Math 55—Fall 2025—Haiman
Homework 3

Problems from Rosen

Section 2.3

12. (c,d) (justify your answers).

    12.c) Answer: The derivative of $f(n) = n^3$ is $f'(n) = 3n^2 \geq 0$ for all $\mathbb{Z}$ Since the function strictly monotone, it is always injective (one-to-one)

    12.d) Answer: $f(n) = \lfloor f(n/2) \rfloor$ is not one-to-one. A counter example is $f(2) = f(3)$ but $2 \neq 3$, hence the function is not one-to-one.

14. (b,d) (justify your answers).

    14.b) Answer: $f(m,n) = m^2 - n^2 \implies f(m,n) = (m+n)(m-n)$ In the difference of squares equation, the two factors (m+n) and (m-n) are either both even or both odd.

    case 1: If the two factors are both odd, then their product is also odd.

    case 2: If the two factors are both even, then their product can be described as $4k$, $k \in \mathbb{Z}$

    This means the image of f(m, n) is the set of all odd integers and all integers divisible by 4. This leaves out the integers like 2, 6, 10 which are in the form of 4k + 2, $k \in \mathbb{Z}$. Therefore f(m, n) is not onto.

    14.d) Answer: $f(m,n) = |m| - |n|$ is onto for $\mathbb{Z}$. For any integer k, if $k \geq 0$, then we can choose m to be k and n to be 0. If $k < 0$ then we can choose k to be 0 and n to be k, so the image would contain all possible integers.

34. (a, b, c)

    a.) Answer: If $f \circ g$ is onto, then f must also be onto. Let $c \in C$. Since $f \circ g$ is onto, there exists an $a \in A$ such that $(f \circ g)(a) = c$. Let $b = g(a) \in B$. Then $f(b) = c$. Hence every $c \in C$ has a preimage under f, so f is onto.

    b.) Answer: We can try to prove the contrapositive – Suppose g is not one-to-one, then there exists $a_1 \neq a_2$ in A such that $g(a_1) = g(a_2)$.

    $$(f \circ g)(a_1) = f(g(a_1)) = f(g(a_2)) = (f \circ g)(a_2),$$

    so $f \circ g$ is not one-to-one. Therefore, if $f \circ g$ is injective, g must be injective.

    c.) Answer: We can prove the contrapositive that if g is not injective, then $f \circ g$ is not either, which makes it not a bijection. Suppose g is not onto.

Then there exists some $b \in B$ with no preimage in A. In particular, we can find distinct $x, y \in A$ such that $g(x) = g(y)$. Applying f to both sides gives

$$f(g(x)) = f(g(y)),$$

so $f \circ g$ is not injective and therefore cannot be a bijection.

Now suppose $f \circ g$ is bijective. Then it is also surjective. For any $y \in C$, there exists some $a \in A$ such that $(f \circ g)(a) = f(g(a)) = y$. Let $x = g(a)$. Then $f(x) = y$. This shows that every $y \in C$ has a preimage under f, so f is surjective.

40. To find the condition, we first compute the expressions for the composite functions $f(g(x))$ and $g(f(x))$.

    First calculate $f \circ g$:
    $f(g(x)) = f(cx + d)$
    Substitute cx+d into the expression for f(x):
    $f(g(x)) = a(cx + d) + b = acx + ad + b$

    Then we calculate $g \circ f$
    $g(f(x)) = g(ax + b)$
    Substitute ax+b into the expression for g(x):
    $g(f(x)) = c(ax + b) + d = cax + cb + d$

    For $f \circ g = g \circ f$, the expressions must be equal for all values of x:
    $acx + ad + b = cax + cb + d$

    Since the multiplication of constants is commutative (ac=ca), the terms with x on both sides are identical and cancel each other out.
    $acx + ad + b = acx + cb + d$ This leaves us with the condition on the constant terms: $ad + b = cb + d$.

50. *We want to show that if $x \in \mathbb{R}$, then if x is an integer, $\lceil x \rceil$ - $\lfloor x \rfloor = 0$, and otherwise 1 if x is not an integer.*

    Case 1: x is an integer. Let's say $x = n$ for some integer n. The largest closest integer to n and the smallest closest integer to n are both n itself. Therefore $\lceil x \rceil$ - $\lfloor x \rfloor = 0$.

    Case 2: x is not an integer. Let's say the largest closest integer to x is n. The smallest closest integer to x is than n - 1. This means $\lceil x \rceil$ - $\lfloor x \rfloor = n - (n - 1) = 1$.

