MATH 310 Homework 7

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Question 1

Find all subgroups of $G = \langle a \rangle$ where $|a| = 45$. Describe the containments between these subgroups.

Proof.
The unique subgroups of G are:

$$< a^{45} >$$
$$< a^{3} >$$
$$< a^{5} >$$
$$< a^{15} >$$
$$< a^{9} >$$
$$< e >$$

With the containment realtion:

$$< e > \subseteq < a^9 > \subseteq < a^3 > \subseteq < a^{45} >$$
$$< e > \subseteq < a^{15} > \subseteq < a^5 > \subseteq < a^{45} >$$

Note that :

$$< a^{15} > \subseteq < a^3 >$$

As well. □

Question 2

Find all generators of $\mathbb{Z}_{48}$.

Proof.
To start there are a total of:

$$\phi(48) = \phi(2^4)\phi(3) = 8(2) = 16$$

Geneators since this is how many times we obtain a denominator of one when solving for the order of each element. The 16 relatively prime numbers to 48 are contained in the unit group of 48 therefore generators for $\mathbb{Z}_{48}$ are the elements of $U(48)$. □

Question 3

Let $left(G_1, \circ$ and $(G_2, \bullet)$ be two groups with the respective group operations $\circ$ and $\bullet$. Show that the cartesian product $G_1 \times G_2$ is a group with the following operation:

$$(a_1, b_1) \diamond (a_2, b_2) := (a_1 \circ a_2, b_1 \bullet b_2).$$

Proof.
To show that $G_1 \times G_2$ is a group we prove the following:

1. closure
   To demonstrate closure under the operation $(a_1, b_1) \diamond (a_2, b_2) := (a_1 \circ a_2, b_1 \bullet b_2)$. Since $(G_1, \circ)$ and $(G_2, \bullet)$ are closed under their respective operators we have the fact that for any ordered pair produced by the cartersian product with our new element $a_1 \circ a_2 \in G_1$ and $b_1 \bullet b_2 \in G_2$ This implies that for all elements produced by our operator we obtain a new element of from the cartesian product of the two sets which is the desired meaning of closure in this context.

2. **Assoicativity**

   $$\text{Let } (a_1, b_1), (a_2, b_2), (a_3, b_3) \in G_1 \times G_2$$

   $$(a_1, b_1)((a_2, b_2)(a_3, b_3)) = ((a_1, b_1)(a_2, b_2))(a_3, b_3)$$

   $$(a_1, b_1)(a_2 \circ a_3, b_2 \bullet b_3) = (a_1 \circ a_2, b_1 \bullet b_2)(a_3, b_3)$$

   $$(a_1 \circ a_2 \circ a_3, b_1 \bullet b_2 \bullet b_3) = (a_1 \circ a_2 \circ a_3, b_1 \bullet b_2 \bullet b_3)$$

3. **Identity**
   Consider the composition of the following :

   $$(a_1, b_1) \diamond \left(a_1^{-1}, b_2^{-1}\right):$$

   By definition of the operator we obtain:

   $$\left(a_1 \circ a_1^{-1}, a_1^{-1} \bullet b_1^{-1}\right) = (e, e)$$

4. **Inverse**
   Consider an element of $G_1 \times G_2$:

   $$(c, d) \in G_1 \times G_2$$

   By definition

   $$(c, d) = (a_1 \circ a_2, b_1 \bullet b_2)$$

   Since $a_1 \circ a_2 \in G_2$ so is c, and likewise since $b_1 \bullet b_2 \in G_2$ so is d , Since G_1 and G_2 are groups there exists an inverse for each element in each group so we can construct an inverse:

   $$(c^{-1}, d^{-1}) \in G_1 \times G_2$$

Proving the existence of an inverse:

$$(c,d) \diamond (c^{-1}, d^{-1}) = (e,e)$$

Thus $G_1 \times G_2$ is a group under $(a_1,b_1) \diamond (a_2,b_2) := (a_1 \circ a_2, b_1 \bullet b_2)$ $\square$

Question 4

List the elements in the group $\mathbb{Z}_2 \times \mathbb{Z}_3$. Show that this group is cyclic.

Proof.
The elements of $\mathbb{Z}_2 \times \mathbb{Z}_3$ are:

$$\mathbb{Z}_2 \times \mathbb{Z}_3 = \{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2)\}$$

Assuming that the group operator over the ordered pairs is element wise addidtion with respect to the original modular base: the generator of this set is:

$$< (1,1) >= \{(1,1),(0,2),(1,0),(0,1),(1,2),(0,0)$$

$\square$

Question 5

List the elements in the group $\mathbb{Z}_2 \times \mathbb{Z}_2$. Show that this group is not cyclic. Argue that by now we know two different abelian groups with four elements

Proof.
The elements of this group are as follows:

$$\mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0,0),(0,1),(1,0),(1,1)\}$$

To demonstrate that this is not a cycic group we will show that each subgroup is not a generator:

$$\begin{aligned}
&< (0,0) >= \{(0,0)\} \\
&< (0,1) >= \{(0,0),(1,1)\} \\
&< (1,1) >= \{(0,0),(0,1)\} \\
&< (1,0) >= \{(0,0),(1,0)\}
\end{aligned}$$

This group itself is a non cyclic abelian group since the elements involved in the construction of the set come from abelian groups (integers mod n ,+). This group has four elements so it is one of our order 4 abelian groups .

