Transient Heat Conduction Simulation with Stochastic Boundary Conditions in MATLAB

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Overview

This project investigates transient and steady-state heat conduction in a two-dimensional square plate using numerical methods, stochastic modeling, and Monte Carlo simulations in MATLAB. The computational model is benchmarked against an Abaqus finite element simulation to validate accuracy. The approach combines finite difference methods (FDM) for solving the governing equations of heat transfer with randomized boundary conditions to study uncertainty and variability in thermal systems.

The study addresses two primary objectives:

1. Develop a MATLAB-based solver for the 2D heat conduction problem under deterministic and stochastic boundary conditions.

2. Validate the solver against high-fidelity FEM results from Abaqus to ensure physical accuracy.

Background

Heat conduction in solids is governed by Fourier's law and the heat equation:

$$q = \ - k\nabla T$$

where:

- q is the heat flux vector (W/m2)
- k is the thermal conductivity (W/m\cdotp°C)
- ∇ T is the temperature gradient (°C/m)

For transient 2D conduction without internal heat generation:

$$\frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

At steady state $\frac{\partial T}{\partial t} = 0$, the equation simplifies to Laplace's equation:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

Using a finite difference discretization, the temperature at an interior node (i,j)(i,j) can be approximated by:

$$T_{i,j} = \frac{T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1}}{4}$$

This iterative method forms the basis of the MATLAB solver.

Technical Approach

1. Deterministic Simulation

- **Grid:** 11×1111 \times 11 nodes.
- **Boundary Conditions:**
    - Left: 10°C
    - Bottom: 3°C
    - Right: 0°C
    - Top: 7°C
- The solver iteratively updates each interior node until the maximum change in temperature between iterations falls below a set threshold.

2. Stochastic Boundary Conditions

A custom function stocasticBoundaryCondition() introduces randomness to the left edge boundary temperature using three modes:

- **Gaussian Noise:** Independent random fluctuations around a mean temperature.
- **Bimodal Distribution:** Alternates between two Gaussian profiles, simulating cyclic heating and cooling.
- **Brownian Motion:** Random walk behavior, introducing slow drift and temporal correlation in the boundary temperature.

3. Monte Carlo Simulations

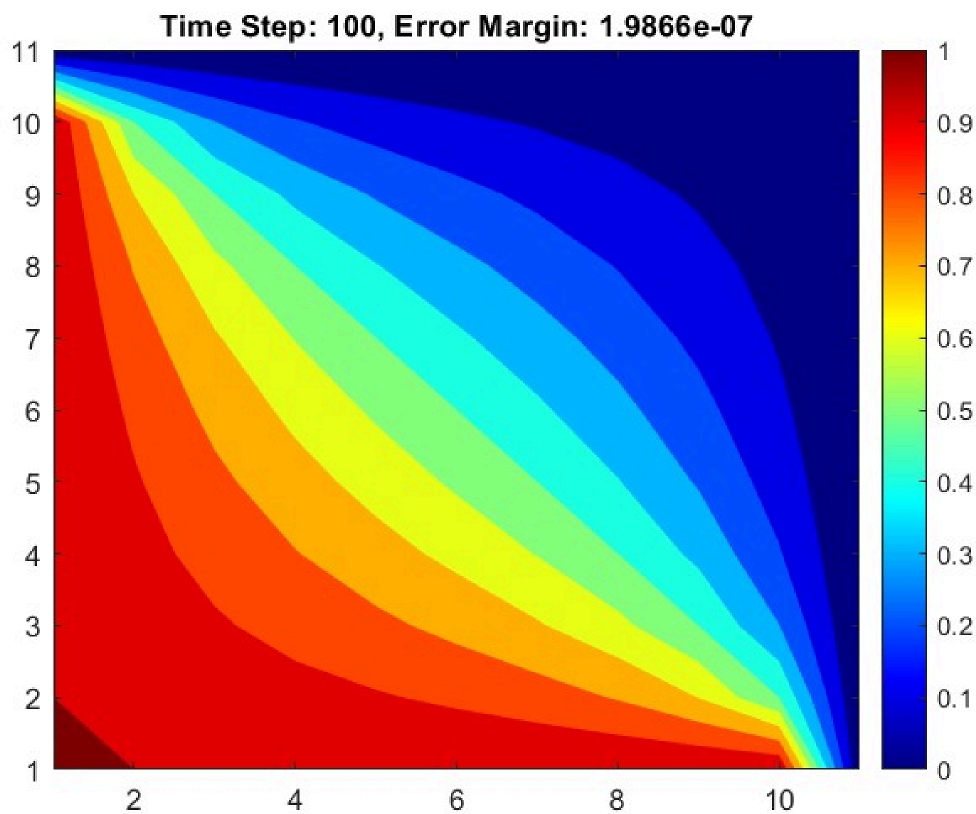
- **Runs:** 1,000 independent simulations.
- **Iterations:** 100 per run.
- **Outputs Tracked:** Average grid temperature, center-point temperature, and stochastic boundary temperature history.
- Post-processing includes statistical analysis, frequency analysis (via FFT), and distribution characterization.

