Homework 1 Responses

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1. Slutsky's theorem:
   If
   $$X_n \xrightarrow{D} X \tag{1}$$
   and
   $$Z_n \xrightarrow{P} z_0 \tag{2}$$
   for some constant $z_0 \in \mathbb{R}$, then:

   a) Prove: $X_n + Z_n \xrightarrow{D} X + z_0$

   I must show that,
   $$\lim_{n\to\infty} F_{X_n+Z_n}(a) = F_{X+z_0}(a).$$
   By definition,
   $$F_{X_n+Z_n}(a) = P(X_n + Z_n \leq a)$$
   and from the Law of Total Probability we have that,
   $$\begin{aligned} P(X_n + Z_n \leq a) = \quad & P(X_n + Z_n \leq a \big| |Z_n - z_0| \leq \epsilon) P(|Z_n - z_0| \leq \epsilon) \\ & + P(X_n + Z_n \leq a \big| |Z_n - z_0| > \epsilon) P(|Z_n - z_0| > \epsilon). \end{aligned}$$

   Taking the limit we have,
   $$\begin{aligned} \lim_{n\to\infty} P(X_n + Z_n \leq a) = \quad & P(X_n + Z_n \leq a \big| |Z_n - z_0| \leq \epsilon) \cdot 1 \\ & + P(X_n + Z_n \leq a \big| |Z_n - z_0| > \epsilon) \cdot 0 \quad \text{(by assumption 1)} \\ = \quad & P(X_n + Z_n \leq a \big| |Z_n - z_0| \leq \epsilon) \end{aligned}$$
   Note that,
   $$\begin{gathered} |Z_n - z_0| \leq \epsilon \equiv z_0 - \epsilon \leq Z_n \leq z_0 + \epsilon \\ \Rightarrow \\ X_n + z_0 - \epsilon \leq X_n + Z_n \leq X_n + z_0 + \epsilon \end{gathered}$$

   Thus,
   $$\begin{aligned} \lim_{n\to\infty} P(X_n + Z_n \leq a \big| |Z_n - z_0| \leq \epsilon) & \leq P(X_n + z_0 - \epsilon \leq a) && \text{(by assumption 2)} \\ & \leq \lim_{n\to\infty} P(X_n \leq a + \epsilon - z_0) \\ & = \lim_{n\to\infty} F_{X_n}(a + \epsilon - z_0) && \text{(def. of CDF)} \\ & = F_X(a + \epsilon - z_0) && \text{(by assumption 1)} \end{aligned}$$

The first inequality is true because the $-\epsilon$ shifts the distribution of $X_n - z_0$ away from a, which means more probability mass occurs before a. By the same reasoning, we have that,

$$F_X(a-\epsilon-z_0) \leq \lim_{n\to\infty} P(X_n + Z_n \leq a)$$

giving

$$F_X(a-\epsilon-z_0) \leq \lim_{n\to\infty} P(X_n + Z_n \leq a) \leq F_X(a+\epsilon-z_0)$$

Therefore, in the limit as ϵ approaches 0 we have that,

$$\begin{gathered}\lim_{\epsilon\to 0} F_X(a-\epsilon-z_0) \leq \lim_{n\to\infty} P(X_n + Z_n \leq a) \leq \lim_{\epsilon\to 0} F_X(a+\epsilon-z_0) \\ \equiv \\ F_X(a-z_0) \leq \lim_{n\to\infty} P(X_n + Z_n \leq a) \leq F_X(a-z_0)\end{gathered}$$

Thus we have:

$$\lim_{n\to\infty} P(X_n + Z_n \leq a) = F_X(a-z_0) \tag{3}$$

Putting together all the peices, we get:

$$\begin{aligned}\lim_{n\to\infty} F_{X_n+Z_n}(a) &= \lim_{n\to\infty} P(X_n + Z_n \leq a) \\ &= F_X(a-z_0) && \text{(by equation 3)} \\ &= P(X < a - z_0) \\ &= P(X + z_0 < a) \\ &= F_{X+z_0}(a)\end{aligned}$$

