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Abstract

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Contents

1 Introduction

Motivation

Consider a typical workflow for a end-user in a trading desk: a price is needed for a particular product, a pricing function is invoked, which takes in a collection of (calibrated) input parameters, which in turn invokes the underlying numerical method, and finally the outputs, being the price and sensitivities are returned. Arguably, a key business objective for derivatives quants at investment banks and market makers is to streamline and improve this process: for any given product and the given *market generator model* (in other words, the modelling assumptions in terms of volatility or underlying processes), produce a pricing function that outputs its *prices* and its *risks* (sensitivities), to a high degree of accuracy and with minimal latency.

To accomplish this, derivatives quants have broadly three families of numerical methods: analytic and semi-analytic form solutions, Monte Carlo (MC) Methods, and Partial Differential Equation solvers (e.g. finite-difference methods). Analytic expression and expansions are only available for few market models and payoffs, for example the Black-Scholes formula for European options, the Heston characteristic transform, and SABR expansion. For many payoffs and market models, this leaves MC and PDE methods, which given a set of input model parameters (for example the beta and rho in SABR), can return a collection of prices over some time discretisation and state or strike discretisation. However, we should note that MC and PDE method are themselves approximation methods with approximation error, given the need for an appropriaate state and time discretisation.

Approximation error to the 'true' model price can be reduced to some extent with greater computational effort. The computational effort may grow very large when the underlying dimensionality of the problem increases, for example with more complex payoffs and volatility models. When the *risk* sensitivities (which may also represent the hedging strategy) also need to be computed, the computational effort may grow even larger, and in addition the convergence rates in price and sensitivities may not necessarily be the same.

In the Monte Carlo and PDE settings, the solutions are only for one set of model parameters, and as model parameters change the numerical method must be reinvoked. Thus for models whose pricing depends on these schemes, calibration and other downstream tasks becomes expensive. In particular, for the pricing. Hence towards some practical applications, there may be acceptable tradeoffs between accuracy in exchange for speed, for example in the context for electronic market-making or risk / VaR calculations.

Neural-networks may be one method to accomplish this. Neural networks, in this setup, may be viewed as as a model agnostic (in the sense of volatility or market generator models) and method agnostic (MC, PDE) extension to existing numeric pricing methods. Neural Networks present several desirable properties: universal approximation, and ability to 'learn' non-parametric data-driven relationships, sensitivities can be obtained quickly with automatic differentiation, and a fast inference time can be obtained, amortising a computationally expensive training period. A vast literature on the application of neural networks toward derivatives modelling has emerged, for example [4]

In addition to any pricing function returning an accurate price for the market model and payoff, another key requisite for any pricing function is adherence to no-arbitrage, and similarly, smoothness of the pricing function and sensitivities in all inputs. Naturally, if *static* arbitrage opportunities exists, this presents the possibility of a loss to the trading desk. In terms of smoothness, an exotics / OTC desk may have to price derivatives beyond standard strikes and maturities, and a non-smooth pricing function is likely to admit static arbitrage. In addition, non-smooth sensitivities functional may also present P&L impact from hedging. On key concern, is that a standard neural network construction may not necessarily adhere to no-arbitrage and smoothness conditions.

The aim of this dissertation is to investigate how to construct a production-level neural network to approximate derivatives pricing and risks under any general payoff and market generator model. We aim to investigate:

- How to construct and train a neural network efficiently, in terms of both approximation error and minimal no-arbitrage violations? What is the empirical computational complexity required / convergence rate?
- How do neural network approximations perform under various market models, and similarly, under higher dimensionalities in terms of the number of inputs (Black-Scholes, Heston, SABR, Rough Vol)?
- How well does it perform under different payoffs (Calls and Digitals, Asians, Barriers), and exercise policy (Bermudan, European)

Article Structure

In section two, we review the relevant pricing problem formulations. In section three, we review neural networks. In section four, we review the relevant literature on applying neural networks for. In section five, we focus on numerical settings, examining how to construct and efficiently train neural networks that are able to approximate European and Bermudan payoffs. Finally, we conclude our observations in

2 Preliminaries

In this section, we review the formulation of derivatives pricing and risk calculations.

Problem Setup

Consider the problem of computing derivative prices and sensitivities with respect to some input parameters $\mathbf{X}$. First, consider a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. Suppose there is a $\mathcal{F}_t$-adapted d-dimensional stochastic process $\mathbf{S}_t \in \mathbb{R}^d, t \in [0, T]$, representing the value of the underlying assets or cash flows through time whose evolution is determined by a fixed vector of parameters $\mathbf{X} \in \mathbb{R}^f$ (or $\mathbf{X}$ is $\mathcal{F}_t$) measurable. Thus in effect, we consider $\mathbf{S}_t(\mathbf{X})$.

Suppose that there exists some equivalent local martingale measure $\mathbb{Q}$

In the most general case, consider a payoff determined by a continuous discounted payoff and a terminal discounted payoff:

$$f(\tau, \mathbf{S_t}, \mathbf{X}) = \mathbb{E}^{\mathbb{Q}}\left[\int_t^T h_1(u, S_u)dt + h_2(S_T)|\mathbf{S}_t, \mathbf{X}\right]$$

Remark: In a ‘real-life‘ scenario, consider only the case of cash flows at discrete times $\mathbb{E}^{\mathbb{Q}}\left[\sum_{i=1}^n h_i(t_i, \mathbf{S}_i)|\mathbf{S}_{t_0}, \mathbf{X}\right]$. Thus without any loss of generality, we consider pricing a single payoff; then for any derivatives, for example interest rate caps and floors , we can be priced by decomposing them as the sum of payoffs.

$$f(\tau, \mathbf{S_t}, \mathbf{X}) = \mathbb{E}^{\mathbb{Q}}\left[h_2(S_T)|\mathbf{S}_t, \mathbf{X}\right] \tag{1}$$

Suppose f is sufficiently smooth, in particular C^1 with respect to time-to-maturity $\tau = T - t$ and C^2 with respect to each $\mathbf{S}_t$. Then by Feynman-Kac, the corresponding (parametric) PDE is given by:

$$f_\tau = \mathcal{L}f, f(0, s, x) = h(s) \quad \forall \tau \in [0, T], \mathbf{X} \in \mathcal{X}, s \in \mathcal{S} \tag{2}$$

subject to some initial (in terms of time-to-maturity) condition corresponding to the European Payoff, and $\mathcal{L}$ is the generator of $\mathbf{S}$

Thus f denotes the true value function, which represents the no-arbitrage price of the , which can be expressed as a conditional expectation or solution to a PDE.

In general, both the pricing function f and $\frac{\partial f}{\partial \theta}$ in (5), cannot be obtained in closed-form with the exception of several cases. Thus in practice we have the approximation g:

$$g(T-t, \mathbf{X}) \approx f(T-t, \mathbf{X}), \quad f(T-t, \mathbf{X}) = g(\mathbf{X}) + \epsilon$$

Where ϵ denotes the error, which may itself depend on θ and the choice of numerical method. We look for a approximating function g such that ϵ is small over a relevant range of parameters $\mathbf{X}$.

In addition to the value f, the sensitivities with respect to the parameters $\mathbf{X}$ are also of interest. Thus we also require

$$\frac{\partial g}{\partial \mathbf{X}} \approx \frac{\partial f}{\partial \mathbf{X}}, \dots, \frac{\partial^d g}{\partial \mathbf{X}}$$

Path-Dependent Payoffs: Payoffs may also be path dependent, and depend on the entire history of $\mathbf{S_t}$:

$$f(\tau, (\mathbf{S}_u)_{u\in[0,t]}, \mathbf{X}) = \mathbb{E}^{\mathbb{Q}}\left[h_2((\mathbf{S_u})_{u\in[0,T]})|(\mathbf{S}_u)_{u\in[0,t]}\mathbf{X}\right] \tag{3}$$

More generally, we also consider the underlying parameters $\mathbf{X}$ which determine $\mathbf{S}_t$

and that $h(\mathbf{X}_t)$ is a $\mathcal{F}_T$-measurable random variable representing the (discounted) total sum of cash flows, as some function of the time t state $\mathbf{X}$, under some risk-neutral measure $\mathbb{Q}$.

Stochastic Control approach

For example, if $\theta = S_0$, then $\frac{\partial f}{\partial S_0}$ gives the sensitivity to the underlying, and the hedging strategy.

American

$$\sup_{\tau} \mathbb{E}[\int_t^{\tau} B_{t,T} h_1(X_t)dt + h_2(X_\tau)|X] \tag{4}$$

$$f(T-t, \mathbf{X}_t) = \mathbb{E}^{\mathbb{Q}}[h(\mathbf{X}_t)|\mathbf{X}] \tag{5}$$

Then f is the conditional expectation and value function for a derivative with latest cash flow at time T. Under some risk-neutral measure $\mathbb{Q}$, or equivalently the assumptions of no-arbitrage and market completeness, then f is the pricing function with respect to the parameters θ.

