Stat 134 Fall 2025: Homework 7 – SOLUTIONS

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1. *[#20 from Chapter 4]* Let $X \sim \text{Bin}(100, 0.9)$. For each part, either construct a coupling that works or explain why it is impossible. Here Y is on the same probability space as X (not necessarily independent).

(a) $Y \sim \text{Pois}(0.01)$ with $\mathbb{P}(X \geq Y) = 1$?
Answer: *Impossible.* If $\mathbb{P}(X \geq Y) = 1$, then $Y \leq 100$ a.s. since $X \leq 100$. But Pois(0.01) assigns positive mass to all nonnegative integers; in particular $\mathbb{P}(Y > 100) > 0$. Hence $\mathbb{P}(X \geq Y) < 1$.

(b) $Y \sim \text{Bin}(100, 0.5)$ with $\mathbb{P}(X \geq Y) = 1$?
Answer: *Possible.* Couple trial-by-trial with i.i.d. $U_i \sim \text{Unif}(0,1)$ for $i = 1, \dots, 100$. Set

$$X_i = \mathbf{1}\{U_i \leq 0.9\}, \qquad Y_i = \mathbf{1}\{U_i \leq 0.5\}.$$

Then $X_i \sim \text{Bern}(0.9)$, $Y_i \sim \text{Bern}(0.5)$ and $Y_i \leq X_i$ pointwise, so $X = \sum X_i \geq Y = \sum Y_i$ a.s., with $Y \sim \text{Bin}(100, 0.5)$.

(c) $Y \sim \text{Bin}(100, 0.5)$ with $\mathbb{P}(X \leq Y) = 1$?
Answer: *Impossible.* If $X \leq Y$ a.s., then $\mathbb{E}[X] \leq \mathbb{E}[Y]$. But $\mathbb{E}[X] = 90$ and $\mathbb{E}[Y] = 50$, a contradiction.

2. *[#64 from Chapter 4]* If $X \sim \text{Geom}(p)$ on $\{1, 2, \dots\}$, find $\mathbb{E}(e^{tX})$.

Answer: For $|(1-p)e^t| < 1$ (i.e. $t < -\ln(1-p)$),

$$\mathbb{E}\big(e^{tX}\big) = \sum_{k=1}^{\infty} e^{tk} p(1-p)^{k-1} = \frac{pe^t}{1-(1-p)e^t}.$$

3. *[#22 from Chapter 4]* Raindrops arrive at rate 20 drops/in^2/min. For a 5 in^2 region over t minutes, suggest a distribution and compute $\mathbb{P}(\text{no drops in 3 seconds})$.

Answer: By the Poisson model for independent counts in space-time, $N \sim \text{Pois}(\lambda)$ with $\lambda = 20 \times 5 \times t = 100t$. For $t = 3/60 = 0.05$ minutes, $\lambda = 5$, so

$$\mathbb{P}(N = 0) = e^{-5} \approx 0.0067.$$

4. *[#29 from Chapter 4]* Let $X \sim \text{Geom}(p)$ on $\{1, 2, \dots\}$ and define $f(x) = \mathbb{P}(X = x)$. Find $\mathbb{E}[f(X)]$.

Answer: For integers $k \geq 1$, $f(k) = p(1-p)^{k-1}$. Hence

$$\mathbb{E}[f(X)] = \sum_{k=1}^{\infty} \underbrace{p(1-p)^{k-1}}_{P(X=k)} \cdot \underbrace{p(1-p)^{k-1}}_{f(k)} = p^2 \sum_{k=1}^{\infty} (1-p)^{2k-2} = \frac{p^2}{1-(1-p)^2} = \frac{p}{2-p}.$$

5. *[#32 from Chapter 4]* *Memoryless* A discrete distribution is memoryless if

$$\mathbb{P}(X \geq j+k \mid X \geq j) = \mathbb{P}(X \geq k) \quad \text{for all integers } j, k \geq 0.$$

(a) Express $\mathbb{P}(X \geq j+k)$ in terms of F and/or p_j.
Answer: Write $S(m) = \mathbb{P}(X \geq m) = 1 - F(m-1)$. The property gives $S(j+k) = S(j)S(k)$, i.e.

$$1 - F(j+k-1) = \big(1 - F(j-1)\big)\big(1 - F(k-1)\big).$$

(b) Name a discrete distribution with the memoryless property and justify.
Answer: The geometric distribution. If $X \sim \text{Geom}(p)$ on $\{1, 2, \dots\}$, then $S(m) = \mathbb{P}(X \geq m) = (1-p)^{m-1}$ and $S(j+k) = S(j)S(k)$. Uniqueness follows from $S(m+1) = c\,S(m)$ for all m, forcing $S(m) = c^{m-1}$ and thus geometric.

6. $X \sim \text{Unif}[2, 10]$.