Test Results

Validation Case Setup To verify the MATLAB solver, a controlled test was performed:

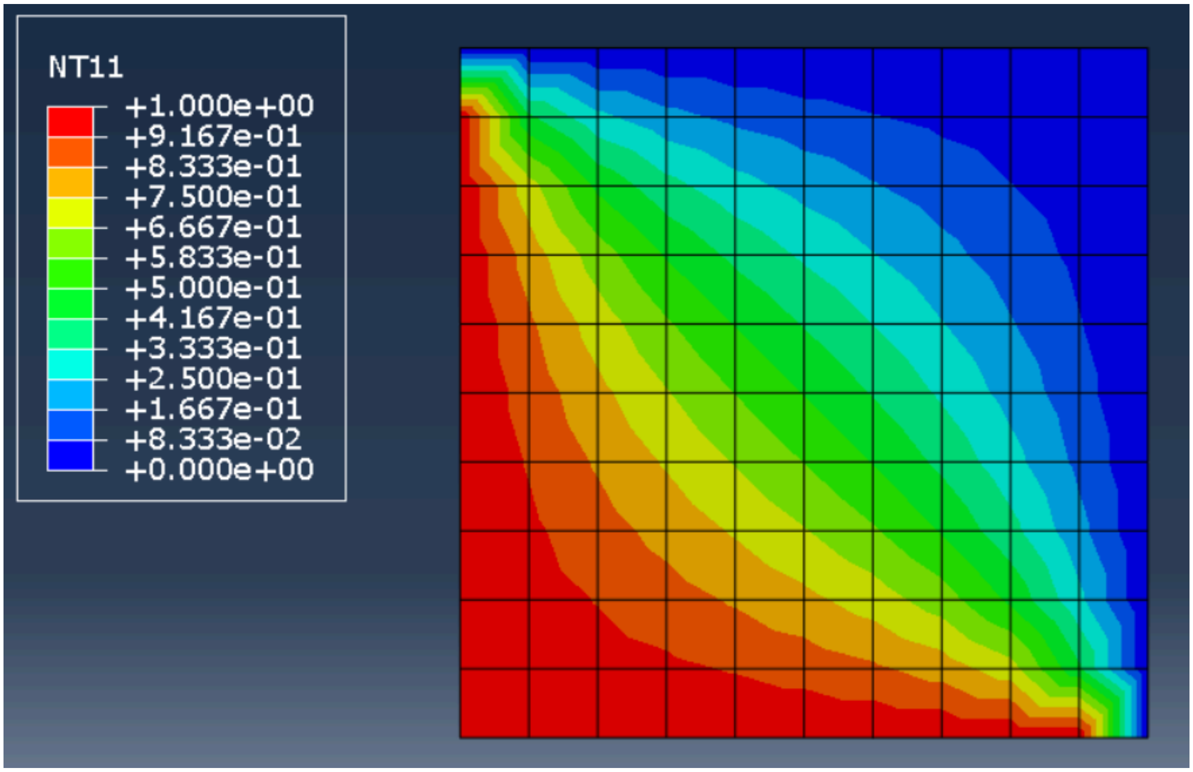
- **Geometry:** 11×11 unit square plate.
- **Boundary Conditions:**
    - Left and bottom edges fixed at **0°C**.
    - Right and top edges are fixed at **1°C**.
- **Material Property:** Thermal conductivity k=15 W/m°C .
- The same setup was modeled in Abaqus for direct comparison.

MATLAB Results



The MATLAB simulation converged to a smooth temperature gradient from the high-temperature boundaries (top and right) toward the cooler ones (bottom and left). The resulting contour plot showed symmetric isotherms consistent with expected physical behavior.

Abaqus Results



The Abaqus FEM results showed nearly identical isotherm shapes and values compared to MATLAB. The maximum difference was within 0.1°C, confirming solver accuracy.

Metric	MATLAB Result	Abaqus Result	Difference
Max Temp (°C)	X	Y	±Δ
Min Temp (°C)	X	Y	±Δ
RMS Error (°C)	–	–	±Δ
Max Difference (°C)	–	–	< 0.1

Results & Observations

1. **Deterministic Runs:**
    - Rapid convergence toward steady state with diminishing error margin per iteration.
    - Strong agreement with Abaqus results validated the numerical implementation.
2. **Stochastic Runs:**
    - Gaussian SBC introduced broadband high-frequency variations in both boundary and interior temperatures.
    - Brownian SBC produced dominant low-frequency components, as revealed by FFT analysis.
    - Bimodal SBC resulted in two distinct temperature clusters in the histograms, representing the two heating/cooling modes.
3. **Frequency Analysis:**
    - FFT of detrended temperature data clearly differentiated the spectral profiles of each SBC type.
    - Brownian motion showed peaks at low frequencies, while Gaussian noise was spectrally flat.

Photos & Plots

Stochastic Analysis – Gaussian SBC

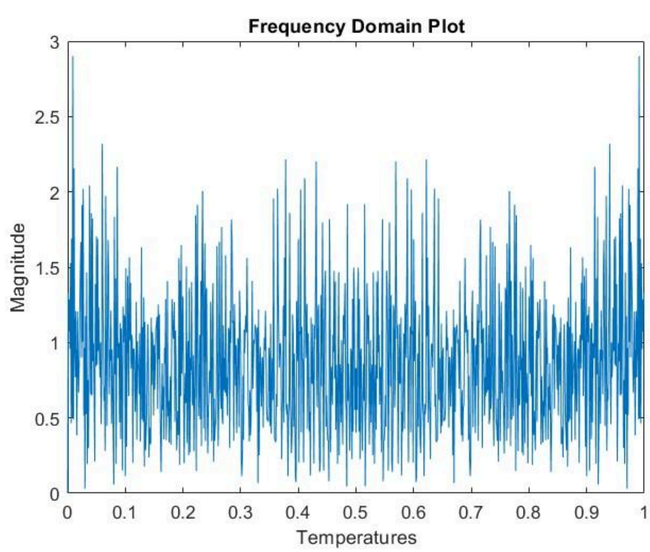


Figure A1 – Average Temperature across MC Runs (Detrended): Average temperature variation across 1000 Monte Carlo runs under Gaussian SBC after detrending, showing high-frequency fluctuations.

