C3 June 2014 Exam paper - answers and explanations

Q1 Curve C has equation $y = f(x)$,

$$f(x) = \frac{4x+1}{x-2}, \quad x > 2$$

Show that:

a)

$$f'(x) = \frac{-9}{(x-2)^2}$$

Given P is a point such that $f'(x) = -1$

b) Find the coordinates of P

a)

We can use the quotient rule: $\frac{vu'-uv'}{v^2}$ to differentiate $\frac{u}{v}$.

$$f'(x) = \frac{(x-2)(4) - (4x+1)(1)}{(x-2)^2}$$

$$= \frac{4x-8-4x-1}{(x-2)^2} = \frac{-9}{(x-2)^2}$$

b) x coordinate of P satisfies $f'(x) = -1$. So solve

$$f'(x) = \frac{-9}{(x-2)^2} = -1$$
$$\implies 9 = (x-2)^2$$
$$\implies \pm 3 = x - 2$$
$$\implies x = 2 \pm 3 \quad \text{but told } x > 2$$
$$\implies x = 2 + 3 = 5$$

Now sub back into to $f(x)$ to find y coordinate

$$f(5) = \frac{(4)(5)+1}{5-2} = frac213 = 7$$
$$\implies P = (5, 7)$$

Q2

a)

$$\begin{aligned}
&2\ln(2x+1) - 10 = 0 \\
&\implies \ln(2x+1) = \frac{10}{5} = 2 \\
&\implies 2x + 1 = e^5 \\
&\implies x = \frac{e^5 - 1}{2}
\end{aligned}$$

b)

$$\begin{aligned}
&3^x e^{4x} = e^7 \\
&\implies (3e^4)^x = e^7 \\
&\implies \ln(3e^4)^x) = 7\ln(e) = 7 \\
&\implies x\ln(3e^4) = 7 \\
&\implies x = \frac{7}{\ln(3e^4)} \\
&\implies x = \frac{7}{\ln(3) + 4\ln(e)} \\
&\implies x = \frac{7}{\ln(3) + 4}
\end{aligned}$$

Q3 a) verify (means plug in)

$$8y\tan(2y) = 8(\pi/8)\tan(2\pi/8) = \pi\tan(\pi/4) = \pi$$

Since $\tan(\pi/4) = 1$. So $(\pi, \pi/8)$ is on curve

b) Implicit differentiation. Use d/dx on both sides, and use chain rule for expressions with y in them.

$$\begin{aligned} x = 8y\tan(2y) \implies \frac{d}{dx}(x) &= \frac{d}{dx}(8y\tan(2y)) \\ \implies 1 &= (\tan(2y))(8\frac{dy}{dx}) + 8y(2)\frac{dy}{dx}\sec^2(2y) \\ \implies 1 &= \frac{dy}{dx}\left(8\tan(2y) + 16y\sec^2(2y)\right) \end{aligned}$$

So, rearranging

$$\frac{dy}{dx} = \frac{1}{(8\tan(2y) + 16y\sec^2(2y))}$$

$$= \frac{1}{(8\tan(2y) + 16y(\tan^2(2y) + 1))}$$

So the gradient at $P = (\pi, \pi/8)$ is

$$\left.\frac{dy}{dx}\right|_P = \frac{1}{(8\tan(\pi/4) + 16(\pi/8)(\tan^2(\pi/4) + 1))} = \frac{1}{8 + 4\pi}$$

Now use the straight line form $y - y_1 = m(x - x_1)$ where (x_1, y_1) is a point in the line.

$$\begin{aligned} y - \frac{\pi}{8} &= \frac{1}{8 + 4\pi}(x - \pi) \\ \implies (8y - \pi)(2 + \pi) &= 2x - 2\pi \\ \implies 16y + 8\pi y - \cancel{2\pi} - \pi^2 &= 2x - \cancel{2\pi} \\ \implies 4(\pi + 2)y &= x + \frac{\pi^2}{2} \end{aligned}$$

So $a = 4(\pi + 2)$ and $b = \pi^2/2$.

Q4 As per hand-drawn diagrams. Take each transformation one at a time, start from those transformation nearest to the function f itself.