$\blacksquare$

b) Prove: $X_n \cdot Z_n \xrightarrow{D} X \cdot z_0$

This has nearly the identical reasoning as in part a), except with a multiplicative factor instead of an additive one.
I will start where the proofs diverge:
Note that,

$$\begin{gathered}|Z_n - z_0| \leq \epsilon \equiv z_0 - \epsilon \leq Z_n \leq z_0 + \epsilon \\ \Rightarrow \\ X_n(z_0-\epsilon) \leq X_n \cdot Z_n \leq X_n(z_0+\epsilon), \qquad X_n = x \geq 0 \\ X_n(z_0-\epsilon) \geq X_n \cdot Z_n \geq X_n(z_0+\epsilon), \qquad X_n = x < 0\end{gathered}$$

Wlog, I will prove the case where $X_n \geq 0$, but the same reasoning applies to the case where $X_n < 0$. Fixing ϵ, this gives:

$$\begin{aligned}\lim_{n\to\infty} P(X_n \cdot Z_n \leq a \big| |Z_n - z_0| \leq \epsilon) &\leq \lim_{n\to\infty} P(X_n(z_0-\epsilon) \leq a) && \text{(by assumption 2)} \\ &= \lim_{n\to\infty} F_{X_n}(a/(z_0-\epsilon)) && \text{(def. of CDF)} \\ &= F_X(a/(z_0-\epsilon)) && \text{(by assumption 1)}\end{aligned}$$

Note that we do not need to consider the case when $z_0 - \epsilon = 0$. This is because ϵ can be arbitrarily small by the definition of convergence in probability. We can therefore always assume $z_0 - \epsilon > 0$ since we can just limit our consideration to values of ϵ smaller than z_0.

When combined with the other inequality, this gives:

$$F_X(a/(z_0 + \epsilon)) \leq \lim_{n \to \infty} P(X_n + Z_n \leq a) \leq F_X(a/(z_0 - \epsilon))$$

The rest of the proof is exactly the same as in part a), so I will omit it.

c) Prove: $X_n/Z_n \xrightarrow{D} X/z_0, \quad z_0 \neq 0$

With the assumption that $z_0 \neq 0$, the proof of this is nearly identical to part b) since the only edge case is handled by the assumption. I will therefore omit this since I do not have time to write it again and it will be overly repititious.

2. Poisson problems

(a) Prove that if $Y \sim \text{Poisson}(\lambda) \to \mathbb{E}[Y] = \lambda$ and $\text{Var}(Y) = \lambda$

The first part can be shown directly by the definition of $\mathbb{E}[Y]$:

$$\begin{aligned}
\mathbb{E}[Y] &= \sum_{-\infty}^{\infty} y \lambda^y \frac{e^{-\lambda}}{y!} && \text{(by definition)} \\
&= \sum_{y=0}^{\infty} y \lambda^y \frac{e^{-\lambda}}{y!} && \text{(Poisson region of interest)} \\
&= e^{-\lambda} \sum_{y=0}^{\infty} \frac{y \lambda^y}{y!} && \text{(Algebra)} \\
&= e^{-\lambda} \lambda \sum_{y=1}^{\infty} \frac{\lambda^{y-1}}{(y-1)!} && \text{(Algebra)} \\
&= e^{-\lambda} \lambda \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} && \text{(Substitute } k = y - 1\text{)} \\
&= e^{-\lambda} \lambda e^{\lambda} && \text{(Taylor expansion for } e^x\text{)} \\
&= \lambda
\end{aligned}$$