One way to find another order 4 abelian group is to consider the cartesian product of an abelian group with two elements and another abelian group with with two elements: Consider = the unit group of 3 :

$$U(3) \times U(3) = \{(1,1),(1,2),(2,1),(2,2))$$

Which is an abelian group of order 4 $\square$

Question 6

Prove that neither $\mathbb{Z}_2 \times \mathbb{Z}$ nor $\mathbb{Z} \times \mathbb{Z}$ are cyclic groups.

Subproof 1: $\mathbb{Z}_2 \times \mathbb{Z}$

1. *Proof.*

$$\mathbb{Z}_2 = \{0,1\}$$

$$\mathbb{Z}_2 \times \mathbb{Z} = \{(0,k),(1,k) : k \in \mathbb{Z}\}$$

By the definition of composition of groups under a cartesian product given in problem 3.
Suppose that $\mathbb{Z}_2 \times \mathbb{Z}$ is cyclic, then there exists (a,b)$\in \mathbb{Z}_2 \times \mathbb{Z}$ such that :

$$<(a,b)>= \mathbb{Z}_2 \times \mathbb{Z}$$

Consider the following:
Since (a,b)$\in \mathbb{Z}_2 \times \mathbb{Z}$ It is either of the form (1,k) or (0,k) where k is an integer.
Case 1: (a,b) is of the form (1,k):
If (a,b) is of the form (1,k) then the set generated by (a,b) does not contain (0,k) $\forall k \in \mathbb{Z}$ which is a contradiction
Case 2: (a,b) is of the form (0,k):
If (a,b) is of the form (0,k) then the set generated by (a,b) does not contain (1,k) $\forall k \in \mathbb{Z}$ which is a contradiction
Thus in either case we arrive to a contradiction meaning there does not exist a generator for$\mathbb{Z}_2 \times \mathbb{Z}$ implying it is not cyclic. □

Subproof 1: $\mathbb{Z} \times \mathbb{Z}$

2. *Proof.*
To demonstrate that $\mathbb{Z} \times \mathbb{Z}$ is not a cyclic group we will proceed with a proof by cotradiction levergaining the fact that if a group is cyclic then there exists a generator:
Suppose that $\mathbb{Z} \times \mathbb{Z}$ is cyclic, then by definition there exists an ordered pair in $\mathbb{Z} \times \mathbb{Z}$ (a,b) such that :

$$<(a,b)>= \mathbb{Z} \times \mathbb{Z}$$

If $a = 0$ then $(1,0)$ is not in this set, which leads to a contradiction since (a,b) is a supposedly a generator. So we have $a \neq 0$.
If $b = 0$ then $(0,1)$ is not in this set, which leads to a contradiction since (a,b) is a supposedly a generator. So we have $b \neq 0$.
Consider the element

$$(a,-b) \in \mathbb{Z} \times \mathbb{Z}$$

There is an integer $k \in \mathbb{Z}$ with $(ka,kb) = (a,-b)$, and since $a,b \neq 0$ this gives $k = 1$ and $k = -1$, which is a contradiction.

Together we have shown that element (0,1) cannot be generated with this construction □

Question 7

Let G be a group and let $C_1 = \langle a \rangle$ and $C_2 = \langle b \rangle$ be two cyclic subgroups with orders n and m, respectively. Prove that if $\gcd(n,m) = 1$ then $C_1 \cap C_2 = \{e\}$.

Proof.
If $| < a > | = n$ this implies $a^n = e$ likewise If $| < b > | = m$ this implies $a^m = e$. This knowledge together allows us to build representations of the elements of the group:

$$C_1 = \{e, a, a^1, a^2, \cdots, a^{n-1}\}$$

$$C_2 = \{e, b, b^1, b^2, \cdots, b^{m-1}\}$$

Suppose there exists another element in the intersection of C_1 and C_2, Since we have the property that:

$$|a^k| = \frac{n}{\gcd(n,k)}$$

This means that:

$$|a^k| \gcd(n,k) = n \to |a^k||n$$

So this means that since $x \in C_1$ that

$$|x||n$$

as $x = a^l, l \in [1, n-1]$
Likewise since $x \in C_2$ that

$$|x||m$$

as $x = b^i, i \in [1, m-1]$
But gcd(n,m)=1, meaning that the only time this could be true is when x=1=e $\square$