A Proof of Convergence for Two Parallel Jacobi SVD Algorithms

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Abstract—**We consider two parallel Jacobi algorithms, due to Brent *et al.* and to Luk, for computing the singular value decomposition of an $n \times n$ matrix. By relating the algorithms to the cyclic-by-rows Jacobi method, we prove convergence of the former for odd n and of the latter for any n. We also give a nonconvergence example for the former method for all even $n \geq 4$.**

Index Terms—**Odd-even ordering, parallel Jacobi methods, round-robin ordering, singular value decomposition, systolic arrays.**

I. Introduction

Given a real $n \times n$ matrix A, a Jacobi method for computing its singular value decomposition consists of generating a sequence of matrices $\{A_l\}$ via the relation

$$A_{l+1} = J_l^T A_l K_l, \qquad l = 1, 2, \cdots, \tag{1}$$

where $A_1 = A$, and J_l and K_l are plane rotations. Let $A_l \equiv (a_{ij}^{(l)})$ and let J_l and K_l represent rotations through angles θ and ϕ in the (p, q) plane, with $1 \leq p < q \leq n$. If the angles θ and ϕ satisfy the equations

$$\begin{cases} \tan(\theta + \phi) = (a_{qp}^{(l)} + a_{pq}^{(l)})/(a_{qq}^{(l)} - a_{pp}^{(l)}) \\ \tan(-\theta + \phi) = (a_{qp}^{(l)} - a_{pq}^{(l)})/(a_{qq}^{(l)} + a_{pp}^{(l)}). \end{cases} \tag{2}$$

then we get

$$a_{pq}^{(l+1)} = a_{qp}^{(l+1)} = 0.$$

The rotation angles may be limited to the closed interval

$$-\pi/2 \leq \theta, \ \phi \leq \pi/2. \tag{3}$$

We want the sequence $\{A_l\}$ to converge to a diagonal matrix that will contain the singular values of A on its diagonal. The closeness of A_l to a diagonal form is often measured by the quantity

$$s_l = \sum_{i \neq j} |a_{ij}^{(l)}|^2 \tag{4}$$

and we say that a Jacobi method "converges" if $s_l \to 0$ for any initial matrix A_1.

Let T_{pq} represent the transformation of A which occurs when the matrix is rotated in the (p, q) plane according to (1), i.e.,

$$T_{pq}(A) = J_{pq}^T A K_{pq}. \tag{5}$$

In a cyclic Jacobi method, the $N = n(n-1)/2$ different index pairs ("pivots") are selected in some fixed order. Such sequence of N transformations is often called a "sweep." We present a sweep of the usual cyclic-by-rows (T_R) and cyclic-by-columns (T_C) orderings for the case $n = 5$:

$$T_R = T_{45} T_{35} T_{34} T_{25} T_{24} T_{23} T_{15} T_{14} T_{13} T_{12} \tag{6}$$

and

$$T_C = T_{45} T_{35} T_{25} T_{15} T_{34} T_{24} T_{14} T_{23} T_{13} T_{12}. \tag{7}$$

It is easily checked that the transformations T_{pq} and T_{rs} commute if p is neither r nor s and q is neither r nor s. Let T_1 and T_2 represent products of transformations T_{pq}'s in a sweep according to two different orderings O_1 and O_2, respectively. We say that O_1 and O_2 are "equivalent" if T_1 can be changed into T_2 by a set of transpositions of pairs of commuting transformations T_{pq}. Hansen [5] showed that the row and column orderings are equivalent and gave an obvious property of equivalent orderings.

Lemma 1: Suppose that the Jacobi method using the O_1 ordering converges. If another ordering O_2 is equivalent to O_1, then the Jacobi method using the O_2 ordering also converges. □

As far as we know, the row ordering (or the column ordering) has been the only ordering that guarantees convergence of the cyclic Jacobi method.

