MATH 335 Homework 3

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1. Deterimine which one of the following sets is a group under addition: Addition is a binary operator by the proof provided in class, therefore we will validate the properties of each group accepting that addition is binary operator in these contexts.

   a) the set of rational numbers in lowest terms whose denominators are odd

   Proof. Let G be the following set:

$$G = \left\{ \frac{m}{n}, n = 2k+1, n \neq 0, k, m \in \mathbb{Z} \right\}$$

   (a) Proof of associativity over operator let a,b,c∈G where:

$$a = \frac{a_m}{a_n} \quad b = \frac{b_m}{b_n} \quad c = \frac{c_m}{c_n}$$

$$\left(\frac{a_m}{a_n} + \frac{b_m}{b_n}\right) + \frac{c_m}{c_n} = \frac{a_m}{a_n} + \left(\frac{b_m}{b_n} + \frac{c_m}{c_n}\right)$$

$$\frac{a_m b_n + b_m a_n}{a_n b_n} + \frac{c_m}{c_n} = \frac{a_m}{a_n} + \left(\frac{b_m c_n + c_m b_n}{b_n c_n}\right)$$

$$\frac{c_n(a_m b_n + b_m a_n) + c_m a_n b_n}{a_n b_n c_n} = \frac{a_n(b_m c_n + c_m b_n) + a_m b_n c_n}{a_n b_n c_n}$$

$$\frac{c_n a_m b_n + c_n b_m a_n + c_m a_n b_n}{a_n b_n c_n} = \frac{c_n a_m b_n + c_n b_m a_n + c_m a_n b_n}{a_n b_n c_n}$$

   The denominator $a_n b_n c_n$ is the product of three odd numbers which is also odd preserving the construction of G.

   (b) Proof of existence of identity element
   The Identity element for this set is the element: $\frac{0}{1} = 0$

   (c) Proof of existence of inverse of operator
   For all g∈ G the additive inverse is the element $-g$

   □

   b) the set of rational numbers in lowest terms whose denominators are even

Proof. Let G be the following set:

$$G = \left\{ \frac{m}{n}, n = 2k, n \neq 0, k, m \in \mathbb{Z} \right\}$$

(a) Proof of associativity over operator let a,b,c∈G where:

$$a = \frac{a_m}{a_n} \quad b = \frac{b_m}{b_n} \quad c = \frac{c_m}{c_n}$$

$$\left(\frac{a_m}{a_n} + \frac{b_m}{b_n} \right) + \frac{c_m}{c_n} = \frac{a_m}{a_n} + \left(\frac{b_m}{b_n} + \frac{c_m}{c_n} \right)$$

$$\frac{a_m b_n + b_m a_n}{a_n b_n} + \frac{c_m}{c_n} = \frac{a_m}{a_n} + \left(\frac{b_m c_n + c_m b_n}{b_n c_n} \right)$$

$$\frac{c_n(a_m b_n + b_m a_n) + c_m a_n b_n}{a_n b_n c_n} = \frac{a_n(b_m c_n + c_m b_n) + a_m b_n c_n}{a_n b_n c_n}$$

$$\frac{c_n a_m b_n + c_n b_m a_n + c_m a_n b_n}{a_n b_n c_n} = \frac{c_n a_m b_n + c_n b_m a_n + c_m a_n b_n}{a_n b_n c_n}$$

The denominator $a_n b_n c_n$ is the product of three even numbers which is also even preserving the construction of G.

(b) Proof of existence of identity element
The Identity element for this set is the element: $\frac{0}{2} = 0$

(c) Proof of existence of inverse of operator
For all g$\in G$ the additive inverse is the element $-g$

□

c) the set of rational numbers of absolute value < 1
Let G be the following set:

$$G = \left\{ \frac{m}{n}, n \neq 0, n, m \in \mathbb{Z}, |\frac{m}{n}| < 1 \right\}$$

Proof. (a) Proof of associativity over operator let a,b,c∈G where:

$$a = \frac{a_m}{a_n} \quad b = \frac{b_m}{b_n} \quad c = \frac{c_m}{c_n}$$

$$\left(\frac{a_m}{a_n}+\frac{b_m}{b_n}\right)+\frac{c_m}{c_n}=\frac{a_m}{a_n}+\left(\frac{b_m}{b_n}+\frac{c_m}{c_n}\right)$$

$$\frac{a_mb_n+b_ma_n}{a_nb_n}+\frac{c_m}{c_n}=\frac{a_m}{a_n}+\left(\frac{b_mc_n+c_mb_n}{b_nc_n}\right)$$

$$\frac{c_n(a_mb_n+b_ma_n)+c_ma_nb_n}{a_nb_nc_n}=\frac{a_n(b_mc_n+c_mb_n)+a_mb_nc_n}{a_nb_nc_n}$$

$$\frac{c_na_mb_n+c_nb_ma_n+c_ma_nb_n}{a_nb_nc_n}=\frac{c_na_mb_n+c_nb_ma_n+c_ma_nb_n}{a_nb_nc_n}$$

(b) Proof of existence of identity element

(c) Proof of existence of inverse of operator

□

d) the set of rational numbers of absolute value ≥ 1 together with 0.
Let G be the following set:

$$G=\left\{\frac{m}{n},n\neq 0,k,m\in\mathbb{Z},|\frac{m}{n}|>1\right\}\cup\{0\}$$

Proof. (a) Proof of associativity over operator

(b) Proof of existence of identity element

(c) Proof of existence of inverse of operator

□

2. Let $G=\{x\in\mathbb{R} : 0\leq x<1\}$ and for $x,y\in G$ let $x\cdot y$ be the fractional part of $x+y$ (i.e. $x\cdot y=x+y-[x+y]$ where $[a]$ is the greatest integer less than or equal to a). Prove that $\cdot$ is a binary operation on G and that G is a group.

Proof. Proof that $\cdot$ is a binary operator

□

Proof. Proof that G is a group

(a) Proof of associativity over operator

(b) Proof of existence of identity element

(c) Proof of existence of inverse of operator

□

3. Let $G=\{z\in\mathbb{C} : z^n=1$ for some nonnegative integer $n\}$. Prove that G is a group under multiplication (called the groups of *roots of unity* in $\mathbb{C}$).

Proof. (a) Proof of associativity over operator

(b) Proof of existence of identity element

(c) Proof of existence of inverse of operator

□

4. Let $G = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$.

a) Prove that G is a group under addition.

Proof. (a) Proof of associativity over operator

Let

$$a = a_1 + b_1\sqrt{2} \quad b = a_2 + b_2\sqrt{2}) \quad c = a_3 + b_3\sqrt{2}$$

$$(a+b)+c = a+(b+c)$$
$$(a_1 + b_1\sqrt{2} + a_2 + b_2\sqrt{2}) + a_3 + b_3\sqrt{2} = a_1 + (b_1\sqrt{2} + a_2 + b_2\sqrt{2} + a_3 + b_3\sqrt{2})$$
$$a_1 + b_1\sqrt{2} + a_2 + b_2\sqrt{2} + a_3 + b_3\sqrt{2} = a_1 + b_1\sqrt{2} + a_2 + b_2\sqrt{2} + a_3 + b_3\sqrt{2}$$

(b) Proof of existence of identity element

(c) Proof of existence of inverse of operator

□

b) Prove that the nonzero elements of G are a group under multiplication ["Rationalize the denominators" to find the inverses].

Proof. (a) Proof of associativity over operator

(b) Proof of existence of identity element

(c) Proof of existence of inverse of operator

□