1. Verify that if you have an image I with all zeroes except for a single location with value 1, that $(f * I)(a, b) = f$. Try it for a 3x3 filter f and a 5x5 image I with a 1 in the middle and zeros everywhere else. Note that the result $(f * I)(a, b)$ might be a bit offset from $f(a, b)$ depending on your indexing.

$$I = \begin{array}{c|ccccc|} & -2 & -1 & 0 & 1 & 2 \\ \hline -2 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array} \qquad f = \begin{array}{c|ccc|} & -1 & 0 & 1 \\ \hline -1 & (-1,-1) & (0,-1) & (1,-1) \\ 0 & (-1,0) & (0,0) & (1,0) \\ 1 & (-1,1) & (0,1) & (1,1) \\ \hline \end{array}$$

$$(f * I)(a,b) = \sum_{i=-k}^{k} \sum_{j=-k}^{k} f(i,j) I(a-i, b-j)$$

Let's assume that $k = 1$ and the (a,b) is at center which is $(a,b) = 1$; $a = 0$, $b = 0$

Also, the convolution image has size 3X3 after the iteration.

Since the middle has number 1 which is at $(0,0)$ and everywhere is 0. $(0,0)$ will decide convolution image.

Therefore, $(-1,-1)$ $(0,-1)$ $(1,-1)$, $(-1,0)$ $(0,0)$ $(1,0)$ $(-1,1)$ $(0,1)$ $(1,1)$ as (a,b) will decide the convolution image.

i.e) $(f * I)(-1,-1) = \sum_{i=-1}^{1} \sum_{j=-1}^{1} f(i,j) I(-1-i, -1-j)$

$= f(-1,-1) I(0,0) + f(0,-1) I(-1,0) + \dots + f(1,1) I(-2,-2)$

$= f(-1,-1)$

$(f * I)(0,-1) = f(0,-1)$ $\Rightarrow$ So final image would be

$\vdots$

$(f * I)(1,1) = f(1,1)$

f(-1,-1)	f(0,-1)	...
...	...	...
...	...	f(1,1)

$\therefore (f * I)(a,b) = f$.

2. Verify that for the same filter and image $(f \otimes I)(a, b)$ is f flipped top to bottom and side to side.

$$I = \begin{array}{c|ccccc} & -2 & -1 & 0 & 1 & 2 \\ \hline -2 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \end{array} \qquad f = \begin{array}{c|ccc} & -1 & 0 & 1 \\ \hline -1 & (-1,-1) & (0,-1) & (1,-1) \\ 0 & (-1,0) & (0,0) & (1,0) \\ 1 & (-1,1) & (0,1) & (1,1) \end{array}$$

$$(f \otimes I)(a,b) = \sum_{i=-k}^{k} \sum_{j=-k}^{k} f(i,j) I(a+i, b+j)$$

Let's assume that $k=1$ and the (a,b) is at center which is $(a,b)=1$; $a=0$, $b=0$

Also, the filterred image has size 3X3 after the iteration.

Since the middle has number 1 which is at $(0,0)$ and everywhere is 0. $(0,0)$ will decide filtered image.

Therefore, $(-1,-1)$ $(0,-1)$ $(1,-1)$, $(-1,0)$ $(0,0)$ $(1,0)$ $(-1,1)$ $(0,1)$ $(1,1)$ as (a,b) will decide the filterred image.

i.e) $(f \otimes I)(-1,-1) = \sum_{i=-1}^{1} \sum_{j=-1}^{1} f(i,j) I(a+i, b+j)$

$$= f(-1,-1) I(-2,-2) + \ldots + f(1,1) I(0,0) = f(1,1)$$

$\vdots$

$(f \otimes I)(1,1) = f(-1,-1) I(0,0) + 0 + 0 \ldots + 0 = f(-1,-1)$

Therefore, final filterred image would be

f(1,1)	...	...
...	...	...
f(1,-1)	f(0,-1)	f(-1,-1)

$\therefore$ the image will fillipped top to down and side to side

3. (*fixed*) Show that the filter $F = \begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix}$ is separable by finding s v and w so that $F = vw^T$. Then show that the filter $F = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$ is **not** separable by showing that you cannot find s v and w so that $F = vw^T$.

