On Bulks and Sculptures

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When talking about sculptures and bulks, Christopher and I found out that the definitions seem to be wrong.

The intuition of sculpting is “take a cube of some dimension, and sculpt at it, taking off bits and pieces here and there”. Hence bulks are “geometric” cubes; but they can’t be combinatorial cubes, because you can’t take bits and pieces away from combinatorial cubes. Rather, bulks should be *subdivisions* of combinatorial cubes.

Figure 1 shows a two-dimensional example. On the left, the bulk plus the pieces we want to sculpt away (in red); on the right, the subdivision we need for the combinatorics. The resulting scuplture consists of 17 squares (plus their 1- and 0-dimensional faces).

Hence the following definitions, where an *n-cube* is an image of the Yoneda embedding, and we use the prefix “n” for “new” to distinguish this from the old definitions:

- An *nbulk* is a subdivided n-cube.
- An *nsculpture* is any subset of an nbulk.

Now these definitions are problematic, because they are geometric in nature. We want combinatorial definitions, and subdivisions are not combinatorial (they’re not precubical morphisms).

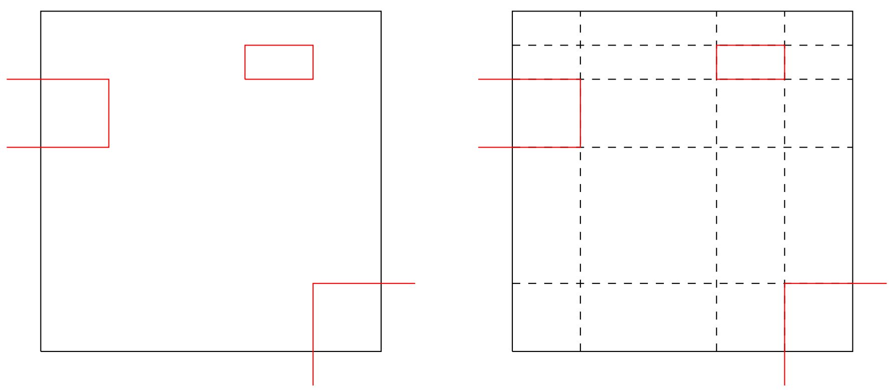


Figure 1: Sculpting

So we'd like definitions of the form "A precubical set X is an nbulk if *blabla*"; "A precubical set X is an nsculpture if *blibli*".

We've talked with the local expert, Emmanuel Haucourt, about this: It should be easy enough to write up what "*blabla*" is; but there is nobody who seems to know the precise "*blibli*".

Note the famous example of the "broken box" (Fig. 4 in `main3`): we proved in Oslo that it's not a sculpture (in the old sense), but this proof doesn't work anymore. It might well be an nsculpture.

There is a notion of *loop-free* for precubical sets; essentially it means that for any $x \in X$, once you leave x, you never see x again. Nsculptures are obviously loop-free, but is this enough?

1 Properties

An obvious question is whether any nbulk can be obtained as a sculpture, i.e. as a subset of a higher-dimensional cube.

Below we show that

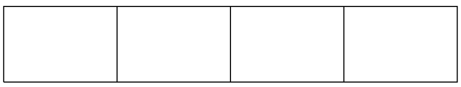
- cubes which are subdivided in one direction only are sculptures, but
- the simple 2×3 two-dimensional grid is not a sculpture.

Intuitively, it is strange that such a simple grid should not be a sculpture; this adds weight to the proposal that nsculptures are what we should be using instead.

Lemma 1 *Any sequence of k n-cubes is an $(n+k-1)$-dimensional sculpture.* □

By such a sequence, we mean k n-cubes which are connected through $n-1$-faces *in the same direction*

Proof (sketch) We show this for $n = 2$ and $k = 4$. Hence we are looking at a $k \times 1$–2-grid as so:



Using notation inspired by Chu-spaces (over 3), we can label the squares as follows:

$tt000$	$1tt00$	$11tt0$	$111tt$

Hence the three faces between the squares are $1t000$, $11t00$, and $111t0$ in that order. We have shown that this is a 5-dimensional sculpture.

This can readily be generalized to all n and k. ■

Lemma 2 *The* 3×2–2-*grid is* not *a sculpture.* □

PROOF (SKETCH) We start by labeling the squares as so (the number of 0s at the end is unimportant; this works in any dimension):

$t1t00$	$11tt0$	
$tt000$	$1tt00$	$1t1t0$

Note that we don't have much choice for the labeling:

- $1tt00$ and $t1t00$ are the only possible neighbors of $tt000$;
- $11tt0$ is the only square which is an upper neighbor of both $1tt00$ and $t1t00$;
- $1t1t0$ is the only other upper neighbor of $1tt00$ (we've already use $11tt0$).
- (We could have tried to extend the 2×2-grid upwards instead of to the right, using $t11t0$ as upper neighbor of $t1t00$; this does not change the argument.)

Now we have a problem: $11tt0$ and $1t1t0$ have a common upper face, namely $111t0$. So what we have constructed looks like this:

$t1t00$	$11tt0$	
$tt000$	$1tt00$	$1t1t0$

and there is no space for the 6th square. ■