

CE 311K: Taylor series and Newton Raphson

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1 Catenary vs Parabola

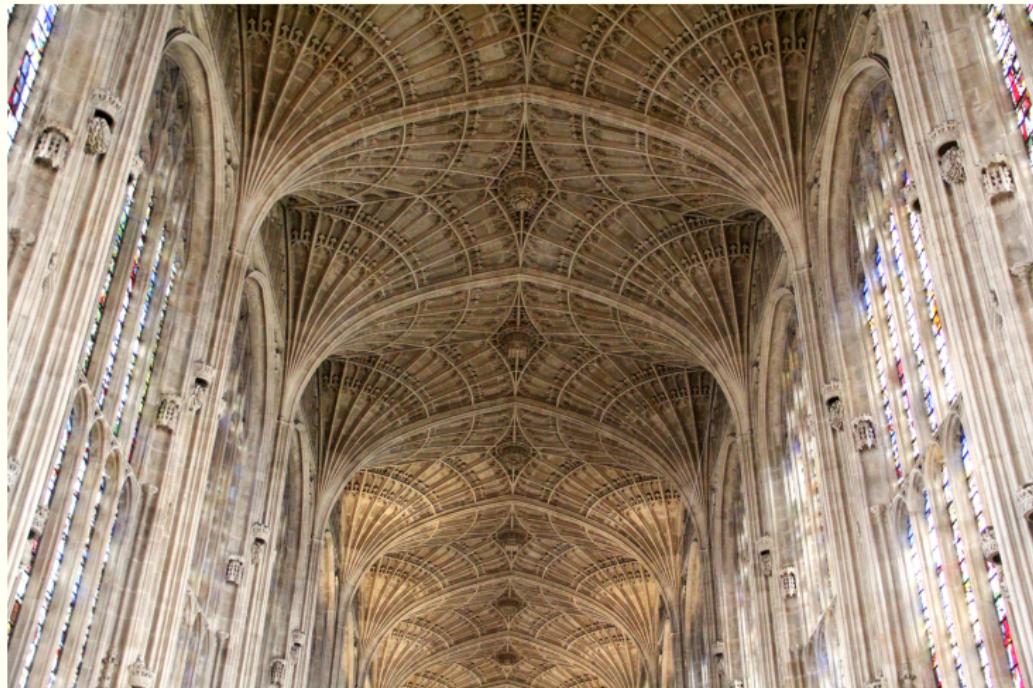
2 Taylor series

3 Newton Raphson

The fan vaults of King's college



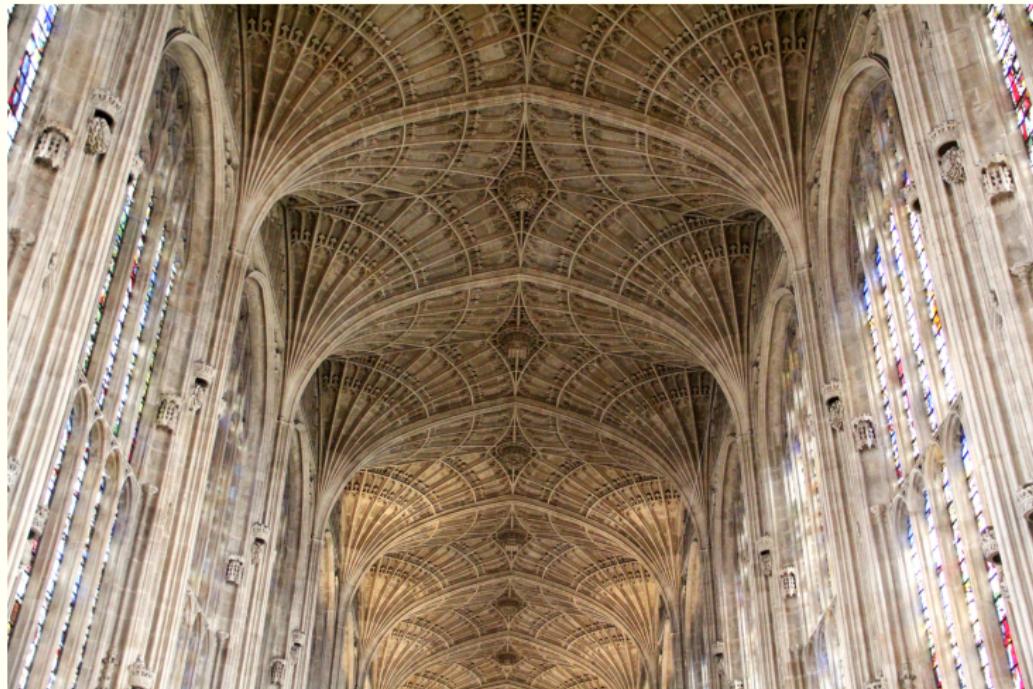
② What is the thickness of the ceiling?



