order n is the minimum order of a k-uniform hypergraph H such that $\omega(H) = \omega(\mathcal{G})$ and that for every r-coloring of the vertices of H there exists a monochromatic, induced copy of $\mathcal{G}$. Dudek et al. [10] showed that $F(r, \mathcal{G})$ for hypergraphs $\mathcal{G}$ is *almost* quadratic, with the bound $F(r, \mathcal{G}) \leq cn^2(\log n)^2$ for any k-uniform hypergraph $\mathcal{G}$ on n vertices. In doing so, Dudek et al. also showed that for every pair of positive integeers k, n there exists a constant c such that $F(r, \mathcal{G}) \leq cr^2$ for a hypergraph $\mathcal{G}$ on n vertices and any non-negative number of colors r. Dudek et al. [11] showed a special case of this bound with $\mathcal{G} = \mathcal{K}^k_n$, the complete hypergraph on n vertices which is k-uniform, shown in the following theorem.

Theorem 108. [11] $F(r, \mathcal{K}^k_n) \leq cr(\log r)^{\frac{1}{k-2}}$ for some constant $c = \mathcal{O}(k, n)$.

It may be intuitive to assume that hypergraphs with small clique numbers have linear or sub-quadratic Folkman numbers. However, Dudek et al. [10] have shown this is not the case with Theorem 109.

Theorem 109. [10] For all natural numbers $r \geq 1$ and $k \geq 3$ there are constants c and d such that for every n there exists a k-uniform hypergraph $\mathcal{G}$ of order n and clique number $\omega(\mathcal{G}) \leq d$ such that

$$F(r, \mathcal{G}) \geq cn^2 \frac{\overbrace{\log\log \ldots \log\log n}^{k-1}}{\underbrace{\log\log \cdots \log n}_{k-2}}.$$

This result can be restated in terms of the asymptotic value of r, as is done in Theorem 110.

Theorem 110. For every k and n there is a constant C such that for any k-uniform hypergraph $\mathcal{G}$ of order n and any number of colors r

$$F(r, \mathcal{G}) \leq Cr^2.$$

Remark 111. Mubayi and Dudek [11] showed that when $\mathcal{G}$ is the complete k-uniform hypergraph the following holds true:

$$F_v(r, \mathcal{K}_n^k) \leq Cr(\log r)^{\frac{1}{k-2}},$$

for a constant $C = C(k, n)$.

Dudek also went on to show results for 3-uniform ($k = 3$) hypergraphs, drawing on results for hypergraph Ramsey numbers $R_k(s, t)$, which is the minimum integer n such that every 2-coloring of $\mathcal{K}_n^k$ contains a monochromatic $\mathcal{K}_s^k$ or $\mathcal{K}_t^k$. In particular, using the result of Conlon, Fox, and Sudakov [5], which states that $R_e(4, t) \geq 2^{t \log t}$ for some constant c, Dudek found the result of Theorem 112.

Theorem 112. [10] $F(r, \mathcal{G}_n^3 = \mathcal{R}_{n/2}^3 \cup \bar{\mathcal{K}}_{n/2}^3) = \Theta(n^{2+o(1)})$.

Remark 113. Dudek12-2 finish - not sure about the asymptotics, more details warranted.

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