MATH 301 Homework 11

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Chapter 10: 2,10,22,30,32,20,42,

Problem 10.2

Prove that 1+2+3+4...+n=$\dfrac{n^2+n}{2}$ for all $n \in \mathbb{Z}^+$
Proof by induction:
Base case:n=3

$$1+2+3=\frac{9+3}{2}$$

$$6=6$$

Inductive Hypothesis:

$$P(k)=\sum_{n=1}^{k} n=\frac{k^2+k}{2}$$

Proving P(k+1)

$$P(k+1)=\sum_{n=1}^{k+1} n=\frac{(k+1)^2+k+1)}{2}$$
$$=\sum_{n=1}^{k} n+(k+1)=\frac{(k^2+3k+2)}{2}$$
$$=\frac{k^2+k}{2}+(k+1)=\frac{(k^2+3k+2)}{2}$$
$$\frac{(k^2+3k+2)}{2}=\frac{(k^2+3k+2)}{2}$$

It follows that the hypothesis holds for all k in the positive integers

Problem 10.10

Prove that $3|(5^{2n}-1)\forall n \geq 0, n \in \mathbb{Z}$
Proof by induction:
Base Case:n=2

$$5^(4) - 1 = 624 = 208(3)$$

Inductive Hypothesis

$$P(k) = 3|5^{2n} - 1 \therefore 5^{2n} - 1 = 3(x), x \in \mathbb{Z}$$

Proving P(k+1)

$$\begin{aligned}
&P(k+1) = 5^{2(k+1)} - 1 \\
&= (5^{2k+2}) - 1 \\
&= (5^{2k}25) - 1 \\
&= 25(5^2k) - 1 \\
&= 25(3x+1) - 1 \\
&= 75x + 24 \\
&= 3(25x+8)
\end{aligned}$$

Therefore it is shown that 3 divides P(k+1) which implies 3 divides P(k) for all k

Problem 10.20

Prove that $(1+2+3+..+n)^2 = 1^3 + 2^3 + 3^3 ... + n^3$
Prove that

$$\left(\sum_{i=1}^{n} i\right)^2 = \sum_{i=1}^{n} i^3$$

Recall from problem 10.2 that we proved

$$\sum_{i=1}^{n} i = \frac{n^2+n}{2}$$

So we now have to consider:

$$\left(\frac{n^2+n}{2}\right)^2 = \sum_{i=1}^{n} i^3$$

$$\frac{(n^2+n)^2}{4} = \sum_{i=1}^{n} i^3$$

$$\frac{(n^4+2n^3+n^2}{4} = \sum_{i=1}^{n} i^3$$

Poof by induction:
Base case:
let n =3:

$$\frac{(3^4+2(3^3)+3^2}{4} = \sum_{i=1}^{3} i^3$$

$$\frac{144}{4} = 1+8+27$$

$$36 = 36$$

Inductive hypothesis:

$$P(k) = \sum_{i=1}^{k} i^3 = \frac{(k^4+2k^3+k^2}{4}$$

Proving P(k+1)

$$P(k+1) = \sum_{i=1}^{k+1} i^3 = \frac{((k+1)^4+2(k+1)^3+(k+1)^2}{4}$$

$$P(k+1) = \sum_{i=1}^{k} i^3 + (k+1)^3 = \frac{(k^4+4k^3+6k^2+4k+1)+2(k^3+3k^2+3k+1)+(k^2+2k+1)}{4}$$

$$P(k+1) = \sum_{i=1}^{k} i^3 + (k+1)^3 = \frac{k^4+6k^3+13k^2+12k+4}{4}$$

$$P(k+1) = \sum_{i=1}^{k} i^3 + (k+1)^3 = \frac{k^4+6k^3+13k^2+12k+4}{4}$$

$$P(k+1) = \frac{(k^4+2k^3+k^2}{4} + (k+1)^3 = \frac{k^4+6k^3+13k^2+12k+4}{4}$$

$$P(k+1) = \frac{(k^4+2k^3+k^2+4((k+1)^3)}{4} = \frac{k^4+6k^3+13k^2+12k+4}{4}$$

$$P(k+1) = \frac{(k^4+2k^3+k^2+4(k^3+3k^2+3k+1)}{4} = \frac{k^4+6k^3+13k^2+12k+4}{4}$$

