MATH6105 Complex numbers: class questions

Jan 2017

In these notes we will make use of the following:

The complex conjugate of a complex number $z = x + iy$, is written

$$\bar{z} = x - iy, \quad \text{the sign of the imaginary part is flipped (Ch.5,s2)} \tag{1}$$

The trigonometric (or polar) representation of a complex number z

$$z = r(\cos\theta + i\sin\theta), \quad r > 0, \quad \text{(Ch.5,s2 of notes)} \tag{2}$$

The exponential representation of a complex number z

$$z = re^{i\theta}, \quad r > 0, \quad \text{(Ch.5,s5 of notes)} \tag{3}$$

Euler's identity

$$e^{i\theta} = \cos\theta + i\sin\theta, \quad \text{(Ch.5,s4 of notes)} \tag{4}$$

and, De Moivre's theorem

$$(\cos + i\sin\theta)^n = \cos n\theta + i\sin n\theta \quad \text{(Ch.5,s9 of notes)} \tag{5}$$

1 Argument and modulus

A complex number is a number that consists of two parts: a real part and a so-called imaginary part. The imaginary part is written as a multiple of the imaginary number $i = \sqrt{-1}$, by convention. We often use z or w to denote complex numbers.

Complex numbers can be written using three different formats:

- Cartesian
- Trigonometric (polar form)
- Exponential (sometimes called argument-modulus form)

The Cartesian form of a complex number z (also called the rectangular form) is simply the real and imaginary parts of the number written in decimal notation,

$$\boxed{z = a + ib, \quad a, b \text{ a real numbers }, i = \sqrt{-1}}$$

For example, $z = 2.5 + 0.4i$, or $z = \pi - \sqrt{3}i$ are complex numbers written in Cartesian form. This form is often the most convenient for plotting complex numbers in the complex plane, which consists of two axes: a horizontal axis to plot the real coordinate and a vertical axis to plot the imaginary coordinate.

Real numbers (represented on the real number line) are 1-dimensional. Since complex numbers are 2-dimensional (we need 2 pieces of information to describe them), we must use a plane (which is 2-dimensional) to represent them. We refer to this plane as either the *complex plane* or equivalently as the *Argand diagram*.

Imagine selecting a complex number as a point in the plane, and then drawing a line from the origin to the point. The 'fixed position' from which we measure the angle θ is the *positive real axis* - which is the right-hand side of the horizontal axis in the complex plane.

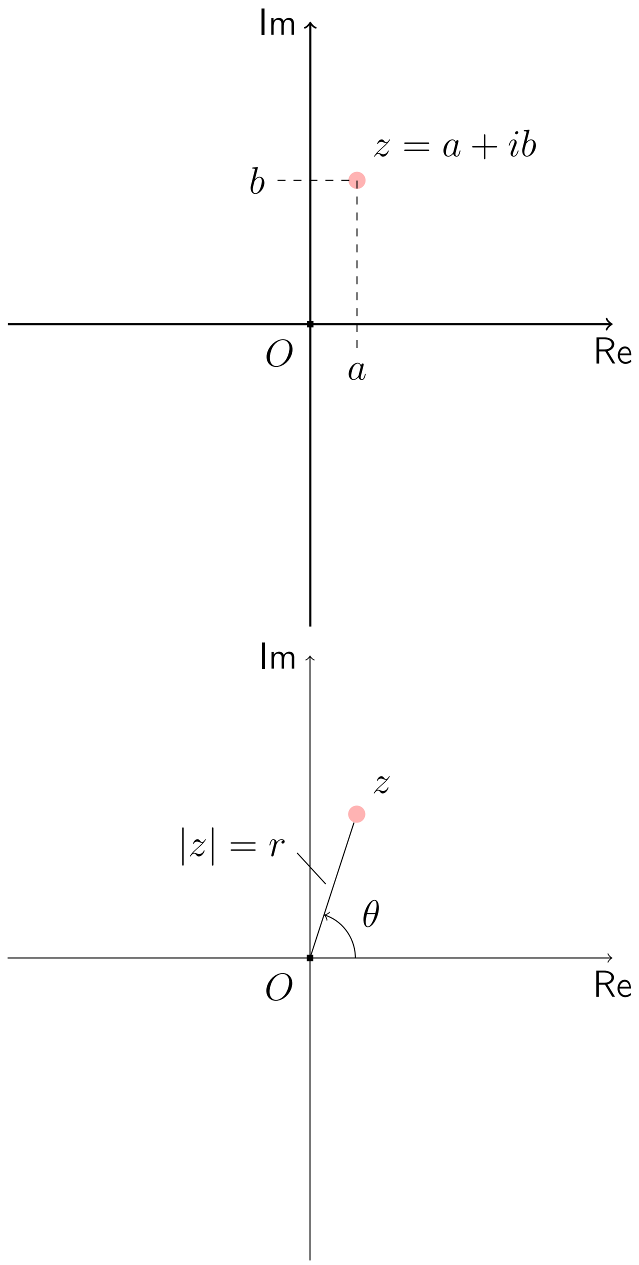


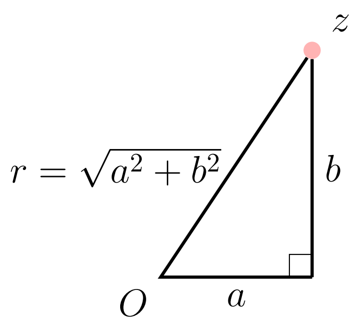
Figure 1: Top plot shows the complex number $z = a + ib$ as a point in the complex plane. The lower plot indicates how we define the modulus (r) and the argument (θ) for the same z, by drawing a line from the origin O to z and measuring the angle from the positive real axis to the line Oz.

The trigonometric (polar) and exponential forms require a little more notation, which will introduce now.

For the complex number $z = a + ib$ we define the *modulus*, r, as

$$\boxed{r = |z| = \sqrt{a^2 + b^2}}$$

Using Pythagoras, we can easily see that the modulus of a complex number is simply the length of the line from the origin O to z in the complex plane (see Figure 1).



The *argument* (or principle argument) of a complex number z is denoted by $\arg(z)$, and is defined as the angle θ between the positive real axis and the line Oz (see Figure 1). By convention we measure the angle as *positive* when θ is measure **anti-clockwise** from the positive real axis (that is, the complex number lies in the upper half-plane), and *negative* when θ is measure **clockwise** from the positive real axis (the complex number lies in the lower half-plane). Figure 2A shows $\arg z_1$ and $\arg z_4$ in the complex plane, where z_1 and z_4 are quadrants 1 and 4 respectively.

