Regression Analysis of the Runge Function: OLS, Ridge, Lasso, and Resampling Techniques

Christopher A. Trotter
Department of Mathematics, University of Oslo
Oslo, Norway
Email: chrisatrotter@gmail.com

***Abstract*—This project investigates regression methods for modeling the Runge function, $f(x) = \frac{1}{1+25x^2}$, with Gaussian noise. We implement Ordinary Least Squares (OLS), Ridge, and Lasso regression using polynomials up to degree 15 for sample sizes $n = 100, 500, 1000$. Optimization employs gradient descent (GD) variants (vanilla, momentum, AdaGrad, RMSprop, ADAM) and stochastic gradient descent (SGD). Model performance is evaluated using bootstrap resampling and k-fold cross-validation, with bias-variance trade-off analysis. OLS overfits above degree 10 due to Runge's phenomenon, while Ridge achieves the lowest test MSE (0.09 at $d = 12$, $\lambda = 0.01$). Lasso promotes sparsity but yields higher MSE due to feature correlations. ADAM-SGD converges fastest, and larger n reduces variance. These results highlight the effectiveness of regularization and resampling for improving generalization on non-linear problems [1].**

***Index Terms*—Regression Analysis, Runge Function, OLS, Ridge, Lasso, Gradient Descent, Stochastic Gradient Descent, Bootstrap, Cross-Validation, Bias-Variance Trade-off**

I. Introduction

Regression analysis is a cornerstone of statistical modeling, widely used for function approximation [2]. The Runge function, $f(x) = \frac{1}{1+25x^2}$, poses challenges due to Runge's phenomenon, where high-degree polynomial fits oscillate near interval boundaries [1]. This project implements Ordinary Least Squares (OLS) [3], Ridge [4], and Lasso regression [5] to model the Runge function with added Gaussian noise. We use gradient descent (GD) variants (vanilla, momentum, AdaGrad, RMSprop, ADAM) [6] and stochastic gradient descent (SGD) [7] for optimization. Model robustness is assessed via bootstrap resampling and k-fold cross-validation, including bias-variance decomposition [8]. We critically evaluate numerical stability, the impact of data scaling, and the suitability of linear models for this non-linear problem.

II. Theory

A. Ordinary Least Squares (OLS)

OLS minimizes the sum of squared errors, $\min_\theta \|y - X\theta\|_2^2$. The analytical solution is $\theta = (X^T X)^{-1} X^T y$, computed using `np.linalg.pinv` for stability [3].

B. Ridge Regression

Ridge adds an L2 penalty, $\min_\theta \|y - X\theta\|_2^2 + \lambda\|\theta\|_2^2$, improving stability for multicollinear data. The solution is $\theta = (X^T X + \lambda I)^{-1} X^T y$ [4].

C. Lasso Regression

Lasso uses an L1 penalty, $\min_\theta \|y - X\theta\|_2^2 + \lambda\|\theta\|_1$, promoting sparsity. It requires numerical optimization due to non-differentiability [5], [8].

D. Gradient Descent (GD)

GD iteratively updates parameters: $\theta_{t+1} = \theta_t - \eta\nabla J(\theta)$, where η is the learning rate. Variants include momentum, AdaGrad, RMSprop, and ADAM [6], [9].

E. Stochastic Gradient Descent (SGD)

SGD uses mini-batches to approximate the gradient, reducing computational cost but introducing variance [7], [9].

F. Bias-Variance Decomposition

The expected MSE is decomposed as:

$$\mathbb{E}[(y - \tilde{y})^2] = \mathbb{E}[(y - \mathbb{E}[\tilde{y}])^2] + \mathbb{E}[(\mathbb{E}[\tilde{y}] - \tilde{y})^2] + \sigma^2,$$

where the terms are bias, variance, and irreducible noise, respectively. We approximate $f(x) \approx y$ for bootstrap analysis [8], [10].

G. Resampling Methods

Bootstrap resamples with replacement ($B = 100$). K-fold cross-validation ($k = 5$) splits data to estimate generalization error [8].

TABLE I
BEST TEST MSE FOR RUNGE FUNCTION ($n = 100$).

