MATH 335 Lecture 24

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Definition

A subgroup H of a group G is called a normal subgroup if $gH = Hg \quad \forall \in G$

Equivilantly H is a normal subgroup if and only if: $gHg^{-1} = H$ for all g in G.

Example:
Every subgroup of an abelian group is normal.
Example
Let $G = S_3$ and $H =< (12) >= \{e, (12)\}$ We showed last time that H is not normal . In general we can prove this by showing tha gHg^{-1} is not equal to H.
All subgroups of index two must automatically be a normal subgroup.
Homework hint: Given a group G consider the following set : $Z(G) = \{g \in G : ga = ag \forall a \in G\}$ Show that Z(G) is a normal subgroup of G. This is always true the abelian subgroup of a group is always a normal subgroup of G.

Motivation: Factor Groups

Let G be a group and H is a subgroup of G. We consider all distinct left cosets of H in G. The set formed by all left cosets of G is denoted: G/H We wish to make this set of all distinct left cosets of H in G into a group
Intuitivley it feels like this can be accomplished by the following let a and b the representatives of two left cosets in H then::

$$(aH)(bH) = abH$$

is our binary operation. but this is not well defined because selection of different representatives leads to different outcomes meaning that it fails as the binary operation. If H is a normal subgroup then this binary operation is sufficient

Definition

Let G be a gorup and N a norma subgroup of G
If $aN = a^*N$ and $bN = b^*N$ then $abN = a^*b^*N$

Proof. Let $a^* = an$ for some n in N and $b^* = bn$ for some other n in N. Then we have

$$a^*b^*N = an_1bn_2N = an_1bN = an_1Nb = aNb = abN$$

We are able to convert a left coset to a right coset since we know that N is a norla sugroup giving us the ability to commuete right and left subgroups as we please throughout our proofs. □

Theorem:

If N is a normalsubgroup of the group G then the binary operation defined by $(aN)(bN) = (ab)N$ makes G/N the set of distinct left cosets

To prove this statement we must prove the three propeties that form a group all three can be proven using the extended properties of the original group. This group is called a factor group or quotient group

If G is finite and N is a normal subgroup then the factor group: G/N is the index of G given N.

$$|G/N| = \frac{|G|}{|N|}$$