Higher Dimensional Modal Logic

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The main purpose is to provide concrete relationships between highly expressive concurrency models coming from two different schools of thought: the higher dimensional automata, a state-based approach by Pratt and van Glabbeek; and the configuration structures and unrestricted event structures, an event-based approach by van Glabbeek and Plotkin. In this respect, we will define the method of sculpting, described by Pratt, in categorical terms to better understand the event-state duality defined by Pratt. These investigations of sculptures in category theory are intended to provide a better understanding of the relationship between ST-structures, Higher Dimensional Automata and Sculptures in terms of expresiveness.

1 Introduction

In highly concurrency models, Higher Dimensional Automata(HDA) is a state-based approach by Pratt and van Glabbeek and is a geometric model of concurrency that has an attractive aspect which is the automata-like presentation. This aspect is opposed to the event-based model of concurrency, like (prime, flow, (non-)stable) event structures[Pratt MC, Pratt HDA RV, VGlab Exp] or configuration structures such as ST[CJ STs]. Here, I only treat HDAs and ST-structures(ST) which are models of processes that perform actions a, b, c, ... whose internal structure is not further examined. Furthermore, I only study the representation of processes, not representation of operators on processes. I limit myself to models that take a fully asynchronous view on parallelism: whenever a number of actions can happen simultaneously, this must be because they are causally independent, and hence they can also happen in any order.

Part of the contribution of this paper is a definition of sculptures in relation to ST-structures and HDAs that may be expressive up to isomorphism. In the latter, we will further discuss these relations.

By comparing HDAs through ST, as done in [ST-structures], then we can identify non-isomorphic and non-bisimilar HDAs, as shown by Figure 6 in [ST-structure], such that non-isomorphic HDAs can be collapsed into isomorphic ST. Providing us with an example of the perfect duality paradox described by Pratt[HDA revisited]. It follow that the more interesting figure of figure 6(left) in [ST-structure] is due to subleties that evaporate when considering processes up to isomorphism, and even hereditary history preserving bisimular equivalence. We will compare Sculptures to ST by showing that isomorphism holds and investigate HDAs embedded into Sculptures.

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2 Higher Dimensional Automata

We recall the definition of higher dimensional automata (HDA) following the terminology of [VGlab Exp, VP Trans] and also following additional notions including the restriction to acyclic and non-degenerate HDAS defined in [ST-structure]. A logical study of HDAs can be found in [CJ Modal HDA].

For an intuitive understading of the HDA model consider the standard example from [VGlab Exp] pictured in Figure 4. It represents a HDA that models two concurrent events which are labelled by a and b (we can also have the same label a for both events, giving rise to the notion of *autoconcurrency*). The HDA consists of four states q_0^1 to q_0^4, and four transitions between them such that it is the same as the classical picture for the interleaving concurrency, but in the case of HDA there is also a square q_2. Traversing through the interior of the square means that both events are executing. It is the idea of "fillilng in the holes" of traditional automata as described by Pratt in [Pratt HDA Revisited]. When traversing on the lower transition it means that event one is executing but event two has not started yet, whereas, when traversing through the upper transition it means that event one is executing and event two has finished already. In the state q_0^4 both events have finished and in state q_0^1 no event has started yet such that these are the two states where no event is executing.

Similar, HDAs allow to represent three concurrent events through a cube, or more event through hypercubes. We will follow the terminology of [CJ ST-structures] for HDAs, which is also defined in set theoretical terms in [VGlab Exp].

Definition 2.1. ***(higher dimensional automata [3, Def.1])***

A cubical set $H = (Q, \bar{s}, \bar{t})$ is formed of a family of sets $Q=\cup_{n=0}^{\infty} Q_n$ with all sets Q_n disjoint, and for each n, a family of maps s_i, $t_i : Q_n \to Q_{n-1}$, with $1 \leq i \leq i \leq n$, which respects the following cubical laws:

$$\alpha_i \circ \beta_j = \beta_{j-1} \circ \alpha_i,\ 1 \leq i < j \leq n \text{ and } \alpha, \beta \in \{s,t\}. \tag{1}$$

In H, the $\bar{s}$ and $\bar{t}$ denote the collection of all the maps from all the families (i.e., for all n).A higher dimensional automaton $(Q, \bar{s}, \bar{t}, l, F)$ over an alphabet σ is a cubical set together with a labelling function $l:Q_1 \to \sigma$ which respects $l(s_i(q)) = l(t_i(q))$ for all $q \in Q_2$ and $i \in \{1,2\}$; and with $I \in Q_0$ initial and $F \subset Q_0$ final cells.

