Math 2271 - Ordinary Differential Equations

Lab Assignmnent

816000772 - Christopher Sahadeo

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1. (a) Required to solve the IVP:

$$y' + 2y = 2 - e^{-4x}, \quad y(0) = 1 \tag{1}$$

This is a first order ode of the form

$$\frac{dy}{dx} + p(x)y(x) = g(x)$$

The integrating factor here is

$$e^{\int 2dx} = e^{2x}$$

Multiply through by the integrating factor, we have

$$\begin{aligned} e^{2x}\left(\frac{dy}{dx}\right) + 2e^{2x}y &= 2e^{2x} - e^{-2x} \\ \frac{d}{dx}\left(ye^{2x}\right) &= 2e^{2x} - e^{-2x} \end{aligned}$$

Integrating we have

$$\begin{aligned} ye^2x &= \int 2e^{2x} - e^{-2x}dx \\ &= e^{2x} + \frac{1}{2}e^{-2x} + C \end{aligned}$$

Where C is a constant. Hence

$$y(x) = 1 + \frac{1}{2}e^{-4x} + Ce^{-2x}$$

Now using the condition $y(0) = 1$, we have

$$1 = 1 + \frac{1}{2} + C \implies C = \frac{1}{2}$$

and

$$\boxed{y(x) = 1 + \frac{1}{2}e^{-4x} - \frac{1}{2}e^{-2x}}$$

(b)

$$y(0.3) = 0.876 \quad [3 \text{ d.p.}]$$

(c) Click here for google sheets working

Consider the IVP in equation 1 over the interval $[0, 0.5]$

Let

$$f(x, y) = \frac{dy}{dx} = -2y + 2 - e^{-4x}$$

Take $N = 5$ subintervals

There are 6 points of step size $h = 0.1$

The points are $x_0 = 0.0$, $x_1 = 0.1$, $x_2 = 0.2$, $x_3 = 0.3$, $x_4 = 0.4$ and $x_5 = 0.5$

Apply Euler's method:

$$\begin{aligned}
y(x_{j+1}) &\approx y(x_j) + h \cdot f((x_j,\ y(x_j)),\ j = 0, 1, \ldots, N-1 \\
y(x_1) &\approx y(x_0) + h \cdot (-2y(x_0) + 2 - e^{-4x_0}) \\
y(0.1) &\approx y(0) + 0.1 \times (-2(1) + 2 - e^{-4(0)}) = 0.90000 \\
y(0.2) &\approx y(0.1) + 0.1 \times (-2(0.90000)) + 2 - e^{-4(0.1)}) = 0.85297 \\
y(0.3) &\approx y(0.2) + 0.1 \times (-2(0.85297)) + 2 - e^{-4(0.2)}) = 0.837 \quad [3 \text{ d.p.}]
\end{aligned}$$

(d) `EulerFirst.m`

```matlab
function [x,y]=EulerFirst(a,l,h,y0,f)

%     MATLAB code for using Euler's MEthod to solve the general 1st order IVP
%     dy/dx = f(x, y) on the interval (a, a+l) with y(a) = y0
%
%     Inputs:     a and l define the interval [a, a+l]
%                 h is the interval step
%                 y0 is the value of y(a)
%                 f is the function f(x,y) which must be defined separately
%
%     Outputs:    x stores the discretized values of x in the interval [a, a+l]
%                 y stores the corresponding approximate values of the solution

    x=a:h:(a+l);
    N = length(x);
    y=zeros(1,N);
    y(1)=y0;
    for i=1:N-1
        y(i+1)=y(i)+h*feval(f,x(i),y(i));
    end

end
```

`f.m`

```matlab
function [u] = f(x,y)
    u = -2.*y + 2 - exp(-4.*x);
end
```

(e) `Script1.m`

```
h = [0.1 0.01 0.001];
figure;
title('A plot of the numerical solution of y versus x');
xlabel('x');
ylabel('y');
hold on;

for i = 1:length(h)
    [x, y] = EulerFirst(0, 0.5, h(i), 1, 'f');
    plot(x, y)
    legendInfo{i} = ['h = ', num2str(h(i))];
end
legend(legendInfo)

% legend('h = ')
```