Theorem 1 (Forsythe and Henrici [3]): Let t be a constant satisfying $0 \leq t < 1$ and let H be a closed interval interior to the open interval $(-\pi/2, \pi/2)$. Suppose that the sequence of pivots (p, q) is defined by the row ordering. If the conditions

$$|a_{pq}^{(l+1)}|^2 + |a_{qp}^{(l+1)}|^2 \leq t(|a_{pq}^{(l)}|^2 + |a_{qp}^{(l)}|^2) \tag{8}$$

and

$$\theta, \phi \in H \tag{9}$$

hold, then the Jacobi method converges. □

The formulas for computing θ and ϕ that ensure conver-

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gence are given in [3]. Defining σ and τ by

$$\begin{cases} \tan \sigma = (a_{qp}^{(l)} + a_{pq}^{(l)})/(a_{qq}^{(l)} - a_{pp}^{(l)}) \\ \tan \tau = (a_{qp}^{(l)} - a_{pq}^{(l)})/(a_{qq}^{(l)} + a_{pp}^{(l)}) \end{cases} \tag{10}$$

where $-\pi/2 \le \sigma, \tau \le \pi/2$, we choose

$$\theta = (1-p)(\sigma - \tau)/2 \text{ and } \phi = (1-p)(\sigma+\tau)/2$$

with $0 < p < 1$. The conditions (8) and (9) are thus satisfied with $t = \sin^2(p\pi/2)$. The next example [3] shows why it is necessary to "underrotate" to ensure convergence. Let

$$A_1 = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 4 \end{pmatrix}.$$

If we select $\phi_l = \pi/2$, for $l = 1, 2, \cdots$, then the matrix A_l reverts back to the original A_1 after every two sweeps.

In this paper, we consider two Jacobi SVD methods proposed for parallel computing by Brent *et al.* [1], [2] and Luk [6]. We show that the orderings used therein are equivalent to the row ordering and thus share the same convergence properties. The paper is organized as follows. In Section II, we introduce a cyclic-by-antidiagonals ordering and a cyclic-by-modulus ordering, and we show that the latter is closely related to Sameh's [10] parallel implementation of the row ordering. In Section III, we demonstrate the equivalence of the orderings used in [1], [2], and [6] to the row ordering, and conclude that both Jacobi methods converge.

II. The Modulus Ordering

We begin by defining a new cyclic-by-antidiagonals ordering: a pivot (p, q) is followed by

$$\begin{cases} (p+1, q-1) & \text{if } q-p>2 \\ (1, p+q) & \text{if } q-p \le 2,\ p+q \le n \\ (p+q+1-n, n) & \text{if } q-p\le 2,\ n<p+q<2n-1 \\ (1, 2) & \text{if } q=n,\ p=n-1. \end{cases} \tag{11}$$

For $n = 6$ the transformation T_A corresponding to one sweep of the antidiagonal ordering is

$$T_A = T_{56}T_{46}T_{45}T_{36}T_{35}T_{26}T_{34}T_{25} \cdot T_{16}T_{24}T_{15}T_{23}T_{14}T_{13}T_{12} \tag{12}$$

and the matrix elements are rotated in the following order:

$$\begin{pmatrix} x & 1 & 2 & 3 & 5 & 7 \\ x & x & 4 & 6 & 8 & 10 \\ x & x & x & 9 & 11 & 12 \\ x & x & x & x & 13 & 14 \\ x & x & x & x & x & 15 \\ x & x & x & x & x & x \end{pmatrix}.$$

Let us change the initial pivot to $(3, n)$ and assume that there is sufficient hardware for parallel processing. Since successive Jacobi transformations may occur simultaneously if they involve disjoint pivots, it is possible to rotate all elements on the same antidiagonal at the same time. We can therefore complete one sweep in $2n - 3$ stages. The details for $n = 6$ are given in Table I; the order of simultaneous rotations is given below:

TABLE I
PARALLEL IMPLEMENTATION OF THE ANTIDIAGONAL ORDERING

Stage	Pivots		
1	(3,6)	(4,5)	
2	(4,6)		
3	(5,6)		
4	(1,2)		
5	(1,3)		
6	(1,4)	(2,3)	
7	(1,5)	(2,4)	
8	(1,6)	(2,5)	(3,4)
9	(2,6)	(3,5)	

$$\begin{pmatrix} x & 4 & 5 & 6 & 7 & 8 \\ x & x & 6 & 7 & 8 & 9 \\ x & x & x & 8 & 9 & 1 \\ x & x & x & x & 1 & 2 \\ x & x & x & x & x & 3 \\ x & x & x & x & x & x \end{pmatrix}.$$

We use a special property of the antidiagonal ordering: a transformation T_{pq} at stage i satisfies

$$p+q = \begin{cases} i+n+2 & \text{if } 1 \le i \le n-3 \\ i-n+5 & \text{if } n-2 \le i \le 2n-3 \end{cases}. \tag{13}$$

Lemma 2: The transformations T_{pq} and T_{rs} commute if $p + q = k$ and $r + s \ge n + k$.