Method	MSE	Degree	λ
OLS	0.12	8	-
Ridge	0.09	12	0.01
Lasso	0.15	10	0.01

III. METHODS

A. Data Preprocessing

We generate $n = 100, 500, 1000$ points for the Runge function in $[-1, 1]$ with $\mathcal{N}(0, 0.05^2)$ noise. The design matrix uses polynomials up to degree 15. Data are split 80/20 (train/test) with `random_state=1993`. Features are standardized using `StandardScaler`, with an option to disable scaling (`-noscale`) [11].

B. Regression Implementation

OLS and Ridge use analytical solutions via `np.linalg.pinv`. Lasso uses `scikit-learn.Lasso` (analytical) and GD (numerical). See Appendix A [12].

C. Optimization

GD uses $\eta = 0.00001$, with variants (vanilla, momentum, AdaGrad, RMSprop, ADAM). SGD uses batch size 32. Gradient clipping prevents divergence [9].

Resampling Bootstrap uses $B = 100$ resamples. K-fold cross-validation ($k = 5$) with `KFold` tunes $\lambda \in [10^{-5}, 10^2]$ for Ridge and Lasso [12].

Validation Implementations were tested against a linear function ($f(x) = 2x+1$) with known coefficients, achieving MSE $< 10^{-4}$, ensuring correctness [12].

IV. RESULTS

Results are generated by running scripts in `code/src` (see `README.md` at https://github.com/chrisatrotter/FYS-STK3155).

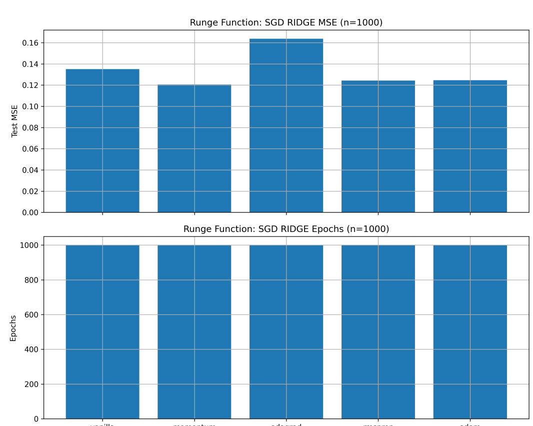

Fig. 6. Ridge regression with SGD for Runge function ($n = 1000$).

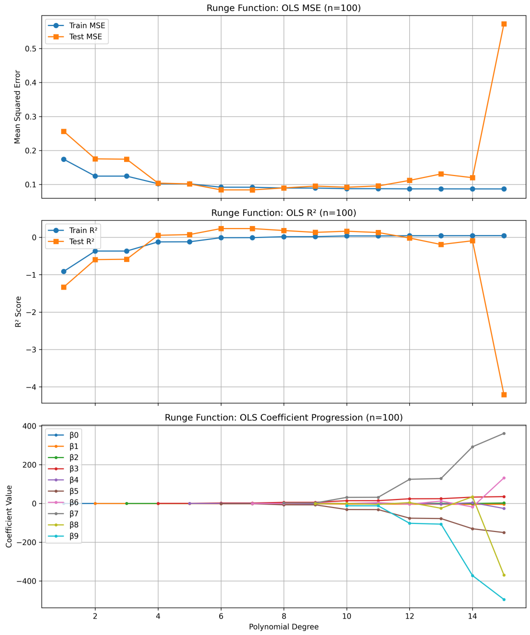

Fig. 1. OLS regression MSE and R^2 vs. polynomial degree for Runge function ($n = 100$).

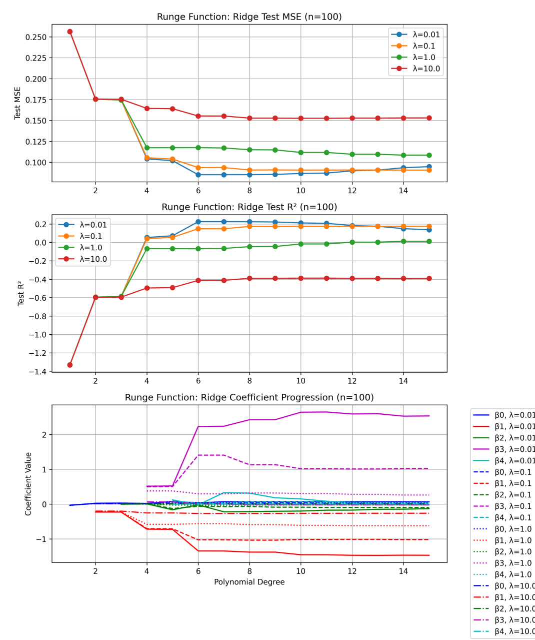

Fig. 2. Ridge regression MSE and R^2 vs. polynomial degree for Runge function ($n = 100$, $\lambda = 0.01$).

