Lecture 1: Natural Deduction

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1 Introduction

Lecture outline:

- Propositions, connectives, judgments
- Natural Deduction
- Proof terms
- Counting proofs
- Harmony

2 Propositions, connectives, judgments

Judgment: $\Gamma \vdash A \ \mathsf{true}$

3 Inference rules for propositional natural deduction

(Approximately Gentzen's NJ)

Conjunction:

$$\frac{\Gamma \vdash A \ \mathsf{true} \quad \Gamma \vdash B \ \mathsf{true}}{\Gamma \vdash A \wedge B \ \mathsf{true}} \wedge I \qquad \frac{\Gamma \vdash A \wedge B \ \mathsf{true}}{\Gamma \vdash A \ \mathsf{true}} \wedge E_1 \qquad \frac{\Gamma \vdash A \wedge B \ \mathsf{true}}{\Gamma \vdash B \ \mathsf{true}} \wedge E_2$$

Disjunction:

$$\frac{\Gamma \vdash A \ \mathsf{true}}{\Gamma \vdash A \vee B \ \mathsf{true}} \vee I_1 \qquad \frac{\Gamma \vdash B \ \mathsf{true}}{\Gamma \vdash A \vee B \ \mathsf{true}} \vee I_2$$

$$\frac{\Gamma \vdash A \vee B \ \mathsf{true} \quad \Gamma, A \ \mathsf{true} \vdash C \ \mathsf{true} \quad \Gamma, B \ \mathsf{true} \vdash C \ \mathsf{true}}{\Gamma \vdash C \ \mathsf{true}} \vee E$$

(The hypothetical judgment and hypothesis rule)

$$\frac{A \ \mathsf{true} \in \Gamma}{\Gamma \vdash A \ \mathsf{true}} \ \mathsf{hyp}$$

Implication:

$$\frac{\Gamma, A \ \mathsf{true} \vdash B \ \mathsf{true}}{\Gamma \vdash A \supset B \ \mathsf{true}} \supset I \qquad \frac{\Gamma \vdash A \supset B \ \mathsf{true} \quad \Gamma \vdash A \ \mathsf{true}}{\Gamma \vdash B \ \mathsf{true}} \supset E$$

Truth and Falsehood:

$$\frac{}{\top \ \mathsf{true}} \ \top I \qquad \text{(no } \top\text{E)} \qquad \text{(no } \bot\text{I)} \qquad \frac{\Gamma \vdash \bot \ \mathsf{true}}{\Gamma \vdash C \ \mathsf{true}} \ \bot E$$

Negation:

$$\neg A := A \supset \bot$$

4 Examples

Each of the following formulas ϕ can be shown $\cdot \vdash \phi$ true, i.e. they hold in the empty context, and can be regarded as "theorems" of the system we have defined.

- $(A \supset B) \wedge A \supset B$
- $(A \supset (B \vee C)) \supset (A \wedge \neg B) \supset C$
- $(A \vee B) \wedge C \supset (A \wedge C) \vee (B \wedge C)$

5 Soundness and Completeness

What are some properties we believe this logic should have?

What stops us from defining the rules some other way?

What are the "guiding principles"? What does it even mean to be a logic, or a logical connective?

We won't be able to answer these questions in full yet, but we can at least start with one proposed notion of soundness: *It is not possible to derive* $\cdot \vdash \bot$ true.

How would we go about demonstrating such a fact?

6 Counting Proofs

Here is a related question: how many proofs are there of

$$(A \supset B) \supset (A \wedge B) \supset B$$

?

There are at least two. However, there are in fact infinitely many.

7 Proof Terms

For the remaining discussion, it will be convenient to have a more compact notation for proofs. We therefore introduce *proof terms*.

We make a small change to three of our rules. First, for $\supset I$ and $\vee E$, we add labels to the hypotheses:

$$\frac{\Gamma, x{:}A \ \mathsf{true} \vdash B \ \mathsf{true}}{\Gamma \vdash A \supset B \ \mathsf{true}} \supset I^x$$

$$\frac{\Gamma \vdash A \vee B \ \mathsf{true} \quad \Gamma, x{:}A \ \mathsf{true} \vdash C \ \mathsf{true} \quad \Gamma, y{:}B \ \mathsf{true} \vdash C \ \mathsf{true}}{\Gamma \vdash C \ \mathsf{true}} \vee E^{x,y}$$

Then, we rename "the" hypothesis rule to its corresponding label:

$$\frac{x{:}A \ \mathsf{true} \in \Gamma}{\Gamma \vdash A \ \mathsf{true}} \ x$$

These changes have the effect of allowing us to refer uniquely to hypotheses that appear in the context Γ, and with them we can abbreviate derivations of the preceding proofs.

In one more step, we apply the following syntactic translation:

$\supset I^x(M)$	$\lambda x.\, M$ (function abstraction)
$\supset E(M, N)$	$M\ N$ (function application)
$\wedge I(M, N)$	(M, N) (pairs)
$\wedge E_i(M)$	$\pi_i M$ (projection)
$\vee I_i(M)$	$\mathsf{in}_i\, M$ (injection)
$\vee E^{x,y}(M, N, P)$	$\mathsf{case}(M, x.\, N, y.\, P)$ (case analysis)
$\top I$	$()$ (unit)
$\bot E(M)$	abort, M (error)

This gives us a somewhat familiar-looking syntax akin to functional programming—more specifically, the simply-typed λ calculus (STLC).

8 Harmony: Local Soundness and Completeness

8.1 (Internal) Soundness and Completeness

Soundness: the logic is not vacuous. That is, there are some propositions that are not true.

In particular, we can't write a closed proof of $\bot$.

Completeness: the hypothetical judgment captures deduction in the logic. That is, even if we only allow the use of hypotheses at atomic propositions, then we can still construct a proof of each proposition by assuming it.

8.2 Local Soundness as Proof Reduction

Soundness essentially states that a logic's proofs do not admit *more information* than what is used to construct them. The proof amounts to showing that *circuitous* steps in proofs can be eliminated. Soundness is a *global* property of the logic: any rule could interact in some unexpected ways with all of the other rules, so we can't just check them in isolation.

However, there is a weaker notion of soundness that we can check for just a single propositional connective: if a proof has circuitousness by virtue of *immediately* introducing and then eliminating a connective, we can eliminate such a redundancy. We demonstrate this by identifying "circuitous" proofs and showing how they can be rewritten to avoid the unnecessary steps.

Conjunction:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ A\ \mathsf{true}\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ B\ \mathsf{true}\end{array}}{A \wedge B\ \mathsf{true}}\wedge I}{A\ \mathsf{true}}\wedge E_1 \quad \Longrightarrow \quad \begin{array}{c}\mathcal{D}_1\\ A\ \mathsf{true}\end{array}$$

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ A\ \mathsf{true}\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ B\ \mathsf{true}\end{array}}{A \wedge B\ \mathsf{true}}\wedge I}{B\ \mathsf{true}}\wedge E_2 \quad \Longrightarrow \quad \begin{array}{c}\mathcal{D}_2\\ B\ \mathsf{true}\end{array}$$

Implication:

$$\dfrac{\dfrac{\begin{array}{c}\overline{A\ \mathsf{true}}\ u\\ \vdots\\ B\ \mathsf{true}\end{array}}{A \supset B\ \mathsf{true}}\supset I^u \quad \begin{array}{c}\mathcal{D}\\ A\end{array}}{B\ \mathsf{true}}\supset E \quad \Longrightarrow \quad \begin{array}{c}\overline{A\ \mathsf{true}}\ u\\ \mathcal{D}\\ B\ \mathsf{true}\end{array}$$