MATH 425 Lecture 18

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Problem 1

Find an orthogonal basis for the column space of the matrix A:

$$\begin{bmatrix} -1 & 6 & 6 \\ 3 & -8 & 3 \\ 1 & -2 & 6 \\ 1 & -4 & -3 \end{bmatrix}$$

For this problem I will use gramn schmidt orthogonalization to obtain an orthonormal basis for the columns of A.

$$\text{Let } u_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}$$

$$\text{Let } u_2 = \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} - \frac{\begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\text{Let } u_3 = \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} - \frac{\begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} - \frac{v_3 \cdot u_2}{u_2 \cdot u_2} u_2 = \begin{bmatrix} -1 \\ 1 \\ 3 \\ -1 \end{bmatrix}$$

Thus the set :

$$\left\{ \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ \\ 1 \\ \\ 3 \\ \\ -1 \end{bmatrix} \right\}$$

Forms an orthogonal basis for the given vectors. To validate this claim I have checked that all three vectors in the described set are orthogonal to one another computationally.

Problem 2

2. Find an orthonormal basis for the column space of the matrix $A = \begin{bmatrix} 3 & -3 & 0 \\ -4 & 14 & 10 \\ 5 & -7 & -2 \end{bmatrix}$.

First note that A $\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ Let $\{v_1, v_2, v_3\}$ denote the columns of A. The columns space of A is $\{v_1, v_2\}$ as these are the linearly independent columns of A. To obtain an orthonormal basis for A we will first compute an orthogonal basis then normalize. The orthogonal basis of the column space of A can be found using gram schmidt.

$$u_1 = v_1 \qquad\qquad u_2 = v_2 - \frac{v_2 \cdot u_1}{u_1 \cdot u_1} u_1 = \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix}$$

Normalizing both:

$$e_1 = \frac{u_1}{||u_1||} = \frac{u_1}{5\sqrt{2}}$$
$$e_2 = \frac{u_2}{||u_2||} = \frac{u_2}{3\sqrt{6}}$$

Thus: $\{\frac{1}{5\sqrt{2}}u_1, \frac{1}{3\sqrt{6}}u_2\}$ form an orthonormal basis for the column space of A. also note that the dotproduct of e_1 and e_2 is zero.

Problem 3

3. Let $\mathbf{u}_1, \ldots, \mathbf{u}_p$ be an orthogonal basis for the subspace W of $\mathbb{R}^n$, and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be defined by $T(\mathbf{x}) = \text{proj}_W \mathbf{x}$. Show that T is a linear transformation.

Let $u_1, ...u_p$ be an orthogonal basis for the subspace W of $\mathbb{R}^n$ and let T: $\mathbb{R}^n \mapsto \mathbb{R}^n$ ve defined by T(x)=proj$_W x$ Show that T is a linear transformation.

$$T(x) = \frac{x \cdot u_1}{||u_1||} u_1 + ... + \frac{x \cdot u_n}{||u_n||} u_n$$

$$T(cx + y) = \frac{(cx_1 + y_1) \cdot u_1}{||u_1||} u_1 + ... + \frac{(cx_p + y_p) \cdot u_p}{||u_p||} u_p$$

$$T(cx + y) = \frac{(cx_1 \cdot u_1 + y_1 \cdot u_1)}{||u_1||} u_1 + ... + \frac{(cx_p \cdot u_p + y_p \cdot u_p)}{||u_p||} u_p$$

$$T(cx + y) = \frac{(cx_1 \cdot u_1)}{||u_1||} u_1 + \frac{y_1 \cdot u_1}{||u_1||} u_1 .... + \frac{(cx_1 \cdot u_p)}{||u_p||} u_p + \frac{y_p \cdot u_p}{||u_p||} u_p$$

$$T(cx + y) = T(cx + y) = \frac{(cx_1 \cdot u_1)}{||u_1||} u_1 .... + \frac{(cx_1 \cdot u_p)}{||u_p||} u_p + T(y)$$

$$T(cx + y) = cT(x) + T(y)$$

Problem 4

4. Let $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \\ 1 & 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}$. Find (a) the orthogonal projection of $\mathbf{b}$ onto Col A and (b) a least-squares solution of $A\mathbf{x} = \mathbf{b}$.

Note that the columns of A already form an orthogonal basis ! The two columns are linearly independent and are orthogonal to one another. This means that to compute the orthogonal projection of b onto colA we can proceed normally:

$$proj_b(ColA) = \frac{b \cdot a_1}{a_1 \cdot a_1} a_1 + \frac{b \cdot a_2}{a_2 \cdot a_2} a_2 = 3a_1 + \frac{1}{2} a_2 = \begin{bmatrix} 4 \\ -1 \\ 4 \end{bmatrix}$$

To find the least squares solution we must compute:

$$A^T A x = A^T b$$

$$\begin{bmatrix} 3 & 0 \\ 0 & 24 \end{bmatrix} x = \begin{bmatrix} 9 \\ 12 \end{bmatrix}$$

$$x = \begin{bmatrix} 3 \\ \frac{1}{2} \end{bmatrix}$$

Meaning that the least squares solution is $3a_1 + \frac{1}{2} a_2$ which is the same as shown above during th computation of the orthgonal projection of b onto the columns space.

