ETF portfolio optimizer – formal specification

Decision variables

$$w = (w_1, \dots, w_N)^\top \in \mathbb{R}^N \quad \text{(portfolio weights on the } N \text{ ETFs)}$$

Parameters

$\Sigma \in \mathbb{R}^{N\times N}$	(covariance matrix of ETF returns)
$\rho_{ij} \in [-1, 1]$	(pairwise correlation between ETF i and j)
$R = (\rho_{ij})_{i,j} \in \mathbb{R}^{N\times N}$	(correlation matrix)
$\mu \in \mathbb{R}^N$	(expected / historical mean returns of ETFs)
$\mathrm{TER} \in \mathbb{R}^N$	(total expense ratio or cost vector)
$A \in \mathbb{R}^{C\times N}$	(country exposure matrix; $A_{c,i}$ = fraction of ETF i's holdings in country c)
$E^{\min}, E^{\max} \in \mathbb{R}^C$	(lower/upper bounds for country exposures)
$w^{\max} \in \mathbb{R}^N$	(per-ETF maximum weight vector)
$\alpha, \beta, \gamma, \delta, \lambda \in \mathbb{R}_{\geq 0}$	(tuning hyperparameters)
$C_{\text{off}} = R - I$	(correlation matrix with zeros on diagonal)

Objective function (to minimize)

We combine variance, a correlation penalty, a (negative) return reward, expense penalty and a concentration (L2) penalty.

$$\min_w \quad \underbrace{\alpha\, w^\top \Sigma\, w}_{\text{portfolio variance (risk)}} + \underbrace{\beta\, w^\top C_{\text{off}}\, w}_{\text{correlation penalty (off-diagonal)}} - \underbrace{\gamma\, \mu^\top w}_{\text{return reward}} + \underbrace{\delta\, \mathrm{TER}^\top w}_{\text{cost/fee penalty}} + \underbrace{\lambda\, \|w\|_2^2}_{\text{concentration (regularizer)}} \tag{1}$$

Notes:

- The first term $w^\top \Sigma w$ is the usual portfolio variance.
- $C_{\text{off}} = R - I$ zeros the diagonal so the correlation penalty only counts cross-ETF correlations; equivalently one can use $\sum_{i<j} \rho_{ij} w_i w_j$.
- The third term is negative because higher expected return should reduce the objective (we are minimizing).
- $\lambda \|w\|_2^2$ encourages weight spreading (discourages concentration).
- All hyperparameters $\alpha, \beta, \gamma, \delta, \lambda$ control trade-offs; scale them consistently with units of the terms.

Constraints

(full investment)	$\sum_{i=1}^{N} w_i = 1$	(2)
(long-only / bounds)	$0 \leq w_i \leq w_i^{\max}, \quad i = 1, \dots, N$	(3)
(country exposure bounds)	$E(w) \;=\; Aw \quad \text{and} \quad E^{\min} \leq Aw \leq E^{\max}$	(4)
(optionally: cardinality / sparsity)	$\|w\|_0 \leq k \quad$ (non-convex; optional)	(5)

Remarks:

- Constraint (3) enforces long-only exposure. To allow shorting, replace the left bound with a negative lower bound.
- Constraint (4) enforces that the portfolio's country exposures (computed from ETF-level country exposures A) stay inside specified limits.
- The cardinality constraint (5) is optional and makes the problem mixed-integer / non-convex; exclude it if you want a convex QP.

Equivalent explicit pairwise correlation term

If you prefer the pairwise-sum form instead of matrix form:

$$\sum_{i<j} \beta\, \rho_{ij}\, w_i w_j$$

which is identical to $\frac{1}{2}\beta \sum_{i \neq j} \rho_{ij} w_i w_j$ and equals $\beta\, w^\top C_{\text{off}} w$ when C_{off} has zeros on the diagonal and contains ρ_{ij} off-diagonal.

Compact matrix summary

$$\boxed{\min_{w}\ \alpha\, w^\top \Sigma w \;+\; \beta\, w^\top C_{\text{off}} w \;-\; \gamma\, \mu^\top w \;+\; \delta\, \text{TER}^\top w \;+\; \lambda\, w^\top w}$$

subject to $\mathbf{1}^\top w = 1,\ 0 \le w \le w^{\max},\ E^{\min} \le Aw \le E^{\max}$.

This formulation is a *convex quadratic program* as long as the combined quadratic matrix

$$Q \;=\; \alpha\Sigma \;+\; \beta C_{\text{off}} \;+\; \lambda I$$

is positive semidefinite. With $\Sigma \succeq 0$ and $\lambda \ge 0$, and moderate β, convexity typically holds; if you add cardinality or other integer constraints the problem becomes non-convex / mixed-integer.