Lecture 8: Linear Logic

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1 Introduction

Lecture outline:

- Inference rules and examples
- Invertibility
- Polarizing linear logic

We have so far only seen polarity and focusing in the context of intuitionistic logic. Historically, however, these concepts first arose in developing proof search for *linear logic*. In this lecture, we introduce a sequent calculus for linear logic and examine the polarity of its connectives. In the next lecture, we will focus it.

2 Linear Sequent Calculus

Intuitionistic logic can be described as a "logic of knowledge". Its formulas are propositions that carry a notion of truth, and once proven, remain immutably true. This property is called the *persistence* or *monotonicity* of the logic.

In contrast, the formulas of linear logic behave like *resources*, which can be expended in deriving other resources. As a preview, consider the linear analog of implication: $A \multimap B$, which can be interpreted as a mechanism for exchanging the resource A for resource B, after which A is no longer usable. Here, the logic is no longer persistent nor monotonic; formulas are ephemeral and once derived, may be consumed.

Below, we present the formulas and inference rules of a sequent calculus for linear logic:

Formulas:

$$
\begin{array}{lr}
A ::= A \mathbin{\&} B & \text{“}A \textit{ with } B\text{”} \\
\quad\mid A \otimes B & \text{“}A \textit{ tensor } B\text{”} \\
\quad\mid A \multimap B & \text{“}A \textit{ lolli } B\text{”} \\
\quad\mid A \oplus B & \text{“}A \textit{ plus } B\text{”} \\
\quad\mid \top & \text{unit of } \& \\
\quad\mid \mathbb{1} & \text{unit of } \otimes \\
\quad\mid \mathbb{0} & \text{unit of } \oplus
\end{array}
$$

With Rules:

$$
\frac{\Delta \Rightarrow A \quad \Delta \Rightarrow B}{\Delta \Rightarrow A \& B}\ \&R \qquad \frac{\Delta, A \Rightarrow C}{\Delta, A \& B \Rightarrow C}\ \&L_1 \qquad \frac{\Delta, B \Rightarrow C}{\Delta, A \& B \Rightarrow C}\ \&L_2
$$

Tensor rules:

$$
\frac{\Delta_1 \Rightarrow A \quad \Delta_2 \Rightarrow B}{\Delta_1, \Delta_2 \Rightarrow A \otimes B}\ \otimes R \qquad \frac{\Delta, A, B \Rightarrow C}{\Delta, A \otimes B \Rightarrow C}\ \otimes L
$$

Lolli rules:

$$
\frac{\Delta, A \Rightarrow B}{\Delta \Rightarrow A \multimap B}\ \multimap R \qquad \frac{\Delta_1 \Rightarrow A \quad \Delta_2, B \Rightarrow C}{\Delta_1, \Delta_2, A \multimap B \Rightarrow C}\ \multimap L
$$

Plus rules:

$$
\frac{\Delta \Rightarrow A}{\Delta \Rightarrow A \oplus B}\ \oplus R_1 \qquad \frac{\Delta \Rightarrow B}{\Delta \Rightarrow A \oplus B}\ \oplus R_2 \qquad \frac{\Delta, A \Rightarrow C \quad \Delta, B \Rightarrow C}{\Delta, A \oplus B \Rightarrow C}\ \oplus L
$$

Top rule:

$$
\frac{}{\Delta \Rightarrow \top}\ \top R \qquad \text{(no left rule)}
$$

$\mathbb{1}$ **rules**:

$$
\frac{\Delta = \cdot}{\Delta \Rightarrow \mathbb{1}}\ \mathbb{1}R \qquad \frac{\Delta \Rightarrow C}{\Delta, \mathbb{1} \Rightarrow C}\ \mathbb{1}L
$$

$\mathbb{0}$ **rule**:

$$
\frac{}{\Delta, \mathbb{0} \Rightarrow C}\ \mathbb{0}L \qquad \text{(no right rule)}
$$

Identity rule:

$$
\frac{}{P \Rightarrow P}\ \text{id}
$$

We now dive a little deeper into how linear logic departs from intuitionistic logic, focusing on the meaning of sequents and certain connectives, as well as corresponding inference rules.

