ETH zürich

A Wasserstein Subsequence Kernel for Time Series

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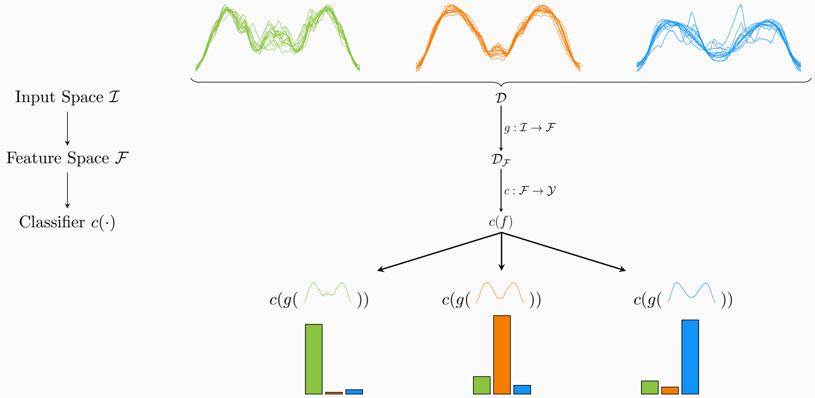
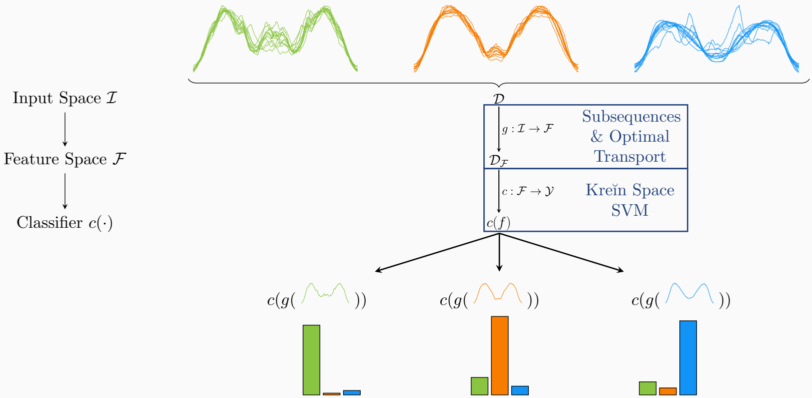
Motivation

Time Series Classification in a Nutshell



Time Series Classification in a Nutshell

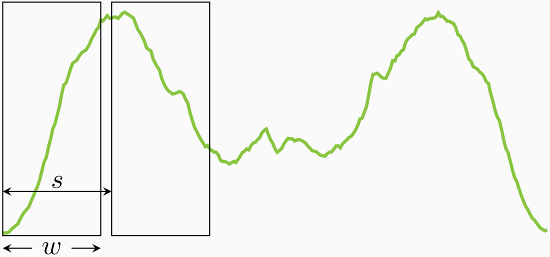


Feature Space: Subsequences and Optimal Transport

Subsequence Extraction

The Sliding Window Approach

- Window Size w
- Stride s



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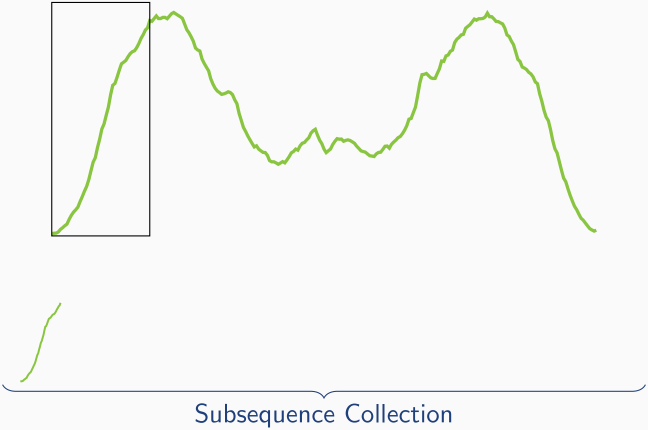

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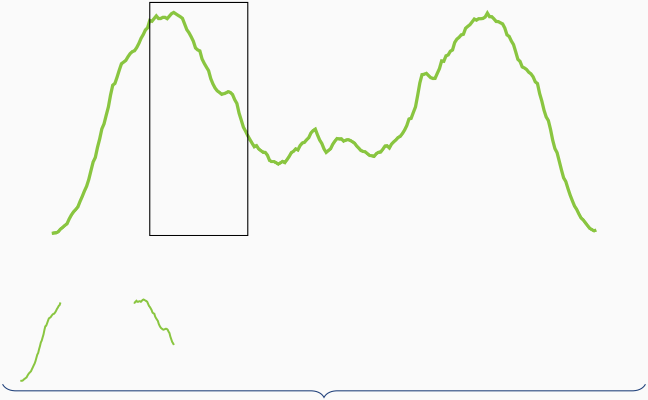

