Performance of Covert Networks

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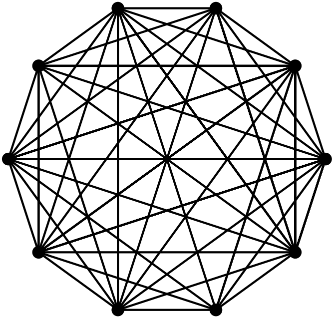
What is the best design?

Idea 1.

You could have every member in contact with every other member.



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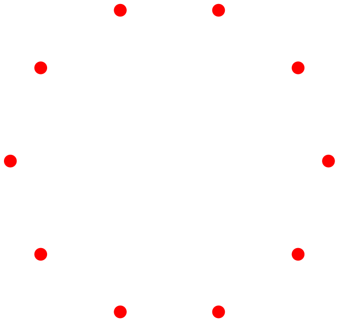
You could have every member in contact with every other member.

But then the subversion of any member betrays the entire organization!



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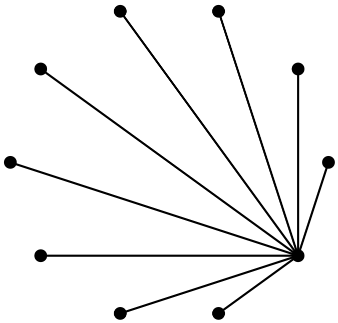
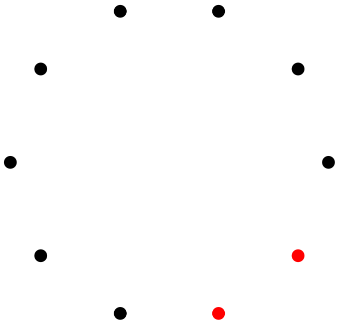
You could have one member in contact with all others, none of whom are in contact with each other.



Idea 2.

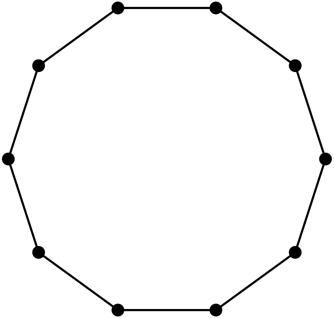
You could have one member in contact with all others, none of whom are in contact with each other.

But then the subversion of any member leaves the organization totally disconnected!



Idea 3.

You could have every member in contact with exactly two others.



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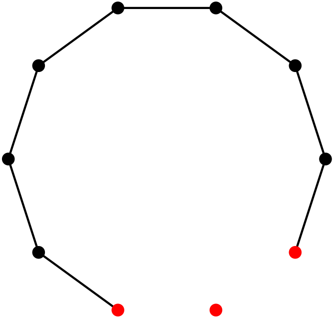

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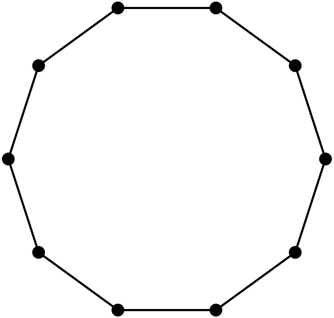

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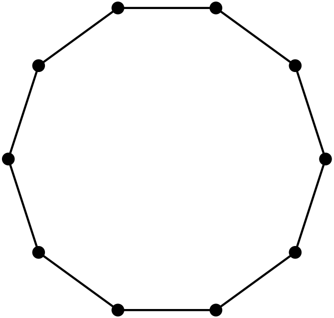
We need a measure of *performance* that takes into account the conflicting demands of security and information.



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$$\begin{aligned} \text{COVERT NETWORK} &\rightarrow \text{CONNECTED GRAPH} \\ \text{“SPIES”} &\rightarrow \text{VERTICES} \\ \text{MUTUAL AQUAINTANCE} &\rightarrow \text{EDGE} \\ \text{CAPTURE} &\rightarrow \text{DELETE CLOSED NBD} \end{aligned}$$

$$\text{PERFORMANCE} = \text{INFORMATION} \times \text{SECURITY}$$

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$$perf(G) = \binom{n}{2} \frac{1}{n^2 D(G)} \cdot \sum_{v \in V(G)} \frac{n - deg(v) - 1}{\Omega(G - N[v])}$$

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$$perf(G) = \frac{1}{n} \cdot \frac{\text{average vertices/components upon deleting a closed nbd}}{\text{average distance between vertices}}$$

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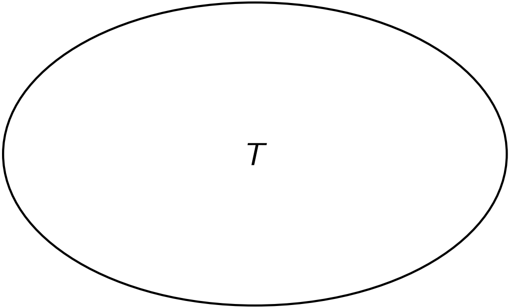
Our main result is to prove this conjecture.

