1 hh-bisimulation for HDAs

Two higher dimensional automata H_A and H_B(with I_A and I_B the initial cells) are history-preserving bisimulation equivalent(hh-bisimilar), denoted $H_A \overset{hh}{\sim} H_B$, if there exists a binary relation R between their paths starting at I_A, respectively I_B, that respects the following:

1. if $\pi_A R \pi_B$ and $\pi_A \xrightarrow[a_i]{a} \pi_A^{'}$ then $\exists \pi_B^{'}$ with $\pi_B \xrightarrow[a_i]{a} \pi_B^{'}$ and $\pi_A^{'} R \pi_B^{'}$

2. if $\pi_A R \pi_B$ and $\pi_B \xrightarrow[a_i]{a} \pi_B^{'}$ then $\exists \pi_A^{'}$ with $\pi A \xrightarrow[a_i]{a} \pi_A^{'}$ and $\pi_A^{'} R \pi_B^{'}$

3. if $\pi_A R \pi_B$ and $\pi_A \overset{l}{\longleftrightarrow} \pi_A^{'}$ then $\exists \pi_B^{'}$ with $\pi_B \overset{l}{\longleftrightarrow} \pi_B^{'}$ and $\pi_A^{'} R \pi_B^{'}$

4. if $\pi_A R \pi_B$ and $\pi_B \overset{l}{\longleftrightarrow} \pi_B^{'}$ then $\exists \pi_A^{'}$ with $\pi A \overset{l}{\longleftrightarrow} \pi_A^{'}$ and $\pi_A^{'} R \pi_B^{'}$

Here A and B is a hereditary history-preserving bisimulation if:

1. if $\pi_A R \pi_B$ and $\pi_A^{'} \xrightarrow[a_i]{a} \pi A$ then $\exists \pi_B^{'}$ with $\pi_B^{'} \xrightarrow[a_i]{a} \pi_B$ and $\pi_A^{'} R \pi_B^{'}$

2. if $\pi_A R \pi_B$ and $\pi_B^{'} \xrightarrow[a_i]{a} \pi_B$ then $\exists \pi_A^{'}$ with $\pi_A^{'} \xrightarrow[a_i]{a} \pi A$ and $\pi_A^{'} R \pi_B^{'}$

To explain what bisimulation of HDAs are then we need to understand histories for HDAs. Furthermore, to understand what a history is we need to know what a path is.

Note: Hereditary is the notion of backwards mapping that is history-preserving. History is often preserved with forward mapping, and not backward.

1.1 Paths in a HDA

A single step in a HDA is either

1. $q_{n-1} \overset{si}{\rightarrow} q_n with s_i(q_n) = q_{n-1}$ or

2. $q_n \overset{ti}{\rightarrow} q_{n-1} with t_i(q_n) = q_{n-1}$

where $q_n \in Q_n$ and $q_{n-1} \in Q_{n-1}$ and $1 \leq i \leq n$.

A path $\pi \triangleq q^0 \xrightarrow{a^1} q^1 \xrightarrow{a^2} q^2 \xrightarrow{a^3} ...$, is a sequence of single steps $q^j \xrightarrow{a^{j+1}} q^{j+1}$, with $a^j \in \{s, t\}$. We say that $q \in \pi$ iff $q = q^j$ appears in one of the steps in π. The first cell in a path is denoted st(π) and the ending cell in a finite path is en(π). Now that we have defined what a step in a HDA is and what defines a path, it is time to explain what a history is.

