Consider all 100 digit numbers, i.e. those between 0 to $10^{100} - 1$, inclusive. For each number, take the product of non-zero digits (treat the product of digits of 0 as 1), and sum across all the numbers. What's the last digit?

GCD

The greatest common divisor of two numbers, $gcd(a, b)$, is the largest positive integer that is a divisor of both numbers. For example, $gcd(8, 12) = 4$ since $8/4 = 2$ and $12/4 = 3$. Pairs of numbers for which $gcd(a, b) = 1$ are called *coprime*.

Geometrically, an $a \times b$ rectangle grid can be covered with square tiles of side length c only if c is a common divisor of of a and b

Computing the last m digits of a^x

Numbers raised to a power follow a pattern in their digits. For instance, in decimal to get the last m digits a^x we want to compute

$$\begin{aligned} a^x \mod 10 & \quad \text{last digit} \\ a^x \mod 100 = a^x \mod 10^2 & \quad \text{last 2 digits} \\ a^x \mod 10^m & \quad \text{last m digits} \end{aligned}$$

There are known patterns in numbers raised to a power behave. Number that end in

The last digit of powers of 1 is always	1
The last digits of powers of 2 repeat in a cycle of	$4, 8, 6, 2$
The last digits of powers of 3 repeat in a cycle of	$9, 7, 1, 3$
The last digits of powers of 4 repeat in a cycle of	$6, 4$
The last digit of powers of 5 is always	5
The last digit of powers of 6 is always	6
The last digits of powers of 7 repeat in a cycle of	$9, 3, 1, 7$
The last digits of powers of 8 repeat in a cycle of	$4, 2, 6, 8$
The last digits of powers of 9 repeat in a cycle of	$1, 9$

For example, $21^3 = 21 \times 441 = 9261$. Consider a number with the last digit 2, such as 12. The last digit of each number each time 12 is raised to a power is shown.

$12^2 = 14\mathbf{4}, 12^3 = 172\mathbf{8}, 12^4 = 2073\mathbf{6}, 12^5 = 24883\mathbf{2}, 12^6 = 298598\mathbf{4}, 12^7 = 3583180\mathbf{8}, 12^8 = 42998169\mathbf{6}$.

The last digit follows a periodic pattern: $4, 8, 6, 2, 4, 8, 6, 2, ...$ Table above shows the pattern, and the table below summarises the period for each last digit.

Digit	Period
$0, 1, 5, 6$	1
$2, 3, 7, 8$	4
$4, 9$	2

To simplify modular calculations we can use Euler's totient function, or the Chinese remainder theorem.

Euler's totient

Often denoted by ϕ the totient function counts the number of natural numbers less than or equal to n that are coprime to n. For example $\phi(15) = 8$, since starting with

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$$

We eliminate numbers that are multiples of 3 and 5 since $15 = 3 \times 5$. This leaves

$$\{1, 2, \cancel{3}, 4, \cancel{5}, \cancel{6}, 7, 8, \cancel{9}, \cancel{10}, 11, \cancel{12}, 13, 14, \cancel{15}\}$$

leaving 8 numbers. For $\phi(n)$, each number, a, that remains in the depleted set satisfies $gcd(a, n) = 1$.

To compute $\phi(n)$ we can express n in prime factors.

$$\begin{aligned}\phi(n) = \phi\left(p_1^{f1} ... p_k^{fk}\right) &= \prod_{i=1}^{k} \phi\left(p_i^{fi}\right) \\ &= \prod_{i=1}^{k} \left(p_i^{fi} - p_i^{fi-1}\right)\end{aligned}$$

We can use Eulers theorem to reduce large exponents

If $gcd(a, n) = 1$ and $\phi(n)$ denotes Eulers totient, then

$$a^{\phi(n)} = 1 \mod n$$

Example find last 2 digits of 33^{42}.

We need to compute $33^{42} \mod 100$ to find the last two digits. In the above, $n = 100$. So

$$\phi(100) = \phi\left(2^2 5^2\right) = (2^2 - 2^1)(5^2 - 5^1) = (2)(20) = 40$$

Need to check if $33 = 3 * 11$ is coprime to 100, which it is, therefore $gcd(33, 100) = 1$. Therefore

$$\begin{aligned}33^{42} = 33^{40+2} = 33^{40} 33^2 = 33^{\phi(100)} 33^2 \mod 100 &= 33^2 \mod 100 \\ &= 1089 \mod 100 = 89\end{aligned}$$

So last 2 digits are 89.

Using the Chinese remainder

When finding the last m digits of a^x, the idea is to express $10^m = 2^m 5^m$ and then find

$$u = a^x \mod 2^m, \quad v = a^x \mod 5^m$$

and combine these to find the last m digits by solving for y satisfying

$$y = u \mod 2^m, \quad y = v \mod 5^m$$

Example find last 2 digits of 34^{540}

$$34^{540} \mod 4 = (2*17)^{540} = 2^{540}17^{540} = 0 \mod 4$$
$$34^{540} \mod 25 = 9^{540} \mod 25 = 9^{\phi(25)*27} = 1 \mod 25$$

So $u = 0$ and $v = 1$. Now we solve

$$y = 0 \mod 4, \quad y = 1 \mod 25$$

So the answer is $y = 76 = 3*25+1 = 4*19$. So last 2 digits are 76.

Multinomial expansions

The expansion

$$(x_1 + x_2 + ... + x_m)^n = \sum_{k_1+...+k_m=n} \binom{n}{k_1, k_2, ..., k_m} \prod_{i=1}^{m} x_i^{k_i}$$

where $\binom{n}{k_1, k_2, ..., k_m} = \dfrac{n!}{k_1!k_2!...k_m!}$,

and each $k_i \geq 0$

Answering the question

We are asked to find the sum of the product of the digits over all 0 to 100 digit numbers. We are told to treat 0 as 1 for the purposes of multiplication. Scaling down, consider 3 digit numbers and let the operator D compute the product of the digits, according to the above.

$$D(101) = 1*1*1 = 1^3$$
$$D(050) = 1*5*1 = 1^2 5^1$$
$$D(722) = 7*2*2 = 2^2 7^1$$

The possible digits are 0 to 9, and it is clear each such product may be written as

$$1^{k_1}1^{k_2}2^{k_3}...9^{k_{10}}$$

Clearly the number of such combinations are simply the numerous ways of partitioning 100 into 10 non-negative summands. Therefore,

$$\sum_{k_1+...+k_{10}=100} \binom{100}{k_1, k_2, ..., k_{10}} \prod_{i=1}^{10} x_i^{k_i}, \quad \text{where } x_1 = 1, x_2 = 1, x_3 = 2, ..., x_{10} = 9$$

By the multinomial theorem, this simply equals

$$\left(\sum_{i=1}^{10} x_i\right)^{100} = (1+1+2+3+...+9)^{100} = 46^{100}$$

The last digit is clearly 6, as the last digit of 46 is 6 (recall from the table above, the digits 0,1,5,6 have period 1 when raised to powers). In order to show algebraicly,

$$
\begin{aligned}
&46 = 2 * 23 \implies 46^{100} \mod 10 = 2^{100}23^{100} \mod 10 \\
&\text{since } \phi(10) = 4 \quad \text{and } 100 = 4 * 25 \\
&2^{100}23^{100} = (2^4)^{25}(23^4)^{25} = (2^4)^{25} \mod 10 \quad \text{by Euler} \\
&2^4 = 16 = 6 \mod 10
\end{aligned}
$$

Therefore the last digit is 6.