MATH 370 Homework 4

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1. 9.8 (no proofs needed)

   (a)
   $$\lim_{n\to\infty} n^3 = \infty$$
   (b)
   $$\lim_{n\to\infty} (-n^3) = -\infty$$
   (c)
   $$\lim_{n\to\infty} (-n^n) = NOTEXIST$$
   (d)
   $$\lim_{n\to\infty} (1.01)^n = \infty$$
   (e)
   $$\lim_{n\to\infty} (n^n) = \infty$$

2. 9.17 (proof needed):
   Give a formal proof that
   $$\lim_{n\to\infty} (n^2) = +\infty$$
   using definition 9.8: **Definition 9.8**
   For a squence(s_n) we write
   $$\lim_{n\to\infty} (s_n) = +\infty$$
   provided:
   $$\forall_{M>0}\exists_N : n > N \to s_n > M$$

   Proof. To prove this fact we must choose some value for N such that the sequence s_n for any n greater than N s_n is greater than M. To do this let us consider the relationship between M and N. To satisfy the requirement under the definition of the function, let N=$\sqrt{m}$ then we have the following for n greater than our choice:
   $$\begin{aligned} &n > N \\ &n > \sqrt{M} \\ &n^2 > M \\ &s_n < M \end{aligned}$$

   So we have proven that with a selection of $N = \sqrt{M}$ then $n > N$ implies $s_n > M$ □

3. 9.18

Proof. (a) Verify:

$$\sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}, a \neq 1$$

Proof. Base case: n=1:

$$\begin{aligned}
a^0 + a^1 &= \frac{1-a^2}{1-a} \\
1 + a &= \frac{1-a^2}{1-a} \\
1 + a &= \frac{(1+a)(1-a)}{1-a} \\
1 + a &= 1 + a
\end{aligned}$$

$$P(k) : \sum_{i=0}^{k} a^i = \frac{1-a^{k+1}}{1-a}, a \neq 1$$

P(k+1)

$$\sum_{i=0}^{k+1} a^i = \frac{1-a^{k+2}}{1-a}$$

$$\sum_{i=0}^{k} a^i + a^{k+1} = \frac{1-a^{k+2}}{1-a}$$

$$\frac{1-a^{k+1}}{1-a} + a^{k+1} = \frac{1-a^{k+2}}{1-a}$$

$$\frac{1-a^{k+1}+a^{k+1}(1-a)}{1-a} = \frac{1-a^{k+2}}{1-a}$$

$$\frac{1-a^{k+1}+a^{k+1}-a^{k+2}}{1-a} = \frac{1-a^{k+2}}{1-a}$$

$$\frac{1-a^{k+2}}{1-a} = \frac{1-a^{k+2}}{1-a}$$

□

(b) **Find** $\lim_{n\to\infty}(\sum_{k=0}^{n} a^k), |a| < 1$
Find

$$\lim_{n\to\infty} \frac{1-a^{n+1}}{1-a}$$

$$\lim_{n\to\infty} \frac{1}{1-a} 1 - a^{n+1}$$

$$\lim_{n\to\infty} \left(\frac{1}{1-a}\right) \lim_{n\to\infty} (1 - a^{n+1})$$
$$\lim_{n\to\infty} \left(\frac{1}{1-a}\right) \lim_{n\to\infty} (1) - \lim_{n\to\infty} (a^{n+1})$$
$$\lim_{n\to\infty} \frac{1}{1-a} = \frac{1}{1-a}$$

(c) **Calculate** $\lim_{n\to\infty}(\sum_{k=0}^{n} \frac{1}{3^k})$

$$\sum_{k=0}^{n} \frac{1}{3^k} = \sum_{k=0}^{n} \frac{1}{3}^k$$

By part b we know that this sequence can be solved in the limit as:

$$\frac{1}{1-a}$$

thus:

$$\lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{3^k} = \frac{1}{\frac{2}{3}} = \frac{3}{2}$$

(d) **What is** $\lim_{n\to\infty}(\sum_{k=0}^{n} a^k), a \geq 1$

$$\lim_{n\to\infty} \left(\frac{1}{1-a} - \frac{1-a^{n+1}}{1-a}\right)$$
$$\left(\frac{1}{1-a} - \frac{\lim_{n\to\infty} 1 - a^{n+1}}{1-a}\right)$$
$$\left(\frac{1}{1-a} - \frac{\infty}{1-a}\right) = \infty$$

□

4. 10.1

Proof. Which of the sequences are increasing? decreasing? Bounded?

(a)

$$s_n = \frac{1}{n}$$

The sequence above converges to zero, and thus by theorem 9.1 is bounded. The sequence is decreasing as $s_n \geq s_{n+1} \forall n$

(b)

$$s_n = \frac{(-1^n)}{n^2}$$

This sequence is bounded but does not converge and is not monotonic

(c)

$$s_n = n^5$$

This sequence is increasing, is not bounded and does not converge

(d)

$$s_n = sin(\frac{n\pi}{7})$$

This sequence is bounded but does not converge and is not monotonic

(e)

$$s_n = (-2)^n$$

This sequence is not bounded, and is not monotonic

(f)

$$s_n = \frac{n}{3^n}$$

This sequence is decreasing and is bounded it converges as well.

□

5. 10.6

Proof.

Let (s_n) be a sequence such that:

$$|s_{n+1} - s_n| < 2^{-n} \quad \forall n \in \mathbb{N}$$

Prove s_n is a Cauchy sequence and hence a convergent sequence.

Proof. To prove that s_n is a Cauchy sequence let us prove then that

$$\forall \varepsilon > 0 \exists N : \quad m,n > N |s_n - s_m| < \varepsilon$$

□

Is the result in (a) true if we only assume $|s_{n+1} - s_n| < \frac{1}{n}$? □