MATH 335 Homework 3

Chris Camano: ccamano@sfsu.edu

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1. Let $A = \{a,b,c\}$ and $B = \{1,2,3,4\}$.

   a) Determine $A \times B$.

   $AB = \{(a,1),(a,2),(a,3),(a,4),(b,1),(b,2),(b,3),(b,4),(c,1),(b,2),(b,3),(b,4)\}$

   b) Draw $B \times \mathbb{R}$.

   c) Prove that if C and D are finite sets then $|C \times D| = |C| \cdot |D|$.

   Proof. By the definition of the cartesian product for any fixed $c_o \in C$ there exist$|D|$ corresponding ordered pairs of the form $(c_0,d), d \in D$.

   Since there are $|C|$ possible choies for c_0 and $|D|$ corresponding ordered pairs of the form (c_0,d) then this implies $\exists \quad |C| \cdot |D|$ unique ordered pairs, thus $|C \times D = |C||D|$ as the cartesian product is the set of all unique ordered pairs between two sets. □

2. Determine which one of the following are equivalence relations. Justify.

   a) $x \sim y$ in $\mathbb{R}$ if $x \geq y$.

   (a) **Reflexivity**

   Proof. Prove: $x \sim x$
   if $x \sim x$ then:

   $$x \geq x \rightarrow x = x$$

   So we have shown the relation satisfies Reflexivity. □

   (b) **Symmetry**

   Proof. Prove $x \sim y \rightarrow y \sim x$
   if $x \sim y$ then $x \geq y$ from this inequality alone we cannot show that $y > x$ however it could be the case that $x = y$ in which case the statement $y \geq x$ would hold true staisfying symmetry.

However, we cannot conclude if it is the case that $y = x$ given $x \geq y$ and thus we cannot prove the symmetritricity of the relation.

$\square$

(c) **Transitivity**

Proof. Prove if $x \sim y$ and $y \sim z$ then $x \sim z$
if $x \sim y$ then $x \geq y$, if $y \sim z$ then $y \geq z$ then:

$$x \geq y \geq z$$

$$x \geq z$$

Therefore $x \sim z$ $\square$

This relation is not an equivilance relation .

b) $m \sim n$ in $\mathbb{Z}$ if $mn > 0$.

(a) **Reflexivity**

Proof. Prove: $m \sim m$ if $m \sim m$ then this implies

$$m^2 > 0$$

Consider the case of m =0, then this relation implies:

$$0 > 0$$

Which is not true, thus since Reflexivity fails for this relation this implies that the relation is not an equivilance relation. $\square$

(b) **Symmetry**

Proof. Prove $x \sim y \to y \sim x$
If $x \sim y$ then :

$$xy > 0$$

by commutatitivty of integer multiplication this also gives:

$$yx > 0$$

and thus we have that $y \sim x$ so y is related to x $\square$

This relation is not an equivilance relation .

c) $x \sim y$ in $\mathbb{R}$ if $|x - y| \leq 4$.

(a) **Reflexivity**

Proof. Prove: $x \sim x$
if $x \sim x$ then this implies :

$$|x - x| \leq 4$$
$$|0| \leq 4$$
$$0 \leq 4$$

$\square$

(b) **Symmetry**

Proof. Prove $x \sim y \rightarrow y \sim x$
if $x \sim y$ then :

$$|x-y| \leq 4$$

if $y \sim x$ then :

$$|y-x| \leq 4$$

For all x, y in the real numbers $|x-y|$ =$|y-x|$ therefore by this reason y is symmetric to x. □

(c) **Transitivity**

Proof. Prove if $x \sim y$ and $y \sim z$ then $x \sim z$
Consider the following hypothetical: given x,y,z $\in \mathbb{R}$ with x=4 y= 8, z= 12
Then if $x \sim y$ this implies:

$$|4-8| \leq 4$$
$$4 \leq 4$$

$$|8-12| \leq \text{“}4\text{“}$$
$$4 \leq 4$$

$$|4-12| \leq 4$$
$$8 \leq 4$$

so we have shown that the relation does not satisfy Transitivity. □

This relation is not an equivilance relation .