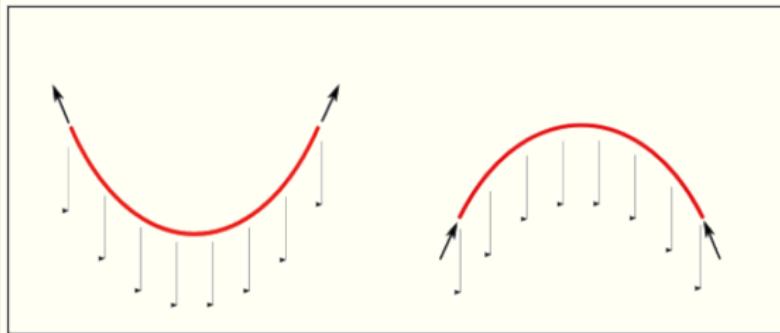
From arches to fan vaults



The fan vaults of King's college



Catenary



A [catenary curve](#) (left) and a catenary arch, also a catenary curve (right). One points up, and one points down, but the curves are the same.

Sagrada Familia



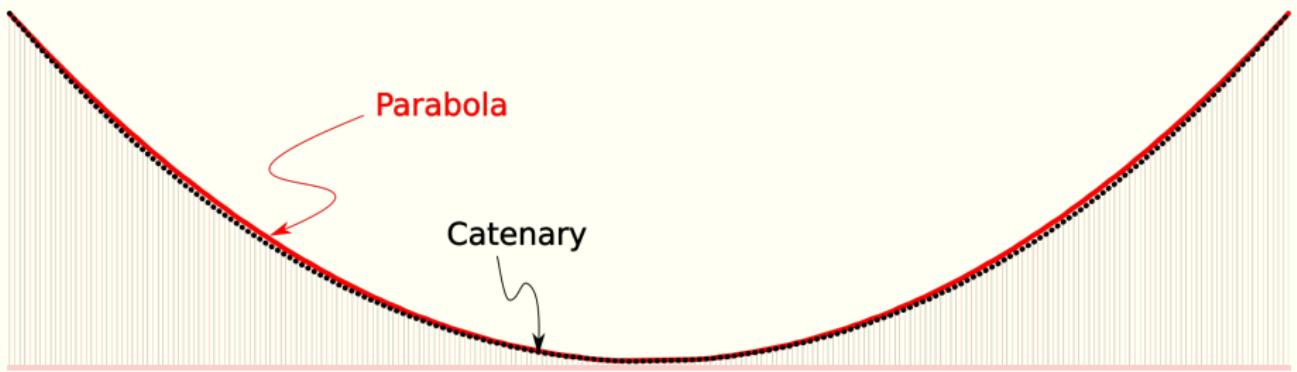
Catenary, catenary . . . everywhere



② What is this shape?

