Partial Derivatives of a Two Variable Function

In case of one variable function $y = f(x)$, we have

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

But it does not apply to the case of two variable function $z = f(x, y)$ since there are two independent variables.

To find the derivatives of $f(x, y)$, we need to calculate the derivatives with respect to each independent variable separately. We call it ***"partial derivative"***.

Partial Derivatives

How to calculate the partial derivative?

To find a partial derivative of $f(x, y)$ with respect to one variable, we consider another input variable as a constant. For example, to find the partial derivative of $f(x, y)$ with respect to x, we considers y as a constant. Then, we take an ordinary derivative of $f(x, y)$ with respect to x as in the case of one variable function.

Notations for Partial Derivatives

Let $z = f(x, y)$.

The partial derivative of $f(x, y)$ with respect to x is denoted by

$$\frac{\partial z}{\partial x} \quad \text{or} \quad \frac{\partial f}{\partial x} \quad \text{or} \quad f_x.$$

If we want to evaluate the partial derivative at (x_0, y_0), we use the notations:

$$\left.\frac{\partial z}{\partial x}\right|_{(x_0, y_0)} \quad \text{or} \quad \frac{\partial f}{\partial x}(x_0, y_0) \quad \text{or} \quad f_x(x_0, y_0).$$

Similarly, the partial derivative of $f(x, y)$ with respect to y is denoted by

$$\frac{\partial z}{\partial y} \quad \text{or} \quad \frac{\partial f}{\partial y} \quad \text{or} \quad f_y.$$

Partial Derivative with Respect to x

Definition The partial derivative of $f(x, y)$ with respect to x at (x_0, y_0) is defined as

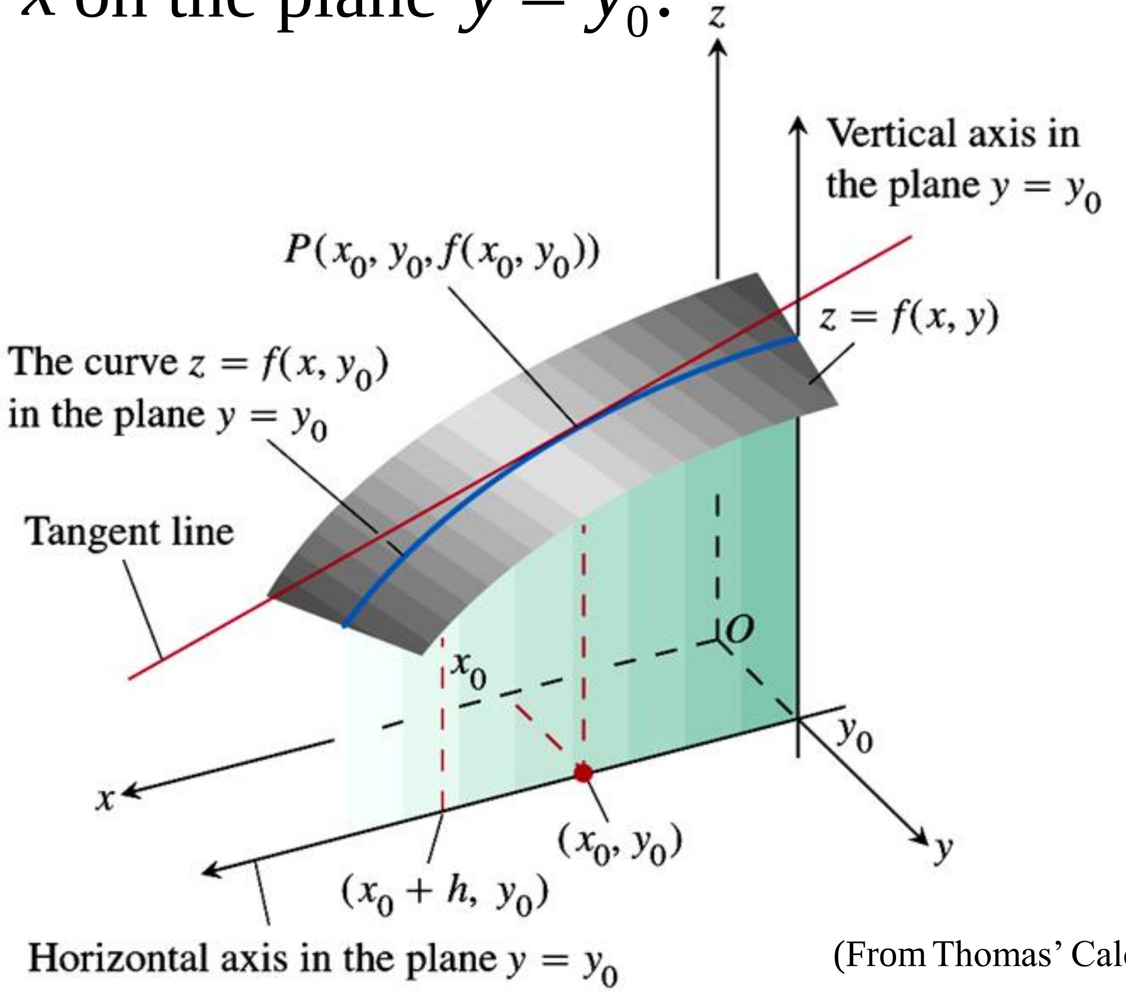
$$\left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)} = \left.\frac{d}{dx} f(x, y_0)\right|_{x=x_0} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}.$$

Example Let $f(x, y) = x^3 - 3x^2 y + 3xy^2 - y^3$. Find $f_x(1, 0)$.

