Lecture 10: Focused Linear Logic

Chris Martens

October 8, 2025

1 Introduction

Lecture outline:

- Warmup: linear logic
- Focusing as a proof technique
- Polarizing linear logic propositions
- Focusing linear logic

2 Warmup

Prove the following sequent in linear logic:

$$\cdot \Rightarrow (p\&q) \multimap (r\&s) \multimap (p \otimes r)$$

Here is one solution:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p \Rightarrow p}\,\mathsf{id} \quad \dfrac{}{r \Rightarrow r}\,\mathsf{id}}{p, r \Rightarrow p \otimes r}\,\otimes R}{p, r\&s \Rightarrow p \otimes r}\,\&L_1}{p\&q, r\&s \Rightarrow p \otimes r}\,\&L_1}{p\&q \Rightarrow (r\&s) \multimap p \otimes r}\,\multimap R}{\cdot \Rightarrow (p\&q) \multimap (r\&s) \multimap (p \otimes r)}\,\multimap R
$$

Exercise 1. *Both $\otimes R$ and $\&L$ are non-invertible rules. Give another proof of the same proposition ($\cdot \Rightarrow (p\&q) \multimap (r\&s) \multimap (p \otimes r)$) where they are used in the other order.*

3 Focusing as a proof technique

Last time, we claimed that linear logic was *more general* than affine logic, which permits weakening (but not contraction). To make this claim precise, we could give an encoding of sequents in affine logic into linear. Perhaps as follows:

$$\ulcorner \Gamma \Rightarrow^{\mathsf{aff}} A \urcorner = \Gamma \Rightarrow A \otimes \top$$

We then need to show that this translation is sound in the sense that translated sequents admit weakening. This would mean: whenever $\Gamma \Rightarrow A \otimes \top$, it is also the case that for a arbitrary proposition B, we have $\Gamma, B \Rightarrow A \otimes \top$.

Unfortunately, we cannot only consider the case where our input derivation has the form

$$\frac{\begin{array}{c}\mathcal{D}_1 \\ \Gamma_1 \Rightarrow A\end{array} \quad \begin{array}{c}\mathcal{D}_2 \\ \Gamma_2 \Rightarrow \top\end{array} \quad \Gamma = \Gamma_1, \Gamma_2}{\Gamma \Rightarrow A \otimes \top} \otimes R$$

If we could, then the proof would be done by constructing a proof of the weakened sequent as:

$$\frac{\begin{array}{c}\mathcal{D}_1 \\ \Gamma_1 \Rightarrow A\end{array} \quad \dfrac{}{\Gamma_2, B \Rightarrow \top} \top R \quad \Gamma = \Gamma_1, \Gamma_2, B}{\Gamma, B \Rightarrow A \otimes \top} \otimes R$$

But without further information, we would also have to consider every possible left rule... a focused logic drastically reduces the search space of possible proofs, allowing us to make this argument by considering the structure of the derivation at the level of alternating focus-inversion phases.

(TODO: make this argument more carefully. The proof still isn't trivial.)

4 Polarizing Linear Logic Propositions

We observe that each connective has exactly one of its *left rules* or *right rules* invertible; this gives us an unambiguous polarization for each connective.

Furthermore, if we choose to maintain focus as long as possible, we can deterministically assign polarity to each subformula.

This gives us a way to stratify propositions into positives and negatives, with *shift* operators that embed each polarity into the other:

- Positives $A^+ ::= A^+ \otimes B^+ \mid \mathbb{1} \mid A^+ \oplus B^+ \mid \mathbb{0} \mid p^+ \mid \downarrow A^-$
- Negatives $A^- ::= A^- \& B^- \mid \top \mid A^+ \multimap B^- \mid p^- \mid \uparrow A^+$

5 Focused Linear Logic

The following development is based on Simmons' "structural focalization" for intuitionistic logic [Simmons(2014)], including the idea of suspended atomic propositions, which are more fully explained in that text.

Previously, when writing down rules for a focused calculus, we had premises like $\Gamma \Rightarrow A$ stable to indicate when it was permissible to move from an inversion phase to a focus phase. We would like to write down a calculus with a more syntactic criterion for stability. We do this by introducing an ordered *inversion context* consisting of propositions that must be decomposed before transitioning into a focus phase. Stability can then be defined as the inversion context being empty and the goal proposition being "stable", which is a syntactic property applying only to positive propositions and suspended negative atoms (i.e. propositions with no further right-inversions).