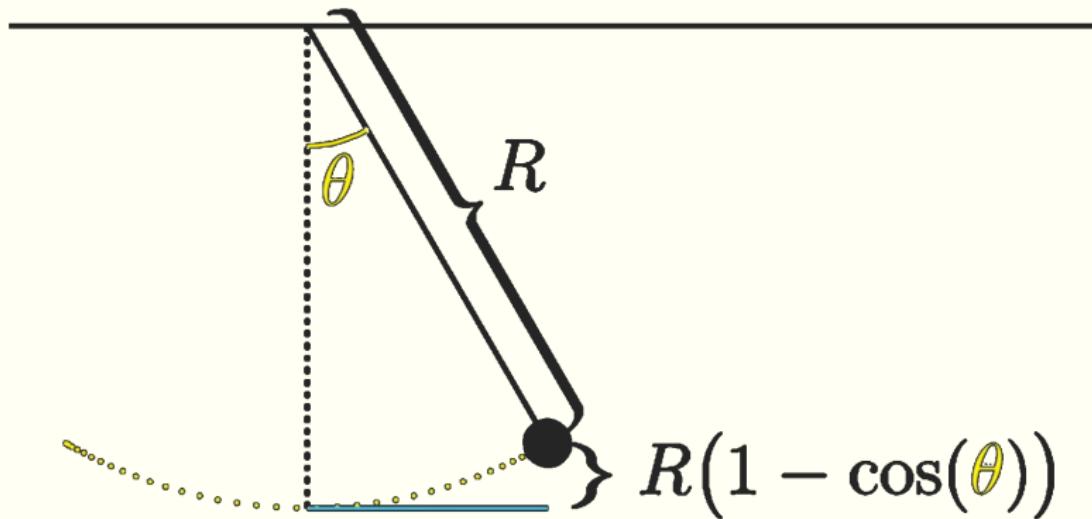


Catenary vs Parabola



Potential energy of a simple pendulum

We need to know how high the weight of the pendulum is above its lowest point



1 Catenary vs Parabola

2 Taylor series

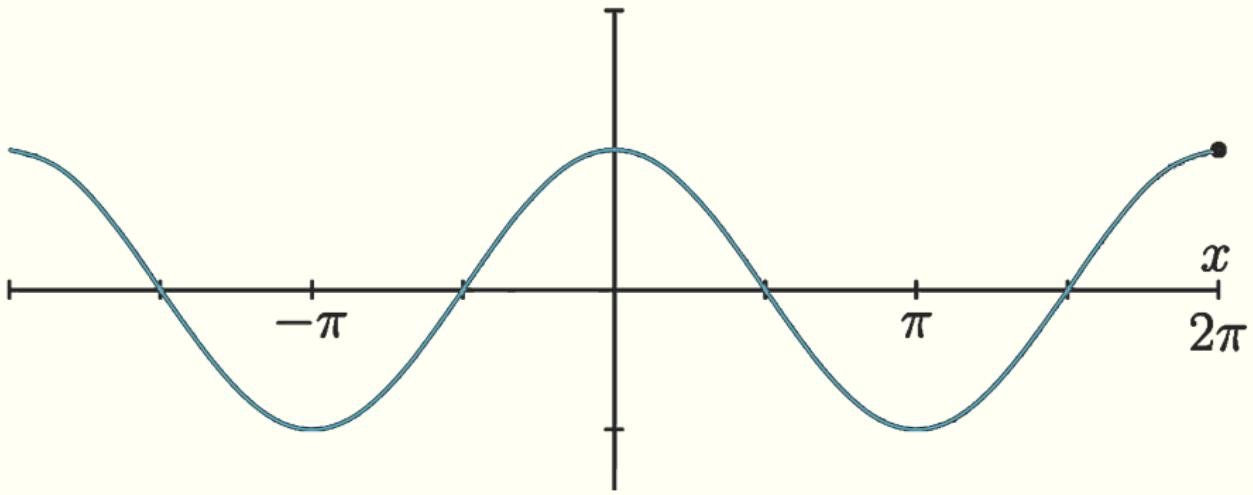
3 Newton Raphson

Taylor series

- Taylor series is one the best tools maths has to offer for approximating functions.
- Taylor series is about taking non-polynomial functions and finding polynomials that approximate at some input.
- The motive here is the polynomials tend to be much easier to deal with than other functions, they are easier to compute, take derivatives, integrate, just easier overall.
- $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