1.2 Histories for a HDA

In a HDA two paths are adjacent, denoted $\pi \overset{adj}{\longleftrightarrow} \pi^{'}$, if one can be obtained from the other by replacing, for $q, q^{'} \in Q$ and $i < j$,

1. A segment $\xrightarrow{s_i} q \xrightarrow{s_j}$ by $\xrightarrow{s_{j-1}} q^{'} \xrightarrow{s_i}$, or
2. A segment $\xrightarrow{t_j} q \xrightarrow{t_i}$ by $\xrightarrow{t_i} q^{'} \xrightarrow{t_{j-1}}$, or
3. A segment $\xrightarrow{s_i} q \xrightarrow{t_j}$ by $\xrightarrow{t_{j-1}} q^{'} \xrightarrow{s_i}$, or
4. A segment $\xrightarrow{s_j} q \xrightarrow{t_i}$ by $\xrightarrow{t_i} q^{'} \xrightarrow{s_{j-1}}$

Two finite paths are l-adjecent, denoted $\pi \overset{l}{\longleftrightarrow} \pi^{'}$, when the segment replacement happens at position $l+1$, i.e., q is the $l+1$ cell in the path.

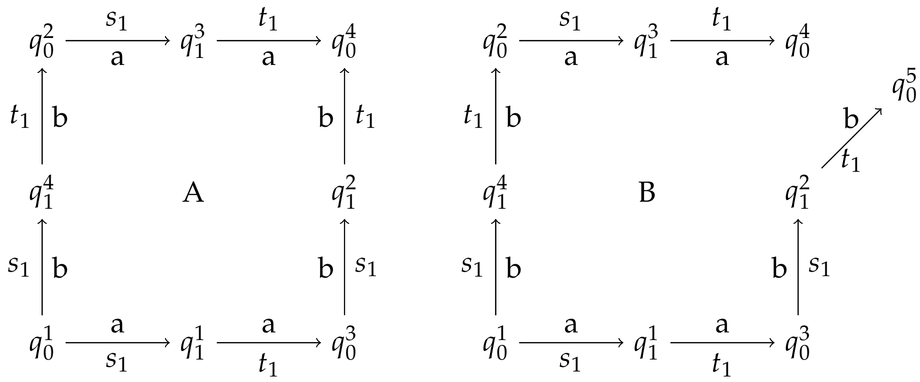
Homotopy is the reflexive and transitive closure of adjacency. Two homotopic paths are denoted $\pi \overset{hom}{\longleftrightarrow} \pi^{'}$ and share their respective start and end cells. The homotopy class(equivalence class) of a rooted path is denoted $[\overleftrightarrow{\pi}]$. A homotopy class with end cell q is said to be a history of q.

One cell may have several histories, as the case with the interleaving square HDA from Figure 4.

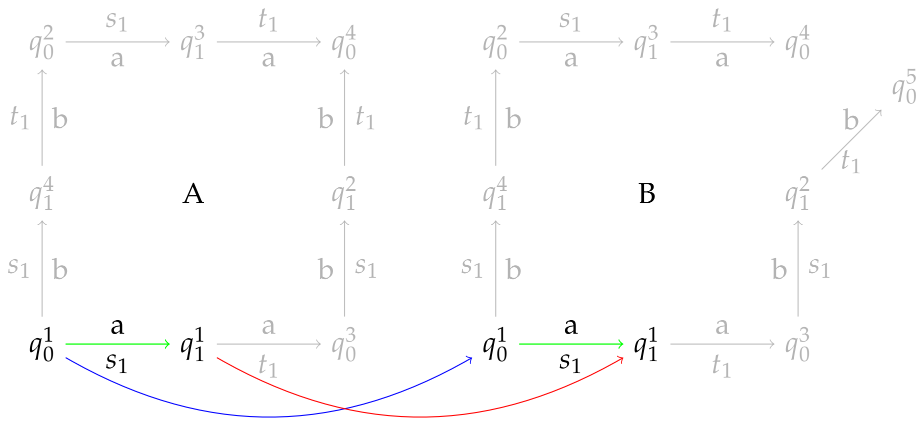
Whenever a cell has a unique history we use the notation $[\overleftrightarrow{q}]$, instead of $[\overleftrightarrow{\pi}]$ with en(π) = q.

