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Pricing and Trading the Moments of Distributions

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DECLARATION

This project is the result of my own work. Material from the published or unpublished work of others, which is referred to in the project, is credited to the author in question in the text. This thesis is approximately 9,750 words in length.

ABSTRACT

The following series of investigations assess the use of moments to describe the distributions of returns, whether they offer an improvement to prevailing risk models and whether potential lies in trading the moments of distributions. Models assessing asset allocation with the inclusion of the moments are considered, finding great precision and conciseness, potentially offering better representation of investor preferences. The moments also offer great flexibility as one chooses how many moments terms to include. Style investing trends are considered quantitatively through the pricing model developed by Mhiri and Prigent (2010), assessing how prevalent factor combinations perform and whether other combinations can be unearthed. Most prominent was the combination of reversal with investment factors. A similar approach to asset class allocation showed weak performance of equities in conjunction with other classes. Most optimal combinations consisted solely of short-term fixed income, agriculture and gold. Moment trading is explored, testing the ability of risk-neutral distributions to predict their respective physical distributions, and whether markets display overreaction to market events or exhibit trends. Some predictability is demonstrated, with somewhat temperamental findings of over- and under-reaction, varying depending upon the day's stock returns. R^2 values up to 25% are found to predict daily moment movements. Stress testing through machine learning methods is conducted to scour for deeper relationships; offering reinforcement of findings but little improvement.

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Chapter One

Introduction

The prime goal in finance is the allocation of assets. Countless studies and professionals assess the performance of asset managers and investment strategies each day. Many of these will make inferences about the relative performance of investments based solely upon the mean and variance of returns, or alpha and beta using the CAPM. Whilst maintaining simplicity and comparability, these methods are in a large part flawed; they make highly restrictive assumptions and respond terribly to more complex strategies and environments.

Since Markowitz's mathematical definition and proof of the concept of diversification in 1952 (Markowitz, 1952), many have sought to follow the mean-variance (MV) ideology in search of superior returns distributions. This thesis seeks to extend the Markowitz theory to include the consideration of the two next central moments of the return's distribution-skew and kurtosis. Much of the progress made has been in the field of narrowing the utility functions of investors and generating optimisation techniques for this utility. Markowitz was first to optimise for the first two central moments, then this was extended to include the third by Konno, Shirakawa, and Yamazaki (1993).

Proven by Scott and Horvath (1980), the sign for preferences regarding the central moments was found through Taylor expansion of wealth maximisation to alternate, starting with a preference for a higher mean return. This shows a positive preference for higher mean and skew, but negative preference for variance and kurtosis of returns.

An important factor to consider with multiple competing risk factors is what diversification looks like. Under multidimensional risk, with different investors having varying utility functions, there will be no universally optimal portfolio such as the market portfolio anymore. Investors will value different factors in different proportions. Some are more willing to accept greater deviation around the mean but dislike tail events more than another. For example, a pension fund will seek to reduce tail risk more than a hedge fund as pension funds must consistently pay members. This is especially apparent for a defined benefit scheme.

If there is anything this seeks to portray, it is that the same assets can have different values to different investors. Be this due to different investors' preferences over different types of risk or the different changes the asset imposes on the investors' distribution of portfolio returns through the inclusion of this asset.

This thesis will set out first to map out the development from the MV framework into the higher moments. Then, the moments themselves will be assessed for their usefulness in extending the asset allocation problem, with intent to utilise in quantitative investment strategy. Next, the potential for pricing assets will be discussed and how this would then translate into a concept of portfolio pricing.

Chapter Two

Moments and Distributions

Moments in mathematics are used to describe the shape of a distribution. In statistics, the moments derive metrics to quantify probability distributions. The first four central moments are commonly known as mean, variance, skew and kurtosis. Central moment n can be defined as below for a random variable X with $E\left(|X|^n\right) < \infty$ (Severini, 2005; Kim et al., 2014, p.155):

$$CM_n = E\left(X - \bar{X}\right)^n \tag{2.1}$$

Mean and variance are commonly seen in returns assessment, though rarely are the skew or kurtosis considered. The moments provide great potential, given that they can be derived for any asset, whilst infinite moments can also be derived, theoretically. The Hausdorff moment problem specifies that any probability density function (PDF) can be derived from its moments. This was proven in Tagliani (1999). Here the specifics are not necessary, the point is to prove that the moments stand as one of the best methods of decomposing distributions. It is also a flexible method in allowing the choice of accuracy with the number of central moments considered, tending to unrivalled accuracy when infinite moments are considered.

Moments have been utilised to describe some of the more advanced risks posed when investing, for instance in the hedge fund industry and smart beta strategies (Gang and Qian, 2016). A majority of such use LTCM as a prime example of the put-like returns distributions often masked by similar strategies (Chan et al., 2005; De Los Rios and Garcia, 2011; Kozhan, Neuberger, and Schneider, 2013). These risks usually manifest themselves as negative skew.

Mixed moments are another branch of moments that has earned some attention in asset pricing and allocation, notably being used for the H-CAPM of Dittmar (2002). This extension of the CAPM and other models such as Black (2014), Hasan and Kamil (2014), and Ando and Hodoshima (2006). The mixed moments are attractive for asset pricing as they directly measure the relationships between distributions of random variables. The mixed moment corresponding to variance, covariance, is commonly seen in asset pricing. The consideration of higher mixed moments then allows for better asset return co-movement quantification. Such relationships are massively important for diversification.

Taking the equations of Christie-David and Chaudhry (2001, p.59) and modifying to include the different possible combinations of variables gives:

$$\gamma_{ijk} = \frac{E\left[(\tilde{R}_i - \bar{R}_i)(\tilde{R}_j - \bar{R}_j)(\tilde{R}_k - \bar{R}_k)\right]}{E\left(\tilde{R}_i - \bar{R}_i\right)^3} \tag{2.2}$$

$$\delta_{ijkl} = \frac{E\left[(\tilde{R}_i - \bar{R}_i)(\tilde{R}_j - \bar{R}_j)(\tilde{R}_k - \bar{R}_k)(\tilde{R}_l - \bar{R}_l)\right]}{E\left(\tilde{R}_i - \bar{R}_i\right)^4} \tag{2.3}$$

where the above is modified to show that coskew and cokurtosis take 3 and 4 arguments, respectively. The arguments i, j, k, l represent different assets chosen, which do not have to all be different. $\tilde{R}$ represent sample returns, while $\bar{R}$ are mean asset returns. Coskew actually measures the co-movement between up to 3 assets and cokurtosis up to 4

Generalising these formulae gives a formula for any of the mixed moments.

$$\mu_{\boldsymbol{i}} = \frac{E\left[\prod_{a=1}^{n}(\tilde{R}_{i_a} - \bar{R}_{i_a})\right]}{E\left(\tilde{R}_{i_1} - \bar{R}_{i_1}\right)^n} \tag{2.4}$$

where $\boldsymbol{i}$ is a vector of $i_1, ..., i_n$ assets, which do not have to be distinct

To give some empirical understanding of (2.2) and (2.3); there are many ways to use both. Coskewsness and cokurtosis can represent extreme movements between two assets with γ_{ijj}, δ_{ijjj} or δ_{iijj}, where the direction of co-movement is considered in the first two.

The important result to be taken here is that these higher moments are able to specifically measure the extreme movements between multiple assets. An important factor in investor risk aversion is tail risk, estimated with metrics such as value-at-risk (VaR) and conditional value-at-risk (CVaR). Tail risk has been found to be very important in empirical behavioural findings, such as prospect theory (Kahneman and Tversky, 1979). This, therefore, brings these higher moments into the light as an important consideration, and an important risk to be addressed.

In attempting to quantitatively represent distributions of returns, there are difficulties that must be passed on the mathematical front. Namely, there is some disagreement even in the definition of kurtosis. Dodge and Rousson (1999) define the central moments as:

$$\mu_r(X) = \int_{-\infty}^{+\infty} (x - \mu)^r dF(x) \tag{2.5}$$

Whilst the usual description of kurtosis is the peakedness of a distribution, Darlington (1970) argued that kurtosis represents the bimodality of a variable. This claim, whilst having some credit, was shown not to be so robust by Moors (1986) and Hildebrand (1971). An interesting summary to the argument is given by Moors (1986, p.284), pointing out that kurtosis is increased both by more values being closer to the mean and by more values being in the tails. This should be extended to suggest that, in both cases, more results that are more extreme are now in this distribution. This is what the kurtosis is showing.

Many difficulties are found once one starts to derive fourth-order moments (and above) and their estimation. Dodge and Rousson (1999) document a thorough set of cases, where the first three moments, central moments and cumulants have simple definitions. In each case, the definition for the fourth condition is relatively complicated and messy. This is what creates the difficulty comprehending what the fourth and higher moments are actually describing. The scale of this difficulty spans the discussions of Darlington (1970), Moors (1986), and Ruppert (1987). This, therefore, requires precaution ensuring the correct definitions and understanding of additivity when combining assets to form portfolios.

Chapter Three

Extending the Markowitz Model

Note: in each portfolio optimisation, it should be assumed if not explicitly stated that short-selling is not permitted. This is done to avoid trivial solutions where assets are given infinite long/short weights and eliminate the insights the models set out to provide.

The concept that made MV optimisation revolutionary was that it brought the concept of diversification into mathematical reality. It was able to illustrate this two-dimensionally and in an easily comprehensible way since risk is quantified only by the variance of returns. The author suggests this is not the stage at which we should have halted inquiry. The most serious flaw in the Markowitz model is that returns follow a normal distribution, even for a diversified portfolio. It is widely accepted that returns almost never follow a normal distribution (and rarely even a skew-normal, log-normal etc). This is hence why preferences over higher moments must be included. Much of the progress in quantifying these and bringing them into asset allocation lies in utility theory. The challenge from here is then internalising the economics for investors. Two-dimensional simplicity will almost certainly be lost.

First attempts to extend the optimisation added skew as a further constraint. Konno, Shirakawa, and Yamazaki (1993) followed the Taylor series approach to wealth management seen in Scott and Horvath (1980), stopping at the fourth term, as is shown in equation (3.1).