Similarly, for $\mathrm{Var}(Y)$ we have:

$$
\begin{aligned}
\mathrm{Var}(Y) &= \mathbb{E}[Y^2] - \mathbb{E}[Y]^2 && \text{(by definition)}\\
&= \sum_{y=0}^{\infty} y^2 \lambda^y \frac{e^{-\lambda}}{y!} - \lambda^2\\
&= e^{-\lambda}\lambda \sum_{y=1}^{\infty} y \frac{\lambda^{y-1}}{(y-1)!} - \lambda^2 && \text{(Above steps)}\\
&= e^{-\lambda}\lambda \sum_{y=1}^{\infty} ((y-1)+1) \frac{\lambda^{y-1}}{(y-1)!} - \lambda^2\\
&= e^{-\lambda}\lambda \sum_{y=1}^{\infty} \Big((y-1) \frac{\lambda^{y-1}}{(y-1)!} + \frac{\lambda^{y-1}}{(y-1)!} \Big) - \lambda^2\\
&= e^{-\lambda}\lambda \Big(\sum_{y=1}^{\infty} (y-1) \frac{\lambda^{y-1}}{(y-1)!} + \sum_{y=1}^{\infty} \frac{\lambda^{y-1}}{(y-1)!} \Big) - \lambda^2\\
&= e^{-\lambda}\lambda \Big(\sum_{y=1}^{\infty} (y-1) \frac{\lambda^{y-1}}{(y-1)!} + e^{\lambda} \Big) - \lambda^2 && \text{(Same as above)}\\
&= e^{-\lambda}\lambda \Big(\lambda \sum_{y=2}^{\infty} \frac{\lambda^{y-2}}{(y-2)!} + e^{\lambda} \Big) - \lambda^2\\
&= e^{-\lambda}\lambda \Big(\lambda e^{\lambda} + e^{\lambda} \Big) - \lambda^2\\
&= \lambda^2 + \lambda - \lambda^2\\
&= \lambda
\end{aligned}
$$

(b) Decide which estimator is better λ, $\bar{Y}_n$ or s_n^2

We have shown in class that both $\bar{Y}_n$ and s_n^2 are unbiased, so here we will calculate the variance.

i. $\text{Var}(\bar{Y}_n)$:

$$\begin{aligned}
\text{Var}(\bar{Y}_n) &= \mathbb{E}[\bar{Y}_n^2] - \mathbb{E}[\bar{Y}_n]^2 \\
&= \frac{1}{n^2}\mathbb{E}[\sum_{i=1}^{n} Y_i \sum_{j=1}^{n} Y_j] - \lambda^2 \\
&= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} \mathbb{E}[Y_i Y_j] - \lambda^2 \\
&= \frac{1}{n^2}\Big(\sum_{i=j} \mathbb{E}[Y_i Y_j] + \sum_{i\neq j} \mathbb{E}[Y_i Y_j]\Big) - \lambda^2 \\
&= \frac{1}{n^2}\Big(\sum_{i=1}^{n} \mathbb{E}[Y_i^2] + \sum_{i\neq j} \mathbb{E}[Y_i]\mathbb{E}[Y_j]\Big) - \lambda^2 && \text{(by independence)} \\
&= \frac{1}{n^2}\Big(\sum_{i=1}^{n} (\text{Var}(Y_i) + \mathbb{E}[Y_i]^2) + n(n-1)\lambda^2\Big) - \lambda^2 \\
& && (\mathbb{E}[Y_i^2] = \text{Var}(Y_i) + \mathbb{E}[Y_i]^2, \text{ by def. of Var}(\cdots)) \\
&= \frac{1}{n^2}\Big(n(\lambda + \lambda^2) + n(n-1)\lambda^2\Big) - \lambda^2 \\
&= \frac{1}{n}\lambda + \frac{1}{n}\lambda^2 + \lambda^2 - \frac{1}{n}\lambda^2 - \lambda^2 \\
&= \frac{1}{n}\lambda
\end{aligned}$$

ii. $\text{Var}(s_n^2)$:
From slide 29 of lecture we know that:

$$\text{Var}(s_n^2) = \frac{\mu_4}{n}$$

Online I found that μ_4 for the Poisson distribution is $\lambda^4 + 7\lambda^3 + 6\lambda^2 + \lambda$ from taking the fourth derivative of the MGF of the Poisson evaluated at zero. This gives

$$\text{Var}(s_n^2) = \frac{\lambda^4 + 7\lambda^3 + 6\lambda^2 + \lambda}{n} \geq \frac{1}{n}\lambda = \text{Var}(\bar{Y}_n)$$

Therefore I would choose $\bar{Y}_n$ as $\hat{\lambda}$.

