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Abstract

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Contents

1 Introduction

Motivation

Consider a typical workflow for a end-user in a trading desk: a price is needed for a particular product, a pricing function is invoked, which takes in a collection of (calibrated) input parameters, which in turn invokes the underlying numerical method, and finally the outputs, being the price and sensitivities are returned.

Arguably, one key business objective for derivatives quants at investment banks and market makers is to streamline and improve this process: for any given product and the given *market generator model* (in other words, the modelling assumptions in terms of volatility or underlying processes), produce a pricing function that outputs its *prices*, its *sensitivities*, and the *hedging strategy* as accurately as possible, and as quickly as possible.

To accomplish this, derivatives quants have broadly three families of numerical methods: analytic and semi-analytic form solutions (e.g. Black Scholes, Characteristic Transform), Monte Carlo Methods, and PDE / finite-difference methods. Each of these methods has their own drawbacks and limitations, and with the exception of few true closed-form expressions, are themselves approximation methods.

Approximation error to the 'true' model price can be reduced to some extent with greater computational effort. The computational effort may grow very large when the underlying dimensionality of the problem increases, for example with more complex payoffs and volatility models. When the *risk* sensitivities (which may also represent the hedging strategy) also need to be computed, the computational effort may grow even larger, given the convergence rates in price and sensitivities may not necessarily be the same.

Another issue with Monte Carlo and PDEs, in addition, the solutions are only for one set of model parameters (for example the beta and rho in SABR), and as model parameters change the numerical method must be reinvoked. Thus for models whose pricing depends on these schemes, the calibration task becomes expensive.

Towards the calibration task and other practical applications, it may be acceptable to trade off some accuracy in exchange for speed, for example in the context for electronic market-making or risk / VaR calculations.

Neural-networks may be one method to accomplish this. Neural networks, in this setup, may be viewed as as a model agnostic (in the sense of volatility or market generator models) and method agnostic (MC, PDE) extension to existing numeric pricing methods. Neural Networks present several desirable properties: universal approximation, and ability to 'learn' non-parametric data-driven relationships, sensitivities can be obtained quickly with automatic differentiation, and a fast inference time can be obtained, amortising a computationally expensive training period.

A requisite for any pricing function is adherence to no-arbitrage, and similarly, smoothness of the pricing function and sensitivities in all inputs. Naturally, if *static* arbitrage opportunities exists, this presents the possibility of a loss to the trading desk. In terms of smoothness, an exotics / OTC desk may have to price derivatives beyond standard strikes and maturities, and a non-smooth pricing function is likely to admit static arbitrage. In addition, non-smooth sensitivities functional may also present P&L impact from hedging. On key concern, is that a standard neural network construction may not necessarily adhere to no-arbitrage and smoothness conditions.

The aim of this dissertation is to investigate how to construct a production-level neural network to approximate derivatives pricing and risks under any general payoff and market generator model. We aim to investigate:

- How to construct and train a neural network efficiently, in terms of both approximation error and minimal no-arbitrage violations?
- How do neural network approximations perform under various market models (Black-Scholes, Heston, SABR, Rough Vol)?
- How well does it perform in high dimensionality?
- How well does it perform under different payoffs (Calls and Digitals, Asians, Barriers), and exercise policy (Bermudan, European)
- What is the empirical computational complexity required / convergence rate?

Article Structure

In section one, we review the relevant pricing problem formulations. In section two, we review neural networks and the relevant literature. In section three, we focus on numerical settings, examining how to construct and efficiently train neural networks that are able to approximate European and Bermudan payoffs.

2 Preliminaries

In this section, we review the formulation of derivatives pricing and risk calculations.

Problem Setup

Consider the problem of computing derivative prices and sensitivities with respect to some input parameters $\mathbf{X}$. First, consider a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. Suppose there is a $\mathcal{F}_t$-adapted d-dimensional stochastic process $\mathbf{S}_t \in \mathbb{R}^d, t \in [0, T]$, representing the value of the underlying assets or cash flows through time whose evolution is determined by a fixed vector of parameters $\mathbf{X} \in \mathbb{R}^f$ (or $\mathbf{X}$ is $\mathcal{F}_t$) measurable. Thus in effect, we consider $\mathbf{S}_t(\mathbf{X})$.

Suppose that there exists some equivalent local martingale measure $\mathbb{Q}$

In the most general case, consider a payoff determined by a continuous discounted payoff and a terminal discounted payoff:

$$f(\tau, \mathbf{S_t}, \mathbf{X}) = \mathbb{E}^{\mathbb{Q}}\left[\int_t^T h_1(u, S_u)dt + h_2(S_T)|\mathbf{S}_t, \mathbf{X}\right]$$

Remark: In a ‘real-life‘ scenario, consider only the case of cash flows at discrete times $\mathbb{E}^{\mathbb{Q}}\left[\sum_{i=1}^n h_i(t_i, \mathbf{S}_i)|\mathbf{S}_{t_0}, \mathbf{X}\right]$. Thus without any loss of generality, we consider pricing a single payoff; then for any derivatives, for example interest rate caps and floors , we can be priced by decomposing them as the sum of payoffs.

$$f(\tau, \mathbf{S_t}, \mathbf{X}) = \mathbb{E}^{\mathbb{Q}}\left[h_2(S_T)|\mathbf{S}_t, \mathbf{X}\right] \tag{1}$$

Suppose f is sufficiently smooth, in particular C^1 with respect to time-to-maturity $\tau = T - t$ and C^2 with respect to each $\mathbf{S}_t$. Then by Feynman-Kac, the corresponding (parametric) PDE is given by:

$$f_\tau = \mathcal{L}f, f(0, s, x) = h(s) \quad \forall \tau \in [0, T], \mathbf{X} \in \mathcal{X}, s \in \mathcal{S} \tag{2}$$

subject to some initial (in terms of time-to-maturity) condition corresponding to the European Payoff, and $\mathcal{L}$ is the generator of $\mathbf{S}$

Thus f denotes the true value function, which represents the no-arbitrage price of the , which can be expressed as a conditional expectation or solution to a PDE.