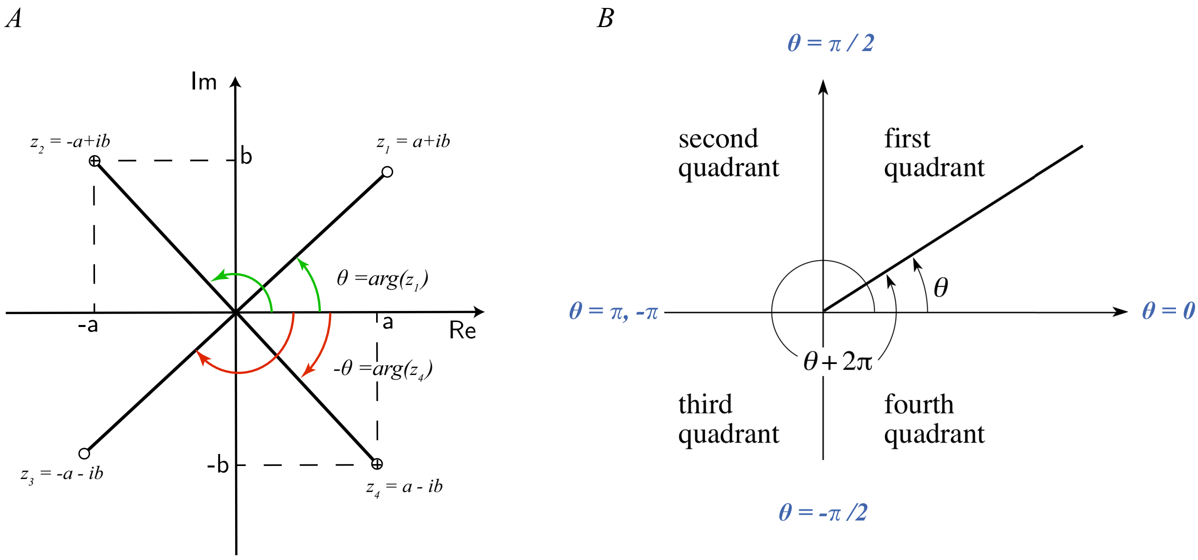


Figure 2: The complex plane

In order to compute $\arg z$, we need to understand the $\arctan$ function and its range.

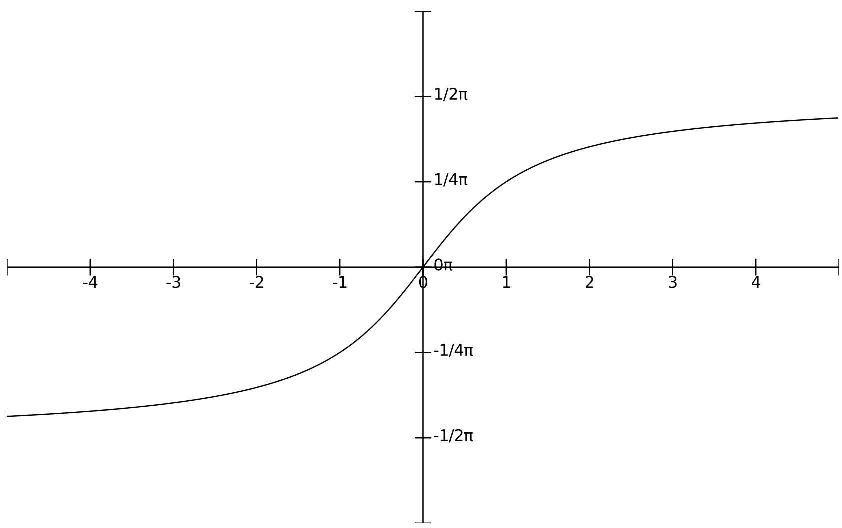


Figure 3: The principle value of the $\arctan$. Its range is $(-\pi/2, \pi/2)$. Notice that this is an odd function.

Formally, we define $\arg z = \arctan\left(\frac{Im(z)}{Re(z)}\right)$, but we must remember to adjust any numerical answer to take account of the quadrant where the complex number lies. From Figure 2, we can see that for any complex number z,

$$-\pi < \arg z \leq \pi$$

But, we can see that if z lies in quadrant 2 or 3 (as z_2 and z_3 do in Figure 2A) then $\arg z$ will be numerically larger than $\pi/2$. The full definition of $\arg(z) = \arg(x + iy)$ is therefore

$$\boxed{\arg z = \begin{cases} \arctan\left(Im(z)/Re(z)\right) \quad \text{when} & Re(z) > 0 \\ \pi + \arctan\left(Im(z)/Re(z)\right) \quad \text{when} & Im(z) \geq 0, Re(z) < 0 \\ \arctan\left(Im(z)/Re(z)\right) - \pi \quad \text{when} & Im(z) < 0, Re(z) < 0 \\ \pi/2 \quad \text{when} & Im(z) > 0, Re(z) = 0 \\ -\pi/2 \quad \text{when} & Im(z) < 0, Re(z) = 0 \\ \text{undefined} \quad \text{when} & Im(z) = 0, Re(z) = 0 \end{cases}}$$

Examples

1. Find $\arg(-1+\sqrt{3}i)$, and $\left|-1+\sqrt{3}i\right|$

Answer:

Note, the complex number is in quadrant 2.

Since $Re(-1+\sqrt{3}i) = -1 < 0$ and $Im(-1+\sqrt{3}i) = \sqrt{3} > 0$

So,

$\arg(-1+\sqrt{3}i) = \pi + \arctan(\sqrt{3}/(-1)) = \pi + \arctan(-\sqrt{3})$

$= \pi - \arctan(\sqrt{3})$ (Why? See below for this step)

$= \pi - (\pi/3) = 2\pi/3$

For $\left|-1+\sqrt{3}i\right| = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2$

2. Find $\arg(-\sqrt{3}-i)$ and $\left|-\sqrt{3}-i\right|$.

Answer:

Note, the complex number is in quadrant 3.

Since $Re(-\sqrt{3}-i) = -\sqrt{3} < 0$ and $Im(-\sqrt{3}-i) = -1 < 0$

So,

$\arg(-\sqrt{3}-i) = \arctan((-1)/(-\sqrt{3})) - \pi = \arctan(1/\sqrt{3}) - \pi$

$= (\pi/6) - \pi$

$= -5\pi/6$

For $\left|-\sqrt{3}-i\right| = \sqrt{(-\sqrt{3})^2 + (-1)^2} = \sqrt{3+1} = \sqrt{4} = 2$

In Example 1 we used $\arctan(-\sqrt{3}) = -\arctan(\sqrt{3})$. This is true for any **odd function**. Look again at Figure 3 and it is easy to convince ourselves that $\arctan$ is indeed an odd function. Recall the following, A function, f, is an odd function if and only if

$$f(-x) = -f(x) \quad \text{for any real } x$$

(see printed notes on functions)

To answer any question on arguments, you should know the following special values of $\tan$ (as a minimum)

$$\tan(0) = 0, \;\; \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}, \;\; \tan\left(\frac{\pi}{4}\right) = 1, \;\; \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$$

$$\lim_{x\to\pi/2} \tan(x) = \infty, \tan(\pi) = 0, \tan(x + k\pi) = \tan(x), \text{ for any } k \in \mathbb{Z}$$

But we do not have to use horizontal and vertical coordinates to describe a complex number. Any point in a plane can be uniquely determined by

- specifying how far the point is from the origin (call this r) together with,
- specifying the angle (call this θ) the line joining the origin to the point in the plane makes with some fixed position.