1.3 Visualization of bisimulation of HDAs

To begin with our step-by-step visualization of a bisimulation of HDAs, we will use the following model as an example:

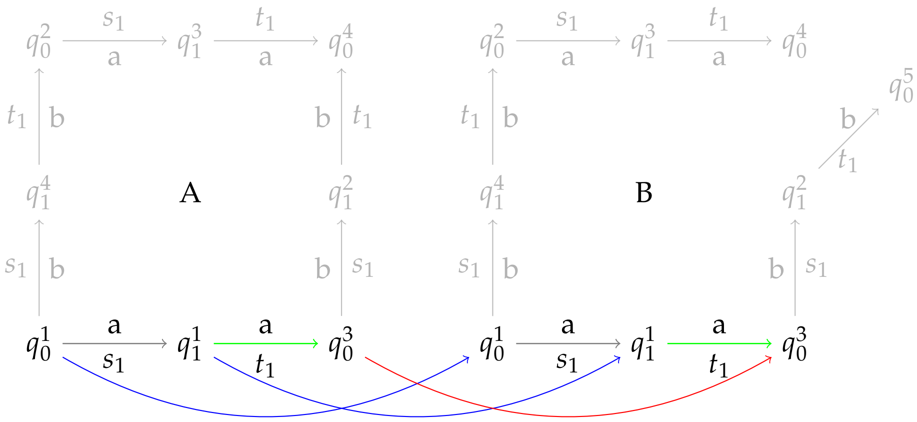


When we are to show bisimulation of a HDA, then we need to show that the conditions for history-preservation above hold and also check for hereditary conditions if we are to show that backwards mapping is preserved.



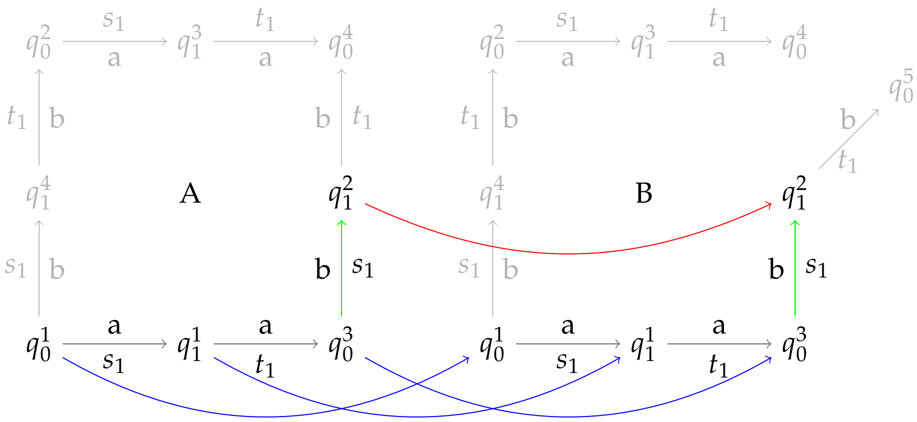
From the figure above, we have that $\pi_A \triangleq q_0^1$ and $\pi'_A \triangleq q_1^1$, and the same for π_B and π'_B. From the first condition if there is a binary relation, $\pi_A R \pi_B$, then there is a step in A from $q_0^1 \xrightarrow[s_1]{a} q_1^1 (\pi_A \xrightarrow[a]{a_i} \pi'_A)$. If there is such a step in A, then there also must exists a step in B such that $q_0^1 \xrightarrow[s_1]{a} q_1^1 (\pi_B \xrightarrow[a]{a_i} \pi'_B)$. From this we also get that there is a binary relation $\pi'_A R \pi'_B$. This shows that the first condition holds, and we can also see that the second condition holds since it is the same just that the arrow from A to B point in the opposite direction.