In general, both the pricing function f and $\frac{\partial f}{\partial \theta}$ in (5), cannot be obtained in closed-form with the exception of several cases. Thus in practice we have the approximation g:

$$g(T - t, \mathbf{X}) \approx f(T - t, \mathbf{X}), \quad f(T - t, \mathbf{X}) = g(\mathbf{X}) + \epsilon$$

Where ϵ denotes the error, which may itself depend on θ and the choice of numerical method. We look for a approximating function g such that ϵ is small over a relevant range of parameters $\mathbf{X}$.

In addition to the value f, the sensitivities with respect to the parameters $\mathbf{X}$ are also of interest. Thus we also require

$$\frac{\partial g}{\partial \mathbf{X}} \approx \frac{\partial f}{\partial \mathbf{X}}, \dots, \frac{\partial^d g}{\partial \mathbf{X}}$$

Path-Dependent Payoffs: Payoffs may also be path dependent, and depend on the entire history of $\mathbf{S_t}$:

$$f(\tau, (\mathbf{S}_u)_{u\in[0,t]}, \mathbf{X}) = \mathbb{E}^{\mathbb{Q}}\left[h_2((\mathbf{S_u})_{u\in[0,T]})|(\mathbf{S}_u)_{u\in[0,t]}\mathbf{X}\right] \tag{3}$$

More generally, we also consider the underlying parameters $\mathbf{X}$ which determine $\mathbf{S}_t$

and that $h(\mathbf{X}_t)$ is a $\mathcal{F}_T$-measurable random variable representing the (discounted) total sum of cash flows, as some function of the time t state $\mathbf{X}$, under some risk-neutral measure $\mathbb{Q}$.

Stochastic Control approach

For example, if $\theta = S_0$, then $\frac{\partial f}{\partial S_0}$ gives the sensitivity to the underlying, and the hedging strategy.

American

$$\sup_{\tau} \mathbb{E}[\int_t^{\tau} B_{t,T} h_1(X_t)dt + h_2(X_\tau)|X] \tag{4}$$

$$f(T-t, \mathbf{X}_t) = \mathbb{E}^{\mathbb{Q}}[h(\mathbf{X}_t)|\mathbf{X}] \tag{5}$$

Then f is the conditional expectation and value function for a derivative with latest cash flow at time T. Under some risk-neutral measure $\mathbb{Q}$, or equivalently the assumptions of no-arbitrage and market completeness, then f is the pricing function with respect to the parameters θ.

Longstaff-Schwartz

Suppose $\mathbb{E}[g(X_n)^2] < \infty$ the payoff is L^2 integrable.

[21] introduced the Longstaff-Schwartz method, or Least Squares Monte Carlo for valuing American Options. In the one step case, consider

$$f(X_t(\theta)) = \mathbb{E}^{\mathbb{Q}}[h(X_T(\theta))|X_t(\theta)] \tag{6}$$

Where X is some appropriately chosen (Markov) state process such that $f(X_t(\theta))$. Then let g denote an approximation for the value function. In [21], they consider basis functions such as Chebyshev and Legendre polynomials.

$$g(X_t) = \sum_{b=1}^{B} \phi_b(X)$$

In the multi-period case , we consider for $t_0 < t_i < \dots t_n = T$:

$$f(X_{t_i}) = \max\{\mathbb{E}^{\mathbb{Q}}[f(X_{t_{i+1}})\}|X_{t_i}], h(X_{t_i})\}, \quad f(X_T) = h(X_T)$$

Where h is the exercise value for the pay

[21] argues that as the number of basis functions B to infinity, g approximates the true value function, and as the number of timesteps N goes to infinity. Further to this, in the Monte Carlo setting, we also require the number of samples M to go to infinity.

$$f(S_t) = \mathbb{E}[(S_T - K)^+|S_t] \tag{Black-Scholes}$$

$$\mathbb{E}[(f(S_T) - \mathbb{E}[g(S_t)|S_t])^2] = 0$$

Then by the tower property of expectation

$$\mathbb{E}[(g(S_T) - \mathbb{E}[g(S_t)|S_t])^2] = 0$$

Adjoint Automatic Differentiation

The adjoint automatic differentiation method was first brought to [**giles2004**]

In the most general case, consider the pricing function in the form of 5. Given regularity constraints

$$\frac{\partial f}{\partial \theta} = \frac{\partial}{\partial \theta}\mathbb{E}^{\mathbb{Q}}[]$$

The sensitivities of f with respect to θ, can be computed through the application of the chain rule backwards:

$$\frac{\partial S}{\partial \theta} \cdot \frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial \theta}$$

where the partials are evaluated in reverse order. Programming language wise, it can be implemented via computional graphs and implementing *adjoint* operators.