(f) Command Window

```
>> Script1
```

Plot

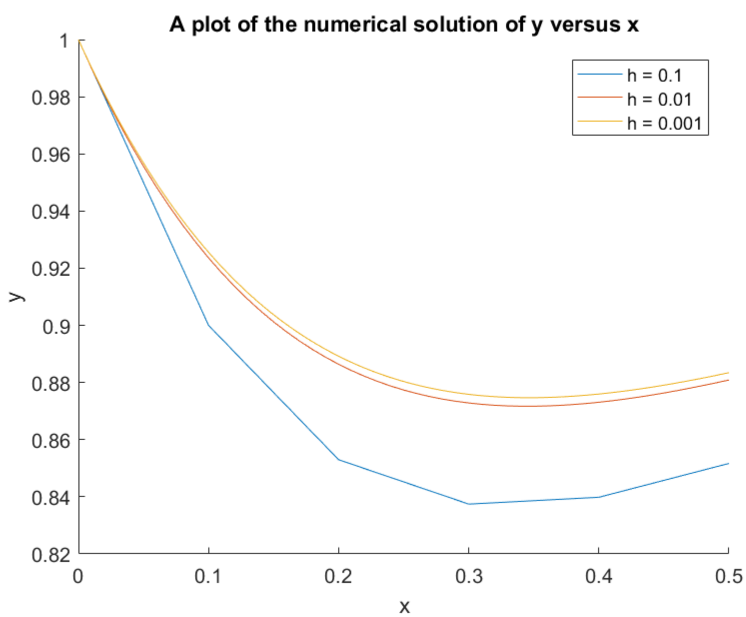

Figure 1: A plot of the numerical solution of y vs x

(g) The graph in figure 1 is concave upward which implies that Euler's Method has produced an underestimate of the actual solution.

(h) Setup

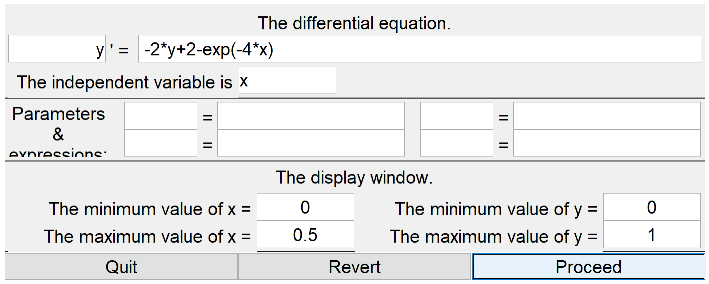

Figure 2: dfield9 setup

Direction Field

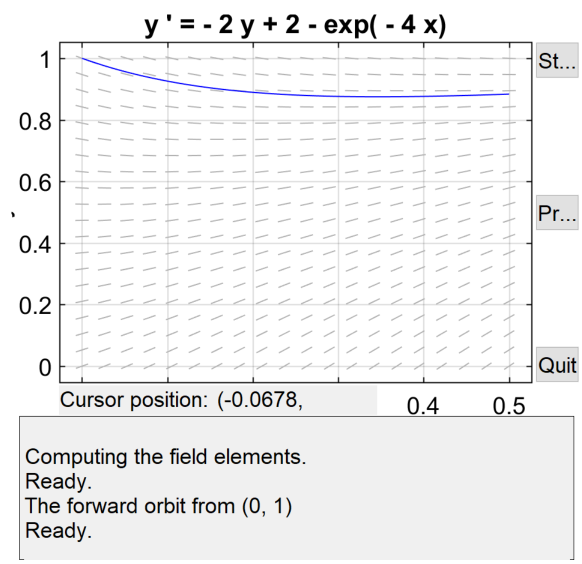

Figure 3: Direction Field

2. (a) Required to solve the IVP:

