Bond Portfolio Optimization: A Dynamic Heteroskedastic Factor Model Approach

João F. Caldeira
Departamento de Economia
Universidade Federal do Rio Grande do Sul

Guilherme V. Moura
Departamento de Economia
Universidade Federal de Santa Catarina

André A. P. Santos
Departamento de Economia
Universidade Federal de Santa Catarina

Abstract

In this paper we use Markowitz's approach to optimize bond portfolios. We derive closed form expressions for the vector of expected bond returns and for their conditional covariance based on a general class of dynamic heteroskedastic factor models. These estimators are then used as inputs to obtain mean-variance and minimum variance optimal bond portfolios. An empirical examination using the dynamic version of the Nelson & Siegel yield curve model and Svensson's four factor model is applied involving a data set of 14 future contracts of Brazilian's interbank rate with different maturities indicates that the optimized bond portfolios deliver an improved risk-adjusted performance in comparison to the benchmarks.

Keywords: portfolio optimization, yield curve, dynamic conditional correlation (DCC), forecast.

JEL C53, E43, G17

1. Introduction

The portfolio optimization approach developed by Markowitz (1952), based on mean-variance relationship between assets is one of the cornerstones of modern finance theory. In this context, individuals decide their allocations in risky assets based on a trade-off between expected return and risk. This approach is now widely used to assist managers in portfolio construction and development of quantitative allocations (**?**). However, these applications are restricted, in most cases, to the construction of portfolios of stocks, see, eg, Brandt (2009), **?** and DeMiguel *et al.* (2009a) for recent applications. Thus, much is known about the strengths, weaknesses and performance of optimized stock portfolios using mean-variance approach, but not about fixed-income securities portfolio optimization.

The literature shows few references suggesting the use of mean-variance approach to bond portfolio selection (see, for example, Wilhelm, 1992; Korn & Koziol, 2006; Puhle, 2008). In practice, fixed-income portfolios tend to be selected in order to approximate the duration of a benchmark or to replicate the performance of this benchmark in terms of return and volatility (Fabozzi & Fong, 1994). In this sense, the subjective opinion of managers about the evolution of the term structure of interest rates will largely determine the composition of the portfolio relative to the benchmark. Models for the term structure of interest rates, as for example, Vasicek (1977), Cox *et al.* (1985), Nelson & Siegel (1987), Svensson (1994), Hull & White (1990), Heath *et al.* (1992), tend to be applied only to risk management or for pricing derivatives, but not directly to form a portfolio.

At least two reasons can be cited to justify the slow development of the optimization of fixed income portfolios based on mean-variance approach. The first reason is the relative stability and low historical volatility of this asset class, which discouraged the use of a sophisticated methodology for active risk management of fixed income portfolios. However, this situation is changing rapidly in recent years, even in markets where assets have low probability of default (Korn & Koziol, 2006). The turbulence in global markets brought great volatility to bond prices, which increases the importance of using portfolio optimization approaches that not only take into account the risk-return trade-off in bond returns, but also allow the possibility of risk diversification across different maturities.

According to Wilhelm (1992), Korn & Koziol (2006) and Puhle (2008), the Markowitz approach to portfolio selection has not been applied to fixed income due to difficulties in modeling returns and covariance matrix of bonds. Fabozzi & Fong (1994) argue that if it were possible to compute a covariance matrix relating various bonds, the process of portfolio optimization using fixed income could be similar to that of stock portfolios. However, fixed-income securities have finite maturities and promise to pay face value at

maturity. Thus, the end of this year price of a bond with two years to maturity is a random variable. However, the price of that same bond in two years is a deterministic quantity (disregarding the risk of default) given by its face value. This implies that the statistical properties of price and return of a fixed income security depends on their maturity. Thus, both price and return of bonds are non-ergodic processes, and traditional statistical techniques cannot be used to directly model the expected return and volatility of these assets (Meucci, 2009, pg. 110).

In this paper, a new approach to obtain estimates for the vector of expected returns and for the covariance matrix of returns on bonds will be presented. These estimates are used as inputs to the mean-variance optimization of a portfolio of bonds. The proposed approach builds on heteroskedastic dynamic factor models, as proposed by Santos & Moura (2011) for a portfolio of stocks, but now applied to the term structure of interest rates. According to Korn & Koziol (2006), the great advantage of using factor models for the term structure of interest rates is the possibility to model fixed maturity yields. This allows the estimation of the conditional distribution of yields, since these are invariants (Meucci, 2009). Therefore, we use dynamic factor models for the term structure, which were successfully used to predict the yield curve as, for example, the dynamic version of the Nelson & Siegel model proposed by Diebold & Li (2006), and the four factor model proposed by Svensson (1994)[1]. Moreover, unlike the approaches of Korn & Koziol (2006) and Puhle (2008), we allow for the presence of conditional heteroscedasticity. To this end, we use a parsimonious multivariate GARCH specification that allows the estimation and forecast of conditional covariance matrices for problems of high dimension.

It is noteworthy that the approach for portfolio optimization of fixed income proposed here differs in several respects to existing approaches. Wilhelm (1992), Korn & Koziol (2006) and Puhle (2008), for example, propose the use of mean-variance paradigm for selection of bond portfolios using the model of Vasicek (1977) for the yield curve. However, the Vasicek model uses only one factor to explain all the variability in the yield curve, and does not produce good fit or good forecasts of yields (see, for example, Duffee, 2002). The approach proposed here, on the other hand, is based on dynamic factor models which, besides allowing the derivation of the moments of the returns of bonds, are widely used for forecasting the yield curve (see, for example , BIS, 2005; ANBIMA, 2009).

Furthermore, Wilhelm (1992), Korn & Koziol (2006) and Puhle (2008) use homoscedastic factor models, ignoring the persistence in volatility of returns of bonds. Here, the conditional heteroscedasticity is modeled

---

[1]See, eg, De Pooter (2007), Diebold & Rudebusch (2011), ?,Caldeira *et al.* (2010b), Rezende & Ferreira (2011), and Caldeira & Torrent (2011) for an analysis of the predictive performance of factor models for the term structure.

explicitly, and the mean-variance optimization are based on estimates not only of the vector of expected returns, but also on estimates of the conditional covariance matrix of returns of bonds.

An empirical application involving a fixed maturity database of daily closing prices of future contracts for the Brazilian interbank rate (DI) is used to illustrate the applicability of the proposed approach. More specifically, DI-future contracts traded on the BM&F Bovespa with maturities 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, 42 and 48 months are used. Based on the estimates for the vector of expected returns on these bonds and their conditional covariance matrix, optimal mean-variance and minimum variance portfolios are constructed and their out-of-sample performance are then compared to benchmarks used in the Brazilian fixed income market. The results indicate that the proposed approach generates optimal mean-variance and minimum variance portfolios with superior risk-adjusted performance when compared with the benchmark. Moreover, the results are shown to be robust against different i) econometric specifications used to model the yield curve, ii) econometric specifications for the dynamics of factors, iii) econometric specifications for the covariance matrix of returns and iv) frequency of rebalancing optimized portfolios.

The paper is organized as follows. Section 2 describes the factor models used for modeling the term structure, as well as the econometric specification for the conditional heteroscedasticity of the factors. Section 3 discusses an estimation procedure in multi-steps for the proposed model. Section 4 discusses the Markowitz mean-variance optimization for fixed income portfolio, and bring an empirical application. Finally, Section 5 brings the final considerations.

2. Bond Protfolio Optimization Using Yield Curve Models

In this section we propose the use of factor models of the yield curve to perform bond portfolio optimization according to the mean-variance approach proposed by Markowitz (1952). Factor models for the term structure of interest rates allow us to find closed form expressions for the expected yields, as well as for their conditional covariance matrix. From these moments, it is shown how to compute the distribution of prices and returns of bonds, which will later be used as an input for portfolio optimization. We focus on models widely used by both the academic community, and by market participants[2].

2.1. Dynamic Factor Models for the Yield Curve

The general dynamic factor model increasingly plays a major role in econometrics, allowing explain a large panel of time series in terms of a small number of unobserved common factors (see, for example, Fama

---

[2]See BIS (2005), ? and ANBIMA (2009) for a discussion about the use of factor models for the yield curve by central banks and financial institutions.

& French, 1993; **?**). The relationship between the time series and the common factors is generally assumed to be linear. In this case its is customary to refer to the weights of the individual factors as the factor loadings. In many applications where dynamic factor models are considered, the individual series in the panel have a natural ordering in terms of one or more variables or indicators. For example, if we consider a time series panel of bond yields we can order the yields according to time to maturity of the bond. For this type of data set, is often natural to assume that the factor loadings for a specific factor are a relatively smooth function of the variable that is used to order the time series. Many models for the yield curve can be viewed as dynamic factor models with a set of restrictions imposed on the factor loadings. Almost aways, these restrictions imply smoothness of the factor loadings when viewed as a function of time to maturity.

We consider a daily time series panel of ID-future yields for a set of N maturities $\tau_1, \ldots, \tau_N$. The yield at time t of the ID-future with maturity τ_i is denoted by $y_t(\tau_i)$ for $t = 1, \ldots, T$. The $N \times 1$ vector of all yields at time t is given by

$$y_t = \begin{bmatrix} y_t(\tau_1) \\ \vdots \\ y_t(\tau_N) \end{bmatrix}, \quad t = 1, \ldots, T.$$

The general dynamic factor model is given by

$$y_t = \Lambda(\lambda) f_t + \varepsilon_t, \quad \varepsilon_t \sim \text{NID}(0, \Sigma_t), \quad t = 1, \ldots, T, \tag{1}$$

where $\Lambda(\lambda)$ iis the $N \times K$ factor loadings matrix. f_t is an $K-$dimensional stochastic process, ε_t is the $N \times 1$ disturbance vector and Σ_t is an $N \times N$ conditional covariance matrix of the disturbances. We restrict the covariance matrix Σ_t to be diagonal. This means that the covariance between the yields of different maturities is explained solely by the common latent factor f_t. The state vector is modeled by the dynamic stochastic process

$$f_t = \mu + \Upsilon f_{t-1} + R \eta_t, \quad \eta_t \sim \text{NID}(0, \Omega_t), \quad t = 1, \ldots, T, \tag{2}$$

where μ is a $K \times 1$ vector of constants, Υ is the $K \times K$ transition matrix, R is the $K \times K$ selection matrix that allows inclusion of individuals states, and Ω_t is the conditional covariance matrix of disturbance vector η_t, which are independent of the residues ε_t $\forall t$.