3. Define the relation $(x_1, y_1) \sim (x_2, y_2)$ in $\mathbb{R}^2$ if $x_1^2 + y_1^2 = x_2^2 + y_2^2$. Show that this is an equivalence relation. Then describe the equivalence classes.

(a) **Reflexivity**

Proof. Prove: $x \sim x$
if $x \sim x$ then we have that:

$$x_1^2 + y_1^2 = x_1^2 + y_1^2$$

Which is true. □

(b) **Symmetry**

Proof. Prove $x \sim y \rightarrow y \sim x$ if $x \sim y$ then we have that:

$$x_1^2 + y_1^2 = x_2^2 + y_2^2$$

which by the symmetric property of equality proves that $y \sim x$ □

(c) **Transitivity**

Proof. Prove if $x \sim y$ and $y \sim z$ then $x \sim z$
if $x \sim y$ then

$$x_1^2 + y_1^2 = x_2^2 + y_2^2$$

if $y \sim z$ then we have:

$$x_2^2 + y_2^2 = x_3^2 + y_3^2$$

performing a substitution gives:

$$x_1^2 + y_1^2 = x_3^2 + y_3^2$$

and thus $x \sim z$ □

This allows us to conclude that this relation is an equivilance realtion. The equivilance classes of this relation take the following structure:

$$[a = (x_1, y_1)] = \{b \in \mathbb{R} : a \sim b\} = \{(x_2, y_2) \in \mathbb{R}^2 : (x_1, y_1) \sim (x_2, y_2)\}$$

$$[(x_1, y_1)] = \{(x_2, y_2) \in \mathbb{R}^2 : (x_1, y_1) \sim (x_2, y_2)\}$$

so for some fixed ordered pair in $\mathbb{R}^2$ the equivilance class is the set of all other ordered pairs satisfying the relation. These relations consitute circles formed by the points related centered at (x_1, y_1)or rather (x_i, y_i) for some arbitrary location in $\mathbb{R}^2$

In other words each equivilance class is a circle in $\mathbb{R}^2$ centered at the point you are drawing a relation with with all elements of the equivilance class defining the perimeter of the circle.

4. Define a relation on $\mathbb{R}^2 \setminus \{(0,0)\}$ by letting $(x_1, y_1) \sim (x_2, y_2)$ if there exists a nonzero λ such that $(x_1, y_1) = (\lambda x_2, \lambda y_2)$. Prove that $\sim$ defines an equivalence relation on $\mathbb{R}^2 \setminus \{(0,0)\}$. What are the equivalence classes ?

   (a) **Reflexivity**

   Proof. Prove: $x \sim x$
   If $x \sim x$ then we have the following relation:

   $$(x_1, y_1) = \lambda(x_2, y_2)$$

   which is true for the case of λ=1 proving Reflexivity. □

   (b) **Symmetry**

   Proof. Prove $x \sim y \rightarrow y \sim x$
   If $x \sim y$ then:

   $$(x_1, y_1) = \lambda_1(x_2, y_2)$$

   Solving for (x_2, y_2) we obtain:

   $$\frac{1}{\lambda_1}(x_1, y_1) = (x_2, y_2)$$

   So we have shown that for $\lambda_2 = \frac{1}{\lambda_1}$ that $y \sim x$ and thus have proven symmetritricity. □

   (c) **Transitivity**

Proof. Prove if $x \sim y$ and $y \sim z$ then $x \sim z$
If $x \sim y$ then :

$$(x_1, y_1) = \lambda_1 (x_2, y_{2)}$$

If $y \sim z$ then:

$$(x_2, y_2) = \lambda_2 (x_3, y_3)$$

substituting for (x_2, y_2):

$$(x_1, y_1) = \lambda_1 \lambda_2 (x_3, y_3)$$

so we have that $(x_1, y_1) \sim (x_3, y_3)$ for $\lambda_3 = \lambda_1 \lambda_2$ Proving Transitivity. □

The equivilance classes are the set of all scalar multiples of the vector formed by the original selection of (x_1, y_1). This is to say that each equivilance class is a line in $\mathbb{R}$ where the elements on the line represent the span of the original vector chosen.