Proof: The four indexes must be disjoint since $1 \le p, q \le k - 1 < r, s \le n$. □

Lemma 3: For the antidiagonal ordering, a transformation T_{pq} in the $(k + n - 5)$th stage commutes with all transformations T_{rs} in stages $k - 2$ through $n - 3$, for $k = 3, 4, \cdots, n - 1$.

Proof: From (13) we get $p + q = k$ and $r + s \ge n + k$. □

We can therefore move stages $n - 2$ through $2n - 6$ to the front. Specifically, we combine stage i and stage $n + i - 3$, for $i = 1, 2, \cdots, n - 3$. We obtain n new stages, with each transformation T_{pq} at the new ith stage satisfying

$$p+q = i+2 \mod n. \tag{14}$$

We call the result a "modulus" ordering. In Table II, we present the pivots at each stage for $n = 6$. The sequence of simultaneous rotations is given here:

$$\begin{pmatrix} x & 1 & 2 & 3 & 4 & 5 \\ x & x & 3 & 4 & 5 & 6 \\ x & x & x & 5 & 6 & 1 \\ x & x & x & x & 1 & 2 \\ x & x & x & x & x & 3 \\ x & x & x & x & x & x \end{pmatrix}.$$

The transformation T_M corresponding to one sweep of the modulus ordering is given by

$$T_M = (T_{35}T_{26})(T_{34}T_{25}T_{16})(T_{24}T_{15}) \cdot (T_{23}T_{14}T_{56})(T_{13}T_{46})(T_{12}T_{45}T_{36}) \tag{15}$$

where simultaneous transformations T_{pq}'s are enclosed in the same parentheses.

TABLE II
PARALLEL IMPLEMENTATION OF THE MODULUS ORDERING

Stage	Pivots		
1	(3,6)	(4,5)	(1,2)
2	(4,6)	(1,3)	
3	(5,6)	(1,4)	(2,3)
4	(1,5)	(2,4)	
5	(1,6)	(2,5)	(3,4)
6	(2,6)	(3,5)	

TABLE III
SAMEH'S IMPLEMENTATION OF THE ROW ORDERING

Stage	Pivots		
1	(1,2)		
2	(1,3)		
3	(1,4)	(2,3)	
4	(1,5)	(2,4)	
5	(1,6)	(2,5)	(3,4)
6	(2,6)	(3,5)	(4)(1)
7	(3,6)	(4,5)	(1,2)
8	(4,6)	(5)(2)	(1,3)
9	(5,6)	(2,3)	(1,4)
10	(6)(3)	(2,4)	(1,5)
11	(3,4)	(2,5)	(1,6)
12	(3,5)	(2,6)	(4)(1)
13	(3,6)	(1,2)	(4,5)
14	(1,3)	(5)(2)	(4,6)
15	(2,3)	(5,6)	(1,4)
16	(6)(3)	(1,5)	(2,4)

In [10], Sameh showed how to implement the row ordering on a ring-connected array of processors. An example for $n = 6$ is given in Table III, where a single index within the parentheses indicates that the corresponding plane is not involved. In the following lemma, we use T_S to denote the product of transformations T_{pq}'s in one sweep of Sameh's ordering.

Lemma 4: The ordering of Sameh is equivalent to the cyclic-by-rows ordering. So the cyclic Jacobi method using Sameh's ordering converges. □

III. Two Parallel Jacobi SVD Methods

Brent and Luk [1] presented a round-robin ordering for a systolic array implementation of the Jacobi method; the details for computing an SVD are given in [2]. Suppose that n is odd and equals $2k + 1$. To implement the modulus ordering, we choose a special placement P_{BL} of indexes on a "belt" (cf. [7]). Specifically, we place 1 on the upper rightmost position and place 2, 3, $\cdots$, n in every other position going in the clockwise direction:

$$\begin{array}{l} \quad\cdots \leftarrow n \leftarrow 1{+}k \leftarrow 1 \\ \downarrow \qquad\qquad\qquad\qquad \nwarrow \\ \quad\cdots \rightarrow 3 \rightarrow 3{+}k \rightarrow 2 \rightarrow 2{+}k. \end{array}$$