No-Arbitrage Constraints

We consider the conditions for our approximating function g to be free of static arbitrage in the case of European calls. Suppose that g is continuously and and once-differentiable with respect to $\tau = T - t$ and twice differentiable with respect to K or $x = S/K > 0, y = \log(S/K)$ omitting other arguments. We note that: $\frac{\partial X}{\partial K} = \frac{S}{-K^2} = \frac{-X}{K}, \frac{\partial y}{\partial K} = \frac{-1}{K}$. Then:

$$\frac{\partial g}{\partial x}\frac{\partial x}{\partial K} = \frac{-x}{K}\frac{\partial g}{\partial X}$$
$$\frac{\partial g}{\partial y}\frac{\partial y}{\partial K} = -\frac{1}{K}\frac{\partial g}{\partial y}$$
$$\frac{\partial}{\partial K}(-\frac{x}{K}\frac{\partial g}{\partial x}) = \frac{1}{K^2}(x\frac{\partial g}{\partial X} + x^2\frac{\partial^2 g}{\partial X^2})$$
$$\frac{\partial}{\partial K}(-\frac{1}{K}\frac{\partial g}{\partial y}) = \frac{1}{K^2}(\frac{\partial g}{\partial y} + \frac{\partial^2 g}{\partial y^2})$$

Then in terms of K, x, y we have:

No Static Arbitrage for European Calls: From [11] [5]

$\frac{\partial g}{\partial \tau} \geq 0$	carry, time-value
$\frac{\partial g}{\partial K} \leq 0, \frac{\partial g}{\partial x} \geq 0$	monotonically decreasing (increasing) in strike (moneyness / underlying)
$\frac{\partial^2 g}{\partial K^2} \geq 0, \frac{\partial^2 g}{\partial x} \geq 0, \frac{\partial^2 g}{\partial y} \geq 0$	convexity in strike, moneyness
$g(T-t, K) \geq (S_t - K)^+ \geq 0$	intrinsic value
$\lim_{K\to\infty} g(K, T-t) = 0$	1

We also consider no static arbitrage for European digitals. From the Breeden-Litzenberger formula we have:

Breeden-Litzenberg formula. We now have an additional approach: solve for the risk-neutral transition density $p(y, T; s, t)$ and obtain the price of any european payoff via numerical integration.

Since the payoff of a European digital $\mathbb{Q}[]$

$\frac{\partial g}{\partial \tau} \geq 0$	carry, time-value
$\frac{\partial g}{\partial K} \leq 0, \frac{\partial g}{\partial x} \geq 0$	monotonically decreasing in strike, increasing in moneyness/underlying
$\frac{\partial^2 g}{\partial K^2} \geq 0, \frac{\partial^2 g}{\partial x} \geq 0, \frac{\partial^2 g}{y} \geq 0$	convexity in strike
$g(T-t, K) \geq (S_t - K)^+ \geq 0$	intrinsic value
$\lim_{K\to\infty} g(K, T-t) = 0$	1

$$\int_{\mathbb{R}} (y-K)^+ p(y, T; s, t) dy = \int_K^\infty (y-K) p(y, T; s, t) dy$$

No Dynamic Arbitrage: No dynamic arbitrage indicates the lack of a replication strategy with zero initial wealth that leads to positive wealth $\mathbb{P}$-almost surely. For Dynamic Arbitrage, a sufficient condition may be that the function g satisfies the relevant pricing PDE.

$$\mathcal{L}g = g_\tau, \qquad \text{for all } \tau, K$$

If $\mathcal{L}g \neq 0$, the dynamic arbitrage strategy is given by longing the payoff if $\mathcal{L}g$ and shorting the replicating portfolio, and $\mathcal{L}g < 0$, and shorting the payoff if $\mathcal{L}g < 0$ and longing the replicating strategy.

No arbitrage bounds for non-vanilla payoffs: For non-european options, we can consider replication, and in some case lower bounds. For example, in the case of Americans we must have a value greater than the equivalent european $V^A(t,s) \geq B^B(t,s)V^E(t,s)$, and for barries, we must have .

3 Neural Networks

We aim to train a neural network to approximate a pricing function, over as wide a range as possible under a given market generator model, that has a minimal approximation error and minimal static and dynamic arbitrage.

Neural Network Definition

A comprehensive outline of neural networks is better in [14], and a reference on practical implementations using `Tensorflow/Keras` can be found in [7]. An overview of neural networks and their applications towards finance can be found in [10].

We begin by considering a standard feed-forward neural network architecture . Consider some input $\mathbf{X} \in \mathbb{R}^{N \times d}$, representing N observations or samples of a d-dimensional vector. A n-layer feed-forward neural network with n layers can be characterised by :

$$\mathbf{Z}_1 = g_1(\mathbf{X}\mathbf{W}_1 + \mathbf{b}_1\mathbf{1}^\top) \tag{7}$$

$$\mathbf{Z}_2 = g_2(\mathbf{Z}_1\mathbf{W}_2 + \mathbf{b}_2\mathbf{1}^\top) \tag{8}$$

$$f(\mathbf{X};\theta) = g_{n+1}(\mathbf{Z}_n\mathbf{W}_{n+1} + \mathbf{b}_n\mathbf{1}^\top) \tag{feed-forward neural network}$$

Where $\mathbf{W}_i \in \mathbb{R}^{H_{i-1} \times H_i}, i = 1, \dots n-1$ denotes the weights for the n-th hidden layer, $b_i \in \mathbb{R}^{H_i}$ denotes the bias term for the n-th hidden layer, and $g_i, i = 1, \dots, n$, is the activation function for the i-th hidden layer. The activation function is applied element-wise to the inputs, such that $g(\mathbf{Z})_{ij} = g(z_{ij})$, and introduces non-linearities. Given the neural network f is a composition of functions, we can regard any general neural network as a composition of neural networks.

Typically in a regression setting the final layer is the identity (or linear) activation $g(z_{ij}) = z_{ij} \in \mathbb{R}^n$, such that the final value may attain any real value, although a final activation function could also be applied. For example, if the output is known to take values between $y_i \in [0, 1]$, then sigmoidal activation could be considered. Regardless, the neural network may be seen as 'learning' a collection of basis functions of 'latent representation' $\mathbf{Z}_n$ from inputs $\mathbf{X}$, for the given objective.