With a utility function dependent on the return gained on a portfolio, R(x). The Taylor series up to the third term is:

$$U(R(x)) = U(\bar{R}(x)) + \frac{1}{2}U''(\bar{R}(x))[R(x) - \bar{R}(x)]^2 + \frac{1}{6}U'''(\bar{R}(x))[R(x) - \bar{R}(x)]^3 \quad (3.1)$$

With their challenge to optimise utility function (3.1), Konno, Shirakawa, and Yamazaki (1993) simply maximise skew, setting the mean return and variance of returns as constraints, seen below (3.2).

$$\begin{aligned} \max_x \quad & \gamma[R(x)] \\ \text{s.t.} \quad & E[R(x)] = r \\ & V[R(x)] = \sigma^2 \\ & x^T I = 1 \end{aligned} \quad (3.2)$$

Usually, variance is minimised for a series of mean return values. Now, with an added dimension of plotting out different variances for each optimised value of skew, a 3D plot can be made, known as the efficient surface. This does not, however, give any indication as to preferences between different portfolios and frontiers along the efficient surface formed. Without this, the method is only able to display the options available for investors, though does not give any indication as to which portfolios then offer the best risk-return trade-offs. For example, the market portfolio offered by the CAPM is constant for all investors- there is no incorporation of preferences between moments.

Mhiri and Prigent (2010) progressed this model to include consideration of kurtosis, as well as proposing a method to include preferences between moments. Their proposal was to start with a straightforward utility function approach, as shown below.

$$
\begin{aligned}
\max_{X} \quad & U(R) = \bar{R} - a\,\mathrm{Var}(R) + b\,\mathrm{Skew}(R) - c\,\mathrm{Kurt}(R) \\
\text{s.t.} \quad & X^T I = 1 \\
& X \geq 0 \\
& a, b, c > 0
\end{aligned}
\tag{3.3}
$$

This follows economic principle, though is likely too general to be applied universally. A utility function or family of utility functions must be defined for investors to define their values for a, b and c. Further, restrictions on linearity between central moments may lack practical dexterity. Their next progression was to normalise these coefficients, given the likely non-linear outcomes and growing range of values to be expected for skew and kurtosis relative to variance (not ignoring that the coefficient for mean is fixed at 1).

What Mhiri and Prigent (2010) proposed to overcome this difference in empirical results for the moments was to use a two-stage polynomial goal programming (PGP) process. The first process requires portfolio optimisation on each of the first four moments independently:

$$\begin{aligned} \max_{x} \quad & R(x) = X^T \bar{R} \\ \text{s.t.} \quad & X^T I = 1 \\ & X \geq 0 \end{aligned} \tag{3.4}$$

$$\begin{aligned} \min_{x} \quad & \text{Var}(x) = X^T \boldsymbol{\Sigma} X \\ \text{s.t.} \quad & X^T I = 1 \\ & X \geq 0 \end{aligned} \tag{3.5}$$

$$\begin{aligned} \max_{x} \quad & \text{Skew}(x) = \frac{E[X^T (R - \bar{R})]^3}{\sigma_x^3} \\ \text{s.t.} \quad & X^T I = 1 \\ & X \geq 0 \end{aligned} \tag{3.6}$$

$$\begin{aligned} \min_{x} \quad & \text{Kurt}(x) = \frac{E[X^T (R - \bar{R})]^4}{\sigma_x^4} \\ \text{s.t.} \quad & X^T I = 1 \\ & X \geq 0 \end{aligned} \tag{3.7}$$

where Σ is the covariance matrix for X

The next step requires the use of the optimised values for each moment. The number of deviations from the unrestricted optimal are used, displacing any coefficients. In place, powers instead represent investor-specific preferences between the moments. The optimised values from (3.4), (3.5), (3.6) and (3.7) are denoted by R^*, V^*, S^*, K^*, respectively.

The resulting utility function and optimisation comes out as follows:

$$
\begin{aligned}
\min_{\boldsymbol{d}} \quad & Z = \left|\frac{d_1}{R^*}\right|^{\lambda_1} + \left|\frac{d_2}{V^*}\right|^{\lambda_2} + \left|\frac{d_3}{S^*}\right|^{\lambda_3} + \left|\frac{d_4}{K^*}\right|^{\lambda_4} \\
\text{s.t.} \quad & R^* = \quad d_1 + X^T \bar{R} \\
& V^* = -d_2 + X^T V X \\
& S^* = \quad d_3 + \frac{E[X^T(R-\bar{R})]^3}{\sigma_x^3} \\
& K^* = -d_4 + \frac{E[X^T(R-\bar{R})]^4}{\sigma_x^4} \\
& d_i \geq 0,\ \forall i = \{1,2,3,4\} \\
& X^T I = 1 \\
& X \geq 0
\end{aligned}
\tag{3.8}
$$

where $\boldsymbol{\lambda} = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} \geq 0$

From here investors can adjust their preferences between moments with $\boldsymbol{\lambda}$. Whilst there is still work to be done regarding calibration of preferences, the variation in coefficient values to be expected has been vastly reduced. Differences in scale between moments have been removed and so the model becomes far more intuitive, whilst the restrictions on signs in equation (3.3) are dropped.

Mhiri and Prigent (2010) and Lam, Jaaman, and Isa (2013) tested the model, both suggesting it performed better than MV optimisation. Mhiri and Prigent (2010) went further, finding the model superior to the utility function approach described in equation (3.3) when the power utility function $U(x) = \frac{x^{1-\gamma}}{1-\gamma}$ is used, with $\gamma > 1$. Given that the MV model is a special case of the PGP model ($\boldsymbol{\lambda} = \{1,1,0,0\}$), the PGP model weakly dominates it. What must be answered is whether the addition of complexity is useful. Both sets of authors conclude that the model outperforms the others presented to optimise the multi-objective problem. Their range of testing covers both the international and single-market allocation problem. Mhiri and Prigent (2010) test allocation amongst 18 economies, whilst Lam, Jaaman, and Isa (2013) test for the Malaysian stock market (KLCI). The two studies find allocations that better represent risk-averse investor preferences, namely the minimisation of tail risk. In making improvements to the skew and kurtosis of optimised portfolios, relatively small increases in variance entail. An interesting feature found is that mean returns are increased for the majority of the PGP portfolios relative to the MV benchmarks.

Models have been previously established to attempt to price assets and value risk using the moments. Of course the most important is the MV method of Markowitz (1952) and the CAPM derived through this framework. Hung, Shackleton, and Xu (2004) investigated a set of extensions of the CAPM to consider the higher moments. These include primarily the consideration of the direction of returns. Something discussed by Markowitz himself using semi-variance (Markowitz, 1991, p.476). The first of these extensions is taken from Pettengill, Sundaram, and Mathur (1995), where an intercept dummy is included in the usual CAPM equation. This distinguishes between positive and negative returns of the market portfolio relative to the risk-free rate in a given month (Pettengill, Sundaram, and Mathur, 1995, pp.107-108). Dittmar (2002) instead added terms for higher powers of excess market returns, looking as follows:

$$R_{it} - R_{ft} = \alpha_i + \beta_i \left(R_{mt} - R_{ft}\right) + \gamma_i \left(R_{mt} - R_{ft}\right)^2 + \delta_i \left(R_{mt} - R_{ft}\right)^3 + \varepsilon_t \tag{3.9}$$

The intent with such a model was to consider not just the covariance but also the coskew and cokurtosis between an asset and the market, extending the findings of Harvey and Siddique (2000) that coskew picks up much of the variation left unexplained by beta.

The next step was to use the coefficients from (3.9) to collate the two ideas into a model considering higher co-moments (Hung, Shackleton, and Xu, 2004, pp.92-93).

$$R_{it} - R_{ft} = \eta_0 + \eta_\beta^\pm D^\pm \beta_i + \eta_\gamma^\pm D^\pm \gamma_i + \eta_\delta^\pm D^\pm \delta_i + \varepsilon_i \tag{3.10}$$

Another consideration made by Hung, Shackleton, and Xu (2004) was to include the Fama-French factors for size and value to test between the significance of the CAPM, Fama-French 3 factor model and a co-moments model. This extended the equation to look as follows:

$$R_{it} - R_{ft} = \eta_0 + \eta_\beta^\pm D^\pm \beta_i + \eta_\gamma^\pm D^\pm \gamma_i + \eta_\delta^\pm D^\pm \delta_i + \eta_s^\pm D^\pm s_i + \eta_h^\pm D^\pm h_i + \varepsilon_i \tag{3.11}$$

Where D is the dummy for market direction and η are the premia on each of the variables for each market direction. β, γ, δ, s and h represent the covariance (beta), coskew, cokurtosis, size and value, respectively.

The findings applying (3.9), (3.10) and (3.11) were several. First, the model was found to have very different coefficients in a down-market. This was much more prominent when comparing size-sorted than with beta-sorted decile portfolios. The signs for coefficients changed as expected in most cases, though the absolute value of most coefficients increased dramatically, with t-values increasing similarly. Hence, standard errors remained similar. Down-market models, therefore, gave higher model significance- an interesting observation regarding the behaviour of these risk premia. Another finding was that the size factor reacts far more asymmetrically than value. The size coefficient both increased by around five times and changed sign, while the value coefficient only doubled and remained positive. This was mostly put down to the relatively high returns provided by small stocks during a down market (Hung, Shackleton, and Xu, 2004, p.110). Regarding the coefficients for the co-moments added to test the HCAPM; these were often found insignificant. This is attributed to a lack of data in classifying these portfolios, given data was grouped monthly, where much more data is needed to classify these factors (Hung, Shackleton, and Xu, 2004, p.110).

An interesting and far simpler model proposed by Stacey (2008) optimised a combination of the second and fourth central moments. The idea to run a far simpler mean-kurtosis (MK) optimisation was dismissed as complicating MV optimisation, offering very similar results (Stacey, 2008, p.193). This is not necessarily so robust, given an increased concentration on reducing extreme results may offer a more conservative outcome- kurtosis is not simply variance squared. Instead, what Stacey (2008, p.194) suggested was the minimisation of variance relative to kurtosis. The optimisation is set out in (3.12) for clarity.

$$\begin{aligned} \min_{\omega} \quad & \Omega = \frac{\text{Var}(R(\omega))^2}{\text{Kurtosis}(R(\omega))} \\ \text{s.t.} \quad & \omega^T I = 1 \\ & \omega \geq 0 \end{aligned} \tag{3.12}$$

Whilst counter-intuitive, the method seeks to take advantage of the imperfections in MV optimisation by choosing stocks that are able to offer more extreme results more often, with the same standard deviation. The improvement desired is that these extreme movements should, on average, be positive. What is found is that efficient portfolios using Ω contain fewer assets, a factor promoted for reducing transaction costs. Stacey (2008, p.199) applauds this method for providing a leptokurtic, highly peaked returns distribution. Whilst desirable, the tails are then much thicker, with more extreme results. From Figure 3.1, it is not so clear the extent of extreme outcomes in the Stacey (2008) method. Though, what is clear is the tendency for more downside tail events than upside, where the shape of the distribution generally appears undesirable. The counter-intuitive approach does not show much promise, whilst optimising its inverse seems an interesting avenue for reducing dimensionality in optimising investor preferences. This, it is suggested, is the great innovation brought by mean-(conditional) value-at-risk optimisation.

