Assignment 1: Natural Deduction and Sequent Calculus

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1 Lecture 1 Exercises

1.1 Natural Deduction

- $(A \supset B) \wedge A \supset B$
- $(A \supset (B \vee C)) \supset (A \wedge \neg B) \supset C$
- $(A \vee B) \wedge C \supset (A \wedge C) \vee (B \wedge C)$

Exercise 1. *Typeset derivations for these proofs.*

Exercise 2. *In class, there was a discussion of the meaning of $\vdash$, the notation for the hypothetical judgment. Recall that we give the hypothetical judgment $B_1\ \mathsf{true}, \ldots, B_n\ \mathsf{true} \vdash A\ \mathsf{true}$ its meaning via "scoped axioms" of the form*

$$\frac{}{B_i\ \mathsf{true}}$$

in the ambient metalogic, which are permitted to be used when proving A true.

In Gentzen's original notation, he elided the explicit mention of Γ, instead writing

$$\frac{\begin{array}{c}[A\ \mathsf{true}]\\ B\ \mathsf{true}\end{array}}{A \supset B\ \mathsf{true}} \supset I$$

When we write Γ explicitly, it is also very tempting to treat it as "mere" syntax that can be manipulated by inference rules (e.g. Weakening, Contraction, and Exchange; or an equational theory on contexts).

These presentational choices impact both metatheory and pedagogy. For this exercise, decide for yourself on an approach that you might try using to present natural deduction to a audience unfamiliar with it: would you include Γ explicitly or use Gentzen's (or a similar) notation? How would you write the hypothetical rule? Justify your choices. (If you like the presentation from lecture/these notes, that's fine, but you still have to argue for it.)

1.2 Counting Proofs

There are infinitely many proofs of

$$\cdot \vdash (A \supset B) \supset (A \wedge B) \supset B \ \mathsf{true}$$

Exercise 3. *Write two distinct proofs of the above judgment.*

Exercise 4. *Informally describe a procedure for generating infinitely many such proofs.*

1.3 Proof Terms

Exercise 5. *Rewrite the derivations from Exercise 1 in proof term notation.*

Exercise 6. *Revisit your decision from Exercise 2. How do your choices affect the information recorded by the proof term and its relationship to the derivation? If necessary, design a modification to the proof term syntax to account for your decision.*

2 Lecture 2 Exercises

Exercise 7. *Show that the rules for implication are locally sound and complete, and show how they relate to β-reduction and η-expansion.*

3 Lecture 3 Exercises

3.1 Sequent Calculus

- $A \vee B \supset A \vee B$
- $(A \supset B) \wedge A \supset B$
- $(A \supset (B \vee C)) \supset (A \wedge \neg B) \supset C$
- $(A \vee B) \wedge C \supset (A \wedge C) \vee (B \wedge C)$

Exercise 8. *Typeset proofs of $\cdot \Rightarrow A$ (in sequent calculus) for each formula A above.*

Exercise 9. *State and prove a corresponding theorem for Contraction. If you are unable to make the proof rigorous, discuss why and what might be done to address it.*

3.2 Non-provability

In sequent calculus, it is easy to demonstrate that certain sequents are *not* provable. We will work through some examples:

- Soundness: no proof of $\cdot \Rightarrow \bot$
- Disjunction property: if $\cdot \Rightarrow A \vee B$, then $\cdot \Rightarrow A$ or $\cdot \Rightarrow B$.
- Non-provability of excluded middle (LEM): there is no proof of $\cdot \Rightarrow A \vee \neg A$ for arbitrary A.
- Non-provability of double-negation elimination (DNE): there is no proof of $\cdot \Rightarrow \neg\neg A \implies A$.

Exercise 10. *Typeset proofs of the above non-provability arguments.*

Exercise 11. *Find another classically valid, but not intuitionistically valid, proposition A and demonstrate that $\cdot \Rightarrow A$ is not provable in general.*

Exercise 12. *Why do we keep qualifying each of these statements with "in general" or "for arbitrary A"? Explain how the situation changes when we are allowed to talk about specific propositions A.*

Exercise 13. *Discuss why the Cut and Identity metatheorem statements correspond to soundness and completeness for sequent calculus.*

4 Lecture 4 Exercises

Exercise 14. *Come up with a proof term assignment for sequent calculus proofs. Re-express the cases of the proof above as translating natural deduction proof terms to sequent calculus proof terms.*

Try "running" this translation on a non-normal STLC program, such as $x : A \vdash \pi_1((\lambda y.y)\ x, ())$. Document any observations or hypotheses you have about the results, and any other experiments you might want to run to test them.