MATH 370 Homework 7

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Definition

A group homomorphism is a function:

$$\phi : (G, \circ) \to (H, *)$$

ϕ is a homomorphism if $\forall a, b \in (\phi(a * b)) = \phi(a) \circ \phi(b)$

Definition: Kernel of a homomorphism

$$ker(\phi) = \{g \in G : \phi(g) = e_H\} \subset G$$

Example: $\phi : S_3 \mapsto \{1, -1\}$ $ker(\phi) = \{\sigma \in S_3 | \phi(\sigma) = 1$

Claim Given a homomorphism ϕ ker(ϕ) is a normal subgroup of G}

Proof. First we prove that ker ϕ is a subroup of G, which means we must prove that if $a, b \in ker(\phi)$ then $ab^{-1} \in ker(\phi)$ and $\phi(a * b^{-1}) = e_H$ □

Proof. $\phi(a * b^{-1}) = \phi(a) \circ \phi(b^{1-})$

$$\phi(a) \circ \phi(b^{1-})$$
$$e_H \circ e_H$$
$$e_H$$

□

Proposition

kerϕ is normal in (G,*)
we prove this if $\forall g \in G$ and $\forall a \in ker(\phi)$

$$gag^{-1} \in ker(\phi)$$

$$\phi(gag^{-1}) = \phi(g) \circ \phi(a) \circ \phi(g^{1-})$$
$$\phi(g) \circ e_H \circ \phi(g^{-1})$$
$$e_H$$

So we have proven that $ker(\phi)$ is a normal subgroup in G.

Given a homomorphism the kernel of a homomorphism is normal in G. Given any normal subgroup there exists a homomorphism of phi sucht aht kernel of phi is equal to that normal subgroup:

Theorem

Every normal subgroup is a kernel of some homomorphism, this will be proven later

Theorem

If $\phi(G,*) \to (H,\circ)$ is a homomorphism then $ker(\phi) = \{e_G\}$ if and only if ϕ is injective.

Proof. We wish to show that the function is injective, thus for a,b, $\in G$ if

$$\phi(a) = \phi(b) \mapsto a = b$$

□