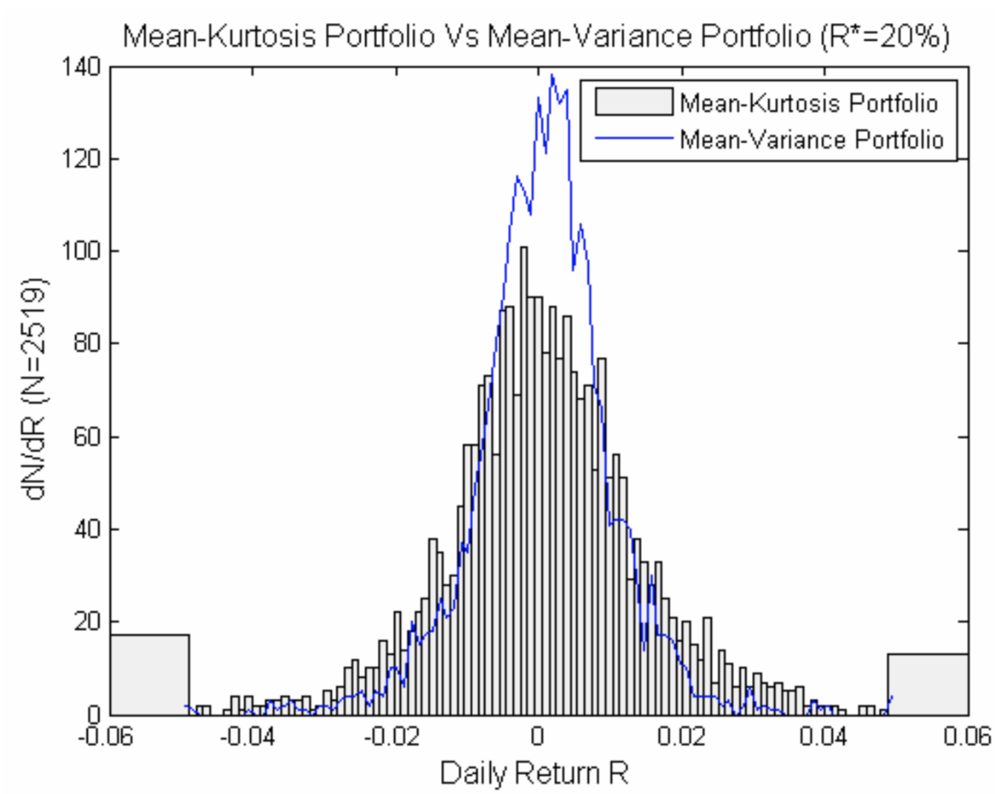

Figure 3.1 A comparison of the returns distributions for the mean-variance and Stacey (2008) Ω methods, taken from Stacey (2008, p.198, Figure 3)

Chapter Four

A Portfolio Pricing Model Based on the Moments

Following the method of Mhiri and Prigent (2010), it should be asked whether such a model provides practical use in pricing assets. A model is now suggested, which takes into consideration the weighting of assets in prior and potential portfolios to determine whether the re-weighting would improve asset allocation. From here, the decision can be made as to whether the new portfolio is preferred to the existing portfolio, given a set of preferences ($\boldsymbol{\lambda}$). It should be noted that this method can be implemented for any group of assets, from any asset class. The only requirement is a set of returns estimations, including how these interact when combining different assets. This, therefore, includes estimates of the co-moments.

For reference, the optimisation function from (3.8) is repeated below.

$$Z = \left| \frac{d_1}{R^*} \right|^{\lambda_1} + \left| \frac{d_2}{V^*} \right|^{\lambda_2} + \left| \frac{d_3}{S^*} \right|^{\lambda_3} + \left| \frac{d_4}{K^*} \right|^{\lambda_4} \tag{4.1}$$

where $\boldsymbol{\lambda} = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} \geq 0$

With a given asset universe covering both the existing and potential portfolio, the optimisations set out in (3.4), (3.5), (3.6) and (3.7) must be run to derive the denominators in

(3.8). From here a score can be worked out for the existing portfolio and potential portfolio. In simple terms, if the 'Z' (4.1) of the potential portfolio is below that of the existing portfolio, the re-weighting should be made to improve the investor's returns distribution. The model works similarly to the CAPM in that it works comparatively, giving an under/overpriced decision, but comparing two portfolios. Absolute pricing would not be so easily derived, given the price would represent the collective price of a whole portfolio and still depends on the comparison of portfolios. In short, there are endogeneity issues around an absolute price and the weighting given to the asset. Presenting comparable portfolios of one with and one without a chosen asset would allow it to be isolated and priced with a given basket and weighting of other assets. This basket would have to maintain equal relative weightings in the two portfolios to isolate the effect of adding the chosen asset. Great difficulty is then presented as it depends upon the weighting it is given in the portfolio, as that would affect the moments, and in a non-linear way. This would hence give different prices for different weightings of the chosen asset. Whilst appearing unusual, this reasoning highlights the change in perception the model hopes to convey; an asset should not be priced on just the value it returns, but how it alters one's overall portfolio returns. Further issues are then presented, regardless of whether relative or absolute valuation is applied, with reference to the float of portfolio weightings. Considerations should then be made as to the time between rebalancing, and the rate at which the portfolio optimality might change. In fact, the model is perfectly equipped for making rebalancing decisions, given the Z can be calculated and assessed for the portfolio pre- and post-rebalancing to decide whether it is worthwhile. Transaction costs could potentially be incorporated by including the cash lost as an asset in the portfolio. This would reduce the mean return but have potentially misleading effects on the other moments, given a constant zero return allocation stabilises the portfolio return. Another option could be to model the costs of rebalancing as a short on the portfolio return, given this is the opportunity cost of the money spent rebalancing and adversely impacts all the moments and confronts the trade-off presented.

The missing piece from the process explained is the definition of investor preferences, $\boldsymbol{\lambda}$. This is investor-specific and therefore requires some form of calibration on their part. This does not necessarily stand as a limiting factor to the model but instead adds flexibility whilst maintaining intuitive simplicity. The process could follow something akin to a behavioural survey run by almost every wealth manager, allowing estimation of preferences over these more advanced risks. An interesting idea could be the reverse engineering of one's $\boldsymbol{\lambda}$ value with the use of a benchmark portfolio which is seen as optimal to the investor. With this portfolio modelled as the optimal, a minimisation can then be run to find the $\boldsymbol{\lambda}$ that gives the benchmark portfolio a Z as close to the optimal as possible. This can be defined in relative or absolute terms. This optimisation for relative Z's is set out in (4.2) for the chosen benchmark portfolio and optimal portfolio under $\boldsymbol{\lambda}$.

$$\begin{aligned} \min_{\boldsymbol{\lambda}} \quad & \frac{Z^B_{\boldsymbol{\lambda}}}{Z^*_{\boldsymbol{\lambda}}} \\ \text{s.t.} \quad & \boldsymbol{\lambda} \geq 0 \end{aligned} \tag{4.2}$$

where Z is the Mhiri and Prigent (2010) (4.1) Z and B is the benchmark portfolio representing a portfolio deemed 'optimum' for an investor. Each Z would be calculated on the available investment universe, depending solely upon the $\boldsymbol{\lambda}$ variable

Finding one's own $\boldsymbol{\lambda}$ value this way presents a way to transition to a more quantitative approach. Further, the results of this $\boldsymbol{\lambda}$-finding process may also improve understanding of allocation decisions. If the $\boldsymbol{\lambda}$ from (4.2) finds preferences far from expectations, investors can be made aware of this issue, where the model is then in place to suggest allocation improvements for a $\boldsymbol{\lambda}$ closer to the investor's expectations for their preferences. Something that should be considered when applying a $\boldsymbol{\lambda}$-finding process is the abundance of local minima offering very different preferences than the global maximum. Given the non-linear and unbounded environment, it is difficult to understand these fully, and near-impossible to predict. What this process seeks to achieve is tying together the transition to a more quantitative approach, seeking to calibrate the model for individuals or expose poor asset allocations relative to one's own risk perception between central moments. Estimating one's own $\boldsymbol{\lambda}$ will not be an exact science, but an intuitive approach allowing assessment of whether the optimal $\boldsymbol{\lambda}^*$ found through (4.2), representing a prediction of one's risk profile, lies far from expectations.

The potential this model provides is a valuable step towards the general and quantitative inclusion of higher moments in asset allocation. It also allows for the assessment of common methods of diversification, especially those concerning tail events.

First, the merits of this pricing method and the benefits it provides can be assessed in the world of factor investing and smart beta. It is commonly understood that the purpose of these factors is to try to segregate the market into different exposures. Asset managers will then likely alter their factor exposure to try to time their relative performance, usually based on macro factors. This is clear in products offered by many of the largest fund providers, who provide specific multifactor and rotational multi-factor ETFs for clients seeking more than just a simple index-tracking strategy (see GSLC: Goldman Sachs Active Beta US, USMC: Principal US Mega-Cap Multifactor ETF, VFMF: Vanguard US Multifactor ETF, IFSW: iShares Edge MSCI World Multifactor ETF). Notable progress in factor investing has been the benefits of combining them to make a portfolio worth more than the sum of its parts, such as in combining momentum and value (Asness, Moskowitz, and Pedersen, 2013). The intent of which is to reduce downside risk associated with momentum crashes (Barroso and Santa-Clara, 2015; Kent and Moskowitz, 2016). Harvey and Siddique (2000) also linked these points, finding momentum stocks showed lower skew, and so present more downside risk than losers. They also found the progress made using the Fama-French 3 factor model rather than the CAPM was to add consideration of coskew through size and value factors (Harvey and Siddique, 2000, p.1293). This modelling hopes to remove qualitative categorisations to discover the best ways to manage these risks.

Data loaded from the Kenneth French data library (French, 2020) gives daily returns for indicators of multiple factors, dating back to 1963. This is taken from the CRSP database and relates only to the US equity market. French (2020) covers the factors: value (HML), size/growth (SMB), profitability (RMW), investment (CMA), Asness (2-12 month) momentum (MOM), long-term reversal (LTR), short-term reversal (STR) and the market (MKT). Quality (QUAL) factor data is also included from AQR's data library (AQR Capital Management, 2020). All factors used are set out in Table 4.1. Running the optimisations for the optimal moments allows testing of the Mhiri and Prigent (2010) pricing model proposed, as well as insight into which multi-factor combinations may be deemed of particular interest. The credit of those factor combinations suggested in the mainstream can also be tested in relation to the optima found with the model.