Problem 5

5. Let $A = \begin{bmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -5 \\ 8 \\ 1 \end{bmatrix}$. Find the least-squares solution of $A\,\mathbf{x} = \mathbf{b}$.

$$A^T A x = A^T b$$

$$\begin{bmatrix} 12 & 8 \\ 8 & 10 \end{bmatrix} x = A^t b$$

$$\begin{bmatrix} 12 & 8 \\ 8 & 10 \end{bmatrix} x = \begin{bmatrix} -24 \\ -2 \end{bmatrix}$$

$$x = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

Thus $-4a_1 + 3a_2 = \begin{bmatrix} -5 \\ 8 \\ 1 \end{bmatrix}$ is the least squares solution. Also note that this means that b is in the columns space of A and could have been found simply by reducing the augmented matrix [A|b]. You got me!

Problem 6

6. Let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \qquad \text{and } \mathbf{b} = \begin{bmatrix} 7 \\ 2 \\ 3 \\ 6 \\ 5 \\ 4 \end{bmatrix}.$$

Describe all least-squares solutions of the equation $A\mathbf{x} = \mathbf{b}$.

After solving the equation $A^T A x = A^T b$ you obtain the following:

$$\begin{bmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The solution $\hat{x}$ takes the form $\begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} x_3$ since the variable x_3 is free.

Problem 7

**** possibly wrong 7. Let

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 4 \\ 1 & -1 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \\ 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix},$$

be the factorization $A = QR$ and let $\mathbf{b} = \begin{bmatrix} -1 \\ 6 \\ 5 \\ 7 \end{bmatrix}$. Use the QR factorization to find the least-squares solution of $A\mathbf{x} = \mathbf{b}$.

$$\begin{aligned} &QRx = b \\ &Q^T QRx = Q^T b \\ &Rx = Q^T b \end{aligned}$$

$$b = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix} x = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -1 \\ 6 \\ 5 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 & \frac{17}{2} \\ 0 & 5 & \frac{9}{10} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{29}{10} \\ 0 & 1 & \frac{9}{10} \end{bmatrix}$$

meaning that the least squares solution for b is $\hat{x} = \frac{29}{10} a_1 + \frac{9}{2} a_2$ where a_1, a_2 are the columns of A

Problem 8

8. A healthy child's systolic blood pressure p (in millimeters of mercury) and weight w (in pounds) are approximately related by the equation

$$\beta_0 + \beta_1 \ln w = p.$$

Use the following experimental data to estimate the systolic blood pressure of a healthy child weighing 100 pounds.

w	44	61	81	113	131
$\ln w$	3.78	4.11	4.41	4.73	4.88
p	91	98	103	110	112

To solve this problem we will use least squares regression with a linear line of regression with the following:

$$X = \begin{bmatrix} 1 & 3.78 \\ 1 & 4.11 \\ 1 & 4.41 \\ 1 & 4.73 \\ 1 & 4.88 \end{bmatrix} \; y = \begin{bmatrix} 91 \\ 98 \\ 103 \\ 110 \\ 112 \end{bmatrix}$$

$$X^T X\beta = X^T y$$

$$\begin{bmatrix} 5 & 21.91 \\ 21.91 & 96.8159 \end{bmatrix} \beta = \begin{bmatrix} 514 \\ 2267.85 \end{bmatrix}$$

reduing we get: $\beta = \begin{bmatrix} 18.5492 \\ 19.2266 \end{bmatrix}$ implying that ρ=18.5492+19.2266ln(w). To predict the blood pressure of a young boy weighing 100 pounds we can use this function evaluated at w=100.

$$\rho = 18.5492 + 19.2266 ln(100) = 107.091$$

So a young boy weighing 100 pounds has approximately a systolic blood pressure of 107 ml of mercury.

Problem 9

9. To measure the takeoff performance of an airplane, the horizontal position of the plane was measured every second, from $t = 0$ to $t = 12$. The positions (in feet) were: 0, 8.8, 29.9, 62.0, 104.7, 159.1, 222.0, 294.5, 380.4, 471.1, 571.7, , 686.8, 809.2 .

(a) Find the least-squares cubic curve $y = \beta_0 + \beta_1 t + \beta_2 t^2 + \beta_3 t^3$ for these data.

(b) Use the result of (a) to estimate the velocity of the plane when $t = 4.5$ seconds.

To solve this problem we will use least squares regression with a linear line of regression with the following:

$$X = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \\ 1 & 5 & 25 & 125 \\ 1 & 6 & 36 & 216 \\ 1 & 7 & 49 & 343 \\ 1 & 8 & 64 & 512 \\ 1 & 9 & 81 & 729 \\ 1 & 10 & 100 & 1000 \\ 1 & 11 & 121 & 1331 \\ 1 & 12 & 144 & 1728 \end{bmatrix} \quad y = \begin{bmatrix} 0 \\ 8.8 \\ 29.9 \\ 62.0 \\ 104.7 \\ 159.1 \\ 222.0 \\ 294.5 \\ 380.4 \\ 471.1 \\ 571.1 \\ 686.8 \\ 809.2 \end{bmatrix}$$

Solving for beta in the following equation $X^T X\beta = X^T y$
:Using python gives the following vector:

$$\begin{bmatrix} -.86 \\ 4.7 \\ 5.56 \\ -.027 \end{bmatrix}$$

Thus the equation of the cubic curve is:

$$y = -.8558 + 4.7025t + 5.5554t^2 - .0274t^3$$

The equation evaluated at t=4.5 is yeilds a horizontal position of 130.42ft.

To find the velocity at 4.5 seconds we can compute the time derivative of position which gives.

$$v = 4.7025 + 11.11086t - .0822t^2$$

evaluated at 4.5 seconds the veloicty is

$$53.0368$$

feet per second