Americans Longstaff-Schwartz

Suppose $\mathbb{E}[g(X_n)^2] < \infty$ the payoff is L^2 integrable.

[23] introduced the Longstaff-Schwartz method, or Least Squares Monte Carlo for valuing American Options. In the one step case, consider

$$f(X_t(\theta)) = \mathbb{E}^{\mathbb{Q}}[h(X_T(\theta))|X_t(\theta)] \tag{6}$$

Where X is some appropriately chosen (Markov) state process such that $f(X_t(\theta))$. Then let g denote an approximation for the value function. In [23], they consider basis functions such as Chebyshev and Legendre polynomials.

$$g(X_t) = \sum_{b=1}^{B} \phi_b(X)$$

In the multi-period case , we consider for $t_0 < t_i < \dots t_n = T$:

$$f(X_{t_i}) = \max\{\mathbb{E}^{\mathbb{Q}}[f(X_{t_{i+1}})\}|X_{t_i}], h(X_{t_i})\}, \quad f(X_T) = h(X_T)$$

Where h is the exercise value for the pay

[23] argues that as the number of basis functions B to infinity, g approximates the true value function, and as the number of timesteps N goes to infinity. Further to this, in the Monte Carlo setting, we also require the number of samples M to go to infinity.

$$f(S_t) = \mathbb{E}[(S_T - K)^+ | S_t] \tag{Black-Scholes}$$

$$\mathbb{E}[(f(S_T) - \mathbb{E}[g(S_t)|S_t])^2] = 0$$

Then by the tower property of expectation

$$\mathbb{E}[(g(S_T) - \mathbb{E}[g(S_t)|S_t])^2] = 0$$

Adjoint Automatic Differentiation

The adjoint automatic differentiation method was first brought to [**giles2004**]

In the most general case, consider the pricing function in the form of 5. Given regularity constraints

$$\frac{\partial f}{\partial \theta} = \frac{\partial}{\partial \theta} \mathbb{E}^{\mathbb{Q}}[]$$

The sensitivities of f with respect to θ, can be computed through the application of the chain rule backwards:

$$\frac{\partial S}{\partial \theta} \cdot \frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial \theta}$$

where the partials are evaluated in reverse order. Programming language wise, it can be implemented via computional graphs and implementing *adjoint* operators.

No-Arbitrage Constraints

We consider the conditions for our approximating function g to be free of static arbitrage in the case of European calls. Suppose that $g \in C^{1,2}$ is continuously and and once-differentiable with respect to $\tau = T - t$ and twice differentiable with respect to K. Equivalently, we could consider moneyness $x = S/K > 0$, or log-moneyness $y = \log(S/K)$, which may be a more convenient representation in some cases, and assume g is twice differentiable with respect to:

$$\begin{aligned}
\frac{\partial X}{\partial K} &= \frac{S}{-K^2} = \frac{-X}{K} && , \quad \frac{\partial y}{\partial K} = \frac{-1}{K} \\
\frac{\partial g}{\partial K} &= \frac{\partial g}{\partial x}\frac{\partial x}{\partial K} = \frac{-x}{K}\frac{\partial g}{\partial X} && , \quad \frac{\partial g}{\partial y}\frac{\partial y}{\partial K} = -\frac{1}{K}\frac{\partial g}{\partial y} \\
\frac{\partial}{\partial K}(-\frac{x}{K}\frac{\partial g}{\partial x}) &= \frac{1}{K^2}(x\frac{\partial g}{\partial X} + x^2\frac{\partial^2 g}{\partial X^2}) \\
\frac{\partial}{\partial K}(-\frac{1}{K}\frac{\partial g}{\partial y}) &= \frac{1}{K^2}(\frac{\partial g}{\partial y} + \frac{\partial^2 g}{\partial y^2})
\end{aligned}$$

Then in terms of K, x, y we have:

No Static Arbitrage for European Calls: From [11] [5]

$\frac{\partial g}{\partial \tau} \geq 0$	carry, time-value
$\frac{\partial g}{\partial K} \leq 0, \frac{\partial g}{\partial x} \geq 0$	monotonically decreasing (increasing) in strike (moneyness / underlying)
$\frac{\partial^2 g}{\partial K^2} \geq 0, \frac{\partial^2 g}{\partial x} \geq 0, \frac{\partial^2 g}{\partial y} \geq 0$	convexity in strike, moneyness
$g(T-t, K) \geq (S_t - K)^+ \geq 0$	intrinsic value
$\lim_{K\to\infty} g(K, T-t) = 0$	1

We also consider no static arbitrage for European digitals. From the Breeden-Litzenberger formula we have:

Breeden-Litzenberg formula. We now have an additional approach: solve for the risk-neutral transition density $p(y, T; s, t)$ and obtain the price of any european payoff via numerical integration.

Since the payoff of a European digital $\mathbb{Q}[]$

$\frac{\partial g}{\partial \tau} \geq 0$	carry, time-value
$\frac{\partial g}{\partial K} \leq 0, \frac{\partial g}{\partial x} \geq 0$	monotonically decreasing in strike, increasing in moneyness/underlying
$\frac{\partial^2 g}{\partial K^2} \geq 0, \frac{\partial^2 g}{\partial x} \geq 0, \frac{\partial^2 g}{y} \geq 0$	convexity in strike
$g(T-t, K) \geq (S_t - K)^+ \geq 0$	intrinsic value
$\lim_{K\to\infty} g(K, T-t) = 0$	1

$$\int_{\mathbb{R}} (y-K)^+ p(y, T; s, t) dy = \int_K^\infty (y-K) p(y, T; s, t) dy$$

In the context of put-call parity, we can simply price one of the European payoffs and obtain the rest via replication, or integration from the derived transition density.

No Dynamic Arbitrage: No dynamic arbitrage indicates the lack of a replication strategy with zero initial wealth that leads to positive wealth $\mathbb{P}$-almost surely. For Dynamic Arbitrage, a sufficient condition may be that the function g satisfies the relevant pricing PDE.

$$\mathcal{L}g = g_\tau, \qquad \text{for all } \tau, K$$

If $\mathcal{L}g \neq 0$, the dynamic arbitrage strategy is given by longing the payoff if $\mathcal{L}g$ and shorting the replicating portfolio, and $\mathcal{L}g < 0$, and shorting the payoff if $\mathcal{L}g < 0$ and longing the replicating strategy.

No arbitrage bounds for non-vanilla payoffs: For non-european options, we can consider replication, and in some case lower bounds. For example, in the case of Americans we must have a value greater than the equivalent european $V^A(t, s) \geq B^B(t, s)V^E(t, s)$, and for barries, we must have $V_{down} + V_{up} =$

One question is how to incorporate these no-arbitrage constraints into a neural network approximation.

3 Neural Networks

We aim to train a neural network to approximate a pricing function, over as wide a range as possible under a given market generator model, that has a minimal approximation error and minimal static and dynamic arbitrage.

Neural Network Definition

A comprehensive outline of neural networks is better in [14], and a reference on practical implementations using `Tensorflow/Keras` can be found in [7]. An overview of neural networks and their applications towards finance can be found in [10].

We begin by considering a standard feed-forward neural network architecture . Consider some input $\mathbf{X} \in \mathbb{R}^{N \times d}$, representing N observations or samples of a d-dimensional vector. A n-layer feed-forward neural network with n layers can be characterised by :

$$
\begin{aligned}
\mathbf{Z}_1 &= g_1(\mathbf{X}\mathbf{W}_1 + \mathbf{b}_1\mathbf{1}^\top) && (7)\\
\mathbf{Z}_2 &= g_2(\mathbf{Z}_1\mathbf{W}_2 + \mathbf{b}_2\mathbf{1}^\top) && (8)\\
g(\mathbf{X};\theta) &= g_{n+1}(\mathbf{Z}_n\mathbf{W}_{n+1} + \mathbf{b}_n\mathbf{1}^\top) && \text{(feed-forward neural network)}
\end{aligned}
$$

Where $\mathbf{W}_i \in \mathbb{R}^{H_{i-1}\times H_i}, i = 1, \ldots n-1$ denotes the weights for the n-th hidden layer, $b_i \in \mathbb{R}^{H_i}$ denotes the bias term for the n-th hidden layer, and $g_i, i = 1, \ldots, n$, is the activation function for the i-th hidden layer. The activation function is applied element-wise to the inputs, such that $g(\mathbf{Z})_{ij} = g(z_{ij})$, and introduces non-linearities. Given the neural network f is a composition of functions, we can regard any general neural network as a composition of neural networks.

Typically in a regression setting the final layer is the identity (or linear) activation $g(z_{ij}) = z_{ij} \in \mathbb{R}^n$, such that the final value may attain any real value, although a final activation function could also be applied. For example, if the output is known to take values between $y_i \in [0, 1]$, then sigmoidal activation $g_i(x) = \frac{1}{1+e^{-x}}$ could be considered. Regardless, the neural network may be seen as 'learning' a collection of basis functions of 'latent representation' $\mathbf{Z}_n$ from inputs $\mathbf{X}$, for the given objective.

