Lecture 4: Relating Natural Deduction to Sequent Calculus via Cut Admissibility

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1 Introduction

Lecture outline:

- Cut Admissibility proof
- Natural Deduction to Sequent Calculus via Cut Elimination
- Verifications and Uses

This lecture sources extensively from Frank Pfenning's 2017 lecture notes on Sequent Calculus and Cut Elimination.

2 Cut Admissibility

Recall our statement of the Cut theorem.

Theorem 1 (Cut). *If* $\Gamma \Rightarrow A$ *and* $\Gamma, A \Rightarrow C$ *then* $\Gamma \Rightarrow C$.

Proof. By induction on the lexicographic ordering of A followed by the unordered pair of $\mathcal{D}$ and $\mathcal{E}$, where $\mathcal{D}$ is the derivation of the first assumption and $\mathcal{E}$ is the derivation of the second.

The cases for the proof fall into three categories:

- The so-called *principal cuts*, in which the cut formula A is the *principal formula* for the outermost rule of both $\mathcal{D}$ and $\mathcal{E}$. The principal formula of a rule is the one it matches against, e.g. $A \wedge B$ in $\wedge L$ and $\wedge R$.
- The cut formula is not the principal formula of the outermost rule comprising $\mathcal{D}$.
- The cut formula is not the principal formula of the outermost rule comprising $\mathcal{E}$.

In class, we discuss three cases.

- Principal cut for $\supset$.
- Side formula cut (1) for $\vee L$.
- Side formula cut (2) for $\wedge R$.

$\square$

3 From Natural Deduction to Sequent Calculus

Theorem 2 (ND to SC)**.** *If* $\Gamma \vdash A$ true *then* $\ulcorner\Gamma\urcorner \Rightarrow A$*. (*$\ulcorner\Gamma\urcorner$ *turns each* A_i $\mathsf{true} \in \Gamma$ *into a bare* A_i*).*

Proof. By induction on the structure of the derivation of $\Gamma \vdash A$ true.

In class, we cover the $\supset E$ case:

$$\mathcal{D} = \dfrac{\overset{\mathcal{D}'}{\Gamma \vdash A \supset B \ \mathsf{true}} \quad \overset{\mathcal{E}}{\Gamma \vdash A \ \mathsf{true}}}{\Gamma \vdash B \ \mathsf{true}} \supset E$$

Need to show: $\Gamma \Rightarrow B$.

By IH on $\mathcal{D}'$, we have $\mathcal{D}^* : \Gamma \Rightarrow A \supset B$. By IH on $\mathcal{E}$, and then weakening, we have $\mathcal{E}^* : \Gamma, A \supset B \Rightarrow A$.

In sequent calculus, construct the following derivation $\mathcal{F}$:

$$\dfrac{\overset{\mathcal{E}^*}{\Gamma, A \supset B \Rightarrow A} \quad \dfrac{}{\Gamma, A \supset B, B \Rightarrow B}\ \mathsf{id}_B}{\Gamma, A \supset B \Rightarrow B} \supset L$$

By cut admissibility on $\mathcal{D}^*$ and $\mathcal{F}$, with cut formula $A \supset B$, we have $\Gamma \Rightarrow B$ as needed.

The introduction rules from natural deduction all follow directly via their corresponding right rules in sequent calculus.

The other elimination rules proceed similarly to the shown case, using cut on the principal formula of the rule to connect its inductively assumed proof on the right to a derivation we can construct that uses it on the left. $\square$

Exercise 1. *Come up with a proof term assignment for sequent calculus proofs. Re-express the cases of the proof above as translating natural deduction proof terms to sequent calculus proof terms.*

Try "running" this translation on a non-normal STLC program, such as $x : A \vdash \pi_1((\lambda y.y)\ x, ())$*. Document any observations or hypotheses you have about the results, and any other experiments you might want to run to test them.*

3.1 Remarks: Soundness of ND and Normalization for STLC

Because the sequent calculus allows us to establish its consistency by easy inspection (see previous lecture), the preceding development—cut admissibility and the translation from natural deduction—combine to give us the consistency of natural deduction. To see why, suppose there exists a proof of $\cdot \vdash \bot\ \mathsf{true}$. Then by **??**, there must be a cut-free proof of the sequent $\cdot \Rightarrow \bot$, but we previously ruled this out, so the premise cannot hold.

