

MATH 370 Homework 4

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1. 9.8 (no proofs needed)

(a)

$$\lim_{n \rightarrow \infty} n^3 = \infty$$

(b)

$$\lim_{n \rightarrow \infty} (-n^3) = -\infty$$

(c)

$$\lim_{n \rightarrow \infty} (-n^n) = \text{NOT EXIST}$$

(d)

$$\lim_{n \rightarrow \infty} (1.01)^n = \infty$$

(e)

$$\lim_{n \rightarrow \infty} (n^n) = \infty$$

2. 9.17 (proof needed):

Give a formal proof that

$$\lim_{n \rightarrow \infty} (n^2) = +\infty$$

using definition 9.8: **Definition 9.8**

For a sequence (s_n) we write

$$\lim_{n \rightarrow \infty} (s_n) = +\infty$$

provided:

$$\forall M > 0 \exists N : n > N \rightarrow s_n > M$$

Proof. To prove this fact we must choose some value for N such that the sequence s_n for any n greater than N s_n is greater than M . To do this let us consider the relationship between M and N . To satisfy the requirement under the definition of the function, let $N = \sqrt{M}$ then we have the following for n greater than our choice:

$$n > N$$

$$n > \sqrt{M}$$

$$n^2 > M$$

$$s_n > M$$

So we have proven that with a selection of $N = \sqrt{M}$ then $n > N$ implies $s_n > M$

□

3. 9.18

Proof. (a) Verify:

$$\sum_{k=0}^n a^k = \frac{1-a^{n+1}}{1-a}, a \neq 1$$

Proof. Base case: $n=1$:

$$a^0 + a^1 = \frac{1-a^2}{1-a}$$

$$1+a = \frac{1-a^2}{1-a}$$

$$1+a = \frac{(1+a)(1-a)}{1-a}$$

$$1+a = 1+a$$

$$P(k) : \sum_{i=0}^k a^i = \frac{1-a^{k+1}}{1-a}, a \neq 1$$

$P(k+1)$

$$\sum_{i=0}^{k+1} a^i = \frac{1-a^{k+2}}{1-a}$$

$$\sum_{i=0}^k a^i + a^{k+1} = \frac{1-a^{k+2}}{1-a}$$

$$\frac{1-a^{k+1}}{1-a} + a^{k+1} = \frac{1-a^{k+2}}{1-a}$$

$$\frac{1-a^{k+1} + a^{k+1}(1-a)}{1-a} = \frac{1-a^{k+2}}{1-a}$$

$$\frac{1-a^{k+1} + a^{k+1} - a^{k+2}}{1-a} = \frac{1-a^{k+2}}{1-a}$$

$$\frac{1-a^{k+2}}{1-a} = \frac{1-a^{k+2}}{1-a}$$

□

(b) **Find** $\lim_{n \rightarrow \infty} (\sum_{k=0}^n a^k), |a| < 1$
Find

$$\lim_{n \rightarrow \infty} \frac{1-a^{n+1}}{1-a}$$

$$\lim_{n \rightarrow \infty} \frac{1}{1-a} (1 - a^{n+1})$$

$$\lim_{n \rightarrow \infty} \left(\frac{1}{1-a} \right) \lim_{n \rightarrow \infty} (1 - a^{n+1})$$

$$\lim_{n \rightarrow \infty} \left(\frac{1}{1-a} \right) \lim_{n \rightarrow \infty} (1) - \lim_{n \rightarrow \infty} (a^{n+1})$$

$$\lim_{n \rightarrow \infty} \frac{1}{1-a} = \frac{1}{1-a}$$

(c) **Calculate** $\lim_{n \rightarrow \infty} \left(\sum_{k=0}^n \frac{1}{3^k} \right)$

$$\sum_{k=0}^n \frac{1}{3^k} = \sum_{k=0}^n \left(\frac{1}{3} \right)^k$$

By part b we know that this sequence can be solved in the limit as:

$$\frac{1}{1-a}$$

thus:

$$\lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{1}{3^k} = \frac{1}{\frac{2}{3}} = \frac{3}{2}$$

