MATH 370 Homework 1

Chris Camano: ccamano@sfsu.edu

August 31, 2022

α

Problem 1

Proof. Show that:

$$\sqrt{3}-\sqrt{2} \notin \mathbb{Q}$$

Let S=$\sqrt{3}-\sqrt{2}$

$$S^2 = 3 - 2\sqrt{3}\sqrt{2} + 2$$

$$\frac{5-S^2}{2} = \sqrt{6}$$

Proof. Suppose that $\left(\frac{p}{q}\right)^2 = 6 \quad p, q \in \mathbb{Z}$ and gcd(p,q)=1 (realtively prime).
this implies:

$$p^2 = 6q^2$$

If $a|bc$ then $a|b$ or $a|c$. This implies that either $6|p$ or $6|p$ meaning $6|p$.
If $6|p$ this implies $p = 6k, k \in \mathbb{N}$.
Returning to the original problem we now have:

$$6k^2 = q^2$$

Which implies that q is divisible by 6 this contradicts the original statement that the rational number $\frac{p}{q}$ is realtively prime.

$\square$

Therefore $\sqrt{6}$ is not rational and inturn $\frac{5-S^2}{2}$ is not rational by equivilancy. $\square$

Problem 2

$1^2 + 2^2 + ..n^2 = \frac{1}{6}n(n+1)(2n+1)$

Proof. Suppose

$$1^2 + 2^2 + ..n^2 = \frac{1}{6}n(n+1)(2n+1)$$

is true:
Base Case:n=2:

$$1 + 4 = \frac{1}{6}(2)(3)(5)$$
$$5 = 5$$

. P(k):

$$\sum_{i=1}^{k} i^2 = \frac{(k)(k+1)(2k+1)}{6}$$

P(k+1):

$$\sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k+2)(2(k+1)+1)}{6}$$

$$\sum_{i=1}^{k} i^2 + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$
$$\frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$
$$\frac{2k^3 + 9k^2 + 13k + 6}{6} = \frac{2k^3 + 9k^2 + 13k + 6}{6}$$

□

Problem 3

Use induction to show”

$$\sum_{k=1}^{n} k^4 = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$$

Proof. Base Case: n=1:

$$1^4 = \frac{1^5}{5} + \frac{1^4}{2} + \frac{1^3}{3} - \frac{1}{30}$$
$$1 = 1$$

P(k):

$$\sum_{i=1}^{k} i^4 = \frac{k^5}{5} + \frac{k^4}{2} + \frac{k^3}{3} - \frac{k}{30}$$

P(k+1):

$$\sum_{i=1}^{k+1} i^4 = \frac{(k+1)^5}{5} + \frac{(k+1)^4}{2} + \frac{(k+1)^3}{3} - \frac{k+1}{30}$$

$$\sum_{i=1}^{k+1} i^4 = \frac{(k+1)^5}{5} + \frac{(k+1)^4}{2} + \frac{(k+1)^3}{3} - \frac{k+1}{30}$$

$$\sum_{i=1}^{k} i^4 + (k+1)^4 = \frac{(k+1)^5}{5} + \frac{(k+1)^4}{2} + \frac{(k+1)^3}{3} - \frac{k+1}{30}$$

$$\frac{k^5}{5} + \frac{k^4}{2} + \frac{k^3}{3} - \frac{k}{30} + (k+1)^4 = \frac{(k+1)^5}{5} + \frac{(k+1)^4}{2} + \frac{(k+1)^3}{3} - \frac{k+1}{30}$$

$$\frac{6k^5 + 15k^4 + 10k^3 - k}{30} + (k+1)^4 = \frac{6k^5 + 45k^4 + 130k^3 + 180k^2 + 119k + 30}{30}$$

$$\frac{6k^5 + 45k^4 + 130k^3 + 180k^2 + 119k + 30}{30} = \frac{6k^5 + 45k^4 + 130k^3 + 180k^2 + 119k + 30}{30}$$

□

Problem 4

Proof. Show that: $\sqrt{2+\sqrt{2}}$.

Suppose that $\sqrt{2+\sqrt{2}} = \frac{p}{q}$ where p and q are relatively prime

$$2+\sqrt{2}=\frac{p^2}{q^2} \quad p,q \in \mathbb{Z}$$

$$\sqrt{2}=\frac{p^2-2q^2}{q^2}$$

$$2=\frac{(p^2-2q^2)^2}{q^4}$$

$$2q^4=(p^2-2q^2)^2$$

$$2q^4=p^4-4p^2q^2+4q^4$$

$$p^4=-4p^2q^2+4q^4-2q^4$$

$$p^4=2(-2p^2q^2+2q^4-q^4)$$

$$\therefore 2|p \to p=2k, k \in \mathbb{Z}$$

$$2=\frac{((2k)^2-2q^2)^2}{q^4}$$

$$2=\frac{(4k^2-2q^2)^2}{q^4}$$

$$2q^4=(4k^2-2q^2)^2$$

$$2q^4=16k^4-16k^2q^2+4q^2$$

$$q^4=8k^4-8k^2q^2+2q^2$$

$$q^4=2(4k^4-4k^2q^2+q^2)$$

$$\therefore 2|q$$

$2|p$and $2|q$ which contradicts the notion that p and q are relatively prime. $\square$

Problem 5

Proof. Find an nsuch that $n^{10} < 2^n$. Then use induction to show that the inequality remains true for all numbers greater or equal to the one that you found, $n > 1$

Base case: n =59 :

$$59^{10} < 2^{59}$$

$$5.1111675e + 17 < 5.7646075e + 17$$

P(k):

$$k^{10} < 2^k$$

P(k+1):

$$k^{10} < 2^k$$

$$2k^{10} < 2^{k+1}$$

$$k^{10} + k^{10} < 2^{k+1}$$

$$k^{10} + \sum_{i=1}^{9} \binom{10}{i} k^{10-i} + k(k^9 - \sum_{i=1}^{9} \binom{10}{i} k^{10-i-1}) < 2^{k+1}$$

the final term of $(k+1)^{10}$ is

$$\binom{10}{10} x^{10-10} = 1$$

For all $k \geq 59$

$$k(k^9 - \sum_{i=1}^{9} \binom{10}{i} k^{10-i-1}) > 1$$

so

$$k^{10} + \sum_{i=1}^{9} \binom{10}{i} k^{10-i} + 1 < k^{10} + \sum_{i=1}^{9} \binom{10}{i} k^{10-i} + k(k^9 - \sum_{i=1}^{9} \binom{10}{i} k^{10-i-1}) < 2^{k+1}$$

$$(k+1)^{10} < x^{10} + \sum_{i=1}^{9} \binom{10}{i} k^{10-i} + k(k^9 - \sum_{i=1}^{9} \binom{10}{i} k^{10-i-1}) < 2^{k+1}$$

$$(k+1)^{10} < 2^{k+1}$$

$\square$