Lecture 3: Sequent Calculus

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1 Introduction

Lecture outline:

- Sequent calculus: left and right rules
- Statement of Cut and Identity properties
- Proof: Admissibility of Cut
- Proof: ND into sequent calculus (via Cut)
- Identity

This paper sources extensively from (TODO cite) Gentzen'35 and Frank Pfenning's 2017 lecture notes on Sequent Calculus and Cut Elimination.

2 Sequent calculus: left and right rules for connectives

The calculus presented herein corresponds approximately to Gentzen's LJ.

We will write the judgment as $\Gamma \Rightarrow A$, where Γ is an unordered list of propositions $A_1, \ldots, A_n$ and A is a proposition. To be more precise, we can think of the assumptions in Γ as judgments of the form A hyp and the conclusion as a judgment of the form A conc. However, this treatment represents a divergence from Gentzen, who treated the sequent $\Gamma \Rightarrow A$ as a syntactic structure.

For each connective, we will define *left rules* that define a proposition's meaning on the left side of the sequent and *right rules* that determine its meaning on the right.

Conjunction:

$$\frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow A}{\Gamma \Rightarrow A \wedge B}\ \wedge R \qquad \frac{\Gamma, A \wedge B, A \Rightarrow C}{\Gamma, A \wedge B \Rightarrow C}\ \wedge L_1 \qquad \frac{\Gamma, A \wedge B, B \Rightarrow C}{\Gamma, A \wedge B \Rightarrow C}\ \wedge L_2$$

Disjunction:

$$\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \vee B}\ \vee R_1 \qquad \frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \vee B}\ \vee R_2$$

$$\frac{\Gamma, A \vee B, A \Rightarrow C \quad \Gamma, A \vee B, B \Rightarrow C}{\Gamma, A \vee B \Rightarrow C} \vee L$$

Implication:

$$\frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \supset B} \supset R \qquad \frac{\Gamma, A \supset B \Rightarrow A \quad \Gamma, A \supset B, B \Rightarrow C}{\Gamma, A \supset B \Rightarrow C} \supset L$$

Truth and Falsehood:

$$\frac{}{\Gamma \Rightarrow \top} \top R \qquad \text{(no } \top\text{L)} \qquad \text{(no } \bot\text{R)} \qquad \frac{}{\Gamma, \bot \Rightarrow C} \bot L$$

Negation $\neg A$ is, as before, defined as $A \supset \bot$.

Finally, we need something analogous to the hypothesis rule from natural deduction in order to make use of things on the left of the sequent on the right. We call this the "identity rule" and sometimes refer to its conclusion as an "initial sequent."

$$\frac{}{\Gamma, A \Rightarrow A} \mathsf{id}$$

2.1 Proof Examples

We will go through some subset of the following as examples in class.

- $A \vee B \supset A \vee B$
- $(A \supset B) \wedge A \supset B$
- $(A \supset (B \vee C)) \supset (A \wedge \neg B) \supset C$
- $(A \vee B) \wedge C \supset (A \wedge C) \vee (B \wedge C)$

Exercise 1. *Typeset proofs of* $\cdot \Rightarrow A$ *for each formula* A *above.*

3 Observations

3.1 Structural Rules

Some presentations of sequent calculus include the so-called "structural rules", weakening (or thinning) and contraction.

$$\frac{\Gamma \Rightarrow C}{\Gamma, A \Rightarrow C} \mathsf{wk} \qquad \frac{\Gamma, A, A \Rightarrow C}{\Gamma, A \Rightarrow C} \mathsf{contr}$$

However, it will be to our advantage to prove these rules *admissible* instead of explicitly including them.

Proving a rule admissible means: assume there is some derivation for premises, and show that there is a derivation of the conclusion.

We can come up with a theorem statement corresponding to the rule to show it admissible.