Compared with standard basis regression, there is no need to specify the basis functions $\mathbf{Z}_n$ explicitly.

Furthermore, several universal approximation.

Theorem 1 (Hornik (1990)) *Let $\mathcal{N}_{d_0,d_1,\sigma}$ be the set of neural networks mapping from $\mathbb{R}^{d_0} \to \mathbb{R}^{d_1}$, with activation function $\sigma : \mathbb{R} \to \mathbb{R}$. Suppose $F \in C^n$ is continuously n-times differentiable. Then if $\sigma \in C^n(\mathbb{R})$ is continuously n-times differentiable, then $\mathcal{N}_{d_0,d_1,\sigma}$ arbitrarily approximates F and its derivatives up to order n.*

Thus there exists some neural network that arbitrarily approximates. However, the universal approximation theorem only proves existence of such a neural network, and not how to construct it.

Neural Networks in their capacity as function approximators can be characterised as

$$\arg\min_{\theta\in\Theta} L(f(\mathbf{X}), g)$$

the minimizer of some loss function L, over a space of feasible parameters Θ.

Training Neural Networks and Practical Issues

In general, given some loss function $L(g(\mathbf{X}), f(\mathbf{X}))$, and sample data: $\mathbf{X}_j, j = 1, \ldots, N$

$$\arg\min_{\theta\in\Theta} \mathbb{E}[\|\mathbf{y} - f(\mathbf{X};\theta)\|^2]$$

$$\theta_{t+1} \leftarrow \theta_t - \eta_t \nabla_\theta L, \quad \eta > 0 \tag{9}$$

The gradient of the loss with respect to the parameters θ are obtained via AAD. and updated via backpropogation.

This loss function is generally non-convex with respect to the parameters θ; as such there are no guarantees of convergence to a global minima except in.

In practice the gradients with respect to the neural network parameters $\nabla_\theta L$ also need to be estimated. Mini-batch stochastic gradient descent

$$\sum \theta_{t+1} \leftarrow \theta_t - \eta_t \nabla_\theta L, \quad \eta > 0 \tag{10}$$

In practice, techniques such as:
batchnormalisation: Samples are normalised $(\mathbf{X}_i \ominus \mu_i)/\sigma_i$
Dropout:,
learning rate scheduling: have been found to lead to empirically faster training

Neural Network Architectures

The universal approximation theorem of [**hornik**] only suggests that there *exists*. Empirically, architectures that have been selected for domain-specific problems as opposed to . We consider several hidden layer or block architectures we will utilise in later sections.

Gated unit: In the context, this can help learn feature interactions, and also facilitate feature *selection*.

$$\mathbf{Z}_1 = g_1(\mathbf{X}\mathbf{W}_1 + \mathbf{b}_1\mathbf{1}^\top) \tag{11}$$
$$\mathbf{Z}_2 = g_2(\mathbf{X}\mathbf{W}_2 + \mathbf{b}_2\mathbf{1}^\top) \tag{12}$$
$$\mathbf{Z}_3 = \mathbf{Z}_1 \otimes \mathbf{Z}_2 \tag{gated block}$$

Where $\otimes$ denotes element-wise multiplication. [30] uses gated units to construct a neural network that has the requisite monotonicity and convexity constraints with respect to strike and moneyness.

Residual block: Residual blocks allow for the *flow* of information from earlier layers to later layers

$$\begin{aligned}
\mathbf{Z}_1 &= g_1(\mathbf{X}\mathbf{W}_1 + \mathbf{b}_1\mathbf{1}^\top) && (13)\\
\mathbf{Z}_2 &= g_2(\mathbf{Z}_1\mathbf{W}_2 + \mathbf{b_2}\mathbf{1}^\top) && (14)\\
\mathbf{Z}_3 &= g_3(\begin{pmatrix}\mathbf{Z}_1 & \mathbf{Z}_2\end{pmatrix}\mathbf{W}_3 + \mathbf{b}_3\mathbf{1}^\top) && \text{(gated block)}
\end{aligned}$$

Recurrent neural networks: Recurrent neural networks capture sequential, or temporal dependencies.

$$\begin{aligned}
\mathbf{Z}_1 &= g_1(\mathbf{X}\mathbf{W}_1 + \mathbf{b}\mathbf{1}^\top) && (15)\\
\mathbf{Z}_2 &= g_2(\mathbf{Z}_1\mathbf{W}_2 + \mathbf{b}\mathbf{1}^\top) && (16)\\
\mathbf{y} = f(\mathbf{X};\theta) &= g_n(\mathbf{Z}_n\mathbf{W}_n + \mathbf{b}\mathbf{1}^\top) && \text{(gated block)}
\end{aligned}$$

Recurrent neural networks may be useful in a non-markovian setting, for example, in pricing path dependent payoffs. Furthermore, they may be useful even for European payoffs, when additional market frictions such as transaction costs or market impact are included, as in [4].

AutoML: Aside from defining specific architectures, on the machine learning side there is the aspect of architecture. In effect, we consider some space of neural networks Θ, for example in terms of the number of hidden layers or hidden units.

4 Neural Networks for Derivatives Pricing

Having defined the pricing and sensitivities and neural networks, in this section

Modelling Choices, Literature Review

A machine learning setup generally consists of several components: the modelling objective, the dataset, the model, training, and inference.

The first question is how to approach modelling for the pricing approximation task. To date, various approaches utilising neural networks for derivatives pricing and hedging have been presented, of which a comprehensive review of methods can be further found in [26]:

Supervised: Direct approximation of model prices, with automatic differentiation for sensitivities, as in [19]. A natural choice could be to directly learn a mapping between parameters, and prices or which appears to be a standard approach.

| [16] | |

Neural PDEs: Use of neural networks to solve (high-dimensional) PDEs as in []. Given that in some cases, pricing problems can be formulated as. In addition, in our application we would need to solve the PDE over a range of model parameters.

Neural SDEs: Represent the dynamics of a SDE $\mathbf{S}_t$ as a nueral network, as in [13] and Generative Adversarial Networks, and calibrate by minimising the MC pricing error between observed prices $\|P(T,K) - \mathbb{E}[h(S_T,K)]\|$. This approach has the potential advantage of allowing more realistic dynamics to be captured; however, being a Monte-Carlo based approach, the actual inference time may be slow.

Model hedging strategy; Approximation for a hedging or replicating strategy; In effect this can be seen as pricing via replication or stochastic control [4]. The *Deep Hedging* approach of [4] considers approximating the replicating strategy. Thus in this case, the neural network represents the delta.

$$y - \sum_{i=1}^{N} \Delta_i(X_{t_i})(S_{i+1} - S_i)$$

Learn the hedging strategy with a neural network, and price can be obtained via superhedging. However the drawback is that this hedging strategy is tuned to a single payoff and requires MC simulations for pricing.

We focus on the supervised learning approach. In addition, the hedging and neural PDE approaches can also be incorporated to some extent, by including them in the loss function for the supervised task.

However, even within the supervised learning ask. In effect, the choice of input-output pairs $(\mathbf{X}, \mathbf{y})$.

Model risk-neutral implied distribution: Let $g(T, y/K; t, X)$ be a estimate for the conditional density at time T. Thus in effect, we approximation the price of a European digital and obtain (in the 1D case).

$$g(T-t, X) \approx DF_{t,T} \int_K^{\infty} (\frac{y}{K} - 1)^+ g(T, y; t, X) dy$$

Model Control Variate Residuals:

Supervised - Calibration: learn mappings for implied volatilities/prices to parameters, or vice-versa [17]. [17] also considers a grid based approach, where a fix set of strikes are predicted, and an image based approach where a fixed collection of maturities and strikes are modelled.

Dataset Construction

Machine Learning Approach: In the machine learning framework, typically three datasets are constructed: A *train* dataset, which, the model is directly trained. In theory, given that $\mathbf{X}$ is sampled from the same distribution.

In [18], their ys are based on sample Monte Carlo payoffs. However, given that in our application we must also consider

In the general case of a standard diffusion process, we could consider

Data Generation Method. Train, Val: Hypercube / Grid sampling vs Random sampling, Random sampling within boundary

In the case of machine learning Quasi Monte Carlo, Sobol Sequences, Control Variates

Test: Another random sample within boundary, random sample outside of boundary

In a real life setting, next-day prices could be used, although this does not necessarily make sense.

Data Preprocessing

The dataset may also contain arbitrageable prices, for example in Monte Carlo simulations. In the Longstaff-Schwartz paper, they simply set out-of-the money paths to weight zero:

$$\sum_{j=1}^{N} w_j L(f(X_j), g(X_j))$$

In the Longstaff-Schwartz case, paths for which $f(X_j)$ have weight zero.

In this same style, [9]. Preprocessing, remove arbitrage out of the money paths.

Dataset Quality and Risk: The risk of the neural network is also present through *data risk*. Dataset quality For example, how does the NN perform on unseen parameter ranges?, An issue is that the NN may be able to , e.g. when the market regime shifts. Thus the NN must likely be trained on a very large grid of parameters.