Proof

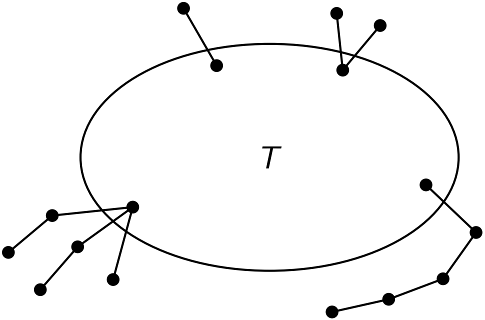
Let T be a tree with maximum performance.



Proof

Let T be a tree with maximum performance.

We consider the *periphery* of T, the collection of leaves together with the path from each leaf to the nearest vertex of degree greater than 2.



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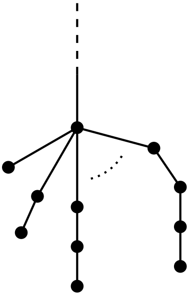
The performance of T does not decrease if we rearrange a peripheral superstar into a 2-balanced superstar.



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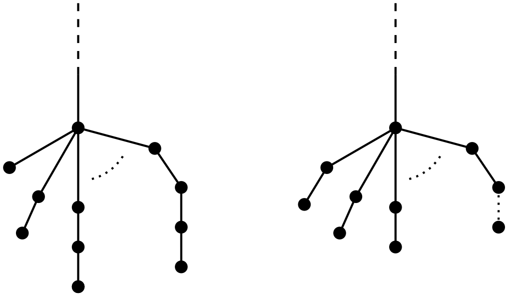

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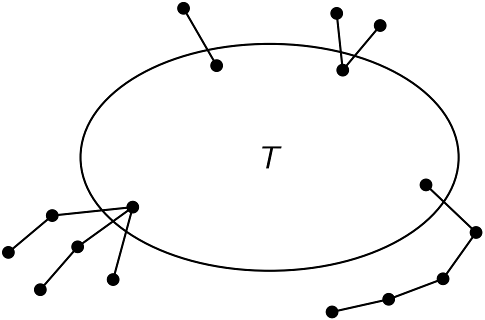

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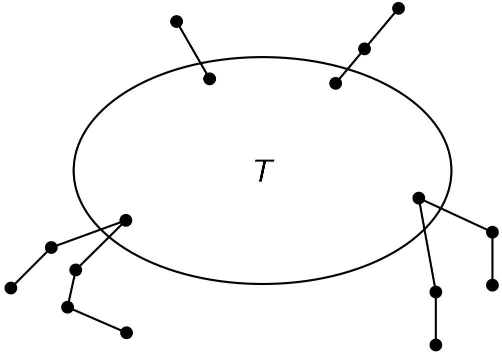

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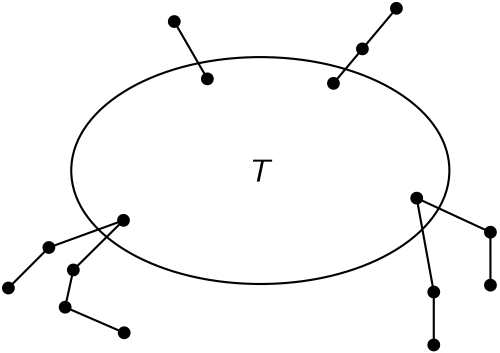
If there are any long paths in T, we can remove vertices from the middle of these paths and adjoin them to the nearest peripheral superstar.