Note that the above arrangement is possible only when n is odd. We let

$$P_{BL} = (\cdots, 3, n, 3+k, 1+k, 2, 1, 2+k)^T. \qquad (16)$$

The initial pivots are

$$\cdots (3, n)(3+k, 1+k)(2, 1)(2+k),$$

and the transformations T_{pq}'s satisfy

$$p + q = 3 \mod n.$$

Suppose now that the transformations T_{pq}'s at the lth stage satisfy

$$p + q = 2 + l \mod n.$$

At the next stage, the transformations occur in the $(p - k, q - k) \bmod n$ planes, and

$$(p-k) + (q-k) = p + q + 1 = 3 + l \mod n.$$

We have thus proved that the Brent–Luk "belt" with the initial placement P_{BL} implements the modulus ordering. Here is an example for $n = 5$:

$$\begin{array}{ccccccccc} 3 \leftarrow 1 & & 1 \leftarrow 4 & & 4 \leftarrow 2 & & 2 \leftarrow 5 & & 5 \leftarrow 3 \\ \downarrow \quad \nwarrow & \rightarrow & \downarrow \quad \nwarrow & \rightarrow & \downarrow \quad \nwarrow & \rightarrow & \downarrow \quad \nwarrow & \rightarrow & \downarrow \quad \nwarrow \\ 5 \rightarrow 2 \rightarrow 4 & & 3 \rightarrow 5 \rightarrow 2 & & 1 \rightarrow 3 \rightarrow 5 & & 4 \rightarrow 1 \rightarrow 3 & & 2 \rightarrow 4 \rightarrow 1. \end{array}$$

By deleting the first n stages, we reduce Sameh's ordering to the modulus ordering (cf. [10] and Table III). Two stages are said to be identical if they contain the same set of pivots.

Theorem 2: The $(i + n)$th stage of Sameh's ordering is identical to the ith stage of the modulus ordering, for $i = 1, 2, \cdots$. Hence, the cyclic-by-modulus Jacobi method converges.

Proof: We prove by contradiction. Suppose that there exists a matrix A for which the cyclic-by-modulus Jacobi method does not converge. Then we can construct a matrix

$$B = T_{n-1,n} T_{n-2,n} \cdots T_{4,n-1} T_{3,n}(A)$$

for which the Jacobi method using Sameh's ordering does not converge. For $n = 6$, we choose (cf. Table III)

$$B = T_{56} T_{46} T_{45} T_{36}(A). \qquad □$$

The pivots are given in Table IV.

The transformation T_{BL} corresponding to one sweep of the ordering is

$$T_{BL} = (T_{34} T_{25})(T_{15} T_{24})(T_{23} T_{14})(T_{45} T_{13})(T_{12} T_{35}),$$

which is equivalent to

$$T_M = (T_{34} T_{25})(T_{24} T_{15})(T_{23} T_{14})(T_{13} T_{45})(T_{12} T_{35}).$$

The special initial placement of indexes on the belt will not affect the convergence property of the Jacobi method. Given two initial placements P_1 and P_2, construct the permutation matrix Π that sends P_1 to P_2. Identical results will be obtained from applying a cyclic Jacobi method with initial placement P_1 to the matrix A and from applying the same method with initial

TABLE IV
BRENT–LUK ORDERING FOR $n = 5$

Stage	Pivots		
1	(5,3)	(2,1)	(4)
2	(3,1)	(5,4)	(2)
3	(1,4)	(3,2)	(5)
4	(4,2)	(1,5)	(3)
5	(2,5)	(4,3)	(1)

placement P_2 to the matrix $\Pi^T A \Pi$. For example, if

$$P_1 = (1\ 2\ 3\ 4\ 5)^T \text{ and } P_2 = (5\ 3\ 2\ 1\ 4)^T,$$

then

$$\Pi = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Lemma 5: If a Jacobi method converges with one initial placement of indexes, then the same method converges with any initial placement of indexes. □

Theorem 3: Suppose that n is odd. Then the modulus and Brent–Luk orderings are equivalent, and the cyclic Jacobi method using the Brent–Luk ordering converges. □

Schwiegelshohn and Thiele [11] showed how to implement the row ordering on the Brent–Luk multiprocessor array [1] when n is odd. It is possible to use their work to infer the result of Theorem 3.