In [18] and the standard Longstaff-Schwartz approach,

In the case of hedging, FBSE, and American / Bermudan options modelling, The neural network approximation approach is more closely aligns with a Monte Carlo method. Thus there is error from both the neural network approximation g and the simulation of state-payoff pairs X_j, y_j

For example, consider the case of very deep in-the-money or out-of-the money strikes for a European all. A proposed solution of is to sample more from these regions, or similarly to set a greater weight w_j in a weighted loss function

Swaption

Neural Network Construction for Derivatives Pricing and Risks

We characterise handcrafted neural networks as described in [26]: in short, we can embed prior knowledge into the construction of the neural network, in the choice of an appropriate loss function or architecture.

Activation	$f(x)$	$f'(x)$	$f''(x)$
ReLU	$\max\{x, 0\}$	$1_{x>0}$	0
ELU	$\alpha(e^x - 1)1_{x<0} + x1_{x>0}$	$1_{x>0} + \alpha e^x 1_{x<0}$	$\alpha e^x 1_{x<0}$
Sigmoid	$\frac{1}{1+e^{-x}}$	$\frac{e^{-x}}{(1+e^{-x})^2}$	$\frac{-e^{-x}(1+e^{-x})}{(1+e^{-x})^4}$
SoftPlus	$\log(1 + e^x)$	$\frac{1}{1+e^{-x}}$	$\frac{e^{-x}}{(1+e^{-x})^2}$
Swish	$\frac{x}{1+e^{-x}}$	$\frac{(x-1)e^{-x}}{(1+e^{-x})^2}$	
GeLU	$x\Phi(x)$	$x\phi(x) + \Phi(x)$	$\phi(x) + x\phi''(x)$
tanh	$\frac{e^x - e^{-x}}{e^x + e^{-x}}$		
RBF	$e^{-\frac{x^2}{2}}$	$-xe^{-\frac{x^2}{2}}$	$(x^2 - 1)e^{-\frac{x^2}{2}}$

Table 1: Activation Functions

[20] propose a modified elu function with: $R(z) = \alpha(e^z - 1)1_{z\leq 0} +$

Smoothness. Firstly, to obtain smooth (or at least, continuous) approximations for the d-th order partial derivatives, we require $g(\cdot)$ to be C^d continuous d-time differentiable with respect to its inputs [20]. Given the loss functions in 19, 20, may contain even higher-order partial derivative terms, we may require even further smoothness constraints.

In order for the neural network output g to be C^d, one method could be to constrain all intermediate activation functions g_i to be sufficiently smooth, such that g as a composition of smoothness will be a smooth function. [6] uses this approach.

Hard constraints [6] *softplus* activation for the strike and sigmoid activation for the time-to-maturity, with non-negative weight constraints. Thus this guarantees that the neural network is non-negative, monotonic in strike and time, and convex in strike. **Asymptotics**: [1] highlights that neural networks are generally able to interpolate within the domain of training, but unable to extrapolate. This is also have particular relevance, given models have asymptotics, and payoff asymptotics correspond to no-arbitrage bounds.

The proposed solution of [1] uses a control-variate-like two-step procedure: first, they fit a *control variate function* , in their case cubic spline, with the appropriate asymptotics, then they fit a neural network with vanishing asymptotics to the residual of the original function and the control variate.

Their proposed architecture consists of the following gaussian kernel, or radial basis (RBF) activation

$$g_i(\mathbf{x}) = \exp(-\frac{1}{2}\|\mathbf{x}\mathbf{W}_i + \mathbf{b}_i\|^2) \tag{17}$$

Such that if any $x_i \to \pm\infty$ we have $g_i(\mathbf{x}) \to 0$. Thus if our neural network consists only of RBF activations, we can obtain the target asymptotics.

Loss Functions for Derivatives Pricing and Risks

We now consider a choice of specific loss functions for the supervised learning task.

In the most simple case, we consider direct approximation, for example with mean squared error as a loss function. For a single sample: X_i, y_i

$$L_{price}(y_i, g(X_i)) = (y_i - X_i)^2 \tag{18}$$

The proposed method from [18] is to consider a joint loss function, an approach they describe as *differential machine learning.*

$$L_{price}(g(X_i), f(X_i)) + \lambda L_{greeks}\left(\frac{\partial g(X_i)}{\partial X}, \frac{\partial g(X_i)}{\partial X}\right), \lambda \geq 0 \tag{19}$$

The partial derivative $\frac{\partial f(X_i)}{\partial X}$ can be obtained at some extra cost given AAD. [18] argues that the λ, but in practice this could be determined by trial-and-error and prior knowledge of the relative scales.

A further extension could be to consider the, which is a *Neural PDE* approach

$$L_{price}(g(X_i), f(X_i)) + \lambda_1 L_{greeks}\left(\frac{\partial g(X_i)}{\partial X}, \frac{\partial g(X_i)}{\partial X}\right) + \lambda_2 L_{PDE}(\mathcal{L}g) \tag{20}$$

Where $\mathcal{L}$ denotes a PDE operator and $\lambda_1, \lambda_2 \geq 0$ are weights for each loss function. For example, in the case of black scholes, we add the differential:

$$\mathcal{L}g = \frac{\partial g}{\partial t} + rs\frac{\partial g}{\partial s} + \frac{\sigma^2 s^2}{2}\frac{\partial^2 g}{\partial s^2} - rg$$

[29] argues that the inclusion of a PDE loss term leads to self-consistency, given that if the neural network approximation satisfies the pricing PDE (in their application, the Dupire Local Volatility PDE), $\mathcal{L}g = 0$ there is no dynamic arbitrage.

Finally, we can also consider any no-arbitrage constraints. An alternative is a *soft penalty* approach [20]. For example, we could consider any of the no-arbitrage constraints, and include a loss term if the . One such example could be a penalty $((S-K)^+ - g(S))^+$, i.e. a penalty for being beneath the intrinsic call bound. let $L_{cons_1}, \ldots L_{cons_P}$ denote P soft penalties. In the most general case, the loss function consists of:

$$L_{price} + \lambda_1 L_{greeks} + \lambda_2 L_{PDE}(\mathcal{L}g) + \sum_{i=1}^{P} \lambda_i L_{cons_i} \tag{21}$$

In practice, the inclusion of higher order differentials increases the total training time. In the subsequent sections, we explore whether the inclusion of each different penalty leads to better results, in terms of error and no-arbitrage.

Interpreting and Monitoring Neural Networks

We only use neural networks as an approximating function as an overlay on top of some market generator model and numerical method, which may alleviate some of the *black-box* issues [8]. However, the neural network may still produce unexpected outputs. We can evaluate and interpret neural networks to some extent. In the simple case, we can use graphical methods, or evaluate behaviour around boundary conditions. In addition, we can leverage machine learning interpretability methods [25], which [3] explores in the context of using deep neural networks for Heston calibration.

- Dependency plots against one or two dependent variables, boundary conditions.
- First order partial derivatives, Second order (Hessian), higher order derivatives.
- Output of penultimate layer (basis functions or latent representation)
- Machine-learning interpretabilty methods: Shapley Values, LIME

In terms of monitoring and deploying the neural network. In the training period, we obtain an estimate of the pricing errors for some range of parameters. We could define a region of parameters $\mathcal{X} \subseteq \mathbb{R}^d$ for which the maximum error of the neural network is under some ϵ. These could be stored as simply $2d$ linear constraints $d_i < X_i < u_i$ for each parameter. If the pricer is evoked outside this region, we simply revert to the original numerical method. If however, the market parameters have been consistently away the training region of parameters, in other words, *distribution drift*, or the contract structure of a derivative chaanges, then this necessitates retraining of the neural network.

Alternative Methods

Alternative basis functions: For function approximation, a natural comparison could be made with standard polynomial or basis regression as in [23]. [2] explored the use of Tensor Methods. [28] discussed the use of the Karhunone-louvre basis and signatures. Chebyschev Methods.

In terms of alternative machine learning models. [22] explored the use of Gausisan Processes. Some papers have looked at gradient boosting

[13] suggests initialising multiple random seeds, to obtain a confidence interval on prices

$$P(\theta^-), P(\theta^+)$$

5 Numerical Experiments

In our numerical experiments, our general workflow is as follows:

- We consider a forward value processes F_t which is a $\mathbb{Q}$-local martingale with zero drift. In practice, if F_t is not a local martingale, we could consider some appropriate change of numéraire, or consider the chain rule / AAD to obtain the relevant sensitiviies,
- We consider exploiting homogeneity, or other properties for an initial dimensionality reduction of the problem.
- We then construct a dataset, consider multiple neural networks and train them, then evaluate them.