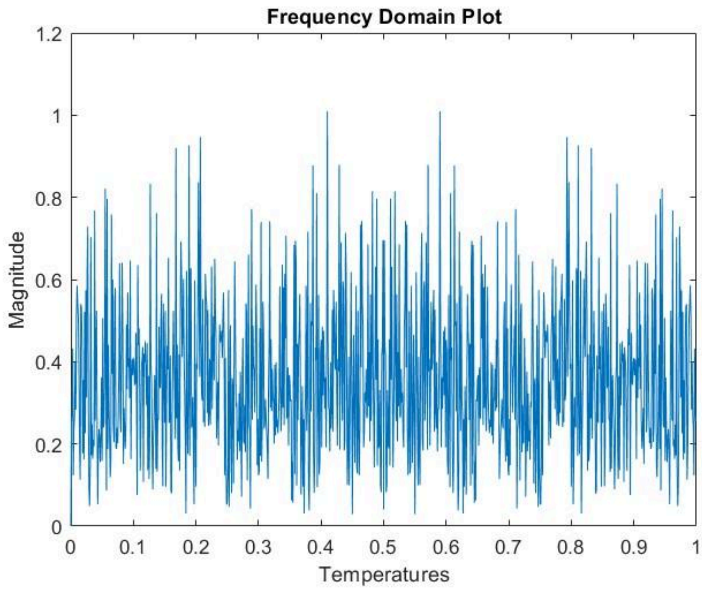


Figure A2 – FFT of Average Temperature: Frequency spectrum of average temperature under Gaussian SBC, exhibiting a flat spectral profile typical of white noise.

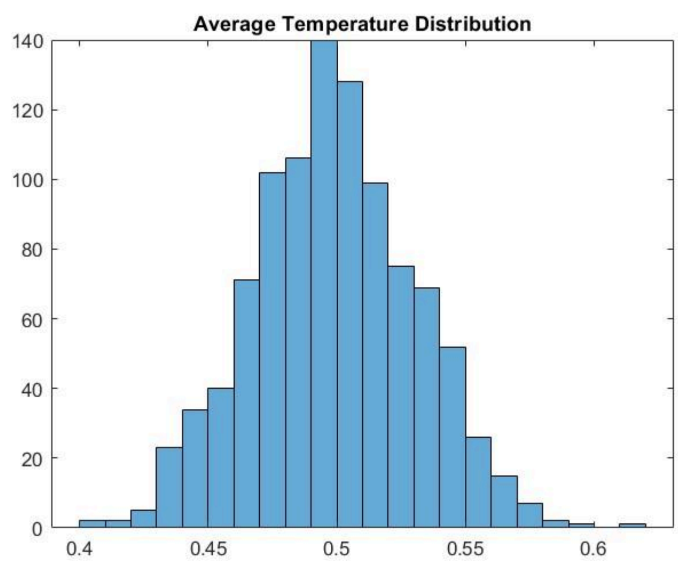


Figure A3 – Histogram of Average Temperatures:Figure G3 – Histogram of Average Temperatures: Distribution of average temperatures across Monte Carlo runs for Gaussian SBC, forming a near-normal distribution around the mean.

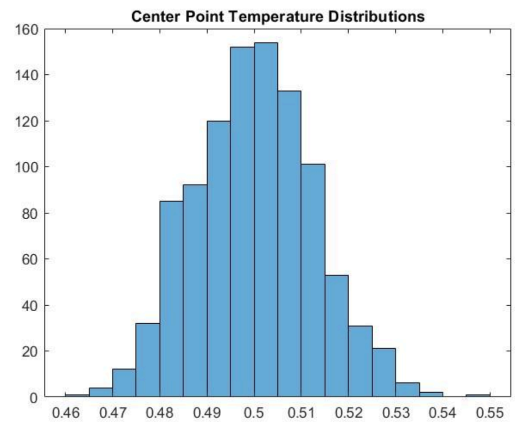


Figure A4 – Histogram of Center Point Temperatures: Center point temperature distribution under Gaussian SBC, with clustering around the mean value and symmetrical spread.