Subsequence Collection

Subsequence Extraction

The Sliding Window Approach

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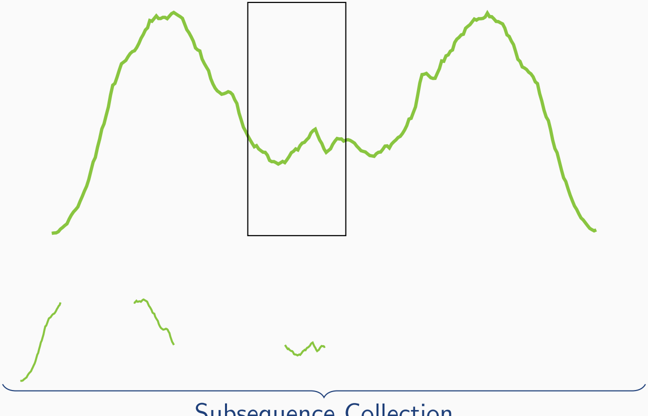

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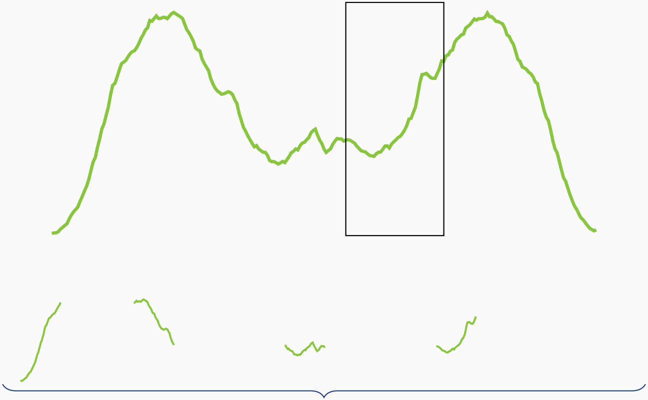

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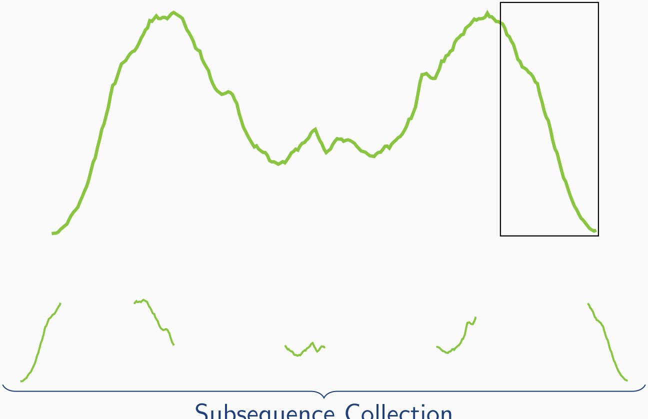
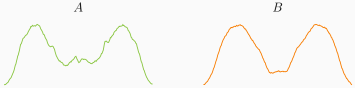
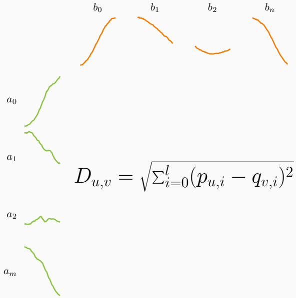


