Analyzing Performance of Covert Networks Using a Toughness-like Measure

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- vertices represent members of covert group
- edges indicate direct communication
- subvert a vertex means delete the closed neighborhood of vertex (following Gunther and Hartnell)
- ASSUMPTION: each vertex is equally likely to be subverted (group members are uniformly exposed)

Measuring Secrecy and Information: Lindelauf

- SECRECY of graph G is measured as the expected fraction of the network to survive after subversion of a vertex:

$$\text{TS} = \frac{1}{|V|} \cdot \sum_{v_i \in V} \frac{|V| - |N[v_i]|}{|V|}$$

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- INFORMATION is measured by the normalized reciprocal of total distance:

$$\text{INF} = \frac{|V| \cdot (|V| - 1)}{\sum_{v_i \in V} td(v_i)}$$

where $td(v)$ is the sum of distances of all vertices of G from v.

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- But if any vertex of a star is subverted, the resulting graph either has no vertices s or has $p-1$ isolated vertices.

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- $$\mathrm{SEC} = \frac{1}{|V|^2} \cdot \sum_{v_i \in V} \frac{|V| - |N[v_i]|}{\Omega(G - N[v_i])}$$
- A variation on the parameter *toughness* defined by Chvátal.

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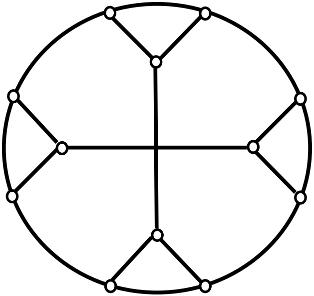
- PERF*(P_n) $= \dfrac{3(n^2+n-8)}{2n^2(n+1)}$
- For large n, PERF*(C_n) $>$ PERF*(P_n) $>$ PERF*($Star_n$)

The Barbell Graph



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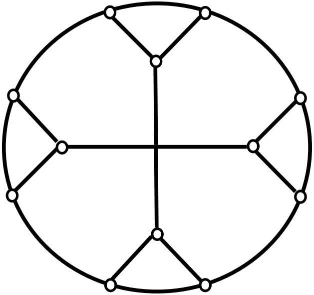

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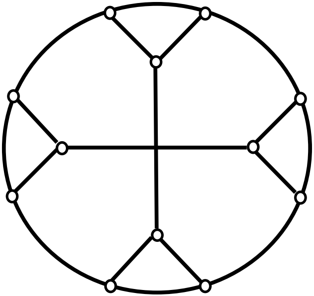

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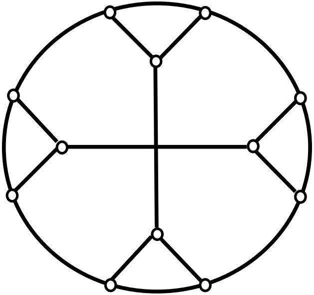

- When 6 divides $|V|$, barbell graph is inflation of $C_{|V|/3}$ with diagonals
- Modified barbell graphs exist for all $|V| > 6$
- Barbells and modified barbells achieve maximum toughness (Ferland & Doty)
- Barbells achieve maximum neighbor connectivity (Gunther & Hartnell)

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- For $n \geq 8$,
  PERF*($Barbell_n$) $>$ PERF*(C_n) $>$ PERF*(P_n) $>$ PERF*($Star_n$)
- There are order 12 graphs with higher PERF* than $Barbell_{12}$.

Maximum Performance among Trees

- A tree in which exactly one vertex has degree larger than 2 is called a ray tree. The vertex whose degree is larger than 2 is called the center. Note that the number of leaves in the tree equals the degree of the center.

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- A balanced ray tree is a ray tree in which

$$|dist(v_i, c) - dist(v_j, c)| \leq 1$$

  for all leaves v_i, v_j.

Ray Trees–Some Results

Lemma

Among ray trees with n vertices and maximum degree Δ,

- *balanced ray tree has maximum* SEC*.*
- *balanced ray tree has maximum* INF*.*
- *balanced ray tree has maximum* PERF**.*

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Lemma

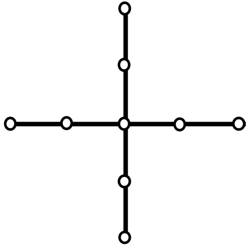
Among balanced ray trees with n vertices,

- *maximum* SEC *occurs when max degree* $\Delta = \lfloor \frac{p-1}{3} \rfloor$.
- *maximum* PERF* *occurs when max degree* $\Delta = \lfloor \frac{p-1}{2} \rfloor$.

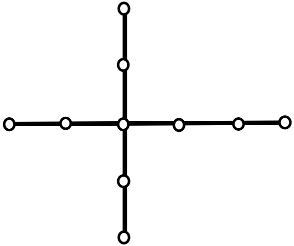
A Conjecture that Should be a Small Theorem

Conjecture

Among trees with n vertices, the balanced ray tree with max degree $\Delta = \lfloor \frac{p-1}{2} \rfloor$ *has maximum* PERF* .



Odd number of vertices



Even number of vertices