Lecture 1: Natural Deduction

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September 15, 2025

1 Introduction

Lecture outline:

- Propositions, connectives, judgments
- Natural Deduction
- Proof terms
- Counting proofs
- Harmony

2 Propositions, connectives, judgments

Judgment: $\Gamma \vdash A \ \mathsf{true}$

3 Inference rules for propositional natural deduction

(Approximately Gentzen's NJ)

Conjunction:

$$\frac{\Gamma \vdash A \ \mathsf{true} \quad \Gamma \vdash B \ \mathsf{true}}{\Gamma \vdash A \wedge B \ \mathsf{true}} \wedge I \qquad \frac{\Gamma \vdash A \wedge B \ \mathsf{true}}{\Gamma \vdash A \ \mathsf{true}} \wedge E_1 \qquad \frac{\Gamma \vdash A \wedge B \ \mathsf{true}}{\Gamma \vdash B \ \mathsf{true}} \wedge E_2$$

Disjunction:

$$\frac{\Gamma \vdash A \ \mathsf{true}}{\Gamma \vdash A \vee B \ \mathsf{true}} \vee I_1 \qquad \frac{\Gamma \vdash B \ \mathsf{true}}{\Gamma \vdash A \vee B \ \mathsf{true}} \vee I_2$$

$$\frac{\Gamma \vdash A \vee B \ \mathsf{true} \quad \Gamma, A \ \mathsf{true} \vdash C \ \mathsf{true} \quad \Gamma, B \ \mathsf{true} \vdash C \ \mathsf{true}}{\Gamma \vdash C \ \mathsf{true}} \vee E$$

(The hypothetical judgment and hypothesis rule)

$$\frac{A\ \mathsf{true} \in \Gamma}{\Gamma \vdash A\ \mathsf{true}}\ \mathsf{hyp}$$

Implication:

$$\frac{\Gamma, A\ \mathsf{true} \vdash B\ \mathsf{true}}{\Gamma \vdash A \supset B\ \mathsf{true}} \supset I \qquad \frac{\Gamma \vdash A \supset B\ \mathsf{true} \quad \Gamma \vdash A\ \mathsf{true}}{\Gamma \vdash B\ \mathsf{true}} \supset E$$

Truth and Falsehood:

$$\frac{}{\top\ \mathsf{true}}\ \top I \qquad (\text{no } \top\text{E}) \qquad (\text{no } \bot\text{I}) \qquad \frac{\Gamma \vdash \bot\ \mathsf{true}}{\Gamma \vdash C\ \mathsf{true}}\ \bot E$$

Negation:

$$\neg A := A \supset \bot$$

4 Examples

Each of the following formulas ϕ can be shown $\cdot \vdash \phi\ \mathsf{true}$, i.e. they hold in the empty context, and can be regarded as "theorems" of the system we have defined.

- $(A \supset B) \wedge A \supset B$
- $(A \supset (B \vee C)) \supset (A \wedge \neg B) \supset C$
- $(A \vee B) \wedge C \supset (A \wedge C) \vee (B \wedge C)$

Exercise 1. *Typeset derivations for these proofs.*

Exercise 2. *In class, there was a discussion of the meaning of* $\vdash$*, the notation for the hypothetical judgment. Recall that we give the hypothetical judgment* $B_1\ \mathsf{true}, \ldots, B_n\ \mathsf{true} \vdash A\ \mathsf{true}$ *its meaning via "scoped axioms" of the form*

$$\frac{}{B_i\ \mathsf{true}}$$

in the ambient metalogic, which are permitted to be used when proving $A\ \mathsf{true}$.

In Gentzen's original notation, he elided the explicit mention of Γ*, instead writing*

$$\frac{\begin{matrix}[A\ \mathsf{true}]\\ B\ \mathsf{true}\end{matrix}}{A \supset B\ \mathsf{true}} \supset I$$

When we write Γ *explicitly, it is also very tempting to treat it as "mere" syntax that can be manipulated by inference rules (e.g. Weakening, Contraction, and Exchange; or an equational theory on contexts).*

These presentational choices impact both metatheory and pedagogy. For this exercise, decide for yourself on an approach that you might try using to present *natural deduction to a audience unfamiliar with it: would you include Γ explicitly or use Gentzen's (or a similar) notation? How would you write the hypothetical rule? Justify your choices. (If you like the presentation from lecture/these notes, that's fine, but you still have to argue for it.)*

5 Soundness and Completeness

What are some properties we believe this logic should have?

What stops us from defining the rules some other way?

What are the "guiding principles"? What does it even mean to be a logic, or a logical connective?

We won't be able to answer these questions in full yet, but we can at least start with one proposed notion of soundness: *It is not possible to derive* $\cdot \vdash \bot \ \mathsf{true}$.

How would we go about demonstrating such a fact?

6 Counting Proofs

Here is a related question: how many proofs are there of

$$\cdot \vdash (A \supset B) \supset (A \wedge B) \supset B \ \mathsf{true}$$

?

There are at least two. However, there are in fact infinitely many.

Exercise 3. *Write two distinct proofs of the above judgment.*

Exercise 4. *Informally describe a procedure for generating infinitely many such proofs.*

7 Proof Terms

For the remaining discussion, it will be convenient to have a more compact notation for proofs. We therefore introduce *proof terms.*

We make a small change to three of our rules. First, for $\supset I$ and $\vee E$, we add labels to the hypotheses:

$$\frac{\Gamma, x{:}A \ \mathsf{true} \vdash B \ \mathsf{true}}{\Gamma \vdash A \supset B \ \mathsf{true}} \supset I^x$$

$$\frac{\Gamma \vdash A \vee B \ \mathsf{true} \quad \Gamma, x{:}A \ \mathsf{true} \vdash C \ \mathsf{true} \quad \Gamma, y{:}B \ \mathsf{true} \vdash C \ \mathsf{true}}{\Gamma \vdash C \ \mathsf{true}} \vee E^{x,y}$$

Then, we rename "the" hypothesis rule to its corresponding label:

$$\frac{x{:}A \ \mathsf{true} \in \Gamma}{\Gamma \vdash A \ \mathsf{true}}\ x$$

These changes have the effect of allowing us to refer uniquely to hypotheses that appear in the context Γ, and with them we can abbreviate derivations of the preceding proofs.

In one more step, we apply the following syntactic translation:

$\supset I^x(M)$	$\lambda x.\ M$ (function abstraction)
$\supset E(M, N)$	$M\ N$ (function application)
$\wedge I(M, N)$	(M, N) (pairs)
$\wedge E_i(M)$	$\pi_i M$ (projection)
$\vee I_i(M)$	$\mathsf{in}_i\ M$ (injection)
$\vee E^{x,y}(M, N, P)$	$\mathsf{case}(M, x.\,N, y.\,P)$ (case analysis)
$\top I$	$()$ (unit)
$\bot E(M)$	abort, M (error)

This gives us a somewhat familiar-looking syntax akin to functional programming—more specifically, the simply-typed λ calculus (STLC).

Exercise 5. *Rewrite the derivations from Exercise 1 in proof term notation.*

Exercise 6. *Revisit your decision from Exercise 2. How do your choices affect the information recorded by the proof term and its relationship to the derivation? If necessary, design a modification to the proof term syntax to account for your decision.*