The metrics we aim to evaluate our experiments on are:

- Time Complexity: Dataset Construction (Size (Timesteps, Number of MC paths), method), Training time, Inference time of price + sensitivities
- Model Complexity (No. of parameters, (width, depth). The choice of Model Architecture: Standard, Differential, Neural PDE
- Errors (L1, L2 , Linf) in Price, First Order, Second Order sensitivities, against closed form if available.
- No-arbitrage violations, asymptotic behaviour, extrapolation behaviour
- Comparison vs MC, PDE, other approximation methods
- Applications: Pathwise Hedging Error ($V_T - \sum_{i=1}^{N} \Delta_{t_i} \cdot (S_{t_i+1} - S_{t_i}) - P_0$) (Loss, CVaR / Quantiles), or explained PNL

Geometric Brownian Motion / Black-Scholes

We consider the most simple case: European Call pricing in 1D Black-Scholes

SDE and MC: We have

$$\begin{aligned}
dF_t = F_t \sigma dW_t, &\quad F_t = F_0 \exp(-\frac{\sigma^2}{2} t + \sigma\sqrt{t}\frac{W_t}{\sqrt{t}}) \\
dX_t = d(F_t/K) &= \frac{F_t}{K}\sigma dW_t = X_t \sigma dW_t \\
d\log(F_t) &= \frac{-\sigma^2 t}{2} dt + \sigma W_t \\
h(S_T) &= \left(\frac{F_T}{K} - 1\right)^+ = (X_T - 1)^+ = (\exp(y_T) - 1)^+
\end{aligned}$$

In the case of Black-Scholes, we can exploit positive homogeneity

$$\mathbb{E}^{\mathbb{Q}}[(\frac{F_t}{K} - 1)^+] = \frac{\mathbb{E}^{\mathbb{Q}}[(F_T - K)^+ | S_t, K, \sigma]}{K} \tag{22}$$

$$\lambda C(F_t, K, \sigma, \tau) = C(\lambda F_t, \lambda K, \sigma, \tau) \tag{23}$$

Thus we can eliminate one of F_t, K by letting $\lambda = \frac{1}{K}$ and fixing. We can further eliminate one of σ, τ by noting that in the Black-Scholes formula:

$$d_{\pm} = \frac{\log(F_t/K)}{\sigma\sqrt{\tau}} \pm \frac{(\sigma\sqrt{\tau})^2}{2}$$

$$C(\frac{F_t}{K}, 1, \sigma, \tau) = \frac{F_t}{K}\Phi(d_+) - \Phi(d_-)$$

The σ and τ terms are grouped together as
PDE: The corresponding PDE is given by:

$$V_\tau + \frac{s^2\sigma^2 V_{ss}}{2} = 0 \tag{24}$$

$$V_{\sigma\sqrt{\tau}} + \frac{V_y}{} \tag{25}$$

Dataset:

$y = \log(F_t/K)$	$\sigma\sqrt{\tau}$
Log-moneyness	Time-scaled Implied Volatility
$[-3, 3]$	$[0, 3]$

We consider the second example from [18], 1D Black-Scholes for a single time-to-maturity $T - t$ with respect to different levels of the underlying S_t.

We parameterise in terms of moneyness $\frac{S_T}{K}$ and exploit the homogeneity of Black-Scholes. $\frac{\partial f}{\partial x}\frac{\partial x}{\partial s} =$

The lack. Thus we consider a parameterisation with respect to moneyness and time-scaled implied volatility

$$f(S_t/K, \sigma\sqrt{T-t})$$

In this case, we can leverage the homogeneity of Black-Scholes, and reduce our problem to the process $\log(S_t/K)$ instead of S_t and time-scaled implied volatility $\sigma\sqrt{\tau}$ instead of τ

$$V_t + rsV_s - + \frac{V_{ss}\sigma^2 s^2}{2} - rV = 0$$

$$\frac{\partial\sigma\sqrt{\tau}}{\partial t} = \frac{\sigma^2}{2\sqrt{\tau}}, \frac{\partial \log(S/K)}{\partial s} = \frac{1}{s},$$

$$-V_{\sigma\sqrt{\tau}}\frac{\sigma^2}{2\sigma\sqrt{\tau}} - \frac{\sigma^2}{2}V_x + \frac{\sigma^2 V_{xx}}{2} = 0$$

$$V_t + rsV_s + \frac{s^2 V_{ss}\sigma^2}{2} - rV = 0$$

$x = \log(S/K), \frac{\partial x}{\partial s} = \frac{1}{s}, \frac{\partial^2 x}{\partial s^2} = \frac{-1}{s^2}$

$$V_{\sigma\sqrt{T-t}}\frac{\sigma}{2\sqrt{T-t}} + rsV_x\frac{\partial x}{\partial s} + \frac{s^2\sigma^2}{2}[V_{xx}(\frac{\partial x}{\partial s})^2 + V_x\frac{\partial^2 x}{\partial s^2}] - rV = 0$$

$$V_{\sigma\sqrt{T-t}}\frac{\sigma}{2\sqrt{T-t}} + rsV_x\frac{\partial x}{\partial s} + \frac{s^2\sigma^2}{2}[V_{xx}(\frac{\partial x}{\partial s})^2 + V_x\frac{\partial^2 x}{\partial s^2}] - rV = 0$$

$$-V_\sigma\sqrt{T-t}\frac{\sigma}{2\sqrt{T-t}} + (r - \frac{\sigma^2}{2})V_x + \frac{\sigma^2 V_{xx}}{2} - rV = 0$$

$$\frac{-V_\sigma\sqrt{T-t}}{\sigma\sqrt{T-t}} + (\frac{r}{2\sigma^2} - 1)V_x + V_{xx} - \frac{r}{2\sigma^2}V = 0$$

$$V_t + rsV_s + \frac{s^2 V_{ss}\sigma^2}{2} - rV = 0$$

$x = \log(S/K), \frac{\partial x}{\partial s} = \frac{1}{s}, \frac{\partial^2 x}{\partial s^2} = \frac{-1}{s^2}$

$$V_{\sigma\sqrt{T-t}}\frac{\sigma}{2\sqrt{T-t}} + rsV_x\frac{\partial x}{\partial s} + \frac{s^2\sigma^2}{2}[V_{xx}(\frac{\partial x}{\partial s})^2 + V_x\frac{\partial^2 x}{\partial s^2}] - rV = 0$$

$$V_{\sigma\sqrt{T-t}}\frac{\sigma}{2\sqrt{T-t}} + rsV_x\frac{\partial x}{\partial s} + \frac{s^2\sigma^2}{2}[V_{xx}(\frac{\partial x}{\partial s})^2 + V_x\frac{\partial^2 x}{\partial s^2}] - rV = 0$$

$$-V_\sigma\sqrt{T-t}\frac{\sigma}{2\sqrt{T-t}} + (r - \frac{\sigma^2}{2})V_x + \frac{\sigma^2 V_{xx}}{2} - rV = 0$$

$$\frac{-V_\sigma\sqrt{T-t}}{\sigma\sqrt{T-t}} + (\frac{r}{2\sigma^2} - 1)V_x + V_{xx} - \frac{r}{2\sigma^2}V = 0$$

$V_K = V_x\frac{\partial x}{\partial K} = -\frac{V_x}{K}, V_{KK} = \frac{V_{xx}}{K^2} + \frac{V_x}{K^2}$
Thus we require: $V_x \geq 0, V_{xx} \geq 0, V_\tau \geq 0$
$V(-\infty, \sqrt{\tau}) = 0, V(x, 0) = (x-1)^+$

$$x = \frac{\log(S/K)}{\sigma\sqrt{T-t}}, \frac{\partial}{\partial S} = \frac{1}{sy}, \frac{\partial}{t} = \frac{\log(S/K)\sigma^2}{2y}$$

SABR

We consider the SABR of [15]

$$dF_t = F_t^{\beta}\sigma_t dW_{1,t} \tag{26}$$

$$d\sigma_t = \xi\sigma_t dW_{2,t}, dW_{2,t} = \alpha\sigma_t[\rho dW_{1,t} + \sqrt{1-\rho^2}dW_{1,t}^{\top}] \tag{27}$$

Consider also, the normalised forward or moneyness process F_t/K, and the log-forward process

$$d(F_t/K) = \frac{\sigma_t}{K}F_t^{\beta}dW_{1,t} = (\sigma_t K^{\beta-1})(F_t/K)^{\beta}dW_{1,t}$$

$$d\log F_t = \frac{1}{F_t}dF_t + \frac{-1}{2F_t^2}(dF_t)^2 = \sigma_t F_{t-1}^{\beta-1}dW_{1,t} - \frac{\sigma_t^2 F_t^{2\beta-2}}{2}dt$$

$$dX_t = d\log F_t = \sigma_t \exp((\beta-1)X_t)dW_{1,t} - \frac{\sigma_t^2}{2}\exp(2(\beta-1)X_t)dt$$

In the case of SABR (and CEV), we have the following relationship for call prices, we have the relationship:

$$C(\lambda F_t, \lambda K, \lambda^{1-\beta}\sigma_t, \beta, \rho, \xi) = \lambda C(F_t, K, \sigma_t, \beta, \rho, \xi)$$

Thus we can fix one of F_t, K. However, it may be difficult for a neural network to learn the scaling relationship for the volatility input.