(a) Density and $\mathbb{P}(X \in [a, b] \subseteq [2, 10])$?
Answer: $f_X(x) = \dfrac{1}{8}$ for $x \in [2, 10]$ and 0 otherwise. For $[a, b] \subseteq [2, 10]$,

$$\mathbb{P}(X \in [a, b]) = \frac{b-a}{8}.$$

(b) Compute $\mathbb{P}(X > 5)$, $\mathbb{P}(5 < X < 7)$, and $\mathbb{P}(X^2 - 12X + 35 > 0)$.
Answer: $\mathbb{P}(X > 5) = \dfrac{10-5}{8} = \dfrac{5}{8}$; $\quad \mathbb{P}(5 < X < 7) = \dfrac{7-5}{8} = \dfrac{1}{4}$. Since $x^2 - 12x + 35 = (x-5)(x-7) > 0$ on $(-\infty, 5) \cup (7, \infty)$, within $[2, 10]$ the favorable length is $3 + 3 = 6$, so $\mathbb{P} = 6/8 = 3/4$.

7. Y on $[2, 10]$ with density $f_Y(y) = c\,y$.

(a) Find c. $\quad$ **Answer:** $1 = \int_2^{10} cy\,dy = c\left[\frac{y^2}{2}\right]_2^{10} = c \cdot 48 \Rightarrow c = \dfrac{1}{48}$.

(b) For $[a, b] \subseteq [2, 10]$, compute $\mathbb{P}(Y \in [a, b])$. $\quad$ **Answer:** $\mathbb{P}(Y \in [a, b]) = \displaystyle\int_a^b \frac{y}{48}\,dy = \frac{b^2 - a^2}{96}$.

(c) Compute $\mathbb{P}(Y > 5)$, $\mathbb{P}(5 < Y < 7)$, and $\mathbb{P}(Y^2 - 12Y + 35 > 0)$.
Answer: $\mathbb{P}(Y > 5) = \dfrac{100 - 25}{96} = \dfrac{75}{96} = \dfrac{25}{32}$; $\quad \mathbb{P}(5 < Y < 7) = \dfrac{49 - 25}{96} = \dfrac{24}{96} = \dfrac{1}{4}$; and over $[2, 5] \cup [7, 10]$ we get $\dfrac{25 - 4}{96} + \dfrac{100 - 49}{96} = \dfrac{21}{96} + \dfrac{51}{96} = \dfrac{72}{96} = \dfrac{3}{4}$.

8. *[#1 from Chapter 5] Rayleigh* If X has PDF $f(x) = xe^{-x^2/2}$ for $x > 0$:

(a) $\mathbb{P}(1 < X < 3)$? $\quad$ **Answer:** The CDF is $F(x) = 1 - e^{-x^2/2}$. Thus

$$\mathbb{P}(1 < X < 3) = F(3) - F(1) = e^{-1/2} - e^{-9/2}.$$

(b) Quartiles: find q_j with $\mathbb{P}(X \leq q_j) = j/4$. $\quad$ **Answer:** Solve $1 - e^{-q^2/2} = j/4 \Rightarrow q_j = \sqrt{-2\ln(1 - j/4)}$ for $j = 1, 2, 3$.

9. *[#3 from Chapter 5]* Let F be a CDF with PDF $f = F'$.

(a) Show $g(x) = 2F(x)f(x)$ is a valid PDF.
Answer: $g(x) \geq 0$. Also

$$\int_{-\infty}^{\infty} 2F(x)f(x)\,dx = \int 2F\,dF = \left[F(x)^2\right]_{-\infty}^{\infty} = 1 - 0 = 1.$$

(b) Show $h(x) = \frac{1}{2}f(-x) + \frac{1}{2}f(x)$ is a valid PDF.
Answer:

$$\int_{-\infty}^{\infty} h(x)\,dx = \tfrac{1}{2}\int f(x)\,dx + \tfrac{1}{2}\int f(-x)\,dx = \tfrac{1}{2} \cdot 1 + \tfrac{1}{2} \cdot 1 = 1,$$

using the substitution $u = -x$ for the second integral.

10. *[#5 from Chapter 5]* Random radius $R \sim \text{Unif}(0,1)$; area $A = \pi R^2$.

(a) Mean and variance of A (without first finding CDF/PDF).
Answer: $\mathbb{E}[A] = \pi\,\mathbb{E}[R^2] = \pi \cdot \frac{1}{3} = \dfrac{\pi}{3}$, since $\mathbb{E}[R^2] = \int_0^1 r^2 dr = 1/3$. Also $\text{Var}(A) = \pi^2\,\text{Var}(R^2)$, and $\mathbb{E}[R^4] = \int_0^1 r^4 dr = \frac{1}{5}$, so $\text{Var}(R^2) = \frac{1}{5} - \left(\frac{1}{3}\right)^2 = \frac{4}{45}$. Hence $\text{Var}(A) = \dfrac{4\pi^2}{45}$.

(b) CDF or PDF of A.
Answer: For $a \in [0, \pi]$,

$$F_A(a) = \mathbb{P}(A \le a) = \mathbb{P}\Big(R \le \sqrt{a/\pi}\Big) = \sqrt{a/\pi}.$$

Thus $f_A(a) = F_A'(a) = \dfrac{1}{2\sqrt{\pi a}}, \quad 0 < a < \pi.$