Lecture 11: Classical Logic as Computation

Mingtong Lin

November 03, 2025

1 Introduction

Lecture outline:

- Judgmental classical logic: proof rules, proof terms, and operational semantics
- Classical reasoning: Double Negation Elimination, Law of the Excluded Middle, proof by contradiction

2 Judgmental Classical Logic (JCL)

2.1 Judgments and Syntax

In [Lovas and Crary(2006)], the authors present a judgmental formulation of natural deduction for classical logic, JCL. There are three forms of judgment in JCL:

- $\Gamma \vdash A\ \mathsf{true}$ from Γ, we can prove A true
- $\Gamma \vdash A\ \mathsf{false}$ from Γ, we can prove A false
- $\Gamma \vdash \#$ from Γ, we can prove contradiction

In this note, we will present a simplified version of JCL. The syntax is defined as the following:

$$
\begin{array}{ll}
A, B ::= A \times B \mid A + B \mid \mathbf{1} \mid \mathbf{0} \mid \neg A & \text{propositions} \\
\Gamma ::= \cdot \mid \Gamma, A\ \mathsf{true} \mid \Gamma, A\ \mathsf{false} & \text{contexts}
\end{array}
$$

Note that the above syntax does not include a rule for implications. In classical logic, we normally have two choices of encoding implications:

$$
\begin{aligned}
A \supset B &\triangleq \neg A + B \\
&\triangleq \neg(A \times \neg B)
\end{aligned}
$$

We will use the first encoding, $\neg A + B$, through out this note. For example, $A \to A \times A$ is translated to $\neg A + (A \times A)$.

2.2 Logical Rules

The logical rules of JCL are presented as the following. Note that many T rules are exactly the same introduction rules in NJ.

$$\boxed{\Gamma \vdash A \ \mathsf{true}}$$

$$\frac{}{\Gamma, A \ \mathsf{true} \vdash A \ \mathsf{true}} \ \mathsf{hypT} \qquad \frac{\Gamma, A \ \mathsf{false} \vdash \#}{\Gamma \vdash A \ \mathsf{true}} \ \#E_T$$

$$\frac{}{\Gamma \vdash \mathbf{1} \ \mathsf{true}} \ \mathbf{1}T \qquad \frac{\Gamma \vdash A \ \mathsf{true} \quad \Gamma \vdash B \ \mathsf{true}}{\Gamma \vdash A \times B \ \mathsf{true}} \ \times T$$

$$\frac{\Gamma \vdash A \ \mathsf{true}}{\Gamma \vdash A + B \ \mathsf{true}} \ +T_1 \qquad \frac{\Gamma \vdash B \ \mathsf{true}}{\Gamma \vdash A + B \ \mathsf{true}} \ +T_2$$

$$\frac{\Gamma \vdash A \ \mathsf{false}}{\Gamma \vdash \neg A \ \mathsf{true}} \ \neg T$$

$$\boxed{\Gamma \vdash A \ \mathsf{false}}$$

$$\frac{}{\Gamma, A \ \mathsf{false} \vdash A \ \mathsf{false}} \ \mathsf{hypF} \qquad \frac{\Gamma, A \ \mathsf{true} \vdash \#}{\Gamma \vdash A \ \mathsf{false}} \ \#E_F$$

$$\frac{\Gamma \vdash A \ \mathsf{false}}{\Gamma \vdash A \times B \ \mathsf{false}} \ \times F_1 \qquad \frac{\Gamma \vdash B \ \mathsf{false}}{\Gamma \vdash A \times B \ \mathsf{false}} \ \times F_2$$

$$\frac{}{\Gamma \vdash \mathbf{0} \ \mathsf{false}} \ \mathbf{0}F \qquad \frac{\Gamma \vdash A \ \mathsf{false} \quad \Gamma \vdash B \ \mathsf{false}}{\Gamma \vdash A + B \ \mathsf{false}} \ +F$$

$$\frac{\Gamma \vdash A \ \mathsf{true}}{\Gamma \vdash \neg A \ \mathsf{false}} \ \neg F \qquad \text{(no } \mathbf{1}F\text{)}$$

$$\boxed{\Gamma \vdash \#}$$

$$\frac{\Gamma \vdash A \ \mathsf{true} \quad \Gamma \vdash A \ \mathsf{false}}{\Gamma \vdash \#} \ \#I$$