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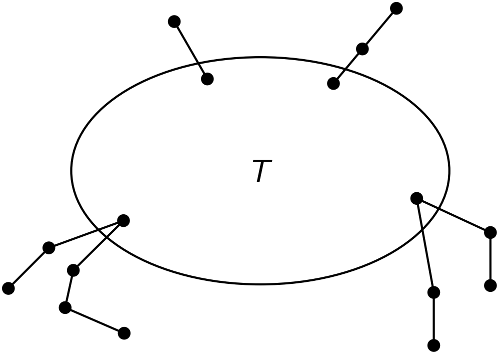
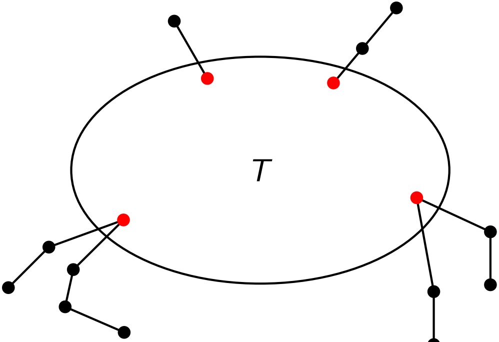
(Repeating the previous step if neccessary.)



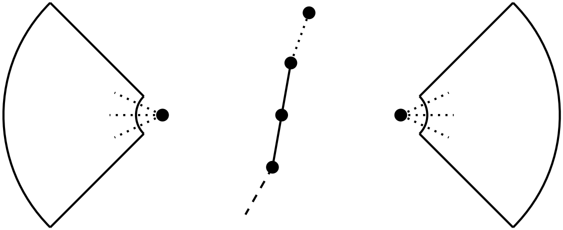
Proof

Claim: Some pair $\{u, v\}$ of red vertices satisfy

$$deg(u) \geq deg(v)$$
$$D(u) \leq D(v)$$



Proof, con't.



Low degree, High D

High degree, Low D

We remove an arm from the left and attach it to the right.

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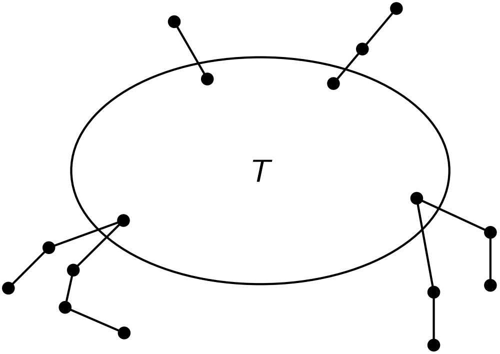

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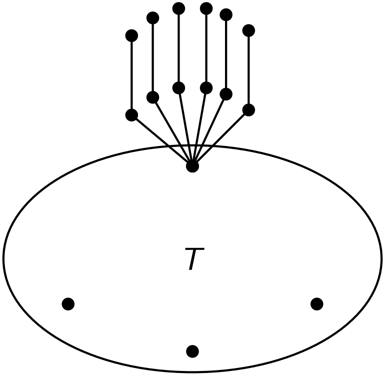

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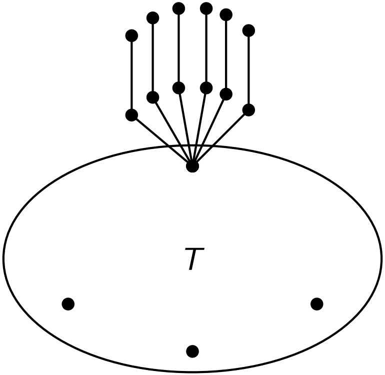
If there are remaining vertices of degree > 2, there must also be other peripheral superstars, and we can continue inductively.



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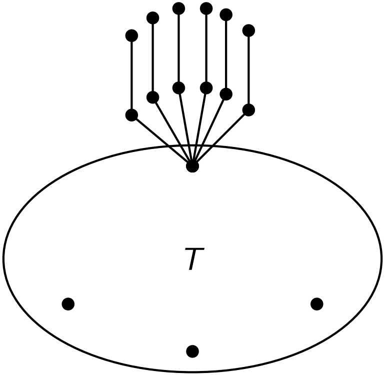
Otherwise we have arrived at a 2-balanced superstar.



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Can we do better with a general graph?

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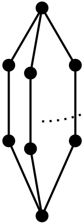

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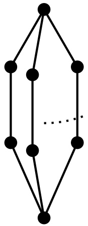

As $n \longrightarrow \infty$, the performance of the birdcage graph is asymptotically $\frac{2}{5}$.

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Conjecture

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Questions?