

# MATH 370 Homework 1

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## Problem 1

*Proof.* Show that:

$$\sqrt{3} - \sqrt{2} \notin \mathbb{Q}$$

Let  $S = \sqrt{3} - \sqrt{2}$

$$S^2 = 3 - 2\sqrt{3}\sqrt{2} + 2$$

$$\frac{5 - S^2}{2} = \sqrt{6}$$

*Proof.* Suppose that  $\left(\frac{p}{q}\right)^2 = 6$   $p, q \in \mathbb{Z}$  and  $\gcd(p, q) = 1$  (relatively prime).  
this implies:

$$p^2 = 6q^2$$

If  $a|bc$  then  $a|b$  or  $a|c$ . This implies that either  $6|p$  or  $6|q$  meaning  $6|p$ .

If  $6|p$  this implies  $p = 6k, k \in \mathbb{N}$ .

Returning to the original problem we now have:

$$6k^2 = q^2$$

Which implies that  $q$  is divisible by 6 this contradicts the original statement that the rational number  $\frac{p}{q}$  is relatively prime. □

Therefore  $\sqrt{6}$  is not rational and hence  $\frac{5 - S^2}{2}$  is not rational by equivariance. □

## Problem 2

$$1^2 + 2^2 + \dots + n^2 = \frac{1}{6}n(n+1)(2n+1)$$

*Proof.* Suppose

$$1^2 + 2^2 + \dots + n^2 = \frac{1}{6}n(n+1)(2n+1)$$

is true:

Base Case:  $n=2$ :

$$\begin{aligned} 1 + 4 &= \frac{1}{6}(2)(3)(5) \\ 5 &= 5 \end{aligned}$$

.  $P(k)$ :

$$\sum_{i=1}^k i^2 = \frac{(k)(k+1)(2k+1)}{6}$$

$P(k+1)$ :

$$\sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k+2)(2(k+1)+1)}{6}$$

$$\begin{aligned} \sum_{i=1}^k i^2 + (k+1)^2 &= \frac{(k+1)(k+2)(2k+3)}{6} \\ \frac{k(k+1)(2k+1)}{6} + (k+1)^2 &= \frac{(k+1)(k+2)(2k+3)}{6} \\ \frac{2k^3 + 9k^2 + 13k + 6}{6} &= \frac{2k^3 + 9k^2 + 13k + 6}{6} \end{aligned}$$

□

## Problem 3

Use induction to show

$$\sum_{k=1}^n k^4 = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$$

*Proof.* Base Case:  $n=1$ :

$$1^4 = \frac{1^5}{5} + \frac{1^4}{2} + \frac{1^3}{3} - \frac{1}{30}$$

$$1 = 1$$

$P(k)$ :

$$\sum_{i=1}^k i^4 = \frac{k^5}{5} + \frac{k^4}{2} + \frac{k^3}{3} - \frac{k}{30}$$

$P(k+1)$ :

$$\sum_{i=1}^{k+1} i^4 = \frac{(k+1)^5}{5} + \frac{(k+1)^4}{2} + \frac{(k+1)^3}{3} - \frac{k+1}{30}$$

$$\sum_{i=1}^{k+1} i^4 = \frac{(k+1)^5}{5} + \frac{(k+1)^4}{2} + \frac{(k+1)^3}{3} - \frac{k+1}{30}$$

$$\sum_{i=1}^k i^4 + (k+1)^4 = \frac{(k+1)^5}{5} + \frac{(k+1)^4}{2} + \frac{(k+1)^3}{3} - \frac{k+1}{30}$$

$$\frac{k^5}{5} + \frac{k^4}{2} + \frac{k^3}{3} - \frac{k}{30} + (k+1)^4 = \frac{(k+1)^5}{5} + \frac{(k+1)^4}{2} + \frac{(k+1)^3}{3} - \frac{k+1}{30}$$

$$\frac{6k^5 + 15k^4 + 10k^3 - k}{30} + (k+1)^4 = \frac{6k^5 + 45k^4 + 130k^3 + 180k^2 + 119k + 30}{30}$$

$$\frac{6k^5 + 45k^4 + 130k^3 + 180k^2 + 119k + 30}{30} = \frac{6k^5 + 45k^4 + 130k^3 + 180k^2 + 119k + 30}{30}$$

□

## Problem 4

*Proof.* Show that:  $\sqrt{2 + \sqrt{2}}$ .

Suppose that  $\sqrt{2 + \sqrt{2}} = \frac{p}{q}$  where  $p$  and  $q$  are relatively prime

$$2 + \sqrt{2} = \frac{p^2}{q^2} \quad p, q \in \mathbb{Z}$$

$$\sqrt{2} = \frac{p^2 - 2q^2}{q^2}$$

$$2 = \frac{(p^2 - 2q^2)^2}{q^4}$$

$$2q^4 = (p^2 - 2q^2)^2$$

$$2q^4 = p^4 - 4p^2q^2 + 4q^4$$

$$p^4 = -4p^2q^2 + 4q^4 - 2q^4$$

$$p^4 = 2(-2p^2q^2 + 2q^4 - q^4)$$

$$\therefore 2|p \rightarrow p = 2k, k \in \mathbb{Z}$$

$$2 = \frac{((2k)^2 - 2q^2)^2}{q^4}$$

$$2 = \frac{(4k^2 - 2q^2)^2}{q^4}$$

$$2q^4 = (4k^2 - 2q^2)^2$$

$$2q^4 = 16k^4 - 16k^2q^2 + 4q^2$$

$$q^4 = 8k^4 - 8k^2q^2 + 2q^2$$

$$q^4 = 2(4k^4 - 4k^2q^2 + q^2)$$

$$\therefore 2|q$$

$2|p$  and  $2|q$  which contradicts the notion that  $p$  and  $q$  are relatively prime.

□

## Problem 5

*Proof.* Find an  $n$  such that  $n^{10} < 2^n$ . Then use induction to show that the inequality remains true for all numbers greater or equal to the one that you found,  $n > 1$

Base case:  $n = 59$  :

$$59^{10} = 2^{59}$$

$$5.1111675e + 17 = 5.7646075e + 17$$

$P(k)$ :

$$k^{10} < 2^k$$
$$x^{10} + \sum_{i=1}^k \binom{10}{i} x^{10-i} < 2^k$$

$P(k+1)$ :

□