The dynamic specification for f_t is general. All vector autorregressive moving average models can be

formulated (see **?**). In the specific case of modeling of yield curves the typical dynamic specification for f_t is the vector autoregressive process of lag order 1 (This choice is the same as in related studies such Diebold *et al.* , 2006; Caldeira *et al.* , 2010b).

The alternative model specifications are the two main variants of the original formulation of Nelson & Siegel (1987) model, namely the the dynamic Nelson-Siegel model proposed by Diebold & Li (2006), and the extension of Svensson (1994). The different Nelson-Siegel specifications that we consider are all nested and can therefore be captured in one general formulations in (1) and (2) with different restrictions imposed on the loading matrix $\Lambda(\lambda)$. For some models, restrictions on the dynamics of the factors and the mean vector μ are also required. The first model for the term structure is based on the seminal paper of Nelson & Siegel (1987) in which the yield curve is approximated by a weighted sum of three smooth functions. The form of these three functions depends on a single parameter. Diebold & Li (2006) use the Nelson-Siegel framework to develop a two-step procedure for the forecasting of future yields. The second specification considered is the four-factor Svensson (1994) model. Svensson (1994) proposes to increase the flexibility and fit of the Nelson-Siegel model by adding a second hump-shape factor with a separate decay parameter. The model of Nelson & Siegel (1987) and its extension by Svensson (1994) are widely used by central banks and other market participants as a model for the term structure of interest rates (**?**BIS, 2005).

2.2. Dynamic Nelson-Siegel Model

In an important contribution Nelson & Siegel (1987) have shown that the term structure can surprisingly well be fitted by a linear combination of three smooth functions. Although the Nelson-Siegel model was in essence designed to be a static model which does not account for the intertemporal evolution of the term structure, Diebold & Li (2006) show that the coefficients in f_t can be interpreted as three latent dynamic factors. The Nelson-Siegel yield curve, denoted by $g_{NS}(\tau)$ is then given by

$$g_{NS}(\tau) = \xi_1 + \lambda^S \cdot \xi_2 + \lambda^C \cdot \xi_3, \tag{3}$$

where

$$\lambda^S(\tau) = \frac{1 - \exp(-\lambda\tau)}{\lambda\tau}, \qquad \lambda^C(\tau) = \frac{1 - \exp(-\lambda\tau)}{\lambda\tau} - \exp(-\lambda\tau) \tag{4}$$

and where λ, ξ_1, ξ_2 and ξ_3 are treated as parameters. The yield curve depends on these parameters which can be estimated by least squares method based on nonlinear regression model:

$$y_t(\tau_i) = g_{NS}(\tau_i) + u_i, \qquad i = 1, \ldots, N, \quad t = 1, \ldots, n$$

where u_{it} is a noise with zero mean and possibly different variances for different maturities τ_i. One of the attractions of the Nelson-Siegel model is that the parameters ξ has a clear interpretation. The parameter ξ_1 clearly controls the *level* of the yield curve. The parameter ξ_2 can be associated with the *slope* of the yield curve since its loading $\lambda^S(\tau)$ is high for short maturity τ and low for a long maturity. The loadings $\lambda^C(\tau)$ for different time to maturities τ forma U-shaped function and therefore ξ_3 can be interpreted as the *curvature* parameter of the yield curve. The decomposition of the yield curve in to level, slope and curvature factors has been highlighted by Litterman & Scheinkman (1991).

The Nelson-Siegel yield curve can also be incorporated in a dynamic factor model by treating the ξ parameters as factors and to let them evolve as a time-varying process. We obtain

$$y_t = \Lambda_{NS} f_t + \varepsilon_t, \qquad \varepsilon_t \sim \text{NID}\,(0, \Sigma_t)\,, \tag{5}$$

where f_t is a 3×1 vector ($K = 3$) and Σ_t is a diagonal conditional covariances matrix. The loading matrix Λ_{NS} consists of the three columns $(1, \ldots, 1)'$, $\lambda^S(\tau)$ and $\lambda^C(\tau)$ respectively. This dynamic factor representation of the Nelson-Siegel model is proposed by Diebold *et al.* (2006).

2.2.1. Svensson Model

Svensson (1994) included another exponential term, similar to the third, but with a different decaying parameter. The model is one extension Nelson-Siegel model described by equations (3) and (4) with the inclusion of a fourth component, ξ_4 with decay parameter independent , λ_2. The fourth component, $\left[\lambda_2^{C_2} = \frac{1 - \exp(-\lambda_2 \tau)}{\lambda_2 \tau} - \exp(-\lambda_2 \tau)\right]$, introduces a second medium-term component to the model. The Svensson Nelson-Siegel model can more easily fit term structure shapes with more that one local maximum or minimum along the maturity spectrum. Like the Nelson-Siegel model, the Svensson model can be seen as a dynamic factor model:

$$y_t = \Lambda_{SV} f_t + \varepsilon_t, \qquad \varepsilon_t \sim \text{NID}\,(0, \Sigma_t)\,, \tag{6}$$

where the 4×1 vector ($K = 3$) f_t represents the *level*, *slope* and *curvature* factors and is modelled as a VAR(1) process as in (2) while Σ_t is a diagonal conditional variance matrix. The i-th row of the loading matrix Λ_{SV} is given by $\left[1, \quad \lambda_1^S(\tau_i), \quad \lambda_1^{c_1}(\tau_i), \quad \lambda_2^{C_2}(\tau_i)\right]$. The resulting dynamic Svensson model can also be regarded as a smooth dynamic factor model in which the smoothness in the loading matrix is determined by the functional form (4) and parameters λ_1 and λ_2. The different model specifications yield curve factor to be considered are all nested and can therefore be captured by the general structure represented by the equations (1) and (2).

2.3. Conditional covariance of factor models for the yield curve

Forecasting volatility of interest rates remains an important challenge in finance[3]. A rich body of literature has shown that the volatility of the yield curve is, at least partially, related to the shape of the yield curve. For example, volatility of interest rates is usually high when interest rates are high and when the yield curve exhibits higher curvature (see Cox *et al.* (1985), **?**, and **?**, among others). This suggests that the shape of the yield curve is a potentially useful instrument for forecasting volatility.

Notwithstanding the sizeable amount of studies dealing with fitting and forecasting the yield curve, only very recently has attention been turned to the fitting and forecast of interest rates along with the conditional heteroskedasticity of the term structure of interest rates[4]. Some ways to overcome this issue were proposed by Bianchi *et al.* (2009), Haustsch & Ou (2010), Koopman *et al.* (2010) and Caldeira *et al.* (2010a). In most cases, the models adopt the assumption of constant volatility for all maturities throughout the sample period. These assumptions are related to the greater estimation difficulty in the presence of time- varying volatilities and decay (weight) factors. This issue is particularly important because the assumption of constant interest rate volatility in these models often has remarkable practical implications for risk management policies, which can be too simple, neglecting the risk of a time-varying volatility structure. Interest rate hedging and arbitrage operations could be influenced by the presence of time-varying volatility as, in these operations, it is necessary to compensate for the market price of interest rate risk. Another important implication is that in the presence of conditional volatilities the confidence intervals of fits and forecasts derived from these models will be calculated incorrectly in finite samples.

In this paper, the effects of time-varying volatility are incorporated using multivariate GARCH models of the type proposed by Santos & Moura (2011)[5]. To model Ω_t, the conditional covariance matrix of the factors in (2), alternative specifications can be considered, including not only multivariate GARCH models as well as multivariate stochastic volatility models (see Harvey *et al.* , 1994; Aguilar & West, 2000; Chib *et al.* , 2009). In this paper, we consider the dynamic conditional correlation model (dynamic conditional correlation - DCC) proposed by Engle (2002), which is given by:

$$\Omega_t = D_t \Psi_t D_t \tag{7}$$

where D_t is a $K \times K$ diagonal matrix with diagonal elements given by $h_{f_{kt}}$, which is the conditional variance

---

[3]See Poon & Granger (2003) or **?** for recent surveys of volatility forecasting.

[4]See, for instance, Filipovic (2009), for a review on interest rate modeling.

[5]See Bauwens *et al.* (2006) and Silvennoinen & Teräsvirta (2009) for a detailed review of multivariate GARCH models.

of the k-th factor, and Ψ_t is a symmetric correlation matrix with typical element $\rho_{ij,t}$, where $\rho_{ii,t} = 1$, $i, j = 1, \ldots, K$. In the DCC model, the conditional correlation $\rho_{ij,t}$ is given by:

$$\rho_{ij,t} = \frac{q_{ij,t}}{\sqrt{q_{ii,t} q_{jj,t}}} \tag{8}$$

where $q_{ij,t}$, $i, j = 1, \ldots, K$, are elements of the $K \times K$ matrix Q_t, which follows a GARCH type dynamics:

$$Q_t = (1 - \alpha - \beta) \bar{Q} + \alpha z_{t-1} z'_{t-1} + \beta Q_{t-1} \tag{9}$$

where $z_{f_t} = (z_{f_{1t}}, \ldots, z_{f_{kt}})$ is the standardized return vector of factors, whose elements are $z_{f_{it}} = f_{it} / \sqrt{h_{f_{it}}}$, $\bar{Q}$ is the unconditional covariance matrix z_t, α e β are non negative scalar parameters satisfying $\alpha + \beta < 1$.

