Math 55 — HW 8: Questions & Step–by–Step Solutions

(Rosen §5.1: 6, 16, 22, 62, 64, 76; Rosen §5.2: 8, 10, 14; Hutchings §4: 2)

Rosen §5.1 — Mathematical Induction

1. (6) Prove that for all positive integers n,

$$1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1.$$

Proof (by induction on n). **Base case ($n = 1$).** LHS $= 1 \cdot 1! = 1$, RHS $= (1+1)! - 1 = 2! - 1 = 2 - 1 = 1$. Holds.

Inductive hypothesis. Assume for some $n \geq 1$,

$$\sum_{k=1}^{n} k\, k! = (n+1)! - 1.$$

Inductive step ($n \to n+1$).

$$\begin{aligned}
\sum_{k=1}^{n+1} k\, k! &= \Big(\sum_{k=1}^{n} k\, k!\Big) + (n+1)(n+1)! \\
&= (n+1)! - 1 + (n+1)(n+1)! \quad \text{(by IH)} \\
&= (n+1)!\big(1 + n + 1\big) - 1 \\
&= (n+2)(n+1)! - 1 \\
&= (n+2)! - 1.
\end{aligned}$$

Thus the statement holds for $n+1$. By induction, true for all $n \geq 1$.

2. (16) Prove that for all $n \geq 1$,

$$\sum_{k=1}^{n} k(k+1)(k+2) = \frac{n(n+1)(n+2)(n+3)}{4}.$$

Proof (by induction on n). **Base case ($n = 1$).** LHS $= 1 \cdot 2 \cdot 3 = 6$. RHS $= \frac{1 \cdot 2 \cdot 3 \cdot 4}{4} = 6$. Holds.

Inductive hypothesis. Assume

$$\sum_{k=1}^{n} k(k+1)(k+2) = \frac{n(n+1)(n+2)(n+3)}{4}.$$

Inductive step.

$$\begin{aligned}
\sum_{k=1}^{n+1} k(k+1)(k+2) &= \frac{n(n+1)(n+2)(n+3)}{4} + (n+1)(n+2)(n+3) \\
&= \frac{(n+1)(n+2)(n+3)}{4}\big(n+4\big) \\
&= \frac{(n+1)(n+2)(n+3)(n+4)}{4}.
\end{aligned}$$

This equals the RHS with $n \mapsto n+1$.

3. (22) Determine all nonnegative integers n for which $n^2 \le n!$ and prove it.

Solution. Direct check:

$$n = 0: \ 0 \le 1; \quad n = 1: \ 1 \le 1; \quad n = 2: \ 4 \le 2 \text{ (false)}; \quad n = 3: \ 9 \le 6 \text{ (false)}; \quad n = 4: \ 16 \le 24 \text{ (true)}.$$

We claim $n^2 \le n!$ for $n = 0, 1$ and all $n \ge 4$.

Proof (by induction for $n \ge 4$). **Base** $n = 4$ holds as above. **IH:** Assume $n^2 \le n!$ for some $n \ge 4$. **Step:**

$$(n+1)^2 \le (n+1) \cdot n! \quad \text{since } n! \ge n^2 \text{ and } n+1 \ge 5 > 1,$$

and $(n+1)\, n! = (n+1)!$. Hence $(n+1)^2 \le (n+1)!$.

4. (62) Show that n lines in general position (no two parallel, no three concurrent) divide the plane into

$$R_n = \frac{n^2+n+2}{2}$$

regions.

Proof (by induction on n). **Base** $n = 0$. No lines: $R_0 = 1$ region, and $\frac{0+0+2}{2} = 1$. Holds.

Inductive hypothesis. Suppose $R_{n-1} = 1 + \frac{(n-1)n}{2}$.

Inductive step. The n-th line intersects the previous $n-1$ lines at distinct points, hence is cut into n segments and creates n new regions:

$$R_n = R_{n-1} + n = \left(1 + \frac{(n-1)n}{2}\right) + n = 1 + \frac{n(n+1)}{2} = \frac{n^2+n+2}{2}.$$

5. (64) If p is prime and $p \mid a_1 a_2 \cdots a_n$, then $p \mid a_i$ for some i.

Proof (by induction on n). **Base** $n = 2$. Euclid's Lemma: if p prime and $p \mid ab$, then $p \mid a$ or $p \mid b$.

Inductive hypothesis. Assume the statement is true for $n-1$.

Inductive step. If $p \mid (a_1 \cdots a_{n-1})a_n$, then by the $n = 2$ case either $p \mid a_n$ or $p \mid a_1 \cdots a_{n-1}$. In the latter case, by the IH, $p \mid a_i$ for some $i \le n-1$. Hence $p \mid a_i$ for some i.