Subsequence Distance Matrix

Given Two Time Series



we Construct the Subsequence Distance Matrix D

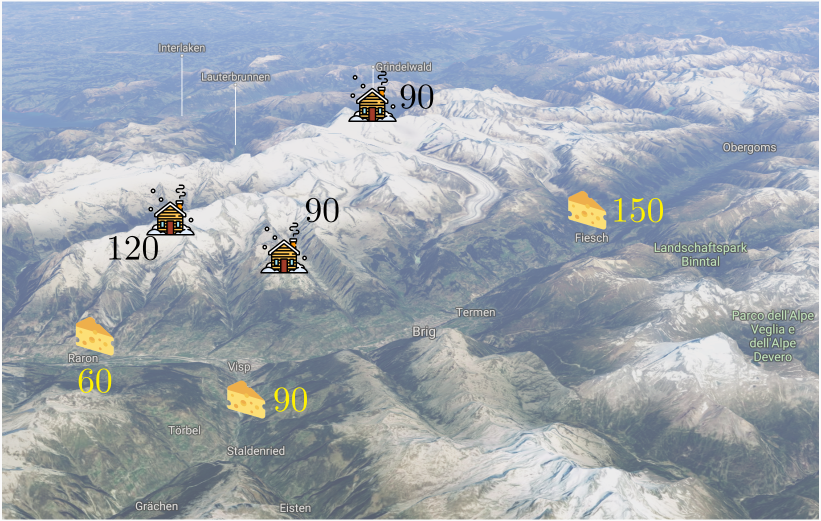


$$D_{u,v} = \sqrt{\Sigma_{i=0}^{l} (p_{u,i} - q_{v,i})^2}$$

and obtain $\quad W_1 (A, B) = \min_{P \in \Gamma(A,B)} \langle D, P \rangle_{\mathrm{F}}$

Optimal Transport and the Wasserstein Distance

The Swiss Kantorovich Problem



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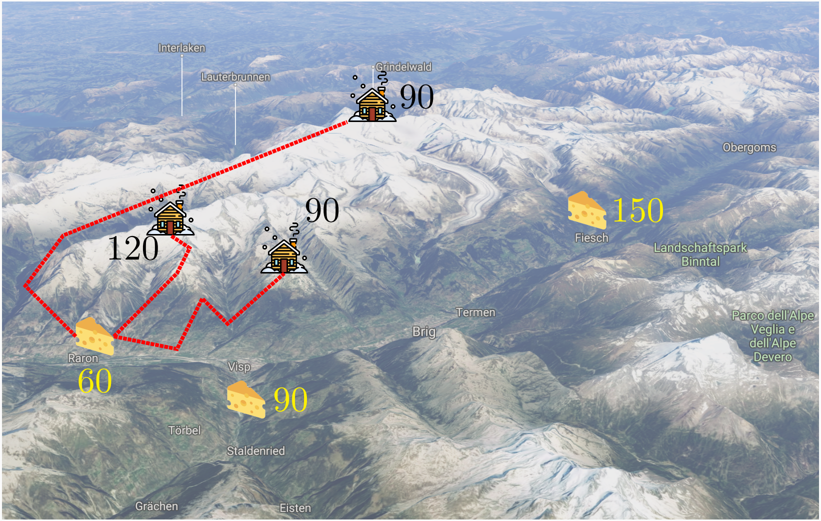

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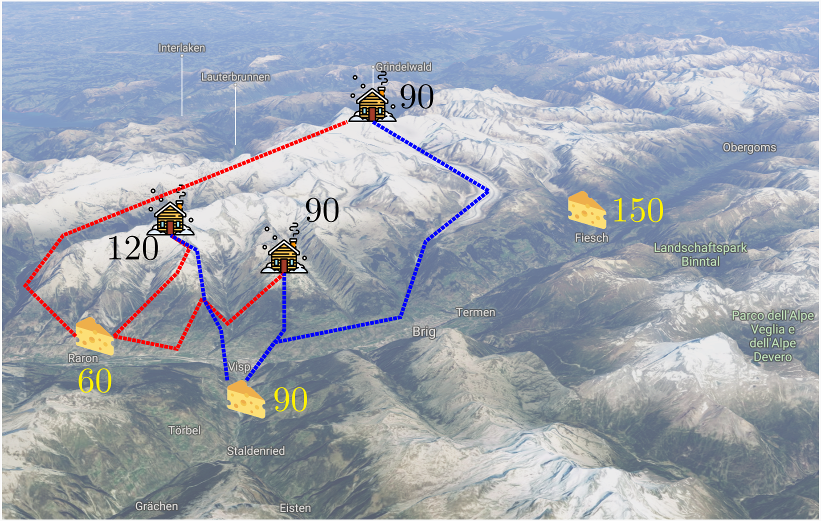