To model the conditional variance of the measurement errors ε_t in (1), it is assumed that Σ_t is a diagonal matrix with diagonal elements given by $h_{t\varepsilon_i}$, where $h_{t\varepsilon_i}$ is the conditional variance of ε_i. Moreover, a procedure similar to that used in Cappiello *et al.* (2006) is used and alternative specifications of the univariate GARCH type are used to model $h_{t\varepsilon_i}$. In particular, we consider the GARCH model of Bollerslev (1986), the GJR-GARCH model asymmetric Glosten *et al.* (1993), the exponential GARCH (EGARCH) of Nelson (1991), the model threshold GARCH (TGARCH) of Zakoian (1994), the asymmetric GARCH model exponent (APARCH) of Ding *et al.* (1993), asymmetric GARCH (AGARCH) of Engle (1990), and model non-linear asymmetric GARCH (NAGARCH) of Engle & Ng (1993). In all models, their simplest form is adopted in which the conditional variance depends on one lag of past returns and lagged conditional variances. Appendix 2 lists the specifications of each of these models. The same procedure is applied to the choice of the GARCH specification for the conditional variance of the factors in (7). In all cases, the choice of the specification used is based on Akaike Information Criterion (AIC).

2.4. Mapping the moments of yields into the moments of returns

The Markowitz approach to portfolio optimization requires estimates of the expected return of each bond, as well as the covariance matrix of returns. However, the factor models for the term structure of interest rates previously presented model only yields. Nevertheless, it is possible to obtain expressions for the expected return and for the conditional covariance matrix of returns based on the distribution of the expected yields. The following proposition defines this distribution[6].

---

[6]The first appendix presents statements of all propositions.

Proposition 1. Given the system formed by equations (1) and (2), the distribution of expected yields $y_t|Y_{t-1}$ is $N(\mu_{y_t}, \Sigma_{y_t})$ with $\mu_t = \lambda(\lambda) f_{t|t-1}$ and $\Sigma_{y_t} = \lambda \Omega_{t|t-1} \lambda' + \Sigma_{t|t-1}$, where $f_{t|t-1}$ is a one-step-ahead forecasta of the factors and $\Sigma_{t|t-1}$ and $\Omega_{t|t-1}$ are one-step-ahead forecasts of the conditional covariance matrices in (1) and (2), respectively.

Based on these formulas, it is possible to derive the distribution of expected prices of bonds with fixed maturities. Considering that $P_t(\tau)$ denote the price of a t-period discount bond, i.e., the present value at time t of \$1 receivable t periods ahead, and let $y_t(\tau)$ denote its continuously compounded zero-coupon nominal yield to maturity, we obtain the vector of expected prices $P_{t|t-1}$:

$$P_{t|t-1} = \exp\left(-\tau \otimes y_t\right), \tag{10}$$

where $\otimes$ is the element by element multiplication and τ is the vector of maturities. Since $y_t|y_{t-1}$ follows a Normal distribution, $P_t|y_{t-1}$ has a log-normal distribution with mean given by:

$$\mu_{p_t} = \exp\left\{\left[-\tau \otimes \mu_{y_t} + \frac{\tau^2}{2} \otimes diag\left(\Sigma_{y_t}\right)\right]\right\}, \tag{11}$$

where $diag\left(\Sigma_{y_t}\right)$ is a vector containing the diagonal elements of Σ_{y_t}. The covariance of prices, Σ_{p_t}, has elements given by:

$$\sigma_{p_t}^{2^{i,j}} = \exp\left\{\left[-\tau^i \mu_{y_t}^i - \tau^j \mu_{y_t}^j + 0.5\left(\tau^{i^2} \sigma_{y_t}^{2^{i,i}} + \tau^{j^2} \sigma_{y_t}^{2^{j,j}}\right)\right]\right\} \cdot \left[\exp\left(\tau^i \tau^j \sigma_{y_t}^{2^{i,j}}\right) - 1\right] \tag{12}$$

Using the formula for log-returns, one can find a closed form expression for the vector of expected returns of bonds as well as for their conditional covariance matrix. Proposition 2 defines these expressions.

Proposition 2. Given the system (1)-(2) and Proposition 1, the vector of expected returns, $\mu_{r_{t|t-a}}$, and their conditional covariance matrix $\Sigma_{r_{t|t-1}}$, which is positive-definite $\forall t$, are given by:

$$\mu_{r_{t|t-1}} = -\tau \otimes \mu_t + \tau \otimes y_{t-1}, \tag{13}$$

$$\Sigma_{r_{t|t-1}} = \tau'\tau \otimes \left[\underbrace{\Lambda \Omega_{t|t-1} \Lambda' + \Sigma_{t|t-1}}_{\Sigma_{y_t}}\right]. \tag{14}$$

10

The results in Proposition 2 show that it is possible to obtain closed form expressions for the expected returns and their covariance matrix based on yield curve models such as the ones by Nelson & Siegel (1987) and Svensson (1994). These estimates are key ingredients to the problem of portfolio selection based on mean-variance paradigm proposed by Markowitz, as discussed in section 4.

As pointed out by Litterman & Scheinkman (1991), the return on fixed maturity zero-coupon bond can be decomposed into two parts. The first part is a result of the capitalization received due to ageing of the bond and the second part is attributed to the change in price of zero coupon bonds with fixed maturities. Furthermore, Litterman & Scheinkman (1991) point out that the first part is deterministic, while the second part is subject to uncertainty regarding the changes in prices. Clearly, portfolio optimization is only concerned with the second part.

However, for comparison with other benchmarks, it is also necessary to calculate the deterministic part of the return. The total return will be given by the income generated by the capitalization based on the interest rate on the bond, plus capital appreciation given by the variation in prices of bonds with fixed maturities. Following Jones $et\ al.$ (1998) and de Goeij & Marquering (2006), the total return on a bond with fixed maturity τ between periods t and $t + p$ is given by:

$$R_{t,t+p}(\tau) = \frac{P_t}{P_{t-p}} - 1 + \frac{p}{252} y_{t-p} = \exp(r_{t,t+p}) - 1 + \frac{p}{252} y_{t-p}, \tag{15}$$

where p is given on weekdays and $r_{t,t+p}$ is the part of the return generated by changes in yields of fixed maturities from period t to $t + p$[7].

3. Estimation procedure

In this section, a procedure for estimating the parameters of the yield curve, as well as the parameters of the volatility models is presented. The estimation is conducted in a multi-stage procedure, where the parameters of the factor model are first estimated, and from the volatility models are then estimated based on the residuals of the factor model.

3.1. Estimation of the yield curve models

The most straightforward approach to estimate the factors and parameters of the system (1) and (2) consists of a two-step procedure proposed by (Diebold & Li, 2006). In the first step, the measurement

---

[7]See equation (19) in the appendix for details concerning the calculation of $r_{t,t+p}$.

equation is treated as a cross section for each period of time, and OLS is employed to estimate the factors for all time periods individually. In the second step, the dynamic specification of the factors are estimated. To simplify the estimation procedure, Diebold & Li (2006) suggest reducing the parameter vector by setting the value of λ_t on a priori specified value, which is held fixed, rather than treating it as an unknown parameter.

The first step produces time series for the K factors, $\left\{\{\beta_{k,t}\}_{t=1}^{T}\right\}_{k=1}^{K}$. The next step is to estimate the factor dynamics of the state equations. We estimate separate AR(1) models for each factor, thus assuming that Υ in (2) are diagonal, as well as a joint VAR(1) by assuming that Υ in (2) are full matrix instead.

The choice of the decay parameters for the models of Nelson-Siegel and Svensson is restricted to the interval between 0.04 and 0.5, since these values correspond to a maximum of the curvature loadings at 48 months and in 6 months, which are the highest and lowest maturities of the database, respectively. Given these boundaries, we construct the set $\Phi = \{0.04 + 0.001j\}_{j=1}^{491}$. Given $\lambda_j \in \Phi$ and the correspondent matrix of factor loadings $\Lambda(\lambda_j)$, the vector of factors f_t is estimated by OLS for each period t. The decay parameter $\widehat{\lambda} \in \Phi$ is chosen to minimize the sum of the Root Mean Squared Error (RMSE). More specifically, $\widehat{\lambda}$ is chosen to minimize the difference between the adjusted yield, $\widehat{y}_t$, and the observed yield, y_t. The optimization problem can be represented as:

$$\widehat{\lambda} = \arg\min_{\lambda \in \Phi} \sqrt{\frac{1}{TN} \sum_{t=1}^{T} \sum_{i=1}^{N} \left(y_t(\tau_i) - \widehat{y}_t(\tau_i, \lambda, f_{t|t-1})\right)^2}$$

where T is the number of yield curves in the sample.

In the case of Svensson model, the problem is similar, except that in this case it is necessary to find two parameters $(\widehat{\lambda}_1, \widehat{\lambda}_2)$ set $\Theta = \{(\lambda_1^t, \lambda_2^t) | \lambda_1 \in \Phi, \lambda_2 \in \Phi\}$. Then, (λ_1, λ_2) solve the following problem:

$$\left(\widehat{\lambda}_1, \widehat{\lambda}_2\right) = \arg\min_{(\lambda_1, \lambda_2) \in \Theta} \sqrt{\frac{1}{TN} \sum_{t=1}^{T} \sum_{i=1}^{N} \left(y_t(\tau_i) - \widehat{y}_t(\tau_i, \lambda_1, \lambda_2, f_{t|t-1})\right)^2}.$$

A multicolinearity problem arises when the decay parameters λ_1 and λ_2 assume similar values. When this happens, the Svensson model reduces to the three-factor base model but with a curvature factor equal to the sum of β_{3t} and β_{4t}. In this case β_{3t} and β_{4t} will have the same factor loading, we have two perfectly collinear regressors. This extreme case was noticed before, for instance in ? or De Pooter (2007). Only the sum of these parameters can then still be estimated efficiently, not the individual parameters. To circumvent this problem, De Pooter (2007) proposes to replace the last term of $\lambda_2^{C_2}$, i.e., $-\exp\left(\frac{-\tau}{\lambda_{2t}\tau}\right)$ for $-\exp\left(\frac{-2\tau}{\lambda_{2t}\tau}\right)$. This specification, is the one we adopted here.