Geometrical Interpretation of f_x

Let y_0 be a constant. The curve $(x, y_0, f(x, y_0))$ is the intersection between the surface $z = f(x, y)$ and the plane $y = y_0$. Note that curve $(x, y_0, f(x, y_0))$ is a function of only one variable x on the plane $y = y_0$.

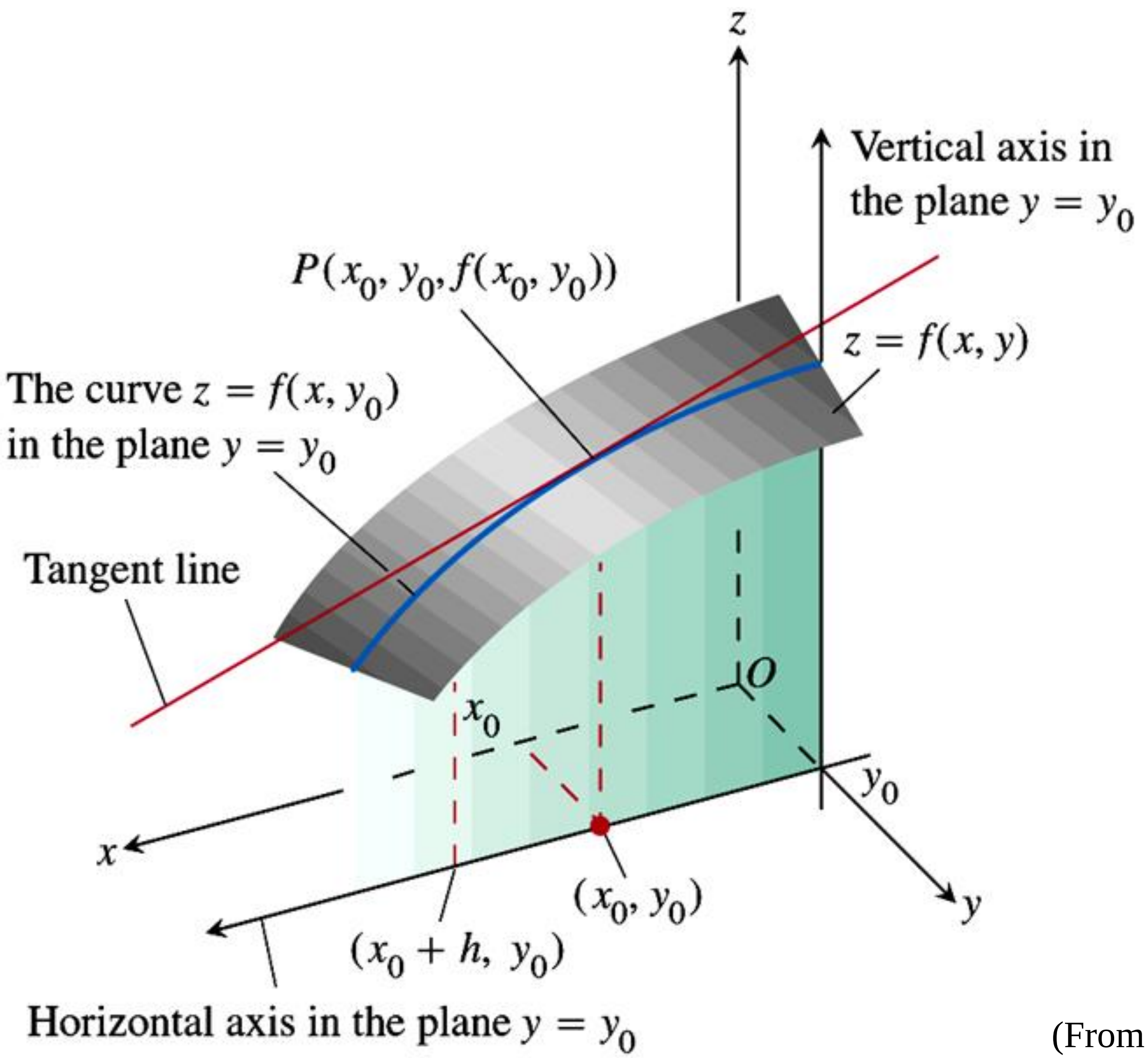
Set $y = y_0$ to be a constant.