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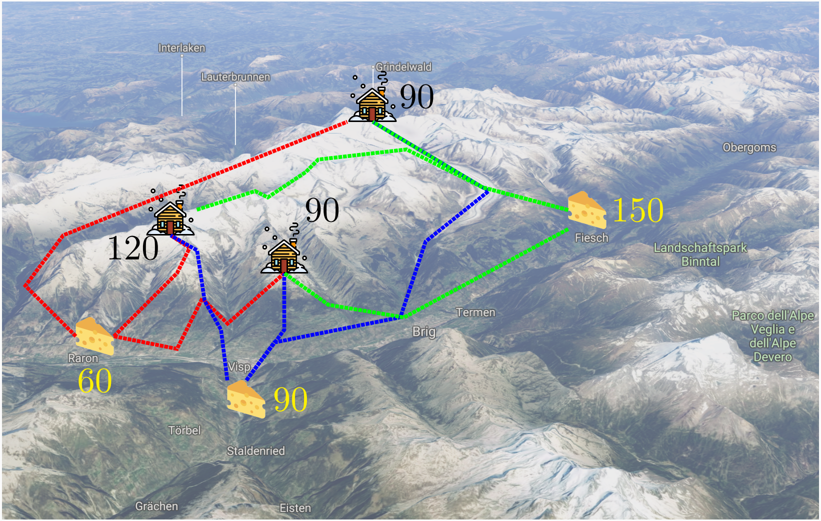
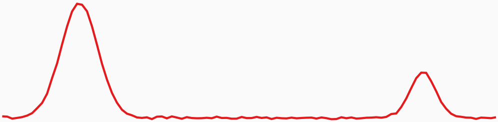


A Wasserstein Kernel for Time Series

Time Series Pairwise Distance Calculation



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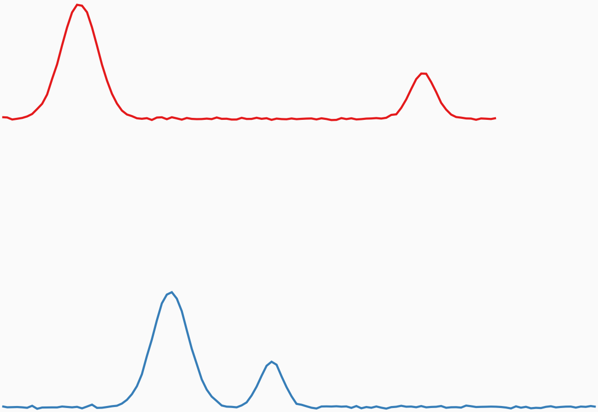

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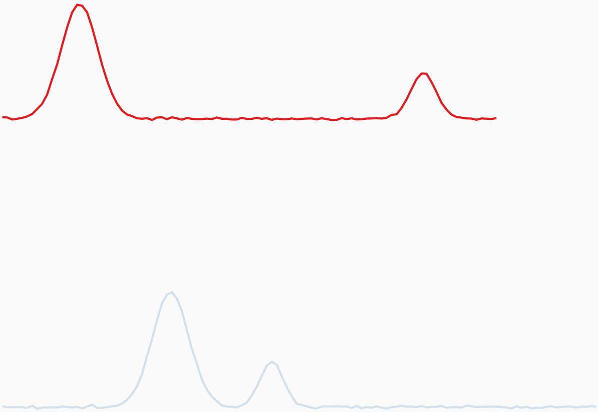

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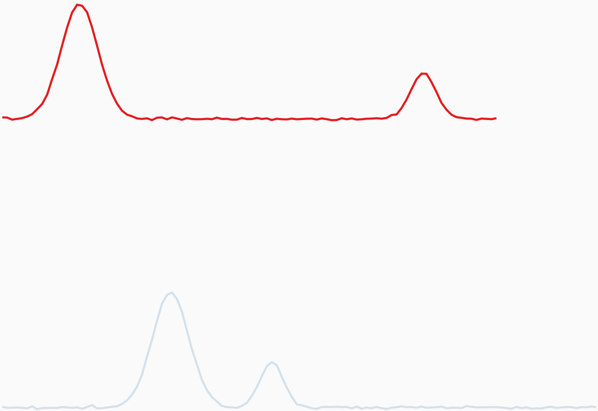

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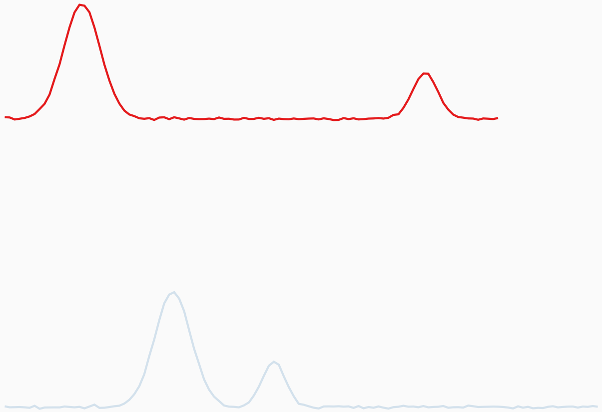


Wasserstein Distance Matrix

From Distance to Kernels

Definition (Wasserstein time series kernel)

