CE 311K: Taylor series and Newton Raphson

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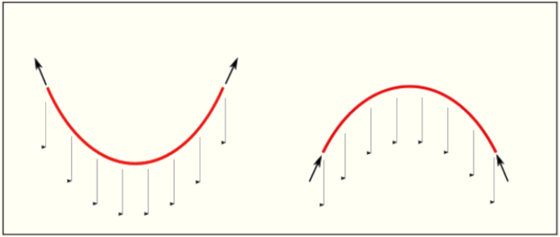
The fan vaults of King's college

? What is the thickness of the ceiling?

From arches to fan vaults

The fan vaults of King's college

Catenary



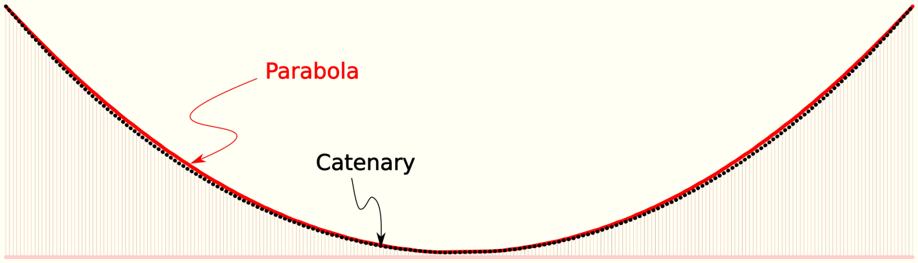
A catenary curve (left) and a catenary arch, also a catenary curve (right). One points up, and one points down, but the curves are the same.

Sagrada Familia

Catenary, caternary . . . everywhere

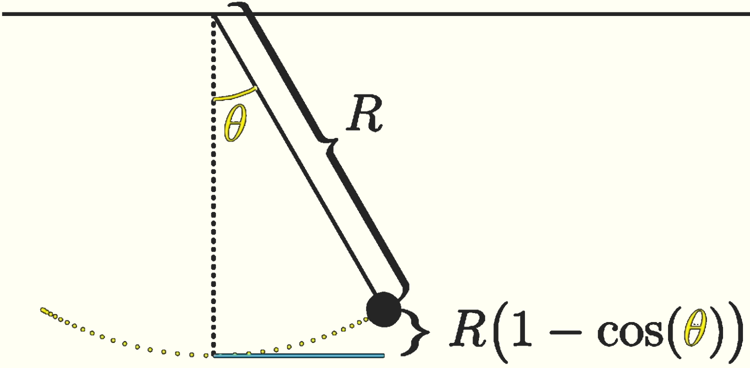
What is this shape?

Catenary vs Parabola



Potential energy of a simple pendulum

We need to know how high the weight of the pendulum is above its lowest point

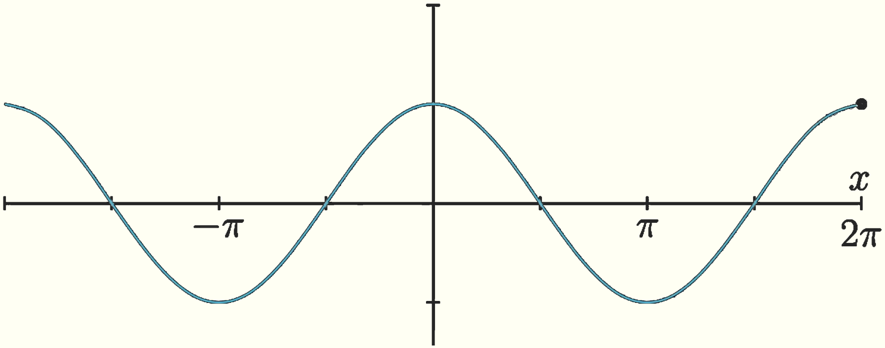


Taylor series

- Taylor series is one the best tools maths has to offer for approximating functions.
- Taylor series is about taking non-polynomial functions and finding polyomials that approximate at some input.
- The motive here is the polynomials tend to be much easier to deal with than other functions, they are easier to compute, take derivatives, integrate, just easier overall.
- $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

Taylor series of cos(x)

$\cos(x)$



Taylor series of cos(x)