$$F = \begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix}$$

$F = VW^T$ $\quad V = 3\times1$ matrix, $\quad W^T = 1\times3$ matrix.

$$V = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \quad W^T = [x, y, z] \quad = \begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix}$$

$$V \cdot W^T = \begin{bmatrix} ax & ay & az \\ bx & by & bz \\ cx & cy & cz \end{bmatrix} \qquad \begin{matrix} a = -1 \\ b = -2 \\ c = -1 \end{matrix} \qquad \begin{matrix} x = 1 \\ y = 0 \\ c = -1 \end{matrix}$$

So it is separable

For the same reason, $F = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$ can not be separable because the common multiplier for 2nd column should be 0 otherwise it would not separate.

i.e) $\begin{bmatrix} ax & ay & az \\ bx & by & bz \\ cx & cy & cz \end{bmatrix} \qquad \begin{matrix} a = 1 \\ b = 1 \\ c = 1 \end{matrix} \qquad \begin{matrix} x = -1 \\ y = ? \\ c = 1 \end{matrix}$

4. Check that if F is a separable filter where $F = vw^T$, then $F \otimes I = v \otimes (w^T \otimes I)$.

$F = V \cdot W^T$ and $F \otimes I = V \otimes (W^T \otimes I)$

$F \otimes I$ means that take filter for W^T first and take V filter after.

Let's assume that $I = 5 \times 5$ image and $F = 3 \times 3$.
Then $V = 3 \times 1$ and $W^T = 1 \times 3$.
So $W^T \otimes I \Rightarrow 5 \times 3$ as a result.
Then $V \otimes (W^T \otimes I) \Rightarrow 3 \times 3$ as a result.
which is same result as $F \otimes I$.

Illustration:

$$I = \underset{5 \times 5}{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}} \otimes \underset{W^T \;\; 1 \times 3}{\begin{bmatrix} a & b & c \end{bmatrix}} \Rightarrow \underset{I \otimes W^T \;\; 5 \times 3}{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ c & b & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

$$\underset{I \otimes W^T \;\; 5 \times 3}{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ c & b & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}} \otimes \underset{V \;\; 3 \times 1}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} \Rightarrow \underset{V \otimes (W^T \otimes I) \;\; 3 \times 3}{\begin{bmatrix} cz & bz & az \\ cy & by & ay \\ ax & bx & cx \end{bmatrix}}$$

And $V \cdot W^T = F$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} a & b & c \end{bmatrix} \Rightarrow \begin{bmatrix} ax & bx & cx \\ ay & by & cy \\ az & bz & cz \end{bmatrix}$$

And $F \otimes I$ will repeat to F filp top to down side to side as I prove #2.

$\therefore$ Therefore if $F = VW^T$ then $F \otimes I = V \otimes (W^T \otimes I)$

5. How many operations (multiplications and additions) does it take to apply the filter F in the previous question with a 10x10 image I (how many operations to compute the image $F \otimes I$? How many operations for $v \otimes (w^T \otimes I)$? It should be fewer!

Let's assume $F = 5\times5$ filter.
Then $v = 5\times1$ and $w^T = 1\times5$.
After filtering process the filtered image becomes 6×6.
Because $10 - 5 + 1 = 6$.

So for $F \otimes I$, each pixel of filtered image would be calculated with 25 multiplication and 24 addition. for 36 times.
which is $(25+24)\cdot(6\cdot6) = 1764$

For $v \otimes (w^T \otimes I)$, $w^T \otimes I$ would take 5 multiplication and 4 addition per time
Each row has 6 tasks for 10 rows.
so for $w^T \otimes I$ would be $(5+4)\cdot(6\cdot10) = 540$

with same logic, $v \otimes (w^T \otimes I)$ would take 5 multiplication and 4 addition per time.
However, it has 6 columns. so $(5+4)\cdot(6\cdot6) = 324$

So $F \otimes I$ will take 1764 computation time.
However, $v \otimes (w^T \otimes I)$ will take $540 + 324 = 864$ computation time.

$v \otimes (w^T \otimes I)$ is much fewer.

6. Implement filtering in python and see if you can get the same result as the built in function (at least for the cases illustrated here)

Yes