Theorem 1 (Weakening)**.** *If* $\Gamma \Rightarrow C$*, then* $\Gamma, A \Rightarrow C$ *with a structurally identical deduction.*

Proof. Add A to Γ in every sequent appearing in the given derivation of $\Gamma \Rightarrow C$. The rules still apply in each case since they are parametric in Γ. The result is a structurally identical derivation of $\Gamma, A \Rightarrow C$. $\square$

(Note that we could make the above more formal by induction over the structure of the derivation, but it would be a very boring proof.)

Exercise 2. *State and prove a corresponding theorem for Contraction. If you are unable to make the proof rigorous, discuss why and what might be done to address it.*

3.2 Non-provability

In sequent calculus, it is easy to demonstrate that certain sequents are *not* provable. We will work through some examples:

- Soundness: no proof of $\cdot \Rightarrow \bot$
- Disjunction property: if $\cdot \Rightarrow A \vee B$, then $\cdot \Rightarrow A$ or $\cdot \Rightarrow B$.
- Non-provability of excluded middle (LEM): there is no proof of $\cdot \Rightarrow A \vee \neg A$ for arbitrary A.
- Non-provability of double-negation elimination (DNE): there is no proof of $\cdot \Rightarrow \neg\neg A \implies A$.

Exercise 3. *Typeset proofs of the above non-provability arguments.*

Exercise 4. *Find another classically valid, but not intuitionistically valid, proposition* A *and demonstrate that* $\cdot \Rightarrow A$ *is not provable in general.*

Exercise 5. *Why do we keep qualifying each of these statements with "in general" or "for arbitrary* A*"? Explain how the situation changes when we are allowed to talk about specific propositions* A*.*

4 Sequent Calculus Metatheory: Cut and Identity

There are two theorems corresponding to internal soundness and completeness for sequent calculus.

The soundness theorem is *cut*:

Proposition 2 (Cut)**.** *If* $\Gamma \Rightarrow A$ *and* $\Gamma, A \Rightarrow C$ *then* $\Gamma \Rightarrow C$*.*

The completeness theorem requires that we restrict the identity rule to atomic propositions. We will call this restricted form of provability $\Rightarrow^{\mathsf{id}}$.

$$\frac{}{\Gamma, P \Rightarrow^{\mathsf{id}} P}\ \mathsf{id}$$

P here is atomic. All other rules in $\Rightarrow^{\mathsf{id}}$ are identical to $\Rightarrow$.

Proposition 3 (Identity)**.** *For all propositions A, $A \Rightarrow^{\mathsf{id}} A$.*

Exercise 6. *Discuss why these metatheorem statements correspond to soundness and completeness for sequent calculus.*

5 Proof of Cut Admissibility

TODO add discussion of induction order.

Theorem 4 (Cut)**.** *If $\Gamma \Rightarrow A$ and $\Gamma, A \Rightarrow C$ then $\Gamma \Rightarrow C$.*

Proof. By induction on the lexicographic ordering of A followed by the unordered pair of $\mathcal{D}$ and $\mathcal{E}$, where $\mathcal{D}$ is the derivation of the first assumption and $\mathcal{E}$ is the derivation of the second.

The cases for the proof fall into three categories:

- The so-called *principal cuts*, in which the cut formula A is the *principal formula* for the outermost rule of both $\mathcal{D}$ and $\mathcal{E}$. The principal formula of a rule is the one it matches against, e.g. $A \wedge B$ in $\wedge L$ and $\wedge R$.
- The cut formula is not the principal formula of the outermost rule comprising $\mathcal{D}$.
- The cut formula is not the principal formula of the outermost rule comprising $\mathcal{E}$.

Order to go through cases:

- Identity rule, three ways.
- Principal cut for $\wedge$.
- Principal cut for $\supset$.
- Side formula cut (1) for $\wedge L$.
- Side formula cut (2) for $\supset R$.
- Side formula cut (2) for $\supset L$.

□

6 Adequacy of Sequent Calculus for Natural Deduction

Statement: if $\Gamma \vdash A$ true then $\ulcorner\Gamma\urcorner \Rightarrow A$.

Proof: by induction on the structure of the derivation of $\Gamma \vdash A$ true.

7 Proof of Identity

TODO exercise?