Section 2.5

2. (c,d,e,f).

   c) Answer: The set of integers with absolute value less than 1,000,000 is the set $S = \{x \in \mathbb{Z} \mid |x| < 1{,}000{,}000\}$. The number of elements in this set is $999{,}999 - (-999{,}999) + 1 = 1{,}999{,}999$. Because the set has a specific, finite number of elements, it is finite.

d) Answer: The set of real numbers between 0 and 2 constitutes the interval $(0, 2)$*. Any non-empty interval of real numbers is uncountable by Cantor's diagonalization argument, which shows it's impossible to create a one-to-one correspondence between the positive integers and the real numbers in the interval.*

e) Answer: The set is $A \times \mathbb{Z}^+$ *where* $A = \{2, 3\}$*. This set is the union of two disjoint countably infinite sets,* $\{2\} \times \mathbb{Z}^+$ *and* $\{3\} \times \mathbb{Z}^+$*, which means the set is countably infinite. We can define a one-to-one correspondence* $f : \mathbb{Z}^+ \to A \times \mathbb{Z}^+$ *by mapping odd positive integers to pairs starting with 2 and even positive integers to pairs starting with 3:*

$$f(n) = \begin{cases} (2, (n+1)/2) & \textit{if } n \textit{ is odd} \\ (3, n/2) & \textit{if } n \textit{ is even} \end{cases}$$

f) Answer: The set of integers that are multiples of 10 is $S = \{10k \mid k \in \mathbb{Z}\} = \{\dots, -20, -10, 0, 10, 20, \dots\}$*. This is an infinite subset of the set of all integers which is countably infinite. A one-to-one correspondence* $f : \mathbb{Z}^+ \to S$ *can be constructed by mapping odd integers to zero and the negative multiples, and even integers to the positive multiples:*

$$f(n) = \begin{cases} 10(n/2) & \textit{if } n \textit{ is even} \\ 10\left(\frac{1-n}{2}\right) & \textit{if } n \textit{ is odd} \end{cases}$$

This function maps $f(1) = 0$*,* $f(2) = 10$*,* $f(3) = -10$*,* $f(4) = 20$*, and so on, covering all elements of* S*.*

10. (and explain).

   a) *$A - B$ is finite.*

   Let $A = \mathbb{R}$ (the set of all real numbers) and let $B = \mathbb{R} - \{0, 1\}$ (the set of real numbers excluding 0 and 1). Both A and B are uncountable.

   The set difference is $A - B = \mathbb{R} - (\mathbb{R} - \{0, 1\}) = \{0, 1\}$.

   The resulting set $\{0, 1\}$ contains two elements, so it is finite.

   b) *$A - B$ is countably infinite.*

   Let $A = \mathbb{R}$ and let B be the set of all irrational numbers. Both A and B are uncountable.

   The set difference is $A - B = \mathbb{R} - (\text{irrational numbers}) = \mathbb{Q}$ (the set of rational numbers).

   The set of rational numbers, $\mathbb{Q}$, is countably infinite.

c) *$A - B$ is uncountable.*

Let $A = [0, \infty)$ (the set of non-negative real numbers) and let $B = (10, \infty)$ (the set of real numbers greater than 10). Both sets are intervals of real numbers and are therefore uncountable.

The set difference is $A - B = [0, \infty) - (10, \infty) = [0, 10]$.

The resulting interval $[0, 10]$ is uncountable.

Additional problem

To show that the formula $f((i,j)) = \frac{(i+j)(i+j+1)}{2} + j$ is correct, we can analyze the diagonal counting method from the diagram. The pairs (i,j) are counted along diagonals where the sum $k = i+j$ is constant. The count moves through diagonals $k = 0, 1, 2, \ldots$ in order.

The formula works by combining two parts that match this process:

*1. The first part, $\frac{(i+j)(i+j+1)}{2}$, is a triangular number formula. It counts the total number of pairs in all the diagonals **before** the one containing (i,j).*

*2. The second part, $+j$, gives the 0-indexed position of the pair (i,j) **within its own diagonal**. This works because the pairs on any diagonal are ordered by increasing j.*

So, the total count for (i,j) is the sum of all pairs in previous diagonals plus its position in its current diagonal, which is exactly what the formula calculates. For example, for the pair $(2,1)$, there are $\frac{(2+1)(2+1+1)}{2} = 6$ pairs in the preceding diagonals (for k=0, 1, 2), and its position on its own diagonal (k=3) is $j = 1$. This gives a final value of $6 + 1 = 7$, which matches the diagram.