MATH 425 Lecture 20

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Test on thursday.

Diagonalization of symmetric matricies.

Find eigenbalues of λ corresponding to eigen vectors.

$$det(A - \lambda \mathbb{I}_n) = 0$$

compute the determinant of the modified matrix and you will obtain a characteristic polynomial whos roots are : $\lambda_1, ..., \lambda_n$ where n is the degree of the characteristic polynomial.
To compute eigenvectors solve for:

$$(A - \lambda_i \mathbb{I}_n)v = \vec{0}$$

where λ_i corresponds to the ith eigenvalue

theorem

If a matrix A is shmmetric then any two eigenvectors from different eigenspaces are orthogonal.

Orthogonal diagonalization

This motivates that a matrix can be factored into the following:

$$A = QDQ^T$$

When Q is an orthogonal basis for A. we can obtain an orthogonal diagonalization when A is symmetric.
This implies A is symmetric as :

$$\begin{aligned}
A^T &= (QDQ^T)^T \\
A^T &= (Q^T)^T D^T Q^T \\
A^T &= QDQ^T \\
A^T &= A
\end{aligned}$$

Spectral Theorem

Let A be a symmetric matrix then:

- Every eigenvalue of A is Real and the dimension of the eigenspace is corresponding to λ equals the multiplicity of the eigenvalue.
- Eigenvectors corresponding to different eigenvalues are orthogonal (eigenspaces are orthgonal.)
- Matrix A is orthogonally diagonalizable.
- The columns of Q form an orthonormal basis.

Singular value decomposition

Given an Mxn matrix A with rank r. can be factored as :

$$A = U_{mxm}\Sigma_{mxn}V_{nxn}^T$$

This is an orthgonal matrix multiplied by a "diagonal matrix" then by another orthogonal matrix.
U is an mxm unitary matrix, sigma is the diagonal matrix of singular the singular values where:

$$\sigma_i = \sqrt{\lambda_i}$$

r=rank(A)=dim(ColA)

Finding V

$$A = U\Sigma V^T$$

so

$$A^T A = (U\Sigma V^T)^T U\Sigma V^T = V\Sigma^T U^T U\Sigma V^T$$
$$= V\Sigma^2 V^T = V\Lambda V^T$$

So we have proven the orthgonal diagonalization of the matrix as well. If you find $A^t A$ and orthgonally diagonlize then square root D. If A is symmetric then the eigenvalues of A the sqaures of the singular values since you are just squaring A when you compute $A^T A$

Computing U in UΣV

Approach number 1:

$$
\begin{aligned}
&A = U\Sigma V^T \\
&AA^T = U\Sigma V^T(U\Sigma V^T)^T \\
&AA^T = U\Sigma^2 U^T \\
&AA^T = U\Lambda U^T
\end{aligned}
$$