MATH 301 Homework 8

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Chapter 7) 4,6,8,12,16,20

7.4

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Given an integer a, then $a^3 + 4a + 5$ is odd if and only if a is even.
If and only if proof
$a^3 + 4a + 5$ **is odd** $\rightarrow$ **a is even**
Contrapositive proof: **if a is odd** $\rightarrow a^3 + 4a + 5$ **is even**

$$\begin{aligned}
&\text{Let } a = 2k + 1, k \in \mathbb{Z} \therefore \\
&a^3 + 4a + 5 = (2k + 1)^2 + 4(2k + 1) + 5 = \\
&4k^2 + 4k + 1 + 8k + 4 + 5 = \\
&4k^2 + 12k + 10 = \\
&2(2k^2 + 6k + 5)
\end{aligned}$$

Thus it can be shown that when a is an odd number the experession $a^3 + 4a + 5$ can be expressed as 2 times some integer of the form: $2k^2 + 6k + 5, k \in \mathbb{Z}$
a is even $\rightarrow a^3 + 4a + 5$ **is odd**

$$\begin{aligned}
&\text{Let } a = 2k, k \in \mathbb{Z} \therefore \\
&a^3 + 4a + 5 = (2k)^3 + 4(2k) + 5 = \\
&8k^3 + 8k + 5 = \\
&8k^3 + 8k + 4 + 1 = \\
&2(4k^3 + 4k + 2) + 1 \\
&\text{Let } m = 4k^3 + 4k + 2 \\
&a^3 + 4a + 5 = 2m + 1
\end{aligned}$$

Meaning when a is even $a^3 + 4a + 5$ is odd.
I found this poof to be realy intuitive, this felt like a good first problem that was an extention of the direct proofs for conditionals we were doing in previous chapters

7.6

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Suppose x,y $\in \mathbb{R}$ Then $x^3 + x^2y = y^2 + xy$ if and only if $y = x^2$ or y=-x
If and only if proof
If $x^3 + x^2y = y^2 + xy \rightarrow \quad y = x^2$ **or** $y = -x$
Direct Proof:

$$
\begin{aligned}
&x^3 + x^2y = y^2 + xy \\
&x^2(x + y) = y(x + y) \\
&x^2(x + y) - y(x + y) = 0 \\
&(x^2 - y)(x + y) = 0 \\
&\therefore y = -x \vee y = x^2
\end{aligned}
$$

If $y = x^2$ **or** $y = -x \rightarrow \quad x^3 + x^2y = y^2 + xy$
Direct Proof:

$$
\begin{aligned}
&\text{Let } y = x^2 \therefore \\
&x^3 + x^2y = x^3 + x^4 \\
&y^2 + xy = (x^2)^2 + xx^2 = x^3 + x^4 \\
&\therefore x^3 + x^2y = y^2 + xy
\end{aligned}
$$

This proof was very fun becuase it got very algebraic compared to some of the other problems in this set. I really enjoyed how for the second direct proof I could prove both sides of the equation once I understood the substitution.

7.8

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Suppose a,b $\in \mathbb{Z}$ Prove that $a \equiv b (\mod 10)$ if and only if $a \equiv b (\mod 2)$ and $a \equiv b (mod 5)$

If $a \equiv b (\mod 10) \rightarrow a \equiv b (\mod 2)$ **and** $a \equiv b (\mod 5)$
If $a \equiv b (\mod 10)$ then $10|(a - b)$. This is to say that the a - b is some multiple of 10 in math:

$$a - b = 10k, k \in \mathbb{Z}$$

$$
\begin{aligned}
&a - b = 5(2m), m \in \mathbb{Z} \therefore \\
&a - b = 5(n), n = 2m \in \mathbb{Z} \therefore \\
&a \equiv b \mod 5
\end{aligned}
$$

Likewise:

$$a - b = 2(5l), l \in \mathbb{Z} \therefore$$
$$a - b = 2(p), p = 5l \in \mathbb{Z} \therefore$$
$$a \equiv b \mod 2$$
$$\blacksquare$$

If $a \equiv b(\mod 2)$ **and** $a \equiv b(\mod 5) \rightarrow a \equiv b(\mod 10)$ If $a \equiv b mod 2$ and $a \equiv b mod 5$ then:

$$2|a - b \quad \wedge \quad 5|a - b$$

If two numbers divide an integer this implies that their product does as well. Therefore $10|a - b, a \equiv b \mod 10$ $\blacksquare$
$Thiswasthefirstproblem I haveeverworkedoninvolvingthemodulooperatorandconsideringthenotionofcon$

7.12

+++
There exists a positive real number x for which $x^2 < \sqrt{x}$
Existential proof: Consider the number $\frac{1}{2}$

$$\left(\frac{1}{2}\right)^2 < \sqrt{\frac{1}{2}}$$
$$.25 < .707107...$$

This proof was very straight forward and I was able to provide an example after considering a few numbers and the byproduct of calculating their square root.

7.16

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Suppose $a, b, \in \mathbb{Z}$. If ab is odd then $a^2 + b^2$ is even.
Direct Proof
If ab is odd then a and b are both odd as there does not exist two even numbers with an odd product or an odd an even number that have an odd product.Proof below:

$$\textbf{Let } a = 2k + 1, k \in \mathbb{Z} \quad b = 2m + 1, m \in \mathbb{Z}$$

$$(2k + 1)(2m + 1) =$$
$$4km + 2m + 2k + 1 =$$
$$2(2km + m + k) + 1$$

Therefore the product of two odd numbers is odd.

$$\begin{aligned}&a^2 + b^2 = (2k+1)^2 + (2m+1)^2 = \\ &4k^2 + 4k + 1 + 4m^2 + 4m + 1 = \\ &2(2k^2 + 2k + 2m^2 + 2m + 1)\end{aligned}$$

$a^2 + b^2$ can be expressed as 2 times some integer therefore it is even.
This problem felt slightly out of place for me since it seemed solvable using a direct proof method. Please let me know if I was supposed to solve this differently.

7.20

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There exists an $n \in \mathbb{N}$ for which $11 | (2^n - 1)$
Existential proof:

Figure 1: Consider the numbers of the form 10k, $k \in \mathbb{W}$(whole numbers) these are solutions to this problem, for example 10,20,30 are solutions. Answers generated using c++ computation with the following code:

```
int main() {

    for (int i = 0.0 ; i < 100; i++) {
        if (((int)pow(2,i) - 1) % 11 == 0) cout << i << endl;
    }

}
```

$$2^{40} - 1 = 1099511627776 \quad \frac{1099511627776}{11} = 99955602525$$

This was a fun problem because many of the exestential proofs can be verified computationally. Attached is my code.