3.2. Estimation of the covariance matrix of yields

To obtain the conditional covariance matrix of factors, $\Omega_{t|t-1}$, a DCC specification in (7) is used. The estimation of the DCC model can be conveniently divided into a part of volatility and a correlation part. The part of the volatility relates to estimating the univariate conditional volatilities of the factors using a GARCH specification. The parameters of univariate volatility models are estimated by quasi maximum likelihood (QML) assuming Gaussian innovations[8]. The part of concerning correlation refers to the estimation of the conditional correlation matrix in (8) and (9). To estimate the parameters of the correlation matrix, we employ the method proposed by CL Engle *et al.* (2008). As highlighted by Engle *et al.* (2008), the CL estimator provides more accurate estimates of the parameters estimated in comparison with two-step procedure proposed by Engle & Sheppard (2001) and Sheppard (2003), especially in large problems.

4. Application to portfolio optimization of fixed income

To illustrate the applicability of the estimators of expected returns and conditional covariances proposed in this paper, we consider the optimization problem of fixed income portfolios in the context of mean-variance (Markowitz, 1952; Korn & Koziol, 2006; Puhle, 2008). The formulation of mean-variance portfolio is given by

$$\min_{w_t} w_t \Sigma_{r_{t|t-1}} w_t - \frac{1}{\delta} w_t' \mu_{r_{t|t-1}}$$

$$\text{sujeito a} \tag{16}$$

$$w_t' \iota = 1$$

where $\mu_{r_{t|t-1}}$ is a one step ahead forecast of the expected returns, $\Sigma_{r_{t|t-1}}$ is a one step ahead prediction of the conditional covariance matrix of returns, w_t is the weight vector to be optimized, ι is a vector with dimension $N \times 1$, and δ is the coefficient of risk aversion[9] In the case where short sales are restricted, a constraint to avoid negative weights is added to equation (16), i.e., $w_t \geq 0$. Previous work showed that adding such a restriction can substantially improve performance, especially reducing the turnover of the portfolio, see Jagannathan & Ma (2003), among others.

A variation of the optimization problem of mean-variance optimal portfolio is the minimum variance. In this formulation, it is considered that individuals are highly risk averse, such that $\delta \to \infty$. The formulation

---

[8]A review of issues related to the estimation of these models, such as choice of initial values, numerical algorithms, accuracy, and asymptotic properties are given by Berkes *et al.* (2003), Robinson & Zaffaroni (2006), Francq & Zakoian (2009) and Zivot (2009). It is important to note that even when the normality assumption is inappropriate, the QML estimator based on maximizing the Gaussian likelihood is consistent and asymptotically normal, since the conditional mean and variance functions of the GARCH model is correctly specified, see Bollerslev & Wooldridge (1992).

[9]In this paper, we follow **?** and assume that $\delta = 1$.

of the optimal minimum variance portfolio is given by

$$\min_{w_t} w_t \Sigma_{r_{t|t-1}} w_t$$
$$\text{sujeito a} \tag{17}$$
$$w'_t \iota = 1.$$

As before, a restriction to avoid negative weights is added to equation (17), or $w_t \geq 0$. In both cases, the optimal weights with short-selling restrictions are obtained by numerical optimization methods.

4.1. Data and implementation details

The database is used consist of a panel of daily time series of zero yields from the closing prices of the ID futures. The future interbank deposit (future DI) contract with maturity τ is a future contract of which the basic asset is the DI interest rate[10] accrued on a daily basis, capitalized between trading period t, and τ. The contract value is set by its value at maturity, $R\$100,000.00$ discounted according to the accrued interest rate, negotiated between the seller and the buyer.

When buying a future DI contract for the DI price at time t and keeping it until maturity τ, the gain or loss is given by:

$$100.000 \left(\frac{\prod_{i=1}^{\zeta(t,\tau)} (1+y_i)^{\frac{1}{252}}}{(1+DI^*)^{\frac{\zeta(t,\tau)}{252}}} - 1 \right),$$

where y_i denotes the DI rate, $(i-1)$ days after the trading day. The function $\zeta(t,\tau)$ represents the number of working days between t and τ.

The DI contract is quite similar to the zero-coupon bond, except for the daily payment of marginal adjustments. Every day the cash flow is the difference between the adjustment price of the current day and the adjustment price of the previous day, indexed by the DI rate of the previous day.

Future DI contracts are negotiated in the BM&F, which determines the number of maturities with authorized contracts. In general, there are around 20 maturities with authorized contracts every day, but not all of them have liquidity. Approximately 10 maturities have contracts with greater liquidity. There exist contracts with monthly maturities for the months at the beginning of each quarter (January, April, July and October). In addition, there are contracts with maturities for the four months that follow the current month.

---

[10]The DI rate is the average daily rate of Brazilian interbank deposits (borowing/lending), calculated by the Clearinghouse for Custody and Settlements (CETIP) for all business days. The DI rate, which is published on a daily basis, is expressed in annually compounded terms, based on 252 business days.

The maturity date is the first working day of the month in which the contract is due.

The data used in this paper consist of daily closing prices observed for yields of future DI contracts. In practice, contracts with all maturities are not observed on a daily basis. Therefore, based on the observed rates for the available maturities, the data were converted to fixed maturities of 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, 42 and 48 months, by means of interpolations using cubic splines originally proposed by McCulloch (1971, 1975)[11]. The database contains the maturities with highest liquidity for January 2006 through December 2010 ($T = 986$ observations), and represent the most liquid DI contracts negotiated during the analyzed period. The assessment of the performance of the model is made by splitting the sample into two parts. One of these parts is used to estimate the model and includes 500 observations are used to estimate the parameters of all models according to the procedures discussed in Section 3. The second part is used to analyze the performance out-of-sample of bond portfolios obtained from the model, with 486 observations. It is worth noting that we use iterated forecasts instead of direct forecasts for the multi-period-ahead predictions.

Table 1 displays the descriptive statistics for the Brazilian interest rate curve. For each of the 15 time series we report mean, standard deviation, minimum, maximum and the last three columns contain sample autocorrelations at displacements of 1, 5, and 21 days. The summary statistics confirm any facts common to yield curve data are clearly present: the sample average curve is upward sloping and concave, volatility is decreasing with maturity, autocorrelations are very high and increasing with maturity. In addition, there is a very high persistence in the yields: the first order autocorrelation for all maturities is above 0.99 for each maturity. Even the 21-days-autocorrelation coefficient can be as high as 0.80.

In Figure 1 we present a three-dimensional plot of the data set and illustrates how yield levels and spreads vary substantially throughout the sample. The data plot suggests the presence of an underlying factor structure. Although the yield series vary heavily over time for each of the maturities, a strong common pattern in the 15 series over time is apparent. For most months, the yield curve is an upward sloping function of time to maturity.For example, last year of the sample is characterized by rising interest rates, especially for the shorter maturities, which respond more directly to the contractionary monetary policy implemented by the Central Bank in the first half of 2010. It is clear from Figure 1 that not only the level of the term structure fluctuates over time but also its slope and curvature. The curve takes on various forms ranging from nearly flat to (inverted) S-type shapes.

---

[11]For further details and applications of this method, see Hagan & West (2006) and Hayden & Ferstl (2010).

Tabela 1: **Descriptive Statistics**

Descriptive statistics of daily yields for different maturities. The last three columns contains autocorrelations with a lag of one day, one week and one month, respectively. The sample period runs from Jan. 2006 to Dec. 2010.

Maturity	τ	Mean	Standard Deviation	Minimum	Maximum	Skewness	Kurtosis	$\widehat{\rho}(1)$	$\widehat{\rho}(5)$	$\widehat{\rho}(21)$
	3	10.82	1.65	8.58	14.34	0.220	2.006	0.999	0.997	0.969
	6	10.88	1.67	8.59	14.52	0.264	2.071	0.999	0.997	0.968
	9	10.94	1.69	8.58	14.69	0.306	2.132	0.999	0.996	0.967
	12	11.09	1.72	8.61	15.32	0.386	2.241	0.999	0.995	0.961
	15	11.34	1.73	8.73	16.04	0.495	2.373	0.998	0.992	0.950
	18	11.60	1.72	8.99	16.40	0.572	2.461	0.998	0.989	0.938
	21	11.85	1.68	9.35	16.92	0.655	2.565	0.997	0.986	0.925
Month	24	12.04	1.61	9.55	17.12	0.718	2.659	0.996	0.982	0.911
	27	12.21	1.55	9.79	17.26	0.805	2.815	0.995	0.979	0.894
	30	12.33	1.49	10.06	17.44	0.912	3.026	0.995	0.975	0.877
	33	12.43	1.45	10.27	17.62	1.005	3.290	0.994	0.972	0.859
	36	12.50	1.41	10.42	17.78	1.085	3.586	0.993	0.968	0.843
	42	12.60	1.32	10.71	17.83	1.281	4.180	0.992	0.961	0.814
	48	12.68	1.24	11.09	17.93	1.465	4.910	0.990	0.955	0.788

Figura 1: **Dynamic Yield Curve**

This chart details the evolution of term structure of interest rates (based on DI futures) for the time horizon of 2006:01-2010:12. The sample consisted of the daily yields for the maturities of 1, 3, 4, 6, 9, 12, 15, 18, 24, 27, 30, 36, 42 and 48 months.