We can therefore use r and θ to represent complex numbers - and this gives rise to the trigonometric (polar) form of a complex number (see Figure 1) .

2 Polar and exponential forms of a complex number

Recall Figure 1, reproduced below

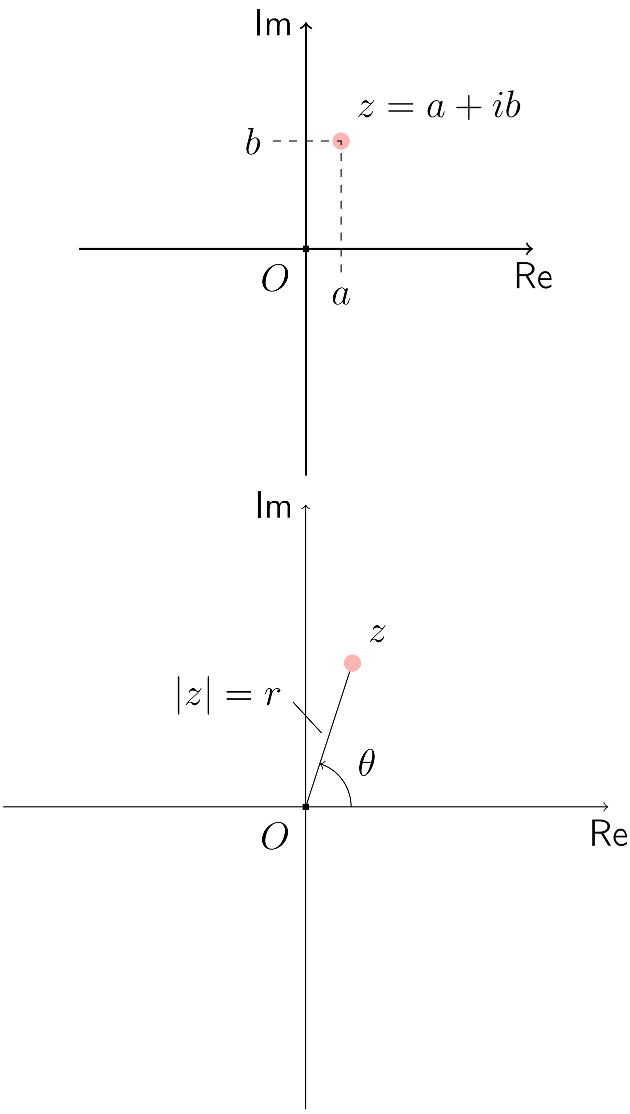
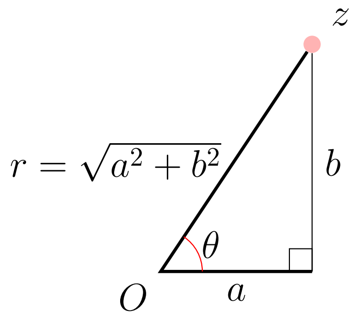


Figure 4: Cartesian, polar and exponential forms of the complex number z.

If we imagine moving the complex number z around the plane, the values of r and θ change accordingly. This means specifying a complex number using r and θ is well-defined.

Lets imagine, we are given the values of r and θ for a complex number z. How do we write this back in Cartesian form $(a+bi)$? Doing this exercise will help us understand where the trigonometric form comes from.



From standard trigonometry (SOH/CAH/TOA) we have

$$\cos\theta = \frac{a}{r} \implies r\cos\theta = a$$

and

$$\sin\theta = \frac{b}{r} \implies r\sin\theta = b$$

This process is sometimes called 'resolving' into rectangular components. Putting this together, we see that we have worked out the Cartesian co-ordinates of z, given r and θ, namely

$$z = a + bi = r\cos\theta + r\sin\theta i = r(\cos\theta + i\sin\theta)$$

This is known as the *trigonometric form* of a complex number. By using Euler's identity,

$$e^{i\theta} = \cos\theta + i\sin\theta$$

we can obtain the exponential form of a complex number, from the trigonometric form

$$\begin{aligned} z =& r(\cos\theta + i\sin\theta) \\ =& re^{i\theta} \end{aligned}$$

In summary we have seen how to define and use concepts such as modulus (length), argument (angle), and the three forms of complex number (Cartesian (rectangular), polar (trigonometric) and exponential). We have also seen how geometric concepts underpin many of the definitions and attributes of complex numbers.

3 Use of De Moivre's theorem: multiple angles to powers

This is covered in Ch.5 s9-10.2 in the notes.

De Moivre's theorem allows us to translate between powers of trigonometric functions and multiple angle expressions.

We will make use of some shorthand $c = \cos\theta$, $s = \sin\theta$, to express multiple angles in terms of powers of trigonometric functions. The procedure we follow is

- We expand the left-hand side of Equation. (5) using the binomial expansion (if you do not know these, see the notes at the end of this document)
- Group real and imaginary terms
- Equate real part (for $\cos n\theta$) and imaginary part (for $\sin n\theta$)

3.1 Worked example 1

➔ Example: Find $\cos 3\theta$, $\sin 3\theta$ in terms of powers of single angles

❒ $(\cos 3\theta + i\sin 3\theta) = (c + is)^3$

❒ Binomial coefficients: $1\;3\;3\;1$

❒ $(c + is)^3 = 1c^3 + 3c^2(is) + 3c(is)^2 + 1(is)^3$

❒ $(c + is)^3 = c^3 - 3cs^2 + i(3c^2s - s^3)$

❒ Equate real and imaginary parts
$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$
$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$

4 Use of De Moivre's theorem: powers to multiple angles

Switching from powers of trigonometric functions to multiple angles is useful for integration.

The key results in this section are the formulas,

$$\cos k\theta = \frac{e^{ik\theta} + e^{-ik\theta}}{2} = \frac{1}{2}\left(z^k + \frac{1}{z^k}\right), \tag{6}$$

$$\sin k\theta = \frac{e^{ik\theta} - e^{-ik\theta}}{2} = \frac{1}{2i}\left(z^k - \frac{1}{z^k}\right). \tag{7}$$

Valid for $k = 0, 1, 2, \ldots$

First, lets see the details of the $k = 1$ case.