6. (76) Prove for all integers $n \ge 1$,

$$\prod_{k=1}^{n} \frac{2k-1}{2k} < \frac{1}{\sqrt{3n}}.$$

Explain why a naive induction attempt fails, then prove the stronger

$$\prod_{k=1}^{n} \frac{2k-1}{2k} \le \frac{1}{\sqrt{3n+1}} \qquad (\star).$$

Why naive induction fails. Let $P(n) : \prod_{k=1}^{n} \frac{2k-1}{2k} < \frac{1}{\sqrt{3n}}$. From $P(n)$,

$$\prod_{k=1}^{n+1} \frac{2k-1}{2k} = \left(\prod_{k=1}^{n} \frac{2k-1}{2k}\right) \frac{2n+1}{2n+2} < \frac{1}{\sqrt{3n}} \cdot \frac{2n+1}{2n+2}.$$

To deduce $P(n+1)$, we would need $\frac{2n+1}{2n+2} \cdot \frac{1}{\sqrt{3n}} \le \frac{1}{\sqrt{3n+3}}$, which is *false* for small n and in general lacks uniform slack.

Proof of ($\star$) *(by induction).* **Base $n = 1$.** $\frac{1}{2} = \frac{1}{\sqrt{4}}$.

Inductive hypothesis. Assume $\prod_{k=1}^{n} \frac{2k-1}{2k} \le (3n+1)^{-1/2}$.

Inductive step.

$$\prod_{k=1}^{n+1} \frac{2k-1}{2k} \le \frac{1}{\sqrt{3n+1}} \cdot \frac{2n+1}{2n+2} \overset{(\dagger)}{\le} \frac{1}{\sqrt{3n+4}},$$

where ($\dagger$) is equivalent (after squaring and cross-multiplying positive terms) to

$$(2n+1)^2(3n+4) \le (2n+2)^2(3n+1),$$

which expands to $12n^3 + 28n^2 + 19n + 4 \le 12n^3 + 28n^2 + 20n + 4$, true since the RHS exceeds the LHS by $n \ge 0$. Thus ($\star$) holds for all $n \ge 1$, and since $(3n+1)^{-1/2} \le (3n)^{-1/2}$, the original inequality follows.

Rosen §5.2 — Induction and Recursion

7. **(8)** Gift certificates of \$25 and \$40. Determine all attainable totals and prove your claim by strong induction.

Claim. A total T is attainable iff T is a multiple of \$5 and $T \ge 140$.

Proof. **Necessity.** Any $T = 25a + 40b = 5(5a + 8b)$ is a multiple of \$5.

Sufficiency (strong induction on T over multiples of \$5). We give bases covering all residues mod \$25 for $T \ge 140$:

$T \pmod{25}$	0	5	10	15	20
base	$150 = 6(25)$	$155 = 3(25) + 2(40)$	$160 = 4(40)$	$140 = 4(25) + 40$	$145 = 25 + 3(40)$

Inductive step. Suppose every multiple of \$5 in $[140, T]$ is attainable. Then $T + 25$ is attainable by adding one \$25 certificate. Hence every multiple of \$5 at least \$140 is attainable.

8. **(10)** A chocolate bar consists of n unit squares in a rectangle. Each break splits one rectangular piece into two along grid lines. Show that exactly $n-1$ breaks are necessary and sufficient to obtain n single squares.

Proof. **Necessity.** Each break increases the number of pieces by exactly 1. Starting from 1 piece ending at n pieces needs at least $n-1$ breaks.

Sufficiency (strong induction on n). Base $n = 1$: 0 breaks $= 1 - 1$. **Inductive step:** Assume any bar with $< n$ squares can be reduced to single squares in (number of squares)-1 breaks. Take a bar with n squares and make one break, producing rectangular pieces with r and s squares ($r + s = n$). By the hypothesis, these can be fully split using $(r-1) + (s-1)$ more breaks. Total $= 1 + (r-1) + (s-1) = n - 1$.

9. **(14)** Begin with a pile of n stones. When a pile of $r + s$ stones is split into r and s, record rs. Show that the sum of all recorded products is always $\dfrac{n(n-1)}{2}$.

Proof (strong induction on n). **Base $n = 1$.** No split, total $0 = \frac{1\cdot 0}{2}$. **Inductive step.** Assume true for all $< n$. First split $n = r + s$ contributes rs. By the hypothesis, finishing the two subpiles contributes $\frac{r(r-1)}{2} + \frac{s(s-1)}{2}$. Total:

$$rs + \frac{r(r-1)}{2} + \frac{s(s-1)}{2} = \frac{(r+s)^2 - (r+s)}{2} = \frac{n(n-1)}{2}.$$

Hutchings §4

10. Guess a formula for

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)}$$

and prove it by induction.

Solution. Note

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}.$$

Claim. $\displaystyle\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}.$

Proof (by induction). **Base $n = 1$:** $1/2 = \frac{1}{2}$. **IH:** Assume $\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$. **Step:**

$$\sum_{k=1}^{n+1} \frac{1}{k(k+1)} = \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} = \frac{n(n+2)+1}{(n+1)(n+2)} = \frac{n+1}{n+2}.$$

Thus the formula holds for all $n \geq 1$.