MATH 425 Lecture 9

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Linear Transformations

Let U and V be vetor spaces over $\mathbb{F}(\mathbb{R}, \mathbb{C}, GF(2))$ let $x, y \in U$ and $c \in \mathbb{F}$

$T : U \mapsto V$ is a linear transformation $\iff$

$$\begin{aligned} T(x+y) &= T(x) + T(y) \\ T(cx) &= cT(x) \\ &\therefore \\ T(cx+y) &= cT(x) + T(y) \end{aligned}$$

A linear operator
A linear operator is a transformation :

$$T : U \mapsto U$$

this is to say linear transformation that takes a vector space to itself

A transformation T is not linear if $T(\vec{0}_U) \neq \vec{0}_V$

Framing Linear Transformations

A linear transformation can be thought of as a function that maps some$x \in \mathbb{F}^n$ where $\mathbb{F}^n$ is the domain to some $y \in \mathbb{F}^m$ where $\mathbb{F}^m$ is the co-domain. The domain and co domain do not have to have different dimensions.

$$\text{if } \dim(U) < \dim(V)$$

The region a linear transformation maps the values of the domain to is called the range r where:

$$r \subset \mathbb{F}^m$$

This is to say the range is a subset of the codomain of the linear transformation

Boateng description
T(x) is the image of x under the action of T the set of all images T (x) is called the range of T.

Additional properties of linear transformations
All matrix-vector multiplications are linear transformations:

$$\text{Let} A \in \mathbb{F}^{mxn}, x, y \in \mathbb{F}^n \wedge c \in \mathbb{F}$$

$$A(cx + y) = cA(x) + A(y)$$

All linear transformations on finite dimensional vector spaces will alwways have a matrix representation

The standard matrix A for the linear transformation T: $\mathbb{F}^n \mapsto \mathbb{F}^m$ is completely determined by what id does to the columns of the nxn identity matrix $\mathbb{I}_n$
Proof:

$$\text{Let } x \in \mathbb{F}^n \rightarrow x = I = x \begin{bmatrix} | & | & & | \\ e_1, & e_2, & .., & e_n \\ | & | & & | \end{bmatrix}$$

Then:

$$\begin{aligned} T(x) &= t(x_1e_1 + ... + x_ne_n) \\ &= x_1T(e_1) + ... + x_nT(e_n) \end{aligned}$$
$$\begin{bmatrix} | & | & & | \\ T(e_1), & T(e_2), & .., & T(e_n) \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ ... \\ x_n \end{bmatrix}$$