Explain also why condition 3 and condition 4 holds.
(**Note:** I need to understand this listing property better, $\pi \xleftrightarrow{l} \pi'$. **Ask CJ**)

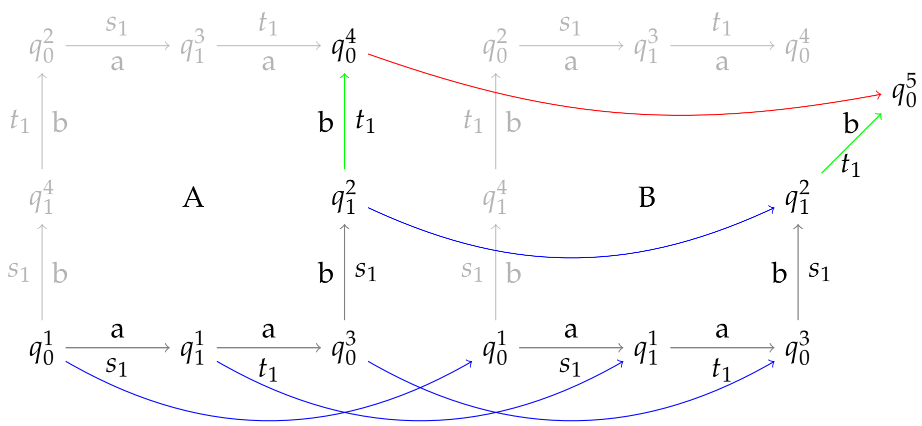


Now we have that $\pi_A \triangleq q_0^1 \xrightarrow[s_1]{\text{a}} q_1^1$ and $\pi'_A \triangleq q_0^3$, and we can see that the next step will be $q_1^1 \xrightarrow[t_1]{a} q_0^3$. The conditions still holds.

(**Note:** show for hereditary property, and show where this property does not hold. With hereditary property does that mean isomorphism? **Ask CJ**)



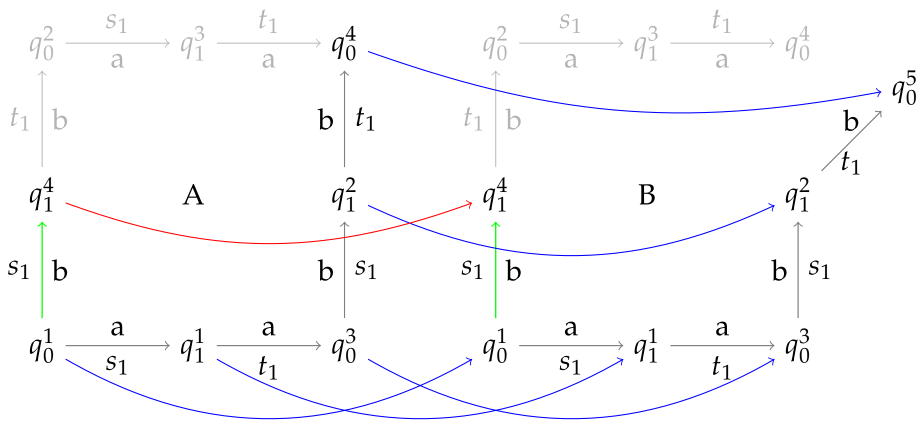
It continous by adding the previous π'_A to π_A, and then finding the next π'_A to then see if the conditions hold. Now we have that $\pi_A \triangleq q_0^1 \xrightarrow[s_1]{\text{a}} q_1^1 \xrightarrow[t_1]{\text{a}} q_0^3$ and $\pi'_A \triangleq q_1^2$, giving $q_0^3 \xrightarrow[s_1]{b} q_1^2$. From the same argument as earlier we see that the conditions still hold. As we have progressed, we have gotten an inuitive notion of how a bisimulation works and we will only show the model and write what π and π' is and the trasition of the next step.



