MATH 425 Lecture 18

Chris Camano: ccamano@sfsu.edu

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Project now due 4/29.

Least Squares Solutions

Least squares solutions can be found without computation of $\hat{b}$

$$b - \hat{b} = b - A\hat{x} \in ColA^{\perp}$$

Which is the same as saying:

$$b - A\hat{x} \in \mathbf{NulA}^T$$

as:

$$\text{ColA}^{\perp} = \text{NulA}^T$$

This implies:

$$A^T(b - A\hat{x}) = 0$$
$$A^Tb - A^TA\hat{x} = 0$$
$$A^TA\hat{x} = A^Tb$$

This equation is largely more useful and does not require the computation of the orthogonal basis of A and the projection of b: $\hat{b}$.
This method is used in the case of an inconsistent solution.

QR decomposition

If A is an nxn matrix linerally independent columns, then A can be factored as A=QR where Q is an mxn matrix whose columns form an orthgonormal basis for the columns space of A and R is an nxn matrix upper triangualar invertible matrix with positive enteries on its diagonal.

$$A = QR$$

$$A = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & ... & v_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & \dots & r_{1n} \\ 0 & r_{22} & \dots & r_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & r_{nn} \end{bmatrix}$$

We can create an orthonormal basis for Col A by applying the Gram Schmidt process to Q. QQ^T is the identity matrix. therfore, :

$$Q^T A = R$$

Algorithm for QR decomposition: (Gram schmidt no shift) given A with linearly independent columns. Algorithm: Classical Gram-Schmidt:
def gram$_schmidt(A):$
$R = np.zeros(A.shape)$
$Q = np.zeros(A.shape)$
$forjinrange(len(A)):$
$v_j = A[j]$
$foriinrange(j-1):$
$r_ij = u[i].transpose() * A[j]$
$v_j = v_i - r_ij * u_i$
$R[i,j] = r_ij$
$r_jj = v_j.magnitude()$
$u_j = v_j/r_jj$
$Q[i][j] = u_j$

least Squares with QR

If A has linearly independent columns then A can be factored as A=QR:

$$A^T Ax = A^t b$$

$$A^T = (QR)^T = R^T Q^T \quad \therefore$$

$$A^T A = R^T Q^T Q^R$$

$$R^T R\hat{x} = R^T Q^T b$$

$$(R^T)^{-1} R^T R\hat{x} = (R^T)^{-1} R^T Q^T b$$

$$R\hat{x} = Q^T b$$

From here we can use back substitution to solve for the solution.

Curve fitting

Given m data points$(x_1, y_1), ..., (x_m, ym)$ if the data points lie on the line then for the line

$$y = \beta_0 + \beta_1 x$$

we have