3. Show that $r_n \xrightarrow{P} \rho$ when finite second order moments exist for the underlying population distribution.

$$r_n = \frac{\sum_{i=1}^{n}(X_i - \bar{X}_n)(Y_i - \bar{Y}_n)}{\sqrt{\big(\sum_{i=1}^{n} X_i - \bar{X}\big)^2}\sqrt{(\sum_{j=1}^{n} Y_j - \bar{Y})^2}}$$

$$\begin{aligned}
r_n &= \frac{\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X}_n)(Y_i - \bar{Y}_n)}{\sqrt{\frac{1}{n}}\sqrt{\frac{1}{n}}\sqrt{\big(\sum_{i=1}^{n} X_i - \bar{X}_n\big)^2}\sqrt{(\sum_{j=1}^{n} Y_j - \bar{Y}_n)^2}} \\
&= \frac{\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X}_n)(Y_i - \bar{Y}_n)}{\sqrt{\cdot\tilde{s}_{Y_n}^2}\sqrt{\cdot\tilde{s}_{X_n}^2}} \qquad (4)
\end{aligned}$$

With $\tilde{s}_{X_n}^2$ and $\tilde{s}_{Y_n}^2$ being the sample variance for X_n and Y_n respectively. By the WLLN, both $\tilde{s}_{Y_n}^2 \xrightarrow{P} \sigma_{Y_n}^2$ and $\tilde{s}_{X_n}^2 \xrightarrow{P} \sigma_{X_n}^2$. By the multivariate continuous mapping theorem then,

$$\sqrt{s_{Y_n}^2}\sqrt{s_{X_n}^2} \xrightarrow{P} \sigma_Y \sigma_X$$

Similarly, the numerator of equation 4 can be written as:

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X_n})(Y_i - \bar{Y_n})) &= \frac{1}{n}(\sum_{i=1}^{n} X_iY_i - \bar{X_n}\bar{Y_n}) \\ &= \frac{1}{n}\sum_{i=1}^{n} X_iY_i - \bar{X_n}\bar{Y_n}\end{aligned}$$

Again, by the WLLN and the multivariate continuous mapping theorem,

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n} X_iY_i &\xrightarrow{P} \mathbb{E}[XY] \\ X_nY_n &\xrightarrow{P} \mathbb{E}[X]\mathbb{E}[Y]\end{aligned}$$

giving,

$$\frac{1}{n}\sum_{i=1}^{n} X_iY_i - X_nY_n \xrightarrow{P} \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = \text{Cov}(X, Y)$$

Thus, for equation 4

$$\frac{\frac{1}{n}(\sum_{i=1}^{n} X_i - \bar{X})(\sum_{j=1}^{n} Y_j - \bar{Y})}{\sqrt{\cdot\tilde{s}^2_{Y_n}}\sqrt{\cdot\tilde{s}^2_{X_n}}} \xrightarrow{P} \frac{\text{Cov}(X, Y)}{\sigma_{X_n}\sigma_{Y_n}} = \rho$$

4. Show that $SE(\hat{\alpha}) \approx \sqrt{\frac{2\alpha(\alpha-1)}{n}}$, where $\hat{\alpha} = \bar{Y_n}/s_n^2$ with n samples drawn from a Gamma distribution with parameters α, β.

   Following the strategic outline from lecture, we will use the multivariate delta method to derive the variance of $\hat{\alpha}$ for large values of n. From this we can easily get the standard error.

