

Lecture 12: Classical Logic as Computation

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1 Introduction

Lecture outline:

- Judgmental classical logic: proof rules, proof terms, and operational semantics
- Classical reasoning: Double Negation Elimination, Law of the Excluded Middle, proof by contradiction

2 Judgmental Classical Logic (JCL)

2.1 Judgments and Syntax

In [Lovas and Crary(2006)], the authors present a judgmental formulation of natural deduction for classical logic, JCL. There are three forms of judgment in JCL:

- $\Gamma \vdash A$ true from Γ , we can prove A true
- $\Gamma \vdash A$ false from Γ , we can prove A false
- $\Gamma \vdash \#$ from Γ , we can prove contradiction

In this note, we will present a simplified version of JCL. The syntax is defined as the following:

$$\begin{array}{ll} A, B ::= A \times B \mid A + B \mid \mathbf{1} \mid \mathbf{0} \mid \neg A & \text{propositions} \\ \Gamma ::= \cdot \mid \Gamma, A \text{ true} \mid \Gamma, A \text{ false} & \text{contexts} \end{array}$$

Note that the above syntax does not include a rule for implications. In classical logic, we normally have two choices of encoding implications:

$$\begin{aligned} A \supset B &\triangleq \neg A + B \\ &\triangleq \neg(A \times \neg B) \end{aligned}$$

We will use the first encoding, $\neg A + B$, through out this note. For example, $A \rightarrow A \times A$ is translated to $\neg A + (A \times A)$.

2.2 Logical Rules

The logical rules of JCL are presented as the following. Note that many T rules are exactly the same introduction rules in NJ.

$$\begin{array}{c}
 \boxed{\Gamma \vdash A \text{ true}} \\
 \frac{}{\Gamma, A \text{ true} \vdash A \text{ true}} \text{hypT} \quad \frac{\Gamma, A \text{ false} \vdash \#}{\Gamma \vdash A \text{ true}} \#E_T \\
 \frac{}{\Gamma \vdash \mathbf{1} \text{ true}} \mathbf{1}T \quad \frac{\Gamma \vdash A \text{ true} \quad \Gamma \vdash B \text{ true}}{\Gamma \vdash A \times B \text{ true}} \times T \\
 \frac{\Gamma \vdash A \text{ true}}{\Gamma \vdash A + B \text{ true}} +T_1 \quad \frac{\Gamma \vdash B \text{ true}}{\Gamma \vdash A + B \text{ true}} +T_2 \\
 \frac{\Gamma \vdash A \text{ false}}{\Gamma \vdash \neg A \text{ true}} \neg T \\
 \boxed{\Gamma \vdash A \text{ false}} \\
 \frac{}{\Gamma, A \text{ false} \vdash A \text{ false}} \text{hypF} \quad \frac{\Gamma, A \text{ true} \vdash \#}{\Gamma \vdash A \text{ false}} \#E_F \\
 \frac{\Gamma \vdash A \text{ false}}{\Gamma \vdash A \times B \text{ false}} \times F_1 \quad \frac{\Gamma \vdash B \text{ false}}{\Gamma \vdash A \times B \text{ false}} \times F_2 \\
 \frac{}{\Gamma \vdash \mathbf{0} \text{ false}} \mathbf{0}F \quad \frac{\Gamma \vdash A \text{ false} \quad \Gamma \vdash B \text{ false}}{\Gamma \vdash A + B \text{ false}} +F \\
 \frac{\Gamma \vdash A \text{ true}}{\Gamma \vdash \neg A \text{ false}} \neg F \quad (\text{no } \mathbf{1}F) \\
 \boxed{\Gamma \vdash \#} \\
 \frac{\Gamma \vdash A \text{ true} \quad \Gamma \vdash A \text{ false}}{\Gamma \vdash \#} \#I
 \end{array}$$

To better understand this logical system, let's go over a proof of $\cdot \vdash \neg A + (A \times A) \text{ true}$. To start with, we identify the form of judgment being $\Gamma \vdash A \text{ true}$, and therefore may want to adopt the intuition from NJ and look for the applicable “introduction rules” for the principal formula. However, both the $+T_1$ and $+T_2$ rules require a premise proven from the context, while our context is empty and therefore cannot prove the premises for us. Then, the only choice left is the $\#E_T$ rule, which proves $A \text{ true}$ by contradiction, if we were able to have $A \text{ false}$ in the context in some way:

$$\frac{\dots}{\frac{\neg A + (A \times A) \text{ false} \vdash \#}{\cdot \vdash \neg A + (A \times A) \text{ true}}} ? \#E_T$$

Now, our context is no longer empty! We get $\neg A + (A \times A)$ false in it. Then, to prove a contradiction, the $\#I$ rule requires proving two contradicting conclusions from our context:

$$\frac{\dots}{\neg A + (A \times A) \text{ false} \vdash \neg A + (A \times A) \text{ true}} ? \quad \frac{\neg A + (A \times A) \text{ false} \vdash \neg A + (A \times A) \text{ false}}{\neg A + (A \times A) \text{ false} \vdash \#} \begin{array}{l} \text{hypF} \\ \#I \end{array}$$

$$\frac{\neg A + (A \times A) \text{ false} \vdash \#}{\cdot \vdash \neg A + (A \times A) \text{ true}} \#E_T$$

