MATH 335 Lecture 21

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New homework on cycle notation short"ish"

Transpositions

1 cycles are called Transpositions. for example:

$$\sigma(a_1, a_2)$$

simply swaps two elements. Any permutation is a product of Transpositions. This was proven last class although the notes are missing refer to the real notes posted on i learn for more details

Theorem

Any permutation in S_n is a product of transpositions. This follows from the fact that any permutation can be writeen as a projduct of disjoint cycles.

$$\sigma = (a_1 a_2 \cdots a_k)(b_1 b_2 \cdots b_l) \cdots (c_1 c_2 \cdots c_m)$$

Cycle notation is generally cylical where the elements contained in the parentheses are moved to the right one index cycling over the parentheses at the boundaries.

Each cycle can be decomposed into a product of 2 cycles, for example:

$$\sigma = (a_1 a_2 \cdots a_k) = (a_1 a_k)(a_1 a_{k-1})(a_1 a_{k-2}) \cdots (a_1 a_3)(a_1 a_2)$$

Example:

$$\sigma = (1572)(364)(89), \quad \sigma \in S_9$$

$$\sigma = (12)(17)(15)(34)(36)(89)$$

decomposed over each element pair from left to right.

Lemma

If the identity element in S_n (the identity permutation)can be written as a product of transpositions then the number of transpositions in that product has to be even.

$$e = (ij)(ij)$$
$$e = (ij)(ij)(kl)(kl)$$

it is not alwways so cue, but the number of transpositions will be even.

Theorem:Parity preserving property of cycles

If you take a permutation inS_n and you express it as a product of transpositions, and then express it in a different way you will retain the parity

Analagously this can be understood as if an element of S_n can be written as a product of transpositions any other representation will have the same parity as the original expression .

Proof. Let a permutation be expressed as follows:

$$\sigma = \tau_1 \tau_2 \cdots \tau_k, \quad \sigma \in S_n$$

Where k is even and τ_i is a transpositions
Suppose that there exists an alternate epxression of sigma as follows:

$$\sigma = \mu_1 \mu_2 \cdots \mu_l$$

Where μ_i is a transpositions
We will now demonstrate that l must also be even.
For any transpositions the inverse is itself:

$$(ij)^{-1} = (ij)$$

since $e = (ij)(ij)$
all transpositions have order 2

$$\begin{aligned}
&\tau_1 \tau_2 \cdots \tau_k = \mu_1 \mu_2 \cdots \mu_l \\
&\tau_1 \tau_2 \cdots \tau_k (\mu_l \mu_{l-1} \cdots \mu_1 \mu_2) = (\mu_1 \mu_2 \cdots \mu_l) \mu_l \mu_{l-1} \cdots \mu_1 = e
\end{aligned}$$

Then by our lemma k+l must be even and since we assume that k is even this implies that l must be even as well.

□

Definition: Even permutation

A permutation in s_n is called even if it is a product of an even number of transpositions

All transpositions are odd
A k cycle is even if k is odd and a k cycle is odd if k is even
A k cycle can be expressed the product as k-1 transpositions

Definition: Set of all even permutations

The set of all even permutations in S_n is denoted by A_n.

Clearly we conclude that $A_n \subset S_n$ the question remains is A_n a subgroup of S_n

This subgroup is called the alternating group.
Example:

$$S_3, A_3 = \{e, \tau, \tau^2\} = \{e, (123), (132)\} = < \tau >$$

The odd permutations do not form a subgroup since the identity is not present

Proof. **Closure**
σ, τ_n the product of two even transpositions is a product of an even number of transpostions thus we know that thier composition is an element of A_n
Existence of inverses
for sigma the inverse is simply the transpostions written in reverse order and since this is even we are still in A_n

□

What happens for S_1

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Corollary

$$[S_n : A_n] = \frac{|S_n|}{|A_n|} = \frac{n!}{\frac{n!}{2}} = 2$$

The two distinct left cosets are A_n itself and composition with a single fixed transposition which generates all of O_n

A subgroup of a group where the left cosets are equal to the rght subgroups are called normal subgroups. Groups that do not contain any normal subgroups are very important those groups are referred to as simple groups. Any group can be built from simple groups. They are analgous to atoms. When n is greater than 5, A_n is a simple group.
Kernals are normal subgroups.