To better understand this logical system, let's go over a proof of $\cdot \vdash \neg A + (A \times A) \ \mathsf{true}$. To start with, we identify the form of judgment being $\Gamma \vdash A \ \mathsf{true}$, and therefore may want to adopt the intuition from NJ and look for the applicable "introduction rules" for the principal formula. However, both the $+T_1$ and $+T_2$ rules require a premise proven from the context, while our context is empty and therefore cannot prove the premises for us. Then, the only choice left is the $\#E_T$ rule, which proves A true by contradiction, if we were able to have A false in the context in some way:

$$\dfrac{\dfrac{\cdots}{\neg A + (A \times A) \ \mathsf{false} \vdash \#} \ ?}{\cdot \vdash \neg A + (A \times A) \ \mathsf{true}} \ \#E_T$$

Now, our context is no longer empty! We get $\neg A + (A \times A)$ false in it. Then, to prove a contradiction, the $\#I$ rule requires proving two contradicting conclusions from our context:

$$\dfrac{\dfrac{\dfrac{\cdots}{\neg A + (A \times A)\ \mathsf{false} \vdash \neg A + (A \times A)\ \mathsf{true}}\ ? \quad \dfrac{}{\neg A + (A \times A)\ \mathsf{false} \vdash \neg A + (A \times A)\ \mathsf{false}}\ \mathsf{hypF}}{\neg A + (A \times A)\ \mathsf{false} \vdash \#}\ \#I}{\cdot \vdash \neg A + (A \times A)\ \mathsf{true}}\ \#E_T$$

Let's "zoom" into the left subtree. Now it seems plausible to use the good old "introduction rules", as our context is no longer empty. Here, we decide to try injecting $\neg A$ true first.

$$\dfrac{\dfrac{\cdots}{\neg A + (A \times A)\ \mathsf{false} \vdash \neg A\ \mathsf{true}}\ ?}{\neg A + (A \times A)\ \mathsf{false} \vdash \neg A + (A \times A)\ \mathsf{true}}\ +T_1$$

Then, after another proof by contradiction, we observe that we are facing the same goal again, except that we now have more information in the context, A true.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\cdots}{u, A\ \mathsf{true} \vdash \neg A + (A \times A)\ \mathsf{true}}\ ? \quad \dfrac{}{u, A\ \mathsf{true} \vdash \neg A + (A \times A)\ \mathsf{false}}\ \mathsf{hypF}}{u, A\ \mathsf{true} \vdash \#}\ \#I}{u \vdash A\ \mathsf{false}}\ \#F}{u \vdash \neg A\ \mathsf{true}}\ \neg T}{u : \neg A + (A \times A)\ \mathsf{false} \vdash \neg A + (A \times A)\ \mathsf{true}}\ +T_1$$

This time, we try injecting $A \times A$ true, since it is trivial to prove $A \times A$ true with A true available.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{u, A\ \mathsf{true} \vdash A\ \mathsf{true}}\ \mathsf{hypT} \quad \dfrac{}{u, A\ \mathsf{true} \vdash A\ \mathsf{true}}\ \mathsf{hypT}}{u, A\ \mathsf{true} \vdash A \times A\ \mathsf{true}}\ \times T}{u, A\ \mathsf{true} \vdash \neg A + (A \times A)\ \mathsf{true}}\ +T_2 \quad \dfrac{}{u, A\ \mathsf{true} \vdash \neg A + (A \times A)\ \mathsf{false}}\ \mathsf{hypF}}{u, A\ \mathsf{true} \vdash \#}\ \#I}{u \vdash A\ \mathsf{false}}\ \#F}{u \vdash \neg A\ \mathsf{true}}\ \neg T}{u : \neg A + (A \times A)\ \mathsf{false} \vdash \neg A + (A \times A)\ \mathsf{true}}\ +T_1$$