(From Thomas' Calculus)

Geometrical Interpretation of f_x

$f_x(x_0, y_0)$ means the slope of the tangent line to the curve $(x, y_0, f(x, y_0))$ at (x_0, y_0).



(From Thomas' Calculus)

Partial Derivative with Respect to y

Definition The partial derivative of $f(x, y)$ with respect to y at (x_0, y_0) is defined as

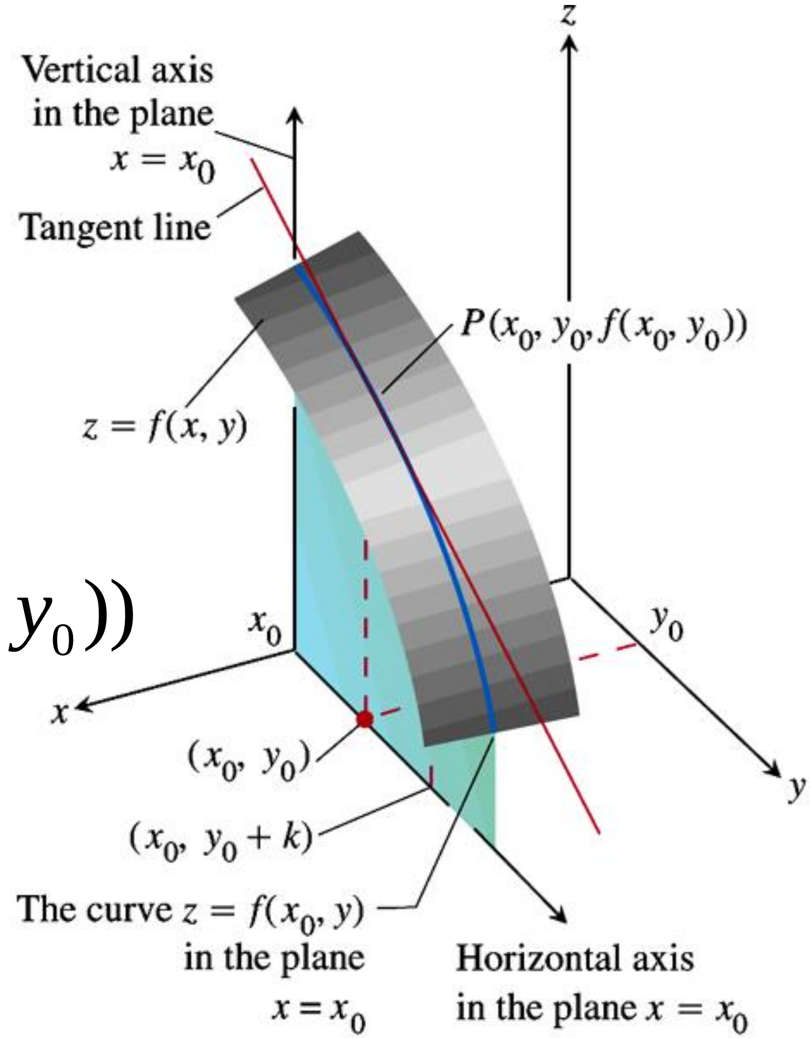
$$\left.\frac{\partial f}{\partial y}\right|_{(x_0, y_0)} = \left.\frac{d}{dy} f(x_0, y)\right|_{y=y_0} = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}.$$

Example Let $f(x, y) = x^3 - 3x^2 y + 3xy^2 - y^3$. Find $f_y(1, 0)$.

Geometrical Interpretation of f_y

$f_y(x_0, y_0)$ means the slope of the tangent line to the curve $(x, y_0, f(x, y_0))$ at (x_0, y_0).