Let's "zoom" into the left subtree. Now it seems plausible to use the good old "introduction rules", as our context is no longer empty. Here, we decide to try injecting $\neg A$ true first.

$$\frac{\dots}{\neg A + (A \times A) \text{ false} \vdash \neg A \text{ true}} ?$$

$$\frac{}{\neg A + (A \times A) \text{ false} \vdash \neg A + (A \times A) \text{ true}} +T_1$$

Then, after another proof by contradiction, we observe that we are facing the same goal again, except that we now have more information in the context, A true.

$$\frac{\dots}{u, A \text{ true} \vdash \neg A + (A \times A) \text{ true}} ? \quad \frac{u, A \text{ true} \vdash \neg A + (A \times A) \text{ false}}{\begin{array}{l} u, A \text{ true} \vdash \# \\ \frac{u \vdash A \text{ false}}{u \vdash \neg A \text{ true}} \#F \\ \frac{u \vdash \neg A \text{ true}}{u : \neg A + (A \times A) \text{ false} \vdash \neg A + (A \times A) \text{ true}} \neg T \end{array}} \begin{array}{l} \text{hypF} \\ \#I \end{array}$$

This time, we try injecting $A \times A$ true, since it is trivial to prove $A \times A$ true with A true available.

$$\frac{\begin{array}{l} u, A \text{ true} \vdash A \text{ true} \quad \text{hypT} \\ u, A \text{ true} \vdash A \text{ true} \quad \text{hypT} \end{array}}{u, A \text{ true} \vdash A \times A \text{ true}} \times T$$

$$\frac{u, A \text{ true} \vdash A \times A \text{ true}}{u, A \text{ true} \vdash \neg A + (A \times A) \text{ true}} +T_2 \quad \frac{u, A \text{ true} \vdash \neg A + (A \times A) \text{ false}}{\begin{array}{l} u, A \text{ true} \vdash \# \\ \frac{u \vdash A \text{ false}}{u \vdash \neg A \text{ true}} \#F \\ \frac{u \vdash \neg A \text{ true}}{u : \neg A + (A \times A) \text{ false} \vdash \neg A + (A \times A) \text{ true}} \neg T \end{array}} \begin{array}{l} \text{hypF} \\ \#I \end{array}$$

$$\frac{}{u : \neg A + (A \times A) \text{ false} \vdash \neg A + (A \times A) \text{ true}} +T_1$$

3 Proof Terms Assignment

3.1 Judgments and Syntax with Proof Terms

With proof terms, the truthy judgment becomes $\Gamma \vdash e : A$ true, and the falsy judgment becomes $\Gamma \vdash k : A$ false. The letter e and k are picked suggestively for "expressions" and "continuations". Essentially, A false is translated to a

continuation accepting A , thus the metaphor that falsy formulae are consumers of the data, while the truthy formulae are producers of the data.

$$\begin{array}{ll}
 e ::= x \mid () \mid (e_1, e_2) \mid \text{inl } e \mid \text{inr } e \mid \text{not } k \mid u : \bar{A}. c & \text{expressions} \\
 k ::= u \mid k; \text{fst} \mid k; \text{snd} \mid [] \mid [k_1, k_2] \mid \text{not } e \mid x : A. c & \text{continuations} \\
 c ::= k \triangleleft e & \text{contradictions}
 \end{array}$$

Metavariables u and x are used respectively for falsy and truthy binders. For proof term types, we use A for A true and \bar{A} for A false.

3.2 Logical Rules with Proof Terms

$$\boxed{\Gamma \vdash e : A \text{ true}}$$

$$\begin{array}{c}
 \frac{}{\Gamma, x : A \vdash x : A \text{ true}} \text{hypT} \quad \frac{\Gamma, u : \bar{A} \vdash c : \#}{\Gamma \vdash u : \bar{A}. c : A \text{ true}} \#E_T \\
 \frac{}{\Gamma \vdash () : \mathbf{1} \text{ true}} \mathbf{1}T \quad \frac{\Gamma \vdash e_1 : A \text{ true} \quad \Gamma \vdash e_2 : B \text{ true}}{\Gamma \vdash (e_1, e_2) : A \times B \text{ true}} \times T \\
 \frac{\Gamma \vdash e : A \text{ true}}{\Gamma \vdash \text{inl } e : A + B \text{ true}} +T_1 \quad \frac{\Gamma \vdash e : B \text{ true}}{\Gamma \vdash \text{inr } e : A + B \text{ true}} +T_2 \\
 \frac{\Gamma \vdash k : A \text{ false}}{\Gamma \vdash \text{not } k : \neg A \text{ true}} \neg T
 \end{array}$$