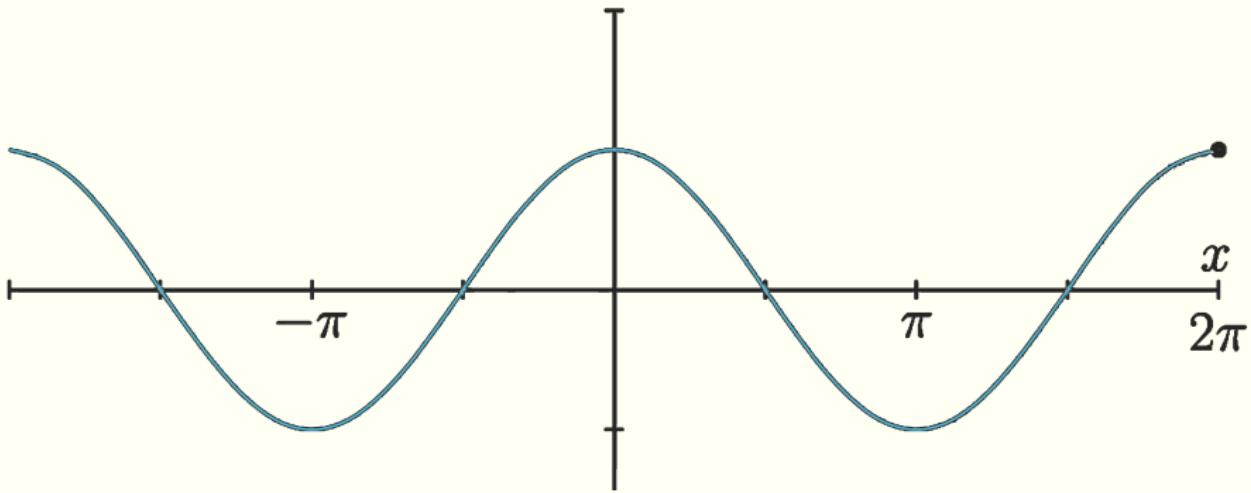
Taylor series of $\cos(x)$

$\cos(x)$



Taylor series of $\cos(x)$

$\cos(x)$



Taylor series of $\cos(x)$

$$\cos(x) \xrightarrow{x=0} 1$$

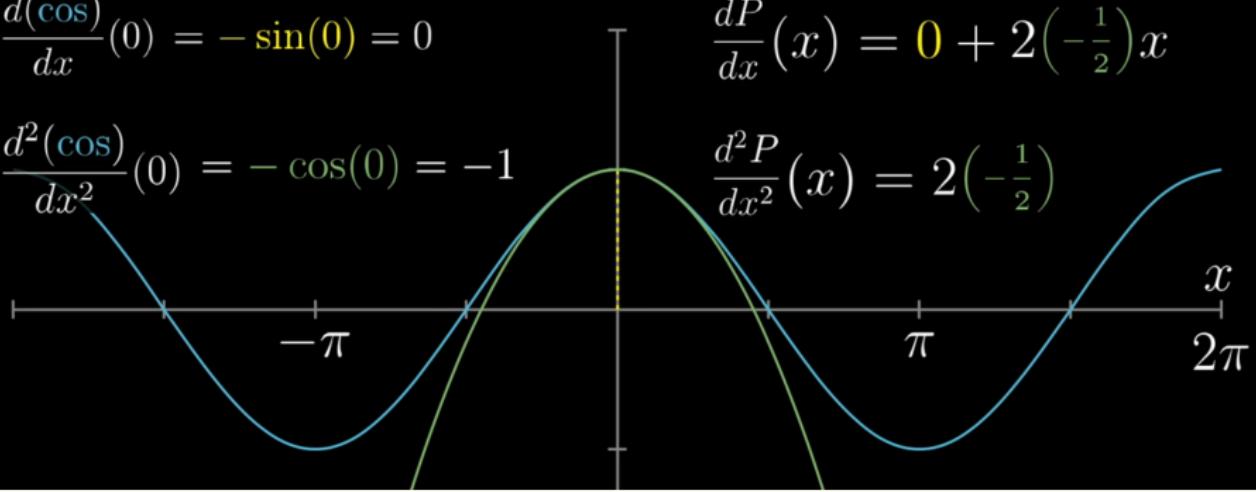
$$\frac{d(\cos)}{dx}(0) = -\sin(0) = 0$$

$$\frac{d^2(\cos)}{dx^2}(0) = -\cos(0) = -1$$

$$P(x) = 1 + 0x + \left(-\frac{1}{2}\right)x^2$$

$$\frac{dP}{dx}(x) = 0 + 2\left(-\frac{1}{2}\right)x$$

$$\frac{d^2P}{dx^2}(x) = 2\left(-\frac{1}{2}\right)$$



Taylor series of cos(x): 4th derivative

$$P(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$$

$$\begin{aligned}\frac{d^4 P}{dx^4}(x) &= 1 \cdot 2 \cdot 3 \cdot 4 \cdot \frac{1}{24} \\ &= 24 \cdot \frac{1}{24}\end{aligned}$$