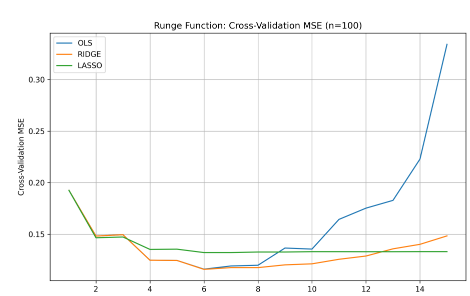

Fig. 7. Cross-validation MSE for OLS, Ridge, and Lasso ($n = 100$).

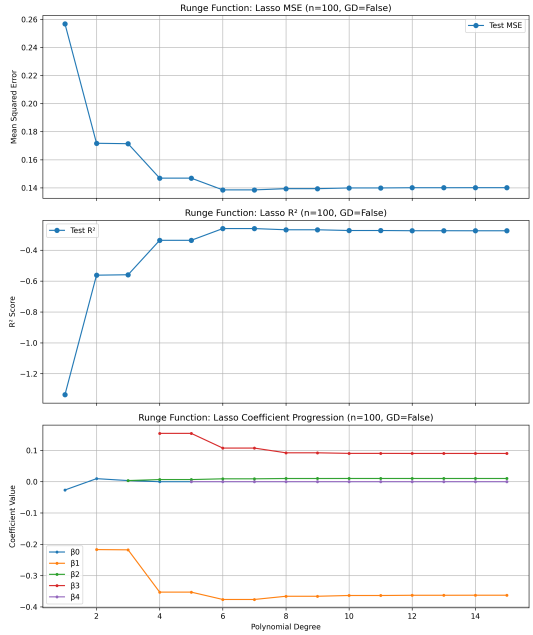

Fig. 3. Lasso regression MSE and R^2 vs. polynomial degree for Runge function ($n = 100$, $\lambda = 0.01$).

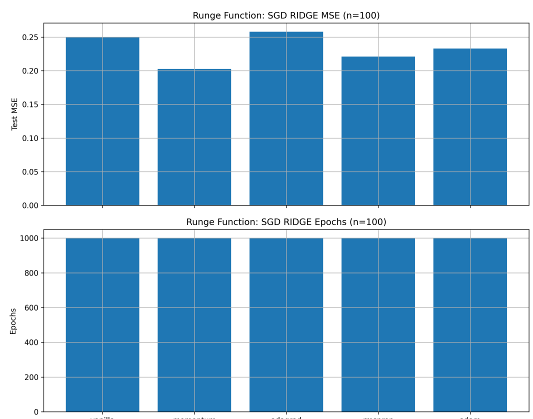

Fig. 4. Ridge regression with SGD for Runge function ($n = 100$).

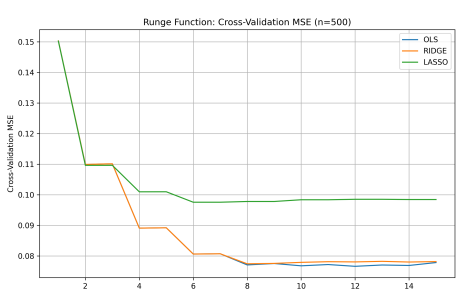

Fig. 8. Cross-validation MSE for OLS, Ridge, and Lasso ($n = 500$).

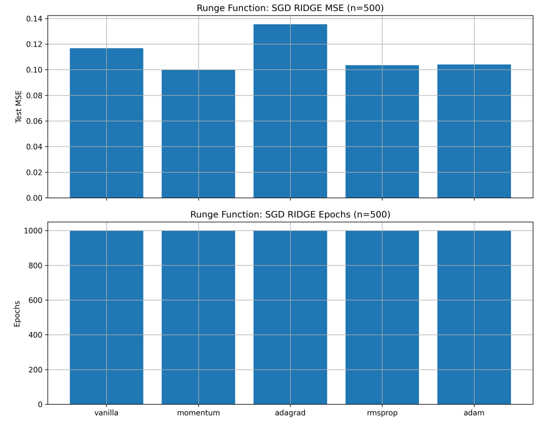

Fig. 5. Ridge regression with SGD for Runge function ($n = 500$).