   Using the information given on slide 124 of lecture, we have:

$$\text{Var}(\bar{Y_n}) = \frac{\alpha}{\beta^2 n}, \text{Var}(s_n^2) = \frac{6\alpha}{\beta^4 n} + \frac{2\alpha^2}{\beta^4 n}, \text{Cov}(\bar{Y_n}, s_n^2) = \frac{2\alpha}{\beta^3 n}$$

$$h(\theta) = \theta_1^2/\theta_2, \theta_1 = \mu^2\bar{Y_n}, \theta_2 = \sigma$$

   Calculating the partial derivatives of $h(\theta)$ gives $h(\theta)'^{\intercal} = \begin{bmatrix}2\beta & -\beta\end{bmatrix}$

$$h(\theta)' = \begin{bmatrix}2\beta \\ -\beta\end{bmatrix}$$

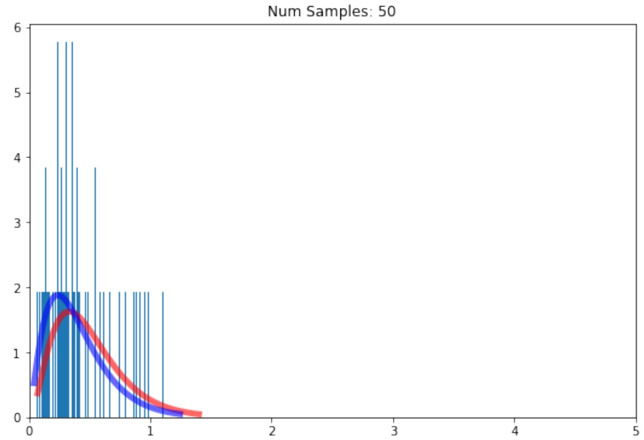
   Because of time constraints I will omit the full matrix calculations for the variance, but as shown in lecture:

$$\text{Var}(\bar{Y_n}^2/s_n^2) = \frac{\alpha(\alpha+1)}{n}$$

   By the continuous mapping theorem, and the fact that $\hat{\alpha}$ is an unbiased estimator (i.e. $\mathbb{E}[\hat{\alpha}] = \alpha$), we have that when n grows large:

$$SE(\hat{\alpha}) \approx \sqrt{\frac{\alpha(\alpha+1)}{n}} \approx \sqrt{\frac{\hat{\alpha}(\hat{\alpha}+1)}{n}}$$

Figure 1: Random Samples from Gamma



5. First plot is in Figure 1. These are the findingstrue $\alpha : 3, \hat{\alpha} : 2.458$, true $\beta : 6, \hat{\beta} : 5.238$. The predicted density is in red and the true density is in blue.

   Second plot is Figure 2.

6. Show that if $X \sim N(\mu, \sigma^2$, then $M_X(t) = e^{\mu t + \sigma^2 t^2/2}$.

   I will show this directly from the definition:

$$
\begin{aligned}
M_X(t) = \mathbb{E}[e^{tX}] &= \int e^{tx} \frac{1}{\sigma\sqrt{2\pi}} e^{-(\frac{x-\mu}{\sqrt{2}\sigma})^2} dx \\
&= \frac{1}{\sigma\sqrt{2\pi}} \int e^{tx - (\frac{x-\mu}{\sqrt{2}\sigma})^2} dx \\
&= \frac{1}{\sigma\sqrt{2\pi}} \int e^{t(u\sigma+\mu) - (\frac{u}{\sqrt{2}})^2} \sigma du && \text{let } u = \tfrac{x-\mu}{\sigma} \\
&= \frac{e^{t\mu}}{\sqrt{2\pi}} \int e^{-\frac{u^2}{2} + t\sigma u} du \\
&= \frac{e^{t\mu}}{\sqrt{2\pi}} \int e^{-\frac{1}{2}(u+t\sigma)^2 + \frac{t^2\sigma^2}{2}} du && \text{(completing the square)} \\
&= e^{t\mu} e^{\frac{t^2\sigma^2}{2}} \int \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(u+t\sigma)^2} du \\
&= e^{t\mu + \frac{t^2\sigma^2}{2}} && \text{(Normal}(u+t\sigma,\, 1) \text{ integrates to 1)}
\end{aligned}
$$