$$y'' + 4y = t^2 + 3e^t,\ y(0) = 0,\ y'(0) = 2$$

We first solve the homogeneous part of the IVP

$$y'' + 4y = 0$$

Assuming solutions of the form $y = e^{rt}$ leads to the characteristic equation

$$r^2 + 4 = 0$$
$$\implies r = \pm 2i$$

The general solution for the homogeneous part is

$$y(t) = C_1 cos(2t) + C_2 sin(2t)$$

where C_1 and C_2 are constants
Now, we wish to find the particular solution $Y(t)$ such that

$$Y'' + 4Y = t^2 + 3e^t$$

Assume a particular solution of the form

$$Y(t) = At^2 + Bt + C + De^t$$

where A, B, C and D are constants
Hence

$$Y' = 2At + B + De^t$$
$$Y'' = 2A + De^t$$

Substituting these into the original equation, we have

$$(2A + De^t) + 4(At^2 + Bt + C + De^t) = t^2 + 3e^t$$
$$\implies (4A)t^2 + (4B)t + (2A + 4C) + (5D)e^t = t^2 + 3e^t$$

Equating coefficients of the two sides, we have

$$4A = 1 \implies A = \frac{1}{4}$$
$$4B = 0 \implies B = 0$$
$$2A + 4C = 0 \implies C = -\frac{1}{8}$$
$$5D = 3 \implies D = \frac{3}{5}$$

giving the particular solution

$$Y(t) = \frac{1}{4}t^2 - \frac{1}{8} + \frac{3}{5}e^t$$

The general solution of the IVP is therefore

$$y(t) = C_1 cos(2t) + C_2 sin(2t) + \frac{1}{4}t^2 - \frac{1}{8} + \frac{3}{5}e^t$$

Now using the first initial condition provided

$$y(0) = 0 = C_1 + 0 + 0 - \frac{1}{8} + \frac{3}{5} \implies C_1 = -\frac{19}{40}$$

Next differentiating we have

$$y' = -2C_1 sin(2t) + 2C_2 cos(2t) + \frac{1}{2}t + \frac{3}{5}e^t$$

Using the second initial condition provided

$$y'(0) = 2 = 0 + 2C_2 + 0 + \frac{3}{5} \implies C_2 = \frac{7}{10}$$

The final solution is therefore

$$\boxed{y(t) = \frac{7}{10} sin(2t) - \frac{19}{40} cos(2t) + \frac{1}{4}t^2 - \frac{1}{8} + \frac{3}{5}e^t}$$

(b) The second order IVP can be solved by first converting it to a system of 2 first order ODEs. Let

$$\frac{dy}{dt} = u$$

This gives

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{du}{dt}$$

Therefore, we have 2 first order ODEs to solve

$$\frac{dy}{dt} = u; \qquad y(0) = 0$$
$$\frac{du}{dt} = -4y + t^2 + 3e^t; \qquad u(0) = 2$$

(c) Writing the system in the form $\vec{x}\,' = A\vec{x} + \vec{g}(t)$

Now

$$\begin{bmatrix} y' \\ u' \end{bmatrix} = \begin{bmatrix} u \\ -4y + t^2 + 3e^t \end{bmatrix}$$
$$\begin{bmatrix} y \\ u \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}\begin{bmatrix} y \\ u \end{bmatrix} + \begin{bmatrix} 0 \\ t^2 + 3e^t \end{bmatrix}, \quad \begin{bmatrix} y(0) \\ u(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