To find the coefficient of n^{th} term:

$$\begin{aligned}\frac{d^8}{dx^8}(c_8 x^8) &= 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot c_8 \\ &= 8!\end{aligned}$$

$$\text{Set } c_8 = \frac{\text{desired derivative value}}{8!}$$

Taylor series of $\cos(x)$

$$\cos(0) = 1$$



$$-\sin(0) = 0$$



$$-\cos(0) = -1$$



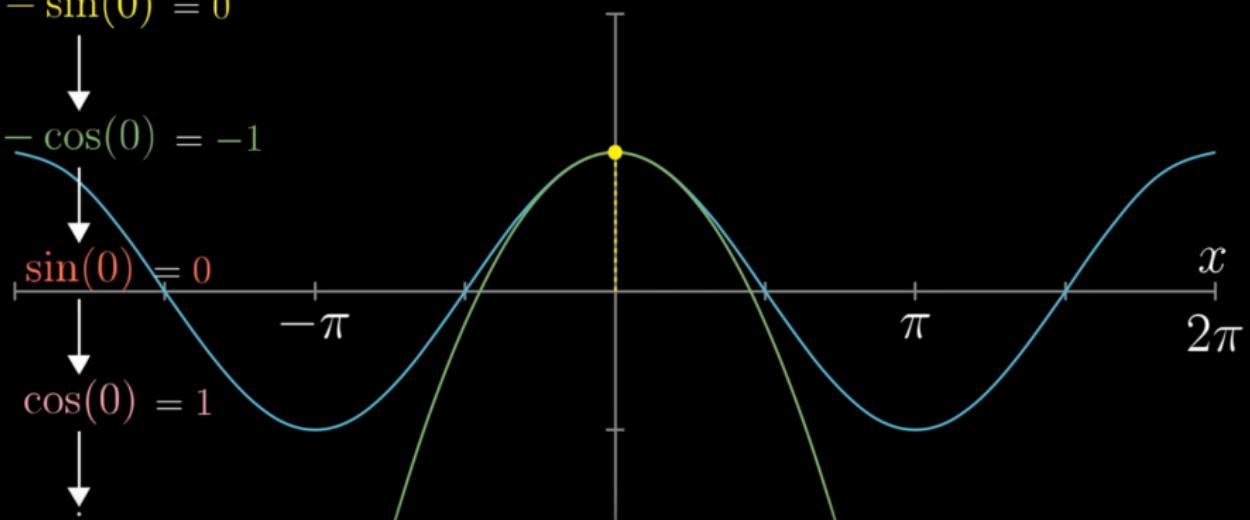
$$\sin(0) = 0$$



$$\cos(0) = 1$$



$$P(x) = 1 + 0\frac{x^1}{1!} + -1\frac{x^2}{2!} + 0\frac{x^3}{3!} + 1\frac{x^4}{4!} + \dots$$



Taylor series: Generalization

$$P(x) = 1 + \left(-\frac{1}{2}\right)x^2 + c_4x^4$$

Doesn't affect previous terms

$$\frac{d^2P}{dx^2}(0) = 2\left(-\frac{1}{2}\right) + 3 \cdot 4c_4(0)^2$$

$$P(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4$$

Controls $P(0)$

Controls $\frac{dP}{dx}(0)$

Controls $\frac{d^2P}{dx^2}(0)$

Controls $\frac{d^3P}{dx^3}(0)$

Controls $\frac{d^4P}{dx^4}(0)$

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graph TD; P0["Controls P(0)"] --> P0Term["P(0)"]; dPdx["Controls dP/dx(0)"] --> dPdxTerm["dP/dx(0)"]; d2Pdx2["Controls d^2P/dx^2(0)"] --> d2Pdx2Term["d^2P/dx^2(0)"]; d3Pdx3["Controls d^3P/dx^3(0)"] --> d3Pdx3Term["d^3P/dx^3(0)"]; d4Pdx4["Controls d^4P/dx^4(0)"] --> d4Pdx4Term["d^4P/dx^4(0)"];
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Taylor series: Generalization