Because these proofs were both constructive, this also means we have a *procedure* for going from a natural deduction proof to some kind of normal form. However, that normal form is still expressed as a sequent calculus derivation. To make the "round trip" back to natural deduction, we need an intermediate calculus that expresses its normal forms.

4 Verifications and Uses

We now introduce a calculus that will serve as a waypoint between natural deduction and sequent calculus, known variously as *verifications and uses* (Pfenning), *intercalation* (Gentzen(?)), *normal natural deduction* (citation needed), and *normal* or *canonical lambda calculus* (TODO cite).

Judgments: $A\uparrow$ (A has a verification); $A\downarrow$ (A may be used)

$$\frac{A\uparrow \quad B\uparrow}{A \wedge B\uparrow}\ \wedge I \qquad \frac{A \wedge B\downarrow}{A\downarrow}\ \wedge E_1 \qquad \frac{A \wedge B\downarrow}{B\downarrow}\ \wedge E_2 \qquad \frac{}{\top\uparrow}\ \top I$$

$$\frac{A\uparrow}{A \vee B\uparrow}\ \vee I_1 \qquad \frac{B\uparrow}{A \vee B\uparrow}\ \vee I_2 \qquad \frac{A \vee B\downarrow \quad A\downarrow \vdash C\uparrow \quad B\downarrow \vdash C\uparrow}{C\uparrow}\ \vee E$$

$$\frac{\bot\downarrow}{C\uparrow}\ \bot E \qquad \frac{A\downarrow \vdash B\uparrow}{A \supset B\uparrow}\ \supset I \qquad \frac{A \supset B\downarrow \quad B\uparrow}{B\downarrow}\ \supset E \qquad \frac{P\downarrow}{P\uparrow}\ \uparrow\downarrow$$

Note that we restrict switching from uses to verifications (the $\uparrow\downarrow$ rule) to atomic propositions P.

4.1 Proof Terms

At the end of lecture, we introduced a proof term language. Because we have two judgments, we have two syntactic categories, R for *atomic* terms, i.e. proofs of $A\downarrow$, and M for *normal* terms, i.e. verifications. We argued (informally) that all $M : A\uparrow$ are in β-short, η-long form, and thus we cannot represent reducible expressions. That is, all verifications are normal by construction.

$$\frac{M_1 : A\uparrow \quad M_2 : B\uparrow}{(M_1, M_2) : A \wedge B\uparrow}\ \wedge I \qquad \frac{R : A \wedge B\downarrow}{\pi_1 R : A\downarrow}\ \wedge E_1 \qquad \frac{R : A \wedge B\downarrow}{\pi_2 R : B\downarrow}\ \wedge E_2$$

$$\frac{}{() : \top\uparrow}\ \top I \qquad \frac{M : A\uparrow}{\mathsf{inl} : A \vee B\uparrow}\ \vee I_1 \qquad \frac{M : B\uparrow}{\mathsf{inr} : A \vee B\uparrow}\ \vee I_2$$

$$\frac{R : A \vee B\downarrow \quad x : A\downarrow \vdash M : C\uparrow \quad y : B\downarrow \vdash N : C\uparrow}{\mathsf{case}(R, x.M, y.N) : C\uparrow} \vee E \qquad \frac{R : \bot\downarrow}{\mathsf{case}(R) : C\uparrow} \bot E$$

$$\frac{x : A\downarrow \vdash M : B\uparrow}{\lambda x.M : A \supset B\uparrow} \supset I \qquad \frac{R : A \supset B\downarrow \quad M : B\uparrow}{R\ M : B\downarrow} \supset E \qquad \frac{R : P\downarrow}{\{R\} : P\uparrow} \uparrow\downarrow$$