(d) `EulerSecond.m`

```
function [x,y] = EulerSecond(a, l, h, y0, u0, f)
%     A second order IVP can be solved by first converting it to a system of 2 first order ODEs
%     substitution with u = dy/dx

%     Inputs:     a and l define the interval [a, a+l]
%                 h is the step size
%                 y0, u0 are the values of y(a) and dy/dx(a)
%                 f is the function f(x,y,dy/dx), defined in another file
%
%      Outputs:   x stores the discretized values of x in the interval [a, a+l]
%                 y stores the corresponding approximate values of y

    %f here is a first order ode
    x=a:h:(a+l);
    N=length(x);
    y=zeros(1,N); % y is the solution to the first order ode
    u=zeros(1,N);
    y(1)=y0;
    u(1)=u0; % the initial value of the second order ode f

    for i=1:N-1
        y(i+1)=y(i)+h*u(i); % finding the solution points of the first ODE
        u(i+1)=u(i)+h*feval(f,x(i),y(i),u(i)); % f is the function 'h': getting the value of the second ...
            ODE by Euler's
    end

    figure;
    hold on;
    title('A plot of the numerical solution of y versus x');
    xlabel('x');
    ylabel('y');
    plot(x,y)

end
```

The function $\frac{du}{dt}$ stored in the function file `h.m`

```
function [ x ] = h( t,y,u )
    % t is the vector of second order ode solution points,
    % u is the first order ode solutions
    x= -4.*y + t.^2 + 3.*exp(t);
end
```

(e) Command Window

```
>> EulerSecond(0,5,0.001,0,2,'h');
```

(f) Plot

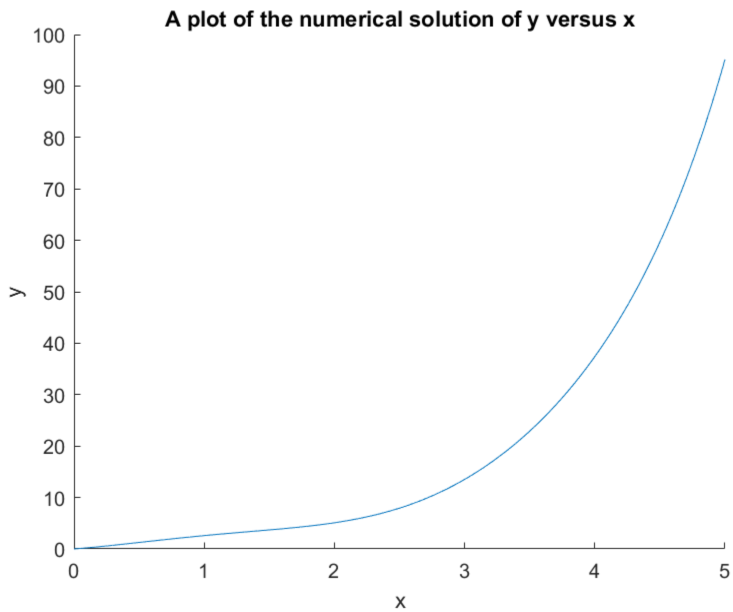

Figure 4: A plot of the numerical solution of y vs x

3. (a)

$$\begin{aligned} u' &= u, & u(0) &= 1 \\ v' &= 2u + v - 2w, & v(0) &= 1 \\ w' &= 3u + 2v + w, & w(0) &= 2 \end{aligned}$$

(b) Solution in assignment appears incorrect

EulerSys.m

```
function [x,Y] = EulerSys(a, b, h, Y0, F)

%     Using Euler's Method to solve a general system of 1st order ODEs
%     dY/dx = F(x,Y) on the interval (a,b)
%
%     Inputs:     a and b define the interval(a,b)
%                 h is the step size
%                 Y0 is a column vector that defines the initial conditions
%                 F is a function that doves the LHS of the system dY/dx = F(x,Y)
%
%     Outputs:    x stores the discretised values of x in the interval (a,b)
%                 Y is a vector that stores the corresponding approximate values

    x=a:h:b;
    N=length(x);
    M=length(Y0);
    Y=zeros(M,N);
    Y(:,1)=Y0;
    for j=1:N-1
        for i=1:M
            Y(i,j+1)=Y(i,j)+h*feval(F,i,x(j),Y(:,j));
        end
    end

%     plotting
    clf;
    for i=1:M
        plot(x,Y(i,:))
        hold on
    end
    title('A plot of the numerical solution of y vs x');
    xlabel('x');
    ylabel('y');
    legend('u','v','w');
    hold off
end
```