4.1 Worked example 2

➔ Example: Let $z = \cos\theta + i\sin\theta$. Show $z + z^{-1} = 2\cos\theta$ and $z - z^{-1} = 2i\sin\theta$. (Recall: z^{-1} means $1/z$)

❒ Use this general result: $z\bar{z} = r^2$, where $r = |z|$ and $\bar{z}$ is the conjugate of z (Ch5.s2)

❒ $z\bar{z} = (\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta) = \cos^2\theta + \sin^2\theta = 1$

❒ Therefore,

$$\frac{1}{z} = \frac{\bar{z}}{r^2} = \frac{\bar{z}}{1} = \bar{z} = \cos\theta - i\sin\theta$$

- Hence, $z+z^{-1} = (\cos\theta + i\sin\theta) + (\cos\theta - i\sin\theta) = 2\cos\theta$
- And, $z - z^{-1} = (\cos\theta + i\sin\theta) - (\cos\theta - i\sin\theta) = 2i\sin\theta$

➔ Another proof using Euler's identity directly. Write $z = e^{i\theta}$

- Then, $e^{-i\theta} = e^{i(-\theta)} = \cos(-\theta) + i\sin(-\theta)$
- But $\cos(-\theta) = \cos(\theta)$ since cos is an even function. And $\sin(-\theta) = -\sin(\theta)$, as sin is an odd function (Ch5.s2).
- So, $e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos(\theta) - i\sin(\theta)$
- Also, since
$$e^{-i\theta} = \frac{1}{e^{i\theta}} = \frac{1}{z}$$
- Therefore
$$z^{-1} = \frac{1}{z} = \cos(\theta) - i\sin(\theta)$$
- Hence, $z+z^{-1} = (\cos\theta + i\sin\theta) + (\cos\theta - i\sin\theta) = 2\cos\theta$
- And, $z - z^{-1} = (\cos\theta + i\sin\theta) - (\cos\theta - i\sin\theta) = 2i\sin\theta$

For the general case when, again let $z = \cos\theta + i\sin\theta$. Then

$(\cos\theta + i\sin\theta)^k = z^k = e^{ik\theta} = \cos k\theta + i\sin k\theta$ using De Moivre's

$(\cos\theta + i\sin\theta)^{-k} = \dfrac{1}{z^k} = e^{-ik\theta} = \cos(-k\theta) + i\sin(-k\theta) = \cos(k\theta) - i\sin(k\theta)$

Therefore,

$$z^k + \frac{1}{z^k} = \cos k\theta + i\sin k\theta + \cos k\theta - i\sin k\theta = 2\cos k\theta,$$
$$z^k - \frac{1}{z^k} = \cos k\theta + i\sin k\theta - \cos k\theta + i\sin k\theta = 2i\sin k\theta.$$

For all $k = 0, 1, 2, 3, ...$

4.2 Worked example 3

➔ Express $\cos^4\theta$ in multiple angles

➔ We will use this result:

$$\cos k\theta = \frac{1}{2}\left(z^k + \frac{1}{z^k}\right)$$

❒

$$\cos^n\theta = (\cos\theta)^n = \frac{1}{2^n}\left(z + \frac{1}{z}\right)^n$$

❒ With $n = 4$,Expand RHS, using binomial coefficients

$$\frac{1}{2^4}\left(z + \frac{1}{z}\right)^4 = \frac{1}{2^4}\left(z^4 + 4z^3\frac{1}{z} + 6z^2\frac{1}{z^2} + 4z\frac{1}{z^3} + \frac{1}{z^4}\right)$$

❒ Simplify and group terms in $z^k + 1/z^k$ pairwise.

$$\begin{aligned}\frac{1}{2^4}\left(z + \frac{1}{z}\right)^4 &= \frac{1}{2^4}\left[z^4 + 4z^2 + 6 + 4\frac{1}{z^2} + \frac{1}{z^4}\right] \\ &= \frac{1}{2^4}\left[\left(z^4 + \frac{1}{z^4}\right) + 4\left(z^2 + \frac{1}{z^2}\right) + 6\right]\end{aligned}$$

❒ To get

$$\cos^4\theta = \frac{1}{2^4}\left(z + \frac{1}{z}\right)^4 = \frac{1}{2^4}\left[2\cos 4\theta + 8\cos 2\theta + 6\right]$$

4.3 Worked example 4

➔ Example: Find $\cos^2\theta\sin^3\theta$ in terms of sines of multiple angles.

➔ We will use

$$\cos k\theta = \frac{1}{2}\left(z^k + \frac{1}{z^k}\right),$$

and

$$\sin k\theta = \frac{1}{2i}\left(z^k - \frac{1}{z^k}\right),$$

❒ Let $z = \cos\theta + i\sin\theta$. Write $\cos^2\theta\sin^3\theta$ as

$$\begin{aligned}
\cos^2\theta\sin^3\theta &= \frac{1}{2^2}\left(z+z^{-1}\right)^2\frac{1}{(2i)^3}\left(z-z^{-1}\right)^3\\
&= \frac{-1}{2^5 i}\left[(z+z^{-1})(z-z^{-1})\right]^2(z-z^{-1}), \text{ since } i^3=-i\\
&= \frac{-1}{2^5 i}(z^2-z^{-2})^2(z-z^{-1})\\
&= \frac{-1}{2^5 i}(z^4-2+z^{-4})(z-z^{-1})\\
&= \frac{-1}{2^5 i}(z^5-z^3-2z+2z^{-1}+z^{-3}-z^{-5})\\
&= \frac{-1}{2^5 i}\left((z^5-z^{-5})-(z^3-z^{-3})-2(z-z^{-1})\right)\\
&= \frac{1}{2^4}\left(\frac{1}{2i}2(z-z^{-1})+\frac{1}{2i}(z^3-z^{-3})-\frac{1}{2i}(z^5-z^{-5})\right)\\
&= \frac{1}{16}(2\sin\theta+\sin 3\theta-\sin 5\theta)
\end{aligned}$$

5 Roots of unity: solutions of $z^n = 1$

This is covered in Ch.5 Section 11 in the notes.

We are told that for any $n = 1, 2, 3$, there are 'exactly n distinct complex solutions' to the equation

$$z^n = 1. \tag{8}$$

Here is a bit more detail on where the n solutions come from, and how to find them.

Our aim will be to determine the modulus, and angles, of the solutions, in the complex plane.