Let T_i and T_j be two time series, and $s_{i1}, \ldots, s_{iU}$ as well as $s_{j1}, \ldots, s_{jV}$ be their respective subsequences. Moreover, let D be a $U \times V$ matrix that contains the pairwise distances of all subsequences, such that $D_{uv} := \operatorname{dist}(s_{iu}, s_{jv})$, where $\operatorname{dist}(\cdot, \cdot)$ denotes the usual Euclidean distance. The optimisation problem

$$W_1(T_i, T_j) := \min_{P \in \Gamma(T_i, T_j)} \langle D, P \rangle_{\mathrm{F}}, \tag{1}$$

yields the optimal transport cost to transform T_i into T_j by means of their subsequences. Then, given $\lambda \in_{>0}$, we can define

$$\mathrm{WTK}(T_i, T_j) := \exp\left(-\lambda W_1(T_i, T_j)\right), \tag{2}$$

which we refer to as our *Wasserstein-based subsequence kernel*;

Wasserstein Time Series Kernel

Algorithm 1 Wasserstein Time Series Kernel

Input: Time series for training and testing $\mathcal{T}_{\text{train}}$, $\mathcal{T}_{\text{test}}$; subsequence length w; kernel weight factor λ

Output: $\mathcal{K}^{\text{train}}, \mathcal{K}^{\text{test}}$

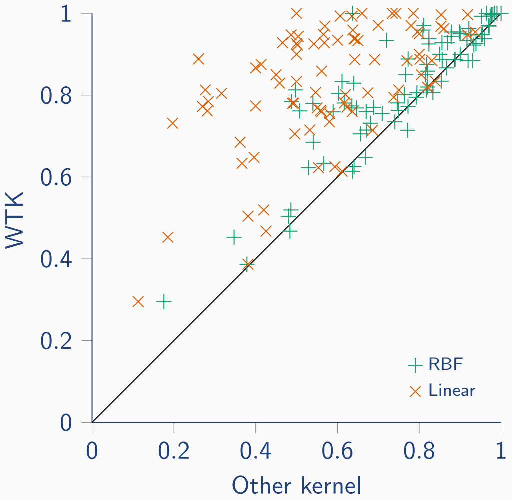
$\mathcal{S}^{\text{train}} \leftarrow \textsc{Subsequences}(\mathcal{T}_{\text{train}}, w)$ // Extract subsequences
$\mathcal{S}^{\text{test}} \leftarrow \textsc{Subsequences}(\mathcal{T}_{\text{test}}, w)$ // Extract subsequences
for $T_i \in \mathcal{T}_{\text{train}}$ **do**
    for $T_j \in \mathcal{T}_{\text{train}}$ **do**
        $\mathcal{D}_{ij}^{\text{train}} \leftarrow \mathrm{W}_1\left(\mathcal{S}_i^{\text{train}}, \mathcal{S}_j^{\text{train}}\right)$ // Wasserstein distance calculation (train)
    end for
    for $T_k \in \mathcal{T}_{\text{test}}$ **do**
        $\mathcal{D}_{ik}^{\text{test}} \leftarrow \mathrm{W}_1\left(\mathcal{S}_i^{\text{train}}, \mathcal{S}_k^{\text{test}}\right)$ // Wasserstein distance calculation (test)
    end for
end for
$\mathcal{K}^{\text{train}} \leftarrow \exp\left(-\lambda \mathcal{D}^{\text{train}}\right)$ // Kernel matrix calculation
$\mathcal{K}^{\text{test}} \leftarrow \exp\left(-\lambda \mathcal{D}^{\text{test}}\right)$ // Kernel matrix calculation
return $\mathcal{K}^{\text{train}}, \mathcal{K}^{\text{test}}$

Results

Experiments

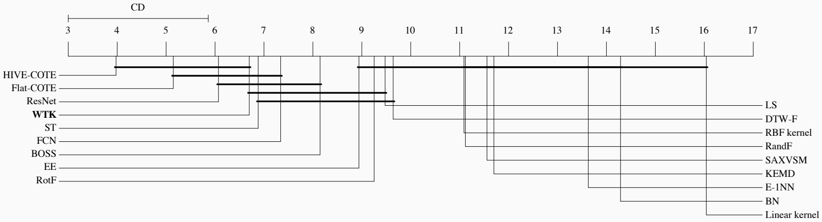
Datasets	*UCR Time Series Archive* 85 datasets predetermined train/test splits
Hyperparameters	Selected via a 5-fold cross-validation on the training set
Evaluation metric	Classification accuracy
Comparison partners	– Other kernels – DTW-1NN – State-of-the-art methods

Comparison with Other Kernels



Critical Difference Plot

The classification performances of methods sharing horizontal bars are not significantly different.



Take aways

Questions?

References