MATH 325 Lecture 19

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Opening Notes exam is on thursday November third, study guide coming friday evening or tomorrow. Short homework will be due tuesday, Submitted on i learn tuesday at 10am.
In class eam will mostly be cyclic groups the take home will be on cosets.

A return to the properties of Cosets

Proposition 1. Let G be a group and H a subgroup of G. Then :

$$g_1H = g_2H \iff g_2 \in H$$

A coset generates its own representatives.

Proof.
$g_1H = g_2H \to g_2 \in g_1H$
Since H is a subgroup when we compute: g_2H We are guarenteed to have at minimum $g_2 \in g_2H$ since $g_2 \circ e = g_2$ So $g_2 \in g_2H$ □

Proof.
$g_1H = g_2H \leftarrow g_2 \in H$
Given $g_2 \in g_1H$ This means that $g2 = g_1\bar{h}, \bar{h} \in H$: Consider the set $g_2H = \{g_2h, h \in H\}$
but we know that $g_2 = g_1\bar{h}$ so :

$$g_2H = \{g_2h, h \in H\} = \{g_1\bar{h}h : h \in H\}$$

Note that since $\bar{h}, h \in H$ that $\bar{h}h \in H$ and it follows that:

$$g_1\bar{h}h \in H$$

Thus:

$$g_2H \subset g_1H$$

We can also redefine g_2 as follows;

$$g_2 = g_1\bar{h} \to g_2\bar{h}^{-1} = g_1$$

So:

$$g_1H = \{g_1h : h \in H\} = \{g_2\bar{h}^{-1}h : h \in H \subset g_2H$$

□

Theorem

Let G be a group and H a subgroup of G then the distinct left cosets of H in G partition G.

Implications:

1. The union of all left cosets of H is G
2. if $g_1H \neq g_2H$ then $g_1H \cap g_2H =$

Proof.
The union of all left cosets of H is G
Given an element: $g \in G$ we show that g is some left coset of H g is in gH since g composed with identity guarentees us the existence of a coset with g.

□

Proof.
if $g_1H \neq g_2H$ **then** $g_1H \cap g_2H =$
We show that if the intersection of $g_1H \cap g_2H$ is not empty then $g_1H = g_2H$ this is the contrapositive of our original argument. Since the intersection is non empty there must exist an element in the intersection. Let g be an element in the intersection of the two cosets
Then:

$$g = g_1h_1, h_1 \in H \quad g = g_2h_2, h_2 \in H$$

Which implies:

$$g_1h_1 = g_2h_2$$

$$g_2 = g_1h_1h_2^{-1},$$

but: $h_1h_2^{-1} \in H$ So we have that $g_2 \in g_1H$ and by the Previous proof we have: $g_2 \in g_1H$ □

If you have an infinite group there are infinite representatives for each coset.
You can have a finite quantity of cosets for an infinite group and infinite subgroup
Example G=$\mathbb{Z}$ $H =< n >$ The cosets are the equivilance classes
The cosets are the elements of: $\mathbb{Z}_n$
The set of cosets does not always form a new group like in this example.

Proposition

Let G be a group and H a subgroup of G, Then for any $g \in G$ there is a bijection between the subgroup H and the left coset gH. In particular, if H is a finite subgroup, then the number of elements in H and the number of elements in its left coset are the same, this is to say:

$$|H| = |gH|$$

Proof.
Let $\phi : H \mapsto gH$ be : $\phi(h) = gh, h \in H,$. We begin by proving surjectivity: Given an element of gH,: gh for some h. Then $\phi(h) = gh$
Next is injectivity: Suppose $\phi(h_1) = \phi(h_2)$ Show $h_1 = h_2$

$$\phi(h_1) = gh_1 \quad \phi(h_2) = gh_2$$

Then by our assumption:

$$gh_1 = gh_2$$

but we can left multiply by g inverse since we are in a group thus :

$$h_1 = h_2$$

$\square$