Functions.m

```
function [ t ] = Functions( i,x,Y )
    u=Y(1); v=Y(2); w=Y(3);
    if i==1
        t= u;
    elseif i==2
        t= 2.*u + v - 2.*w;
    else
        t= 3.*u + 2.*v + w;
    end
end
```

Command line call

```
>> [x,Y] = EulerSys(0, 2, 0.001, [1;1;2], 'Functions');
```

Plot

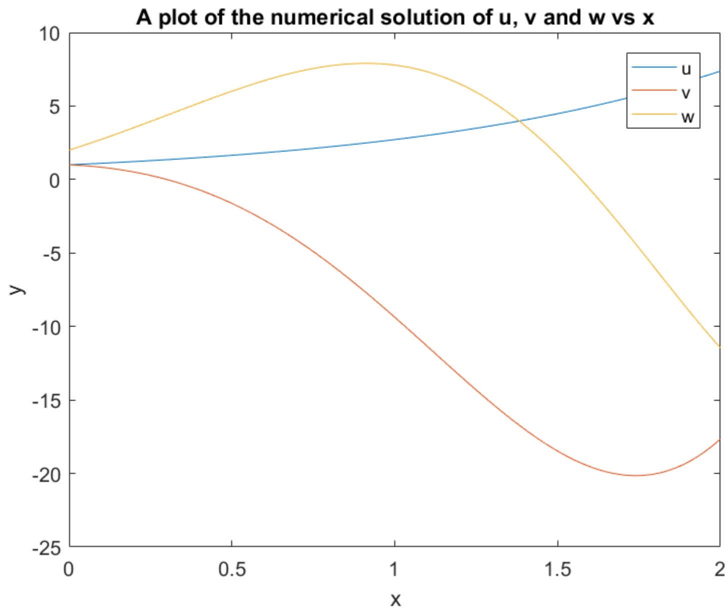

Figure 5: A plot of the numerical solution of u, v, and w vs x

4. (a) Consider

$$\vec{x}\,' = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \vec{x}$$

Assume solutions of the form $\vec{x} = \vec{\xi} e^{\lambda t}$ where λ are eigenvalues and $\vec{\xi}$ are eigenvectors, we get

$$\begin{bmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Seeking first the eigenvalues of the coefficient matrix which satisfies

$$\begin{vmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{vmatrix} = 0$$
$$\implies (1-\lambda)(-2-\lambda) - 4 = 0$$
$$\implies \lambda^2 + \lambda - 6 = 0$$
$$\implies (\lambda+3)(\lambda-2) = 0$$
$$\implies \lambda = 1,\ 3$$

Eigenvalues are $\lambda_1 = -3$ and $\lambda_2 = 2$ (real, distinct, one positive, one negative)
For $\lambda_1 = -3$ we must find the eigenvector $\vec{\xi}^{(1)}$

$$\implies \begin{bmatrix} 1-(-3) & 1 \\ 4 & -2-(-3) \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\implies \begin{bmatrix} 4 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\implies 4\xi_1 + 4\xi_2 = 0$$
$$\implies \xi_2 = -4\xi_1$$
$$\implies \vec{\xi}^{(1)} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

For $\lambda_1 = 2$ we must find the eigenvector $\vec{\xi}^{(2)}$

$$\implies \begin{bmatrix} 1-(2) & 1 \\ 4 & -2-(2) \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\implies \begin{bmatrix} -1 & 1 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\implies -\xi_1 + \xi_2 = 0$$
$$\implies \xi_2 = \xi_1$$
$$\implies \vec{\xi}^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Hence the general solution is

$$\vec{x} = C_1 \begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-3t} + C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}$$

where C_1 and C_2 are constants.