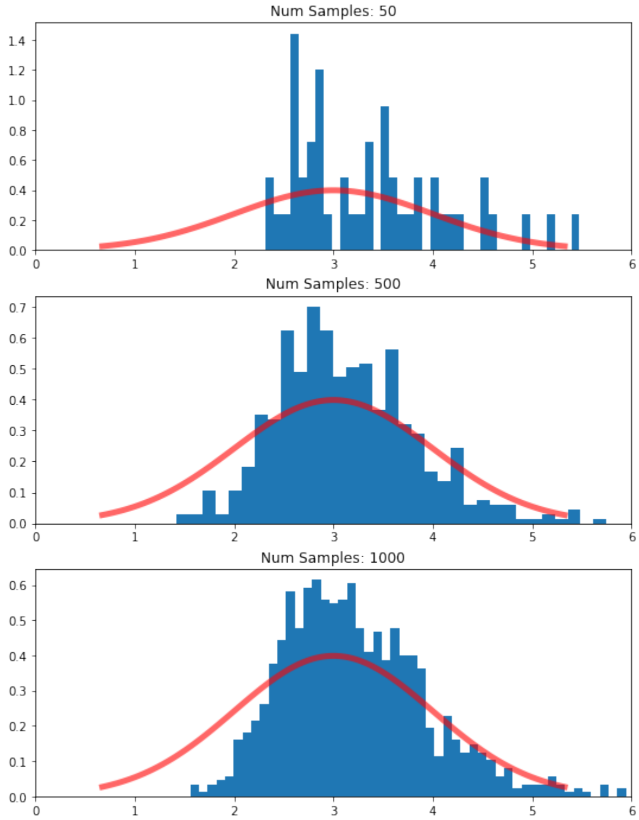
   ■

7. Binomial problems

   (a) Show that for $R \sim Bin(\pi, m)$, $K_{Bin}(t)$ the cumulant generating function $\log(M_{Bin}(t))$ is $m \log(1 - \pi + \pi e^t)$.

   We will start by deriving the MGF of the Binomial distribution,

Figure 2: Distribution of Estimator



$M_{Bin}(t) = (1 - \pi + \pi e^t)^m$.

$$
\begin{aligned}
M_{Bin}(t) = \mathbb{E}[e^{Rt}] &= \sum_{x=0}^{m} e^{xt} \cdot P(R = x) \\
&= \sum_{y=0}^{m} e^{xt} \binom{m}{x} \pi^x (1-\pi)^{m-x} \\
&= \sum_{y=0}^{m} \binom{m}{x} (e^t \pi)^x (1-\pi)^{m-x}
\end{aligned}
$$

Recall from the Binomial Theorem that:

$$
(z + y)^n = \sum_{k=0}^{n} \binom{n}{k} z^k y^{n-k} = \sum_{k=0}^{n} \binom{n}{k} z^{n-k} y^k \tag{5}
$$

By choosing $z := e^t \pi$, $y := 1 - \pi$, $k := x$, and $m := n$, we have that:

$$
\sum_{y=0}^{m} \binom{m}{x} (e^t \pi)^x (1-\pi)^{m-x} = (\pi e^t + 1 - \pi)^m
$$

Thus,

$$
K_{Bin}(t) = log(M_{bin}(t) = m \log(\pi e^t + 1 - \pi)
$$

■

(b) Show that $\lim_{\pi\to 0}\lim_{m\to\infty} K_{Bin}(t)$ implies $m\pi \to \lambda$, for some constant $\lambda > 0$.

$$\begin{aligned}\lim_{\pi\to 0}\lim_{m\to\infty} K_{Bin}(t) &= \lim_{\pi\to 0}\lim_{m\to\infty} m\log(\pi e^t + 1 - \pi) = m\log(1+\pi e^t - \pi)\\ &= \lim_{\pi\to 0}\lim_{m\to\infty} \log\Big((1+\pi(e^t-1))^m\Big)\end{aligned}$$