5.1 Modulus

Assume z_s satisfies Equation. (8), then $z_s^n = 1$. Recall that we can write any complex number in *polar form* (Ch.5,s2), so let us do that here

$$z_s = r(\cos\theta + i\sin\theta), \quad \text{where} \quad r > 0, \quad \text{some } \theta \in \mathbb{R} \tag{9}$$

- Remember that a complex number can be represented in the complex plane (or Argand diagram, same thing) as a point with *horizontal coordinate* $\mathrm{Re}(z_s)$ and vertical coordinate $\mathrm{Im}(z_s)$.

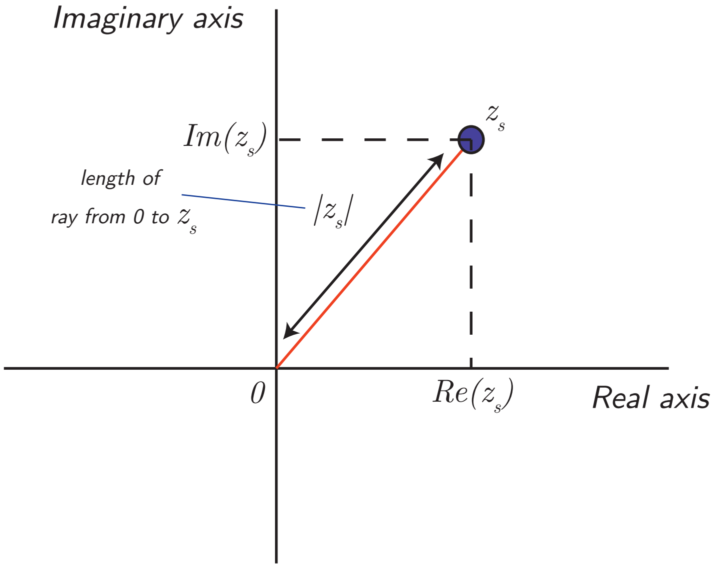


Figure 5: z_s in the complex plane

- The length of the line segment (known as a ray) extending from $0 = 0 + i0$ to z_s is given by the modulus of the complex number, $|z_s|$.

Lets compute the moduls of $|z_s|$, using the polar form in Equation. (9):

$$\begin{aligned}
|z_s| &= |r(\cos\theta + i\sin\theta)| \\
&= |r||\cos\theta + i\sin\theta| \\
&= |r|\sqrt{\cos^2\theta + \sin^2\theta} \\
&= |r| \\
&= r \quad \text{since} \quad r > 0 \quad \text{and} \quad \cos^2\theta + \sin^2\theta = 1 \quad \text{for any } \theta.
\end{aligned}$$

And now, lets compute $|z_s^n|$. We use De Moivre's theorem in Ch.5,s9 in the notes.

$$\begin{aligned}
|z_s^n| &= |\{r(\cos\theta + i\sin\theta)\}^n| \\
&= |r^n(\cos\theta + i\sin\theta)^n| \\
&= |r^n(\cos n\theta + i\sin n\theta)| \quad \text{using De Moivre's Theorem} \\
&= r^n|(\cos n\theta + i\sin n\theta)| \quad \text{since} \quad r^n > 0 \\
&= r^n(\cos^2 n\theta + \sin^2 n\theta) \\
&= r^n \quad \text{using the identity} \quad \cos^2 + \sin^2 = 1.
\end{aligned}$$

Using Equation (8), we know $|z_s^n| = |1| = 1$. Solving $r^n = 1$, gives $r = 1$, because r is a real number and $r > 0$.

Because $r = 1$, we can therefore say that any solution, z_s, to Equation (8) **is on the unit circle (a circle with radius 1) in the complex plane**.

NOTE: To make things easier, we could have used the exponential form of a complex number $z_s = re^{i\theta}$ from the beginning. Do as exercise.

We have demonstrated that all solutions, z_s, to the equation $z^n = 1$ must lie on the unit circle in the complex place - and so it MUST have the form

$$z_s = \cos\theta + i\sin\theta \quad \text{for some } \theta, \tag{10}$$

or, equivalently using Euler's identity (Ch.5,s4)

$$z_s = e^{i\theta} \quad \text{for some } \theta. \tag{11}$$

5.2 Angles (argument)

We have determined any solution to Equation. (8) has the form given by equation (10). We use De Moivre's again to find the angles.

Since $z_s = \cos\theta + i\sin\theta$,

$$\begin{aligned} z_s^n &= (\cos\theta + i\sin\theta)^n \\ &= \cos n\theta + i\sin n\theta \quad \text{using De Moivre's theorem.} \end{aligned}$$

Therefore, we have reduced the problem of solving the equation

$$z^n = 1,$$

to determining the values of θ for which,

$$\cos n\theta + i\sin n\theta = 1,$$

which we can now solve by comparing real and imaginary parts.

Comparing imaginary parts:

$$\sin n\theta = 0 \implies n\theta = k\pi \quad \text{for k an integer,}$$

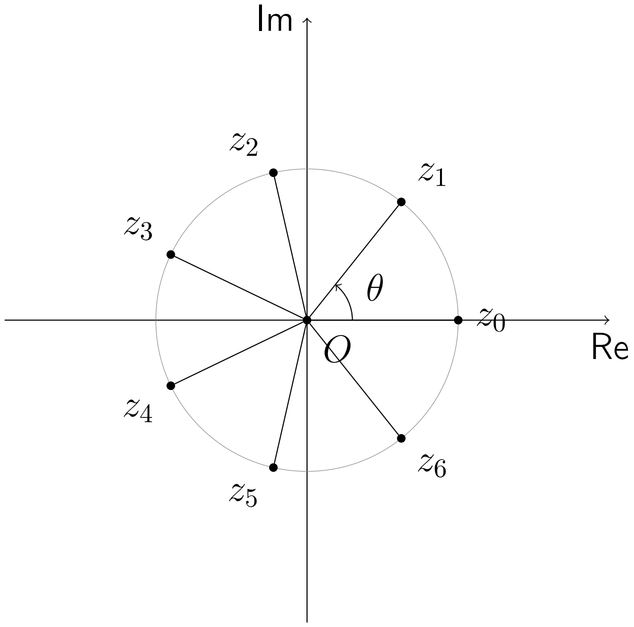
from our knowledge of the graph of sin `http://google.com/#q=sinx`.
Comparing real parts:

$$\cos n\theta = 1 \implies n\theta = k2\pi \quad \text{for k an integer.}$$

from our knowledge of the graph of cos `http://google.com/#q=cosx`.
To satisfy both conditions, we therefore need to take $n\theta = k2\pi$, giving

$$\theta = \frac{2k\pi}{n} \tag{12}$$

Recall that one whole revolution of a circle is 2π radians. This means, if we were to take $k \geq n$, solutions represented in the complex plane would overlap and be repeated. But we are told that there are **exactly** n **distinct** solutions.
We therefore take $k = 0, 1, 2, ..., (n-1)$, and these integers will generate our n **distinct** solutions, each separated by an angle of $\frac{2\pi}{n}$ when plotted in the complex plane.



