1. Let's start with a warm-up exercise for doing proofs where you need to use induction: Prove that for all $n \in \mathbb{N}$

$$1+2+2^2+2^3+\cdots+2^n \;=\; 2^{n+1}-1.$$

Re-expressing the problem statement:
Prove that $\forall n \in \mathbb{N}$

$$1+\sum_{i=1}^{n} 2^i = 2^{n+1}-1$$

Proof. Base case: n=2

$$1+2^1+2^2 = 2^3-1$$
$$7=7$$

Inductive Hypothesis:

$$P(k) = 1+\sum_{i=1}^{k} 2^i = 2^{k+1}-1$$

$$P(k+1):$$

$$1+\sum_{i=1}^{k+1} 2^i = 2^{k+2}-1$$

$$1+\sum_{i=1}^{k} 2^i + 2^{k+1} = 2^{k+2}-1$$

$$2^{k+1}-1+2^{k+1} = 2^{k+2}-1$$

$$2(2^{k+1})-1 = 2^{k+2}-1$$

$$2^{k+2}-1 = 2^{k+2}-1$$

□

2. And here is a second warm-up exercise: if x is a nonnegative real number show that $(1+x)^n - 1 \geq nx$ for $n = 0,1,2,\ldots$.
Re-expressing the problem statment:
let $x \in \mathbb{R}, x \geq 0$, Show that

$$(1+x)^n - 1 \geq nx, \forall n \in \mathbb{W}$$

Proof. Suppose $x \in \mathbb{R}, x \geq 0$

$$\begin{aligned}
&(1+x)^n - 1 \geq nx \\
&(1+x)^n \geq nx + 1 \\
&\sum_{k=0}^{n} \binom{n}{k} x^k \geq nx + 1 \\
&nx + 1 + \sum_{k=2}^{n} \binom{n}{k} x^k \geq nx + 1 \\
&\therefore (1+x)^n \geq nx + 1 \\
&\text{Which implies} \\
&(1+x)^n - 1 \geq nx
\end{aligned}$$

□

3. Show that if p is a prime number, there do not exist nonzero integers a and b such that $a^2 = pb^2$ (i.e. $\sqrt{p}$ is not a rational number).
   If p is a prime number then the only divisors of p are 1 and p
   Let a and be expressed in the prime factorization:

$$a = \prod_{i=1}^{k} P_i^{\alpha_i}$$

$$b = \prod_{i=1}^{k} P_i^{\beta_i}$$

   the original statement is then:

$$\left[\prod_{i=1}^{k} p_i^{\alpha_i}\right]^2 = p \left[\prod_{i=1}^{k} p_i^{\beta_i}\right]^2$$

$$\prod_{i=1}^{k} p_i^{2\alpha_i} = p \prod_{i=1}^{k} p_i^{2\beta_i}$$

   If $a^2 = b^2$ then they would have identical prime factorizations. Since the integer is raised to the second power this implies that for all prime factors, each has an even exponent.
   if p is not a prime factor of b then it would have an odd exponent. If p was a prime factor of b then it would form an odd exponent when combined with the corresponding prime factor since the each prime factor in b^2 has an even exponent. In either case p exists as a prime number with an odd exponent therefore it cannot be equal to a since a has a prime factorization that has eveen exponenets for each prime factor.

4. Compute the gcd of the following pairs of integers:

i) 14 and 39;

$$\begin{aligned}
39 &= 14(2)+11 \\
14 &= 11(1)+3 \\
11 &= 3(3)+2 \\
3 &= 2(1)+1 \\
2 &= 1(2)+0
\end{aligned}$$

gcd(39,14)=1

ii) 234 and 165;

$$\begin{aligned}
234 &= 165(1)+69 \\
165 &= 69(2)+27 \\
69 &= 27(2)+15 \\
27 &= 15(1)+12 \\
15 &= 12(1)+3 \\
12 &= 3(4)+0
\end{aligned}$$

gcd(234,165)=3

iii) 471 and 562.

$$\begin{aligned}
562 &= 471(1)+91 \\
471 &= 91(5)+16 \\
91 &= 16(5)+11 \\
16 &= 11(1)+5 \\
11 &= 5(2)+1 \\
5 &= 1(5)+0
\end{aligned}$$

gcd(562,471)=1

5. Let $a,b,c \in \mathbb{Z}$. Prove that if $\gcd(a,b)=1$ and $a \mid bc$ then $a|c$.

Proof. if $a|bc$ then $bc = ak, k \in \mathbb{Z}$ □

6. Let a and b be positive integers where

$$a = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} \qquad \text{and} \qquad a = p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k}$$

with $p_1,\ldots,p_k$ distinct primes and $\alpha_i, \beta_i \geq 0$ for $i = 1,\ldots,k$. Show that

$$\operatorname{lcm}(a,b) = p_1^{\max(\alpha_1,\beta_1)} p_2^{\max(\alpha_2,\beta_2)} \cdots p_k^{\max(\alpha_k,\beta_k)}.$$

Proof. Let a and b be expressed in a product form as follows:

$$a = \prod_{i=1}^{k} p_i^{\alpha_i}$$

$$b = \prod_{i=1}^{k} p_i^{\beta_i}$$

Let:

$$l = \prod_{i=1}^{k} p_i^{\max(\alpha_i,\beta_i)}$$

for l to be the least common multiple of a and b it must satisfy two conditions.

(a) $a|l$ and $b|l$ (common multiple)

(b) if there exists a multiple e such that $a|e$ and $b|e$ then $l|e$ (least common multiple)

a) We must prove that $l = ak, k \in \mathbb{Z}$ which can be done in the following way:

$$l = ak$$

$$\prod_{i=1}^{k} p_i^{\max(\alpha_i,\beta_i)} = \left[\prod_{i=1}^{k} p_i^{\alpha_i}\right] k$$

$$\prod_{i=1}^{k} p_i^{\max(\alpha_i,\beta_i)} = \left[\prod_{i=1}^{k} p_i^{\alpha_i}\right] \left[\prod_{i=1}^{k} p_i^{\max(\alpha_i,\beta_i)-\alpha_i}\right]$$

By similar reasoning it can be shown that $b|l$.

b) We now prove that if there exists another common multiple that l must be smaller.
Suppose there exists another common multiple e, this is to say, $a|e$, and $b|e$ to prove that l is the least common multiple we must show that e=$lm, m \in \mathbb{Z}$.

We will first express e in its prime factorization:

$$e = \prod_{i=1}^{k} p_i^{\delta_i}$$

□