2.1 Linearity

Since formulas (encoding the ephemeral notion of resource) may be consumed, the semantics of sequents $\Delta \Rightarrow A$ must change. We require that each assumption in the antecedent Δ be used **exactly once** to yield A. This forces our identity rule to deviate from its intuitionistic counterpart, as the context must be restricted to a single occurrence of the atomic formula:

$$\frac{}{P \Rightarrow P}\ \mathrm{id}$$

Additionally, weakening and contraction are not admissible in our sequent calculus – we cannot freely add or dedpulicate assumptions in context since these may leave us with leftover formulas that are never used or missing copies necessary for proving the result:

$$\frac{P \Rightarrow P}{P, Q \Rightarrow P}\ \textit{invalid} \qquad \frac{A, A \Rightarrow A \otimes A}{A \Rightarrow A \otimes A}\ \textit{invalid}$$

The only structural rule we retain is exchange. For this reason, linear logic is referred to as a *substructural* logic.

If we chose to weaken our interpretation of $\Delta \Rightarrow A$ to mean that A can be proved using each assumption in Δ **at most** once, then we would have a relaxation of linear logic known as *affine logic*. In affine logic, we recover weakening, and thus the identity rule above is interchangeable with the familiar rule from intuitionistic SC:

$$\frac{}{\Delta, P \Rightarrow P}\ \mathrm{id}*$$

Intuitionistic vs. Linear Logic. Common terminology used to describe both:

Intuitionistic	Linear
Persistent	-
Monotonic	Non-monotonic
Structural	Substructural
Unrestricted	Restricted

2.2 Connectives

Conjunction. Linear logic splits conjunction into *multiplicative* ($\otimes$) and *additive* ($\&$) conjunction, whose right rules resemble that of intuitionistic $\wedge$:

$$\frac{\Delta_1 \Rightarrow A \qquad \Delta_2 \Rightarrow B}{\Delta_1, \Delta_2 \Rightarrow A \otimes B}\ \otimes R \qquad \frac{\Delta \Rightarrow A \qquad \Delta \Rightarrow B}{\Delta \Rightarrow A \& B}\ \& R$$

Notice that $\otimes R$ requires splitting the context into two subcontexts Δ_1 and Δ_2, each of which proves one of the conjuncts, whereas $\& R$ keeps the context whole in its premises. $A \otimes B$ represents having both A and B together. $\Delta \Rightarrow A \otimes B$

means we can consume Δ to simultaneously yield both A and B. This requires using a portion of Δ to produce A and the remaining portion to produce B.

$A \& B$ represents having A and B as alternatives. $\Delta \Rightarrow A \& B$ means we can choose to transform Δ into A or, alternatively, B. We call this kind of choice *external choice*, which means that when consuming $A \& B$ on the left, we are free to commit to using either A or B:

$$\frac{\Delta, A \Rightarrow C}{\Delta, A \& B \Rightarrow C} \& L_1 \qquad \frac{\Delta, B \Rightarrow C}{\Delta, A \& B \Rightarrow C} \& L_2$$

Disjunction. On the other hand, $A \oplus B$, also known as additive disjunction, represents *internal choice*. When using $A \oplus B$ on the left, it is uncertain whether we have A or B – this choice is made when producing $A \oplus B$ and is "opaque" at the time of use. As a result, in the left rule for $\oplus$, we need to case on each branch of the disjunction:

$$\frac{\Delta, A \oplus C \qquad \Delta, B \oplus C}{\Delta, A \oplus B \Rightarrow C} \oplus L$$

Units. $\top$, $\mathbb{1}$, and $\mathbb{0}$ are the units for $\&$, $\otimes$, and $\oplus$ respectively. *E.g.* $A \& \top$ is interchangeable with A for any A, and likewise for the other connectives with their units.

You may have noticed that there is an asymmetry to the linear connectives: we have multiplicative and additive conjunction, yet only additive disjunction. *Classical linear logic* completes this picture with multiplicative disjunction $\parr$ and its unit $\bot$, as well as a negation operator.

Conjunction vs. Disjunction:

	conjunction		disjunction	
	add.	mult.	add.	mult.
connectives	$\&$	$\otimes$	$\oplus$	$\parr$
units	$\top$	$\mathbb{1}$	$\mathbb{0}$	$\bot$

2.2.1 Proof Example

In intuitionistic and classical logic, implications can be curried:

$$(A \wedge B \supset C) \Rightarrow A \supset B \supset C$$

In linear logic, we can derive a corresponding principle for $\otimes$ and $\multimap$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \Rightarrow A}\,\text{id} \qquad \dfrac{}{B \Rightarrow B}\,\text{id}}{A, B \Rightarrow A \otimes B}\,\otimes R \qquad \dfrac{}{C \Rightarrow C}\,\text{id}}{A \otimes B \multimap C, A, B \Rightarrow C}\,\multimap L}{A \otimes B \multimap C, A \Rightarrow B \multimap C}\,\multimap R}{A \otimes B \multimap C \Rightarrow A \multimap B \multimap C}\,\multimap R$$

Exercise 1. *Prove the other direction.:* $A \multimap B \multimap C \Rightarrow A \otimes B \multimap C$.