Recall that $\lim_{n\to\infty}(1+x/n)^n = e^x$. This gives:

$$\begin{aligned}\lim_{m\to\infty}\lim_{\pi\to 0} \log\Big((1+\pi(e^t-1))^m\Big) &= \lim_{m\to\infty}\lim_{\pi\to 0} \log\Big((1+\frac{m\pi(e^t-1)}{m})^m\Big)\\ &= \lim_{m\to\infty}\lim_{\pi\to 0} \log\Big((1+\frac{\lambda(e^t-1)}{\lambda/\pi})^{(\lambda/\pi)}\Big)\\ &\qquad\qquad (\text{let } \lambda = m\pi)\\ &= \lim_{m\to\infty} \log(e^{\lambda(e^t-1)})\\ &= \lim_{m\to\infty} \lambda(e^t-1)\\ &= \lambda(e^t-1)\end{aligned}$$

■

(c) Show that $\lim_{\pi\to 0}\lim_{m\to\infty} K_{Bin}(t) \xrightarrow{D} Pois(\lambda)$

We showed in part 7b that $\lim_{\pi\to 0}\lim_{m\to\infty} K_{Bin}(t) = \lambda(e^t-1)$. The MGF of the Binomial is just $e^{K_{Bin}(t)}$. The same proof given above holds for $\lim_{\pi\to 0}\lim_{m\to\infty} e^{K_{Bin}(t)}$. This means that in the limits stated, the MGF of the binomial converges to $e^{\lambda(e^t-1)}$. This is exactly the MGF of the Poisson distribution. Thus, by the Uniquess Theorem of MGFs:

$$\lim_{\pi\to 0}\lim_{m\to\infty} P(R=r) \xrightarrow{D} \frac{\lambda^r}{r!}e^{-\lambda} = \text{Poisson}(m\pi = \lambda)$$

(d)

8. Normal problems

(a) If $Z \sim$ Normal(0, 1), derive the density of $Y = Z^2$. We can derive the density of Y using the CDF technique for random variable transformations.

$$\begin{aligned}F_Y(y) &= P(Y \le y)\\ &= P(X^2 \le y)\\ &= P(|X| \le \sqrt{y})\\ &= P(-\sqrt{y} \le X \le \sqrt{y})\\ &= F_X(\sqrt{y}) - F_X(-\sqrt{y})\end{aligned}$$

Now we just need to differentiate the CDF to get the PDF.

$$
\begin{aligned}
F'_Y(y) &= \frac{d}{dy}(F_X(\sqrt{y}) - F_X(-\sqrt{y})) \\
&= \frac{d}{dy}F_X(\sqrt{y}) - \frac{d}{dy}F_X(-\sqrt{y}) \\
&= \frac{f_X(\sqrt{y})}{2\sqrt{y}} + \frac{f_X(-\sqrt{y})}{2\sqrt{y}} \\
&= \frac{1}{2\sqrt{y2\pi}}e^{-\frac{1}{2}(\sqrt{y})^2} + \frac{1}{2\sqrt{y2\pi}}e^{-\frac{1}{2}(-\sqrt{y})^2} \\
&= \frac{1}{2\sqrt{y2\pi}}e^{-\frac{y}{2}} + \frac{1}{2\sqrt{y2\pi}}e^{-\frac{y}{2}} \\
&= \frac{1}{\sqrt{y2\pi}}e^{-\frac{y}{2}}
\end{aligned}
$$

This is a chi-squared distribution

(b) Show that Y is uncorrelated with Z.

To do this we will look at $\mathrm{Cov}(Y, Z)$ which is the numerator of the correlation calculation.

$$
\begin{aligned}
\mathrm{Cov}(Y, Z) &= \mathbb{E}[YZ] - \mathbb{E}[Y]\mathbb{E}[Z] \\
&= \mathbb{E}[YZ] - 0 && (\mathbb{E}[Z] = 0) \\
&= \mathbb{E}[Z^3] - 0 && (\mathbb{E}[Z] = 0) \\
&= 0 - 0 && (\text{odd moments of the normal are } 0)
\end{aligned}
$$

Since $\mathrm{Cov}(Y, Z)$ the correlation is also zero.

9. Did not have time for this.

10. Done