$$f(0)$$

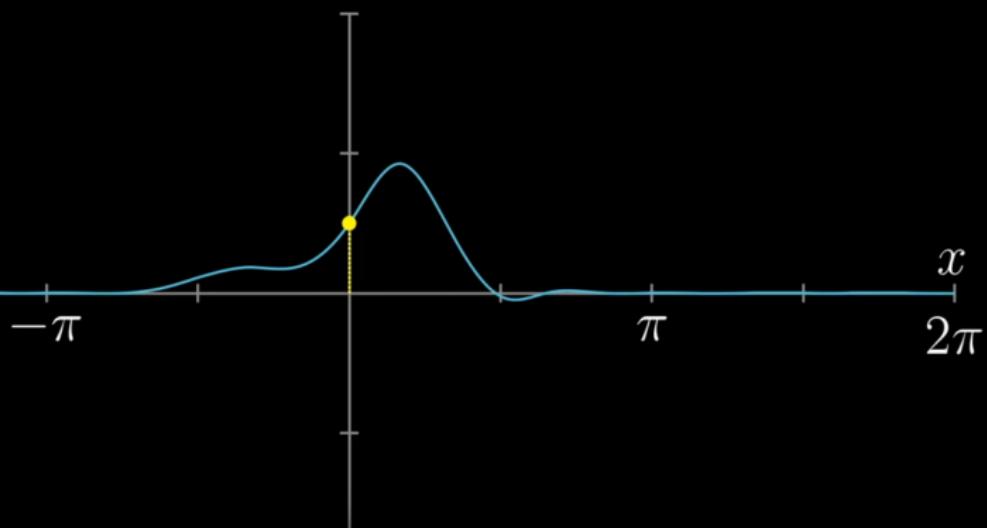
$$\frac{df}{dx}(0)$$

$$\frac{d^2f}{dx^2}(0)$$

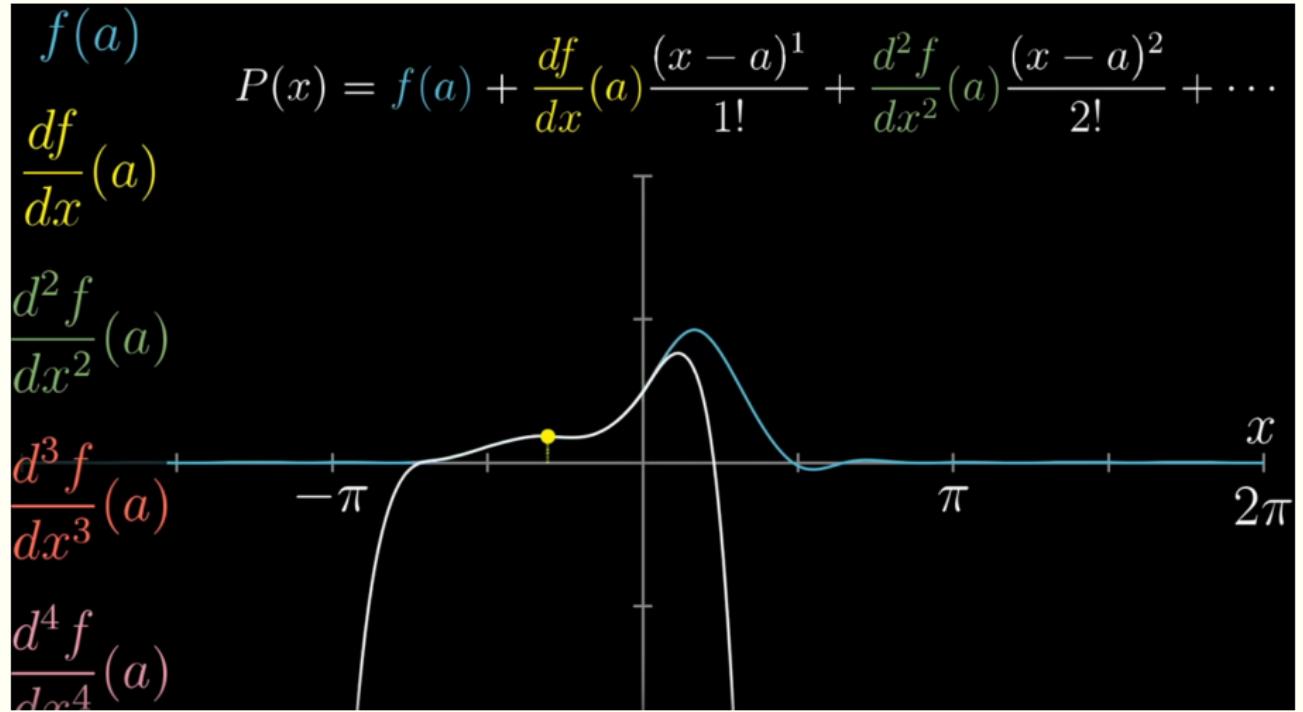
$$\frac{d^3f}{dx^3}(0) +$$

$$\frac{d^4f}{dx^4}(0)$$

$$P(x) = f(0) + \frac{df}{dx}(0) \frac{x^1}{1!} + \frac{d^2f}{dx^2}(0) \frac{x^2}{2!} + \frac{d^3f}{dx^3}(0) \frac{x^3}{3!} + \dots$$



Taylor series: Generalization



Taylor series: Two variables

The Taylor series of f expanded about $(x, y) = (a, b)$ is:

$$\begin{aligned}(x, y) = f(a, b) &+ \frac{\partial f}{\partial x}(a, b) \cdot (x - a) + \frac{\partial f}{\partial y}(a, b) \cdot (y - b) \\ &+ \frac{1}{2!