PDE: The corresponding SABR PDE is given by:

$$-V_\tau + \frac{\sigma^2 F^{2\beta}}{2}V_{FF} + \frac{\sigma^2\xi F^{\beta}}{2}V_{\sigma\sigma} + \frac{\xi^2\sigma^2}{2}V_{F\sigma} = 0, V(F,0) = (F-K)^+ \tag{28}$$

$$-V_\tau + V_y e^{2(\beta-1)y}\frac{\sigma^2}{2} + \frac{\sigma^2\exp(2(\beta-1)y}{2}V_{yy} + \sigma^2\exp((\beta-1)y)V_{y\sigma}, V(y,\tau=0) = (e^y-1) \tag{29}$$

$$-V_\tau + \frac{\sigma^2(F/K)^{2\beta}}{2}V_{FF} + \frac{\sigma^2\xi F^{\beta}}{2}V_{\sigma\sigma} + \frac{\xi^2\sigma^2}{2}V_{F\sigma} = 0, f(F/K,0,\dots) = (F/K-1)^+ \tag{30}$$

$$\tag{31}$$

[**hagan**] provides an asymptotic expansion. However,

F$_t$	K	σ_t	τ	β	ξ	ρ
Forward	Strike	Volatility	Time-to-maturity	CEV	vol-of-vol	correlation
1	[0.5, 5]	[0, 0.2]	[0, 0.3]	[0, 1]	[0, 1]	[0, 0.7]

The asymptotic expansion degrades as σ, ξ, T increase, or for low strikes K (equivalently, high moneyness F/K or $\log(F/K)$)

We simulate a dataset
We generate implied volatilities from the Hagan approximation

$$\begin{aligned} z &= \frac{\xi}{\sigma}(FK)^{(1-\beta)/2} \\ x(z) &= \log\Big((\sqrt{1-2\rho z+z^2}+z-\rho)/(1-\rho)\Big) \end{aligned}$$

We can simulate the volatility process σ_t exactly, and the

$$V_T = V_t \exp((-0.5\alpha^2(T-t)) + \alpha(W_T - W_t)) \tag{32}$$

$$F_{j+1} - F_j = |V_j F_j^{\beta} \Delta W_j| \tag{33}$$

[**mcghee**]

Heston

SDE and MC:

$$\begin{aligned} dF_t &= F_t V_t dW_{1,t} \\ dV_t &= \kappa(\theta - V_t)dt + \xi\sqrt{V_t}dW_{2,t} \\ d[W_1, W_2]_t &= \rho dt \end{aligned}$$

We use the description of the characteristic transform from [12].
PDE:
Dataset:

F$_t/K$	V$_t$	τ	κ	θ	ξ	ρ
Moneyness	Variance	Time-to-maturity	Mean Reversion Speed	Mean Variance Level	Vol-of-Vol	Correlation

For the Heston, we also require the Feller Condition: $2\kappa\theta > \sigma^2$

Asian Option

We consider the case of a geometric asian average option and an arithmetic average asian option in the Black-Scholes setting.

For both cases we generate a dataset by sampling paths $(S_{j,i})$, where j denotes the

Geometric Average: We can obtain a closed-form expression for the geometric average asian option with fixed strike:

$$\frac{1}{N}\sum_{i=1}^{N}\log(S_i) = \log(S_0) - \frac{1}{N}\sum_{i=1}^{N}\frac{\sigma^2}{4}i\Delta t + \frac{\sigma\Delta t}{N}\sum_{i=1}^{N} iZ_{N-i+1} \tag{34}$$

$$= \log(S_0) - \frac{(N+1)\sigma^2\Delta t}{2} + \sqrt{\frac{(N+1)(2N+1)}{6N}}\sigma\Delta t Z \tag{35}$$

$$(\frac{1}{K}\exp(\frac{1}{N}\sum_{i=1}^{n}\log(S_i)) - 1)^+$$

In the continuous time-limit we have, noting that $\int_0^T W_t dt = \int_0^T (T-t)dW_t$

$$\frac{1}{T}\int_0^T \log(S_t)dt = \frac{1}{T}\int_0^T [\log(S_0) - \frac{\sigma^2}{2}t + \sigma W_t]dt = \log(S_0) - \frac{\sigma^2 t^2}{4} + \sigma W_T + \frac{\sigma T}{3}$$

Discrete Time: Consider a discrete-time arithmetic average Asian option on a single underlying

$$(\frac{1}{N}\sum_{i=1}^{n} S_i/K - 1)^+$$

In the continuous time limit consider $Y_t = \frac{1}{K}\int_0^t S_t du, dY_t = S_t dt$

$$f(t,s,y) = \mathbb{E}[e^{-r(T-t}h(S_T, Y_T)|S_t = s, Y_t = y]$$

Then the corresponding PDE is given by:

$$f_t + rsf_s + sf_y + \frac{\sigma^2}{2}f_{ss} - rf = 0, f(T,s,y) = \frac{1}{T}(y/K - 1)^+$$

The replicating strategy is f_s

Multidimensional Example - Arithmetic Brownian Motion / Bachelier Model

We consider the first example from [18], a standard European Call on a basket option where the underlyings have dynamics Arithmetic Brownian Motion.

SDE and MC: Let $\mathbf{S}_t$ be a d-dimensional Arithmetic Brownian motion driven by $\mathbf{W}_t$ a f-dimensional Brownian Motion over $t \in [0,T]$, with covariance $\mathbf{L}\mathbf{L}^\top dt, \mathbf{L} \in \mathbb{R}^{d\times f}$. For simplicity suppose that $\mathbf{S}_t$ is a $\mathbb{Q}$-local martingale. Suppose there is some weight vector $\mathbf{w} \in \mathbb{R}^d$ representing the index weights for a basket option, such that the value of the underlying basket is $\mathbf{w}^\top\mathbf{S}_t$. Then the payoff of the basket call option with payoff $h(\mathbf{w}^\top\mathbf{S}_t, K) = (\mathbf{w}^\top\mathbf{S}_t - K)^+$ is given by:

$$\begin{aligned} d\mathbf{S}_t &= \mathbf{L}d\mathbf{W}_t, \mathbf{S}_t = \mathbf{S}_0 + \mathbf{L}\mathbf{W}_t, \mathbf{S}_t \sim N(\mathbf{S}_0, \mathbf{L}\mathbf{L}^\top t) && (36)\\ d\mathbf{X}_t &= \mathbf{w}^\top d\mathbf{S}_t = \sigma dW_t^\top, \mathbf{X}_t = \mathbf{w}^\top\mathbf{S}_t, \mathbf{X}_t \sim N(\mathbf{w}^\top\mathbf{S}_0, \mathbf{w}^\top\mathbf{L}\mathbf{L}^\top\mathbf{w}) && (37)\\ h(\mathbf{w}, \mathbf{S}_t, K) &= (\mathbf{w}^\top\mathbf{S}_T - K)^+ = (\mathbf{X}_T - K)^+ && (38)\\ f(\tau, \mathbf{S}_t, K, \mathbf{w}) &= \mathbb{E}[(\mathbf{w}^\top\mathbf{S}_t - K)^+|\mathbf{S}_t, \mathbf{w}, \tau, K] && (39)\end{aligned}$$

PDE: Applying Ito's lemma on $\mathbf{X}_t$ and $\mathbf{S}_t$, the corresponding PDE for the price of the basket call option is given by:

$$df(T-t,S_t;\dots) = -\frac{\partial f}{\partial t}dt + \sum_{i=1}^{d} \frac{\partial f}{\partial S_i} dS_{i,t} + \sum_{i=1}^{d}\sum_{j=1}^{d} \frac{1}{2}\frac{\partial V}{\partial S_i S_j} d[S_i,S_j] \tag{40}$$

$$0 = -\frac{\partial f}{\partial \tau} + \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d} \frac{\partial V}{\partial S_i S_j}, f(0,\mathbf{S}) = (\mathbf{w}^\top\mathbf{S} - K)^+ \tag{41}$$

$$df(T-t,X_t;\dots) = -\frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dX_t + \frac{1}{2}d[X]_t \tag{42}$$

$$= \mathbf{w}^\top\mathbf{L}d\mathbf{W}_t + \frac{1}{2}\mathbf{w}^\top\mathbf{L}\mathbf{L}^\top\mathbf{w}dt \tag{43}$$

$$= -\frac{\partial f}{\partial \tau} + \frac{\sigma^2}{2}\frac{\partial f}{\partial x^2}, f(0,\mathbf{X}) = (\mathbf{X} - K)^+ \tag{44}$$

Closed form price: The closed form price is given by the Bachelier normal formula:

For a gaussian random variable with variance a^2 and mean b, We can write:

$$\begin{aligned}
aZ + b - K &= a(Z + \frac{b-K}{a}) \geq 0 \implies Z \geq \frac{K-b}{a} \\
\mathbb{Q}[aZ + b \geq K] &= \Phi(\frac{K-b}{a}) \\
\mathbb{E}[Z \cdot 1_{aZ+b\geq K}] &= \int_{(K-b)/a}^{\infty} z\frac{e^{-z^2/2}}{\sqrt{2\pi}}dz = [\frac{-1}{\sqrt{2\pi}}e^{-z^2/2}]_{(K-b)/a}^{\infty} = \frac{\exp(-((K-b)/a)^2/2)}{\sqrt{2\pi}} \\
\mathbb{E}[(aZ + b - K)^+] &= a\phi(\frac{K-b}{a}) + (b-K)\Phi(\frac{K-b}{a}) \\
&= a\phi(\frac{b-K}{a}) + \frac{b-K}{a}\Phi((\frac{b-K}{a}))]
\end{aligned}$$

In this case, we have $a = \sigma\sqrt{\tau} = \sqrt{\mathbf{w}^\top\mathbf{L}\mathbf{L}^\top\mathbf{w}}\sqrt{\tau}, b = \mathbf{w}^\top\mathbf{S}_0$. Thus the closed-form price for the basket option is given by:

$$f(\mathbf{X}_0,\sigma,\tau,K) = \sigma\sqrt{\tau}[\phi(d_1) + d_1\Phi(d_1)], d_1 = \frac{\mathbf{X}_0 - K}{\sigma\sqrt{\tau}} \tag{45}$$

The analytic greeks are given by, noting that $\frac{\partial d_1}{\partial \tau} = -\frac{d_1}{2\tau}$ and $\frac{\partial d_1}{\partial x} = \frac{1}{\sigma\sqrt{\tau}}$

$$\frac{\partial f}{\partial x} = \sigma\sqrt{\tau}[-\frac{d_1}{\sigma\sqrt{\tau}}\phi(d_1) + \frac{1}{\sigma\sqrt{\tau}}\Phi(d_1) + \phi(d_1)\frac{d_1}{\sigma\sqrt{\tau}}] = \Phi(d_1) \tag{46}$$

$$\frac{\partial f}{\partial \tau} = \frac{f}{2\tau} + \sigma\sqrt{\tau}[\frac{d_1^2}{2\sqrt{\tau}}\phi(d_1) - \frac{d_1^2}{2\sqrt{\tau}}\phi(d_1) - \frac{d_1}{2\sqrt{\tau}}\Phi(d_1)] = \frac{\sigma}{2\sqrt{\tau}}\phi(d_1) \tag{47}$$

$$\frac{\partial f}{\partial x^2} = \frac{1}{\sigma\sqrt{\tau}}\phi(d_1) \tag{48}$$

Experiment: We consider a similar setup as the first code example [1] from [18]. time-to-maturity $\tau = T - t$, strike K, and the covariance $\mathbf{L}\mathbf{L}^\top$ are fixed, and the experiment is repeated with a different dimensionality d and number of sample paths.

In our setup, we consider the following parameter space. We simulate a random covariance matrix by taking the Cholesky decomposition of a matrix of standard normal random variables.

$\mathbf{S_0}$	$d = f$	K	τ	$\mathbf{L}\mathbf{L}^\top$
Underling	No. Assets	Strike	Time-to-maturity	Covariance
$\mathbf{1} + \sqrt{30/250}\mathbf{Z}$	$\{10\}$	$\{ 1 \}$	$\{ 1 \}$	$\{0.2^2 \mathbf{Z}\mathbf{Z}^\top \mathbf{Z}_{ij} \sim N(0,1)\}$

We compare a standard feed-forward neural network trained on the sample payoffs only 18, and a neural network trained on the pathwise differentials 19, and standard Monte Carlo estimation. We also compute the analytic prices and greeks from equations 45-48.

We generate samples $i = 1, \dots, N_{\text{samples}} = 10,000$ samples of $\mathbf{S_{i,0}}, \mathbf{W}_{i,0} \in \mathbb{R}^d$, simulating $\mathbf{S_{i,T}} = \mathbf{S}_{i,t} + \mathbf{L}\mathbf{W_{i,T}}$ exactly from 38. We obtain pathwise differentials $\frac{\partial h}{\partial \mathbf{S}_{i,T}}$ via automatic differentiation, although in this case the pathwise differentials are known analytically: $\mathbf{w}^\top 1_{\mathbf{X}_T \geq 1.0}$. Given the pathwise differentials are not differentiable, we cannot obtain a Hessian estimate from Monte Carlo with AAD.

We use the same neural network architecture and random seed initialisation for the weights for the feed-forward and differential neural network:

- **Inputs**: the values of the underlying $\mathbf{S}_0 \in \mathbb{R}^d$
- **Outputs**: predicted price for strike $K = 1$ $f(\mathbf{S}_0) \approx \mathbb{E}[(\mathbf{w}^\top \mathbf{S}_T - 1)^+ | \mathbf{S}_0]$
- **Architecture**: Standard feed-forward, 4 Hidden layers of 30 hidden units each, softplus activation in all hidden layers, linear output activation
- **Training:** Batch Size of 32, Learning Rate: 10^{-3}. For feed-forward only, Early Stopping with a validation set of 20%, and patience of 50 epochs.
- **Regularisation** : None.

In the neural network case, we do not have an estimate for $\frac{\partial g}{\partial \tau}$. Thus we only verify the pricing error, percentage of no-arbitrage bound violations, and error in the estimate of the sensitivity to the basket factor. We obtain the following results:

In this case, it appears. Indeed, as per [18], the differential neural network achieves lower errors in both the price approximation, and gradients. At least for our particular choice of sample seed and the given neural network architecture, both neural network approaches underperform a MC estimate in all error measures and

Remark: The Bachelier Model has a positive homogeneous relationship. In particular:

[1] Retrieved from: https://github.com/differential-machine-learning/notebooks/blob/master/DifferentialMLTF2.i

	ffn	**differential**	**MC**
pred, L^1	0.03751	0.01069	0.00173
pred, L^2	0.05742	0.01617	0.00174
pred, L^∞	0.55145	0.32498	0.00290
$\lvert\{i : f(X_i) \geq (X_i - K)^+\}\rvert / N_{\text{samples}}$	0.03460	0.02390	0.00000
grad, L^1	0.00326	0.00118	0.00003
grad, L^2	0.00355	0.00150	0.00003
grad, L^∞	0.01184	0.01096	0.00010
$\lvert\{i : \frac{\partial f(X_i)}{\partial X} > 0\}\rvert / N_{\text{samples}}$	0.06300	0.00000	0.00000
Total Time (s)	24.19039	44.79427	140.00

Table 2: Bachelier Basket example

$$f(x, \sigma\sqrt{\tau}) = \mathbb{E}[(x-K)^+] = K\mathbb{E}[(\frac{x}{K} - 1)^+] = Kf(\frac{x}{K}, \sigma\sqrt{\tau}/K)$$

Thus we could possibly eliminate dependency on strike. In addition, we can eliminate the dependence on time-to-maturity τ by noting that the σ and τ terms are grouped together as $\sigma\sqrt{\tau}$.

We note that if we consider the weighted underlying $w_i S_i$, we are effectively as a single variable, we are effectively able to price. However, in this setup we, the neural learns a pricing function the specific strike K, maturity τ, and covariance $\mathbf{L}\mathbf{L}^\top$.

Finally, it be more appropriate to model the basket as a univariate process.

Rough Bergomi

In the rough volatility setting, there is a lack of closed form expressions for prices, hence prices need to be approximated by Monte Carlo. [17] used a two step approach.

MC and SDE:

The Rough Bergomi model has dynamics:

$$1 \tag{49}$$

We use the code implementation for the Turbocharged Rough Bergomi scheme [2] from [24]

Experiment: Our parameter space consists of:

[2] Accessed from: https://github.com/ryanmccrickerd/rough_bergomi

$\mathbf{S_0}$	$\log(K)$	τ	α	ρ	V_0	ξ
Underlying	Strike	Time-to-maturity	Hurst Exponent	correlation	Volatility	Vol-of-vol
$\{1\}$	$\{-0.5 + 0.1i : i = 0,\dots,10\}$	$\{\frac{i}{30} : i = 0,\dots,30\}$	$U[-0.4, 0.5]$	$U[-0.95, -0.7]$	$\{0.235\}$	$U[1.5, 2.5]$

Table 3: Rough Bergomi Parameter Space

We can only analyse error in pricing and greeks with respect to the MC estimates. However, we can analyse no-arbitrage violations for calls as before.

We consider two neural network architectures: a standard feed-forward **??**, a gated neural network **??**, and also compare against cubic spline regression.