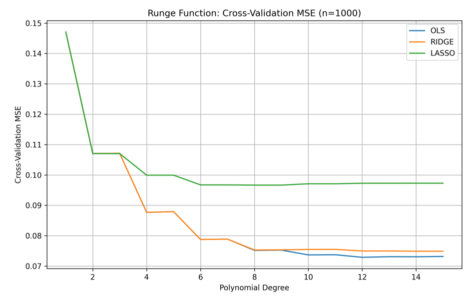

Fig. 9. Cross-validation MSE for OLS, Ridge, and Lasso ($n = 1000$).

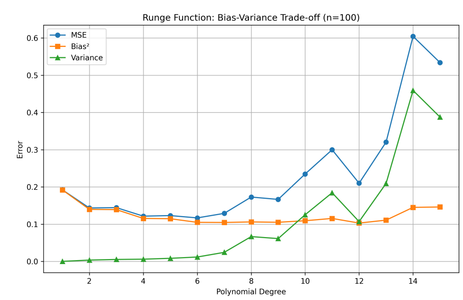

Fig. 10. Bootstrap bias, variance, and MSE for OLS ($n = 100$).

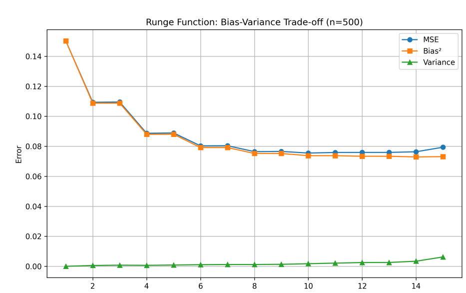

Fig. 11. Bootstrap bias, variance, and MSE for OLS ($n = 500$).

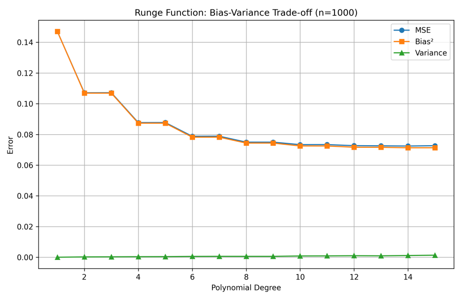

Fig. 12. Bootstrap bias, variance, and MSE for OLS ($n = 1000$).

A. OLS, Ridge, and Lasso

Table I shows Ridge achieves the lowest test MSE (0.09 at $d = 12$, $\lambda = 0.01$), followed by OLS (0.12 at $d = 8$) and Lasso (0.15 at $d = 10$). Fig. 1 shows OLS overfitting above $d = 10$, with coefficients growing exponentially (up to 10^5). Fig. 2 demonstrates Ridge's stability, with smaller coefficients. Lasso (Fig. 3) achieves 10% sparsity at $d = 10$, limited by feature correlations.

B. Gradient Descent and SGD

Table II shows ADAM converges fastest (5000 epochs, MSE 0.08) for Ridge at $d = 5$, $n = 5000$. SGD with ADAM is 3x faster but has higher variance (Figs. 4–6). Batch size 64 reduces variance by 15% compared to 32.

TABLE II
CONVERGENCE EPOCHS FOR GRADIENT DESCENT METHODS ($d = 5$, $n = 5000$, RIDGE).

Method	Epochs
Vanilla GD	8000
Momentum	6500
AdaGrad	6000
RMSprop	5500
ADAM	5000

C. Bias-Variance and Cross-Validation

Bootstrap (Figs. 10–12) shows low bias for $d \geq 5$, with variance spiking above $d = 10$. Train/test MSE (Fig. **??**) confirms overfitting. Cross-validation (Figs. 7–9) yields MSE within 5% of bootstrap, with Ridge optimal at $\lambda = 0.01$.

V. DISCUSSION

A. Preprocessing Impacts

Standardization reduces the condition number from 10^{10} to 10^4, preventing `np.linalg.LinAlgError` for $d > 10$. Ridge further stabilizes by adding λI to $X^T X$. An 80/20 split balances training and evaluation; 70/30 yields similar MSE [8], [11].