16
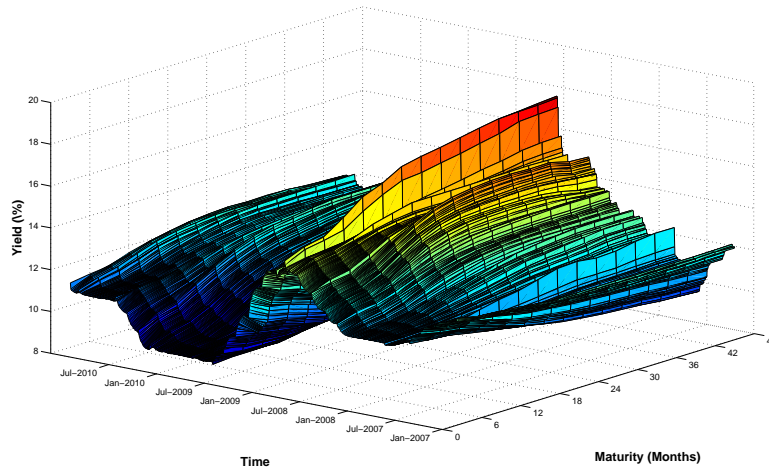
4.2. Performance evaluation

The performance of optimal mean-variance portfolios and of minimum variance portfolios is evaluated in terms of average return ($\hat{\mu}$), average excess return relative to the risk-free rate[12] ($\hat{\mu}_{ex}$), standard deviation (volatility) of returns ($\widehat{\Sigma}$), Sharpe Ratio (SR), and turnover. These statistics are calculated as follows:

$$\hat{\mu} = \frac{1}{T-1} \sum_{t=1}^{T-1} w_t' R_{t+1}$$

$$\hat{\mu}_{ex} = \frac{1}{T-1} \sum_{t=\tau}^{T-1} (w_t' R_{t+1} - CDI_{t+1})$$

$$\hat{\sigma} = \sqrt{\frac{1}{T-1} \sum_{t=1}^{T-1} (w_t' R_{t+1} - \hat{\mu})^2}$$

$$SR = \frac{\hat{\mu}_{ex}}{\hat{\sigma}}$$

$$\text{Turnover} = \frac{1}{T-1} \sum_{t=1}^{T-1} \sum_{j=1}^{N} (|w_{j,t+1} - w_{j,t}|),$$

where $w_{j,t}$ is the weight of the asset j in the portfolio in period t, but before the rebalancing and $w_{j,t+1}$ is the desired weight of the asset j in period $t+1$. As pointed out by DeMiguel *et al.* (2009b), the turnover, as defined above, can be interpreted as the average fraction of wealth traded in each period.

Just as DeMiguel *et al.* (2009a), the stationary bootstrap Politis & Romano (1994) with $B = 1.000$ resampling and block size $b = 5$[13] was used to test the statistical significance of differences between the volatilities and Sharpe ratios of optimal portfolios relative to a benchmark. The p-test values were obtained using the methodology suggested in Ledoit & Wolf (2008, Note 3.2). The benchmark used is the fixed income index IRF-M discussed in Section 4.3.

4.3. Benchmark

For the Brazilian fixed income market, the availability of market indices composed of fixed income securities is recent. However, there has been an improvement with the release of several indices developed by the Brazilian Association of Financial and Capital Markets (ANBIMA). The use of such indexes as benchmarks for industry fixed income funds in Brazil has grown significantly, and its use is widespread among market

---

[12]We consider the risk free rate to be the interbank rate CDI.

[13]We performed extensive robustness tests to define the block size, using values for $b = 5$ to $b = 250$. Regardless of the size of the block, the test results for the variance and Sharpe ratio are similar to those reported here.

participants. In order to meet the needs of different types of investors, there is a set of indices that represent the evolution of the market prices of bonds in accordance with their respective indexes. Among the indices developed, some can be highlighted:

IRF-M: Composed of non-indexed government bonds (NTN-F e LTN) ;

IMA-B: Composed of government bonds, where the principal is indexed to IPCA price index (NTN-B);

IMA-C: Composed of government bonds, where the principal is indexed to IGP-M price index (NTN-C) ;

IMA-S: Composed of government bonds, where the principal is indexed to the SELIC rate (LFT).

Taking into account that the database used is composed of DI-future contracts for fixed-rate and various maturities (less than and greater than one year), we chose to use the fixed income index IRF-M as a benchmark, since this index is also comprised of fixed-bonds of a number of maturity, including maturities lower than 1 year. For more information on this and other benchmarks, see ANBIMA (2011).

4.4. Results

In this section, the results of optimal mean-variance and minimum variance portfolios of DI-futures are presented. All results were obtained using only the observations belonging to the period out-of-sample based on the performance evaluation criteria discussed in section 4.2. To assess the robustness of the results, we considered several econometric specifications to model the vector of expected returns and covariance matrix of the returns. More specifically, we consider two specifications for the factor model (Nelson-Siegel and Svensson) and two specifications for the dynamic factors: VAR(1) and AR(1). The covariance matrices of returns were obtained using the DCC specification presented in section 2.3. Alternatively, the covariance matrices based on the unconditional covariance matrix of the residuals of the factor model, and based on the residuals in the transition equation for the factors is also calculated. These matrices are re-estimated using a rolling window of 500 observations, which allows some variation in the estimates over time.

The compositions of optimized portfolios are recalculated (rebalanced) on a daily basis. However, the transaction costs involved in this rebalancing frequency can degrade the performance of the portfolios and hinder its implementation in practice. Thus, the performance of optimized portfolios is also evaluated for the case of weekly, monthly and quarterly rebalancing. A potentially negative effect of adopting a lower frequency of rebalancing is that the optimal compositions may become outdated.

Tables 2 and 3 bring the results of performance evaluation of optimal mean-variance and minimum variance portfolios, respectively. In addition, the table 4 shows the performance evaluation indexes benchmark. The

statistics of average return, average excess return, standard deviation and Sharpe ratio are annualized. The results in Table 2 show that all the specifications considered for the mean-variance portfolio outperform the benchmark index IRF-M in terms of average return, and excess average return on all rebalancing frequencies. For example, the specification Nelson-Siegel/AR/DCC with quarterly rebalancing frequency generates an average excess return of 21.2%, average return and 11.8%, while the benchmark index generates 2.11% and 1.8%, for these indicators. However, we observed that the standard deviation (risk) of the mean-variance portfolios is statistically higher than that obtained by the benchmark in all specifications, since the standard deviation of the optimized portfolio ranges from 3% to 6%, while the standard deviation of the benchmark index is 1.6%. However, when examining the Sharpe ratio, the optimized portfolios show better risk-adjusted performance than that obtained by the benchmark in most specifications. For example, the specification Nelson-Siegel/AR/DCC generates a Sharpe ratio of 2.3 at quarterly rebalancing, while the reference index generates a Sharpe ratio of 1.1. It was also noted that, as expected, the quarterly rebalancing generates turnover substantially lower compared to the daily rebalancing.

Table 3 presents the results of the portfolios obtained by the criterion of minimum variance. It is observed that for average return and average excess return, results are inferior to the reference index IRF-M. However, the standard deviation of minimum variance portfolios is statistically lower than those obtained by the benchmark. Throughout all the econometric specifications and over all frequencies of rebalancing, the standard deviation of minimum variance portfolios ranges from 0.2% to 0.4%, while the benchmark index has a standard deviation of 1.6%. Consequently, the risk adjusted return of portfolios of minimum variance index measured by Sharpe ratio is statistically higher than that of the reference index in all cases, it varies from 2.1 to 3.4 while the Sharpe ratio of the IRF-M is 1.1. Similar to that obtained with the mean-variance portfolios, we observe that the turnover of mean-variance portfolios and minimum variance falls substantially as the frequency of rebalancing decreases. A comparative analysis of the performance of mean-variance and minimum variance portfolios shows that the latter generates higher Sharpe ratios, and a substantially lower standard deviation. Thus, the results suggest that, in fact, the minimum variance portfolios serve their purpose of generating optimal compositions that are less risky relative to the benchmark and also with respect to other optimized portfolios.

Another point worth mentioning refers to the comparative performance specifications for the covariance matrix of returns. It can be observed in all cases that the risk of the mean-variance and minimum variance portfolios obtained with the unconditional model is always higher than that obtained with the conditional model specification given by the DCC. In the case of mean-variance portfolio, for example, the standard

Tabela 2: Out-of-sample performance of optimal mean-variance portfolios

Performance statistics for mean-variance portfolios using DI future contracts with maturities equal to 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, 42 and 48 months traded at BM & F. The optimal portfolios are rebalanced with daily, weekly, monthly and quarterly. The statistics of returns, standard deviation and Sharpe ratio are annualized. The excess return is calculated using the CDI as a risk-free asset. Stars indicates that the coefficient is statistically different to that obtained by the index benchmark IRF-M at a significance level of 10 %.

Factor Model	Factor Dynamics	Covariance Matrix	Average Return (%)	Average Excess Return (%)	Standard Deviation (%)	Sharpe de Ratio	Turnover
			Daily rebalancing				
Nelson-Siegel	AR	DCC	17.763	8.384	3.785*	2.214*	0.990
Nelson-Siegel	VAR	DCC	17.587	8.207	3.679*	2.230*	0.994
Nelson-Siegel	AR	Unconditional	23.079	13.707	5.663*	2.420*	0.970
Nelson-Siegel	VAR	Unconditional	23.076	13.703	5.663*	2.419*	0.970
Svensson	AR	DCC	15.340	5.960	3.128*	1.905*	1.277
Svensson	VAR	DCC	15.152	5.773	3.050*	1.892*	1.277
Svensson	AR	Unconditional	19.975	10.603	5.460*	1.941*	1.228
Svensson	VAR	Unconditional	19.965	10.593	5.460*	1.939*	1.228
			Weekly rebalancing				
Nelson-Siegel	AR	DCC	17.736	8.357	3.602*	2.319*	0.238
Nelson-Siegel	VAR	DCC	17.587	8.208	3.542*	2.316*	0.239
Nelson-Siegel	AR	Unconditional	22.044	12.672	5.439*	2.329*	0.221
Nelson-Siegel	VAR	Unconditional	22.044	12.672	5.439*	2.329*	0.221
Svensson	AR	DCC	15.186	5.807	2.698*	2.151*	0.298
Svensson	VAR	DCC	15.137	5.758	2.646*	2.175	0.299
Svensson	AR	Unconditional	20.768	11.396	5.037*	2.262*	0.304
Svensson	VAR	Unconditional	20.752	11.380	5.035*	2.259*	0.304
			Monthly rebalancing				
Nelson-Siegel	AR	DCC	16.568	7.189	3.967*	1.811*	0.054
Nelson-Siegel	VAR	DCC	16.517	7.138	3.937*	1.812*	0.054
Nelson-Siegel	AR	Unconditional	20.050	10.678	5.152*	2.072*	0.057
Nelson-Siegel	VAR	Unconditional	20.047	10.674	5.151*	2.071*	0.057
Svensson	AR	DCC	15.206	5.827	2.633*	2.212*	0.073
Svensson	VAR	DCC	15.126	5.746	2.593*	2.215	0.073
Svensson	AR	Unconditional	20.784	11.412	5.067*	2.251*	0.072
Svensson	VAR	Unconditional	20.778	11.406	5.065*	2.251*	0.072
			Quarterly rebalancing				
Nelson-Siegel	AR	DCC	21.231	11.851	5.148*	2.301*	0.023
Nelson-Siegel	VAR	DCC	21.118	11.738	5.095*	2.303*	0.023
Nelson-Siegel	AR	Unconditional	23.842	14.470	5.980*	2.419*	0.025
Nelson-Siegel	VAR	Unconditional	23.842	14.470	5.980*	2.419*	0.025
Svensson	AR	DCC	16.304	6.924	2.928*	2.364*	0.028
Svensson	VAR	DCC	16.268	6.888	2.923*	2.356*	0.028
Svensson	AR	Unconditional	24.075	14.703	6.129*	2.398*	0.021
Svensson	VAR	Unconditional	24.075	14.703	6.129*	2.398*	0.021

Tabela 3: Performance out-of-sample optimal portfolios of minimum variance

Performance statistics for portfolios optimum minimum variance using DI future contracts with maturities equal to 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, 42 and 48 months traded at BM & F. The optimal portfolios are rebalanced with daily, weekly, monthly and quarterly. The statistics of returns, standard deviation and Sharpe ratio are annualized. The excess return is calculated using the CDI as a risk-free asset. Stars indicates that the coefficient is statistically different to that obtained by the index benchmark IRF-M at a significance level of 10 %.

