Homework 5 Responses

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I chose to try and use R for this homework (I always use python and am new to R). Learning the syntax and and proper functions to manipulate data common data structures became very time consuming and I was not able to finish all the problems.

1

1.a

The plot for importance resampling for $n = 100$ can be found in Figure 1. The plot for $n = 10000$ can be found in Figure 2.

1.b

n=100

For $n = 100$, my estimate $\hat{\mu} = -0.119$.
For the t distribution with 3 degrees of freedom, t_3, the mean should be 0.
My s^2 estimate for the variance was 1.25915827778705
For the t_3 distribution this variance is 3.

These do not seem like very good estimate so I would like to explain my calculation. I calculated the mean by using the dot product of my proposal samples and the normalized 'weights', where each weight is the proposal density divided by the t_3 (true) density of a each sample.

For the sample variance I element-wise multiplied the samples by each corresponding weight, then subtracted my mean estimate, square, summed, and divided by $n - 1$.

These calculations seemed correct to me and my resampling looked good

n=10000

The same caluculations as above produces $\hat{\mu} = -0.0499$ and $s^2 = 13.7920400662904$. There was clearly something wrong with my calculation, but I could not figure out what. My resampling plots looked good so I couldn't figure out the issue.

2

I used a burn in of 20 and a thinning stride of 4. The plot of the samples along with the contour plot can be seen in Figure 3. When I calculated the posterior means by hand I got $\mathbb{E}[X1|X2] = \mu_1 - .75(x_2 - 2)$, but this caused a negative correlation in the variables so I manually flipped the sign. Still, I'm unsure where my calculation went wrong.

The trace plot for the first variable (mean=0) is in Figure 4 and the autocorrelation (after thinning) for the same variable is in Figure 5.

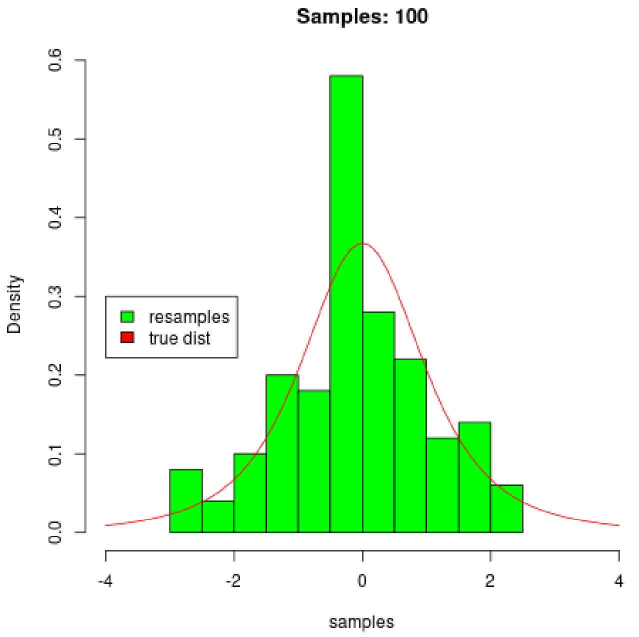

Figure 1: Importance sampling. n=100

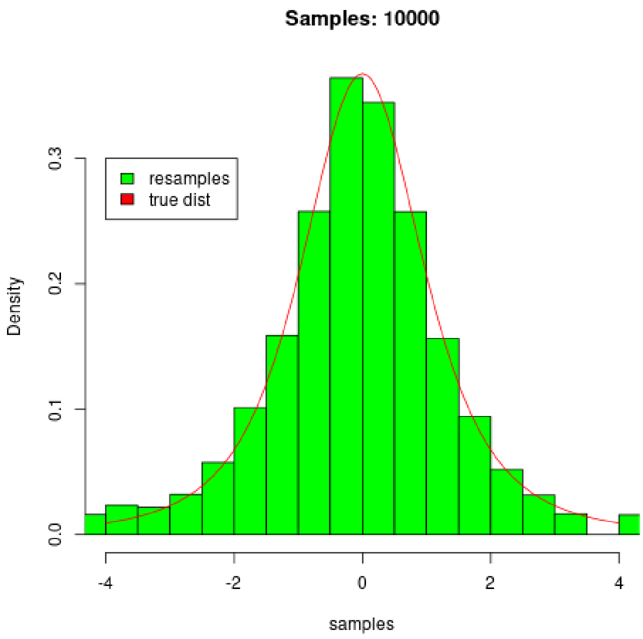

Figure 2: Importance sampling. n=100

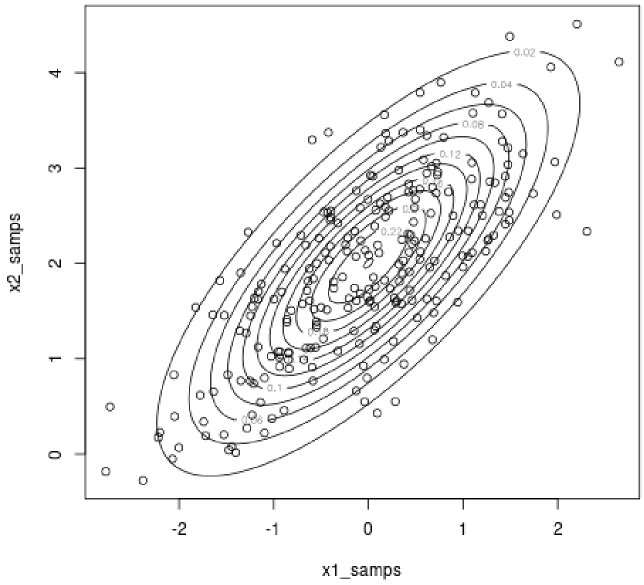

Figure 3: Results of Gibbs sampling.