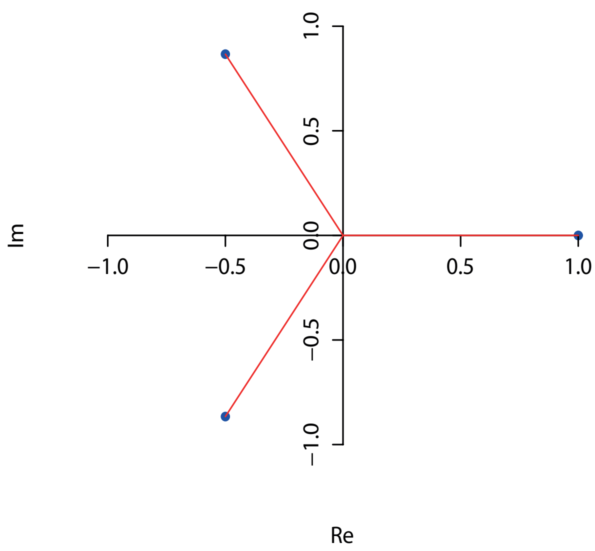
We have deduced that the n distinct complex solutions to $z^n = 1$ lie on the unit circle, and are equally spaced with an angle of $\frac{2\pi}{n}$.

Using the exponential form of a complex number, we can write these n solutions as

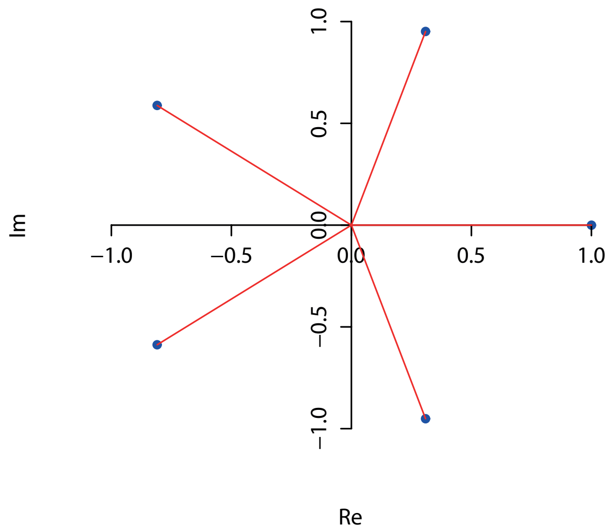
$$z_k = e^{i2\pi k/n} \quad \text{for} \quad k = 0, 1, 2, 3, ..., (n-1). \tag{13}$$

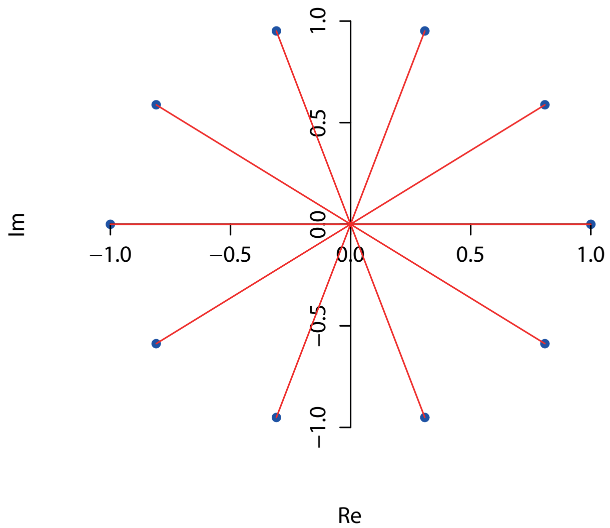
Above, we can see the solutions to $z^7 = 1$, plotted in the complex plane, with $\theta = \frac{2\pi}{7}$.

solutions of z^3=1, seperated by angles of $2\pi/3$



solutions of z^5=1, seperated by angles of $2\pi/5$





solutions of z^{10}=1, seperated by angles of $2\pi/10$

5.3 When complex numbers are multiplied: add the arguments

This follows from Ch.5,s2 (p.4) in the notes.

Let z and w be two complex numbers. We write them first in polar form, and then in exponential form.

$z = r_z(\cos\theta + i\sin\theta)$ and $w = r_w(\cos\phi + i\sin\phi)$. Multiplying z and w, we obtain

$$\begin{aligned}
zw &= r_z(\cos\theta + i\sin\theta)r_w(\cos\phi + i\sin\phi)\\
&= r_z e^{i\theta} r_w e^{i\phi}\\
&= r_z r_w e^{i(\theta+\phi)}\\
&= r_z r_w(\cos(\theta+\phi) + i\sin(\theta+\phi)) \quad \text{back in polar form.}
\end{aligned}$$

Therefore the modulus of the number resulting from the multiplication is the product of the two moduli, while the new argument is the sum of the two original arguments.

Returning to the roots of the equation $z^n = 1$ (roots of unity given by Equation. (13)), we notice a simple pattern to the n solutions (z_0 to z_{n-1}), with an arbitrary solution represented by z_k. Namely,

- $z_0 = 1$

- $z_1 = e^{i2\pi/n} = \omega$. This is a definition for ω.
- By combining the observation concerning multiplication of complex numbers, and the addition of the angles, we can show

- $z_k = \omega z_{k-1}$. If we apply this relation to z_{k-1}, we see

- $z_k = \omega z_{k-1} = \omega\omega z_{k-2} = \omega^2 z_{k-2}$

- If we reapply the above to z_{k-2} and then to z_{k-3}, and so on, until we reach z_0, we obtain

- $z_k = \omega^k z_0 = \omega^k \quad$ for $\quad k = 0, 1, ..., (n-1) \quad$ since $z_0 = 1$

5.4 Sum of the n roots of unity

Consider the equation $z^n = 1$. This implies $z^n - 1 = 0$. But we can write (multiply out the brackets to see)

$$z^n - 1 = (z-1)(z^{n-1} + z^{n-2} + ... + z^2 + z + 1) = 0. \tag{14}$$

Remembering that the distinct solutions to $z^n = 1$ can be written as $\{1, \omega, \omega^2, ..., \omega^{n-1}\}$, with $\omega = e^{i2\pi/n}$ and because these are distinct, $\omega \neq 1$.
Therefore, substituting ω into Equation. (14) we obtain

$$(\omega - 1)(\omega^{n-1} + \omega^{n-2} + ... + \omega^2 + \omega + 1) = 0 \tag{15}$$

Since $\omega \neq 1$, this means

$$\omega^{n-1} + \omega^{n-2} + ... + \omega^2 + \omega + 1 = 0. \tag{16}$$

Therefore the **sum of all n complex roots of unity are equal to zero**.