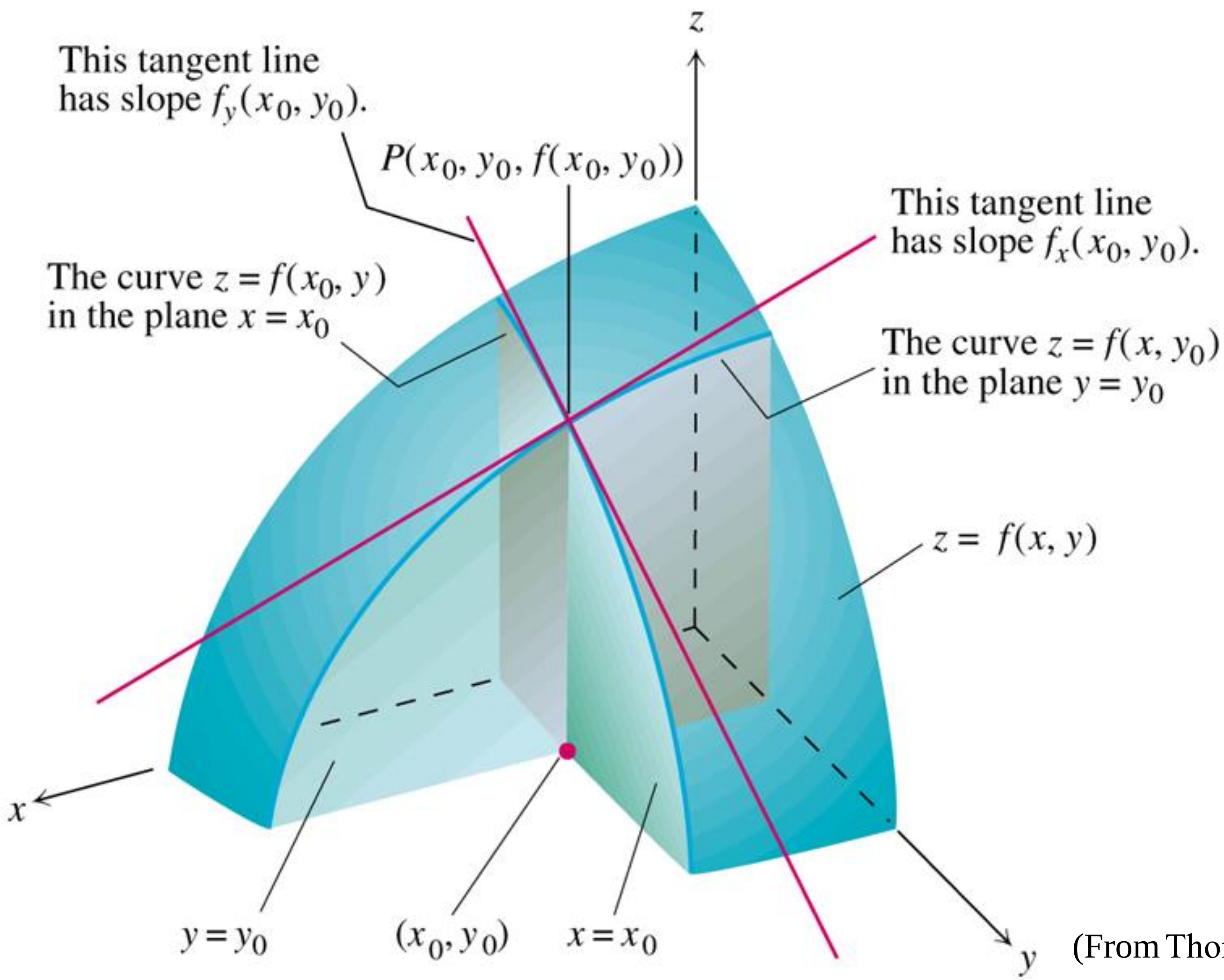
The curve $(x, y_0, f(x, y_0))$ is the intersection between the surface $z = f(x, y)$ and the plane $x = x_0$.
Note that the curve $(x, y_0, f(x, y_0))$ is a function of one variable y.



(From Thomas' Calculus)

Partial Derivatives

There are many tangent lines to the surface $z = f(x, y)$ at $(x_0, y_0, f(x_0, y_0))$ depending on the directions.



(From Thomas' Calculus)

Example: Partial Derivatives

Let $f(x,y)=\dfrac{x}{x^2+y^2}$. Then

$$\frac{\partial f}{\partial x}=$$

$$\frac{\partial f}{\partial y}=$$

Example: Partial Derivatives

Let $f(x, y) = \cos(3x - y^2)$. Then

$$\frac{\partial f}{\partial x} =$$

$$\frac{\partial f}{\partial y} =$$

Implicit Partial Differentiation

In some case, for example $z^2+2z+1=x^2+2xy+y^2$, it's not easy to write $z=f(x,y)$ before calculating the derivatives. In this situation, we can find the partial derivative by taking partial derivative operator $\frac{\partial}{\partial x}$ on both sides of the equation:

$$\frac{\partial}{\partial x}(z^2+2z+1)=\frac{\partial}{\partial x}(x^2+2xy+y^2).$$

$$\left.\begin{aligned}
&\frac{\partial}{\partial x}(z^2)=\frac{\partial(z^2)}{\partial z}\frac{\partial z}{\partial x}=2z\frac{\partial z}{\partial x}\\
&\frac{\partial}{\partial x}(2z)=2\frac{\partial z}{\partial x}\\
&\frac{\partial}{\partial x}(x^2+2xy+y^2)=2x+2y
\end{aligned}\right\}\quad (2z+2)\frac{\partial z}{\partial x}=2x+2y$$

Thus, $\frac{\partial z}{\partial x}=\frac{x+y}{z+1}$.

Example: Implicit Partial Differentiation

Let $xz^2 + e^z = \sin(x^2 + y^2)$. Find $\dfrac{\partial z}{\partial x}$.

Functions of More Than 2 Variables

To find partial derivatives of multivariable functions, we use the same method as in the case of two-variable functions. For example, the case of the partial derivative of $f(x, y, z)$ with respect to x, we consider y and z as constants. Then, we take a usual derivative of $f(x, y, z)$ with respect to x as in the case of one variable function.