$$P(k+1) = \frac{k^4+6k^3+13k^2+12k+4}{4} = \frac{k^4+6k^3+13k^2+12k+4}{4}$$

Thus it follows by mathematical induction that P(k) is true for all natural numbers n.

Problem 10.22

If $n \in \mathbb{N}$ then :

$$\prod_{i=1}^{n} \left[1 - \frac{1}{2^i}\right] \geq \frac{1}{4} + \frac{1}{2^{n+1}}$$

Proof by induction:
Base case:
Let n =1:

$$\prod_{i=1}^{2}\left[1-\frac{1}{2^i}\right] \geq \frac{1}{4}+\frac{1}{2^{2+1}}$$
$$(1-\frac{1}{2})(1-\frac{1}{2^2}) \geq \frac{1}{4}+\frac{1}{2^3}$$
$$\frac{3}{8} \geq \frac{3}{8}$$

Inductive Hypothesis

$$P(k)=\prod_{i=1}^{k}\left[1-\frac{1}{2^i}\right] \geq \frac{1}{4}+\frac{1}{2^{k+1}}$$

Proving P(k+1)

$$P(k+1)=\prod_{i=1}^{k+1}\left[1-\frac{1}{2^i}\right] \geq \frac{1}{4}+\frac{1}{2^{k+2}}$$
$$P(k+1)=\prod_{i=1}^{k}\left[1-\frac{1}{2^i}\right](1-\frac{1}{2^{k+1}}) \geq \frac{1}{4}+\frac{1}{2^{k+2}}$$
$$\left(\frac{1}{4}+\frac{1}{2^{k+1}}\right)(1-\frac{1}{2^{k+1}}) \geq \frac{1}{4}+\frac{1}{2^{k+2}}$$
$$\frac{1}{4}-\frac{1}{4(2^{k+1})}+\frac{1}{2^{k+1}}-\frac{1}{2^{k+1}(2^{k+1})} \geq \frac{1}{4}+\frac{1}{2^{k+2}}$$
$$\frac{1}{4}-\frac{1}{2^{k+1}}\left(-\frac{1}{4)}+1-\frac{1}{(2^{k+1})}\right) \geq \frac{1}{4}+\frac{1}{2^{k+2}}$$
$$\frac{1}{4}-\frac{1}{2^{k+1}}\left(\frac{3}{4}-\frac{1}{(2^{k+1})}\right) \geq \frac{1}{4}+\frac{1}{2^{k+2}}$$

We are proving for values of k past the base case so we know that k must be greater than 1
This implies k > 1 $\therefore \frac{1}{2^{k+1}} < \frac{1}{2^2}$
Since increasing the exponent of the 2 in the denominator would
lower the value of the rational number.switching the sign and concequently the inequality:

$$
\begin{aligned}
&\frac{1}{2^{k+1}} < \frac{1}{2^2} = -\frac{1}{2^{k+1}} > -\frac{1}{4} \\
&\frac{1}{4} - \frac{1}{2^{k+1}}\left(\frac{3}{4} - \frac{1}{4}\right) \geq \frac{1}{4} + \frac{1}{2^{k+2}} \\
&\frac{1}{4} - \frac{1}{2^{k+1}}\left(\frac{1}{2}\right) \geq \frac{1}{4} + \frac{1}{2^{k+2}} \\
&\frac{1}{4} - \frac{1}{2^{k+2}} \geq \frac{1}{4} + \frac{1}{2^{k+2}}
\end{aligned}
$$

Since

Problem 10.30

Problem 10.32

Problem 10.42