

1. Let's start with a warm-up exercise for doing proofs where you need to use induction: Prove that for all $n \in \mathbb{N}$

$$1 + 2 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 1.$$

Re-expressing the problem statement:

Prove that $\forall n \in \mathbb{N}$

$$1 + \sum_{i=1}^n 2^i = 2^{n+1} - 1$$

Proof. Base case: $n=2$

$$1 + 2^1 + 2^2 = 2^3 - 1$$

$$7 = 7$$

Inductive Hypothesis:

$$P(k) = 1 + \sum_{i=1}^k 2^i = 2^{k+1} - 1$$

$P(k+1)$:

$$1 + \sum_{i=1}^{k+1} 2^i = 2^{k+2} - 1$$

$$1 + \sum_{i=1}^k 2^i + 2^{k+1} = 2^{k+2} - 1$$

$$2^{k+1} - 1 + 2^{k+1} = 2^{k+2} - 1$$

$$2(2^{k+1}) - 1 = 2^{k+2} - 1$$

$$2^{k+2} - 1 = 2^{k+2} - 1$$

□

2. And here is a second warm-up exercise: if x is a nonnegative real number show that $(1+x)^n - 1 \geq nx$ for $n = 0, 1, 2, \dots$

Re-expressing the problem statement:

let $x \in \mathbb{R}, x \geq 0$, Show that

$$(1+x)^n - 1 \geq nx, \forall n \in \mathbb{W}$$

Proof. Suppose $x \in \mathbb{R}, x \geq 0$

$$(1+x)^n - 1 \geq nx$$

$$(1+x)^n \geq nx + 1$$

$$\sum_{k=0}^n \binom{n}{k} x^k \geq nx + 1$$

$$nx + 1 + \sum_{k=2}^n \binom{n}{k} x^k \geq nx + 1$$

$$\therefore (1+x)^n \geq nx + 1$$

Which implies

$$(1+x)^n - 1 \geq nx$$

□

3. Show that if p is a prime number, there do not exist nonzero integers a and b such that $a^2 = pb^2$ (i.e. \sqrt{p} is not a rational number).

If p is a prime number then the only divisors of p are 1 and p

Let a and b be expressed in the prime factorization:

$$a = \prod_{i=1}^k p_i^{\alpha_i}$$

$$b = \prod_{i=1}^k p_i^{\beta_i}$$

the original statement is then:

$$\left[\prod_{i=1}^k p_i^{\alpha_i} \right]^2 = p \left[\prod_{i=1}^k p_i^{\beta_i} \right]^2$$

$$\prod_{i=1}^k p_i^{2\alpha_i} = p \prod_{i=1}^k p_i^{2\beta_i}$$

If $a^2 = b^2$ then they would have identical prime factorizations. Since the integer is raised to the second power this implies that for all prime factors, each has an even exponent.

if p is not a prime factor of b then it would have an odd exponent. If p was a prime factor of b then it would form an odd exponent when combined with the corresponding prime factor since the each prime factor in b^2 has an even exponent. In either case p exists as a prime number with an odd exponent therefore it cannot be equal to a since a has a prime factorization that has even exponents for each prime factor.

4. Compute the gcd of the following pairs of integers:

i) 14 and 39;

$$39 = 14(2) + 11$$

$$14 = 11(1) + 3$$

$$11 = 3(3) + 2$$

$$3 = 2(1) + 1$$

$$2 = 1(2) + 0$$

$$\gcd(39, 14) = 1$$

ii) 234 and 165;

$$234 = 165(1) + 69$$

$$165 = 69(2) + 27$$

$$69 = 27(2) + 15$$

$$27 = 15(1) + 12$$

$$15 = 12(1) + 3$$

$$12 = 3(4) + 0$$

$$\gcd(234, 165) = 3$$

iii) 471 and 562.

$$562 = 471(1) + 91$$

$$471 = 91(5) + 16$$

$$91 = 16(5) + 11$$

$$16 = 11(1) + 5$$

$$11 = 5(2) + 1$$

$$5 = 1(5) + 0$$

$$\gcd(562, 471) = 1$$

5. Let $a, b, c \in \mathbb{Z}$. Prove that if $\gcd(a, b) = 1$ and $a \mid bc$ then $a \mid c$.

Proof. if $a \mid bc$ then $bc = ak, k \in \mathbb{Z}$

□

6. Let a and b be positive integers where

$$a = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} \quad \text{and} \quad a = p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k}$$

with p_1, \dots, p_k distinct primes and $\alpha_i, \beta_i \geq 0$ for $i = 1, \dots, k$. Show that

$$\text{lcm}(a, b) = p_1^{\max(\alpha_1, \beta_1)} p_2^{\max(\alpha_2, \beta_2)} \cdots p_k^{\max(\alpha_k, \beta_k)}.$$

Proof. Let a and b be expressed in a product form as follows:

$$a = \prod_{i=1}^k p_i^{\alpha_i}$$

$$b = \prod_{i=1}^k p_i^{\beta_i}$$

Let:

$$l = \prod_{i=1}^k p_i^{\max(\alpha_i, \beta_i)}$$

be the least common multiple of a and b.

(a) a

□