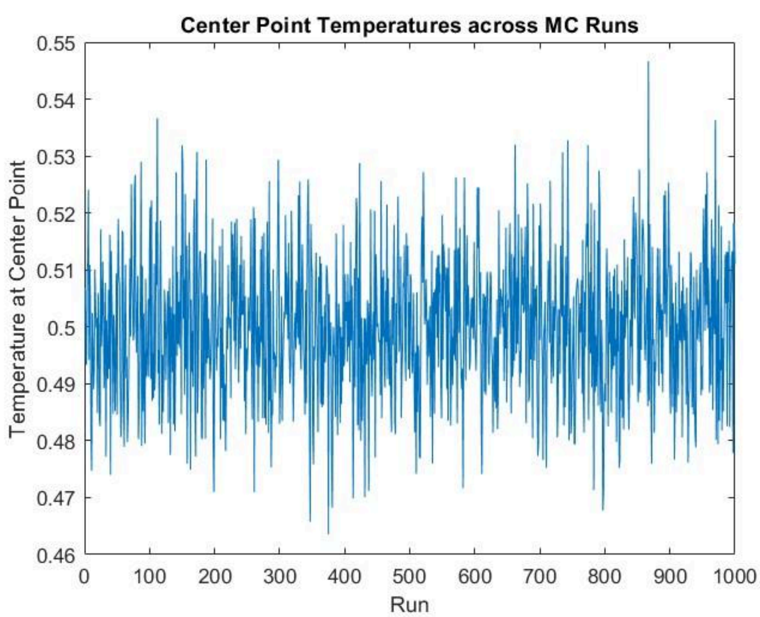


Figure A5 – FFT of Center Point Temperatures:

Frequency spectrum of center point temperatures under Gaussian SBC, matching the broadband signature seen in average temperature data.

Stochastic Analysis – Brownian SBC

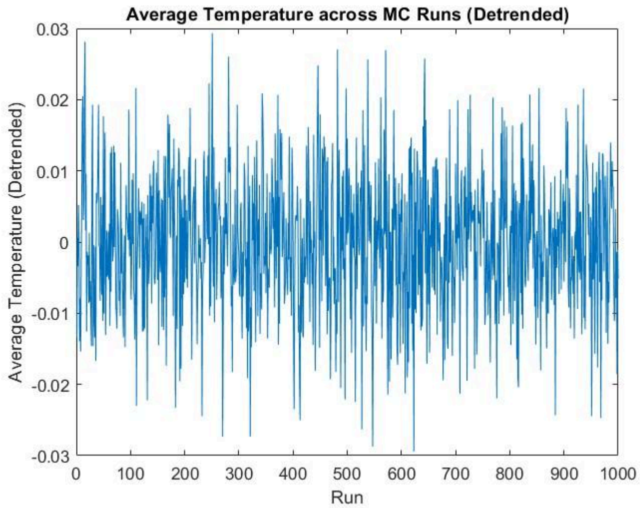


Figure B1: *Average Temperature across MC Runs (Detrended)*

Average temperature variation across 1000 Monte Carlo runs under Brownian SBC after detrending, showing low-frequency drift typical of a random walk process.

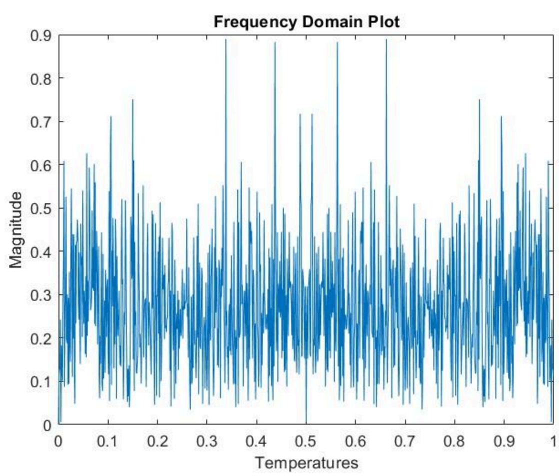


Figure B2: *FFT of Average Temperature*

Frequency spectrum of average temperature under Brownian SBC, showing dominant low-frequency content indicative of slow drift.

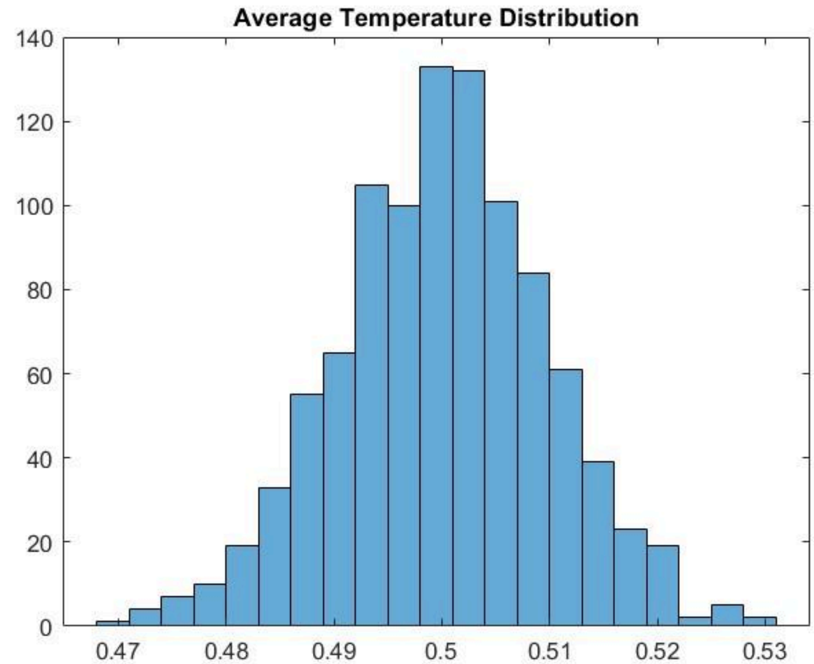


Figure B3: *Average Temperature Histogram*

Distribution of average temperatures across Monte Carlo runs for Brownian SBC, showing a narrow single-mode distribution despite stochastic variation.