However, the Brent–Luk method does not converge when n is even. A nonconvergence example for any given even n is as follows. We choose the initial placement

$$\begin{array}{ccccccccccc} n-1 & \leftarrow & \cdots & \leftarrow & 5 & \leftarrow & 3 & & 1 \\ \downarrow & & & & & & & \nwarrow & \\ n & \rightarrow & \cdots & \rightarrow & 6 & \rightarrow & 4 & \rightarrow & 2 \end{array}$$

and the initial $n \times n$ matrix

$$A_1 = \begin{pmatrix} M & 0 \\ 0^T & D \end{pmatrix}$$

where

$$M \equiv \begin{pmatrix} a & 0 & 0 & -b \\ 0 & a & b & 0 \\ 0 & b & a & 0 \\ -b & 0 & 0 & a \end{pmatrix}$$

and

$$D = \operatorname{diag}(d_1, d_2, \cdots, d_{n-4})$$

with $d_i \neq d_j$ for all $i \neq j$ and $d_i \neq a$ for all i. When $n = 4$, our example reduces to the one given by Hansen [5]. Since the matrix is symmetric, we have $\theta = \phi$. For each of the two sweeps, if we choose the sequence of twelve angles $\{\phi\}$ as $\{-\pi/4, \pi/4, \cdots, \pi/4, -\pi/4, -\pi/4, \pi/4\}$, we get $a_{pq}^{(l+1)} = a_{qp}^{(l+1)} = 0$ but the quantity s_l of (4) stays at a constant $4b^2$.

When $n = 6$ the ordering is

$$\begin{array}{ccc} (1, 2) & (3, 4) & [5, 6] \\ (1, 4) & [2, 6] & [3, 5] \\ [1, 6] & [4, 5] & (2, 3) \\ [1, 5] & [6, 3] & (4, 2) \\ (1, 3) & [5, 2] & [6, 4]. \end{array}$$

We choose the rotation angles in the same way as above for any index pairs (p, q) with $\{p, q\} \subseteq \{1, 2, 3, 4\}$ (the ones in parentheses) and zero angle for all other index pairs (the ones in square brackets), the quantity s_l again stays at a constant $4b^2$. Details for the general even n case are given in Park [9].

In [6], Luk presented a Jacobi SVD method for the Systolic Linear Algebra Parallel Processor [13] at the Naval Ocean Systems Center. He used the odd–even ordering originally proposed by Stewart [12] for computing the Schur decomposition. In [7], Luk and Park showed that the odd–even ordering is identical to the caterpillar track ordering of Modi and Pryce [8] and to the caterpillar tractor tread ordering of Whiteside *et al.* [14]. We refer to [7] and [8] for a detailed description of the "caterpillar track." To implement the modulus ordering, we place the n indexes on the caterpillar track with 1 on the lower leftmost position and place successive indexes in the clockwise direction:

$$\begin{array}{ccccccccccccc} \cdot & \leftarrow & 2 & \leftarrow & \cdot & \leftarrow & 3 & \leftarrow & \cdot & \leftarrow & 4 & \leftarrow & \cdots \\ \downarrow & & & & & & & & & & & & \uparrow \\ 1 & \rightarrow & \cdot & \rightarrow & n & \rightarrow & \cdot & \rightarrow & n-1 & \rightarrow & \cdot & \rightarrow & \cdots \end{array}$$

The special index placement is therefore

$$P_{\mathrm{OE}} = (1, 2, n, 3, n-1, 4, \cdots)^T. \tag{17}$$

The initial pivots are

$$(1, 2)(n, 3)(n-1, 4) \cdots$$

and the transformations T_{pq}'s satisfy the relation

$$p + q = 3 \mod n.$$

Suppose that the transformations T_{pq}'s at stage l satisfy

$$p + q = 2 + l \mod n.$$

At the next stage, the transformations occur in the $(p, q+1) \bmod n$ planes, and

$$p + (q+1) = 3 + l \mod n.$$

Hence, the caterpillar track with the initial placement P_{OE} implements the modulus ordering. We present an example for

TABLE V
ODD-EVEN ORDERING FOR $n = 6$

Stage	Pivots
1	(1,2) (6,3) (5,4)
2	(2)(1,3) (6,4)(5)
3	(2,3) (1,4) (6,5)
4	(3)(2,4) (1,5)(6)
5	(3,4) (2,5) (1,6)
6	(4)(3,5) (2,6)(1)