The elements of Q_0 are called nodes and those of Q_1, Q_2 and Q_3 are edges, square *and* cubes, *respectively. In general, the elements of Q_n are called* n-dimensional hypercubes, *or* n-cells. *An n-dimensional hypercube represents a state of a concurrent system in which n transitions are firing concurrently. Because the dimensions of the hypercube are numbered 1,...,n, these transitions are de facto stored as a list.*

3 ST-structures

We define ST-structures, showing in Section 3 of [CJ ST-structures] that they are a natural extension of configuration structures[VGlab CS:event], and include related notions that stem from the latter. The classical notions of concurrency, causality and conflict are not interdefinable as in the case of event structures or stable configuration structures; but are more loose, as in the case with HDAs.

Definition 3.1. ***(ST-structures)*** *An ST-configuration structure(also called ST-structure) is a tuple ST = (E, ST, l) with ST a set of ST-configurations over E satisfying the constraint:*

$$if(S,\ T) \in ST\ then\ (S,\ S) \in ST, \tag{2}$$

and l:E $\rightarrow \sigma$ a labelling function with σ the set of labels. We often omit the set of events E from the notation when there is no danger of confusion.

The contraint(2) above is a closure, ensuring that we do not represent events that are started but never terminated. The set of all ST-structures is denoted $\mathbb{ST}$.

4 Sculptures

In the cubical set approach to higher dimensional automata an automaton is (possibly infinite) set of cubes of various dimensions. In the Chu space approach one starts instead with a single cube of very large, possibly infinite, dimension and "sculpts" the desired process by removing unwanted faces. The axes of the starting cube constitute the events, initially a discrete or unstructured set. The removal of states has the effect of structuring the event set. For example sculpting renders two events equivalent, or synchronized, after all states distinguishing them have been removed.

The sculpture way of looking at Chu spaces does not reveal the intrinsic symmetry of events and states. An alternative presentation that brings out the symmetry better is as matrix whose rows and columns are indexed by events and states respectively, and whose entries are draw from the set 3={0,1,2}. The column of this matrix constitute the selected faces of the cube.

(use definition from ST-structure to define Sculptures in terms of HDAs)

5 Results/just future plans

5.1 category theory relate sculptures

Let $\square$ be the small category with objects $\{0,1\}^n$ and morphisms freely generated by $\delta_1^0, \dots, \delta_{n+1}^0, \delta_1^1, \dots, \delta_{n+1}^1 :$ $\{0,1\}^n \to \{0,1\}^{n+1}$ given by

$$\delta_k^\nu(x_1, \dots, x_n) = (x_1, \dots, x_{k-1}, \nu, x_k, \dots, x_n)\,, \tag{3}$$

that is, δ_k^ν inserts a ν into the kth position.

The category of *precubical sets* is $\mathfrak{C} = \mathsf{Set}^{\square^{\mathrm{op}}}$, the presheaf category. That is, objects in $\mathfrak{C}$ are functors $\square \to \mathsf{Set}$ and morphisms are natural transformations.

In elementary terms, objects in $\mathfrak{C}$ are graded sets $X = \{X_n\}_{n \in \mathbb{N}}$ together with face maps $\delta_1^0, \dots, \delta_n^0, \delta_1^1, \dots, \delta_n^1 : X_n \to X_{n-1}$ which (automatically) satisfy the precubical identity $\delta_k^\nu \delta_\ell^\mu = \delta_{\ell-1}^\mu \delta_k^\nu$ for $k < \ell$, and the morphisms are graded functions $f = \{f_n\}_{n \in \mathbb{N}}$ for which $f\delta = \delta f$.

By the Yoneda lemma, there is an embedding $\square^{\mathrm{op}} \hookrightarrow \mathfrak{C}$, the images of which are called *bulks*. Hence a bulk is a precubical set X which is generated by a single (possibly infinite-dimensional) cube.

Let $\mathfrak{B} \subseteq \mathfrak{C}$ be the induced subcategory of bulks.

Lemma 5.1. *Let $f : X \to B$ be a morphism in $\mathfrak{C}$ with $B \in \mathfrak{B}$. Then f is an embedding.*

Sketch. In think this follows directly from the Yoneda lemma, but an elementary proof is not difficult. The essence is that in B there are no loops, hence f cannot send two different cubes to the same image (this would create a loop). $\square$

The category $\mathfrak{S}$ of *sculptures* is the arrow category $\mathfrak{C} \to \mathfrak{B}$. That is, objects in $\mathfrak{S}$ are $\mathfrak{C}$-morphisms $X \to B$ with $B \in \mathfrak{B}$, and morphisms in $\mathfrak{S}$ are commutative diagrams

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, hook] & Y \arrow[d, hook] \\ B_1 \arrow[r, hook] & B_2 \end{tikzcd}$$

(Injectivity of $X \to Y$ follows from injectivity of the other three morphisms.)

There is a (forgetful) functor $U : \mathfrak{S} \hookrightarrow \mathfrak{C}$ which "forgets" the embeddings into $\mathfrak{B}$, i.e. maps the above diagram to $X \hookrightarrow Y$.

Problem 1. *Does U have right and/or left adjoints?*

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References