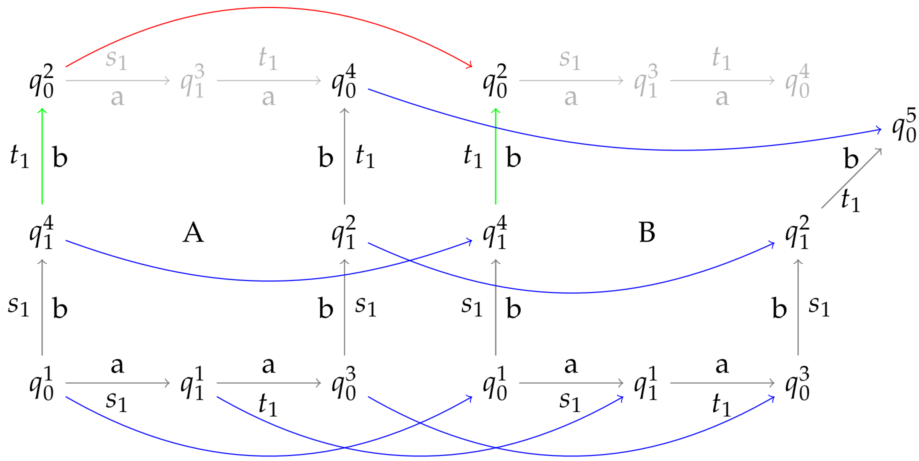
$\pi_A \triangleq q_0^1 \xrightarrow[s_1]{a} q_1^1 \xrightarrow[t_1]{a} q_0^3 \xrightarrow[s_1]{b} q_1^2$ and $\pi'_A \triangleq q_0^5$, giving $q_1^2 \xrightarrow[t_1]{b} q_0^5$.

Since there does not exist a step further such that $q^j \xrightarrow{a^{j+1}} q^{j+1}$, we get that our path is $\pi_A \triangleq q_0^1 \xrightarrow[s_1]{a} q_1^1 \xrightarrow[t_1]{a} q_0^3 \xrightarrow[s_1]{b} q_1^2 \xrightarrow[t_1]{b} q_0^5$, denote as p_A^1.

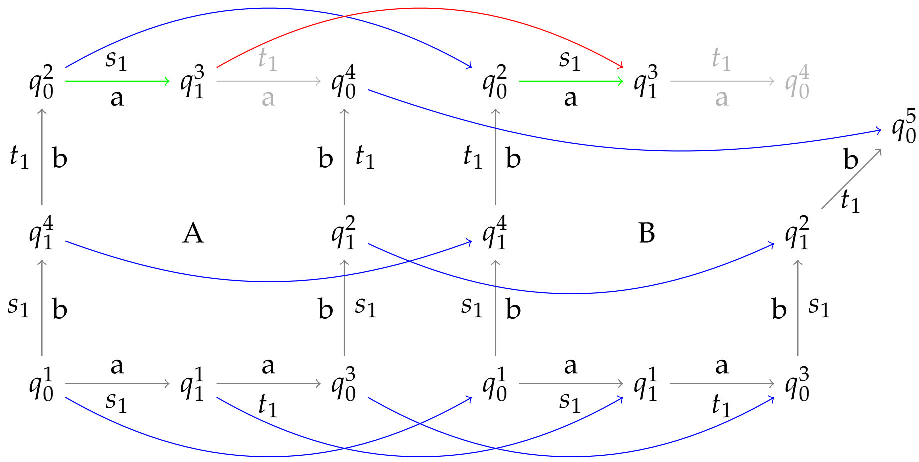
Now we must work along the other path in the model to show that the whole model satisfies all the conditions of bisimulation, without hereditary property in this case. We will still use π_A as we have done earlier, just having it move along a new path.