Table 4.1 Style Investing Factors

Factor	Ticker
Value	HML
Size	SMB
Profitability	RMW
Investment	CMA
Momentum	MOM
Long-Term Reversal	LTR
Short-Term Reversal	STR
Market	MKT
Quality	QUAL

After first running optimisations on this factor universe for the optimal moments $\{M^*, V^*, S^*, K^*\}$, a $\boldsymbol{\lambda}$ must be chosen to assess the quality of the expected returns distribution. To find which factor combinations might offer the most diversification benefits, optimisations are run for a set of $\boldsymbol{\lambda}$ values:

$$\{0,1,0,0\},\{0,0,1,0\},\{0,0,0,1\}\{1,1,0,0\},\{1,0,1,0\},$$
$$\{1,0,0,1\},\{1,1,1,0\},\{1,1,1,1\},\{2,1,1,1\},\{1,2,1,1\},$$
$$\{1,1,2,1\},\{1,1,1,2\},\{1,1,2,2\}$$

These are chosen attempting to concisely cover a wide span of moment preferences likely to be encountered, covering a wide range of the investor universe.

Table 4.2 Concentrated Optimum Portfolios

$\boldsymbol{\lambda}$	Factor	Weight
{0,0,1,0}	Value	.0437
	Market	.0892
	Quality	.1979
	ST Reversal	.6692
{0,0,0,1}	Investment	.2180
	LT Reversal	.6506
	ST Reversal	.1313
{1,0,1,0}	ST Reversal	1.0000
{1,0,0,1}	Investment	.6989
	ST Reversal	.1877

The resulting portfolios are mixed. Some are fairly well-rounded, with close to equal weighting. The rest allocate entirely to between one and four factors. Those with concentrated weightings are given in Table 4.2. What is evident first is that ST reversal and investment frequently appeal and improve the returns distribution when higher risk dimensionality is considered. Allocations to size, investment, quality and ST reversal are of most interest in a suggested smart beta fund following this model. Size and investment qualitatively has its merits for likely allocating to smaller tech firms with a strong backing and hence growth potential. ST reversal appears to be a very high performing strategy, yet relies on equally lower transaction costs to exploit these anomalies, explaining why they probably exist. Therefore, the inclusion of ST reversal should rely on the ability to successfully implement this without being overshadowed by costs.

Table 4.3 All-Factor Optimal and Value, Quality, Momentum Portfolios
Mhiri and Prigent (2010) (4.1) Z of portfolios formed for a sample of $\boldsymbol{\lambda}$ values
The columns show: optimal Z using all factors in Table 4.1, the Z of an equal-weighted portfolio of only value, momentum and quality factors, and the optimal Z through allocating between only value, momentum and quality factors

$\boldsymbol{\lambda}$	Optimal All-Factors	Equal-Weighted HML+MOM+QUAL	Optimal HML+MOM+QUAL
{0,1,0,0}	3.0000	6.9149	5.8355
{0,0,1,0}	3.0000	4.2899	3.8381
{0,0,0,1}	3.0000	3.9417	3.3022
{1,1,0,0}	2.7852	6.7105	5.6579
{1,0,1,0}	2.0239	4.0855	3.6749
{1,0,0,1}	2.7103	3.7373	3.1474
{1,1,1,0}	2.5780	7.0004	5.6920
{1,1,1,1}	2.2535	6.9421	5.1310
{2,1,1,1}	2.1190	6.7795	4.9900
{1,2,1,1}	2.0633	18.3540	10.4737
{1,1,2,1}	2.2116	7.3162	5.0676
{1,1,1,2}	2.0381	6.8872	4.8926
{1,1,2,2}	1.9691	7.2613	4.8385

It is interesting that momentum is not suggested for any portfolio, so a value/quality-momentum combination is not found. It should, however, be considered how well they do still perform compared to these optimal allocations. This could be done simply with an equal-weighted portfolio of the three, and with optimising for an allocation between just value, quality and momentum. The results of this investigation are set out in Table 4.3.

What can first be seen with this factor combination are the strong scores where skew and kurtosis preferences are more strictly imposed. It is also clear that optimising their relative weightings does not always offer a substantial improvement to the equal-weighted combination. What can be taken here is that the combination generally offers a reasonable allocation, with little apparent sensitivity to their weighting. This is probably why this lends itself as an attractive, simple factor combination. Looking further into the merits of this strategy, it is clear the best results are generally gained where skew and kurtosis are of high importance. The reverse is especially evident for variance, shedding light on why the strategy is seen favourably; with a high-variance strategy, this appears risky by basic risk analysis, but this variance is complemented by strong skew and kurtosis features, making it attractive under tail risk measures.

The optimal HML+MOM+QUAL allocations referenced in Table 4.3 generally allocate most of their weights to value and quality, leaving only slight momentum exposure. This again lies contrary to popular belief that the value-momentum combination may offer excellent factor tilt benefits, where much of the value generated may simply derive from tilting between the often-interchanged value and quality factors. Their interchangeability may, therefore, come into question.

Insights such as this are just one expansion on the improvements the model seeks to provide through quantitative modelling. Another area for diversification to be considered is how the model performs in recommending multi-asset allocations.

Asset class ETF trackers were taken from Seeking Alpha (2020), with 16 ETFs selected to provide a broad representation of each asset class. These asset classes cover equities, fixed income, commodities, real estate and private equity/venture capital. Data for the period April 2010-June 2020 was available with complete data for all ETFs chosen, which are listed in Table 4.4. Optimising over the same $\boldsymbol{\lambda}$ values on page 21, the best asset class allocations can be assessed under a variety of preferences.

Table 4.4 Asset Class Representative ETFs

Asset Class	Sub-Class	Ticker
Equities	Global Equities	ACWI
	Small Cap Equities	GWX
	ESG	SUSA
Fixed Income	Corporate Bonds	PICB
	Government Bonds	BWX
	Short-Term Government Bonds	BWZ
	Inflation Protected Bonds	WIP
	Emerging Government Bonds	PCY
Commodities	Agriculture	DBA
	Energy	DBE
	Goldman Sachs Commodities	GSG
	Gold	GLD
Real Estate	US REITs	VNQ
	European REITs	IFEU
	World REITs	RWO
PE/VC	Global Private Equity	PSP

The optimisations on the asset class combinations give unusual results, suggesting most portfolios consist of only government bonds, short-term government bonds, agriculture and gold. Only MV optimisation ($\boldsymbol{\lambda} = \{1, 1, 0, 0\}$) returns another asset class, with an allocation to government bonds, short-term government bonds, emerging market equities and ESG equities. This allocation is interesting, given no equity allocations are suggested for almost any of the preferences tested. The only cases for which equities are suggested are those where mean preferences are more strictly imposed, given ESG offers the highest total return. It is therefore implied that equities, although often offering the highest rates of return, do not offer the most attractive returns distributions, even when aggregated on this scale into a few representative trackers. Whilst not commonly appreciated as a major equity market, the prominence of ESG investing should be appreciated, where it may become its own asset class soon, or part of every asset class. Hence, it was included in the investigation, which also highlighted its likely favourable returns distribution and relationships with other major markets when applied to equities.

Some concerns to present here with such an outcome is that there is potential for unstable results due to tail events in the final months of the sample, caused by COVID-19. Testing with these final months removed gives almost identical results, differing with Goldman Sachs commodities and US REITs being included once each. Whilst being a questionable backtest, this stability stands as proof that the model works to control tail risk that is poorly managed by traditional methods. It also points to stability when applying the model based on far-from-perfect returns predictions; letting returns before COVID-19 act as the predictive returns. Of course, the overlap is significant, though the crash inflicted on almost all markets should be seen as at least equally significant. The extent of the crash can be appreciated in the latter stages of Figure 4.1.