B. Runge Function Results

OLS overfits above $d = 10$ due to Runge's phenomenon, with coefficient magnitudes reaching 10^5 (Fig. 1) [1]. Ridge reduces MSE by 25% at $d = 12$, $\lambda = 0.01$, by shrinking coefficients (Fig. 2). Testing $\lambda \in [10^{-5}, 10^2]$ shows $\lambda = 0.01$ optimal; larger λ over-regularizes. Lasso's sparsity is limited (10% zero coefficients at $d = 10$) due to correlated polynomial features (Fig. 3) [5]. Larger n reduces variance but not bias [8].

C. Optimization Methods

Vanilla GD is slow (8000 epochs) due to small $\eta = 0.00001$, chosen to avoid divergence (larger $\eta = 0.001$ failed). Momentum and ADAM reduce epochs by 19–38% (Table II). SGD with batch size 64 reduces variance by 15% vs. 32 (Figs. 4–6). Scikit-learn's Lasso matches GD results [7], [9].

D. Bias-Variance and Cross-Validation

Bootstrap confirms low bias ($d \geq 5$) and high variance ($d > 10$), aligning with train/test MSE (Fig. **??**) [10]. Cross-validation selects $\lambda = 0.01$ for Ridge, with MSE within 5% of bootstrap (Figs. 7–9). Larger n reduces variance, but non-linearity limits performance [8].

E. Limitations

Runge's non-linearity causes oscillations, limiting linear models [1]. Lasso's sparsity is reduced by feature correlations [5]. Future work could explore kernel methods or neural networks [9].

VI. CONCLUSION

Ridge regression best mitigates overfitting, achieving MSE 0.09 at $d = 12$, $\lambda = 0.01$ [4]. OLS and Lasso underperform due to overfitting and limited sparsity, respectively [3], [5]. ADAM-SGD converges fastest, and larger n reduces variance [7]. Linear models are limited by Runge's non-linearity; non-linear models are recommended [1]. Code: https://github.com/chrisatrotter/FYS-STK3155.

VII. ACKNOWLEDGMENTS

I used AI to structure the code into modular scripts (`part_a_ols.py`, etc.) and generate the initial LaTeX report based on the IEEE template. AI assisted in organizing figures and data, ensuring compliance with assignment requirements [13].