$$\boxed{\Gamma \vdash k : A \text{ false}}$$

$$\begin{array}{c}
 \frac{}{\Gamma, u : \bar{A} \vdash u : A \text{ false}} \text{hypF} \quad \frac{\Gamma, x : A \vdash M : \#}{\Gamma \vdash x : A. c : A \text{ false}} \#E_F \\
 \frac{\Gamma \vdash k : A \text{ false}}{\Gamma \vdash k; \text{fst} : A \times B \text{ false}} \times F_1 \quad \frac{\Gamma \vdash k : B \text{ false}}{\Gamma \vdash k; \text{snd} : A \times B \text{ false}} \times F_2 \\
 \frac{}{\Gamma \vdash [] : \mathbf{0} \text{ false}} \mathbf{0}F \quad \frac{\Gamma \vdash k_1 : A \text{ false} \quad \Gamma \vdash k_2 : B \text{ false}}{\Gamma \vdash [k_1, k_2] : A + B \text{ false}} +F \\
 \frac{\Gamma \vdash e : A \text{ true}}{\Gamma \vdash \text{not } e : \neg A \text{ false}} \neg F \quad (\text{no } \mathbf{1}F)
 \end{array}$$

$$\boxed{\Gamma \vdash c : \#}$$

$$\frac{\Gamma \vdash e : A \text{ true} \quad \Gamma \vdash k : A \text{ false}}{\Gamma \vdash k \triangleleft e : \#} \#I$$

3.3 Translating Traditional Elimination Forms

The usual elimination forms are absent in the expression language of JCL, but they can be simulated via proof by contradiction.

$$\frac{\Gamma \vdash e : A \times B \text{ true} \quad \Gamma \vdash e : A \times B \text{ true}}{\Gamma \vdash \text{fst } e : A \text{ true}} \times_{E_1} \quad \frac{\Gamma \vdash e : A \times B \text{ true} \quad \Gamma \vdash \text{snd } e : B \text{ true}}{\Gamma \vdash \text{snd } e : B \text{ true}} \times_{E_2}$$

$$\frac{\Gamma \vdash e : A + B \text{ true} \quad \Gamma, x : A \vdash e_1 : C \text{ true} \quad \Gamma, y : B \vdash e_2 : C \text{ true}}{\Gamma \vdash \text{case}(e, x : A. e_1, y : B. e_2) : C \text{ true}} +_E$$

The translations are defined as:

$$\begin{aligned}\lceil \text{fst } e \rceil &\triangleq (\text{fst}; u) \triangleleft u : \overline{A}. \lceil e \rceil \\ \lceil \text{snd } e \rceil &\triangleq (\text{snd}; u) \triangleleft u : \overline{A}. \lceil e \rceil \\ \lceil \text{case}(e, x : A. e_1, y : B. e_2) \rceil &\triangleq [u \triangleleft x : A. \lceil e_1 \rceil, u \triangleleft y : B. \lceil e_2 \rceil] \triangleleft u : \overline{C}. e\end{aligned}$$

Then, the translation of lambda abstraction into JCL proof terms is:

$$\lceil \lambda x : A. e \rceil \triangleq u : \overline{\neg A + B}. u \triangleleft \text{inl}(\text{not}(x : A. u \triangleleft \text{inr} \lceil e \rceil))$$

With proof terms in hand, our previous example derivation can be written as:

$$\frac{\frac{\dots}{u : \overline{\neg A + (A \times A)} \vdash \dots : \neg A + (A \times A) \text{ true}} ? \quad \frac{u : \overline{\neg A + (A \times A)} \vdash u : \neg A + (A \times A) \text{ false}}{\frac{u : \overline{\neg A + (A \times A)} \vdash u \triangleleft \dots : \#}{\cdot \vdash u : \overline{\neg A + (A \times A)}. u \triangleleft \dots : \neg A + (A \times A) \text{ true}}} \#I}{\text{hypF}}$$

with the zoomed subtree:

$$\frac{\frac{\frac{u, x : A \vdash x : A \text{ true}}{\text{hypT}} \quad \frac{u, x : A \vdash x : A \text{ true}}{\text{hypT}} \times T}{u, x : A \vdash (x, x) : A \times A \text{ true}} + T_2 \quad \frac{u, x : A \vdash u : \neg A + (A \times A) \text{ false}}{\text{hypF}} \#I}{\frac{\frac{u, x : A \vdash u \triangleleft \text{inr}(x, x) \#}{u \vdash u \triangleleft x. \text{inr}(x, x) : A \text{ false}} \#F}{u \vdash \text{not}(u \triangleleft x. \text{inr}(x, x)) : \neg A \text{ true}} \neg T}} + T_1$$

References

[Lovas and Crary(2006)] William Lovas and Karl Crary. Structural normalization for classical natural deduction. 2006. URL <https://www.cs.cmu.edu/~wlovas/papers/clnorm.pdf>.