The robustness of neural networks lie in the lack of need to specify the basis functions $\mathbf{Z}_n$ explicitly, versus standard basis regression. Furthermore, several universal approximation.

Theorem 1 (Hornik (1990)) *Let $\mathcal{N}_{d_0,d_1,\sigma}$ be the set of neural networks mapping from $\mathbb{R}^{d_0} \to \mathbb{R}^{d_1}$, with activation function $\sigma : \mathbb{R} \to \mathbb{R}$. Suppose $F \in C^n$ is continuously n-times differentiable. Then if $\sigma \in C^n(\mathbb{R})$ is continuously n-times differentiable, then $\mathcal{N}_{d_0,d_1,\sigma}$ arbitrarily approximates F and its derivatives up to order n.*

Thus there exists some neural network that arbitrarily approximates. However, the universal approximation theorem only proves existence of such a neural network, and not how to construct it.

Neural Networks in their capacity as function approximators can be characterised as

$$\arg\min_{\theta\in\Theta} L(f(\mathbf{X}), g)$$

the minimizer of some loss function L, over a space of feasible parameters Θ.

$$\arg\min_{\theta\in\Theta} \mathbb{E}[\|\mathbf{y} - f(\mathbf{X};\theta)\|^2]$$

Training Neural Networks and Practical Issues

In general, given some loss function $L(g(\mathbf{X}), f(\mathbf{X}))$, and sample data: $\mathbf{X}_j, j = 1, \ldots, N$

$$\arg\min_{\theta} L(f, g)$$

$$\theta_{t+1} \leftarrow \theta_t - \eta_t \nabla_\theta L, \quad \eta > 0 \tag{9}$$

This loss function is generally non-convex with respect to the parameters θ; as such there are no guarantees of convergence to a global minima except in.

In practice the gradients with respect to the neural network parameters $\nabla_\theta L$ also need to be estimated. Mini-batch stochastic gradient descent

$$\sum \theta_{t+1} \leftarrow \theta_t - \eta_t \nabla_\theta L, \quad \eta > 0 \tag{10}$$

In practice, techniques such as:

batchnormalisation: Samples are normalised $(\mathbf{X}_i \ominus \mu_i)/\sigma_i$

Dropout:, learning rate scheduling have been found to lead to empirically faster training

The gradients with respect to the parameters θ are obtained via AAD, and updated via backpropogation.

Neural Network Architectures

The universal approximation theorem of [**hornik**] only suggests that there *exists*. Empirically, architectures that have been selected for domain-specific problems as opposed to . We consider several hidden layer or block architectures we will utilise in later sections.

Gated unit: In the context, this can help learn feature interactions, and also facilitate feature *selection*.

$$\mathbf{Z}_1 = g_1(\mathbf{X}\mathbf{W}_1 + \mathbf{b}_1\mathbf{1}^\top) \tag{11}$$
$$\mathbf{Z}_2 = g_2(\mathbf{X}\mathbf{W}_2 + \mathbf{b}_2\mathbf{1}^\top) \tag{12}$$
$$\mathbf{Z}_3 = \mathbf{Z}_1 \otimes \mathbf{Z}_2 \tag{gated block}$$

Where $\otimes$ denotes element-wise multiplication. [26] uses gated units to construct a neural network that has the requisite monotonicity and convexity constraints with respect to strike and moneyness.

Residual block: Residual blocks allow for the *flow* of information from earlier layers to later layers

$$\mathbf{Z}_1 = g_1(\mathbf{X}\mathbf{W}_1 + \mathbf{b}_1\mathbf{1}^\top) \tag{13}$$

$$\mathbf{Z}_2 = g_2(\mathbf{Z}_1\mathbf{W}_2 + \mathbf{b_2}\mathbf{1}^\top) \tag{14}$$

$$\mathbf{Z}_3 = g_3(\begin{pmatrix}\mathbf{Z}_1 & \mathbf{Z}_2\end{pmatrix}\mathbf{W}_3 + \mathbf{b}_3\mathbf{1}^\top) \tag{gated block}$$

Recurrent neural networks: Recurrent neural networks capture sequential, or temporal dependencies.

$$\mathbf{Z}_1 = g_1(\mathbf{X}\mathbf{W}_1 + \mathbf{b}\mathbf{1}^\top) \tag{15}$$

$$\mathbf{Z}_2 = g_2(\mathbf{Z}_1\mathbf{W}_2 + \mathbf{b}\mathbf{1}^\top) \tag{16}$$

$$\mathbf{y} = f(\mathbf{X};\theta) = g_n(\mathbf{Z}_n\mathbf{W}_n + \mathbf{b}\mathbf{1}^\top) \tag{gated block}$$

Recurrent neural networks may be useful in a non-markovian setting, for example, in pricing path dependent payoffs. Furthermore, they may be useful even for European payoffs, when additional market frictions such as transaction costs or market impact are included, as in [4].

Modelling Choices, Literature Review

The first question is what to model for the pricing approximation task; in effect, the choice of input-output pairs $(\mathbf{X}, \mathbf{y})$. In the derivatives domain, various approaches utilising neural networks have been presented. A comprehensive review of methods can be further found in [23]:

We focus first on the supervised learning task. In effect, the hedging strategy and neural PDE approaches can also be incorporated to the loss function for a general supervised task. However, even within the supervised learning ask.

Supervised - Price Map: Direct approximation of model prices, with automatic differentiation for sensitivities, as in [18]. A natural choice could be to directly learn a mapping between parameters, and prices or which appears to be a standard approach.

