Math 55—Fall 2025—Haiman
Homework 2
Due on Gradescope Wednesday, Sept. 10, 8pm

Problems from Hutchings

Appendix A Exercises (page 27):

1. List separately the elements and the subsets of {{1, {2}}, {3}}. (There are 2 elements and 4 subsets.)

    The elements in are {1, {2}} and {3}.
    The subsets are the empty set ∅, {{1,{2}}}, {{3}}, and {{1,{2}},{3}}.

5.Which of the following statements are true, and which are false? Why?
(a) {{∅}} ∪ ∅ = {∅, {∅}} False, the union of any set A with the empty set ∅ is simply the set A itself, so no new element should be added to the set.
(b) {{∅}} ∪ {∅} = {∅, {∅}} True, the union of two sets is the set of all elements that are in either set.
(c) {∅, {∅}} ∩ {{∅}, {{∅}}} = {∅} False, the intersection of two sets is the set of all elements that are common to both sets. The only element that appears in both sets is {∅}. Hence, the intersection is the set containing only this common element: {{∅}}.
(d) {∅, {∅}} ∩ {{∅}, {{∅}}} = {{∅}} True, the only element that appears in both sets is {∅}. Hence, the intersection is the set containing only this common element: {{∅}}

7. Show that A∩(B-C) = (A∩B) - (A∩C). Is it always true that A∪(B-C) = (A ∪ B) - (A ∪ C)?

$A \cap (B - C) = (A \cap B) - (A \cap C)$. Let x be an arbitrary element of $A \cap (B - C)$, so $x \in A$ and $x \in (B - C)$. By the definition of set difference $x \in B$ and $x \notin C$. Combining these conditions, we know that ($x \in A$ and $x \in B$) and also that $x \notin C$. From ($x \in A$ and $x \in B$), we can conclude that $x \in (A \cap B)$. Since $x \notin C$, it cannot be an element of $(A \cap C)$. Hence, $x \notin (A \cap C)$. Thus, $A \cap (B - C) \subseteq (A \cap B) - (A \cap C)$. We then try to show that $(A \cap B) - (A \cap C) \subseteq A \cap (B - C)$. Let x be an arbitrary element of $(A \cap B) - (A \cap C)$. By the definition of set difference, this means $x \in (A \cap B)$ and $x \notin (A \cap C)$. From $x \in (A \cap B)$, we know by the definition of intersection that $x \in A$ and $x \in B$. From $x \notin (A \cap C)$, we know it is not true that $(x \in A and x \in C)$. Since we already know $x \in A$, it must be that $x \notin C$. By the definition of intersection, this means $x \in A \cap (B - C)$.
Thus, $(A \cap B) - (A \cap C) \subseteq A \cap (B - C)$. Since we have shown inclusion in both directions, we can conclude that the sets are equal.

$A \cup (B - C) = (A \cup B) - (A \cup C)$ is not always true.

Let set A = {1,3}, set B={2,3}, and set C ={3,4}.

LHS: B - C={2,3} - {3,4}={2}. $A \cup (B - C)$={1,3}∪{2}={1,2,3}.

RHS: A∪B={1,3}∪{2,3}={1,2,3}, A∪C={1,3}∪{3,4}={1,3,4}. (A∪B)-(A∪C)={1,2,3}-{1,3,4}={2}.

{1,2,3} ≠ {2}

This counterexample shows the statement is not always true.

Problems from Rosen

Section 2.1

20. Find two sets A and B such that A∈B and A⊆B.

A=∅, B={∅}

24. For any two sets A and B, if the power set of A is equal to the power set of B (i.e., P(A)=P(B)), then A must be equal to B. We need to prove both A⊆B and B⊆A.

If a set X is a subset of A, then X is an element of the power set of A. Since A⊆A, it follows that $A \in P(A)$. Since $P(A) = P(B)$, and A is an element of P(A), then A must also be an element of P(B) as the two power sets are identical. Hence $A \in P(B)$ and logically $A \subseteq B$. By the same logical argument, B must also be an element of P(A) and $B \subseteq A$ as B is an element of P(B) which is the identical power set to P(A). Therefore, we can conclude that if P(A)=P(B), then A=B.

28. Show that if A⊆C and B⊆D, then A×B⊆C×D

We want to take an arbitrary element from A×B and show that it must also be an element of C×D. Let (a,b) be an arbitrary element of A×B. By the definition of the Cartesian product, if (a,b)∈A×B, then it must be that a∈A and b∈B. We have A⊆C. Since a∈A, it follows that a∈C. We use a similar reasoning for b∈D. Since a∈C and b∈D, the ordered pair (a,b) must be an element of the set C×D, hence A×B⊆C×D.

32. Suppose that A×B=∅, where A and B are sets. What can you conclude?

If the Cartesian product of two sets A and B is the empty set, then you can conclude that at least one of the sets must be the empty set.

Section 2.2

14. Find the sets A and B if A-B={1,5,7,8}, B-A={2,10}, and A∩B={3,6,9}.

The set A consists of all elements that are only in A and all elements that are in both A and B. Take the union of (A-B) and (A∩B): A={1,3,5,6,7,8,9}.
Similarly, set B consists of all elements that are only in B and all elements that are in both A and B. We again take the union of (B-A) and (A∩B), and we get B={2,3,6,9,10}.

32. Can you conclude that A = B if A, B, and C are sets such that a) A∪C=B∪C, b) A∩C=B∩C, c) A∪C=B∪C and A∩C=B∩C?

a) No, Let A be {1}, B={2}, and C = {1, 2}. A∪C={1}∪{1,2}={1,2}. B∪C={2}∪{1,2}={1,2}. A∪C=B∪C, but A is not equal to B.

b) No, Let A be {1}, B={2}, and C = {3}. A∩C={1}∩{3}=∅, and B∩C={2}∩{3}=∅. A∩C=B∩C, but A is not equal to B.

c) Yes, let x be an arbitrary element in A. Since x is an element of A, it is also an element of the union of A and C. A∪C=B∪C. Therefore, x∈B∪C. This means that either x∈B or x∈C.
Case 1: x∈B. If x is in B, then our goal is met

Case 2: x∈C. If x is in C, we assume x∈A, so in this case, we have both x∈A and x∈C. This means, by definition, that x∈A∩C.

A∩C=B∩C. Therefore, x∈B∩C. In both cases, we showed that x∈B, which follows that A⊆B. We can use a symmetrical argument to show that B⊆A. As we proved A⊆B and B⊆A, we can then conclude that A=B.

54. Let $A_i = \{\ldots,-2,-1,0,1,\ldots,i\}$. Find a) $\bigcup_{i=1}^{n} A_i$, b) $\bigcap_{i=1}^{n} A_i$

A_1 ={…,−2,−1,0,1}, A_2 ={…,−2,−1,0,1,2}, A_3 ={…,−2,−1,0,1,2,3}. The sets are nested as we can see (A1 ⊂A2 ⊂…). The union of all sets will just be the largest set in the chain. The largest set in the sequence would be A_n, so $\bigcup_{i=1}^{n} A_i = A_n$. The intersection of all the sets is the smallest set in the sequence which is A_1, so $\bigcap_{i=1}^{n} A_i = A_1$.