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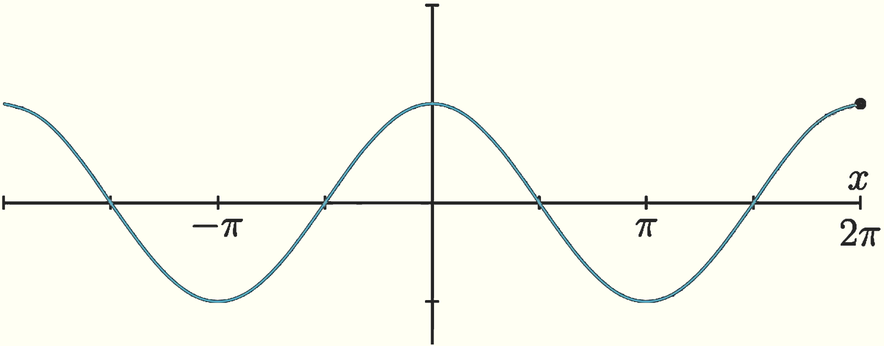
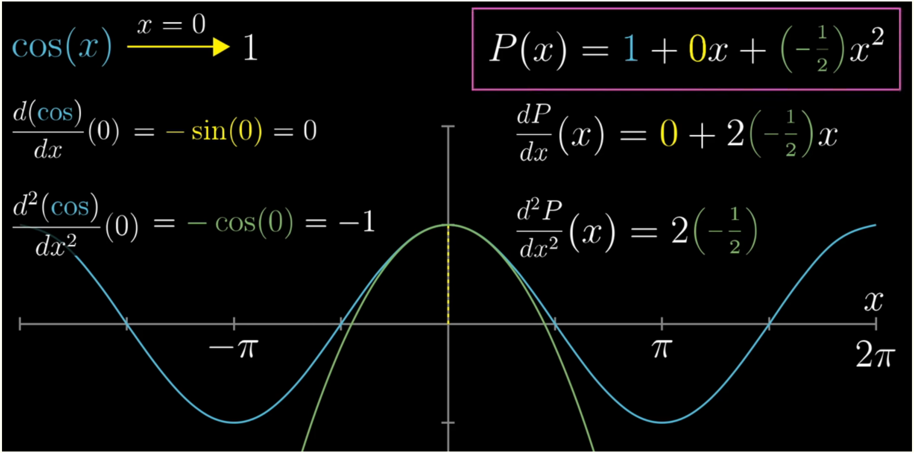


Taylor series of cos(x)



Taylor series of cos(x): 4th derivative

$$P(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$$

$$\frac{d^4P}{dx^4}(x) = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \frac{1}{24}$$

$$= 24 \cdot \frac{1}{24}$$

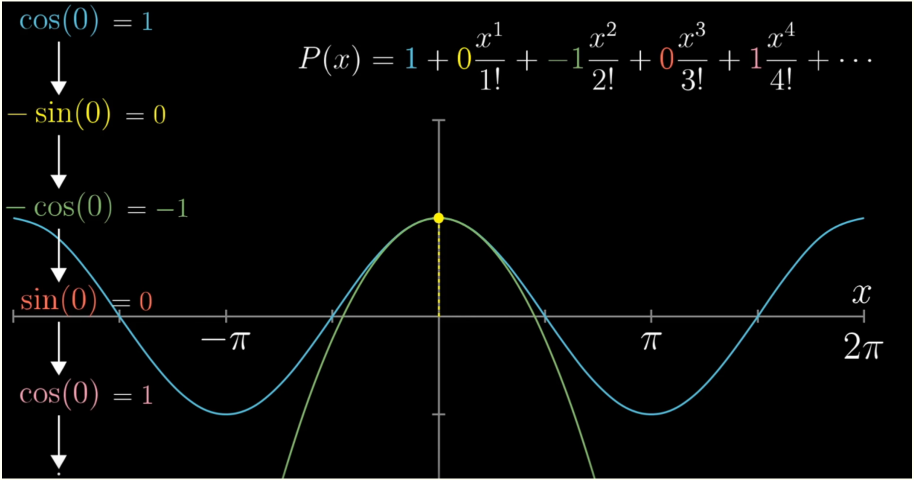
To find the coefficient of n^{th} term:

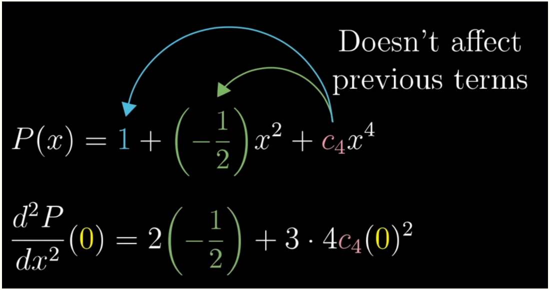
$$\frac{d^8}{dx^8}(c_8 x^8) = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot c_8$$