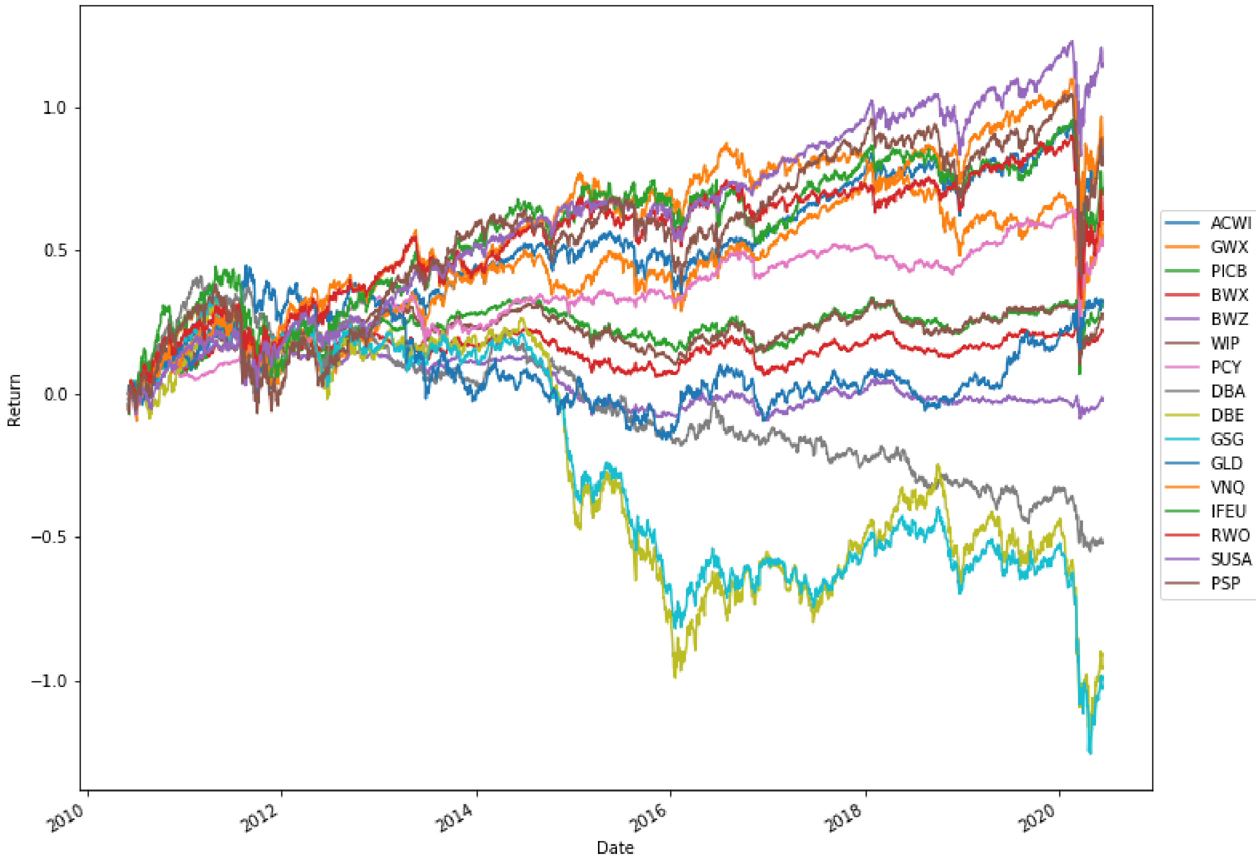

Figure 4.1 ETF Universe and COVID-19 Crash

With the credits of the pricing model tested in this chapter, it may be beneficial to offer an example of an area it may be of use where there is currently more difficulty and qualitative assessment required. A good example is a case where there are minimum or maximum constraints on allocations to an asset, such as in private equity. Adding a target firm to a private equity fund requires capital input for the buyout, which may take up a considerable proportion of the available liquidity. In the absence of private company valuations, a similar process can be simulated for the special case that is Berkshire Hathaway. The most recent change to date for the Berkshire Hathaway portfolio was the addition of a holding in Liberty Sirius XM Group Series C (LSXMK) on 17 June 2020 (CNBC, 2020). Modelling this addition for the chosen $\boldsymbol{\lambda}$ values like an LBO, it can be tested whether this purchase was beneficial. Calculating the Z's for the portfolios before and after the addition of LSXMK allows comparison for whether the Z decreased for each $\boldsymbol{\lambda}$. The results of this investigation are reported in Table 4.5 and find substantial evidence that this was a good investment decision. For all but three of the chosen $\boldsymbol{\lambda}$'s the Z improved. In these three cases, one pair were equal and the others differed by thousandths, representing a very small difference.

Table 4.5 Berkshire Hathaway Portfolio Expansion
Mhiri and Prigent (2010) (4.1) Z of portfolios formed for a sample of $\boldsymbol{\lambda}$ values
Indication of whether the Mhiri and Prigent (2010) model indicates an improvement in asset allocation through the 17 June 2020 trade, performed for each $\boldsymbol{\lambda}$ value

$\boldsymbol{\lambda}$	Before Purchase	After Purchase	Improved Z
{1,0,0,0}	3.5271	3.5305	
{0,1,0,0}	19.8501	19.6062	✓
{0,0,1,0}	4.0343	4.0343	-
{0,0,0,1}	8.9846	8.9766	✓
{1,1,0,0}	19.3772	19.1367	✓
{1,0,1,0}	3.5614	3.5648	
{1,0,0,1}	8.5116	8.5071	✓
{1,1,1,0}	19.4115	19.1710	✓
{1,1,1,1}	24.3961	24.1476	✓
{2,1,1,1}	24.1468	23.8985	✓
{1,2,1,1}	291.4715	283.3070	✓
{1,1,2,1}	24.4316	24.1831	✓
{1,1,1,2}	54.2264	53.8909	✓
{1,1,2,2}	54.2619	53.9263	✓

The main difficulty that should be addressed with the pricing model proposed is the predictability and stability of the parameters affecting the moments of any selected portfolio over time. For further analysis of this issue, the ability to apply the model in a forward-looking manner may be better understood with Chapter 6, which looks at the predictability of the moments.

Chapter Five

Risk-Neutral Moment Dynamics

With European option data, it is possible to derive the implied volatility, as well as the implied distribution, of a stock's returns to option expiry. The process was first discussed by Breeden and Litzenberger (1978), using the cost of butterfly spreads. Through this method, the risk-neutral returns distribution can be extracted for a stock on each date, between that date and the expiry date of available options. Looking at these, one may assess where market expectations are and where they may be misaligned with the physical distribution. To try and quantify these to assess the potential misalignment, the moments stand in place with easy availability, definition and flexibility to understand where these two distributions likely differ.

A similar investigation can assess where markets overreact or underreact to market events. An example would be how the volatility smile skews more during a period of high volatility, where expectations of a large crash rise dramatically and potentially excessively. It is suggested that there may be some opportunity to trade moments using these butterfly spreads like Arrow-Debreu securities, spotting occasions where the market acts inefficiently to an event.

A long butterfly spread can be made using three options, differing only by strike price, whilst these can all be calls or all be puts. Purchases of one at the lowest strike and one at the highest strike are made, meanwhile shorting two options with the middle strike. The model used takes a very simple approach to estimate the distribution of returns, assuming between each set of three strike prices that returns are approximately uniformly distributed when estimating the expected payoff for each spread. At the limit, butterfly spreads approximate the second derivative of the option pricing function with respect to the strike price, as shown in equation (5.1).

$$\lim_{\Delta M \to 0} \frac{P(M,T;\Delta M)}{\Delta M} = \frac{\partial^2 c(X,T)}{\partial X^2} = \frac{\partial^2 p(X,T)}{\partial X^2} \tag{5.1}$$

Where ΔM is the distance $B - A \equiv C - B$ from the reference strike price M (B in Figure 5.1) (Breeden and Litzenberger, 1978, p.627)

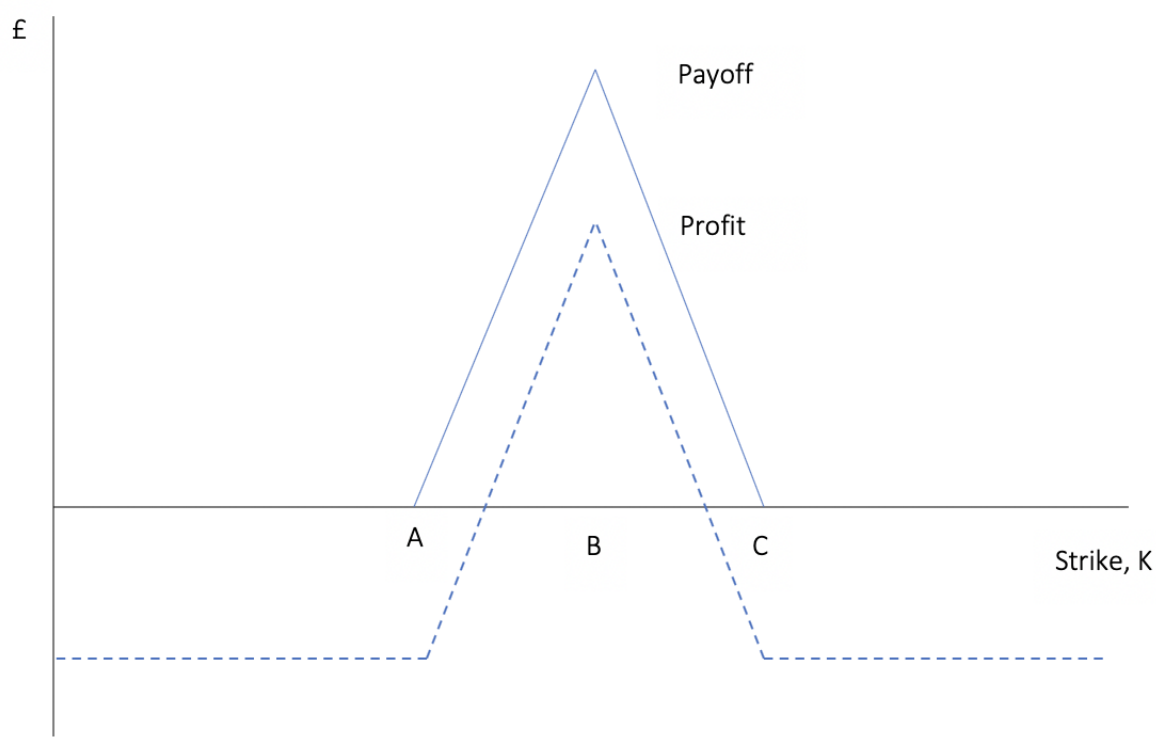

Figure 5.1 Long Butterfly Spread Illustration

The following paragraph clarifies notation on this topic. The strikes of the three options required for each butterfly spread will be referred to as A, B and C. A graph to illustrate these and the butterfly spread dynamics is given in Figure 5.1. It should be noted that in decomposing the butterfly spread, the leg from A to B is a long option spread and the leg from B to C is a short option spread. Both calls and puts are used here in this way, so 'option' can be used interchangeably with either 'call' or 'put'.

This, of course, is much more stable the smaller the gap in strikes available, and generally is not overly large and aided by a wider range of available option strikes. The cost of this butterfly spread is, therefore, the estimated present expected value of the stock being in this region at expiry. This level of simplicity was employed given the scale of data used and time constraints given, where this is one of a few investigations in this project. More advanced methods are proposed in Breeden and Litzenberger (1978) and also by the Bank of England (Blake and Rule, 2015; Bahra, 1997), with some tools available in the RND package available in R (Hamidieh, 2017). The issue with the added sophistication these offer is that the next step in advancement is the fitting of a normal distribution to the implied probabilities. This is, of course, no use for this investigation, which would require advancement again to a non-parametric approach. This further step, as much as it might have improved the investigation, would have taken considerably longer, past the time frame available.

An adjustment was made to the Breeden and Litzenberger (1978) method, allowing the inclusion of option combinations where A to B and B to C are non-equidistant. This presents issues that the payoffs to the long and short option spreads that make up the butterfly spread would not be of the same height. This means the net payoff with final underlying price above C would be non-zero. For example, if $C - B = 2(B - A)$ then the short spread side will be twice the length of the long spread, leaving a payoff of $-(B - A)$ if the spot at expiry is above C. To allow the use of those cases where $B - A \neq C - B$, the weighting on the short leg was changed from -1 to $-\frac{B-A}{C-B}$. This gives a peak payoff of 1 and payoff of 0 outside A or C at expiry. With a uniform distribution assumed, this will give an average payoff of $\frac{B-A}{2}$, so dividing the cost of the butterfly spread by this gives an expected payoff of 1 between A and C. Now each butterfly will represent the cost of gaining an expected payoff of 1 between each triplet of options, allowing its use as a proxy of an Arrow-Debreu security. The only restriction remains the assumption of a uniform distribution between A and C. Given this distance almost always represents a small proportion of the market price, it is not expected to impact the investigation dramatically.

European options data was downloaded using Refinitiv Eikon for the last two years for all those constituents from the FTSE 100 index with available data. This equated to around 40 companies with a total of 475,000 data points, allowing derivation of almost 23,000 expected return distributions. With these distributions, their expected mean, variance, skew and kurtosis could be taken and compared to the physical distribution that can be observed from the actual resulting distribution of returns to expiry. Using these daily stock returns, it is also possible to see how the risk-neutral distribution reacts to stock returns, as this is the area where inefficiencies could first come into play through behavioural factors.