3 Proof Terms Assignment

3.1 Judgments and Syntax with Proof Terms

With proof terms, the truthy judgment becomes $\Gamma \vdash e : A$ true, and the falsy judgment becomes $\Gamma \vdash k : A$ false. The letter e and k are picked suggestively for "expressions" and "continuations". Essentially, A false is translated to a continuation accepting A, thus the metaphor that falsy formulae are consumers of the data, while the truthy formulae are producers of the data.

$$\begin{array}{ll} e ::= x \mid () \mid (e_1, e_2) \mid \mathsf{inl}\, e \mid \mathsf{inr}\, e \mid \mathsf{not}\, k \mid u : \overline{A}.\, c & \text{expressions} \\ k ::= u \mid k; \mathsf{fst} \mid k; \mathsf{snd} \mid [] \mid [k_1, k_2] \mid \mathsf{not}\, e \mid x : A.\, c & \text{continuations} \\ c ::= k \lhd e & \text{contradictions} \end{array}$$

Metavariables u and x are used respectively for falsy and truthy binders. For proof term types, we use A for A true and $\overline{A}$ for A false.

3.2 Logical Rules with Proof Terms

$$\boxed{\Gamma \vdash e : A \ \mathsf{true}}$$

$$\frac{}{\Gamma, x : A \vdash x : A \ \mathsf{true}} \ \mathsf{hypT} \qquad \frac{\Gamma, u : \overline{A} \vdash c : \#}{\Gamma \vdash u : \overline{A}.\, c : A \ \mathsf{true}} \ \#E_T$$

$$\frac{}{\Gamma \vdash () : \mathbf{1} \ \mathsf{true}} \ \mathbf{1}T \qquad \frac{\Gamma \vdash e_1 : A \ \mathsf{true} \quad \Gamma \vdash e_2 : B \ \mathsf{true}}{\Gamma \vdash (e_1, e_2) : A \times B \ \mathsf{true}} \ \times T$$

$$\frac{\Gamma \vdash e : A \ \mathsf{true}}{\Gamma \vdash \mathsf{inl}\, e : A + B \ \mathsf{true}} \ +T_1 \qquad \frac{\Gamma \vdash e : B \ \mathsf{true}}{\Gamma \vdash \mathsf{inr}\, e : A + B \ \mathsf{true}} \ +T_2$$

$$\frac{\Gamma \vdash k : A \ \mathsf{false}}{\Gamma \vdash \mathsf{not}\, k : \neg A \ \mathsf{true}} \ \neg T$$

$$\boxed{\Gamma \vdash k : A \ \mathsf{false}}$$

$$\frac{}{\Gamma, u : \overline{A} \vdash u : A \ \mathsf{false}} \ \mathsf{hypF} \qquad \frac{\Gamma, x : A \vdash M : \#}{\Gamma \vdash x : A.\, c : A \ \mathsf{false}} \ \#E_F$$

$$\frac{\Gamma \vdash k : A \ \mathsf{false}}{\Gamma \vdash k; \mathsf{fst} : A \times B \ \mathsf{false}} \ \times F_1 \qquad \frac{\Gamma \vdash k : B \ \mathsf{false}}{\Gamma \vdash k; \mathsf{snd} : A \times B \ \mathsf{false}} \ \times F_2$$

$$\frac{}{\Gamma \vdash [] : \mathbf{0} \ \mathsf{false}} \ \mathbf{0}F \qquad \frac{\Gamma \vdash k_1 : A \ \mathsf{false} \quad \Gamma \vdash k_2 : B \ \mathsf{false}}{\Gamma \vdash [k_1, k_2] : A + B \ \mathsf{false}} \ +F$$

$$\frac{\Gamma \vdash e : A \ \mathsf{true}}{\Gamma \vdash \mathsf{not}\, e : \neg A \ \mathsf{false}} \ \neg F \qquad (\text{no } \mathbf{1}F)$$

$$\boxed{\Gamma \vdash c : \#}$$

$$\frac{\Gamma \vdash e : A \ \mathsf{true} \quad \Gamma \vdash k : A \ \mathsf{false}}{\Gamma \vdash k \lhd e : \#} \ \#I$$

3.3 Translating Traditional Elimination Forms

The usual elimination forms are absent in the expression language of JCL, but they can be simulated via proof by contradiction.