Factor Model	Factor Dynamics	Covariance Matrix	Average Return (%)	Average Excess Return (%)	Standard Deviation (%)	Sharpe Index	Turnover
			Daily rebalancing				
Nelson-Siegel	AR	DCC	10.078	0.699	0.205*	3.393*	0.073
Nelson-Siegel	VAR	DCC	10.071	0.691	0.206*	3.350*	0.081
Nelson-Siegel	AR	Unconditional	10.317	0.945	0.389*	2.426*	0.002
Nelson-Siegel	VAR	Unconditional	10.311	0.939	0.383*	2.448*	0.002
Svensson	AR	DCC	10.067	0.687	0.223*	3.076*	0.098
Svensson	VAR	DCC	10.048	0.669	0.222*	3.002*	0.098
Svensson	AR	Unconditional	10.192	0.820	0.375*	2.184*	0.001
Svensson	VAR	Unconditional	10.191	0.819	0.371*	2.202*	0.001
			Weekly rebalancing				
Nelson-Siegel	AR	DCC	10.076	0.696	0.206*	3.379*	0.038
Nelson-Siegel	VAR	DCC	10.082	0.702	0.207*	3.389*	0.039
Nelson-Siegel	AR	Unconditional	10.317	0.944	0.389*	2.426*	0.001
Nelson-Siegel	VAR	Unconditional	10.310	0.938	0.383*	2.447*	0.001
Svensson	AR	DCC	10.062	0.683	0.223*	3.051*	0.049
Svensson	VAR	DCC	10.049	0.669	0.222*	3.011*	0.051
Svensson	AR	Unconditional	10.192	0.820	0.375*	2.185*	0.001
Svensson	VAR	Unconditional	10.191	0.819	0.371*	2.203*	0.001
			Monthly rebalancing				
Nelson-Siegel	AR	DCC	10.087	0.708	0.204*	3.457*	0.018
Nelson-Siegel	VAR	DCC	10.088	0.708	0.205*	3.448*	0.018
Nelson-Siegel	AR	Unconditional	10.315	0.942	0.387*	2.429*	0.001
Nelson-Siegel	VAR	Unconditional	10.309	0.937	0.382*	2.452*	0.001
Svensson	AR	DCC	10.036	0.657	0.221*	2.962*	0.023
Svensson	VAR	DCC	10.013	0.633	0.221*	2.860*	0.022
Svensson	AR	Unconditional	10.193	0.821	0.374*	2.190*	0.001
Svensson	VAR	Unconditional	10.191	0.819	0.371*	2.206*	0.001
			Quarterly rebalancing				
Nelson-Siegel	AR	DCC	10.075	0.696	0.206*	3.376*	0.005
Nelson-Siegel	VAR	DCC	10.074	0.695	0.209*	3.308*	0.006
Nelson-Siegel	AR	Unconditional	10.310	0.938	0.379*	2.473*	0.001
Nelson-Siegel	VAR	Unconditional	10.306	0.934	0.373*	2.498*	0.001
Svensson	AR	DCC	10.021	0.642	0.232*	2.759*	0.014
Svensson	VAR	DCC	10.011	0.632	0.230*	2.739*	0.014
Svensson	AR	Unconditional	10.193	0.820	0.372*	2.199*	0.001
Svensson	VAR	Unconditional	10.191	0.819	0.369*	2.216*	0.001

Tabela 4: Performance of strategies *benchmark*

Performance statistics for the benchmark. The indices IRF-M1, IRF-M1+ and IRF-M consist of non-indexed government bonds, whereas the indices IMA-B 5, B-5 + IMA and IMA-B are formed by indexed bonds. The statistics of returns, standard deviation and Sharpe ratio are annualized. The excess return is calculated using the CDI as a risk-free asset.

	Average Return (%)	Average Excess Return (%)	Standard Deviation (%)	Sharpe Index
IRF-M 1	10.288	0.909	0.391	2.324
IRF-M 1+	11.971	2.592	2.835	0.914
IRF-M	11.205	1.826	1.655	1.103
IMA-B 5	12.705	3.326	1.603	2.075
IMA-B 5+	18.925	9.546	5.386	1.772
IMA-B	15.658	6.278	3.269	1.920

deviation of portfolios obtained ranges from 5% to 6.1% model for unconditional and 2.5% to 4% for the conditional model. This result suggests that modeling conditional volatility brings substantial gains in terms of reducing portfolio risk. Similar results were obtained for the minimum variance portfolios.

Figure 2 brings the cumulative returns of portfolios optimized by mean-variance (upper graph) and minimum variance (lower graph) based on specifications Nelson-Siegel/AR/DCC, Nelson-Siegel/AR/Unconditional, Svensson/AR/DCC and Svensson/AR/Unconditional considering a frequency of rebalancing quarterly. Note that, in the case of mean-variance portfolios, the difference in returns relative to the benchmark is positive throughout the period analyzed. Moreover, it is observed that the best performance in terms of cumulative returns was obtained using the specification for the unconditional covariance matrix. This result suggests that the unconditional approach generates portfolios optimized with a different risk-return pattern than that obtained with the conditional approach. For the minimum-variance portfolios, it is observed that the different specifications achieved a similar performance to the benchmark in terms of cumulative returns. However, visual inspection shows that in all cases, optimized portfolios have a variance substantially lower than the variability of the reference index.

In summary, the results reported in Tables 2 and 3 indicate that the proposed methodology to obtain optimal mean-variance and minimum variance portfolios based on the proposed estimates for the vector of expected returns and the covariance matrix of the returns generate a superior out-of-sample performance with respect to the benchmark index in several respects. First, average returns, and cumulative returns of the mean-variance portfolio exceeds the ones obtained by the benchmark in all specifications considered. Second, the standard deviation of portfolios of minimum variance is substantially lower than that obtained by the reference index, so that the risk adjusted return of the optimized portfolio is statistically higher. Moreover,

the results were robust with respect to: i) the factor model used to model the yield curve, ii) the econometric specification for the transition equation of the factors, iii) the specification of the covariance matrix of returns and iv) frequency of rebalancing of portfolios.
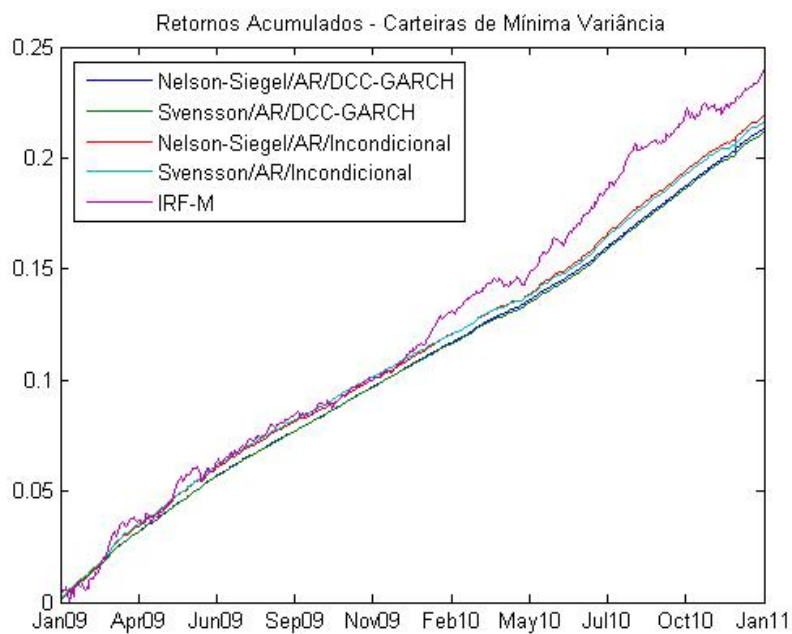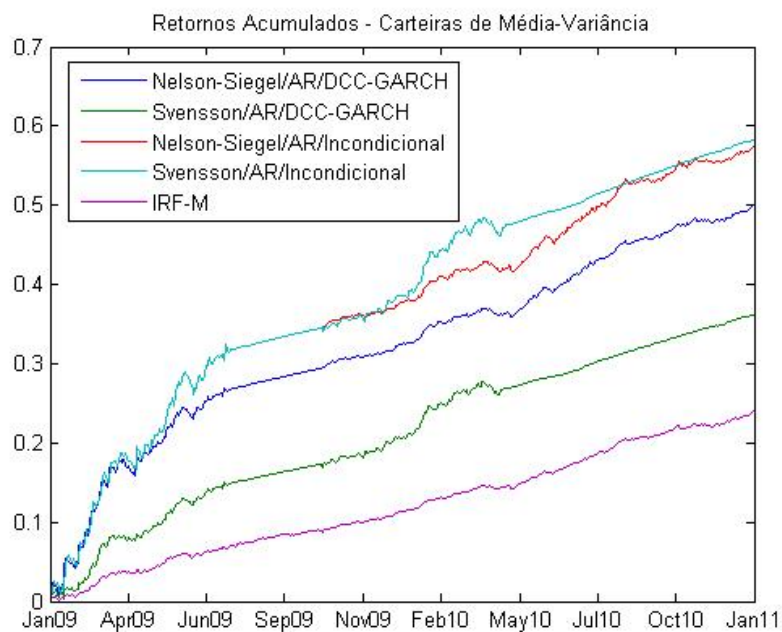
5. Concluding remarks

The use of mean-variance approach introduced by Markowitz (1952) to optimize stock portfolios has been widely used by market participants and documented in the academic literature. However, the use of this methodology for the selection of bonds has not received a lot of attention in the academic literature, nor among managers. In order to address this shortcoming, this paper adopts the mean-variance approach to bond portfolio optimization based on models for the term structure of interest rate, and on recent models for the covariance matrix of yields.