Moments of Risk-Neutral Distributions

To test whether the market overreacts to the realisation of a bad day's trading for a stock, the moments were derived for the risk-neutral distribution for each day, the previous day, the following day and ten days later. This allowed the movement of the distribution to be traced upon realisation of the day's returns and gauging how long the market potentially takes to recover expectations after such an event.

It will first be looked at the relationships over the entire data set, comparing the derived central moments for the risk-neutral distributions with one another, and how these change day by day. Generally, it was found that there was little evidence of a linear relationship between any of the moments and their movement, though significance did increase with moment order. These results are summarised in Table 5.1. The moments display strong autocorrelation, as is to be expected, whilst they do not offer a strong predictor of the moments the next day. What should be noted is that the first four central moments show differing degrees of autocorrelation. The mean risk-neutral return is by far the most stable, with a 99.9% R^2. Variance is almost as stable, with an R^2 of 96%, followed by skew with 93%. This stability then drops off to a 50% R^2 for kurtosis, which is presented visually in Figure 5.2. What should be noted is that the expected value of each risk-neutral moment is descending in absolute value, other than the mean value, which is slightly increasing. This should well be expected, where the risk-neutral distribution becomes more narrow and symmetrical, whilst shifting slowly to the right until expiry. It is unclear why the coefficient for mean return autocorrelation is increasing, at 1.00156. Being so close to 1, it does not present a serious rate of change but is significant from 1 with a t-value of 6.09.

Comparing the Risk-Neutral and Physical Distributions

Looking at the physical distributions for the stock returns for each date and date of option expiry, it is clear that there is no evident relationship, where the best R^2 value found is 3.9% for predicting the mean return. This is quite surprising, given these distributions should be relatively similar if the market had any ability to predict single-stock returns. In mapping the two distributions to one another, it was thought that those distributions close to maturity might have adversely affected the results. When close to expiry it is unlikely the distribution of expected returns is to be particularly smooth, especially given the limits on the size of the spreads covered by each butterfly spread. To test whether this impacted the results, the same regressions were run only on cases with at least 50 days to expiry. The same outcome was found, with very little predictability for the physical distributions. Therefore, it may seem that in this case stock returns follow a randomly random path or a

Table 5.1 Regressions on Risk-Neutral Moment Changes
Regressions assessing ability of the previous day's movement in central moments to predict the next day's movement, for each of the first four central moments

	Intercept	$\Delta M_{t-1,t}$	$\Delta V_{t-1,t}$	$\Delta S_{t-1,t}$	$\Delta K_{t-1,t}$	R^2
$\Delta M_{t,t+1}$	0.000	-0.018	0.001	-0.029 ***	-0.003 ***	0.001
$\Delta V_{t,t+1}$	-0.001	-0.502 ***	-0.193 ***	0.010	0.001	0.018
$\Delta S_{t,t+1}$	-0.001	-0.181 ***	-0.024 *	-0.235 ***	0.022 ***	0.111
$\Delta K_{t,t+1}$	0.003	0.386	0.051 ***	0.246 ***	-0.340	0.134

Significance: $* = 5\%$, $** = 1\%$, $*** = 0.1\%$

M=Mean, V=Variance, S=Skew, K=Kurtosis

random path that is being predicted terribly.

To narrow down analysis to more extreme events, z-scores were worked out for the daily returns of each stock, using the standard deviation from the 5 years preceding the options data. From these values, the results could be narrowed to those trading days with extreme z-scores. Testing the whole data set initially, there was a 5% insignificant relationship between the z-score and the change in the mean of the risk-neutral distribution from the previous to current end-of-day. This appears the movement of the mean is random and unaffected by single-day returns, it follows a random walk with drift. The remaining three moments tested all had 0.1% significant relationships. Variance and skew had negative relationships, whilst kurtosis had a positive relationship with the z-score. For variance, this makes sense for the behavioural features being tested. With a low return comes expectations of higher

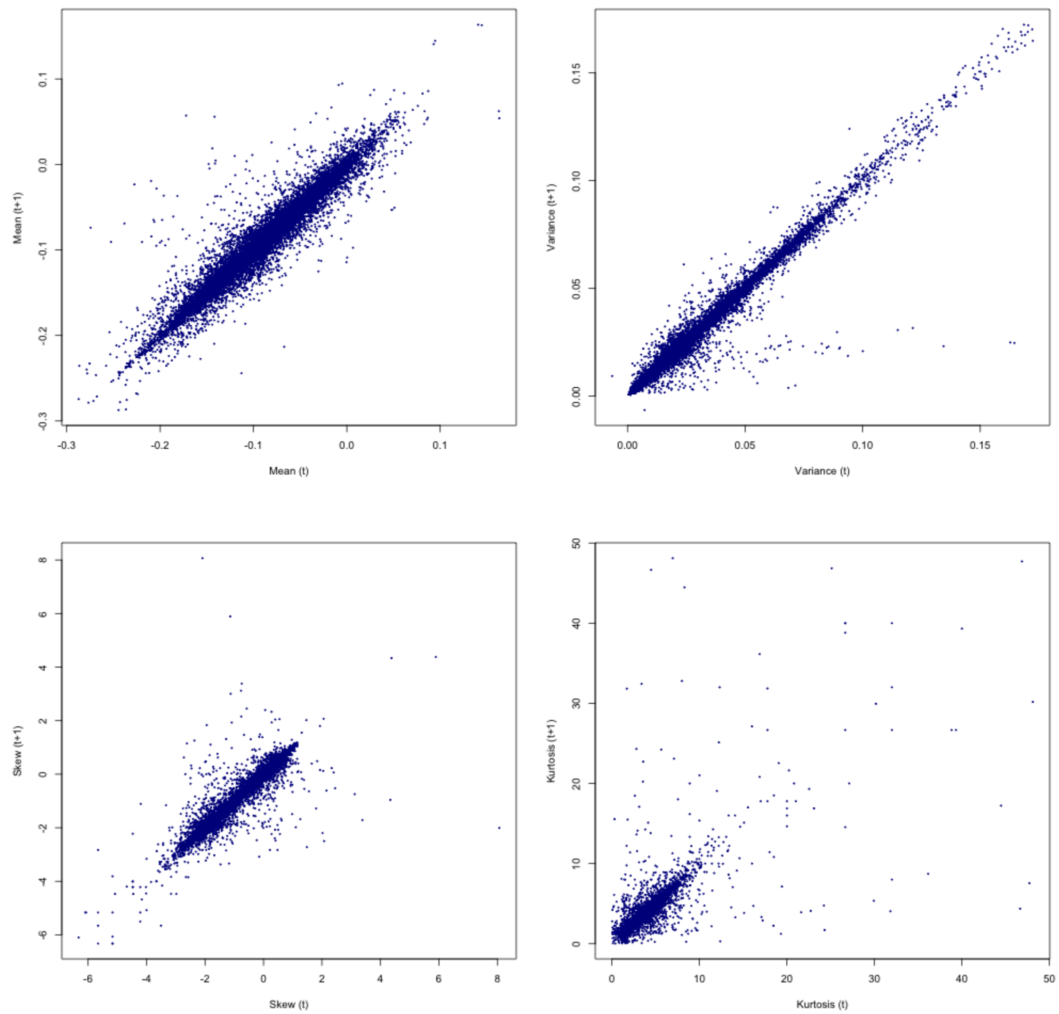

Figure 5.2 Visualisation of Decreasing Moment Autocorrelation Significance

variance. With skew and kurtosis, these results are not as expected. With lower returns comes expectations of higher skew and lower kurtosis. For skew, it may be easier to envision this as lower skew when returns are higher- expectations shift towards a larger downside risk after higher stock returns. Overall, this may be perceived as an expectation of stock price reversal or mean reversion. Explaining these findings is difficult under basic behavioural theories and potentially displays that investors focus the majority of their attention on the implied mean and variance/volatility, spending less time trading and pricing more improbable outcomes on a daily basis.

Combining these terms into one regression changes the results moderately; all coefficients become significant at the 0.1% level and negative. These five regressions are set out in Table 5.2.

Table 5.2 Regression on Z-Score Against Risk-Neutral Moment Changes
Regressand is the Z-score of previous day's return against last 5-years' mean and standard deviation in daily returns. Regressors are changes in each central moment over the last day

	Intercept	$\Delta M_{t-1,t}$	$\Delta V_{t-1,t}$	$\Delta S_{t-1,t}$	$\Delta K_{t-1,t}$	R^2
Z-Score	0.0166	-1.6860	-0.4475	-0.4470	-0.0501	0.036
	***	***	***	***	***	

Significance: $* = 5\%$, $** = 1\%$, $*** = 0.1\%$

M=Mean, V=Variance, S=Skew, K=Kurtosis

The consistency can be compared between the coefficients when tested separately and when combined for the final regression in Table 5.2. The newly significant, much larger coefficient for mean is closer to expectations, showing the z-score to respond much less to changes in risk-neutral mean. Of course, there is still a significant negative relationship here, which is not so easily explained. Upon inspection, it may indicate a change in risk premium, bringing into question how significant this would be on a daily interval. There is a possibility that this is in part down to the relative movement of the distribution due to the change in the 'market' price of the stock. Given that after a negative return the stock price is now lower, each strike price presents a higher implied rate of return than before. The reverse will then be the case after a positive return, meaning all implied returns will decrease. One should also consider that moving a day closer to expiry will increase the absolute value of the implied return if the stock price were to not move for one day.

To narrow down towards extreme events, the top and bottom deciles and percentiles for z-scores were tested separately to assess where changes generally occur in expectations due to relatively large changes to asset value. For the narrowed groups, all coefficients remained very similar, though kurtosis became insignificant. It seems overall, therefore, that variance and kurtosis move as expected, whilst skew moves in the opposite direction to that proposed. Mean was expected to move with the z-score, showing the opposite sign but with relatively small impact. There is difficulty isolating the effects of the first four moments against stock returns but overall there is some potential in exploiting these movements where they are unjustified and simply represent market overreactions- or even underreactions.

Risk-Neutral Moment Dynamics

Looking directly at the dynamics of the central moments, the daily movements of the moments were compared to assess where reversals may be found to support the hypothesis of overreaction. In testing these, lagged values were taken for each of the moments for up to three days prior. Leading values were also taken to act as dependent variables for the next day and the tenth day after. The tenth day was chosen more to reinforce findings for the leading day, testing whether the relationship was lasting. With so many days between the current and tenth day later, there is every likelihood there can be serious influence on the distribution from days between those where stock returns take an extreme value or compound to a large change in stock price. Including this consideration when modelling would be of no practical use as it would require knowledge of future returns, removing the risk that is inherent upon a company event or news story impacting the stock's value. For these reasons, the tenth leading day should not be taken at face value. It is used in this case to try to assess the persistence of reversal, becoming more reliable when specifically looking at extreme outcomes, where similarly extreme outcomes are unlikely to occur in those following days.