[15]	

Calibration: learn mappings for implied volatilities/prices to parameters, or vice-versa [16]. [16] also considers a grid based approach, where a fix set of strikes are predicted, and an image based approach where a fixed collection of maturities and strikes are modelled.

Model risk-neutral implied distribution: Let $g(T, y/K; t, X)$ be a estimate for the conditional density at time T. then

$$f(T-t, X) \approx DF_{t,T}\int_K^\infty (\frac{y}{K} - 1)^+ g(T, y; t, X)dy$$

Neural PDEs: Use of neural networks to solve (high-dimensional) PDEs as in []. Given that in some cases, pricing problems can be formulated as.

Model hedging strategy; Approximation for a hedging or replicating strategy; In effect this can be seen as pricing via replication or stochastic control [4]. The *Deep Hedging* approach of [4] considers approximating the replicating strategy. Thus in this case, the neural network represents the delta.

$$y - \sum_{i=1}^{N} \Delta_i(X_{t_i})(S_{i+1} - S_i)$$

Learn the hedging strategy with a neural network, and price can be obtained via superhedging. However the drawback is that this hedging strategy is tuned to a single payoff, and set of parameters.

A potential drawback of this approach is that a single neural network is tuned to

Neural SDEs: Represent the dynamics of a SDE $\mathbf{S}_t$ as a nueral network, as in [13] and Generative Adversarial Networks, and calibrate by minimising the MC pricing error between observed prices $\|P(T,K) - \mathbb{E}[h(S_T,K)]\|$. This approach has the potential advantage of allowing more realistic dynamics to be captured; however, being a Monte-Carlo based approach, the actual inference time may be slow.

Model Control Variate Residuals: [1] highlights that neural networks are generally able to interpolate within the domain of training, but unable to extrapolate. This is also have particular relevance, given models have asymptotics, and payoff asymptotics correspond to no-arbitrage bounds.

The proposed solution of [1] uses a control-variate-like two-step procedure: first, they fit a *control variate function* , in their case cubic spline, with the appropriate asymptotics, then they fit a neural network with vanishing asymptotics to the residual of the original function and the control variate.

Their proposed architecture consists of the following gaussian, or radial basis activation

$$g_i(\mathbf{x}) = \exp(-\frac{1}{2}\|\mathbf{x}\mathbf{W}_i + \mathbf{b}_i\|^2) \tag{17}$$

Such that if any $x_i \to \pm\infty$ we have $g_i(\mathbf{x}) \to 0$

Dataset Construction

Machine Learning Approach: In the machine learning framework, typically three datasets are constructed: A *train* dataset, which, the model is directly trained. In theory, given that $\mathbf{X}$ is sampled from the same distribution.

Data Generation Method. Train, Val: Hypercube / Grid sampling vs Random sampling, Random sampling within boundary

In the case of machine learning Quasi Monte Carlo, Sobol Sequences, Control Variates

Test: Another random sample within boundary, random sample outside of boundary

In a real life setting, next-day prices could be used, although this does not necessarily make sense.

Data Preprocessing

The dataset may also contain arbitrageable prices, for example in Monte Carlo simulations. In the Longstaff-Schwartz paper, they simply set out-of-the money paths to weight zero:

$$\sum_{j=1}^{N} w_j L(f(X_j), g(X_j))$$

In the Longstaff-Schwartz case, paths for which $f(X_j)$ have weight zero.

In this same style, [9]. Preprocessing, remove arbitrage out of the money paths.

Dataset Quality and Risk: The risk of the neural network is also present through *data risk*. Dataset quality For example, how does the NN perform on unseen parameter ranges?, An issue is that the NN may be able to , e.g. when the market regime shifts. Thus the NN must likely be trained on a very large grid of parameters.

In [17] and the standard Longstaff-Schwartz approach,

In the case of hedging, FBSE, and American / Bermudan options modelling,

The neural network approximation approach is more closely aligns with a Monte Carlo method. Thus there is error from both the neural network approximation g and the simulation of state-payoff pairs X_j, y_j

For example, consider the case of very deep in-the-money or out-of-the money strikes for a European all. A proposed solution of is to sample more from these regions, or similarly to set a greater weight w_j in a weighted loss function

Swaption

Loss Functions for Derivatives Pricing and Risks

We now consider a choice of specific loss functions for the supervised learning task.

The proposed method from [17] is to consider a joint loss function, an approach they describe as *differential machine learning.* Generate state payoff pairs: $X_i, g(X_i)$

$$L_{price}(g(X_i), f(X_i)) + \lambda L_{greeks}\left(\frac{\partial g(X_i)}{\partial X}, \frac{\partial g(X_i)}{\partial X}\right), \lambda \geq 0 \tag{18}$$

Again, $\frac{\partial f(X_i)}{\partial X}$ can be obtained at little extra cost given AAD. [17] argues that the λ, but in practice this could be determined by trial-and-error and prior knowledge.

