HW02: Prove Beta-Binomial conjugate 312581039 智能碩 吳致廣

Assume prior $\sim$ Beta$(p|a,b)$, likelihood $\sim$ Binomial$(X|N,p)$ and sucess m times

$$P(\theta|D) = \frac{\cancel{\binom{N}{m}} p^m (1-p)^{N-m} \times p^{a-1}(1-p)^{b-1} \cancel{\frac{\gamma(a)\gamma(b)}{\gamma(a+b)}}}{\int_0^1 \cancel{\binom{N}{m}} \theta^m (1-\theta)^{m-1} \theta^{a-1}(1-\theta)^{b-1} \cancel{\frac{\gamma(a)\gamma(b)}{\gamma(a+b)}} d\theta} \quad \left(\frac{\text{likelihood} \times \text{prior}}{\text{marginalize}}\right)$$

$$= \frac{p^{m+a-1}(1-p)^{N-m+b-1}}{\int_0^1 \theta^{m+a-1}(1-\theta)^{N-m+b-1} d\theta}$$

marginalize 為 beta function form $\Rightarrow B(a,b) = \int_0^\infty p^{a-1}(1-p)^{b-1} dp = \frac{\gamma(a)\gamma(b)}{\gamma(a+b)}$

$$B(m+a, N-m+b) = \int_0^1 \theta^{m+a-1}(1-\theta)^{N-m+b-1} d\theta = \frac{\gamma(m+a)\gamma(N-m+b)}{\gamma(a+N+b)}$$

代回 posterior

$$\Longrightarrow P(\theta|D) = \frac{p^{m+a-1}(1-p)^{N-m+b-1}}{\int_0^1 \theta^{m+a-1}(1-\theta)^{N-m+b-1} d\theta} = p^{m+a-1}(1-p)^{N-m+b-1} \frac{\gamma(a+N+b)}{\gamma(m+a)\gamma(N-m+b)}$$

根據 beta distribution form $= \theta^{a-1}(1-\theta)^{b-1} \frac{\gamma(a+b)}{\gamma(a)\gamma(b)}$

得出 $P(\theta|D) \sim B(\theta|m+a, N-m+b)$

得證 Beta-Binomial 為 conjugation #.