First investigating the daily changes in the central moments between the prior and following day, it was found that all four were negatively correlated, though for changes to the mean this was statistically insignificant. For variance, skew and kurtosis these findings support the hypothesis of reversal over the daily interval. Looking further by including the changes to all four moments in each regression, it was found that movements in skew and kurtosis were negatively correlated with the next day's change in mean value. It is an interesting case, given change in mean remains unable to provide an indication of future mean movement. It should be appreciated that in these models, between 75% and 99% of the movement of the moments is left unexplained and only small improvements are made in sign prediction. For skew and kurtosis, sign prediction rises to 57% and 55%, respectively. Inclusion of all three previous days' movements for all moments provides little improvement, contributing almost solely to multicollinearity.

Reducing the search to those cases with stock return z-scores in the top and bottom deciles and percentiles, some strange phenomena arise. Firstly, the fit of the models on days with z-scores in the lowest decile and lowest percentile all increase, with improved sign prediction success. The reverse is the case in the highest z-score cases. This suggests there is a more prominent and reliable market reaction in 'bad' days. The peculiar part is that in some cases, these relationships mostly become positive; for skew, this occurs in both the lowest decile and lowest percentile and for mean, this occurs in the lowest percentile. Such unusual results can be seen in Table 5.3, where while the positive coefficient for the lagged change in skew against leading change in skew is insignificant, the positive intercept is significant. These results contradict previous findings, suggesting instead that in these cases the market underreacts to downside events. The story is similar in the top decile and percentile, supporting that these unexpected relationships are present for all large stock price moves. Following this further, the market, therefore, overreacts more on less-extreme days.

Overall, these investigations find a mixture of results. Of course, there are a great number of possibilities to speculate around the cause of these. To avoid this speculation, it

Table 5.3 Regressions on Changes in Risk-Neutral Moments for Bottom Percentile Z-Scores

Regressions from Table 5.1 performed only on those days and stocks with a Z-score in the lowest percentile of all Z-scores. Regressions assessing ability of the previous day's movement in central moments to predict the next day's movement, for each of the first four central moments

	Intercept	$\Delta M_{t-1,t}$	$\Delta V_{t-1,t}$	$\Delta S_{t-1,t}$	$\Delta K_{t-1,t}$	R^2
$\Delta M_{t,t+1}$	-0.002 *	0.203 ***	0.446	0.008	0.001	0.092
$\Delta V_{t,t+1}$	0.000	-0.000	-0.087	0.001	0.000	0.011
$\Delta S_{t,t+1}$	0.039 **	-0.463	-0.343	0.065	0.033	0.019
$\Delta K_{t,t+1}$	-0.102 **	1.657	-0.104	0.039	-0.071	0.033

Significance: $* = 5\%$, $** = 1\%$, $*** = 0.1\%$

M=Mean, V=Variance, S=Skew, K=Kurtosis

should be appreciated that there is not a simple answer nor clear proof of market inefficiency through behavioural factors. There is much work to be done to try to narrow down these events categorically and assess each case more thoroughly, as there can be so many factors causing such events, where some are more unexpected, persistent or industry-specific. The relationships discovered have barely been eternal, as is common in finance. It is therefore unsurprising that mixed results have been found with a process consisting of so many steps, built on one another. There is also question around the isolation of trading on the moments of risk-neutral distributions. Volatility and variance trading has increased in popularity recently amongst those seeking to reduce downside risk, often replicated through delta-hedged trades. To continue this line of development, synthetic products (such as skew and kurtosis swaps) could be offered in a perfect environment. Alternatively, exposure to these could be generated through the trading of vanilla European options in the tails for kurtosis, with an asymmetrical position when trading skew. Another method could be to use volatility/variance swaps. Again, trading opposite positions in the centre of the distribution as in the tails would allow exposure to the relative heights of the expected returns distribution in the centre versus the tails. Adding asymmetry would allow skew exposure. The specifics could be refined quantitatively. With access to the derivatives market, it is expected a very close representative could be generated to isolate these central moments. It is thought skew products could provide some credible benefit, especially with the current rate of quantitative progress and increasing frequency of black swan events.

Chapter Six

Moment Predictability

Machine learning (ML) methods have risen fantastically into prominence over recent decades. It is near impossible to find a company not placing the development of artificial intelligence (AI) amongst their primary objectives. Its application within finance has likewise been pervasive, where forecasting and non-linear optimisation are ubiquitous. Therefore, it is only right to include some applications to assess whether there is an ability to generate or improve trading signals through modern data science.

An important addition to make in the field of forecasting moments- generally, the skew and kurtosis are closely related. In almost all cases tested, the kurtosis forms a U-shape against the skew, closely tracing a quadratic relation between the two. This hence presents an interesting premise that if one is able to predict one of the two, then the key is found to both. Of the two, skew would be far more valuable as the sign of the skew cannot be found from the kurtosis. Questions do then come into play regarding the usefulness of kurtosis in the models discussed, given that a skew component would likely also internalise consideration of kurtosis.

The first effort made assessed the predictability of the central moments of stock returns using feedforward artificial neural networks (ANN). Annual values were taken for each central moment as well as other risk metrics including value-at-risk, conditional value-at-risk, maximum drawdown and semivariance. The first four powers of each moment were also included to capture the relationship between the squared skew and kurtosis and investigate any potential for other similar relationships. There were 20 inputs used covering these variables described, where the previous year's 4 central moments were the outputs. All layers were assigned as dense. This required the input layer have 20 nodes and the output layer 4. The model was optimised for the number of hidden layers (each with 20 nodes) and the dropout proportion at each stage, finding two hidden layers with a dropout of 1/6 gave the best mean squared error (MSE). Though optimal and stable, the model predicted the moments terribly. For mean and skew it was particularly bad, with negative R^2 values, showing the model worse than simply returning the expected value. For variance and kurtosis, there was not a great improvement, with R^2 values of 3.6% and 1.6%, respectively. The optimal model appears to simply predict near-zero values.

With the data in Chapter 5, a long short-term memory (LSTM) model was run using the lagged changes in moments to predict the next day's movement for each central moment. This was equally inconclusive to the model run on the physical data and was similar to the models run in Chapter 5. The instability is clear in Figure 6.1, where the loss function trying to predict changes in risk-neutral skew is completely unstable. This summarises the results of this investigation quite concisely.

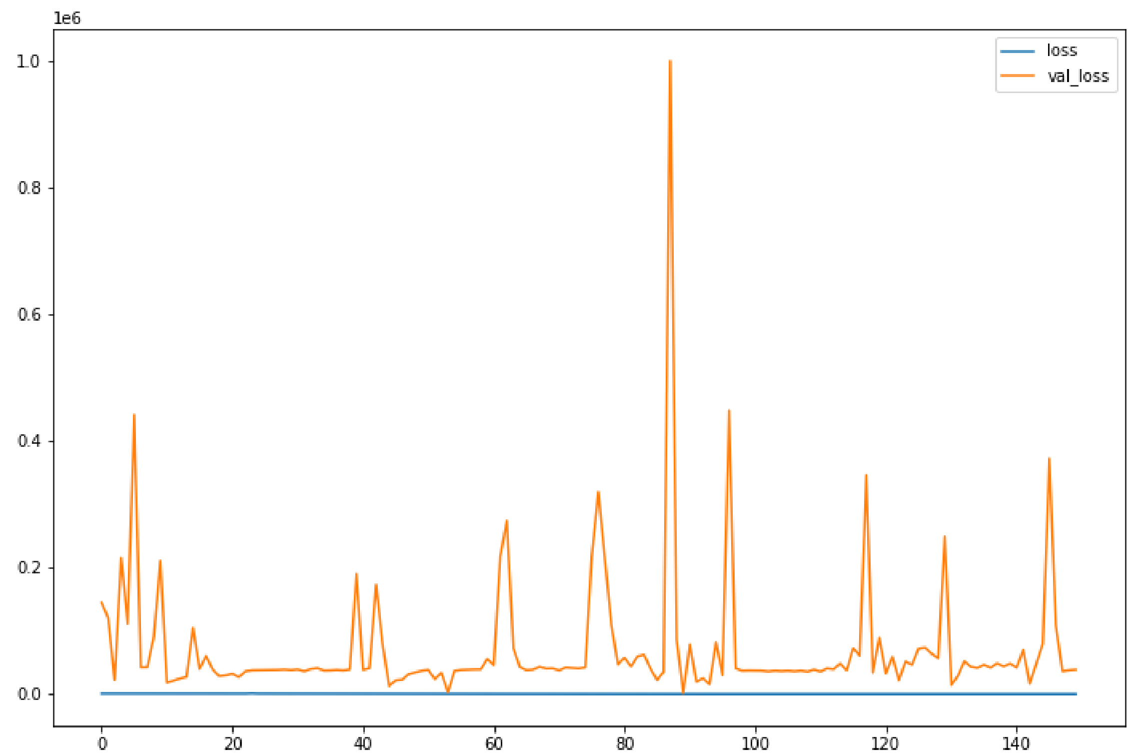

Figure 6.1 Skew LSTM Model Instability

Chapter Seven

Conclusions and Recommendations

This project sought to evaluate the use of the moments of return distributions in asset pricing and investment strategy. Much progress has been made since Markowitz (1952), mostly in the field of computer science. Where MV analysis is mostly used as one of the simplest models for asset pricing, it makes sense to continue this development into at least including skew. Whilst its inclusion adds much better consideration for the asymmetry of asset returns, the primary difficulty is in solving the trade-off between central moments. This requires the addition of a level of complexity and giving up such an intuitive setting. With the ability to run the models and their optimisations quickly, as compared with the time of the first derivation in 1952, little time and sophistication are required to make these improvements now. As a step to advancing the baseline model used in asset pricing, the Fama-French 3 factor model takes the first step beyond the CAPM, including features affecting returns distributions, often depending on the performance of the market. With its opening up the field of factor investing came what is suspected a set of features mimicking different shapes of distributions. To bring these back from a more qualitative to purely quantitative approach is the intention of the pricing model presented in Chapter 4 (p.16), allowing the optimal allocation beyond the restraints of those factors discovered.

With the asset pricing model proposed in Chapter 4, it was found that momentum and equities were not necessarily as attractive as given credit for (p.24). Whilst they generally offer strong average returns, their distribution shape does not combine well with other factors and asset classes. Asset class diversity, therefore, takes precedence over a diversified equity portfolio. Factor investing, recommendations were generally made favouring reversal factors, which mix well with investment stocks. Further, the combination of value and quality was supported, where there has been debate as to combining either or both of these with momentum (Barroso and Santa-Clara, 2015; Dewandarua et al., 2014; Novy-Marx, 2013; Novy-Marx, 2014). It is interesting to note that the inclusion of momentum was not recommended with this combination. Blending these findings brings interesting questions as to the potential benefits of intra-class allocation for non-equities, such as fixed income factor investing (AQR Capital Management, 2016). These concepts could be expanded to allocations within other asset classes, with great potential.

