MATH 425 Homework 6b

Chris Camano: ccamano@sfsu.edu

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Problem 1

Find the singular values of $\begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}$

We begin by computing $A^T A$ which is as follows:

$$\begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}^T \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 25 & 0 \\ 0 & 0 \end{bmatrix}$$

The characteristic polynomial for $A^T A$ is then : $\lambda^2 - 25\lambda$ meaning our eigenvalues are 25 and 0. From here finding the singular values amounts to taking the square root of the eigen values yeilding: 5 and 0 as the final answer.

Problem 2

- Since we have only three singular values for this decomposition this means that the rank of the matrix is 3.
- Since the rank of the matrix is equal to 3 then the basis for ColA is equivilant to the first 3 vectors of U. Since U has only three vectors a basis for ColA is U. A basis for NulA is equivilant to the $v_{r+1} = v_n$ columns of V. In this case this means that a basis for NulA is the last column of V in our decomposition.

Problem 3

Let

$$A = U\Sigma V^T$$

Then:

$$A^{-1} = (U\Sigma V^T)^{-1}$$
$$A^{-1} = V^{T^{-1}}\Sigma^{-1}U^{-1}$$

U and V are orthogonal so U or V tranpose is equal to the inverse:

$$A^{-1} = V^{T^T}\Sigma^{-1}U^T$$
$$A^{-1} = V\Sigma^{-1}U^T$$

Problem 4

$$|det(A)| = |det(U\Sigma V^T)|$$
$$|det(A)| = |det(U)det(\Sigma)det(V^T)|$$
$$|det(A)| = |det(U)\Big(\prod_{i=1}^{r}\sigma_i\Big)det(V^T)|$$

lemma: determinant of an orthogonal matrix is 1 or -1
Since the matrix is orthogonal $AA^T = \mathbb{I}_n$ thus

$$det(AA^T) = det(\mathbb{I}_n)$$
$$det(AA^T) = 1$$

the determinant is a linear transformation so we can expand the left hand side.

$$det(A)det(A^T) = 1$$

det(A)=det(A^T)so we can reexpress this as:

$$det(A)^2 = 1$$

meaning that

$$det(A) = \pm 1$$

for any orthogonal matrix. returning to our proof

$$|det(A)| = |det(U)det(\Sigma)det(V^T)|$$
$$|det(A)| = |\pm 1\Big(\prod_{i=1}^{r}\sigma_i\Big)\pm 1|$$
$$|det(A)| = |\pm 1\Big(\prod_{i=1}^{r}\sigma_i\Big)|$$
$$|det(A)| = \prod_{i=1}^{r}\sigma_i$$

$\blacksquare$

Problem 5

We first start by computing the singular value decomposition of A in a step towards a pseudoinverse description of the minimal solution.
*Values computed with numpy

$$AA^t = \begin{bmatrix} 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \end{bmatrix}$$

The eigenvalues of this matrix are

$$\lambda_1 = 6 \quad \lambda_2 = 2 \quad \lambda_3 = 0 \quad \lambda_4 = 0$$

Thus D is equal to:

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

computing relevant eigenvectors:

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \quad v_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad v_4 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

Normalizing we form the columns of U as:

$$\text{U=}\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

For the columns of V we compute

$$v_i = \frac{1}{\sigma_i} A^T u_i$$

yeilding:

$$v_1 = \begin{bmatrix} \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix} \quad v_2 = \begin{bmatrix} 0 \\ \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

We are missing an additional vector for V so we will select a linearly independent one In this case one can be found that is orthogonal by solving $[v_1, v_2]^T x = 0$ (the nul space):

$$v_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Meaning that

$$V = \begin{bmatrix} \frac{2}{\sqrt{6}} & 0 & -1 \\ \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} & 1 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{6} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{6}} & 0 & -1 \\ \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} & 1 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & 1 \end{bmatrix}^T$$

$$D = \begin{bmatrix} \sqrt{6} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$$

Since the rank of the matrix of 2:

$$U_r = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad V_r = \begin{bmatrix} \frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\hat{x} = V_r D^-1 U_r^T b$$

$$\hat{x} = \begin{bmatrix} \frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{6} & 0 \\ 0 & \sqrt{2} \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}^T \begin{bmatrix} 1 \\ 3 \\ 8 \\ 2 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} \frac{7}{3} \\ \frac{-1}{3} \\ \frac{8}{3} \end{bmatrix}$$