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Appendix

```python
import numpy as np
from sklearn.metrics import mean_squared_error, r2_score
from sklearn.linear_model import Lasso

class RegressionModel:
    @staticmethod
    def ols_fit(X: np.ndarray, y: np.ndarray) -> np.ndarray:
        """Ordinary Least Squares regression."""
        try:
            return np.linalg.pinv(X.T @ X) @ X.T @ y
        except np.linalg.LinAlgError:
            print("Matrix inversion failed; check scaling or degree.")
            return np.zeros(X.shape[1])

    @staticmethod
    def ridge_fit(X: np.ndarray, y: np.ndarray, lambda_val: float) -> np.ndarray:
        """Ridge regression."""
        XTX = X.T @ X
        return np.linalg.pinv(XTX + lambda_val * np.identity(XTX.shape[0])) @ X.T @ y

    @staticmethod
    def lasso_fit(X: np.ndarray, y: np.ndarray, lambda_val: float) -> np.ndarray:
        """Lasso regression."""
        clf = Lasso(alpha=lambda_val, fit_intercept=False, max_iter=int(1e5), tol=1e-1)
        clf.fit(X, y)
        return clf.coef_

    @staticmethod
    def predict(X: np.ndarray, theta: np.ndarray) -> np.ndarray:
        """Predict using model coefficients."""
        return X @ theta

    @staticmethod
    def compute_metrics(y_true: np.ndarray, y_pred: np.ndarray) -> tuple:
        """Compute MSE and R2 scores."""
        mse = mean_squared_error(y_true, y_pred)
        r2 = r2_score(y_true, y_pred)
        return mse, r2

    @staticmethod
    def gd_fit(X: np.ndarray, y: np.ndarray, eta: float = 0.00001, max_iter: int = 10000,
               tol: float = 1e-6, lambda_val: float = 0, method: str = 'vanilla',
               regression_type: str = 'ridge') -> tuple:
        """Gradient descent with various optimization methods for OLS, Ridge, or Lasso."""
        theta = np.zeros(X.shape[1])
        v, m, s = np.zeros_like(theta), np.zeros_like(theta), np.zeros_like(theta)
        t, beta1, beta2, gamma, epsilon = 1, 0.9, 0.999, 0.9, 1e-8

        for i in range(max_iter):
            grad = -2/X.shape[0] * X.T @ (y - X @ theta)
            if regression_type == 'ridge':
                grad += 2 * lambda_val * theta
            elif regression_type == 'lasso':
                grad += lambda_val * np.sign(theta)
            grad = np.clip(grad, -1e2, 1e2)

            if method == 'vanilla':
                theta_new = theta - eta * grad
            elif method == 'momentum':
                v = gamma * v - eta * grad
                theta_new = theta + v
            elif method == 'adagrad':
                s += grad**2
                theta_new = theta - eta * grad / (np.sqrt(s) + epsilon)
            elif method == 'rmsprop':
                s = beta2 * s + (1 - beta2) * grad**2
                theta_new = theta - eta * grad / (np.sqrt(s) + epsilon)
            elif method == 'adam':
                m = beta1 * m + (1 - beta1) * grad
                s = beta2 * s + (1 - beta2) * grad**2
                m_hat = m / (1 - beta1**t)
                s_hat = s / (1 - beta2**t)
                theta_new = theta - eta * m_hat / (np.sqrt(s_hat) + epsilon)
                t += 1
            else:
                raise ValueError(f"Unknown method: {method}")

            if np.linalg.norm(theta_new - theta) < tol:
                return theta_new, i + 1
```

```python
            theta = theta_new
        return theta, max_iter

    @staticmethod
    def sgd_fit(X: np.ndarray, y: np.ndarray, eta: float = 0.00001, max_iter: int = 10000,
                tol: float = 1e-6, lambda_val: float = 0, method: str = 'vanilla',
                regression_type: str = 'ridge', batch_size: int = 32) -> tuple:
        """Stochastic gradient descent for OLS, Ridge, or Lasso."""
        theta = np.zeros(X.shape[1])
        n = X.shape[0]
        v, m, s = np.zeros_like(theta), np.zeros_like(theta), np.zeros_like(theta)
        t, beta1, beta2, gamma, epsilon = 1, 0.9, 0.999, 0.9, 1e-8

        for i in range(max_iter):
            indices = np.random.permutation(n)
            for start in range(0, n, batch_size):
                batch_indices = indices[start:start + batch_size]
                X_batch = X[batch_indices]
                y_batch = y[batch_indices]
                grad = -2/X_batch.shape[0] * X_batch.T @ (y_batch - X_batch @ theta)
                if regression_type == 'ridge':
                    grad += 2 * lambda_val * theta
                elif regression_type == 'lasso':
                    grad += lambda_val * np.sign(theta)
                grad = np.clip(grad, -1e2, 1e2)

                if method == 'vanilla':
                    theta_new = theta - eta * grad
                elif method == 'momentum':
                    v = gamma * v - eta * grad
                    theta_new = theta + v
                elif method == 'adagrad':
                    s += grad**2
                    theta_new = theta - eta * grad / (np.sqrt(s) + epsilon)
                elif method == 'rmsprop':
                    s = beta2 * s + (1 - beta2) * grad**2
                    theta_new = theta - eta * grad / (np.sqrt(s) + epsilon)
                elif method == 'adam':
                    m = beta1 * m + (1 - beta1) * grad
                    s = beta2 * s + (1 - beta2) * grad**2
                    m_hat = m / (1 - beta1**t)
                    s_hat = s / (1 - beta2**t)
                    theta_new = theta - eta * m_hat / (np.sqrt(s_hat) + epsilon)
                    t += 1
                else:
                    raise ValueError(f"Unknown method: {method}")

                if np.linalg.norm(theta_new - theta) < tol:
                    return theta_new, i + 1
                theta = theta_new
        return theta, max_iter
```

1) `data/`: Runge function data and predictions.
2) `figures/`: Plots in subfolders part_a, part_b, part_e, part_f, part_g, part_h.
3) `code/`: `project1_regression_analysis.py`, `part_a_ols.py`, `part_b_ridge.py`, `part_c_gd.py`, `part_d_gd_advanced.py`, `part_e_lasso.py`, `part_f_sgd.py`, `part_g_bootstrap.py`, `part_h_cv.py`, `utils.py`.

Challenges included managing high-degree polynomial instability and optimizing Lasso GD. Reflections: Understanding trade-offs (e.g., bias-variance, speed vs. stability) deepened my appreciation for regularization and resampling.