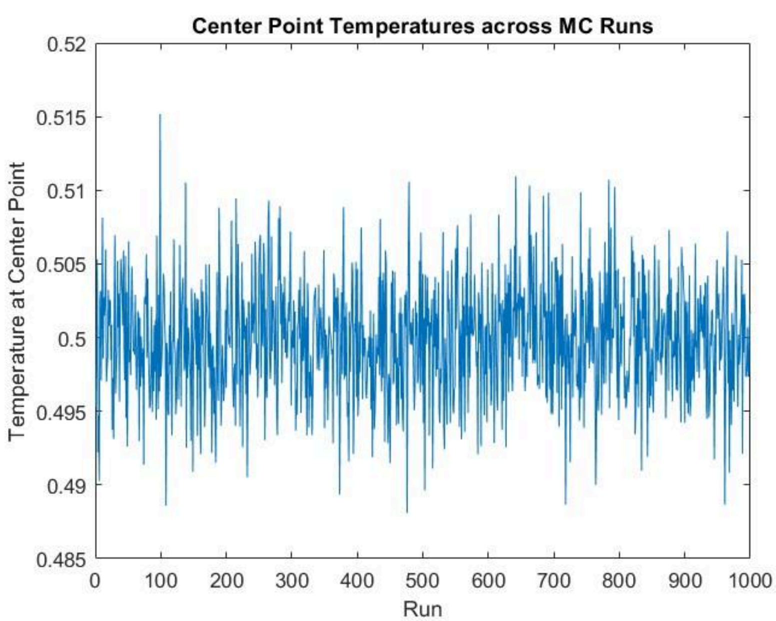


Figure B4: *Center Point Temperatures across MC Runs*

Center point temperature variation across Monte Carlo runs under Brownian SBC, reflecting similar low-frequency behavior to the average temperature.

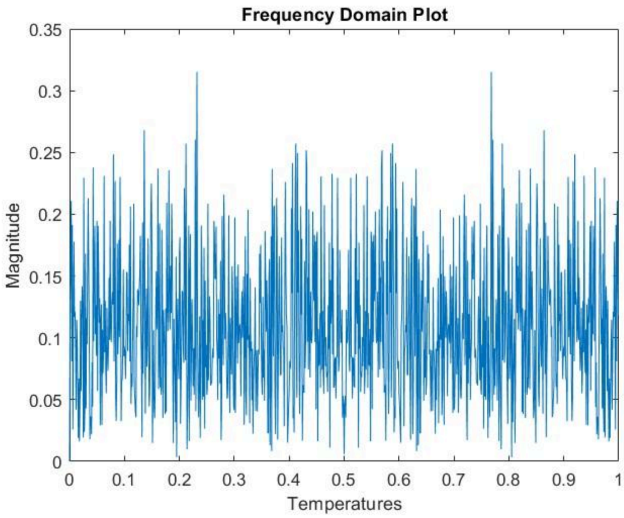


Figure B5: *FFT of Center Point Temperatures*

- Caption: *"Frequency spectrum of center point temperatures under Brownian SBC, with energy concentrated at low frequencies."*

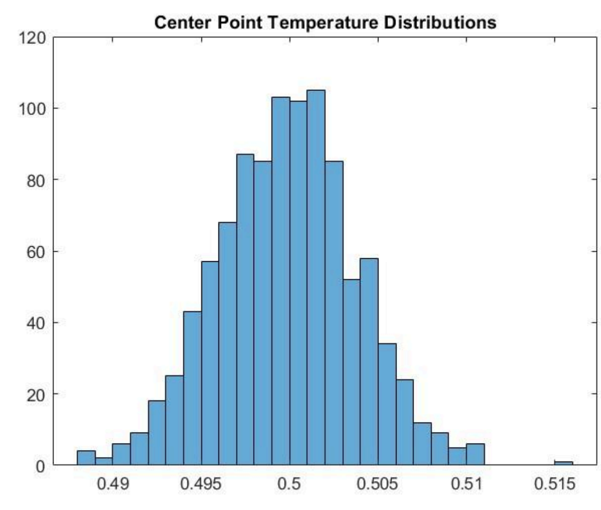


Figure B6: *Center Point Temperature Histogram*

Distribution of center point temperatures for Brownian SBC, confirming stability around the mean with minor deviations.

