MATH 370 Homework 4

Chris Camano: ccamano@sfsu.edu

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1. 9.8 (no proofs needed)

   (a)
   $$\lim_{n\to\infty} n^3 = \infty$$

   (b)
   $$\lim_{n\to\infty}(-n^3) = -\infty$$

   (c)
   $$\lim_{n\to\infty}(-n^n) = NOTEXIST$$

   (d)
   $$\lim_{n\to\infty}(1.01)^n = \infty$$

   (e)
   $$\lim_{n\to\infty}(n^n) = \infty$$

2. 9.17 (proof needed):
   Give a formal proof that
   $$\lim_{n\to\infty}(n^2) = +\infty$$
   using definition 9.8: **Definition 9.8**
   For a squence(s_n) we write
   $$\lim_{n\to\infty}(s_n) = +\infty$$
   provided:
   $$\forall_{M>0}\exists_N : n > N \to s_n > M$$

   Proof. To prove this fact we must choose some value for N such that the sequence s_n for any n greater than N s_n is greater than M. To do this let us consider the relationship between M and N. To satisfy the requirement under the definition of the function, let N=$\sqrt{m}$ then we have the following for n greater than our choice:
   $$\begin{aligned} &n > N \\ &n > \sqrt{M} \\ &n^2 > M \\ &s_n > M \end{aligned}$$
   So we have proven that with a selection of $N = \sqrt{M}$ then $n > N$ implies $s_n > M$ $\square$

3. 9.18

Proof. (a) Verify:

$$\sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}, a \neq 1$$

Proof. Base case: n=1:

$$a^0 + a^1 = \frac{1-a^2}{1-a}$$
$$1+a = \frac{1-a^2}{1-a}$$
$$1+a = \frac{(1+a)(1-a)}{1-a}$$
$$1+a = 1+a$$

$$P(k) : \sum_{i=0}^{k} a^i = \frac{1-a^{k+1}}{1-a}, a \neq 1$$

P(k+1)

$$\sum_{i=0}^{k+1} a^i = \frac{1-a^{k+2}}{1-a}$$

$$\sum_{i=0}^{k} a^i + a^{k+1} = \frac{1-a^{k+2}}{1-a}$$

$$\frac{1-a^{k+1}}{1-a} + a^{k+1} = \frac{1-a^{k+2}}{1-a}$$

$$\frac{1-a^{k+1}+a^{k+1}(1-a)}{1-a} = \frac{1-a^{k+2}}{1-a}$$

$$\frac{1-a^{k+1}+a^{k+1}-a^{k+2}}{1-a} = \frac{1-a^{k+2}}{1-a}$$
$$\frac{1-a^{k+2}}{1-a} = \frac{1-a^{k+2}}{1-a}$$

□

(b) **Find** $\lim_{n\to\infty}(\sum_{k=0}^{n} a^k), |a| < 1$
Find

$$\lim_{n\to\infty} \frac{1-a^{n+1}}{1-a}$$

$$\lim_{n\to\infty} \frac{1}{1-a} 1 - a^{n+1}$$

$$\lim_{n\to\infty} \left(\frac{1}{1-a}\right) \lim_{n\to\infty} (1 - a^{n+1})$$
$$\lim_{n\to\infty} \left(\frac{1}{1-a}\right) \lim_{n\to\infty} (1) - \lim_{n\to\infty} (a^{n+1})$$
$$\lim_{n\to\infty} \frac{1}{1-a} = \frac{1}{1-a}$$

(c) **Calculate** $\lim_{n\to\infty}(\sum_{k=0}^{n} \frac{1}{3^k})$

$$\sum_{k=0}^{n} \frac{1}{3^k} = \sum_{k=0}^{n} \left(\frac{1}{3}\right)^k$$

By part b we know that this sequence can be solved in the limit as:

$$\frac{1}{1-a}$$

thus:

$$\lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{3^k} = \frac{1}{\frac{2}{3}} = \frac{3}{2}$$

(d) **What is** $\lim_{n\to\infty}(\sum_{k=0}^{n} a^k), a \geq 1$

$$\lim_{n\to\infty} \left(\frac{1}{1-a} - \frac{a^{n+1}}{1-a}\right)$$
$$\left(\frac{1}{1-a} - \frac{\lim_{n\to\infty} a^{n+1}}{1-a}\right)$$
$$\left(\frac{1}{1-a} - \frac{\infty}{1-a}\right) = \infty$$

□

4. 10.1

Proof. Which of the sequences are increasing? decreasing? Bounded?

(a)

$$s_n = \frac{1}{n}$$

The sequence above converges to zero, and thus by theorem 9.1 is bounded. The sequence is decreasing as $s_n \geq s_{n+1} \forall n$

(b)

$$s_n = \frac{(-1^n)}{n^2}$$

This sequence is bounded but does not converge and is not monotonic

(c)
$$s_n = n^5$$
This sequence is increasing, is not bounded and does not converge

(d)
$$s_n = sin(\frac{n\pi}{7})$$
This sequence is bounded but does not converge and is not monotonic

(e)
$$s_n = (-2)^n$$
This sequence is not bounded, and is not monotonic

(f)
$$s_n = \frac{n}{3^n}$$
This sequence is decreasing and is bounded it converges as well.

□

5. 10.6

Proof.

Let (s_n) be a sequence such that:
$$|s_{n+1} - s_n| < 2^{-n} \quad \forall n \in \mathbb{N}$$
Prove s_n is a Cauchy sequence and hence a convergent sequence.