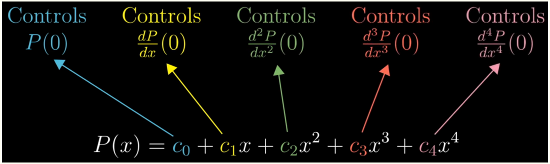
$$= 8!$$

$$Set\ c_8 = \frac{\text{desired derivative value}}{8!}$$

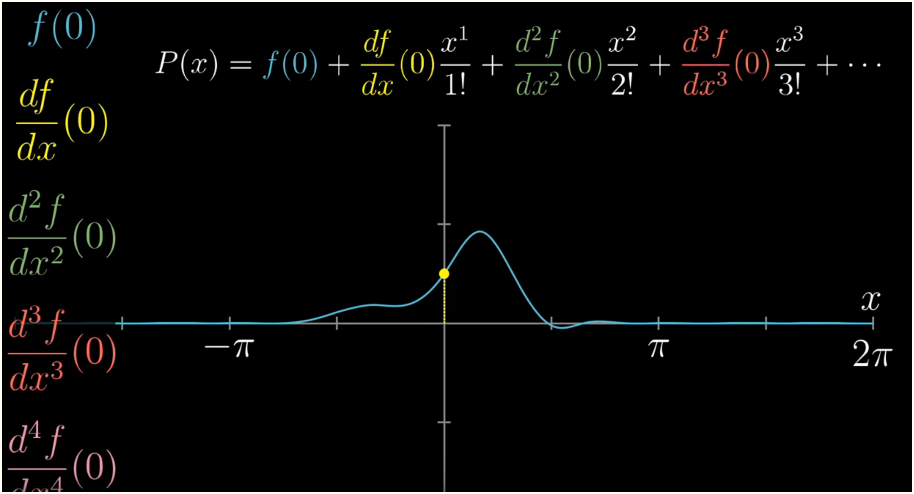
Taylor series of cos(x)



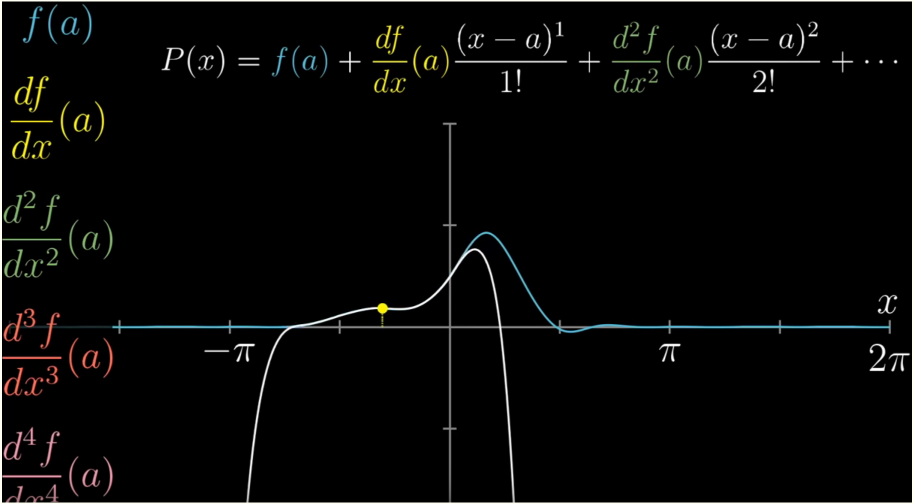

Doesn't affect previous terms
$P(x) = 1 + \left(-\frac{1}{2}\right)x^2 + c_4 x^4$
$\frac{d^2P}{dx^2}(0) = 2\left(-\frac{1}{2}\right) + 3 \cdot 4c_4(0)^2$


Controls $P(0)$
Controls $\frac{dP}{dx}(0)$
Controls $\frac{d^2P}{dx^2}(0)$
Controls $\frac{d^3P}{dx^3}(0)$
Controls $\frac{d^4P}{dx^4}(0)$
$P(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4$