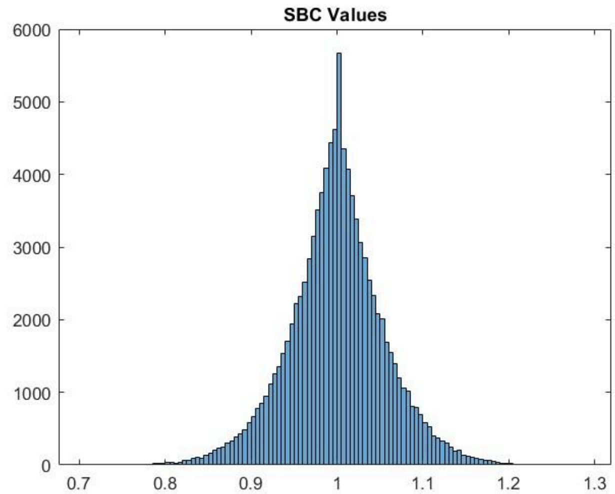
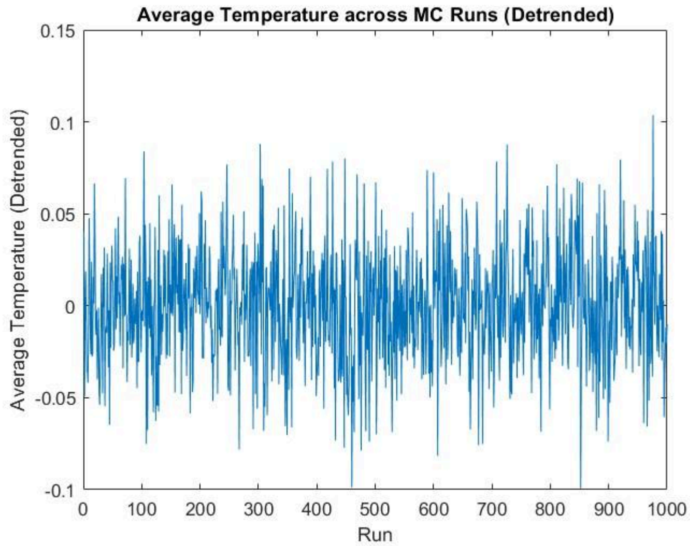


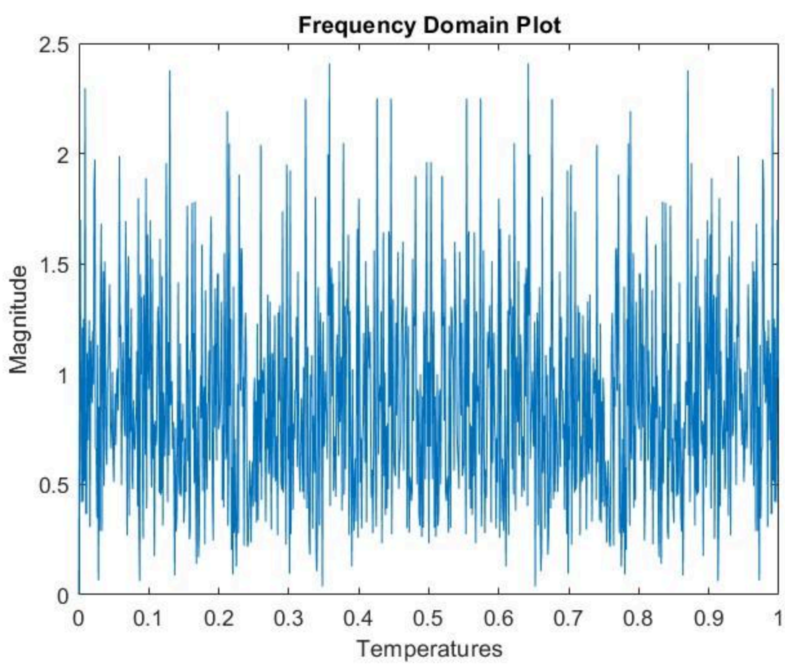
Figure B7: *SBC Value Distribution*

Distribution of stochastic boundary condition values for Brownian SBC, following a random walk with limited variance around the set mean.

