MATH 335 lecture 13

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1 Sub Groups and their properties

Definition 1. Subgroup
Let G be a group. A subset H of G is called a subgroup if H itself is a group, when we restrict the group operation to H.

Let G=S_3 the group operation of this group is function composition, thus to consider subgroups of this group we will first consider which subsets still are verifiable groups under function composition. Consider:

$$\left\{ \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix} \right\}$$

This Group is a subgroup since for all elements we are closed under function composition

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For all subgroups the identity element must be an element of the subgroup Since we still are using the same binary operator.

The subgroup consisting of an element the identity and its inverse always forms a group since H will be closed under the group operation due to the limited selection criteria from the set.

Every group has at minimum two subgroups. The first is the set of just the identity, the trivial subgroup. The next subgroup is the entire group itself. (typically ignored)

If we have the case where an element's inverse is itself the subgroup consisting of that element and the identity forms a subgroup.

Examples include cyclic group about 90 degree rotations in the complex unit circle in C and the general linear group for 2x2 matrices.

Definition 2. SL_2 The special linear group is a subgroup of the general linear group defined as follows.:

$$SL_2 = \{A \in GL_2 : det(A) = 1\}$$

Since we have:

$$\det(AB) = \det A \det B$$

The group is closed under matrix multiplication and is concequently a subgroup of SL_2
We also verify that taking the inverse of a matrix still places us in SL2 after the operation.This can be done by leveraging thefact that the coefficient over the inverse computtation begcomes 1 over 1 when det(A)=1 so we are still within SL_2

Proposition 1. Let G be a group and let h be a subset of G then h is a subgroup of G if and only if:

1. $e \in H$
2. $\forall\, h_1, h_2 \in H \rightarrow h_1 \circ h_2 \in H$
3. if $h \in H$ then $h^{-1} \in H$

Example Consider M_2 the set of all matrices under the binary operator of matrix addition

1. The operator is closed since dimensions are preserved.
2. The operation is associative by extension of the associativty of $\mathbb{R}$
3. Identity element is the zero matrix

4. the inverse element is -A$\forall\, A \in M_2$

Proposition 2. $GL_2 M_2$ since M_2 is all 2x2 matrices. Is GL_2 a subgroup of M_2? Well, the zero matrix is not in GL_2 and concequently we do not have a subgroups

Definition 3. If H is a subgroup of G we write:

$$H \leq G$$