Example Let $f(x, y, z) = x\cos(3y - z^2)$. Then

$$\frac{\partial f}{\partial z} =$$

Second Order Partial Derivatives

We just take partial derivative twice which consists of four possibilities.

$$\frac{\partial^2 f}{\partial x^2} \quad \text{or} \quad f_{xx} \qquad\qquad \frac{\partial^2 f}{\partial y^2} \quad \text{or} \quad f_{yy}$$

$$\frac{\partial^2 f}{\partial x \partial y} \quad \text{or} \quad f_{yx} \qquad\qquad \frac{\partial^2 f}{\partial y \partial x} \quad \text{or} \quad f_{xy}$$

Second Order Partial Derivatives

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

Example Let $f(x, y) = x^3 - 3x^2 y + 3xy^2 - y^3$. Then

$$\frac{\partial^2 f}{\partial x^2} =$$

$$\frac{\partial^2 f}{\partial y^2} =$$

$$\frac{\partial^2 f}{\partial x \partial y} = \qquad \frac{\partial^2 f}{\partial y \partial x} =$$

Example: Second Order Partial Derivatives

Let $f(x,y)=\dfrac{x}{x^2+y^2}$. Then

$$\frac{\partial f}{\partial x}=\frac{y^2-x^2}{\left(x^2+y^2\right)^2},\qquad \frac{\partial f}{\partial y}=\frac{-2xy}{\left(x^2+y^2\right)^2},$$

$$\frac{\partial^2 f}{\partial x^2}=\frac{-2x\left(x^2+y^2\right)^2-2\left(x^2+y^2\right)(2x)(y^2-x^2)}{\left(x^2+y^2\right)^4},$$

$$\frac{\partial^2 f}{\partial y^2}=\frac{-2x\left(x^2+y^2\right)^2-2\left(x^2+y^2\right)(2y)(-2xy)}{\left(x^2+y^2\right)^4},$$

$$\frac{\partial^2 f}{\partial x\partial y}=\frac{-2y\left(x^2+y^2\right)^2-2\left(x^2+y^2\right)(2x)(-2xy)}{\left(x^2+y^2\right)^4}\quad\text{and}$$

$$\frac{\partial^2 f}{\partial y\partial x}=\frac{2y\left(x^2+y^2\right)^2-2\left(x^2+y^2\right)(2y)(y^2-x^2)}{\left(x^2+y^2\right)^4}.$$

The Mixed Derivative Theorem

The 2nd partial derivatives $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ contain $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$.

They are called "Mixed partial derivatives".

Theorem:
If f and its partial derivatives f_x, f_y, f_{xy} and f_{yx} are defined in open region containing (a,b) and all are continuous at (a,b), then

$$f_{xy}(a,b) = f_{yx}(a,b).$$

This theorem implies that if f, f_x, f_y, f_{xy} and f_{yx} are all continuous, then the order of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ in mixed partial derivative does not matter.

Example: The Mixed Derivative Theorem

Let $f(x, y) = \cos(3x - y^2)$. Then

$$\frac{\partial f}{\partial x} = -3\sin\left(3x - y^2\right),$$

$$\frac{\partial^2 f}{\partial y \partial x} = -3\cos\left(3x - y^2\right)(-2y) = 6y\cos\left(3x - y^2\right),$$

$$\frac{\partial f}{\partial y} = 2y\sin\left(3x - y^2\right) \quad \text{and}$$

$$\frac{\partial^2 f}{\partial x \partial y} = 2y\cos\left(3x - y^2\right)(3) = 6y\cos\left(3x - y^2\right).$$

In this case, $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$.

Partial Derivatives of Higher Order

Partial derivative operators $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ may be applied several times such as

$$\frac{\partial^3 f}{\partial y \partial x^2} \quad \text{or} \quad f_{xxy}, \qquad \frac{\partial^4 f}{\partial y^2 \partial x^2} \quad \text{or} \quad f_{xxyy}.$$

Example Let $f(x, y) = \cos(3x - y^2)$. Then

$$\frac{\partial f}{\partial x} = -3\sin\left(3x - y^2\right),$$

$$\frac{\partial^2 f}{\partial x^2} = -9\cos\left(3x - y^2\right) \quad \text{and}$$

$$\frac{\partial^3 f}{\partial y \partial x^2} = 9\sin\left(3x - y^2\right)\left(-2y\right) = -18y\sin\left(3x - y^2\right).$$

Differentiability of a Functions of Two Variables

Definition A function f is said to be *differentiable* at (x_0, y_0) if $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist and it is called a *differentiable function* if it is differentiable everywhere.

Continuity of Partial Derivatives

If f_x and f_y are both continuous in open region R, then f is differentiable everywhere in R.

Differentiability Implies Continuity

If f is differentiable at (x_0, y_0), then f is continuous at (x_0, y_0).