Taylor series: Generalization



Taylor series: Generalization



Taylor series: Two variables

The Taylor series of f expanded about $(x, y) = (a, b)$ is:

$$
\begin{aligned}
(x, y) = f(a, b) + \frac{\partial f}{\partial x}(a, b) \cdot (x - a) + \frac{\partial f}{\partial y}(a, b) \cdot (y - b) \\
+ \frac{1}{2!}[\frac{\partial^2 f}{\partial x^2}(a, b) \cdot (x - a)^2 \\
+2\frac{\partial^2 f}{\partial xy}(a, b) \cdot (x - a)(y - b) + \\
\frac{\partial^2 f}{\partial y^2}(a, b) \cdot (y - b)^2] + \cdots
\end{aligned}
$$

Newton Raphson

Assuming r is a root of f and that f is continuously differentiable in the vicinity of r with $f'(r) \neq 0$, then a sequence (x_n) that converges to r for $n \to \infty$ can be found using the Taylor expansion of f:

$$f(r) = f(x_n + \varepsilon_n) = f(x_n) + f'(x_n)\varepsilon_n + \frac{f''(x_n)}{2!}\varepsilon_n^2 \ldots$$

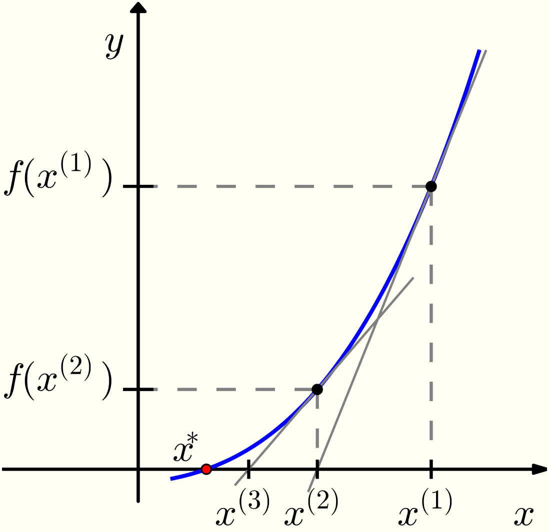
$$\varepsilon_n \approx -\frac{f(x_n)}{f'(x_n)}$$

$$r = x_n + \varepsilon_n \approx x_n - \frac{f(x_n)}{f'(x_n)}$$

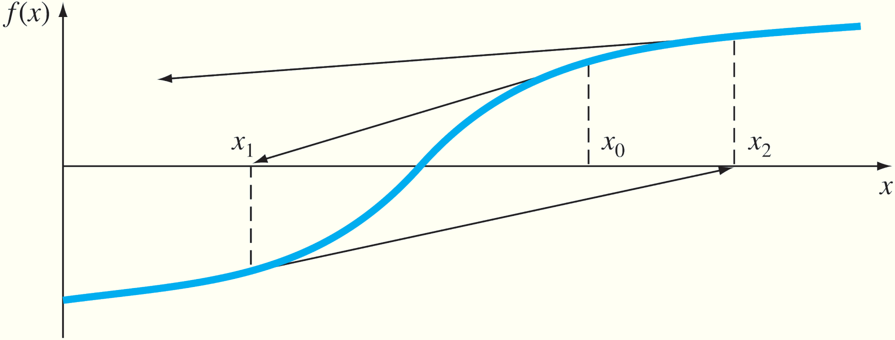
in other words $x_n - \frac{f(x_n)}{f'(x_n)}$ is the next iteration of r, and hence we write:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

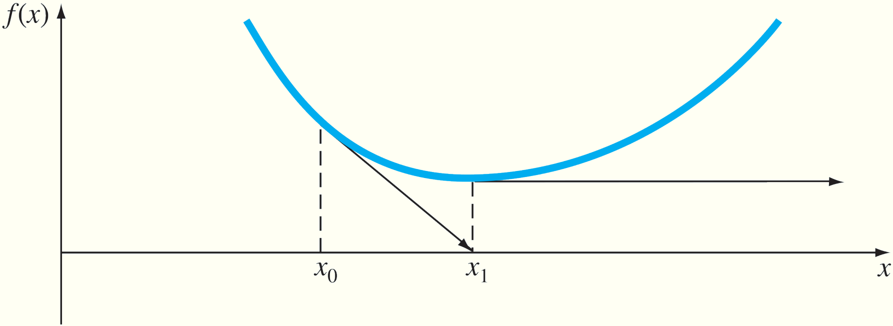
Newton-Raphson graphical expression



Newton-Raphson failure



Newton-Raphson failure



Newton-Raphson failure