A further extension could be to consider the, which is a *Neural PDE* approach

$$L_{price}(g(X_i), f(X_i)) + \lambda_1 L_{greeks}\left(\frac{\partial g(X_i)}{\partial X}, \frac{\partial g(X_i)}{\partial X}\right) + \lambda_2 L_{PDE}(\mathcal{L}g) \tag{19}$$

Where $\mathcal{L}$ denotes a PDE operator and $\lambda_1, \lambda_2 \geq 0$ are weights for each loss function. For example, in the case of black scholes, we add the differential:

$$\mathcal{L}g = \frac{\partial g}{\partial t} + rs\frac{\partial g}{\partial s} + \frac{\sigma^2 s^2}{2}\frac{\partial^2 g}{\partial s^2} - rg$$

Activation	$f(x)$	$f'(x)$	$f''(x)$
ReLU	$\max\{x, 0\}$	$1_{x>0}$	0
ELU	$\alpha(e^x - 1)1_{x<0} + x1_{x>0}$	$1_{x>0} + \alpha e^x 1_{x<0}$	$\alpha e^x 1_{x<0}$
Sigmoid	$\frac{1}{1+e^{-x}}$	$\frac{e^{-x}}{(1+e^{-x})^2}$	$\frac{-e^{-x}(1+e^{-x})}{(1+e^{-x})^4}$
SoftPlus	$\log(1 + e^x)$	$\frac{1}{1+e^{-x}}$	$\frac{e^{-x}}{(1+e^{-x})^2}$
Swish	$\frac{x}{1+e^{-x}}$	$\frac{(x-1)e^{-x}}{(1+e^{-x})^2}$	
GeLU	$x\Phi(x)$	$x\phi(x) + \Phi(x)$	$\phi(x) + x\phi''(x)$
tanh	$\frac{e^x-e^{-x}}{e^x+e^{-x}}$		
RBF	$e^{-\frac{x^2}{2}}$	$-xe^{-\frac{x^2}{2}}$	$(x^2 - 1)e^{-\frac{x^2}{2}}$

Table 1: Caption

[25] argues that the inclusion of a PDE loss term leads to self-consistency, given that if the neural network approximation satisfies the pricing PDE (in their application, the Dupire Local Volatility PDE), $\mathcal{L}g = 0$ there is no dynamic arbitrage.

An alternative is a *soft penalty* approach [19]. For example, we could consider any of the no-arbitrage constraints, and include a loss term if the . One such example could be a penalty $((S - K)^+ - g(S))^+$, i.e. a penalty for being beneath the intrinsic call bound. Hence let $L_1, \dots L_P$ denote P soft penalties. In the most general case, the loss function consists of:

$$L_{price} + \lambda_1 L_{greeks} + \lambda_2 L_{PDE}(\mathcal{L}g) + \sum_{i=1}^{P} \lambda_{2+i} L_i \tag{20}$$

In practice, the inclusion of higher order differentials increases the total training time. In the subsequent sections we explore whether their inclusion leads to better results.

Neural Network Construction for Derivatives Pricing and Risks

We characterise handcrafted neural networks as described in [23]: in short, we can embed prior knowledge into the construction of the neural network, in the choice of an appropriate loss function or architecture.

[19] propose a modified elu function with: $R(z) = \alpha(e^z - 1)1_{z\leq 0} +$

Smoothness. Firstly, to obtain smooth (or at least, continuous) approximations for the d-th order partial derivatives, we require $g(\cdot)$ to be C^d continuous d-time differentiable with respect to its inputs [19]. Given the loss functions in 18, 19, may contain even higher-order partial derivative terms, we may require even further smoothness constraints.

In order for the neural network output g to be C^d, one method could be to constrain all intermediate activation functions g_i to be sufficiently smooth, such that g as a composition of smoothness will be a smooth function. [6] uses this approach.

Hard constraints [6] *softplus* activation for the strike and sigmoid activation for the time-to-maturity, with non-negative weight constraints. Thus this guarantees that the neural network is non-negative, monotonic in strike and time, and convex in strike.

Interpreting and Monitoring Neural Networks

We only use neural networks as an approximating function as an overlay on top of some market generator model and numerical method, which may alleviate some of the *black-box* issues [8]. However, the neural network may still produce unexpected outputs. We can evaluate and interpret neural networks to some extent. In the simple case, we can use graphical methods, or evaluate behaviour around boundary conditions. In addition, we can leverage machine learning interpretability methods [22], which [3] explores in the context of using deep neural networks for Heston calibration.

- Dependency plots against one or two dependent variables, boundary conditions.
- First order partial derivatives, Second order (Hessian), higher order derivatives.
- Output of penultimate layer (basis functions or latent representation)
- Machine-learning interpretabilty methods: Shapley Values, LIME

In terms of monitoring and deploying the neural network. In the training period, we obtain an estimate of the pricing errors for some range of parameters. We could define a region of parameters $\mathcal{X} \subseteq \mathbb{R}^d$ for which the maximum error of the neural network is under some ϵ. These could be stored as simply $2d$ linear constraints $d_i < X_i < u_i$ for each parameter. If the pricer is evoked outside this region, we simply revert to the original numerical method. If in addition, the market parameters have been consistently away the training region of parameters, in other words, *distribution drift* this necessitates retraining of the neural network.

Alternative Methods

Alternative basis functions: For function approximation, a natural comparison could be made with standard polynomial or basis regression as in [21]. [2] explored the use of Tensor Methods. [24] discussed the use of the Karhunone-louvre basis and signatures. Chebyschev Methods.

In terms of alternative machine learning models. [20] explored the use of Gausisan Processes. Some papers have looked at gradient boosting

[13] suggests initialising multiple random seeds, to obtain a confidence interval on prices

$$P(\theta^-), P(\theta^+)$$

4 Numerical Experiments

In general, our workflow for this section is as follows:

- Construct Dataset
- Define Model
- Train Model
- Model Diagnosis
- Use Model for Inference

Market generator models

- Bachelier Model (Arithmetic Brownian Motion) Basket Options
- Black-Scholes (Geometric Brownian Motion), time-scaled vol + delta
- Local Vol
- Fixed Income Examples: Vasciek, Cheyette, CIR, Hull White, LIBOR market model / BGM, HJM, Swaps, Caplets, Floors, Swaptions
- Path-Dependent: Asian, Barriers,
- Optionality: Americans, Bermudans, Callables