Looking at the relationship between the risk-neutral and physical distributions, there is very little in the way of consistency and clarity- indicating great inability to predict the physical distribution shape. There were, however, apparent relationships in the movement of the risk-neutral central moments, dependent upon the day's stock returns. Though ability was found predicting market over/underreaction, the signs for these relationships differed from those hypothesised; finding underreaction on days in outermost deciles and overreaction elsewhere. These discoveries were reinforced through more open-ended and powerful ML techniques. Given the unusual and temperamental results, it appears there are difficulties in the method chosen and data availability. There are often few options expiring on a single date, differing only by their strike price. This, in most cases, gives 10-20 points to make up each distribution and hence makes the model erratic and very sensitive in its tails. There may not even be good coverage of the tails in these cases, where options are not written so far in or out of the money.

Going forward, it is suggested that the model of Mhiri and Prigent (2010), examined in

Chapter 4, provides a far better understanding of one's portfolio and is a better choice in asset allocation than a traditional MV approach. From previous evaluation, such as Bate (2019), there is no definitive way to decide on an optimal method. In this case, the Mhiri and Prigent (2010) model weakly dominates MV optimisation as it is a special case of the Mhiri and Prigent (2010) model. The next question to be covered is whether optimisation of the moments proves superior to the simpler optimisation against another single risk metric, such as mean-conditional value-at-risk optimisation.

Looking further, there is potential for improvement through the inclusion of more central moments, where the fifth central moment would likely assess tail asymmetry. As discussed in Chapter 3, this quickly adds mathematical complication and sensitivity to distribution tails, beyond a practical limit. Alternatively, the inclusion of co-moments in asset allocation models or even in simple trade assessment is promising, where an approximation can easily be made as to the behaviour of one's portfolio during major market fluctuations. This integrates well with the idea of hidden beta, providing deeper understanding with little complication. The model of Hung, Shackleton, and Xu (2004) allowed the inclusion of these co-moments, though it is suggested such complication is not necessary to give them some initial consideration.

With moment trading, it does show some potential in predicting market reaction from its movements, with R^2 values between 0.1% and 4.0% for the mean return and up to 25% for the rest of the central moments. There is room to improve this further with the suggestions made in p.40, along with some adjustments and searches spanning other asset classes. Firstly, far more equity data could be collected to test the hypotheses set out, collecting data for a whole market beyond just FTSE 100 constituents. Next, this approach could be applied for futures, which span a range of asset classes themselves, allowing the flexibility to potentially include any asset class. An added benefit with futures is that they leverage their underlying positions, meaning there will be larger reactions to market events and more clarity in assessing the direction of these movements. Using other asset classes (not necessarily via their futures), there may be more factors and trends to take account of, such as bond duration and yield curve. Including these effects, it would be interesting to investigate the relationships in place and whether they differ between asset classes.

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APPENDICES

Appendix A

Extra Tables and Figures

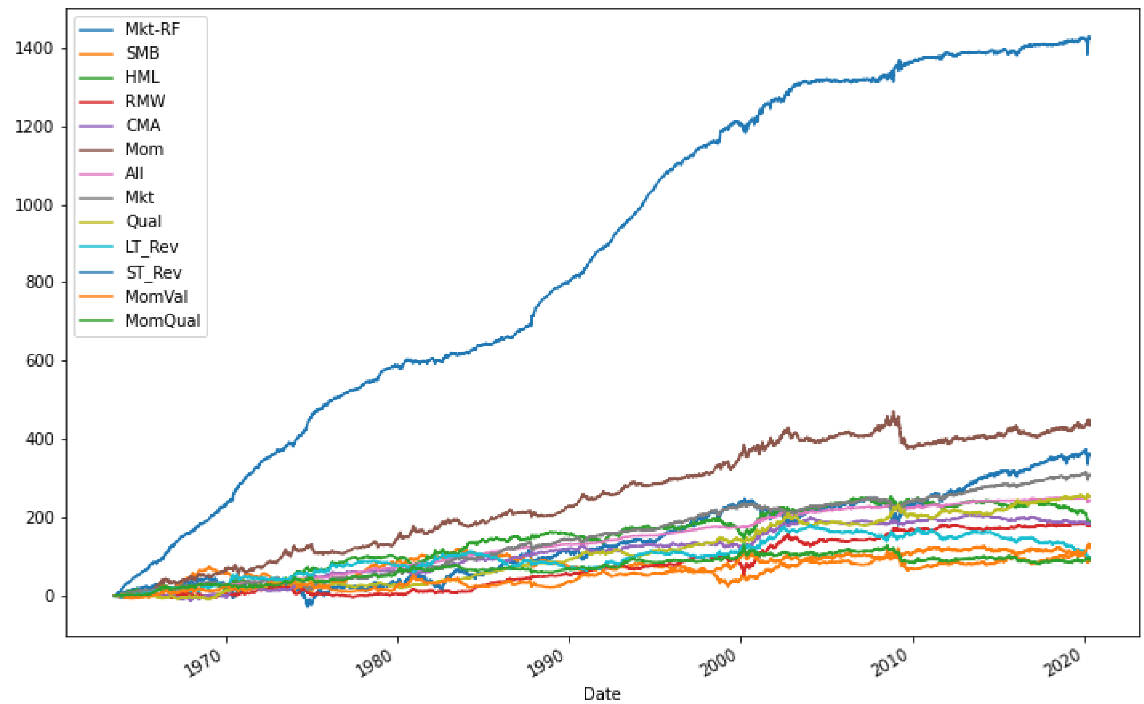

Figure A.1 Cumulative Factor Returns

Table A.1 Berkshire Hathaway Holdings as at 30 June 2020, in USD

Company	Symbol	Holdings	Price ($)	Value ($)	Stake
Apple Inc	AAPL	250866566	354.99	89055122264	0.408346
Amazon.com Inc	AMZN	533300	2682.94	1430811902	0.006561
American Express Company	AXP	151610700	103.00	15615902100	0.071604
Axalta Coating Systems Ltd	AXTA	24070000	22.66	545426200	0.002501
Bank of America Corp	BAC	947760000	25.03	23722432800	0.108775
Biogen Inc	BIIB	643022	264.26	169924994	0.000779
Bank of New York Mellon Corp	BK	88130897	38.71	3411547023	0.015643
Charter Communications Inc	CHTR	5426609	531.30	2883157362	0.013220
Costco Wholesale Corporation	COST	4333363	300.89	1303865593	0.005979
Davita Inc	DVA	38095570	79.55	3030502594	0.013896
Globe Life Inc	GL	6353727	76.95	488919293	0.002242
General Motors Company	GM	74681000	27.58	2059701980	0.009444
Goldman Sachs Group Inc	GS	1920180	204.59	392849626	0.001801
Johnson & Johnson	JNJ	327100	144.27	47190717	0.000216
JPMorgan Chase & Co	JPM	57714433	99.08	5718346022	0.026220
Kraft Heinz Co	KHC	325634818	33.47	10898997358	0.049975
Coca-Cola Co	KO	400000000	47.32	18928000000	0.086791
Kroger Co	KR	18940079	31.67	599832302	0.002750
Liberty Global PLC Class A	LBTYA	19791000	23.57	466473870	0.002139
Liberty Global PLC Class C	LBTYK	7346968	23.03	169200673	0.000776
Liberty Latin America Ltd Class A	LILA	2630792	10.33	27176081	0.000125
Liberty Latin America Ltd Class C	LILAK	1284020	9.99	12827360	0.000059
Liberty Sirius XM Group Series A	LSXMA	14860360	36.94	548941698	0.002517
Liberty Sirius XM Group Series C	LSXMK	42868070	36.42	1561255109	0.007159
Mastercard Inc	MA	4934756	307.50	1517437470	0.006958
Moody's Corporation	MCO	24669778	278.23	6863872333	0.031473
Mondelez International Inc	MDLZ	578000	53.57	30963460	0.000142
M&T Bank Corporation	MTB	5382040	109.69	590355968	0.002707
Occidental Petroleum Corporation	OXY	36207184	20.68	748764565	0.003433
Procter & Gamble Co	PG	315400	121.26	38245404	0.000175
PNC Financial Services Group Inc	PNC	9197984	109.64	1008466966	0.004624
Restoration Hardware Holdings Inc	RH	1708348	255.00	435628740	0.001997
Restaurant Brands International Inc	QSR	8438225	56.50	476759713	0.002186
Sirius XM Holdings Inc	SIRI	132418729	6.11	809078434	0.003710
SPDR S&P 500 ETF Trust	SPY	39400	313.10	12336140	0.000057
StoneCo Ltd	STNE	14166748	39.65	561711558	0.002576
Store Capital Corp	STOR	18621674	25.36	472245653	0.002165
Suncor Energy Inc	SU	14949031	17.68	264298868	0.001212
Synchrony Financial	SYF	20128000	24.33	489714240	0.002245
Teva Pharmaceutical Industries Ltd	TEVA	42789295	12.48	534010402	0.002449
United Parcel Service Inc	UPS	59400	108.26	6430644	0.000029
U.S. Bancorp	USB	149590275	38.65	5781664129	0.026511
Visa Inc	V	10562460	195.55	2065489053	0.009471
VANGUARD IX FUN/S&P 500 ETF SHS NEW	VOO	43000	289.10	12431300	0.000057
Verisign Inc	VRSN	12815613	208.53	2672439779	0.012254
Wells Fargo & Co	WFC	345688918	27.79	9606695031	0.044050

Appendix B

Links to Code

All code can be found at this Github Repository. Please send a request to the author's Gmail if there are any accessibility issues or to access the data used. The repository also holds a copy of the dissertation referred to in the conclusion (p.46, Bate (2019)).

Chapter 4

Factor Optimisations

factors.m

Asset Class Optimisations

AssClass.m

USFrenchMoments.ipynb

AssetClasses.ipynb

Berskshire Hathaway Rebalancing Analysis

BHPortfolios.m

Buffett.ipynb

Chapter 5

Data Collection

EikonAPI.ipynb

Modelling and Analysis

OpInv.r

OpAnalysis.r

Chapter 6

Data Collection

TwentyYear.ipynb

smry.ipynb

CleanUSData.ipynb

Neural Networks

MVSK NN.ipynb

Rolling NN.ipynb

LSTM.ipynb