Q5

$$g(x) = \frac{x}{x+3} + \frac{3(2x+1)}{x^2+x-6}, \quad x > 3$$

Show that:

a)

$$g(x) = \frac{x+1}{x-2}$$

b) Find the range of g

c) Find the exact value of a for which $g(a) = g^{-1}(a)$

a) We first recognise $x^2 + x - 6 = (x+3)(x-2)$. Now we put both fractions over a common denominator

$$g(x) = \frac{x}{x+3} + \frac{3(2x+1)}{x^2+x-6}$$

$$= \frac{x(x-2)}{(x+3)(x-2)} + \frac{3(2x+1)}{(x+3)(x-2)}$$

$$= \frac{x(x-2)+3(2x+1)}{(x+3)(x-2)}$$

$$= \frac{x^2-2x+6x+3}{(x+3)(x-2)} \quad \text{this factorises}$$

$$= \frac{(x+1)\cancel{(x+3)}}{\cancel{(x+3)}(x-2)} = \frac{x+1}{x-2}$$

b) The range of a function is the set of values the function (y value) can take. Here $y = g(x)$. If $x = 3$, then $g(x) = 4$. If we use a little trick to re-write the fraction

$$g(x) = \frac{x+1}{x-2} = \frac{x-2+3}{x-2} = \frac{x-2}{x-2} + \frac{3}{x-2} = 1 + \frac{3}{x-2}$$

then since $x > 3$, $g(x) > 1$ and it is clear that as x increases, $g(x)$ decreases. Therefore, the range of $g(x)$ is given by the interval

$$1 < g(x) < 4 \quad \text{for all } x > 3$$

c) Let

$$y = \frac{x+1}{x-2}$$

Then rearranging for $x = ...$

$$\begin{aligned}
&y(x-2) = (x+1) \\
\implies &yx - 2y - x - 1 = 0 \\
\implies &x(y-1) = 2y + 1 \\
\implies &x = \frac{2y+1}{y-1}
\end{aligned}$$

$$\implies g^{-1}(x) = \frac{2x+1}{x-1} \quad \text{remember to replace } y \text{ with } x$$

We can now equate $g(x)$ and $g^{-1}(x)$

$$\frac{x+1}{x-2} = \frac{2x+1}{x-1}$$

$$\begin{aligned}
&\implies (x+1)(x-1) = (2x+1)(x-2) \\
&\implies x^2 - 1 = 2x^2 - 4x + x - 2 \\
&\implies x^2 - 3x - 1 = 0 \quad \text{this does not factorise}
\end{aligned}$$

$$\implies x = \frac{3 \pm \sqrt{9+4}}{2}$$

$$\implies x = \frac{3 + \sqrt{13}}{2} \quad \text{as } x > 3$$

Q6 a) have to show $y(2.1)$ and $y(2.2)$ have different signs.

$$y(2.1) = -0.22, \quad y(2.2) = 0.546$$

So the curve must cross the x-axis between $2.1 < x < 2.2$.
b) Calculate dy/dx.

$$\frac{dy}{dx} = -2x\sin(1/2x^2) + 3x^2 - 3$$

At R $dy/dx = 0$, so solve.

$$\begin{aligned} 0 &= -2x\sin(1/2x^2) + 3x^2 - 3 \\ 3 + 2x\sin(1/2x^2) &= 3x^2 \\ \sqrt{1 + \frac{2}{3}x\sin(1/2x^2)} &= x \end{aligned}$$

c) We use the iterative formula

$$x_{n+1} = \sqrt{1 + \frac{2}{3}x_n \sin(1/2x_n^2)}$$

With $x_0 = 1.3$ to compute x_1 and x_2. The only thing to remember is to use the EXACT x_n values when computing x_{n+1}. Rounding x_2 to compute x_3 is a common place where people drop marks. Use your calculators memory - or write down the values to 8 decimal places....

Q7. The trick to this question is to rewrite the functions in standard trig (sin, cos, tan) then apply identities/formulas before converting back into cot, etc.

