Revision on Maclaurin series and convergence

The Maclaurin series of a function $f(x)$ is calculated in a mechanical (formal) way - and gives us a very useful representation of a general function $(f(x))$ in terms of an infinite sum of powers of x.
As we know, differentiation and integrating x^n terms is very straightforward - and linear $(d/dx(f(x)+g(x)) = d/dx(f(x)) + d/dx(g(x)))$ over sums.

But equating a function $f(x)$ with its Maclaurin series is only valid **when the series converges**, which in general is some range of x, satisfying $-L < x < L$. This is the reason why finding the range of x for which the Maclaurin series converges is so important.

$$\boxed{f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k, \quad \text{when} \quad -L < x < L}$$

Or stated another way

Let $\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$ be the Maclaurin series of the function $f(x)$. Assume that the series **converges** for x in the range $-L < x < L$, then the Maclaurin series satisfies

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \begin{cases} \pm\infty, & \text{for } x < -L \\ f(x), & \text{for } -L < x < L \\ \pm\infty, & \text{fot } x > L \end{cases}$$

In general the endpoints of in the interval $x = -L$ and $x = L$ need to be tested in each particular case to determine if the series converges for these x.

A simple example is provided by the function $f(x) = 1/(1 - x)$. We have seen previously (see **next few** slides) that the Maclaurin series of $f(x)$ is $1 + x + x^2 + x^3 + \ldots$.. Now, from the function we may compute the value at $x = 2$, $f(2) = 1/(1 - 2) = -1$ and it is very clearly defined here. But from our knowledge of the sum to infinity of a geometric series (first term 1 and ratio x) we know that the series does not converge when $x = 2$, as this gives the following sum $1 + 2 + 2^2 + 2^3 + ...$ which obviously goes to infinity. To determine the range of x for which the series converges we use the **ratio test** to compute L and when it satisfies $L < 1$ (the condition necessary for convergence).

$$L = \lim_{n\to\infty} \left| x^n / x^{n-1} \right| = |x| < 1 \implies -1 < x < 1 \text{ for convergence.}$$

Therefore we can write

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + ..., \quad \text{when } -1 < x < 1$$

Some special functions have a Maclaurin series that converges to the function for all x (such as e^x and $\sin x$). Question 7.4 from HW 7 should now be attempted.

Maclaurin series for *1/(1-x)*

k	$f^k(x)$	$f^k(0)$
0	$f(x) = (1-x)^{-1}$	$(1-0)^{-1} = 1$
1	$f^1(x) = (1-x)^{-2}$	$(1-0)^{-2} = 1$
2	$f^2(x) = 2(1-x)^{-3}$	$2(1-0)^{-3} = 2$
3	$f^3(x) = 2 \times 3(1-x)^{-4}$	$6(1-0)^{-4} = 6$
4	$f^4(x) = 2 \times 3 \times 4(1-x)^{-5}$	$24(1-0)^{-5} = 24$

Note. Not all series converge for all x.

Consider the Maclaurin series for

$$f(x) = \frac{1}{1-x}$$

We can verify that

$$f^k(0) = k!$$

Maclaurin series for *1/(1-x)*

And therefore, the Maclaurin series for $f(x)$ is

$$\sum_{k=0}^{\infty} c_k x^k = \sum_{k=0}^{\infty} \frac{f^k(0)}{k!} x^k = \sum_{k=0}^{\infty} \frac{k!}{k!} x^k = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + ...,$$

Exercises

For each function, find the first 5 terms of the Maclaurin expansion, and the interval for x (given as $|x| < 1/C$) for which the series converges.

1. $\ln(1+x)$
2. $\cosh(x)$

For each function, find the first 2 NON-ZERO terms of the Maclaurin expansion,

1. $\ln(\cos x)$ **Answer**: $-\frac{1}{2}x^2 - \frac{1}{12}x^4$
2. $\frac{1}{2}\ln\left(\frac{1+1/x}{1-1/x}\right)$, assuming that $x > 1$. **Answer**: $\frac{1}{x} + \frac{1}{3x^3}$

k	$f^k(x)$	$f^k(0)$
0	$f(x) = \ln(1+x)$	$\ln(1+0) = 0$
1	$f^1(x) = (1+x)^{-1}$	$(1+0)^{-1} = 1$
2	$f^2(x) = -(1+x)^{-2}$	$-(1+0)^{-2} = -1$
3	$f^3(x) = 2(1+x)^{-3}$	$2(1+0)^{-3} = 2$
4	$f^4(x) = -2 \times 3(1+x)^{-4}$	$-6(1+0)^{-4} = -6$