Composite Functions in Higher Dimensions

Let $w = f(x, y)$, $y = h(t)$ and $x = g(t)$.

Thus,

$$w = f(g(t), h(t))$$

We say that w is a composite function in terms of t,
x and y are called "intermediate variables",
w is called a "dependent variable",
and t is called an "independent variable".

Chain Rule for Functions of Two Independent Variables

Derivative of a composite function can be calculated by applying the chain rule. In case of one-variable functions $y = f(x)$ and $x = g(t)$, we have

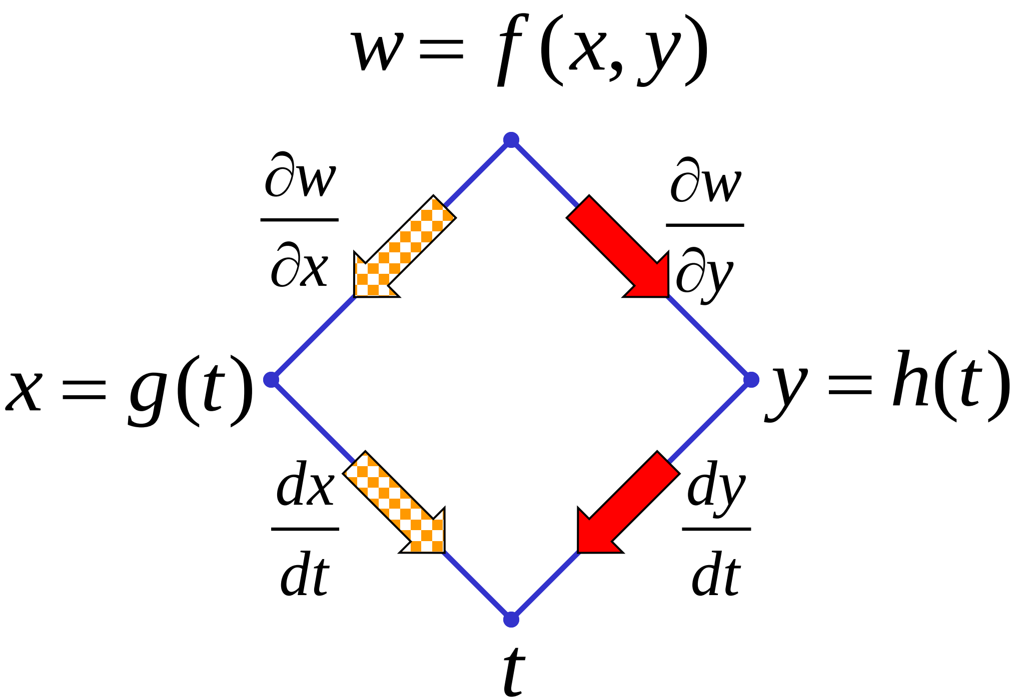
$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}.$$

In case of two-variable functions, let $w = f(x, y)$ and $y = h(t)$ and $x = g(t)$ where f, g and h are differentiable. We have

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Tree Diagram

This diagram shows how to find the derivatives of a composite function.



1. Each node represents each variable.
2. Arrow is the derivative of beginning node with respect to the ending node.
3. Start with the dependent variable *w* and then walk along all branches to *t*.
4. Add them all up.

Get $\frac{\partial w}{\partial x}, \frac{dx}{dt}$

Get $\frac{\partial w}{\partial y}, \frac{dy}{dt}$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}.$$

Example: Chain Rule for Functions of 2 Variables

Let $w = x^2 + 2xy + y^2$, $x = \cos(t)$ and $y = \sin(t)$. Find $\dfrac{dw}{dt}$.

Chain Rule for Functions of 3 Independent Variables

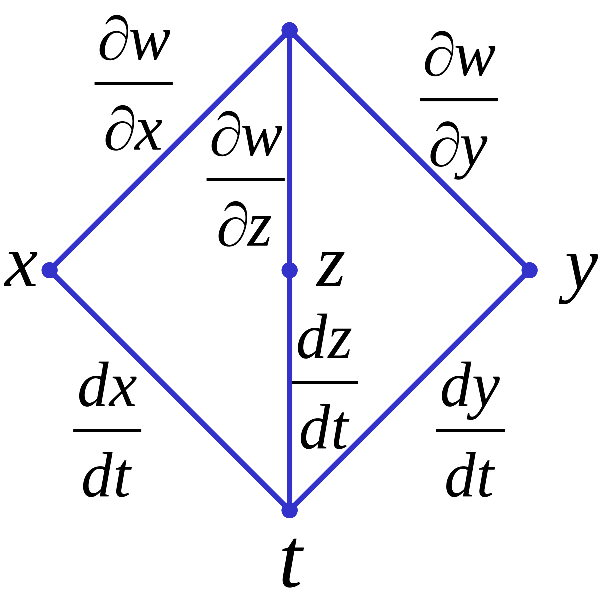
In case of three-variable functions

$$w = f(x, y, z),\ x = g(t),\ y = h(t) \quad \text{and} \quad z = k(t)$$

where f, g, h and k are differentiable. We have

$$\boxed{\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}.}$$

$$w = f(x, y, z)$$



$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}.$$

Example: Chain Rule for Functions of 3 Variables

Let $w = xy + yz + zx,\ x = \cos(t),\ y = \sin(t)$ and $z = t^2$. Then

$$\frac{\partial w}{\partial x} = y + z, \qquad \frac{\partial w}{\partial y} = x + z, \qquad \frac{\partial w}{\partial z} = y + x,$$