We show that factor models for the yield curve simplifies the process of bond portfolio optimization, since it allows the computation of expected return on bonds and their conditional covariance matrix. Closed form expressions for the vector of expected returns and for their conditional covariance matrix based on a general class of heteroscedastic dynamic factor models are developed, and used to obtain optimal mean-variance and minimum variance portfolios of bonds. In particular, we consider the dynamic version of the Nelson & Siegel model proposed by Diebold & Li (2006) and the four-factor model of Svensson (1994).

An application involving a database of daily yields embedded in future contracts on the Brazilian inter-bank rate, which are actively traded on BM&F Bovespa, is used to illustrate the proposed approach. The results show that the optimized bond portfolios exhibit quite attractive risk-return profiles. For example, the portfolios obtained by the mean-variance criterion showed annualized Sharpe ratio between 1.81 and 2.42, versus 1.10 for the benchmark index IRF-M. In practice, an investor bears transaction costs when changing the composition of its portfolio over time, and these costs are a function of frequency and magnitude of changes in the portfolio. Although transaction costs are not taken into account directly, we compute the turnover of portfolios, which shows that there is a stability of optimal compositions. In addition, portfolios obtained by the minimum variance criterion show standard deviation between 0.20 and 0.39, while the reference index (IRF-M) has a standard deviation of 1.66, resulting in Sharpe ratios significantly superior to the benchmark in all specifications considered. Finally, all results were robust with respect to the factor model used to model the yield curve, to the econometric specification of the covariance matrix, to the dynamics of factors, and to the frequency of rebalancing.

Figura 2: **Cumulative Returns**

Retornos Acumulados - Carteiras de Média-Variância

Retornos Acumulados - Carteiras de Mínima Variância

24

References

AGUILAR, O., & WEST, M. 2000. Bayesian dynamic factor models and portfolio allocation. *Journal of Business and Economic Statistics*, **18**(3), 338–357.

ANBIMA. 2009. Títulos Públicos Federais: Critérios de cálculo e Metodologia de Precificação. *Associação Brasileira das Entidades dos Mercados Financeiros e de Capitais*.

ANBIMA. 2011. Índices de Renda Fixa IMA. *Associação Brasileira das Entidades dos Mercados Financeiros e de Capitais*.

BAUWENS, L., LAURENT, S., & ROMBOUTS, J.V.K. 2006. Multivariate GARCH models: a survey. *Journal of Applied Econometrics*, **21**(1), 79–109.

BERKES, I., HORVÁTH, L., & KOKOSZKA, P. 2003. GARCH processes: structure and estimation. *Bernoulli*, **9**(2), 201–227.

BIANCHI, F., MUMTAZ, H., & SURICO, P. 2009. The great moderation of the term structure of UK interest rates. *Journal of Monetary Economics*, **56**(6), 856–871.

BIS. 2005 (Oct.). *Zero-coupon yield curves: technical documentation*. BIS Papers 25. Bank for International Settlements (Department Monetary and Economic).

BOLLERSLEV, T. 1986. Generalized autoregressive conditional heteroskedasticity. *Journal of Econometrics*, **31**(3), 307–327.

BOLLERSLEV, T., & WOOLDRIDGE, J.M. 1992. Quasi-maximum likelihood estimation and inference in dynamic models with time-varying covariances. *Econometric reviews*, **11**(2), 143–172.

BRANDT, M.W. 2009. Portfolio choice problems. *Handbook of financial econometrics*, **1**, 269–336.

CALDEIRA, J. F., LAURINI, M. P., & PORTUGAL, M. S. 2010a. Bayesian Inference Applied to Dynamic Nelson-Siegel Model with Stochastic Volatility. *Brazilian Review of Econometrics*, **30**(1), 123–161.

CALDEIRA, J.F., & TORRENT, H. 2011. Previsão de Curvas de Juros Zero-cupom: Estimação Não-paramétrica de Dados Funcionais. *Texto para discussão, Programa de Pós-Graduação em Economia, Universidade Federal do Rio Grande do Sul.*

CALDEIRA, João F., MOURA, Guilherme V., & PORTUGAL, Marcelo S. 2010b. Efficient Yield Curve Estimation and Forecasting in Brazil. *Economia*, **11**(1).

CAPPIELLO, L., ENGLE, R.F., & SHEPPARD, K. 2006. Asymmetric dynamics in the correlations of global equity and bond returns. *Journal of Financial Econometrics*, **4**(4), 537–572.

CHIB, S., OMORI, Y., & ASAI, M. 2009. *Multivariate stochastic volatility*. Springer Verlag.

COX, J. C., INGERSOLL, J. E., & ROSS, S. A. 1985. A Theory of the Term Structure of Interest Rates. *Econometrica*, **53**(2), 385–407.

DE GOEIJ, PETER, & MARQUERING, WESSEL. 2006. Macroeconomic announcements and asymmetric volatility in bond returns. *Journal of Banking & Finance*, **30**(10), 2659–2680.

DE POOTER, M. 2007. *Examining the Nelson-Siegel Class of Term Structure Models*. Tinbergen Institute Discussion Papers. Tinbergen Institute.

DEMIGUEL, V., GARLAPPI, L., NOGALES, F.J., & UPPAL, R. 2009a. A generalized approach to portfolio optimization: improving performance by constraining portfolio norms. *Management Science*, **55**(5), 798–812.

DEMIGUEL, V., GARLAPPI, L., & UPPAL, R. 2009b. Optimal versus naive diversification: how inefficient is the 1/N portfolio strategy? *Review of Financial Studies*, **22**(5), 1915–1953.

DIEBOLD, F., & LI, C. 2006. Forecasting the term structure of government bond yields. *Journal of Econometrics*, **130**(2), 337–364.

DIEBOLD, F. X., & RUDEBUSCH, G. D. 2011. *The Dynamic Nelson-Siegel Approach to Yield Curve Modeling and Forecasting*. mimeo.

DIEBOLD, F. X., RUDEBUSCH, G. D., & ARUOBA, S. B. 2006. The macroeconomy and the yield curve: a dynamic latent factor approach. *Journal of Econometrics*, **131**(1-2), 309–338.

DING, Z., GRANGER, C.W.J., & ENGLE, R.F. 1993. A long memory property of stock returns and a new model. *Journal of Empirical Finance*, **1**(1), 83–106.

DUFFEE, G. R. 2002. Term Premia and Interest Rate Forecasts in Affine Models. *Journal of Finance*, **57**(1), 405–443.

ENGLE, R. 2002. Dynamic conditional correlation: a simple class of multivariate generalized autoregressive conditional heteroskedasticity models. *Journal of Business & Economic Statistics*, **20**(3), 339–350.

ENGLE, R.F. 1990. Stock volatility and the crash of '87: discussion. *The Review of Financial Studies*, **3**(1), 103–106.

ENGLE, R.F., & NG, V.K. 1993. Measuring and Testing the Impact of News on Volatility. *Journal of Finance*, **48**(5), 1749–78.

ENGLE, R.F., & SHEPPARD, K. 2001. Theoretical and empirical properties of dynamic conditional correlation multivariate GARCH. *NBER Working Paper W8554.*

ENGLE, R.F., SHEPHARD, N., & SHEPPARD, K. 2008. Fitting vast dimensional time-varying covariance models. *Discussion Paper Series n.403, Department of Economics, University of Oxford.*

FABOZZI, F.J., & FONG, G. 1994. *Advanced fixed income portfolio management: the state of the art.* Probus.

FAMA, E.F., & FRENCH, K.R. 1993. Common risk factors in the returns on stocks and bonds. *Journal of financial economics*, **33**(1), 3–56.

FILIPOVIC, D. 2009. *Term Structure Models.* Springer Finance.

FRANCQ, C., & ZAKOIAN, J.M. 2009. A tour in the asymptotic theory of GARCH estimation. *In:* ANDERSEN, T.G., DAVIS, R.A., KREISS, J.-P., & MIKOSCH, T. (eds), *Handbook of Financial Time Series.* Springer Verlag.

GLOSTEN, LAWRENCE, JAGANNATHAN, RAVI, & RUNKLE, DAVID. 1993. On the relation between the expected value and the volatility of the nominal excess return on stocks. *Journal of Finance*, **48**, 1779–1801.

HAGAN, P., & WEST, G. 2006. Interpolation Methods for Curve Construction. *Applied Mathematical Finance*, **13**(2), 89–129.

HARVEY, A., RUIZ, E., & SHEPHARD, N. 1994. Multivariate stochastic variance models. *Review of Economic Studies*, **61**(2), 247–64.

HAUSTSCH, NIKOLAUS, & OU, YANGGUOYI. 2010. Bayesian Inference in a Stochastic Volatility Nelson-Siegel Model. *Computational Statistics and Data Analysis*, **In Press, Corrected Proof**.

HAYDEN, JOSEF, & FERSTL, ROBERT. 2010. Zero-Coupon Yield Curve Estimation with the Package termstrc. *Journal of Statistical Software*, **36**(i01), 1–34.

27

HEATH, DAVID, JARROW, ROBERT, & MORTON, ANDREW. 1992. Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation. *Econometrica*, **60**(1), 77–105.

HULL, J., & WHITE, A. 1990. Pricing interest rate derivative securities. *The Review of Financial Studies*, **3**(4), 573–592.