(d) **What is** $\lim_{n \rightarrow \infty} \left(\sum_{k=0}^n a^k \right), a \geq 1$

$$\begin{aligned} & \lim_{n \rightarrow \infty} \left(\frac{1}{1-a} - \frac{a^{n+1}}{1-a} \right) \\ & \left(\frac{1}{1-a} - \frac{\lim_{n \rightarrow \infty} a^{n+1}}{1-a} \right) \\ & \left(\frac{1}{1-a} - \frac{\infty}{1-a} \right) = \infty \end{aligned}$$

□

4. 10.1

Proof. Which of the sequences are increasing? decreasing? Bounded?

(a)

$$s_n = \frac{1}{n}$$

The sequence above converges to zero, and thus by theorem 9.1 is bounded. The sequence is decreasing as $s_n \geq s_{n+1} \forall n$

(b)

$$s_n = \frac{(-1)^n}{n^2}$$

This sequence is bounded but does not converge and is not monotonic

(c)

$$s_n = n^5$$

This sequence is increasing, is not bounded and does not converge

(d)

$$s_n = \sin\left(\frac{n\pi}{7}\right)$$

This sequence is bounded but does not converge and is not monotonic

(e)

$$s_n = (-2)^n$$

This sequence is not bounded, and is not monotonic

(f)

$$s_n = \frac{n}{3^n}$$

This sequence is decreasing and is bounded it converges as well.

□

5. 10.6

Proof.

Let (s_n) be a sequence such that:

$$|s_{n+1} - s_n| < 2^{-n} \quad \forall n \in \mathbb{N}$$

Prove s_n is a Cauchy sequence and hence a convergent sequence.

Proof. To prove that s_n is a Cauchy sequence let us prove then that

$$\forall \varepsilon > 0 \exists N : \quad m, n > N |s_m - s_n| < \varepsilon$$

In order to go about proving this statement we must first find a way to connect the original statement to a general statment about being cauchy. One such way of doing this is through estbalishing the following corresponce:

$$m = n + k, k \in \mathbb{Z}$$

such that:

$$|s_m - s_n| < \varepsilon \rightarrow |s_{n+k} - s_n| < \varepsilon$$

□

With this new construction we can now leverage the triangle inequality in the following way for each pairwise distance estimate:

$$|s_{n+k} - s_n| \leq \sum_{i=n}^{n+k} |s_{i+1} - s_i|$$

however, note also that we have the original inequality for the case of $k=1$, meaning we can develop this inequality further:

$$|s_{n+k} - s_n| \leq \sum_{i=n}^{n+k} |s_{i+1} - s_i| < \sum_{i=n}^{n+k-1} \frac{1}{2^i}$$

which simplifies to:

$$|s_{n+k} - s_n| < \sum_{i=n}^{n+k-1} \frac{1}{2^i}$$

We would like to use the geometric series formula here to simplify the right hand side however the starting index is problematic, so to correct this we can factor out a term of: $\frac{1}{2^k}$ leaving:

$$|s_{n+k} - s_n| < \frac{1}{2^k} \sum_{i=0}^{n-1} \frac{1}{2^i}$$

Applying formula for sum of a geometric sequence:

$$|s_{n+k} - s_n| < \frac{1}{2^k} \left[\frac{1 - \frac{1}{2^n}}{\frac{1}{2}} \right] < \frac{1}{2^k} \left[\frac{1}{\frac{1}{2}} \right]$$

$$|s_{n+k} - s_n| < \frac{1}{2^{k-1}}$$

From this statement we can now prove that it is cauchy by considering an epsilon as follos:

$$\varepsilon = \frac{1}{2^{k-1}}$$

Which gives a value of N equal to:

$$N = \log_2\left(\frac{1}{\varepsilon}\right) + 1$$

Finally leaving us to the statement:

$$\forall \varepsilon > 0 \exists N = \log_2\left(\frac{1}{\varepsilon}\right) + 1 : n+k, n > N \rightarrow |s_{n+k} - s_n| < \varepsilon$$

Thus since s_n is cauchy by the completeness of the real numbers s_n is also convergent.