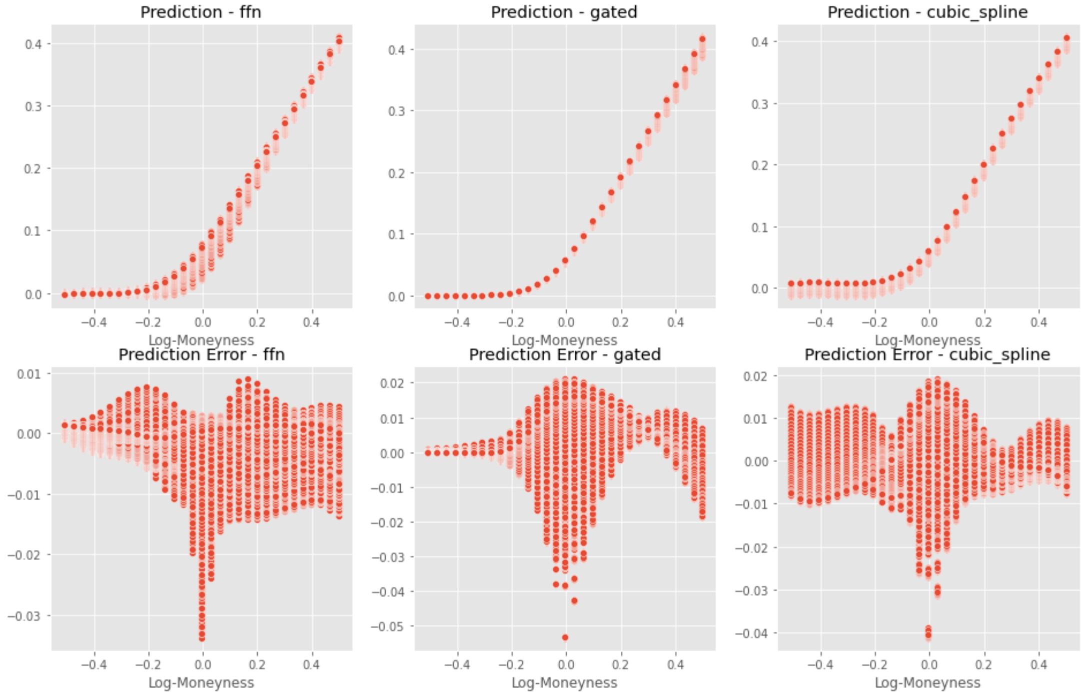
- **Inputs**: $\mathbf{X} = (-\log(K), \tau, \alpha, \rho, \xi)$
- **Outputs**: Predicted call price $\approx \mathbb{E}[(\mathbf{S}_T - K)^+ | \mathbf{X}]$
- **Architecture**: *Feed-Forward*: 2 hidden layers with 100 hidden units each, hidden layers, linear output activation; *Gated*:
- **Training:** Batch Size of 32, Learning Rate: 10^{-3}. For feed-forward only, Early Stopping with a validation set of 20%, and patience of 10 epochs.
- **Regularisation** : None.

We sample $N_{\text{SPACE}} = 100$ random vectors of (α, ρ, ξ) from the distributions above. For each of these, we simulate S_T using [24], with a terminal maturity of $T = 1$ and $N_{\text{Brownian}} = 10000$ paths each, and compute the call prices for the collection of $\tau \times \log(K)$ described above. This produces a dataset of 96100 observations in 18 seconds.

We obtain the following results:

	ffn	gated	polynomial
pred, L^1	**0.00316**	0.00532	0.00447
pred, L^2	**0.00447**	0.00808	0.00601
pred, L^∞	**0.03383**	0.05344	0.04127
$g \geq (S-K)^+$	0.11413	**0.10864**	0.21770
$\frac{\partial g}{\partial K} > 0$	0.05237	**0.00000**	0.12903
$\frac{\partial g}{\partial \tau} < 0$	0.24469	**0.00000**	0.00000
$\frac{\partial g}{\partial K^2} > 0$	0.52610	**0.00000**	0.37017
Total Time (s)	115.34485	122.48292	**0.49137**

Table 4: Rough Bergomi Results



Errors for Rough Bergomi Approximations

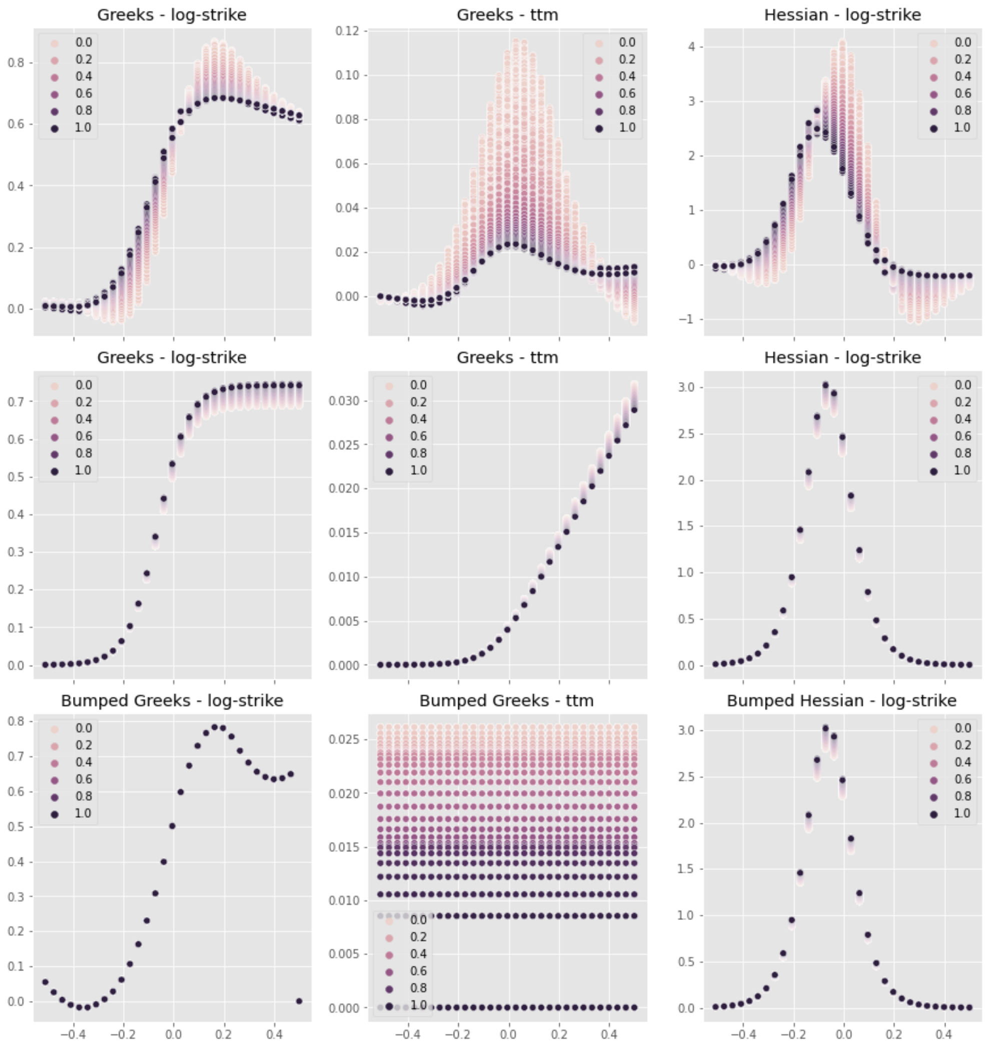


Figure 1: Greeks for Feed-foward, Gated, Cubic-spline respectively

The gated neural network is by construction able to avoid any no-arbitrage in terms of monotonicity in time to maturity in strike, and convexity in strike. The standard neural network however achieves the best result in terms of prediction error. Interestingly, standard cubic spline regression has similar performance to the unconstrained neural network, at a much shorter training time.

Remark: To verify that the neural network pricer has minimal dynamic arbitrage opportunities in this non-Markovian setting, one approach could be to evaluate whether the neural network pricer satisfies the corresponding path-dependent PDE (PPDE). Some literature towards this includes [21] and [27]

Bermudans

TODO: Investigate this aspect further

Remark: A practical concern is that a single neural network can be trained to approximate only one specific exercise schedule.

Fixed Income Example

TODO: simulate rates curve: e.g. Vasciek, Cheyette, CIR, Hull White, LIBOR market model / BGM, HJM and analyse swaption approximation in this example, given the high-dimensional sensitivity to all swaps

Barrier Example

Analyse approximation of barriers in the BS-setting, given the availability of a closed form for the running maximum and minimum.

Spread Payoff

TODO: A 2D example, possibly 2 SABR processes + dependence via copula / correlation matrix

6 Conclusion

Empirical observations thus far:

- Dataset quality and construction likely plays a role in training, convergence, and extrapolation. Cleaner (in terms of estimation error for MC and PDEs) and more data is better, but this increases training time.
- Differential training (whether through explicit greek labels, or the neural PDE term) seems to lead to better results empirically
- Hard constraints can be embedded into a neural network to prevent some no-arbitrage conditions, at the cost of some flexibility.
- Neural networks can be trained to solve complex payoffs and high-dimensional problems / market models, but need to be retrained on new market parameters or contract specifications. In some cases, it can be slower and less accurate than even Monte Carlo.

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