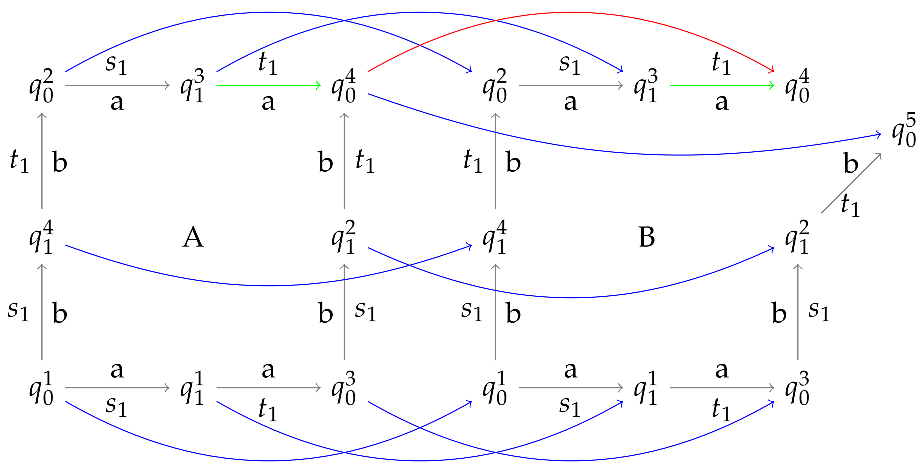
$\pi_A \triangleq q_0^1$ and $\pi'_A \triangleq q_1^4$, giving $q_0^1 \xrightarrow[s_1]{b} q_1^4$.



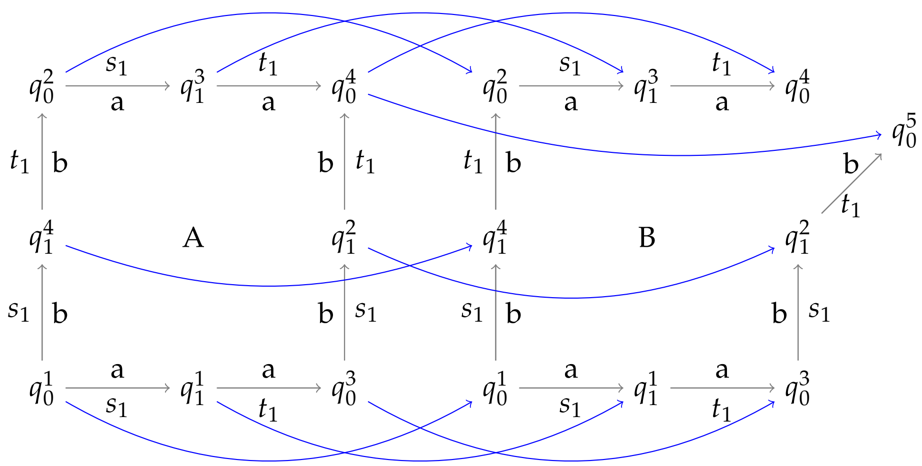
$\pi_A \triangleq q_0^1 \xrightarrow[s_1]{b} q_1^4$ and $\pi'_A \triangleq q_0^2$, giving $q_1^4 \xrightarrow[t_1]{b} q_0^2$.



$\pi_A \triangleq q_0^1 \xrightarrow[s_1]{b} q_1^4 \xrightarrow[t_1]{b} q_0^2$ and $\pi'_A \triangleq q_1^3$, giving $q_0^2 \xrightarrow[s_1]{a} q_1^3$.



$\pi_A \triangleq q_0^1 \xrightarrow[s_1]{b} q_1^4 \xrightarrow[t_1]{b} q_0^2 \xrightarrow[s_1]{a} q_1^3$ and $\pi'_A \triangleq q_0^4$, giving $q_1^3 \xrightarrow[t_1]{a} q_0^4$.



Since there does not exist a step further such that $q^j \xrightarrow{a^{j+1}} q^{j+1}$, we get that our path is $\pi_A \triangleq q_0^1 \xrightarrow[s_1]{b} q_1^4 \xrightarrow[t_1]{b} q_0^2 \xrightarrow[s_1]{a} q_1^3 \xrightarrow[t_1]{a} q_0^4$, denoted as p_A^2.

Each path of the model has been explored and compared based on the conditions that define bisimularity. We can conclude that A and B are bisimular, but not hereditary history-preserving.