Is the result in (a) true if we only assume $|s_{n+1} - s_n| < \frac{1}{n}$?

If we make this assumption then we cannot say that the sequence is cauchy for the following reason: When we start to consider the same set up and get to a sum over $\frac{1}{n}$ we are now working with the harmonic series, or rather a partial sum of said harmonic series. This sequence diverges to infinity (slowly) Meaning we can always pick values greater than some N such that the distance between terms is not fixed in the desired cauchy manner for $|s_{n+k} - s_n|$. \square

6. State Boltzono-Weierstrass Theorem. Apply it to $a_n = \cos(2\pi n)$ and illustrate an example of a subsequence.

The Boltzono-Weierstrass Theorem is as follows: **Every bounded sequence has a convergent subsequence**

Consider the subsequence of $a_n = \cos(2\pi n)$ in which we only consider the values $k \in \mathbb{N}$ then we have the subsequence $\{1, 1, 1, 1, 1, \dots\}$ which has $\lim_{k \rightarrow \infty} a_{nk} = 1$

7. 11.1: Let $a_n = 3 + 2(-1)^n$ State the first five terms of the sequence:

$$a_n = \{1, 5, 1, 5, 1, 5, 1, 5, \dots\}$$

Give a subsequence that is constant with selection function: σ . One example of a subsequence that is constant is the choice of either the odd values or the even values each corresponding with one of the two possible values.

$$s_{nk} = \{1, 1, 1, 1, 1, \dots\} \quad \sigma(k) = 2k + 1$$

Here we are picking the subsequence of the odd values of the sequence giving all ones since every other term is 1.

8. 11.2 Consider the following sequences:

$$a_n = (-1)^n \quad b_n = \frac{1}{n} \quad c_n = n^2 \quad d_n = \frac{6n+4}{7n-3}$$

- (a) For each sequence give an example of a monotone subsequence

$$a_n : a_{nk} = \{-1, -1, -1, -1, \dots\} \quad \sigma(k) = 2k + 1$$

All other sequences are monotone so any of their subsequences should be monotone as well (this could get me in trouble but intuitively makes sense).

- (b) For each sequence give its set of subsequential limits By theorem 11.3 we have that if a sequence converges then every subsequence converges to the same limit. (cool theorem) Three of the sequence above all but the first have a limit thus their subsequence limits are equal to the sequence limits as follows:

$$\lim_{n \rightarrow \infty} b_n = 0 \quad \lim_{n \rightarrow \infty} c_n = \infty \quad \lim_{n \rightarrow \infty} d_n = \frac{6}{7}$$

So for these sequences the subsequence limits are single element sets consisting of just these values. For a_n however we see some special behavior due to the fact that there is an alternating behavior in the sequence between -1 and 1 so the subsequential limits is the set $\{-1, 1\}$

- (c) For each sequence give its limsup and lim inf

$$\liminf a_n = -1 \quad \limsup a_n = 1$$

By theorem 10.7 we have $\limsup = \liminf = \lim a_n$

$$\liminf b_n = \lim b_n = 0 \quad \limsup b_n = \lim a_n = 0$$

$$\liminf c_n = \lim c_n = \infty \quad \limsup c_n = \lim c_n = \infty$$

$$\liminf d_n = \lim d_n = \frac{6}{7} \quad \limsup d_n = \lim d_n = \frac{6}{7}$$

- (d) Which of the sequences converges? Diverges to ∞ ? Diverges to $-\infty$
All sequences except a and b converge. Sequence b diverges to infinity
- (e) Which of the sequences is bounded?
all sequences except c_n are bounded