Metrics:

- Time Complexity (Training, Evaluation of Prices + Derivatives)
- Dataset (Size (Timesteps, Number of MC paths), method)
- Architecture: ResNet, Standard
- Objective: Standard, Differential, PDE, Deep Hedging, Residual / Control-Variate, Neural SDE
- Model Complexity (No. of parameters, (width, depth)
- No. of no-arbitrage violations, asymptotic behaviour, extrapolation behaviour
- Errors (L1, L2 , Linf) in Price, First Order, Second Order sensitivities
- Comparison vs MC, PDE, Closed form, other methods
- Pathwise Hedging Error ($V_T - \sum_{i=1}^{N} \Delta_{t_i} \cdot (S_{t_i+1} - S_{t_i}) - P_0$) (Loss, CVaR / Quantiles)

Arithmetic Brownian Motion / Bachelier Model

We consider the first example from [17], a standard European Call on a basket option where the underlyings have dynamics

$$\begin{aligned} \mathbf{S}_t &= \mathbf{S}_0 + \mathbf{L}\mathbf{W}_t, && \mathbf{S}_t \in \mathbb{R}^d, \mathbf{W}_t \in \mathbb{R}^f \text{ for all } t \in [0,T], \text{ and } \mathbf{L} \in \mathbb{R}^{d\times f} \\ \mathbf{X}_t &= \mathbf{w}^\top \mathbf{S}_t, && \mathbf{w} \in \mathbb{R}^d, \mathbf{X}_t \sim N(\mathbf{w}^\top \mathbf{S}_0, \mathbf{w}^\top \mathbf{L}\mathbf{L}^\top \mathbf{w}) \end{aligned}$$

Where $\mathbf{w}_i = \frac{U_i}{\sum_{i=1}^n U_i}$ is a random weight such that the weights of the basket sum to one, $\mathbf{L}\mathbf{L}^\top$ is the covariance matrix, $\mathbf{S}_0$ denotes the initial values of the underlying assets and

Geometric Brownian Motion / Black-Scholes

We consider the second example from [17], 1D Black-Scholes for a single time-to-maturity $T-t$ with respect to different levels of the underlying S_t.

$$\begin{aligned} \mathbf{S}_T &= \mathbf{S}_t \exp((r - \frac{\sigma^2}{2})(T-t) + \sigma(W_T - W_t)) \\ h(S_T) &= \left(\frac{S_T}{K} - 1\right)^+ \end{aligned}$$

The analytic solution is given by the Black-Scholes formula:

$$d_\pm = \frac{\log(S_t e^{r(T-t)}/K)}{\sigma\sqrt{T-t}} \pm \frac{(\sigma\sqrt{T-t})^2}{2} g(e^{r(T-t)}S_t/K, \sigma\sqrt{T-t}) \quad = \frac{e^{r(T-t)}S_t}{K}\Phi(d_+) - \Phi(d_-)$$

parameter	**range**
$\log(S_t/K)$	[-3, 3]
$\sigma\sqrt{T-t}$	[0, 3]

We parameterise in terms of moneyness $\frac{S_T}{K}$ and exploit the homogeneity of Black-Scholes. $\frac{\partial f}{\partial x}\frac{\partial x}{\partial s} =$

The lack. Thus we consider a parameterisation with respect to moneyness and time-scaled implied volatility

$$f(S_t/K, \sigma\sqrt{T-t})$$

We further extend this to

- Plot of Price, Delta, Vega, Gammaa vs Strike, Time
- Plot of call surface

- Plot of errors

In the most simple case, we could consider a European payoff on some forward (i.e. $r = 0$).

$$dS_t = \sigma S_t dW_t$$

$$f(\tau, S_t, K, \sigma) = \mathbb{E}^{\mathbb{Q}}[(S_T - K)^+ | S_t, K, \sigma]$$

In this case, we can leverage the homogeneity of Black-Scholes, and reduce our problem to the process $\log(S_t/K)$ instead of S_t and time-scaled implied volatility $\sigma\sqrt{\tau}$ instead of τ

$$f(\tau,)$$

In this setup, consider the case of a call option:

$$\mathbb{E}[e^{-r(T-t)}(S_T - K)^+ | S_t] = \mathbb{E}[(S_t \exp(-\frac{\sigma^2(T-t)}{2} + \sigma(W_T - W_t)) - K)^+]$$

If we use Monte Carlo pricing

$$d_\pm = \frac{\log(Se^{rT}/K)}{\sigma\sqrt{T-t}} \pm \frac{(\sigma\sqrt{T-t})^2}{2}$$

$$K[\frac{S}{Ke^{-rT}}\Phi(d_+) - \Phi(d_-)]$$

$$\frac{S_T}{K} = \exp(\log(S_t/K) - 0.5\sigma^2(T-t) + \sigma\sqrt{T-t}Z)$$

$$d_\pm = \frac{\log(F_t/K)}{\sigma\sqrt{T-t}} \pm \frac{(\sigma\sqrt{T-t})}{2}$$

$$K[\frac{F_t}{K}\Phi(d_+) - \Phi(d_-)] = F_t[\Phi(d_+) - \frac{K}{F_t}\Phi(d_-)]$$

$$F_t \mathbb{E}^{\mathbb{Q}}[(1 - \frac{K}{F_T})^+ | F_t/K, \sigma\sqrt{T-t}]$$

$$d_- > 0$$

PDE

$$V_t + rsV_s - + \frac{V_{ss}\sigma^2 s^2}{2} - rV = 0$$

$$\frac{\partial\sigma\sqrt{\tau}}{\partial t} = \frac{\sigma^2}{2\sqrt{\tau}}, \frac{\partial\log(S/K)}{\partial s} = \frac{1}{s},$$

