Introduction

This write-up outlines a project focused on the economic problem of portfolio optimization, which involves determining the optimal allocation of assets in a portfolio subject to various constraints, such as risk tolerance and expected return. This problem was chosen because it has significant implications for investors and financial markets. The methods that will be used to solve this problem include minimization methods (e.g. Newton-Raphson) and interpolation and approximation of functions (e.g. cubic splines).

Background

Portfolio optimization is a well-studied problem in finance and economics, dating back to the work of Harry Markowitz in the 1950s. The literature has developed various methods for analyzing and optimizing portfolios, including mean-variance optimization, risk parity, and factor-based models. This project aims to contribute to this literature by utilizing numerical methods to optimize portfolios, with the potential to provide more flexible and precise solutions.

Model Description

The economic model for this project is based on the mean-variance optimization framework, which seeks to maximize expected return while minimizing risk, as measured by the portfolio's variance. The model will incorporate assumptions such as:

- Investors have rational expectations and aim to maximize their utility
- Asset returns follow a multivariate normal distribution
- Investors have access to a risk-free asset

The key variables and parameters in the model include asset returns, their covariances, risk aversion, and portfolio weights.

Methodology

Methods Overview

The numerical methods used to solve the model include:

1. Minimization methods (e.g. Newton-Raphson): These methods will be employed to find the optimal portfolio weights that minimize the portfolio risk for a given level of expected return.
2. Interpolation and approximation of functions (e.g. cubic splines): These methods will be used to approximate the efficient frontier, which is the set of optimal portfolios that provide the highest expected return for a given level of risk.

Potential limitations or challenges in applying these methods include handling non-normal return distributions and dealing with the curse of dimensionality for large-scale portfolio optimization problems.

Project Steps

1. Data Preparation:
    a. Collect historical price data for the assets in the portfolio.
    b. Calculate the logarithmic returns for each asset.
    c. Compute the mean return and covariance matrix for the assets.

2. Minimization Method (Newton-Raphson):
    a. Define the objective function: This function quantifies the risk-adjusted performance of the portfolio. For instance, the Sharpe ratio, which is the ratio of the portfolio's excess return (i.e., return above the risk-free rate) to its standard deviation, can be used as the objective function.
    b. Calculate the gradient and Hessian matrix of the objective function with respect to the portfolio weights.
    c. Initialize the portfolio weights and set a convergence criterion (e.g., a maximum number of iterations or a tolerance level for changes in weights).
    d. Iterate the Newton-Raphson algorithm:
        i. Update the portfolio weights using the gradient and Hessian matrix.
        ii. Check for convergence. If the convergence criterion is met, stop iterating; otherwise, continue with the next iteration.
    e. Obtain the optimal portfolio weights that maximize the objective function.

3. Interpolation and Approximation of Functions (Cubic Splines):
    a. Define a range of target expected returns or risk levels.
    b. For each target level, use the Newton-Raphson method (as described in step 2) to find the optimal portfolio weights that meet the target while minimizing risk.
    c. Use the optimal portfolio weights and their corresponding risk-return characteristics as data points.
    d. Fit a cubic spline to the data points, approximating the efficient frontier.
    e. Use the fitted cubic spline to interpolate or extrapolate the optimal portfolio weights and risk-return characteristics for any desired target level within the range.

4. Sensitivity Analysis:

a. Perform a sensitivity analysis by perturbing the key model parameters (e.g., expected returns, covariances, and risk aversion) and re-solving the optimization problem using the Newton-Raphson method and cubic splines.
b. Analyze the changes in optimal portfolio weights and risk-return characteristics in response to the perturbations in model parameters.

Expected Results

Preliminary results or expected outcomes of the analysis include:

- The identification of optimal portfolio weights for various levels of risk tolerance
- A comparison of the numerical methods used in terms of their accuracy and computational efficiency
- An examination of the sensitivity of optimal portfolios to changes in model parameters

These results will contribute to the understanding of the portfolio optimization problem and may have practical implications for investment strategies and risk management.

Conclusion

This write-up presents a project that aims to analyze the portfolio optimization problem using numerical methods. By utilizing minimization methods and interpolation and approximation of functions, the project seeks to provide a flexible and precise approach to solving this important economic problem. The results of this analysis have the potential to contribute to the existing literature on portfolio optimization and inform practical investment decisions. Future research could explore alternative optimization methods, incorporate additional constraints, or extend the model to include other asset classes or market frictions.