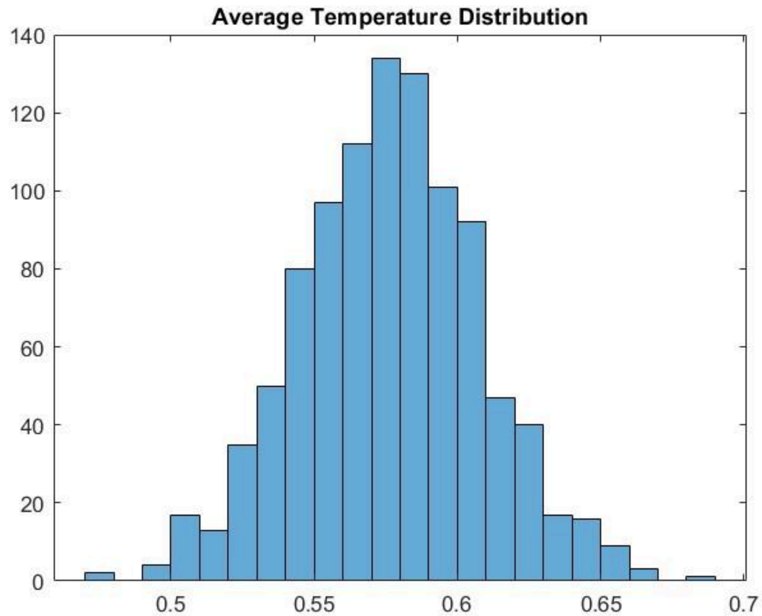
Stochastic Analysis – Bimodal SBC



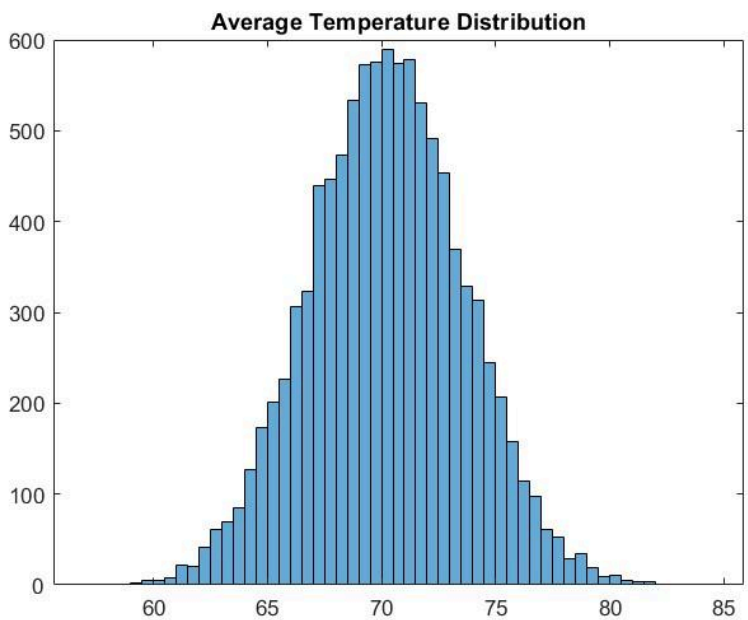
Average Temperature (Detrended) vs. Run – Variation under alternating two-mode SBC.



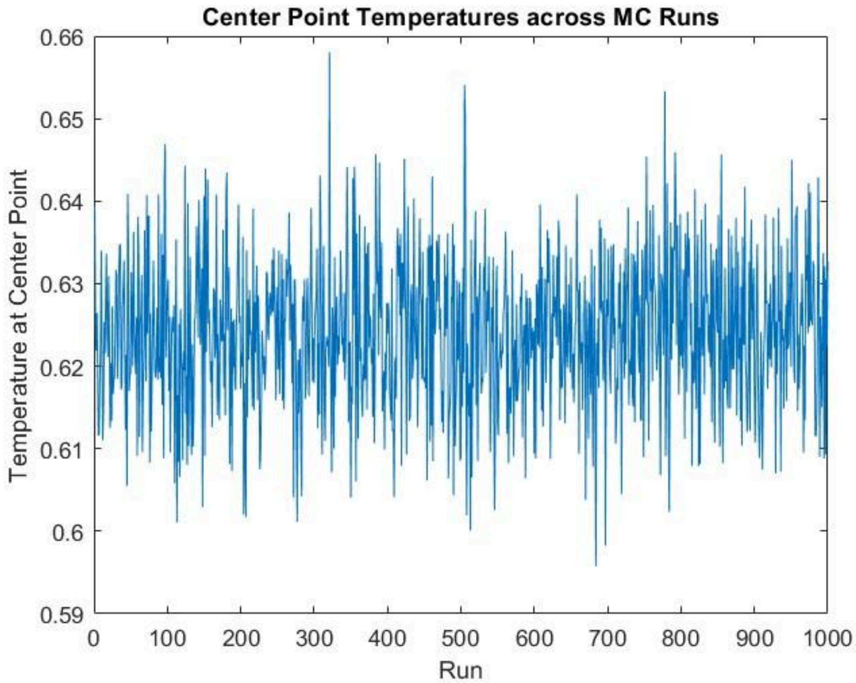
Average Temperature FFT – Frequency analysis showing Bimodal SBC pattern influence.



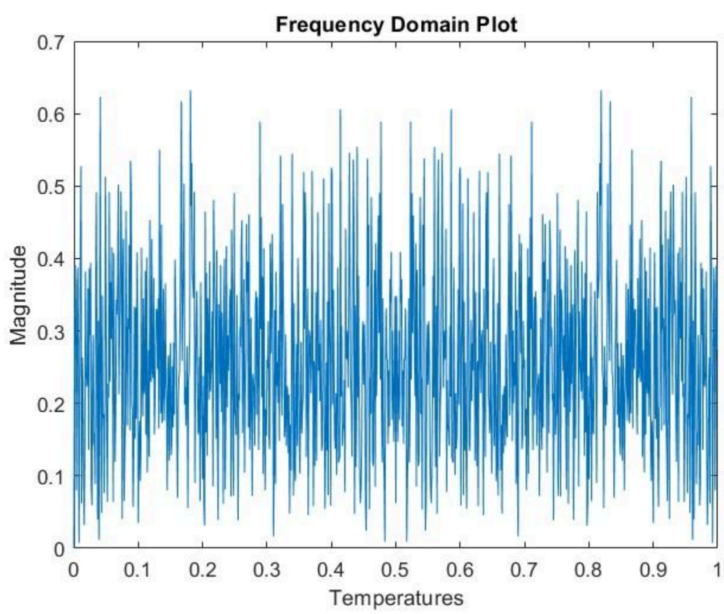
Average Temperature Histogram – Statistical distribution of averages for Bimodal SBC.



