Lecture 2: Harmony

Luis Garcia

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1 Introduction

Lecture outline:

- Harmony
- Normalization for the STLC
- Global Soundness and Completeness
- Sequent Calculus

We want to show that our logic "makes sense". In other words, we want soundness. Last lecture, we posited that a definition of soundness can be showing that it is impossible to prove false. However, that's a bit of a narrow view. What we want to show is that our elimination rules aren't "too strong"—that is, that they don't produce new information we don't already have. Dual to this is showing that our elimination rules aren't "too weak". We will explore what this means with the concept of *harmony*, realized by the notions of *local soundness* and *local completeness*.

2 Local Soundness and Completeness

Local soundness and completeness are heuristics for logic design[1]. Local soundness can be witnessed by a *local reduction*, which is achieved by constructing evidence for a conclusion from evidence for its premises. Here's an example with conjunction:

$$\dfrac{\dfrac{\overset{\mathcal{D}}{\Gamma \vdash A \ \mathsf{true}} \quad \overset{\mathcal{E}}{\Gamma \vdash B \ \mathsf{true}}}{\Gamma \vdash A \wedge B \ \mathsf{true}} \wedge I}{\Gamma \vdash A \ \mathsf{true}} \wedge E_1 \quad \Longrightarrow_R \quad \overset{\mathcal{D}}{\Gamma \vdash A \ \mathsf{true}}$$

[1] Side note: it does not work for *every* logic. However, it's still a good exercise to do when designing a logic.

Local completeness is witnessed by a *local expansion*, which is achieved by applying the elimination rules to a judgment to recover the original judgment. Again, with conjunction:

$$\begin{array}{c}\mathcal{D}\\ \Gamma \vdash A \wedge B \ \mathsf{true}\end{array} \quad \Longrightarrow_E \quad \dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma \vdash A \wedge B \ \mathsf{true}\end{array}}{\Gamma \vdash A \ \mathsf{true}}\ \wedge E_1 \quad \dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma \vdash A \wedge B \ \mathsf{true}\end{array}}{\Gamma \vdash B \ \mathsf{true}}\ \wedge E_2}{\Gamma \vdash A \wedge B \ \mathsf{true}}\ \wedge I$$

We can complete the same exercise for other connectives to show that they are harmonious. Disjunction is a bit more complicated than conjunction. It also gives us a chance to see what happens when local soundness or completeness fails. Consider a "bad" rule for elimination rule for disjunction:

$$\dfrac{\Gamma \vdash A \vee B \ \mathsf{true}}{\Gamma \vdash A \ \mathsf{true}}\ \vee E?$$

Checking for local soundness, we can see that we cannot locally reduce an elimination followed by an introduction:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma \vdash B \ \mathsf{true}\end{array}}{\Gamma \vdash A \vee B \ \mathsf{true}}\ \vee I_2}{\Gamma \vdash A \ \mathsf{true}}\ \vee E?$$

Our elimination rule is too strong: it allows us to conclude A true even though we have no evidence to prove it. To see the contrast with the correct rule, we need to employ the *substitution principle*.

3 Substitution

Definition 1 (Substitution Principle)**.** *If* $\Gamma, A \ \mathsf{true} \vdash C \ \mathsf{true}$ *and* $\Gamma \vdash A \ \mathsf{true}$*, then* $\Gamma \vdash C \ \mathsf{true}$.

This is a fundamental principle of natural deduction that corresponds to variable substitutions in a lambda calculus. Let's see how. The local reduction of disjunction is written below. There are actually two local reductions we could do. The case where we use $\vee I_2$ is symmetric to the one we show here. We write $subst(\mathcal{E}, \mathcal{D})$ to show that we are applying the substitution principle on the derivations.

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma \vdash A \ \mathsf{true}\end{array}}{\Gamma \vdash A \vee B \ \mathsf{true}}\ \vee I_1 \quad \begin{array}{c}\mathcal{E}\\ \Gamma, A \ \mathsf{true} \vdash C \ \mathsf{true}\end{array} \quad \begin{array}{c}\mathcal{F}\\ \Gamma, B \ \mathsf{true} \vdash C \ \mathsf{true}\end{array}}{\Gamma \vdash C \ \mathsf{true}}\ \vee E \quad \Longrightarrow_R \quad \begin{array}{c}subst(\mathcal{E}, \mathcal{D})\\ \Gamma \vdash C \ \mathsf{true}\end{array}$$

For completeness, here's local expansion:

$$\begin{array}{c}\mathcal{D}\\ \Gamma \vdash A \vee B\ \mathsf{true}\end{array} \quad \Longrightarrow_E \quad \dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma \vdash A \vee B\ \mathsf{true}\end{array} \quad \dfrac{\dfrac{}{\Gamma, A\ \mathsf{true} \vdash A\ \mathsf{true}}\,\mathit{hyp}}{\Gamma, A \vdash A \vee B\ \mathsf{true}}\,\vee I_1 \quad \dfrac{\dfrac{}{\Gamma, B\ \mathsf{true} \vdash B\ \mathsf{true}}\,\mathit{hyp}}{\Gamma, B \vdash A \vee B\ \mathsf{true}}\,\vee I_2}{\Gamma \vdash A \vee B\ \mathsf{true}}\,\vee E$$

Now, how does the substitution principle here relate to substitution in programming? Recall that we can condense a proof tree into a proof term in the lambda calculus. For the above tree for the local soundness demonstration, we can write the term:

$$\textbf{case}\ (\mathit{inl}\ M)\ \textbf{of}\ (x \Rightarrow N \mid y \Rightarrow P)$$

where $M : A \vee B$, $x : A$, and $y : B$. Then, the local reduction tells us that we have the following

$$[M/x]N.$$

From the world of programming languages, this is a simple β-reduction rule!

$$\textbf{case}\ (\mathit{inl}\ M)\ \textbf{of}\ (x \Rightarrow N \mid y \Rightarrow P) \mapsto_\beta [M/x]N$$

Similarly, local expansion corresponds to η-expansion rules:

$$M \mapsto_\eta \langle \mathit{inl}\ M, \mathit{inr}\ M \rangle$$

provided that $M : A \vee B$.

Exercise 1. *Show that the rules for implication are locally sound and complete, and show how they relate to β-reduction and η-expansion.*

4 Normalization

By thinking about soundness, we are also working toward a sense for what *normalization* means. The vibe is "there are no detours or circularities in our proofs". By detours, we mean exactly the kinds of things we are doing to show local soundness and completeness: pointless invocations of rules that yield no new information from our hypotheses.