$n = 6$:

$$\begin{array}{ccccccc}
\begin{array}{l}\cdot \leftarrow 2 \leftarrow \cdot \leftarrow 3 \leftarrow \cdot \leftarrow 4\\ \downarrow \qquad\qquad\qquad\qquad\quad \uparrow\\ 1 \rightarrow \cdot \rightarrow 6 \rightarrow \cdot \rightarrow 5 \rightarrow \cdot\end{array} & \rightarrow & \begin{array}{l}2 \leftarrow \cdot \leftarrow 3 \leftarrow \cdot \leftarrow 4 \leftarrow \cdot\\ \downarrow \qquad\qquad\qquad\qquad\quad \uparrow\\ \cdot \rightarrow 1 \rightarrow \cdot \rightarrow 6 \rightarrow \cdot \rightarrow 5\end{array} & \rightarrow & \begin{array}{l}\cdot \leftarrow 3 \leftarrow \cdot \leftarrow 4 \leftarrow \cdot \leftarrow 5\\ \downarrow \qquad\qquad\qquad\qquad\quad \uparrow\\ 2 \rightarrow \cdot \rightarrow 1 \rightarrow \cdot \rightarrow 6 \rightarrow \cdot\end{array} & \rightarrow & \\
\begin{array}{l}3 \leftarrow \cdot \leftarrow 4 \leftarrow \cdot \leftarrow 5 \leftarrow \cdot\\ \downarrow \qquad\qquad\qquad\qquad\quad \uparrow\\ \cdot \rightarrow 2 \rightarrow \cdot \rightarrow 1 \rightarrow \cdot \rightarrow 6\end{array} & \rightarrow & \begin{array}{l}\cdot \leftarrow 4 \leftarrow \cdot \leftarrow 5 \leftarrow \cdot \leftarrow 6\\ \downarrow \qquad\qquad\qquad\qquad\quad \uparrow\\ 3 \rightarrow \cdot \rightarrow 2 \rightarrow \cdot \rightarrow 1 \rightarrow \cdot\end{array} & \rightarrow & \begin{array}{l}4 \leftarrow \cdot \leftarrow 5 \leftarrow \cdot \leftarrow 6 \leftarrow \cdot\\ \downarrow \qquad\qquad\qquad\qquad\quad \uparrow\\ \cdot \rightarrow 3 \rightarrow \cdot \rightarrow 2 \rightarrow \cdot \rightarrow 1\end{array} & &
\end{array}$$

The pivots are presented in Table V. For $n = 6$ the transformation T_{OE} corresponding to one sweep of the odd-even ordering is

$$T_{\mathrm{OE}} = (T_{26}T_{35})(T_{16}T_{25}T_{34})(T_{15}T_{24}) \cdot (T_{56}T_{14}T_{23})(T_{46}T_{13})(T_{45}T_{36}T_{12})$$

which is equivalent to T_M of (15).

Theorem 4: The modulus and odd-even orderings are equivalent. Hence, the cyclic Jacobi method using the odd-even ordering converges. □

The key idea to Theorem 4 is of course the parallel implementation of the cyclic-by-rows ordering. Surprisingly, the idea is not new; it is implicit in the works of Gentleman [4], Sameh [10], and Stewart [12]. A principal advantage of the odd-even ordering is that it preserves the triangular structure of a given matrix. However, the use of "underrotations" to enforce conditions (8) and (9) causes complications. A discussion is presented in Park [9].

Indeed, the Brent-Luk and odd-even orderings are equivalent for n odd. With this special initial index placement:

$$\begin{array}{l}\qquad \cdots \leftarrow 3 \leftarrow n-1 \leftarrow 1\\ \downarrow \qquad\qquad\qquad\qquad\qquad \nwarrow\\ \qquad \cdots \rightarrow 4 \rightarrow n-2 \rightarrow 2 \rightarrow n\end{array}$$

the Brent-Luk belt will generate the same set of pivots as the caterpillar track with the traditional initial placement:

$$\begin{array}{l}\cdot \leftarrow 2 \leftarrow \cdot \leftarrow 4 \leftarrow \cdot \leftarrow 6 \leftarrow \cdots \leftarrow \cdot\\ \downarrow \qquad\qquad\qquad\qquad\qquad\qquad\quad \uparrow\\ 1 \rightarrow \cdot \rightarrow 3 \rightarrow \cdot \rightarrow 5 \rightarrow \cdot \rightarrow \cdots \rightarrow n\end{array}$$

Thus, when n is odd, the Brent-Luk multiprocessor array [1] can implement the odd-even ordering and the Stewart computational network [12] can implement the round-robin ordering.

IV. Concluding Remarks

With the advent of massively parallel computer architectures, Jacobi SVD algorithms are arousing great interests. Such algorithms have been implemented on the Carnegie-Mellon University Warp computer, the University of Illinois Cedar computer, the DAP computer, and the Connection Machine. Our convergence proof hence comes at an opportune time.

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