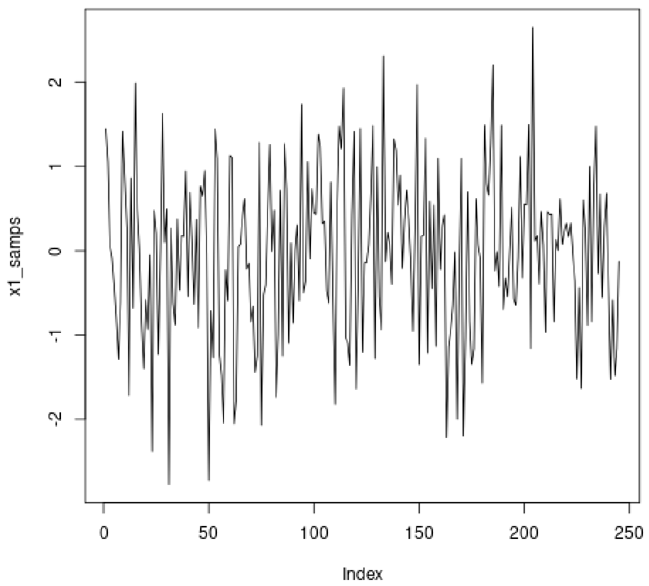

Figure 4: Trace plot of the MCMC samples from the first variable

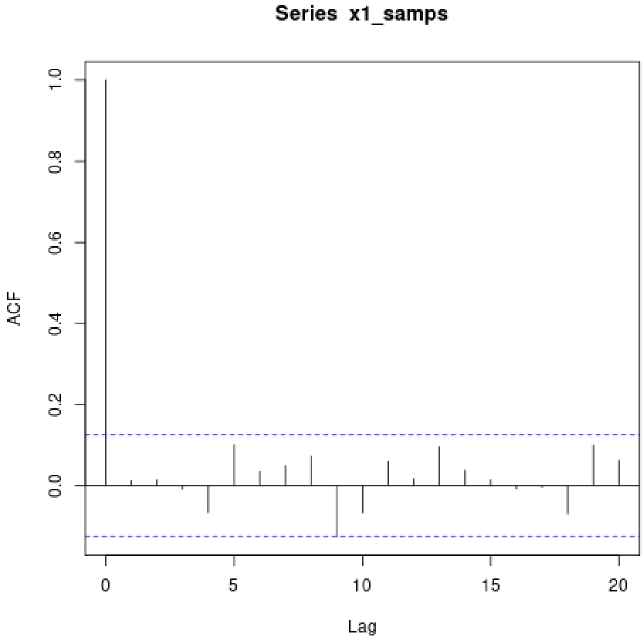

Figure 5: Autocorrelation plot of the MCMC samples of the first variable

3

AR(1) process:

$$y_{t+1} = \rho y_t + \epsilon_{t+1}, \qquad \epsilon \sim \mathcal{N}(0, \sigma^2) \tag{1}$$

3.a

Prompt: *Find conjugate priors for ρ and σ^2.*

Starting with the likelihood we have:

$$\begin{aligned} \mathcal{L}(\theta = \begin{bmatrix} \rho \\ \sigma^2 \end{bmatrix}) &= \prod_{i=1}^{T} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(1-\rho^2)(y_i - \rho y_{i-1})^2}{2\sigma^2}} \\ &= \left(\frac{1}{\sqrt{2\pi\sigma^2}} \right)^T e^{-\frac{(1-\rho^2)}{2\sigma^2} \sum_{i=1}^{T} (y_i - \rho y_{i-1})^2} \end{aligned}$$

To find the conjugate prior for ρ, σ^2 we need to eliminate all factors in the likelihood unrelated to both ρ and σ^2 up to a constant of proportionality and recognize the kernel.

$$\begin{aligned}
\mathcal{L}(\theta = \begin{bmatrix}\rho \\ \sigma^2\end{bmatrix}) &= \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^T e^{-\frac{(1-\rho^2)}{2\sigma^2}\sum_{i=1}^T (y_i - \rho y_{i-1})^2} \\
&\propto \left(\frac{1}{\sqrt{\sigma^2}}\right)^T e^{\frac{\rho^2}{2\sigma^2}\sum_{i=1}^T (y_i - \rho y_{i-1})^2} \\
&= \left(\frac{1}{\sqrt{\sigma^2}}\right)^T e^{\frac{\rho^2}{2\sigma^2}\sum_{i=1}^T y_i^2 - 2\rho y_i y_{i-1} + \rho^2 y_{i-1}^2} \\
&\propto \left(\frac{1}{\sqrt{\sigma^2}}\right)^T e^{\frac{\rho^2}{2\sigma^2}\sum_{i=1}^T \rho^2 y_{i-1}^2 - 2\rho y_i y_{i-1}} \\
&\propto \left(\frac{1}{\sqrt{\sigma^2}}\right)^T e^{\frac{\rho^2}{2\sigma^2}\sum_{i=1}^T \rho^2 y_{i-1}^2 - 2\rho y_i y_{i-1}} && \text{(completing the square)}
\end{aligned}$$