$$\frac{dx}{dt} = -\sin(t), \qquad \frac{dy}{dt} = \cos(t), \qquad \frac{dz}{dt} = 2t,$$

$$\begin{aligned}\frac{dw}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \\ &= (y+z)\big(-\sin(t)\big) + (x+z)\big(\cos(t)\big) + (y+x)(2t) \\ &= (\sin t + t^2)\big(-\sin t\big) + (\cos t + t^2)\big(\cos t\big) + (\sin t + \cos t)(2t).\end{aligned}$$

Functions Defined on Surfaces

Suppose that we have several two-variable functions as intermediate variables

$$w = f(x, y, z),\ x = g(r, s),\ y = h(r, s),\ z = k(r, s).$$

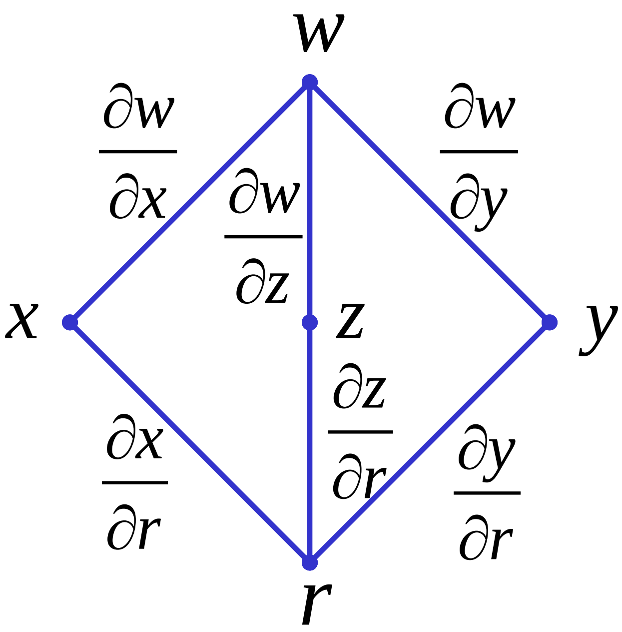
All x, y, z are considered as surfaces while w is a function of all three surfaces. Its partial derivatives are

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

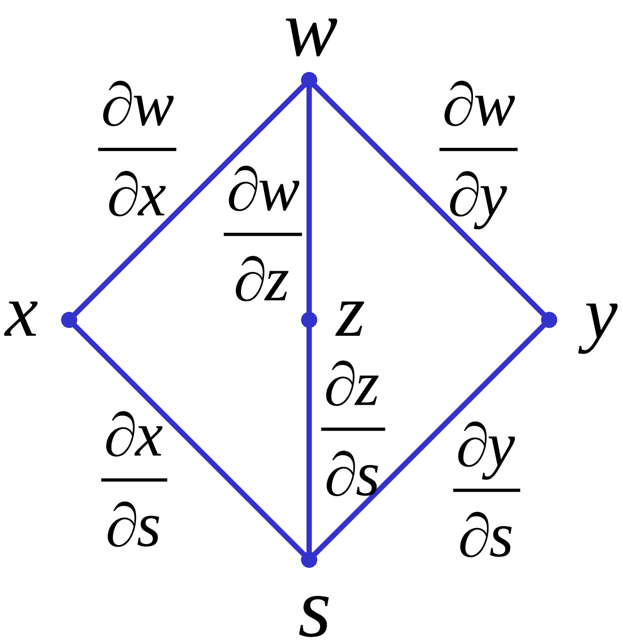
and

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}.$$

Tree Diagram for *f*(*g*(*r*,*s*),*h*(*r*,*s*),*k*(*r*,*s*))



$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}.$$



$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}.$$

Example

Let $w = xz + y^2$, $x = \dfrac{r}{s}$, $y = r^2 + \ln(s)$ and $z = r^2$. Then

$$\frac{\partial w}{\partial x} = z, \qquad \frac{\partial w}{\partial y} = 2y, \qquad \frac{\partial w}{\partial z} = x,$$

$$\frac{\partial x}{\partial r} = \frac{1}{s}, \qquad \frac{\partial y}{\partial r} = 2r, \qquad \frac{\partial z}{\partial r} = 2r,$$

$$\frac{\partial x}{\partial s} = \frac{-r}{s^2}, \qquad \frac{\partial y}{\partial s} = \frac{1}{s}, \qquad \frac{\partial z}{\partial s} = 0,$$