(b) Phase portrait

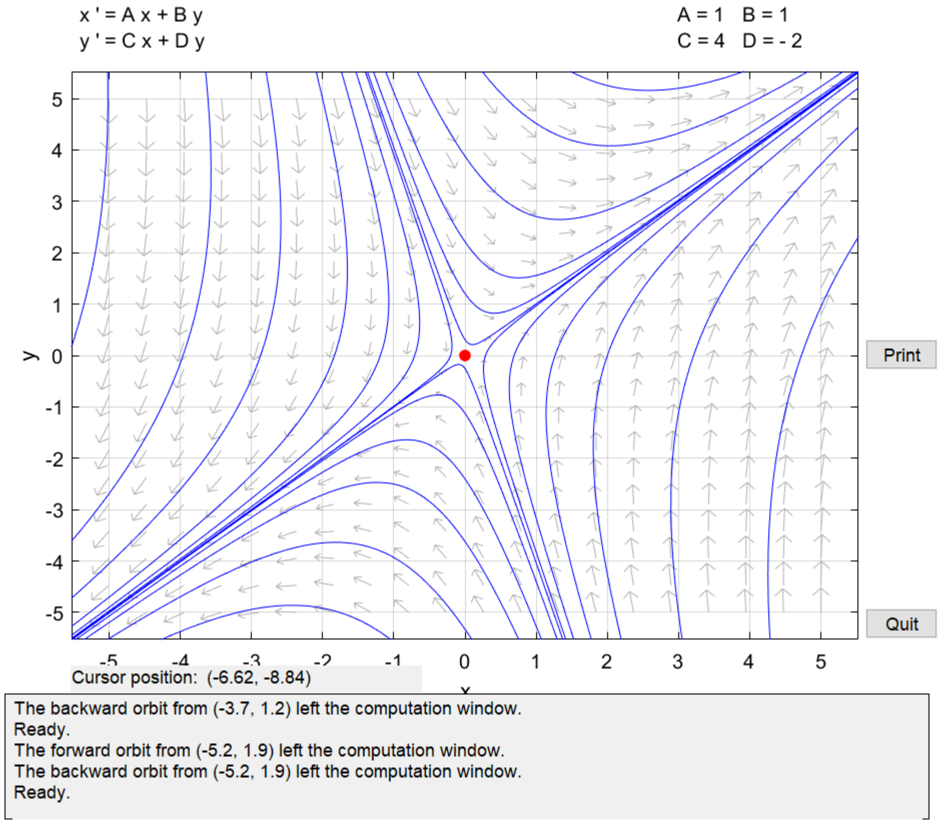

Figure 6: Phase Portrait of the Linear System

Setup

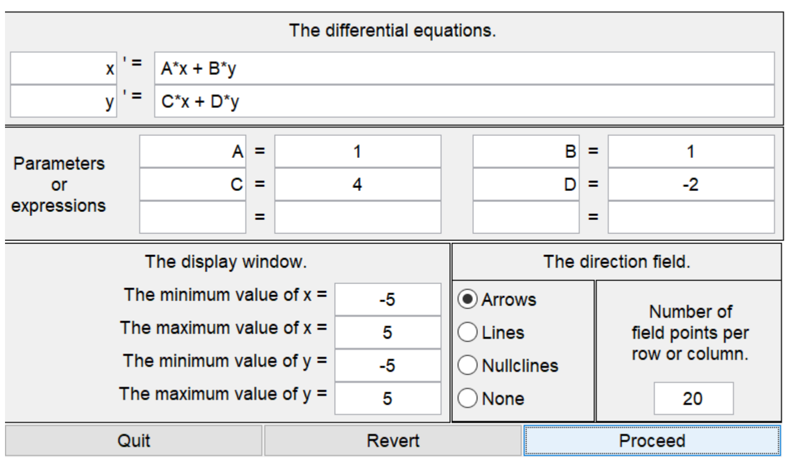

Figure 7: pplane8 setup

(c) The origin is a fixed (*saddle*) point of this system, but there are no trajectories that remain close to the origin as $t \to \infty$. Thus this fixed point is *unstable*.