} \left[\frac{\partial^2 f}{\partial x^2}(a, b) \cdot (x - a)^2 \right. \\ &\quad \left. + 2 \frac{\partial^2 f}{\partial xy}(a, b) \cdot (x - a)(y - b) + \right. \\ &\quad \left. \frac{\partial^2 f}{\partial y^2}(a, b) \cdot (y - b)^2 \right] + \dots\end{aligned}$$

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Newton Raphson

Assuming r is a root of f and that f is continuously differentiable in the vicinity of r with $f'(r) \neq 0$, then a sequence (x_n) that converges to r for $n \rightarrow \infty$ can be found using the Taylor expansion of f :

$$f(r) = f(x_n + \varepsilon_n) = f(x_n) + f'(x_n)\varepsilon_n + \frac{f''(x_n)}{2!}\varepsilon_n^2 \dots$$

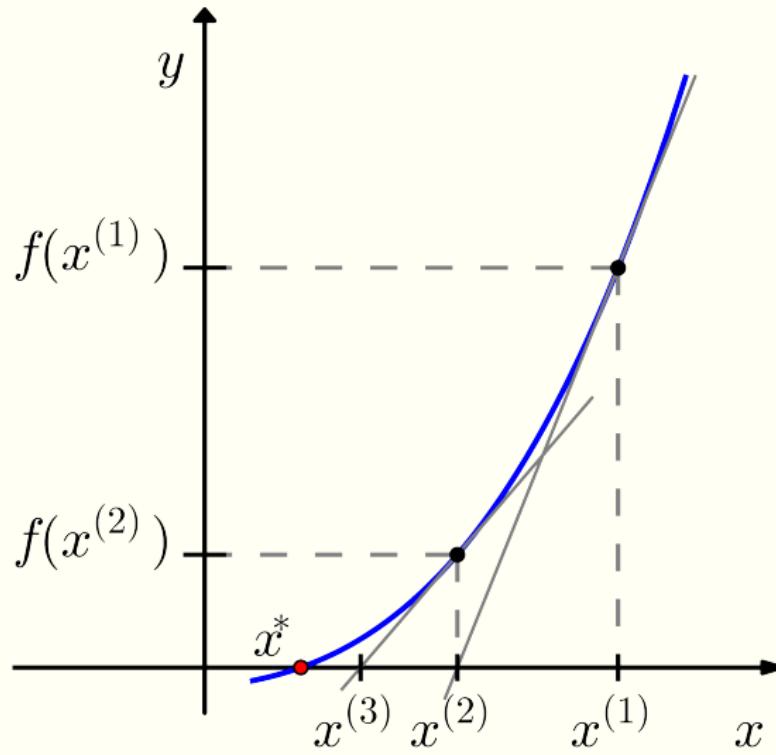
$$\varepsilon_n \approx -\frac{f(x_n)}{f'(x_n)}$$

