MATH 301 Homework 8

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Chapter 7) 4,6,8,12,16,20

7.4

Given an integer a, then $a^3 + 4a + 5$ is odd if and only if a is even.
If and only if proof
$a^3 + 4a + 5$ **is odd** $\rightarrow$ **a is even**
Contrapositive proof: **if a is odd** $\rightarrow a^3 + 4a + 5$ **is even**

$$\begin{aligned}
&\text{Let } a = 2k+1, k \in \mathbb{Z} \therefore \\
&a^3 + 4a + 5 = (2k+1)^2 + 4(2k+1) + 5 = \\
&4k^2 + 4k + 1 + 8k + 4 + 5 = \\
&4k^2 + 12k + 10 = \\
&2(2k^2 + 6k + 5)
\end{aligned}$$

Thus it can be shown that when a is an odd number the experession $a^3 + 4a + 5$ can be expressed as 2 times some integer of the form: $2k^2 + 6k + 5, k \in \mathbb{Z}$
a is even $\rightarrow a^3 + 4a + 5$ **is odd**

$$\begin{aligned}
&\text{Let } a = 2k, k \in \mathbb{Z} \therefore \\
&a^3 + 4a + 5 = (2k)^3 + 4(2k) + 5 = \\
&8k^3 + 8k + 5 = \\
&8k^3 + 8k + 4 + 1 = \\
&2(4k^3 + 4k + 2) + 1 \\
&\text{Let } m = 4k^3 + 4k + 2 \\
&a^3 + 4a + 5 = 2m + 1
\end{aligned}$$

Meaning when a is even $a^3 + 4a + 5$ is odd

7.6

Suppose x,y $\in \mathbb{R}$ Then $x^3 + x^2y = y^2 + xy$ if and only if $y = x^2$ or y=-x
If and only if proof
If $x^3 + x^2y = y^2 + xy \rightarrow \quad y = x^2$ **or** $y = -x$
Direct Proof:

$$\begin{aligned}
&x^3 + x^2y = y^2 + xy \\
&x^2(x+y) = y(x+y) \\
&x^2(x+y) - y(x+y) = 0 \\
&(x^2 - y)(x+y) = 0 \\
&\therefore y = -x \vee y = x^2
\end{aligned}$$

If $y = x^2$ **or** $y = -x \rightarrow \quad x^3 + x^2y = y^2 + xy$
Direct Proof:

$$\begin{aligned}
&\text{Let } y = x^2 \therefore \\
&x^3 + x^2y = x^3 + x^4 \\
&y^2 + xy = (x^2)^2 + xx^2 = x^3 + x^4 \\
&\therefore x^3 + x^2y = y^2 + xy
\end{aligned}$$

7.8

Suppose a,b $\in \mathbb{Z}$ Prove that $a \equiv b(mod 10)$ if and only if $a \equiv b(mod 2)$ and $a \equiv b(mod 5)$

7.12

There exists a positive real number x for which $x^2 < \sqrt{x}$
Existential proof: Consider the number $\dfrac{1}{2}$

$$\left(\frac{1}{2}\right)^2 < \sqrt{\frac{1}{2}}$$

$$.25 < .707107...$$

7.16

Suppose $a, b, \in \mathbb{Z}$. If ab is odd then $a^2 + b^2$ is even.
Direct Proof
If ab is odd then a and b are both odd as there does not exist two even numbers with an odd product or an odd an even number that have an odd product.Proof below:

$$\textbf{Let } a = 2k+1, k \in \mathbb{Z} \quad b = 2m+1, m \in \mathbb{Z}$$

$$\begin{aligned}&(2k+1)(2m+1) = \\ &4km + 2m + 2k + 1 = \\ &2(2km + m + k) + 1\end{aligned}$$

Therefore the product of two odd numbers is odd.

$$\begin{aligned}&a^2 + b^2 = (2k+1)^2 + (2m+1)^2 = \\ &4k^2 + 4k + 1 + 4m^2 + 4m + 1 = \\ &2(2k^2 + 2k + 2m^2 + 2m + 1)\end{aligned}$$

$a^2 + b^2$ can be expressed as 2 times some integer therefore it is even.

7.20

There exists an $n \in \mathbb{N}$ for which $11|(2^n - 1)$
Existential proof:

Figure 1: Consider the numbers of the form 10k, $k \in \mathbb{W}$(whole numbers) these are solutions to this problem, for example 10,20,30 are solutions. Answers generated using c++ computation with the following code:

```
int main() {

    for (int i = 0.0 ; i < 100; i++) {
        if (((int)pow(2,i) - 1) % 11 == 0) cout << i << endl;
    }

}
```

$$2^{40} - 1 = 1099511627776 \quad \frac{1099511627776}{11} = 99955602525$$