(5) The joint posterior distribution with data augmentation, introducing subject-paths as latent variables.

$$\pi(\theta, \mathbf{X}|\mathbf{Y}) \propto \Pr(\mathbf{Y}|\mathbf{X}, \rho) \times \pi(\mathbf{X}|\mathbf{X}(t_1), \beta, \mu) \times \mathbb{P}(\mathbf{X}(t_1)|\mathbf{p}_{t_1}) \times \pi(\beta)\pi(\mu)\pi(\rho)\pi(\mathbf{p}_{t_1}) \tag{1}$$

(6) Let us introduce the variable $\mathcal{I}_\tau^{(-j)}$, the count of the infected individuals at time τ excluding individual j. That is, let

$$\mathcal{I}_\tau^{(-j)} = \sum_{i \neq j} \mathbb{I}(\mathbf{X}_i(\tau) = I)$$

Infinitesimal generator matrix:

$$\mathbf{\Lambda}_m^{(j)}(\theta) = \begin{pmatrix} -\beta I_{\tau_m}^{(-j)} & \beta I_{\tau_m}^{(-j)} & 0 \\ 0 & -\mu & \mu \\ 0 & 0 & -\mu \end{pmatrix} \tag{2}$$

The transition probability matrix for subject j over interval I_m:

$$\mathbf{P}^{(j)}(\tau_{m-1}, \tau_m) = \left(p_{a,b}^{(j)}(\tau_{m-1}, \tau_m)\right)_{a,b \in \mathcal{S}_j}$$

but this really turns out to be the matrix exponential:

$$\mathbf{P}^{(j)}(\tau_{m-1}, \tau_m) = \exp\left[(\tau_m - \tau_{m-1})\mathbf{\Lambda}_m^{(-j)}(\theta)\right]$$

(7) We have the time-inhomogeneous CTMC density over the observation period $[t_1, t_L]$, denoted $\pi(\mathbf{X}_j|\mathbf{x}_{(-\mathbf{j})}, \theta) \equiv \pi(\mathbf{X}_j|\mathbf{\Lambda}^{(-j)}(\theta); \mathcal{I})$. Note that it is only time-inhomogeneous because the transition rate matrix varies as a function of the count of currently infected individuals. Thus, we can decompose the time-inhomogeneous density into a product of time-homogeneous densities for each period of where the number of infected does not change. This is given by:

$$\pi(\mathbf{X}_j|\mathbf{\Lambda}^{(-j)}; \mathcal{I}) = \Pr(\mathbf{X}_j(t_1)|\mathbf{p}_{t_1}) \times \prod_{m=1}^{M} \pi\left(\mathbf{X}_j|\mathbf{x}_j(\tau_{m-1}), \mathbf{\Lambda}_m^{(-j)}(\theta); \mathcal{I}_m\right) \tag{3}$$