MATH 335 Homework 3

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1. Deterimine which one of the following sets is a group under addition: Addition is a binary operator by the proof provided in class, therefore we will validate the properties of each group accepting that addition is binary operator in these contexts.

   only 1 a is a group associativity fails.

   a) the set of rational numbers in lowest terms whose denominators are odd

   Proof. Let G be the following set:

$$G = \left\{ \frac{m}{n}, n = 2k+1, n \neq 0, k, m \in \mathbb{Z} \right\}$$

   (a) Proof of associativity over operator let a,b,c∈G where:

$$a = \frac{a_m}{a_n} \quad b = \frac{b_m}{b_n} \quad c = \frac{c_m}{c_n}$$

$$\left(\frac{a_m}{a_n} + \frac{b_m}{b_n}\right) + \frac{c_m}{c_n} = \frac{a_m}{a_n} + \left(\frac{b_m}{b_n} + \frac{c_m}{c_n}\right)$$

$$\frac{a_m b_n + b_m a_n}{a_n b_n} + \frac{c_m}{c_n} = \frac{a_m}{a_n} + \left(\frac{b_m c_n + c_m b_n}{b_n c_n}\right)$$

$$\frac{c_n(a_m b_n + b_m a_n) + c_m a_n b_n}{a_n b_n c_n} = \frac{a_n(b_m c_n + c_m b_n) + a_m b_n c_n}{a_n b_n c_n}$$

$$\frac{c_n a_m b_n + c_n b_m a_n + c_m a_n b_n}{a_n b_n c_n} = \frac{c_n a_m b_n + c_n b_m a_n + c_m a_n b_n}{a_n b_n c_n}$$

   The denominator $a_n b_n c_n$ is the product of three odd numbers which is also odd preserving the construction of G.

   (b) Proof of existence of identity element
   The Identity element for this set is the element: $\frac{0}{1} = 0$

(c) Proof of existence of inverse of operator
For all g$\in G$ the additive inverse is the element $-g$

□

b) the set of rational numbers in lowest terms whose denominators are even

Proof. This set is not closed under the binary operator, consider the following example:

$$a = \frac{1}{6} \quad b = \frac{1}{6}$$

under addition this yields:

$$\frac{1}{6} + \frac{1}{6} = \frac{1}{3} \notin G$$

Thus since the binary operator fails under closure, this set and operation do **not** form a group □

c) the set of rational numbers of absolute value < 1
Let G be the following set:

$$G = \left\{ \frac{m}{n}, n \neq 0, n, m \in \mathbb{Z}, |\frac{m}{n}| < 1 \right\}$$

Proof. The binary operator of addition fails for this set when dealing with elements whose sum is greater than one. There are infinite examples of this, let us pick an apparent one:

$$a = \frac{1}{2} \quad b = \frac{1}{2}$$

$$a + b = \frac{1}{2} + \frac{1}{2} = 1 \notin G$$

Due to the fact that the binary operator fails for closure this operator and set G do **not** form a group □

d) the set of rational numbers of absolute value ≥ 1 together with 0.
Let G be the following set:

$$G = \left\{ \frac{m}{n}, n \neq 0, k, m \in \mathbb{Z}, |\frac{m}{n}| > 1 \right\} \cup \{0\}$$

Proof. Due to the construction of G taking the absolute value over each of its elements this opens weaknesses in the closure of the binary operator. We can easily pick negative values less than -1 and show that under addition with a positive element that we can arrive at a value less than 1 when the difference between our selection of a and b is satisfactory:

$$a = \frac{-5}{3} \quad b = \frac{3}{2}$$

$$a + b = \frac{-5}{3} + \frac{3}{2} = \frac{-1}{6} \notin G$$

Thus, again, since we fail under closure of the binary operator the set G does **not** form a group under the operator of addition □

2. Let $G = \{x \in \mathbb{R} : 0 \leq x < 1\}$ and for $x, y \in G$ let $x \cdot y$ be the fractional part of $x + y$ (i.e. $x \cdot y = x + y - [x + y]$ where $[a]$ is the greatest integer less than or equal to a). Prove that $\cdot$ is a binary operation on G and that G is a group.

   Proof. Proof that $\cdot$ is a binary operator

   $\square$

   Proof. Proof that G is a group

   (a) Proof of associativity over operator
   (b) Proof of existence of identity element
   (c) Proof of existence of inverse of operator

   $\square$

3. Let $G = \{z \in \mathbb{C} : z^n = 1 \text{ for some nonnegative integer } n\}$. Prove that G is a group under multiplication (called the groups of *roots of unity* in $\mathbb{C}$).

   Proof. (a) Proof of Closure under operator: let :

   $$a = z_1^a = 1 \quad b = z_2^b = 1$$

   Take $z_{1,2} = z_1 z_2$ and $n_{1,2} = ab$

   $$z_{1,2}^{ab} = (z_1 z_2)^{ab} = z_1^{ab} z_2^{ab} = (z_1^a)^b (z_2^b)^a = 1^b 1^a = 1$$

   (b) Proof of associativity over operator
   (c) Proof of existence of identity element
   (d) Proof of existence of inverse of operator

   $\square$

4. Let $G = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$.

   a) Prove that G is a group under addition.

   Proof. (a) Proof of Closure under operator
   (b) Proof of associativity over operator
   Let

   $$a = a_1 + b_1\sqrt{2} \quad b = a_2 + b_2\sqrt{2}) \quad c = a_3 + b_3\sqrt{2}$$

   $$(a + b) + c = a + (b + c)$$
   $$(a_1 + b_1\sqrt{2} + a_2 + b_2\sqrt{2}) + a_3 + b_3\sqrt{2} = a_1 + (b_1\sqrt{2} + a_2 + b_2\sqrt{2} + a_3 + b_3\sqrt{2})$$
   $$a_1 + b_1\sqrt{2} + a_2 + b_2\sqrt{2} + a_3 + b_3\sqrt{2} = a_1 + b_1\sqrt{2} + a_2 + b_2\sqrt{2} + a_3 + b_3\sqrt{2}$$

   (c) Proof of existence of identity element

(d) Proof of existence of inverse of operator

□

b) Prove that the nonzero elements of G are a group under multiplication ["Rationalize the denominators" to find the inverses].

Proof. (a) Proof of Closure under operator

(b) Proof of associativity over operator

(c) Proof of existence of identity element

(d) Proof of existence of inverse of operator

□