In summary,

The n distinct solutions to the equation $z^n = 1$, are given by the complex numbers

$$e^{i2k\pi/n} \quad \text{for} \quad k = 0, 1, 2, ...(n-1).$$

Or, in terms of $\omega = e^{i2\pi/n}$,

$$1, \omega, \omega^2, \omega^3, ..., \omega^{n-1}.$$

The sum of all n distinct roots of unity sum to zero,

$$\omega^{n-1} + \omega^{n-2} + ... + \omega^2 + \omega + 1 = 0.$$

5.5 Worked example 5

Find the distinct solutions satisfying $z^3 = 1$.

We know the solutions will be on the unit circle in the complex plane because the right-hand-side of this equation is 1.
Let $z = e^{i\theta}$ then substituting in to the equation we obtain

$$\begin{aligned}(e^{i\theta})^3 &= e^{i3\theta} \\ &= \cos 3\theta + i \sin 3\theta \quad \text{(using Euler's identity)} \\ &= 1 \quad \text{(using the right hand side of the equation)}\end{aligned}$$

Comparing the real and imaginary parts, we deduce that $\cos 3\theta = 1 \implies 3\theta = 2k\pi$ and $\sin 3\theta = 0 \implies 3\theta = k\pi$, for all integers k. To satisfy both, we therefore take $\theta = 2k\pi/3$.
Let us calculate some of these solutions.

- $k = 0$ gives the solution $z = \cos(0) + i\sin(0) = 1 + 0i = 1$.
- $k = 1$ gives the solution $z = \cos(2\pi/3) + i\sin(2\pi/3) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$
- $k = 2$ gives the solution $z = \cos(4\pi/3) + i\sin(4\pi/3) = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

We know that the equation has exactly 3 distinct solutions, so we can stop there. But just for information, let us compute a few more solutions

- $k = 3$ give the solution $z = \cos(2\pi) + i\sin(2\pi) = 1 + 0i = 1$.
- $k = 4$ gives the solution $z = \cos(8\pi/3) + i\sin(8\pi/3) = \cos(2\pi/3 + 2\pi) + i\sin(2\pi/3 + 2\pi) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

So, the solutions are cyclic, due to the periodicity of the trigonometric functions. Hence the 3 distinct complex solutions to $z^n = 1$ are

$$\{1, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i\}.$$

In class, it was stated that the solutions can be written as $\{1, \omega, \omega^2\}$, where $\omega = e^{i2\pi/3}$. We can see this is true by simply applying Euler's identity

- $\omega = e^{i2\pi/3} = \cos(2\pi/3) + i\sin(2\pi/3)$, the $k = 1$ solution.
- $\omega^2 = e^{i4\pi/3} = \cos(4\pi/3) + i\sin(4\pi/3)$, the $k = 2$ solution.

For the avoidance of any doubt - let us compute ω^2 explicitly, using $\omega = e^{i2\pi/3} = \cos(2\pi/3) + i\sin(2\pi/3)$

$$\begin{aligned}
\omega^2 = \omega\omega &= \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) \\
&= \left(\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right) + i^2\frac{\sqrt{3}}{2}\frac{\sqrt{3}}{2} - 2\frac{1}{2}\frac{\sqrt{3}}{2}i\right) \\
&= \left(\frac{1}{4} - \frac{3}{4} - \frac{\sqrt{3}}{2}i\right) \\
&= \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)
\end{aligned}$$

So, the solutions to $z^3 = 1$ can be written as

$$\{1, \omega, \omega^2\}, \quad \text{where} \quad \omega = e^{i2\pi/3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

5.6 General case: Finding the roots of $z^n = w$, with $w \neq 0$ a complex number

We use the exponential form to represent complex numbers. Let $z = re^{i\theta}$, and $w = Re^{i\phi}$.

Throughout we should keep in mind that w is a *given* complex number, such as $(1+2i)$ or $3e^{i5\pi/8}$, etc.

We begin by substituting in our exponential form for z into the equation we wish to solve

$$\begin{aligned} z^n &= \left(re^{i\theta}\right)^n \\ &= r^n e^{in\theta} \\ &= w \\ &= Re^{i\phi} \end{aligned}$$

By taking the absolute value (modulus) of both sides, we deduce: $r^n = R$, with $R > 0$ and real, and so we simply take the n-th root of R to obtain

$$r = \sqrt[n]{R} = R^{\frac{1}{n}}. \tag{17}$$

Comparing arguments we find

$$n\theta = \phi + 2k\pi \quad \text{because of the periodicity of} \cos, \sin, \tag{18}$$

And therefore,

$$\theta = \frac{\phi}{n} + \frac{2k\pi}{n}. \tag{19}$$

Putting this together, the general solution to $z^n = w$, is

$$\begin{aligned} z_k &= \sqrt[n]{R} e^{i\left(\frac{\phi}{n} + \frac{2k\pi}{n}\right)} \\ &= \sqrt[n]{R} e^{i\frac{\phi}{n}} \omega^k, \end{aligned}$$

where, as before, $\omega = e^{i2k\pi/n}$, and $k = 0, 1, 2, ..., (n-1)$.

5.7 Worked example 6

Find all distinct solutions to $z^4 = 16e^{i4\pi/3}$

Let $z = re^{i\theta}$. According to our formula for the general solution, $R = 16$ and $\phi = 4\pi/3$. Immediately, we can take $r = \sqrt[4]{16} = 16^{1/4} = 2$.
Our next task is to compute ω,

$$\omega = e^{i2\pi/4} = \cos(\pi/2) + i\sin(\pi/2) = 0 + i = i, \tag{20}$$

in this case $\omega = i$. Therefore, the general solution for $k = 0, 1, 2, 3$ is,

$$z_k = 2e^{i\pi/3}\omega^k = 2e^{i\pi/3}i^k. \tag{21}$$

If we wish, we can write these out explicitly, which may aid plotting the complex numbers in the complex plane.

$$z_0 = 2e^{i\pi/3}i^0 = 2(\cos(\pi/3) + i\sin(\pi/3)) = 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = (1 + i\sqrt{3})$$

$$z_1 = 2e^{i\pi/3}i^1 = 2i\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = 2\left(-\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = (-\sqrt{3} + i)$$

$$z_2 = 2e^{i\pi/3}i^2 = 2(-\cos(\pi/3) - i\sin(\pi/3)) = 2\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = (-1 - \sqrt{3}i)$$

$$z_3 = 2e^{i\pi/3}i^3 = 2i\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = 2\left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right) = (\sqrt{3} - i)$$

5.8 Worked example 7

Find all distinct solutions to $z^5 = -32$

Let $z = re^{i\theta}$. The complex number, w is of the form $w = 32(-1 + 0i)$, which we recognise as $w = 32(\cos\pi + i\sin\pi) = 32e^{i\pi}$(remember we are writing w in the exponential form, $Re^{i\phi}$, where $R > 0$). According to our formula, $R = 32 = 2^5$ and $\phi = \pi$. Immediately, we can take $r = \sqrt[5]{32} = 2$.