When asked to *show* we can do so by starting with the expression on the left-hand side (LHS) and manipulating until we get (RHS).

$$\operatorname{cosec}(2x) + \cot(2x) = \frac{1}{\sin(2x)} + \frac{1}{\tan(2x)}$$

1 The is the first identity we use.

$$\tan 2x = \frac{\sin 2x}{\cos 2x} \quad \text{so,} \quad \frac{1}{\tan 2x} = \frac{\cos 2x}{\sin 2x}$$

So, applying this

$$\begin{aligned}\operatorname{cosec}(2x) + \cot(2x) = \frac{1}{\sin(2x)} + \frac{1}{\tan(2x)} &= \frac{1}{\sin(2x)} + \frac{\cos(2x)}{\sin(2x)} \\ &= \frac{1+\cos(2x)}{\sin(2x)}\end{aligned}$$

2 The second identity we use is the double-angle identity for cos.

$$\cos(2x) = 2\cos^2(x) - 1 \implies \cos(2x) + 1 = 2\cos^2(x)$$

So, applying this

$$\begin{aligned}\operatorname{cosec}(2x) + \cot(2x) = \frac{1}{\sin(2x)} + \frac{1}{\tan(2x)} &= \frac{1}{\sin(2x)} + \frac{\cos(2x)}{\sin(2x)} \\ &= \frac{1+\cos(2x)}{\sin(2x)} \\ &= \frac{2\cos^2(x)}{\sin(2x)}\end{aligned}$$

3 The third identity we use is the double-angle identity for sin.

$$\sin(2x) = 2\sin(x)\cos(x)$$

So, applying this

$$\begin{aligned}
\operatorname{cosec}(2x)+\cot(2x) = \frac{1}{\sin(2x)}+\frac{1}{\tan(2x)} &= \frac{1}{\sin(2x)}+\frac{\cos(2x)}{\sin(2x)} \\
&= \frac{1+\cos(2x)}{\sin(2x)} \\
&= \frac{2\cos^2(x)}{\sin(2x)} \\
&= \frac{2\cos^2(x)}{2\sin(x)\cos(x)}, \quad \text{now we cancel terms} \\
&= \frac{\cos(x)}{\sin(x)} \\
&= \frac{1}{\tan(x)} = \cot(x)
\end{aligned}$$

We have shown LHS = RHS, so we are done. b) We now have to use the result we proved in part a)

$$\operatorname{cosec}(4\theta+10)+\cot(4\theta+10)=\sqrt{3}$$

Let $x=\frac{4\theta+10}{2}=2\theta+5$. Then $2x=4\theta+10$. Rewriting this in terms of x gives

$$\begin{aligned}
\operatorname{cosec}(2x)+\cot(2x) &= \sqrt{3}, \quad \text{but using part a)} \\
\implies \cot(x) &= \sqrt{3}, \quad \text{using standard trig functions} \\
\implies \frac{1}{\tan(x)} &= \sqrt{3} \\
\implies \tan(x) &= \frac{1}{\sqrt{3}}
\end{aligned}$$

Therefore

$$x=\tan^{-1}(1/\sqrt{3})=30 \quad \text{and, } x=180+30$$

So in terms of θ

$$2\theta+5=30 \implies \theta=12.5 \quad \text{and, } 2\theta+5=210 \implies \theta=102.5$$

Q8.

a) At the start of the study $t = 0$, so we plug this value in to the equation

$$P = \frac{800e^0}{1+3e^0}\frac{800}{4} = 200$$

b) We are being asked to find the inverse function here. We could find the inverse function first (find an equation for $t = ...$ then plug in $P = 250$, or we can plug in $P = 250$ first and manipulate.

Lets find the inverse function....(hand written notes use $P = 250$ first)

$$\begin{aligned}
&P = \frac{800e^{0.1t}}{1+3e^{0.1t}} \\
&\implies P(1+3e^{0.1t}) = 800e^{0.1t} \\
&\implies P = 800e^{0.1t} - 3Pe^{0.1t} \\
&\implies P = e^{0.1t}(800-3P) \\
&\implies \frac{P}{800-3P} = e^{0.1t} \quad \text{now take logs of both sides} \\
&\implies \ln\left(\frac{P}{800-3P}\right) = 0.1t = \frac{t}{10} \\
&\implies 10\ln\left(\frac{P}{800-3P}\right) = t
\end{aligned}$$

So with $P = 250$,

$$t = 10\ln\left(\frac{250}{800-3\times 250}\right) = 10\ln(250/50) = 10\ln(5)$$

So $a = 10$ and $b = 5$.

c) Different to the hand written notes, we first divide top and bottom by $e^{0.1t}$ to simplify the calculations of derivative

$$\begin{aligned}
P &= \frac{800}{e^{-0.1t}+3} \\
\frac{dP}{dt} &= \frac{(3+e^{-0.1t})(0) - 800(-0.1e^{-0.1t})}{(e^{-0.1t}+3)^2} \\
&= \frac{80e^{-0.1t}}{3+e^{-0.1t}}
\end{aligned}$$