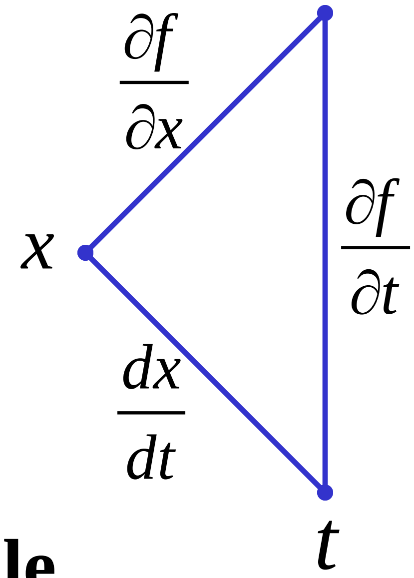
$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} = \frac{z}{r} + 2y(2r) + x(2r)$$

$$= r + 4r(r^2 + \ln s) + 2\frac{r^2}{s},$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} = \frac{-zr}{s^2} + 2\frac{y}{s} + x(0) = \frac{-r^3}{s^2} + 2\frac{r^2 + \ln s}{s}.$$

Functions in a Form of $w = f(t, x(t))$

Suppose that w is a function of one independent and one intermediate variables: $w = f(t, x)$



$$\boxed{\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial t}}$$

Example

Let $w = xt + x^2$ where $x = (t-2)^2$. Find $\frac{dw}{dt}$.

$\frac{\partial f}{\partial x} = t + 2x$, $\frac{\partial f}{\partial t} = 2(t-2)$ and

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial t} = (t+2x)2(t-2) = 2\left(t+2(t-2)^2\right)(t-2).$$

Implicit Functions

Explicit function is a function in a form $y = h(x)$. Then $\dfrac{dy}{dx}$ can be calculated easily. **Implicit function** is a function in a form $F(x, y) = 0$. We may compute $\dfrac{dy}{dx}$ by 2 methods:

<u>Method 1</u>
Rewrite $F(x, y) = 0$ into $y = h(x)$ before computing $\dfrac{dy}{dx}$.
This is sometimes difficult.

<u>Method 2</u>
Use the chain rule for multivariable functions.

Implicit Differentiation

Let y be a function defined implicitly in term of x.

We can find $\dfrac{dy}{dx}$ by following this procedure.

1. Set up $F(x,y)=0.$

2. Differentiate $F(x,y)=0$ with respect to x on both sides:

$$\frac{d}{dx}F(x,y)=\frac{d}{dx}0.$$

Then, we get

$$0=F_x\frac{dx}{dx}+F_y\frac{dy}{dx}=F_x+F_y\frac{dy}{dx}.$$

Therefore,

$$\frac{dy}{dx}=-\frac{F_x}{F_y}.$$

Example: Implicit Differentiation

Suppose that $x^2 + 2xy + y^2 = \sin(xy)$. Find $\dfrac{dy}{dx}$.

Implicit differentiations of a system of equations

- **Example** Let u and v be functions x and y such that

$$uv = x + y \quad \text{and} \quad u - v^2 = x - y.$$

Find $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}.$

Definition Let F and G be functions of u, v. We call

$$\frac{\partial(F,G)}{\partial(u,v)} = \begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix} = F_u G_v - F_v G_u$$

as a Jacobian determinant of $F(u,v)$ and $G(u,v)$.

Jacobian formulas:

Now, we have $F(u,v,x,y)=0$ and $G(u,v,x,y)=0.$

By Cramer's rule: If $F_u G_v - F_v G_u \neq 0,$ then

$$u_x = -\frac{\begin{vmatrix} F_x & F_v \\ G_x & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = -\frac{\frac{\partial(F,G)}{\partial(x,v)}}{\frac{\partial(F,G)}{\partial(u,v)}}, \qquad u_y = -\frac{\begin{vmatrix} F_y & F_v \\ G_y & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = -\frac{\frac{\partial(F,G)}{\partial(y,v)}}{\frac{\partial(F,G)}{\partial(u,v)}},$$

$$v_x = -\frac{\begin{vmatrix} F_u & F_x \\ G_u & G_x \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = -\frac{\frac{\partial(F,G)}{\partial(u,x)}}{\frac{\partial(F,G)}{\partial(u,v)}} \quad \text{and} \quad v_y = -\frac{\begin{vmatrix} F_u & F_y \\ G_u & G_y \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = -\frac{\frac{\partial(F,G)}{\partial(u,y)}}{\frac{\partial(F,G)}{\partial(u,v)}}.$$

Example Let u and v be functions x and y such that

$$uv^2 + xy = x^2 + y \quad \text{and} \quad u^2 - 3v = x^2 + y^2.$$

Find $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}.$