The same relationship fails to hold for & and $\multimap$. Intuitively, being able to prove C using A and then B is not equivalent to doing so with solely one of A or B. If we try to derive the analogous formula for & instead of $\otimes$, we find:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A, B \Rightarrow A}\ \textcolor{red}{\textit{invalid}} \qquad \dfrac{}{A, B \Rightarrow B}\ \textcolor{red}{\textit{invalid}}}{A, B \Rightarrow A \& B}\ \& R \qquad \dfrac{}{C \Rightarrow C}\ \text{id}}{A \& B \multimap C, A, B \Rightarrow C}\ \multimap L}{A \& B \multimap C, A \Rightarrow B \multimap C}\ \multimap R}{A \& B \multimap C \Rightarrow A \multimap B \multimap C}\ \multimap R$$

That is, we cannot prove $A, B \Rightarrow A$ using the identity rule since B is present in the antecedent.

2.3 Vending Machine

To give a concrete interpretation to this logic, let's try to model a simple vending machine selling chips, kitkats, and soda and which accepts dollar bills and loose change. Take atomic formulas to be the basic resources that can be exchanged:

$$P ::= \mathsf{dollar} \mid \mathsf{change} \mid \mathsf{chips} \mid \mathsf{kitkat} \mid \mathsf{soda}$$

Sometimes we have bad days. We may occasionally feed money into the machine without a snack in return. The operation of this machine can be described via:

$$\begin{aligned} \mathsf{dollar} \multimap {} & (((\mathsf{chips} \otimes \mathsf{change}) \oplus \mathbb{1}) \\ & \& \ ((\mathsf{kitkat} \otimes \mathsf{change}) \oplus \mathbb{1}) \\ & \& \ \ (\mathsf{soda} \oplus \mathbb{1})) \end{aligned}$$

That is, providing a dollar, we can choose between:

- a bag of chips and some change, with a chance of losing our money;
- a kitkat and some change, with a chance of losing our money;
- a soda, with a chance of losing our money.

Notice the distinction between & and $\oplus$ in this example. We can freely choose between the different snack selections after inputting our dollar – this is external choice represented using &. Whether we get our snack or our money is sent to the void is a choice internal to the machine and inaccessible to us, the consumer.

3 Invertibility and Polarity

Which of the inference rules are invertible? Starting with &, we find that the right rule can be inverted via a linear cut-elimination (proof omitted, trust this is admissible):

$$\dfrac{\Delta \Rightarrow A\&B \qquad \dfrac{\overline{A \Rightarrow A}\ \text{id}}{A\&B \Rightarrow A}\ \&L}{\Delta \Rightarrow A}\ \text{cut}$$

$\&L_1$ and $\&L_2$ are not invertible, however. For instance, we can derive $A\&B \Rightarrow A$ from $A \Rightarrow A$ using $\&L_1$. But we cannot then invert $\&L_2$:

$$\dfrac{A\&B \Rightarrow A}{B \Rightarrow A}\ \textit{invalid}$$

The opposite holds true of $\otimes$. We have that $\otimes L$ is invertible:

$$\dfrac{\dfrac{\overline{A \Rightarrow A}\ \text{id} \qquad \overline{B \Rightarrow B}\ \text{id}}{A, B \Rightarrow A \otimes B}\ \otimes R \qquad \Delta, A \otimes B \Rightarrow C}{\Delta, A, B \Rightarrow C}\ \text{cut}$$

$\otimes R$ is not invertible, since choosing a splitting of the context is not syntax-directed.

A summary of the invertibility of each rule:

	$\&$	$\otimes$	$\multimap$	$\top$	$\mathbb{1}$	$\oplus$	$\mathbb{0}$
left-invertible?	×	✓	×	×	✓	✓	✓
right-invertible?	✓	×	✓	✓	×	×	×
polarity	-	+	-	-	+	+	+

Unlike in intuitionistic logic, we see that each of our connectives has a clear, indisputable polarity.