So when $t = 10$, dP/dt is

$$\frac{80e^{-1}}{3+e^{-1}} = \frac{80}{3e^{0.1t}+1}$$

After multiplying top and bottom by $e^{0.1t}$ to get fraction in simplest form.

d) When we have a variable in the top and bottom of a fraction, and if we want to find out how the fraction behaves when this variable is large, we divide top and bottom by the variable. Here the variable is $e^{0.1t}$.

$$P = \frac{800}{e^{-0.1t}+3}$$

When t is large, $e^{-0.1t} \to 0$, and $P \to \frac{800}{0+3} = 266.66$. As P is an increasing function, with a maximum value of $P = 266.66$, the population size (P) never reaches $P = 270$.

Q9.

This question uses the multiple angle identities for sin and cos. These are

$$\begin{aligned}
\sin(A+B) &= \sin(A)\cos(B)+\sin(B)\cos(A)\\
\cos(A-B) &= \cos(A)\cos(B)+\sin(A)\sin(B)\\
\sin(A-B) &= \sin(A)\cos(B)-\sin(B)\cos(A)\\
\cos(A+B) &= \cos(A)\cos(B)-\sin(A)\sin(B)
\end{aligned}$$

a) With $A=\theta$ and $B=\alpha$ in the above we are told to use $\sin(\theta-\alpha)$. (Sometimes you will have to decide which of the sum-angle identities to use, and wont be told in the question).

$$\sin(\theta-\alpha)=\cos(\alpha)\sin(\theta)-\sin(\alpha)\cos(\theta)$$

If we multiply both sides by a constant R, we get

$$R\sin(\theta-\alpha)=R\cos(\alpha)\sin(\theta)-R\sin(\alpha)\cos(\theta)$$

Comparing the RHS to

$$2\sin(\theta)-4\cos(\theta)$$

given in the question, we see that

$$R\sin(\theta-\alpha)=\underbrace{R\cos(\alpha)}_{=2}\sin(\theta)-\underbrace{R\sin(\alpha)}_{=4}\cos(\theta)$$

and we can write,

$R\cos(\alpha)=2$ (1)

$R\sin(\alpha)=4$ (2)

We have two equations and two unknowns. If we do (2)/(1) we can find α

$$\begin{aligned}
\frac{R\sin(\alpha)}{R\cos(\alpha)} &= \frac{4}{2}\\
\implies \tan(\alpha)=2 &\implies \alpha=\tan^{-1}(2)=1.107
\end{aligned}$$

If we now do $(2)^2+(1)^2$, we can find R

$$\begin{aligned}
R^2\sin^2(\alpha)+R^2\cos^2(\alpha) &= 4^2+2^2\\
R^2(\sin^2(\alpha)+\cos^2(\alpha)) &= 20\\
R^2=20 \implies R &= \sqrt{20}
\end{aligned}$$

So we have found

$$\sqrt{20}\sin(\theta-1.107)=2\sin(\theta)-4\cos(\theta)$$

b) i)

With

$$H(\theta)=4+5(2\sin(3\theta)-4\cos(3\theta))^2$$

Using part a) we can write $H(\theta)$ as

$$H(\theta)=4+5(\sqrt{20}\sin(3\theta-1.107))^2=4+5(20\sin^2(3\theta-1.107))$$

The max value of H happens when $\sin^2(...)$ attains its max value, which is 1. So the max value of H is $4 + 5 \times 20 = 104$.

ii) The max of H occurs when

$$\begin{aligned}
&\sin^2(3\theta - 1.107) = 1 \\
&\implies \sin(3\theta - 1.107) = \pm 1 = 1 \quad \text{as question says, } \theta \geq 0 \\
&\implies 3\theta - 1.107 = \pi/2 \implies \theta = (\pi/2 + 1.107)/3 = 0.893
\end{aligned}$$

c) i) The minimum value of H occurs when $\sin^2(...) = 0$. Min is therefore

$$4 + 5 \times 0 = 4$$

ii) The minimum occurs when

$$\sin^2(3\theta - 1.107) = 0 \implies \sin(3\theta - 1.107) = 0 \implies 3\theta - 1.107 = 0, \text{ and } \pi, \text{and, } 2\pi$$

The largest value of θ is therefore given by

$$3\theta - 1.107 = 2\pi \implies \theta = (2\pi + 1.107)/3 = 2.463$$