Consider the Maclaurin series for

$$f(x) = \ln(1+x)$$

We can verify that by completing the table,

$$f^k(0) = (-1)^{k-1}(k-1)! \quad \text{for} \quad k = 1, 2, 3, 4, \ldots.$$

$$\text{Maclaurin series:} \quad x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} \ldots$$

Using the ratio test, the series converges to $\ln(1+x)$ when $-1 < x < 1$.

k	$f^k(x)$	$f^k(0)$
0	$f(x) = \cosh(x)$	$\cosh(0) = 1$
1	$f^1(x) = \sinh(x)$	$\sinh(0) = 0$
2	$f^2(x) = \cosh(x)$	$\cosh(0) = 1$
3	$f^3(x) = \sinh(x)$	$\sinh(0) = 0$
4	$f^4(x) = \cosh(x)$	$\cosh(0) = 1$

Consider the Maclaurin series for

$$f(x) = \cosh(x)$$

We can verify that by completing the table,

$f^k(0) = 1$ for $k =$ even, and $f^k(0) = 0$ when k odd

Maclaurin series: $1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \dfrac{x^6}{6!} + \dfrac{x^8}{8!} + \ldots$

Using the ratio test, the series converges to $\cosh(x)$ for all x.

$\boldsymbol{k}$	$\boldsymbol{f^k(x)}$	$\boldsymbol{f^k(0)}$
0	$f(x) = \ln(1-x)$	$\ln(1) = 0$
1	$f^1(x) = -\frac{1}{1-x}$	-1
2	$f^2(x) = -\frac{1}{(1-x)^2}$	-1
3	$f^3(x) = -\frac{2}{(1-x)^3}$	-2
4	$f^4(x) = -\frac{6}{(1-x)^4}$	-6

HW 7 Q4

Recall the Maclaurin expansion is $\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$, so the first 4 terms are (from table above)

$$\ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - ..., \text{ for } -1 < x < 1$$

with k-th term

$$-\frac{1}{k}x^k.$$

To evaluate the sum,

$$\frac{1}{4} - \frac{1}{2(4)^2} + \frac{1}{3(4)^3} - ... + \frac{(-1)^{k+1}}{k(4)^k} + ...$$

we notice that the k-th terms of the expression given and our series expansion for $\ln(1-x)$ are almost the same. Therefore we should try to find a value for x in our Maclaurin expansion that gives the same terms as the expression shown. To do this we equate k-th terms and solve for x

$$-\frac{1}{k}x^k = \frac{1}{k}(-1)^{k+1}\left(\frac{1}{4}\right)^k$$

$$x^k = (-1)^{k+2}\left(\frac{1}{4}\right)^k \quad \text{but} \quad (-1)^{k+2} = (-1)^k(-1)^2 = (-1)^k$$

$$x^k = (-1)^k\left(\frac{1}{4}\right)^k$$

$$x^k = \left(-1\frac{1}{4}\right)^k = \left(-\frac{1}{4}\right)^k \implies x = -1/4$$

Therefore, the sum must be equal to using $x = -\frac{1}{4}$ in the Maclaurin expansion for $\ln(1-x)$. The crucial point is that the Maclaurin series converges to $f(x) = \ln(1-x)$ when $|x| < 1$. i.e. for this range of x the Maclaurin series is **equal** to the function

$$\ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - ..., \text{when } -1 < x < 1$$

and, $x = -1/4$ is in this range of x. Therefore, the sum is equal to

$$\begin{aligned}
&\frac{1}{4} - \frac{1}{2(4)^2} + \frac{1}{3(4)^3} - \frac{1}{4(4)^4} \ldots \\
&= -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - \ldots \Big|_{x=-1/4} \\
&= \ln(1-x)|_{x=-1/4} \\
&= \ln(1-(-1/4)) = \ln(5/4)
\end{aligned}$$

where the notation $|_{x=-1/4}$ means the expression is to be evaluated using the value $x = -1/4$.