$$\frac{\Gamma \vdash e : A \times B\ \mathsf{true}}{\Gamma \vdash \mathsf{fst}\, e : A\ \mathsf{true}}\ \times E_1 \qquad \frac{\Gamma \vdash e : A \times B\ \mathsf{true}}{\Gamma \vdash \mathsf{snd}\, e : B\ \mathsf{true}}\ \times E_2$$

$$\frac{\Gamma \vdash e : A + B\ \mathsf{true} \quad \Gamma, x : A \vdash e_1 : C\ \mathsf{true} \quad \Gamma, y : B \vdash e_2 : C\ \mathsf{true}}{\Gamma \vdash \mathsf{case}(e, x : A.\, e_1, y : B.\, e_2) : C\ \mathsf{true}}\ +E$$

The translations are defined as:

$$\begin{aligned}
\ulcorner \mathsf{fst}\, e \urcorner &\triangleq (\mathsf{fst}\,;u) \lhd u : \overline{A}.\, \ulcorner e \urcorner \\
\ulcorner \mathsf{snd}\, e \urcorner &\triangleq (\mathsf{snd}\,;u) \lhd u : \overline{A}.\, \ulcorner e \urcorner \\
\ulcorner \mathsf{case}(e, x : A.\, e_1, y : B.\, e_2) \urcorner &\triangleq [u \lhd x : A.\, \ulcorner e_1 \urcorner, u \lhd y : B.\, \ulcorner e_2 \urcorner] \lhd u : \overline{C}.\, e
\end{aligned}$$

Then, the translation of lambda abstraction into JCL proof terms is:

$$\ulcorner \lambda x : A.\ e \urcorner \triangleq u : \overline{\neg A + B}.\, u \lhd \mathsf{inl}\,(\mathsf{not}\,(x : A.\, u \lhd \mathsf{inr}\, \ulcorner e \urcorner))$$

With proof terms in hand, our previous example derivation can be written as:

$$\cfrac{\cfrac{\cfrac{\cdots}{u : \overline{\neg A + (A \times A)} \vdash \cdots : \neg A + (A \times A)\ \mathsf{true}}\ ? \qquad \cfrac{}{u : \overline{\neg A + (A \times A)} \vdash u : \neg A + (A \times A)\ \mathsf{false}}\ \mathsf{hypF}}{u : \overline{\neg A + (A \times A)} \vdash u \lhd \cdots : \#}\ \#I}{\cdot \vdash u : \overline{\neg A + (A \times A)}.\, u \lhd \cdots : \neg A + (A \times A)\ \mathsf{true}}\ \#E_T$$

with the zoomed subtree:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{u, x : A \vdash x : A\ \mathsf{true}}\ \mathsf{hypT} \qquad \cfrac{}{u, x : A \vdash x : A\ \mathsf{true}}\ \mathsf{hypT}}{u, x : A \vdash (x, x) : A \times A\ \mathsf{true}}\ \times T}{u, x : A \vdash \mathsf{inr}\,(x, x) : \neg A + (A \times A)\ \mathsf{true}}\ +T_2 \qquad \cfrac{}{u, x : A \vdash u : \neg A + (A \times A)\ \mathsf{false}}\ \mathsf{hypF}}{u, x : A \vdash u \lhd \mathsf{inr}\,(x, x) \#}\ \#I}{u \vdash u \lhd x.\, \mathsf{inr}\,(x, x) : A\ \mathsf{false}}\ \#F}{u \vdash \mathsf{not}\,(u \lhd x.\, \mathsf{inr}\,(x, x)) : \neg A\ \mathsf{true}}\ \neg T}{u : \overline{\neg A + (A \times A)} \vdash \mathsf{inl}\,(\mathsf{not}\,(u \lhd x.\, \mathsf{inr}\,(x, x))) : \neg A + (A \times A)\ \mathsf{true}}\ +T_1$$

References

[Lovas and Crary(2006)] William Lovas and Karl Crary. Structural normalization for classical natural deduction. 2006. URL `https://www.cs.cmu.edu/~wlovas/papers/clnorm.pdf`.