MATH 370 Homework 4

Chris Camano: ccamano@sfsu.edu

September 21, 2022

1. 9.8 (no proofs needed)

   (a)
   $$\lim_{n\to\infty} n^3 = \infty$$

   (b)
   $$\lim_{n\to\infty} (-n^3) = -\infty$$

   (c)
   $$\lim_{n\to\infty} (-n^n) = NOTEXIST$$

   (d)
   $$\lim_{n\to\infty} (1.01)^n = \infty$$

   (e)
   $$\lim_{n\to\infty} (n^n) = \infty$$

2. 9.17 (proof needed):
   Give a formal proof that
   $$\lim_{n\to\infty} (n^2) = +\infty$$
   using definition 9.8: **Definition 9.8**
   For a squence(s_n) we write
   $$\lim_{n\to\infty} (s_n) = +\infty$$
   provided:
   $$\forall_{M>0}\exists_N : n > N \to s_n > M$$

   Proof. To prove this fact we must choose some value for N such that the sequence s_n for any n greater than N s_n is greater than M. To do this let us consider the relationship between M and N. To satisfy the requirement under the definition of the function, let N=$\sqrt{m}$ then we have the following for n greater than our choice:

   $$n > N$$
   $$n > \sqrt{M}$$
   $$n^2 > M$$
   $$s_n > M$$

   So we have proven that with a selection of $N = \sqrt{M}$ then $n > N$ implies $s_n > M$ □

3. 9.18

Proof. (a) Verify:

$$\sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}, a \neq 1$$

Proof. Base case: n=1:

$$\begin{aligned}
a^0 + a^1 &= \frac{1-a^2}{1-a} \\
1+a &= \frac{1-a^2}{1-a} \\
1+a &= \frac{(1+a)(1-a)}{1-a} \\
1+a &= 1+a
\end{aligned}$$

$$P(k) : \sum_{i=0}^{k} a^i = \frac{1-a^{k+1}}{1-a}, a \neq 1$$

P(k+1)

$$\sum_{i=0}^{k+1} a^i = \frac{1-a^{k+2}}{1-a}$$

$$\sum_{i=0}^{k} a^i + a^{k+1} = \frac{1-a^{k+2}}{1-a}$$

$$\frac{1-a^{k+1}}{1-a} + a^{k+1} = \frac{1-a^{k+2}}{1-a}$$

$$\frac{1-a^{k+1} + a^{k+1}(1-a)}{1-a} = \frac{1-a^{k+2}}{1-a}$$

$$\frac{1-a^{k+1} + a^{k+1} - a^{k+2}}{1-a} = \frac{1-a^{k+2}}{1-a}$$

$$\frac{1-a^{k+2}}{1-a} = \frac{1-a^{k+2}}{1-a}$$

□

(b) **Find** $\lim_{n\to\infty}(\sum_{k=0}^{n} a^k), |a| < 1$
Find

$$\lim_{n\to\infty} \frac{1-a^{n+1}}{1-a}$$

$$\lim_{n\to\infty}\frac{1}{1-a}1-a^{n+1}$$

$$\lim_{n\to\infty}\left(\frac{1}{1-a}\right)\lim_{n\to\infty}(1-a^{n+1})$$
$$\lim_{n\to\infty}\left(\frac{1}{1-a}\right)\lim_{n\to\infty}(1)-\lim_{n\to\infty}(a^{n+1})$$
$$\lim_{n\to\infty}\frac{1}{1-a}=\frac{1}{1-a}$$

(c) **Calculate** $\lim_{n\to\infty}(\sum_{k=0}^{n}\frac{1}{3^k})$

$$\sum_{k=0}^{n}\frac{1}{3^k}=\sum_{k=0}^{n}\left(\frac{1}{3}\right)^k$$

By part b we know that this sequence can be solved in the limit as:

$$\frac{1}{1-a}$$

thus:

$$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{3^k}=\frac{1}{\frac{2}{3}}=\frac{3}{2}$$

(d) **What is** $\lim_{n\to\infty}(\sum_{k=0}^{n}a^k), a\geq 1$

$$\lim_{n\to\infty}\left(\frac{1}{1-a}-\frac{a^{n+1}}{1-a}\right)$$
$$\left(\frac{1}{1-a}-\frac{\lim_{n\to\infty}a^{n+1}}{1-a}\right)$$
$$\left(\frac{1}{1-a}-\frac{\infty}{1-a}\right)=\infty$$

□

4. 10.1

Proof. Which of the sequences are increasing? decreasing? Bounded?

(a)

$$s_n=\frac{1}{n}$$

The sequence above converges to zero, and thus by theorem 9.1 is bounded. The sequence is decreasing as $s_n\geq s_{n+1}\forall n$

(b)

$$s_n=\frac{(-1^n)}{n^2}$$

This sequence is bounded but does not converge and is not monotonic

(c)
$$s_n = n^5$$
This sequence is increasing, is not bounded and does not converge

(d)
$$s_n = sin(\frac{n\pi}{7})$$
This sequence is bounded but does not converge and is not monotonic

(e)
$$s_n = (-2)^n$$
This sequence is not bounded, and is not monotonic

(f)
$$s_n = \frac{n}{3^n}$$
This sequence is decreasing and is bounded it converges as well.

□

5. 10.6

Proof.

Let (s_n) be a sequence such that:

$$|s_{n+1} - s_n| < 2^{-n} \quad \forall n \in \mathbb{N}$$

Prove s_n is a Cauchy sequence and hence a convergent sequence.

Proof. To prove that s_n is a Cauchy sequence let us prove then that

$$\forall \varepsilon > 0 \exists N : \quad m,n > N |s_n - s_m| < \varepsilon$$

□

Is the result in (a) true if we only assume $|s_{n+1} - s_n| < \frac{1}{n}$? □