Proof. To prove that s_n is a Cauchy sequence let us prove then that
$$\forall \varepsilon > 0 \exists N : \quad m,n > N |s_m - s_n| < \varepsilon$$
In order to go about proving this statement we must first find a way to connect the original statement to a general statment about being cauchy. One such way of doing this is through estbalishing the following corresponce:
$$m = n + k, k \in \mathbb{Z}$$
such that:
$$|s_m - s_n| < \varepsilon \to |s_{n+k} - s_n| < \varepsilon$$
□

With this new construction we can now leverage the triangle inequality in the following way for each pairwise distance estimate:
$$|s_{n+k} - s_n| \leq \sum_{i=n}^{n+k} |s_{i+1} - s_i|$$
however, note also that we have the original inequality for the case of k=1, meaning we can develop this inequality further:
$$|s_{n+k} - s_n| \leq \sum_{i=n}^{n+k} |s_{i+1} - s_i| < \sum_{i=n}^{n+k-1} \frac{1}{2^i}$$

which simplifies to:

$$|s_{n+k} - s_n| < \sum_{i=n}^{n+k-1} \frac{1}{2^i}$$

We would like to use the geometric series formula here to simplify the right hand side however the starting index is problematic, so to correct this we can factor out a term of: $\frac{1}{2^k}$ leaving:

$$|s_{n+k} - s_n| < \frac{1}{2^n} \sum_{i=0}^{k-1} \frac{1}{2^i}$$

Applying formula for sum of a geometric sequence:

$$|s_{n+k} - s_n| < \frac{1}{2^n} \left[\frac{1 - \frac{1}{2^k}}{\frac{1}{2}} \right] < \frac{1}{2^n} \left[\frac{1}{\frac{1}{2}} \right]$$

$$|s_{n+k} - s_n| < \frac{1}{2^{n-1}}$$

From this statement we can now prove that it is cauchy by considering an epsilon as follos:

$$\varepsilon = \frac{1}{2^{n-1}}$$

Which gives a value of N equal to:

$$N = \log_2(\frac{1}{\varepsilon}) + 1$$

Finally leeaving us to the statement:

$$\forall \varepsilon > 0 \exists N = \log_2(\frac{1}{\varepsilon}) + 1 : n+k, n > N \to |s_{n+k} - s_n| < \varepsilon$$

Thus since s_n is cauchy by the completeness of the real numbers s_n is also convergent.

Is the result in (a) true if we only assume $|s_{n+1} - s_n| < \frac{1}{n}$?

If we make this assumption then we cannot say that the sequence is cauchy for the following reason: When we start to consider the same set up and get to a sum over $\frac{1}{n}$ we are now working with the harmonic series, or rather a partial sum of said harmonic series. This sequence diverges to infinity(slowly) Meaning we can always pick values greater than some N such that the distance between terms is not fixed in the desired cauchy manner for $|s_{n+k} - s_n|$. $\square$

6. State Boltzano-Weierstrass Theorem. Apply it to a_n= cos(2 pi n) and illustrate an example of a subsequence.
   The Boltzano-Weierstrass Theorem is as follows: **Every bounded sequence has a convergent subsequence**
   Consider the subsequence of $a_n = cos(2\pi n)$ in which we only consider the values $k \in \mathbb{N}$ then we have the subsequence $\{1, 1, 1, 1, 1, \}$ which has $lim_{k \to \infty} a_{nk} = 1$

7. 11.1: Let $a_n = 3+2(-1)^n$ State the first five terms of the sequence:

$$a_n = \{1,5,1,5,1,5,1,5,1,5,....\}$$

Give a subsequence that is constant with selection function: σ. One example of a subsequence that is constant is the choice of either the odd values or the even values each corresponding with one of the two possible values.

$$s_{nk} = \{1,1,1,1,1,....,\} \quad \sigma(k) = 2k+1$$

Here we are picking the subsequence of the odd values of the sequence giving all ones since every other term is 1.

8. 11.2 Consider the following sequences:

$$a_n = (-1)^n \quad b_n = \frac{1}{n} \quad c_n = n^2 \quad d_n = \frac{6n+4}{7n-3}$$

(a) For each sequence give an example of a monotone subsequence

$$a_n : a_{nk} = \{-1,-1,-1,-1..\}\sigma(k) = 2k+1$$

All other sequences are monotone so any of their subsequences should be monotone as well (this could get me in trouble but intuitivley makes sense).

(b) For each sequence give its set of subsequential limits By theorem 11.3 we have that if a sequence converges then every subsequence converges to the same limit. (cool theorem) Three of the sequence above all but the first have a limit thus their subsequence limits are equal to the sequence limits as follows:

$$\lim_{n\to\infty} b_n = 0 \quad \lim_{n\to\infty} c_n = \infty \lim_{n\to\infty} d_n = \frac{6}{7}$$

So for these sequences the subsequence limits are single elements sets consisting of just these values. For a_n however we see some special behavior due to the fact that there is an alternating behavior in the sequence between -1 and 1 so the subsequential limits is the set $\{-1,1\}$

(c) For each sequence give its limsup and lim inf

$$\liminf a_n = -1 \quad \limsup a_n = 1$$

By theorem 10.7 we have $\limsup = \liminf = \lim a_n$

$$\liminf b_n = \lim b_n = 0 \quad \limsup b_n = \lim a_b = 0$$

$$\liminf c_n = \lim c_n = \infty \quad \limsup c_n = \lim c_n = \infty$$

$$\liminf d_n = \lim d_n = \frac{6}{7} \quad \limsup d_n = \lim d_n = \frac{6}{7}$$

(d) Which of the sequences converges? Diverges to ∞? Diverges to $-\infty$
An does not converge, b_n converges, c_n diverges, d_n converges

(e) Which of the sequences is bounded ?
all sequences except c_n are boundedjnk,