JAGANNATHAN, R., & MA, T. 2003. Risk reduction in large portfolios: Why imposing the wrong constraints helps. *The Journal of Finance*, **58**(4), 1651–1684.

JONES, CHARLES M., LAMONT, OWEN, & LUMSDAINE, ROBIN L. 1998. Macroeconomic news and bond market volatility. *Journal of Financial Economics*, **47**(3), 315–337.

KOOPMAN, SIEM JAN, MALLEE, MAX I.P., & VAN DER WEL, MICHEL. 2010. Analyzing the Term Structure of Interest Rates using the Dynamic Nelson-Siegel Model with Time-Varying Parameters. *Journal of Business and Economic Statistics*, **28**(3), 329–343.

KORN, O., & KOZIOL, C. 2006. Bond Portfolio Optimization. *The Journal of Fixed Income*, **15**(4), 48–60.

LEDOIT, O., & WOLF, M. 2008. Robust performance hypothesis testing with the Sharpe ratio. *Journal of Empirical Finance*, **15**(5), 850–859.

LITTERMAN, R., & SCHEINKMAN, J. 1991. Common Factors Affecting Bond Returns. *Journal of Fixed Income*, **1**(1), 54–61.

MARKOWITZ, H. 1952. Portfolio Selection. *Journal of Finance*, **7**(1), 77–91.

MCCULLOCH, J. H. 1971. Measuring the Term Structure of Interest Rates. *The Journal of Business*, **44**(1), 19–31.

MCCULLOCH, J. H. 1975. The Tax-Adjusted Yield Curve. *Journal of Finance*, **30**(3), 811–30.

MEUCCI, A. 2009. *Risk and Asset Allocation.* Springer Finance. Springer.

NELSON, C. R. N., & SIEGEL, A. F. 1987. Parsimonious Modeling of Yield Curves. *The Journal of Business*, **60**(4), 473–489.

NELSON, D.B. 1991. Conditional heteroskedasticity in asset returns: a new approach. *Econometrica*, **59**(2), 347–370.

POLITIS, D.N., & ROMANO, J.P. 1994. The stationary bootstrap. *Journal of the American Statistical Association*, **89**(428), 1303–1313.

POON, S.H., & GRANGER, C.W.J. 2003. Forecasting Volatility in Financial Markets: A Review. *Journal of Economic Literature*, **41**(2), 478–539.

PUHLE, M. 2008. *Bond portfolio optimization.* Lecture notes in economics and mathematical systems. Springer.

REZENDE, R. B., & FERREIRA, M. S. 2011. Modeling and Forecasting the Yield Curve by an Extended Nelson-Siegel Class of Models: A Quantile Autoregression Approach. *Journal of Forecasting*, **30**(8), 339–350.

ROBINSON, P.M., & ZAFFARONI, P. 2006. Pseudo-maximum likelihood estimation of ARCH(∞) models. *The Annals of Statistics*, **34**(3), 1049–1074.

SANTOS, A. A. P., & MOURA, G. V. 2011. Dynamic factor multivariate GARCH model. *Working paper, Department of Economics, Universidade Federal de Santa Catarina.*

SHEPPARD, K. 2003. Multi-step estimation of multivariate GARCH models. *In: Proceedings of the International ICSC. Symposium: Advanced Computing in Financial Markets.*

SILVENNOINEN, A., & TERÄSVIRTA, T. 2009. Multivariate GARCH models. *In:* ANDERSEN, T.G., DAVIS, R.A., KREISS, J.-P., & MIKOSCH, T. (eds), *Handbook of Financial Time Series.* Springer Verlag.

SVENSSON, L. O. 1994 (Sept.). *Estimating and Interpreting Forward Interest Rates: Sweden 1992-1994.* IMF Working Papers 94/114. International Monetary Fund.

VASICEK, O. 1977. An equilibrium characterization of the term structure. *Journal of Financial Economics*, **5**(2), 177–188.

WILHELM, J. 1992. Fristigkeitsstruktur und Zinsänderungsrisiko-Vorüberlegungen zu einer Markowitz-Theorie des Bond-Portfolio-Management. *Zeitschrift für betriebswirtschaftliche Forschung*, **44**(1992), 209–246.

ZAKOIAN, J.M. 1994. Threshold heteroskedastic models. *Journal of Economic Dynamics and control*, **18**(5), 931–955.

Zivot, E. 2009. Practical issues in the analysis of univariate GARCH models. *In:* Andersen, T.G., Davis, R.A., Kreiss, J.-P., & Mikosch, T. (eds), *Handbook of Financial Time Series.* Springer Verlag.

Appendix 1

Proof of Proposition 1

Taking the expected value of the factor model for the yield in (1), we have

$$\mu_t = E_{t-1}[y_t] = \mathbf{\Lambda}(\lambda) E_{t-1}[f_t] = \mathbf{\Lambda}(\lambda) f_{t|t-1} \tag{18}$$

where $f_{t|t-1}$ are one step ahead predictions of factors. The corresponding conditional covariance matrix is given by:

$$
\begin{aligned}
\Sigma_{y_t} &= E_{t-1}\left[(y_t - E_{t-1}[y_t])(y_t - E_{t-1}[y_t])\right] \\
&= E_{t-1}\left[(\Lambda f_t + \varepsilon_t - \Lambda E_{t-1}[f_t])(\Lambda f_t + \varepsilon_t - \Lambda E_{t-1}[f_t])'\right] \\
&= E_{t-1}\left[(\Lambda(f_t - E_{t-1}[f_t]) + \varepsilon_t)(\Lambda(f_t - E_{t-1}[f_t]) + \varepsilon_t)'\right] \\
&= E_{t-1}\left[(\Lambda(\mu + \Upsilon f_{t-1} + R\eta_t - \mu - \Upsilon f_{t-1}) + \varepsilon_t)(\Lambda(\mu + \Upsilon f_{t-1} + R\eta_t - \mu - \Upsilon f_{t-1}) + \varepsilon_t)'\right] \\
&= E_{t-1}\left[(\Lambda\eta_t + \varepsilon_t)(\Lambda\eta_t + \varepsilon_t)'\right] \\
&= E_{t-1}\left[\Lambda\eta_t\eta_t'\Lambda' + \varepsilon_t\varepsilon_t'\right] \\
&= \Lambda E_{t-1}\left[\eta_t\eta_t'\right]\Lambda' + E_{t-1}\left[\varepsilon_t\varepsilon_t'\right] \\
&= \Lambda\Omega_{t|t-1}\Lambda' + \Sigma_{t|t-1}.
\end{aligned}
$$

since R is a $K \times K$ identity matrix, and cross-product between η_t and ε_t vanish because of independence. □

Proof of Proposition 2

Using the log-return expression in (12), we get:

$$r_t = \log\left(\frac{P_t}{P_{t-1}}\right) = \log P_t - \log P_{t-1} = -\tau \otimes (y_t - y_{t-1}). \tag{19}$$

Since $y_t|y_{t-1} \sim N(\mu_t, \Sigma_{y_t})$ where μ_t e Σ_{y_t} are defined in Proposition 1, it is known that the expected returns $r_t|y_{t-1} follow N(\mu_{r_t}, \Sigma_{r_t})$ where

$$\mu_{r_{t|t-1}} = -\tau \otimes (E_{t-1}[y_t] - E_{t-1}[y_{t-1}]) = -\tau \otimes \mu_t + \tau \otimes y_{t-1}, \tag{20}$$

$$\Sigma_{r_{t|t-1}} = \tau'\tau \otimes \left[\underbrace{\Lambda\Omega_{t|t-1}\Lambda' + \Sigma_{t|t-1}}_{\Sigma_{y_t}}\right]. \tag{21}$$

31

The positivity of the matrix Σ_{r_t} $\forall t$ can be demonstrated as follows. The first term in brackets, $\Lambda \Omega_t \Lambda'$, is positive-defined as $\Omega_{t|t-1}$ is diagonal and possesses only positive elements on its diagonal. The second term, $\Sigma_{t|t-1}$ is positive-definite for the same reason. Since τ contains only positive elements, $\tau'\tau$ is also a positive-definite matrix. Finally, Schur's product theorem ensures that the Hadamard product between Σ_{y_t} and $\tau'\tau$ is positive definite. $\qquad \square$

Apendix 2:Univariate GARCH models considered

In this appendix we describe the univariate GARCH specifications that were used to model the conditional variance of the factors and the conditional variance of the measurement errors.

GARCH:

$$\sigma_t^2 = \omega + \alpha \epsilon_{t-1}^2 + \beta \sigma_{t-1}^2$$

Glosten-Jagannathan-Runkle GARCH (GJR-GARCH):

$$\sigma_t^2 = \omega + \alpha \epsilon_{t-1}^2 + \gamma I[\epsilon_{t-1} < 0]\epsilon_{t-1}^2 + \beta \sigma_{t-1}^2$$

Exponential GARCH (EGARCH):

$$\ln(\sigma_t^2) = \omega + \alpha \frac{|\epsilon_{t-1}|}{\sqrt{\sigma_{t-1}^2}} + \gamma \frac{\epsilon_{t-1}}{\sqrt{\sigma_{t-1}^2}} + \beta \sigma_{t-1}^2$$

Threshold GARCH (TGARCH):

$$\sigma_t = \omega + \alpha |\epsilon_{t-1}| + \gamma I[\epsilon_{t-1} < 0]|\epsilon_{t-1}| + \beta \sigma_{t-1}$$

Asymmetric power GARCH (APARCH):

$$\sigma_t^\lambda = \omega + \alpha \left(|\epsilon_{t-1}| + \gamma \epsilon_{t-1}\right)^\lambda + \beta \sigma_{t-1}^\lambda$$

Asymmetric GARCH (AGARCH):

$$\sigma_t^2 = \omega + \alpha (\epsilon_{t-1} + \gamma)^2 + \beta \sigma_{t-1}^2$$

Nonlinear asymmetric GARCH (NAGARCH):

$$\sigma_t^2 = \omega + \alpha (\epsilon_{t-1} + \gamma \sqrt{\sigma_{t-1}^2})^2 + \beta \sigma_{t-1}^2$$