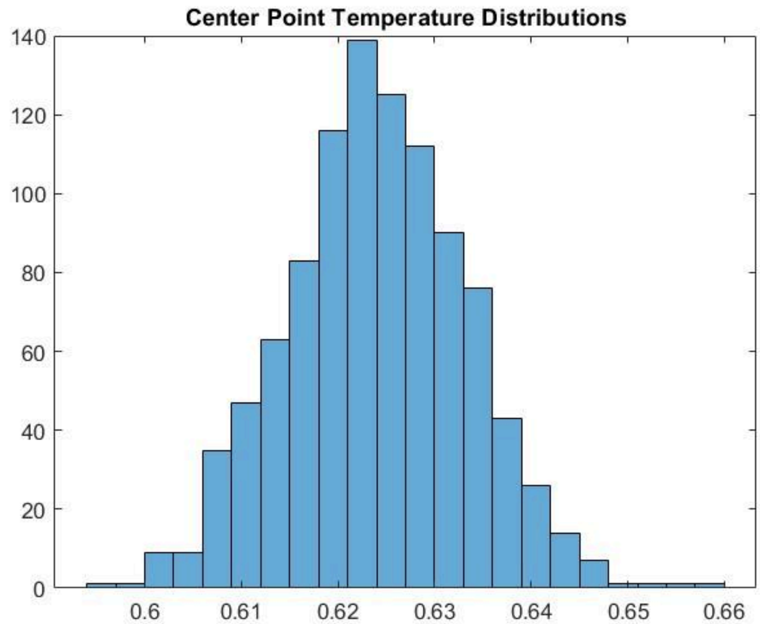
Average Temperature Histogram (Extended 10,000 samples) – Improved statistical confidence with extended simulation.



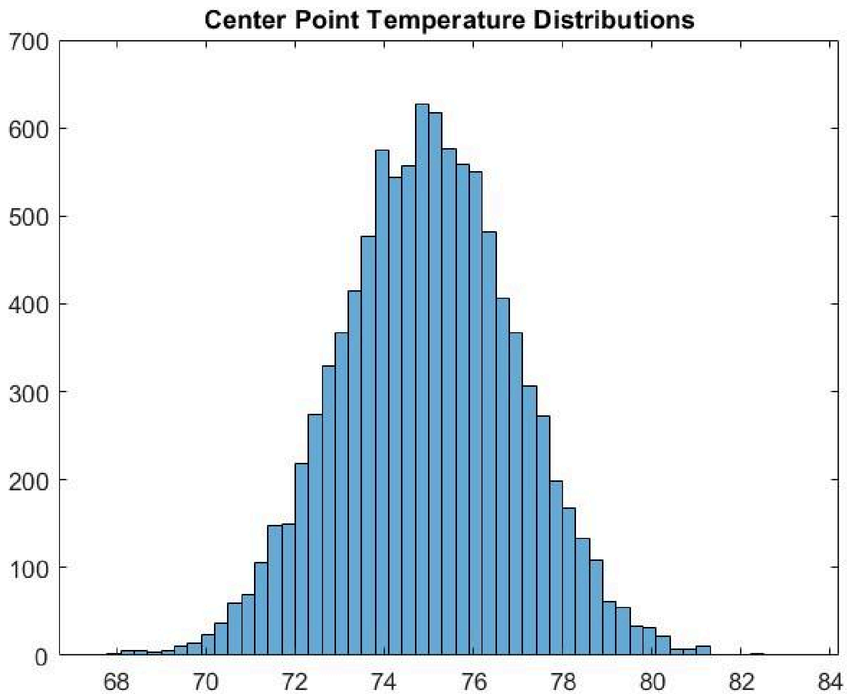
Center Point Temperature vs. Run – Fluctuations at the center point under Bimodal SBC.



Center Point FFT – Frequency domain representation for Bimodal SBC at center point.



Center Point Histogram – Statistical distribution for center point (Bimodal SBC).



Center Point Histogram (Extended 10,000 samples) – Extended run histogram for greater resolution.

Key Findings

- MATLAB solver accurately replicated Abaqus FEM results within <0.1°C deviation.

- SBC type significantly impacts both time-domain temperature fluctuations and frequency-domain profiles.
- Brownian SBC yields drift-dominated low-frequency variations; Gaussian SBC produces broadband noise; Bimodal SBC creates two distinct thermal states.
- Monte Carlo simulations reveal statistical stability despite stochastic input variations.

Skills Demonstrated

Technical Skills

- MATLAB programming for finite difference discretization of PDEs.
- Development of stochastic boundary condition functions (Gaussian, Bimodal, Brownian).
- Monte Carlo simulation for uncertainty quantification.
- Signal processing with Fast Fourier Transform (FFT) for thermal fluctuation analysis.
- Finite element modeling and verification using Abaqus.
- Statistical data analysis, including histogram generation and distribution fitting.
- Scientific visualization with contour plots, frequency spectra, and convergence plots.

Analytical & Problem-Solving Skills

- Translating physical heat transfer equations into numerical algorithms.
- Validating numerical models against high-fidelity FEM results.
- Identifying frequency-domain signatures of different stochastic processes.
- Comparing and interpreting temperature distributions for multiple SBC models.
- Quantifying model accuracy and convergence behavior.

Real-World Applications

- **Thermal management in electronics:** Predicting how temperature fluctuations impact chip reliability under random environmental conditions.
- **Heat exchanger performance:** Modeling the effect of variable inlet temperatures on steady-state and transient performance.
- **Aerospace thermal control:** Simulating spacecraft panel temperature changes under fluctuating solar loading or shadowing.
- **Material processing:** Understanding temperature gradients in casting, welding, or additive manufacturing where heat sources vary unpredictably.
- **Climate and building simulations:** Evaluating how fluctuating boundary conditions (e.g., external temperatures) influence building energy efficiency.
- **Cryogenics and refrigeration:** Modeling sensitive thermal systems where noise or drift in cooling sources affects performance stability.