Inversion contexts have the following grammar and are treated as *ordered sequences* rather than multisets:

$$\Omega ::= \cdot \mid A^+, \Omega$$

The judgments are as follows:

- $\Delta; [A^-] \Rightarrow U$ (Left focus; U stable)
- $\Delta \Rightarrow [A^+]$ (Right focus)
- $\Delta; \Omega \Rightarrow A$ (Inversion)

Right inversion: all right inversion rules require an empty inversion context, forcing left-inversions to happen first.

$$\frac{\Delta; \cdot \Rightarrow A^- \quad \Delta; \cdot \Rightarrow B^-}{\Delta; \cdot \Rightarrow A^- \& B^-} \;\&R \qquad \frac{}{\Delta; \cdot \Rightarrow \top} \;\top R$$

$$\frac{\Delta; A^+ \Rightarrow B^-}{\Delta; \cdot \Rightarrow A^+ \multimap B^-} \multimap R$$

Upshift-right and suspending negative atoms:

$$\frac{\Delta; \cdot \Rightarrow A^+}{\Delta; \cdot \Rightarrow \uparrow A^+} \uparrow R \qquad \frac{\Delta; \cdot \Rightarrow \langle p^- \rangle}{\Delta; \cdot \Rightarrow p^-} \;\mathsf{sus}^-$$

Left inversion: decomposing positive connectives on the left, including shifts (which move negatives to the regular context) and suspending positive atoms.

$$\frac{\Delta; A^+, B^+, \Omega \Rightarrow C}{\Delta; A^+ \otimes B^+, \Omega \Rightarrow C} \otimes L \qquad \frac{\Delta; \Omega \Rightarrow C}{\Delta; \mathbb{1}, \Omega \Rightarrow C} \;\mathbb{1}L$$

$$\frac{\Delta; A^+, \Omega \Rightarrow C \quad \Delta; B^+, \Omega \Rightarrow C}{\Delta; A^+ \oplus B^+, \Omega \Rightarrow C} \oplus L \qquad \frac{}{\Delta; \mathbb{0}, \Omega \Rightarrow C} \;\mathbb{0}L$$

Downshift-left and suspended atomics:

$$\frac{\Delta, A^-; \Omega \Rightarrow C}{\Delta; \downarrow A^-, \Omega \Rightarrow C} \downarrow L \qquad \frac{\Delta, \langle p^+ \rangle; \Omega \Rightarrow C}{\Delta; p^+, \Omega \Rightarrow C} \,\mathsf{sus}^+$$

Moving from inversion to focus:

$$\frac{\Delta \Rightarrow [A^+]}{\Delta; \cdot \Rightarrow A^+} \,\mathsf{foc}R \qquad \frac{U \;\mathsf{stable} \quad \Delta; [A^-] \Rightarrow U}{\Delta, A^-; \cdot \Rightarrow U} \,\mathsf{foc}L \qquad \frac{}{A^+ \;\mathsf{stable}} \qquad \frac{}{\langle p^- \rangle \;\mathsf{stable}}$$

Left focus:

$$\frac{\Delta \Rightarrow [A^+] \quad \Delta; [B^-] \Rightarrow U}{\Delta; [A^+ \multimap B^-] \Rightarrow U} \multimap L \qquad \frac{\Delta; [A^-] \Rightarrow U}{\Delta; [A^- \& B^-] \Rightarrow U} \& L_1 \qquad \Delta; [A^- \& B^-] \Rightarrow U \Delta; [B^-] \Rightarrow U$$

Upshift-left puts us back into left inversion:

$$\frac{\Delta; A^+ \Rightarrow U}{\Delta; [\uparrow A^+] \Rightarrow U} \uparrow L$$

$$\frac{\Delta = \cdot}{\Delta; [p^-] \Rightarrow \langle p^- \rangle} \,\mathsf{id}^-$$

Right focus:

$$\frac{\Delta \Rightarrow [A^+] \quad \Delta \Rightarrow [B^+]}{\Delta \Rightarrow [A^+ \otimes B^+]} \otimes R \qquad \frac{\Delta = \cdot}{\Delta \Rightarrow [\mathbb{1}]} \,\mathbb{1}R$$

$$\frac{\Delta \Rightarrow [A^+]}{\Delta \Rightarrow [A^+ \oplus B^+]} \oplus R_1 \qquad \frac{\Delta \Rightarrow [B^+]}{\Delta \Rightarrow [A^+ \oplus B^+]} \oplus R_2$$

Downshift-right puts us back into right inversion:

$$\frac{\Delta; \cdot \Rightarrow A^-}{\Delta \Rightarrow [\downarrow A^-]} \downarrow R$$

Exercise 2. *Translate your proof of*

$$\cdot \Rightarrow (p \& q) \multimap (r \& s) \multimap (p \otimes r)$$

from the previous exercise, choosing an appropriate polarization for the atomic propositions. Explain how the structure of the proof forces such a polarization.

References

[Simmons(2014)] Robert J Simmons. Structural focalization. *ACM Transactions on Computational Logic (TOCL)*, 15(3):1–33, 2014.