$$r = x_n + \varepsilon_n \approx x_n - \frac{f(x_n)}{f'(x_n)}$$

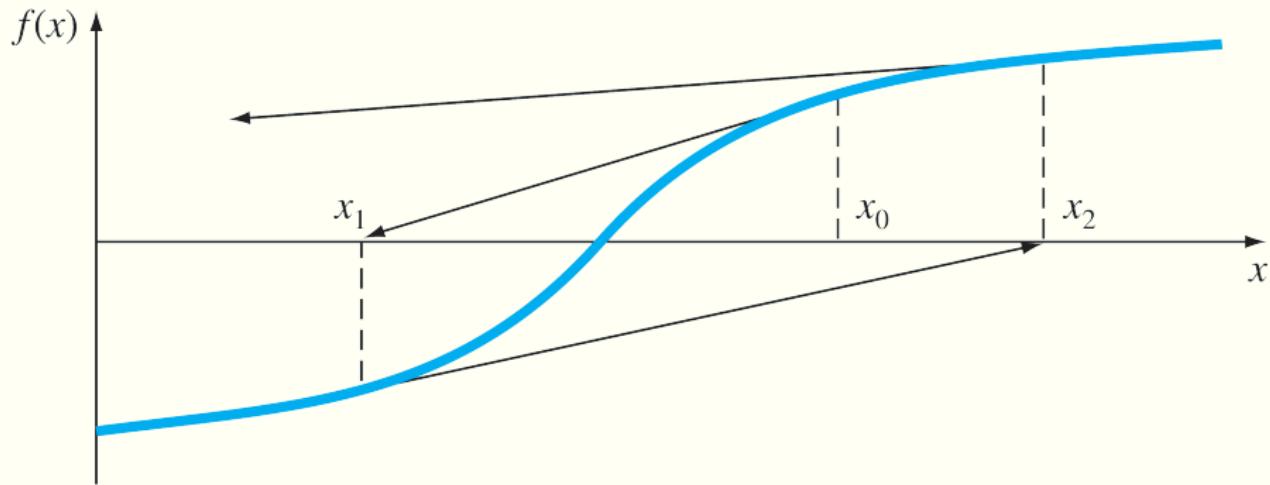
in other words $x_n - \frac{f(x_n)}{f'(x_n)}$ is the next iteration of r , and hence we write:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

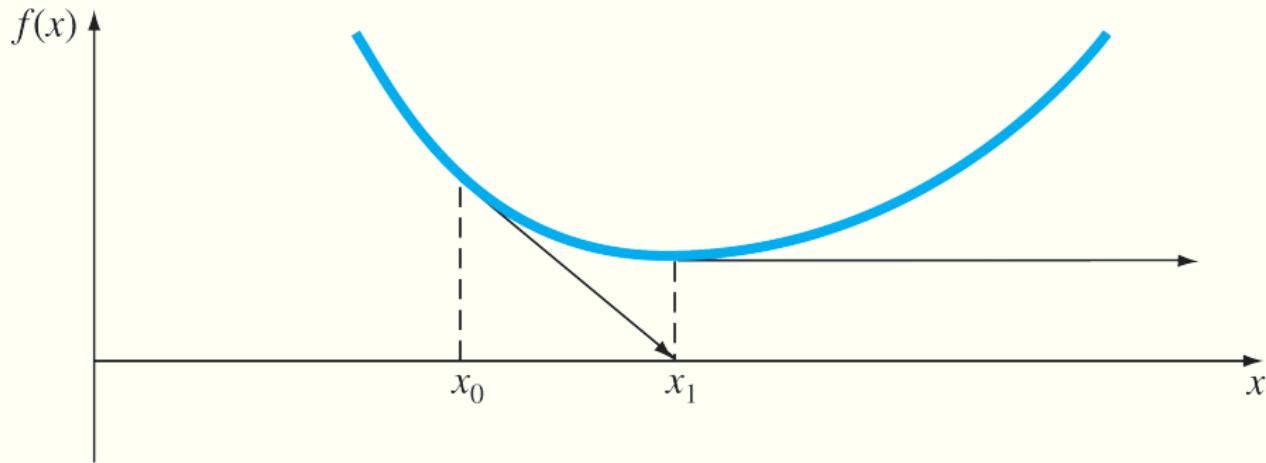
Newton-Raphson graphical expression



Newton-Raphson failure



Newton-Raphson failure



Newton-Raphson failure