Our next task is to compute ω,

$$\omega = e^{i2\pi/5} = \cos(2\pi/5) + i\sin(2\pi/5) = \frac{-1+\sqrt{5}}{4} + \frac{\sqrt{10+2\sqrt{5}}}{4}i, \tag{22}$$

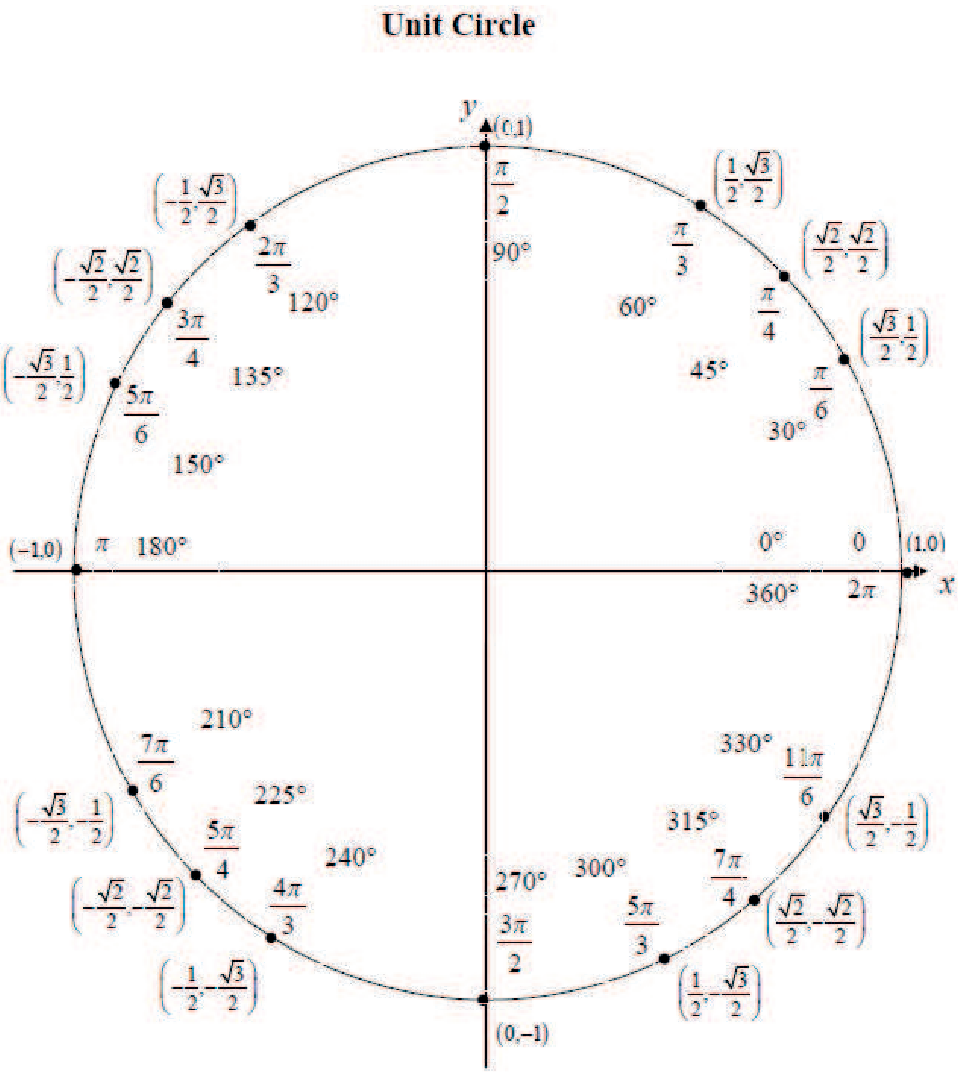
Therefore, the general solution for $k = 0, 1, 2, 3, 4$ is,

$$z_k = 2e^{i\pi/5}\omega^k = 2e^{i\pi/5}e^{i2k\pi/5} = 2e^{i\pi(2k+1)/5}. \tag{23}$$

If we wish, we can write these out explicitly, which may aid plotting the complex numbers in the complex plane - using the below values.

$$\cos(\pi/5) = \frac{1+\sqrt{5}}{4} \quad \text{and} \quad \sin^2(\pi/5) = 1 - \cos^2(\pi/5) \tag{24}$$

6 Values for $\cos$ and $\sin$ around the unit circle



For any ordered pair on the unit circle (x, y) : $\cos\theta = x$ and $\sin\theta = y$

Example

$$\cos\left(\frac{5\pi}{3}\right) = \frac{1}{2} \qquad \sin\left(\frac{5\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

Figure 6: $\cos$ and $\sin$ values around the unit circle

7 Binomial coefficients

Pascal's triangle gives the coefficients of a binomial expansion. Each entry in the triangle is the sum of the two directly above it. Although it is generally known as Pascal's, similar devices were known in China (and elsewhere) hundreds of years before the birth of Pascal.

Starting from row 0, each row, n, gives the binomial coefficients for the expansion $(a+b)^n$.

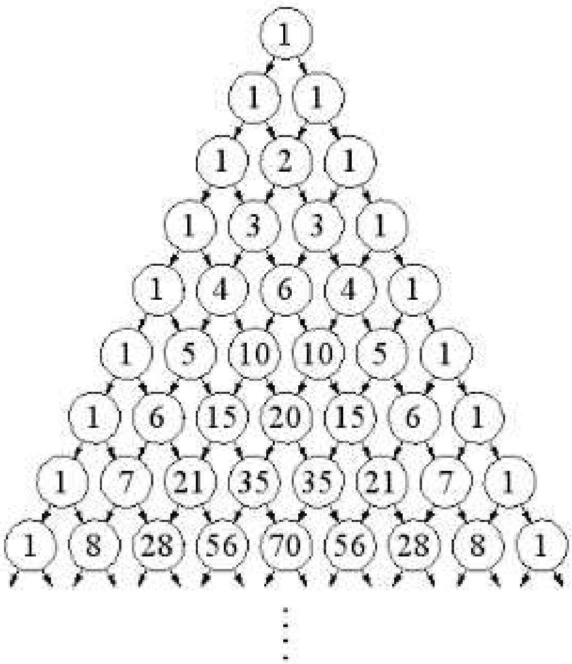


Figure 7: Pascal's triangle