

MATH 370 Homework 1

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Problem 1

Proof. Show that:

$$\sqrt{3} - \sqrt{2} \notin \mathbb{Q}$$

Let $S = \sqrt{3} - \sqrt{2}$

$$S^2 = 3 - 2\sqrt{3}\sqrt{2} + 2$$

$$\frac{5 - S^2}{2} = \sqrt{6}$$

Proof. Suppose that $\left(\frac{p}{q}\right)^2 = 6$ $p, q \in \mathbb{Z}$ and $\gcd(p, q) = 1$ (relatively prime).
this implies:

$$p^2 = 6q^2$$

If $a|bc$ then $a|b$ or $a|c$. This implies that either $6|p$ or $6|q$ meaning $6|p$.

If $6|p$ this implies $p = 6k, k \in \mathbb{N}$.

Returning to the original problem we now have:

$$6k^2 = q^2$$

Which implies that q is divisible by 6 this contradicts the original statement that the rational number $\frac{p}{q}$ is relatively prime. □

Therefore $\sqrt{6}$ is not rational and hence $\frac{5 - S^2}{2}$ is not rational by equivariance. □

Problem 2

$$1^2 + 2^2 + \dots + n^2 = \frac{1}{6}n(n+1)(2n+1)$$

Proof. Suppose

$$1^2 + 2^2 + \dots + n^2 = \frac{1}{6}n(n+1)(2n+1)$$

is true:

Base Case: $n=2$:

$$\begin{aligned} 1 + 4 &= \frac{1}{6}(2)(3)(5) \\ 5 &= 5 \end{aligned}$$

. $P(k)$:

$$\sum_{i=1}^k i^2 = \frac{(k)(k+1)(2k+1)}{6}$$

$P(k+1)$:

$$\sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k+2)(2(k+1)+1)}{6}$$

$$\begin{aligned} \sum_{i=1}^k i^2 + (k+1)^2 &= \frac{(k+1)(k+2)(2k+3)}{6} \\ \frac{k(k+1)(2k+1)}{6} + (k+1)^2 &= \frac{(k+1)(k+2)(2k+3)}{6} \\ \frac{2k^3 + 9k^2 + 13k + 6}{6} &= \frac{2k^3 + 9k^2 + 13k + 6}{6} \end{aligned}$$

□

Problem 3

Use induction to show”

$$\sum_{k=1}^n k^4 = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$$

Proof. Base Case: $n=1$:

$$1^4 = \frac{1^5}{5} + \frac{1^4}{2} + \frac{1^3}{3} - \frac{1}{30}$$

$$1 = 1$$

$P(k)$:

$$\sum_{i=1}^k i^4 = \frac{k^5}{5} + \frac{k^4}{2} + \frac{k^3}{3} - \frac{k}{30}$$

$P(k+1)$:

$$\sum_{i=1}^{k+1} i^4 = \frac{(k+1)^5}{5} + \frac{(k+1)^4}{2} + \frac{(k+1)^3}{3} - \frac{k+1}{30}$$

$$\sum_{i=1}^{k+1} i^4 = \frac{(k+1)^5}{5} + \frac{(k+1)^4}{2} + \frac{(k+1)^3}{3} - \frac{k+1}{30}$$

$$\sum_{i=1}^k i^4 + (k+1)^4 = \frac{(k+1)^5}{5} + \frac{(k+1)^4}{2} + \frac{(k+1)^3}{3} - \frac{k+1}{30}$$

$$\frac{k^5}{5} + \frac{k^4}{2} + \frac{k^3}{3} - \frac{k}{30} + (k+1)^4 = \frac{(k+1)^5}{5} + \frac{(k+1)^4}{2} + \frac{(k+1)^3}{3} - \frac{k+1}{30}$$

$$\frac{6k^5 + 15k^4 + 10k^3 - k}{30} + (k+1)^4 = \frac{6k^5 + 45k^4 + 130k^3 + 180k^2 + 119k + 30}{30}$$

$$\frac{6k^5 + 45k^4 + 130k^3 + 180k^2 + 119k + 30}{30} = \frac{6k^5 + 45k^4 + 130k^3 + 180k^2 + 119k + 30}{30}$$

□

Problem 4

Proof. Show that: $\sqrt{2 + \sqrt{2}}$.

Suppose that $\sqrt{2 + \sqrt{2}} = \frac{p}{q}$ where p and q are relatively prime

$$2 + \sqrt{2} = \frac{p^2}{q^2} \quad p, q \in \mathbb{Z}$$

$$\sqrt{2} = \frac{p^2 - 2q^2}{q^2}$$

$$2 = \frac{(p^2 - 2q^2)^2}{q^4}$$

$$2q^4 = (p^2 - 2q^2)^2$$

$$2q^4 = p^4 - 4p^2q^2 + 4q^4$$

$$p^4 = -4p^2q^2 + 4q^4 - 2q^4$$

$$p^4 = 2(-2p^2q^2 + 2q^4 - q^4)$$

$$\therefore 2|p \rightarrow p = 2k, k \in \mathbb{Z}$$

$$2 = \frac{((2k)^2 - 2q^2)^2}{q^4}$$

$$2 = \frac{(4k^2 - 2q^2)^2}{q^4}$$

$$2q^4 = (4k^2 - 2q^2)^2$$

$$2q^4 = 16k^4 - 16k^2q^2 + 4q^2$$

$$q^4 = 8k^4 - 8k^2q^2 + 2q^2$$

$$q^4 = 2(4k^4 - 4k^2q^2 + q^2)$$

$$\therefore 2|q$$

$2|p$ and $2|q$ which contradicts the notion that p and q are relatively prime.

□

Problem 5

Proof. Find an n such that $n^{10} < 2^n$. Then use induction to show that the inequality remains true for all numbers greater or equal to the one that you found, $n > 1$

Base case: $n = 59$:

$$59^{10} = 2^{59}$$

$$5.1111675e + 17 = 5.7646075e + 17$$

$P(k)$:

$$k^{10} < 2^k$$

$$k^{10} < 2^k$$

$$k^{10} < k^{10} + k^{10} < 2^k$$

$$k^{10} < k^{10} + k^{10} < 2(2^k)$$

□