Rule Sheet: Verifications and Uses

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1 Verifications and Uses

Judgments: $A\uparrow$ (A has a verification); $A\downarrow$ (A may be used)

$$\frac{A\uparrow \quad B\uparrow}{A \wedge B\uparrow}\ \wedge I \qquad \frac{A \wedge B\downarrow}{A\downarrow}\ \wedge E_1 \qquad \frac{A \wedge B\downarrow}{B\downarrow}\ \wedge E_2 \qquad \frac{}{\top\uparrow}\ \top I$$

$$\frac{A\uparrow}{A \vee B\uparrow}\ \vee I_1 \qquad \frac{B\uparrow}{A \vee B\uparrow}\ \vee I_2 \qquad \frac{A \vee B\downarrow \quad A\downarrow \vdash C\uparrow \quad B\downarrow \vdash C\uparrow}{C\uparrow}\ \vee E$$

$$\frac{\bot\downarrow}{C\uparrow}\ \bot E \qquad \frac{A\downarrow \vdash B\uparrow}{A \supset B\uparrow}\ \supset I \qquad \frac{A \supset B\downarrow \quad B\uparrow}{B\downarrow}\ \supset E \qquad \frac{P\downarrow}{P\uparrow}\ \uparrow\downarrow$$

Theorem 1 (Uniform Substitution)**.** *If* $\Gamma, x : A\downarrow \vdash J$ *and* $\Gamma \vdash A\downarrow$ *then* $\Gamma \vdash J$ *(where* J *may be* $C\uparrow$ *or* $C\downarrow$*).*

Proof. Everywhere x us used to justify $A\downarrow$ in the proof of J, use the proof of $\Gamma \vdash A\downarrow$ instead. □

1.1 Proof Terms

$$\frac{M_1 : A\uparrow \quad M_2 : B\uparrow}{(M_1, M_2) : A \wedge B\uparrow}\ \wedge I \qquad \frac{R : A \wedge B\downarrow}{\pi_1 R : A\downarrow}\ \wedge E_1 \qquad \frac{R : A \wedge B\downarrow}{\pi_2 R : B\downarrow}\ \wedge E_2$$

$$\frac{}{() : \top\uparrow}\ \top I \qquad \frac{M : A\uparrow}{\mathsf{inl} : A \vee B\uparrow}\ \vee I_1 \qquad \frac{M : B\uparrow}{\mathsf{inr} : A \vee B\uparrow}\ \vee I_2$$

$$\frac{R : A \vee B\downarrow \quad x : A\downarrow \vdash M : C\uparrow \quad y : B\downarrow \vdash N : C\uparrow}{\mathsf{case}(R, x.M, y.N) : C\uparrow}\ \vee E \qquad \frac{R : \bot\downarrow}{\mathsf{case}(R) : C\uparrow}\ \bot E$$

$$\frac{x : A\downarrow \vdash M : B\uparrow}{\lambda x.M : A \supset B\uparrow}\ \supset I \qquad \frac{R : A \supset B\downarrow \quad M : B\uparrow}{R\ M : B\downarrow}\ \supset E \qquad \frac{R : P\downarrow}{\{R\} : P\uparrow}\ \uparrow\downarrow$$