MATH 335 Lecture 22

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Normal Subgroups/Groups

A normal group or subgroup is a group where the left cosets equals the right cosets for all reprsentatives in a general group.

Let G be a group a subgroup of N of G is called normal if $gN = Ng, \forall g \in G$

Exapmple: $<(1,23)>= N$ generates a 3 element subgroup with 2 distinct cosets. N is a normal subgroup since eN=Ne, $\sigma N = N\sigma$
Note that the subgroup need not be abelian to be normal

Proposition: If G is an abelian group thene very subgroup of G is a normal subgroup

Proof. Let H be a subgroup of G, for $g \in G$ that $gH = Hg$ however since g is abelian this fact is quite obvious.

$$gH = \{gh : h \in H\} = \{hg : h \in H\} = Hg$$

□

We will see that A_n is a normal subgroup of S_n but it is neither abelian nor cyclic

Definition: Let H be a subgroup of group G, and let$g \in G$

Consider the set

$$gHg^{-1} = \{ghg^{-1} : h \in H$$

Example
Let G=$S_3 : H =<\sigma>= \{e,(1)$

$$g = (1\ 2\ 3) \quad gHg^{-1} = (1\ 2\ 3)H(1\ 3\ 2)^{-1} = \{(1\ 2\ 3)e, (1\ 2\ 3)(12)(1\ 3\ 2)\} = \{e, (2\ 3)\}$$

Let N be a sugroup of a group G then the following are equivilant

1. N is Normal
2. $gNg^{-1} \subset N \quad \forall\, g \in G$
3. $gNg^{-1} = N \quad \forall\, g \in G$

Proof. we prove that (1) implies(2) implies(3):

(1) implies (2):

Since N is a normal subgroup this implies that $gN = Ng$ for all elements g in G.

$$gn \in gN, \rightarrow gn \in Ng$$
$$\therefore gn = n^{*}g \text{ for } n^{*} \in N$$
$$gng^{-1} = n^{*} \rightarrow gng^{-1} \in N$$

(2) implies (3):

We wish to show that $gNg^{-1} \subset N \rightarrow gNg^{-1} = N \quad \forall\, g \in G$

We need to show the other side of the set inclusion thus we prove:

$$N \subset gNg^{-1}$$

let n$\in N$, consider $g^{-1}n(g^{-1})^{-1} = g^{-1}ng \in N$

by the previous proof, therefore we sho that $g^{-1}ng = n^{*}$ for $n^{*} \in N$

$$g^{-1}ng = n^{*} \rightarrow n = gn^{*}g^{-1}$$

So we have shown that N is a contained in gNg^{-1} and we conclude that they are equal

(3) implies (1):

$gNg^{-1} = N \quad \forall\, g \in G \rightarrow gN = Ng$

We show this by a set inclusion proof as follows:

gN⊂Ng

$gn \in gN$

$gng^{-1} \in gNg^{-1} = N \rightarrow gng^{-1} = n^{*}, n^{*} \in N$

Right multiplication yeilds:

$$gn = n^{*}g$$

And we conclude $gN \subset Ng$ **Ng⊂gN**

This proof is symmetric □

Theorem: G is always a normal subgroup of itself.