$$-V_{\sigma\sqrt{\tau}}\frac{\sigma^2}{2\sigma\sqrt{\tau}} - \frac{\sigma^2}{2}V_x + \frac{\sigma^2 V_{xx}}{2} = 0$$

$$V_t + rsV_s + \frac{s^2 V_{ss}\sigma^2}{2} - rV = 0$$

$x = \log(S/K), \frac{\partial x}{\partial s} = \frac{1}{s}, \frac{\partial^2 x}{\partial s^2} = \frac{-1}{s^2}$

$$V_{\sigma\sqrt{T-t}}\frac{\sigma}{2\sqrt{T-t}} + rsV_x\frac{\partial x}{\partial s} + \frac{s^2\sigma^2}{2}[V_{xx}(\frac{\partial x}{\partial s})^2 + V_x\frac{\partial^2 x}{\partial s^2}] - rV = 0$$

$$V_{\sigma\sqrt{T-t}}\frac{\sigma}{2\sqrt{T-t}} + rsV_x\frac{\partial x}{\partial s} + \frac{s^2\sigma^2}{2}[V_{xx}(\frac{\partial x}{\partial s})^2 + V_x\frac{\partial^2 x}{\partial s^2}] - rV = 0$$

$$-V_\sigma\sqrt{T-t}\frac{\sigma}{2\sqrt{T-t}} + (r - \frac{\sigma^2}{2})V_x + \frac{\sigma^2 V_{xx}}{2} - rV = 0$$

$$\frac{-V_\sigma\sqrt{T-t}}{\sigma\sqrt{T-t}} + (\frac{r}{2\sigma^2} - 1)V_x + V_{xx} - \frac{r}{2\sigma^2}V = 0$$

$$V_t + rsV_s + \frac{s^2 V_{ss}\sigma^2}{2} - rV = 0$$

$x = \log(S/K), \frac{\partial x}{\partial s} = \frac{1}{s}, \frac{\partial^2 x}{\partial s^2} = \frac{-1}{s^2}$

$$V_{\sigma\sqrt{T-t}}\frac{\sigma}{2\sqrt{T-t}} + rsV_x\frac{\partial x}{\partial s} + \frac{s^2\sigma^2}{2}[V_{xx}(\frac{\partial x}{\partial s})^2 + V_x\frac{\partial^2 x}{\partial s^2}] - rV = 0$$

$$V_{\sigma\sqrt{T-t}}\frac{\sigma}{2\sqrt{T-t}} + rsV_x\frac{\partial x}{\partial s} + \frac{s^2\sigma^2}{2}[V_{xx}(\frac{\partial x}{\partial s})^2 + V_x\frac{\partial^2 x}{\partial s^2}] - rV = 0$$

$$-V_\sigma\sqrt{T-t}\frac{\sigma}{2\sqrt{T-t}} + (r - \frac{\sigma^2}{2})V_x + \frac{\sigma^2 V_{xx}}{2} - rV = 0$$

$$\frac{-V_\sigma\sqrt{T-t}}{\sigma\sqrt{T-t}} + (\frac{r}{2\sigma^2} - 1)V_x + V_{xx} - \frac{r}{2\sigma^2}V = 0$$

$V_K = V_x\frac{\partial x}{\partial K} = -\frac{V_x}{K}, V_{KK} = \frac{V_{xx}}{K^2} + \frac{V_x}{K^2}$
Thus we require: $V_x \geq 0, V_{xx} \geq 0, V_\tau \geq 0$
$V(-\infty, \sqrt{\tau}) = 0, V(x, 0) = (x-1)^+$

$$x = \frac{\log(S/K)}{\sigma\sqrt{T-t}}, \frac{\partial}{\partial S} = \frac{1}{sy}, \frac{\partial}{t} = \frac{\log(S/K)\sigma^2}{2y}$$

Stochastic Volatility

We consider the SABR of Hagan

$$dS_t = S_t^{\beta}\sigma_t dW_{1,t} \tag{21}$$

$$d\sigma_t = \xi\sigma_t dW_{2,t}, dW_{2,t} = \alpha\sigma_t[\rho dW_{1,t} + \sqrt{1-\rho^2}dW_{1,t}^{\top}] \tag{22}$$

F_t/K	σ_t	τ	β	ξ	ρ
Moneyness	Volatility	Time-to-maturity	CEV	vol-of-vol	correlation

Heston

$$dF_t = F_t V_t dW_{1,t}$$
$$dV_t = \kappa(\theta - V_t)dt + \xi\sqrt{V_t}dW_{2,t}$$
$$d[W_1, W_2]_t = \rho dt$$

F_t/K	V_t	τ	κ	θ	ξ	ρ
Moneyness	Variance	Time-to-maturity	Mean Reversion Speed	Mean Variance Level	Vol-of-Vol	Correlation

For the Heston, we also require the Feller Condition: $2\kappa\theta > \sigma^2$
We use the description of the characeristic transform from [12].

American Option

Asian Option

Consider an discrete-time arithmetic average Asian option:

$$(\frac{1}{N}\sum_{i=1}^{n} S_i - K)^+$$

For the continuous time analogue, let $Y_t = \int_0^t S_t du, dY_t = S_t dt$

$$f(t, s, y) = \mathbb{E}[e^{-r(T-t}h(S_T, Y_T)|S_t = s, Y_t = y]$$

Then the corresponding PDE is given by:

$$f_t + rsf_s + sf_y + \frac{\sigma^2}{2}f_{ss} - rf = 0, f(T, s, y) = \frac{1}{T}(y-K)^+$$

The replicating strategy is f_s

5 Appendix

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