Chapter 2
Basic Structures: Sets, Functions, Sequences, Sums and Matrices

2.1 Sets

DEFINITION 1: A set is an unordered collection of objects, called elements or members of the set. A set is said to contain its elements, $a \in A$. Otherwise $a \notin A$

EX1. The set V of all vowels in the English alphabet.

$$V = \{a, e, i, o, u\}$$

EX2. The set of positive integers less than 100 can be denoted by $\{1, 2, 3, \ldots, 99\}$

DEFINITION 2: Two sets are equal if and only if they have the same elements

EX3. $\{1, 3, 3, 4, 5, 7, 7\} = \{1, 3, 4, 5, 7\}$

A Set

- $S = \{a,b,c,d\}$
- Order not important

  $S = \{a,b,c,d\} = \{b,c,a,d\}$
- Each distinct object is either a member or not; listing more than once does not change the set.

  $S = \{a,b,c,d\} = \{a,b,c,b,c,d\}$

- Elipses (**...**) may be used to describe a set without listing all of the members when the pattern is clear.

  $S = \{a,b,c,d, \ldots,z \}$

- Set of all vowels in the English alphabet:

  $V = \{a,e,i,o,u\}$

- Set of all odd positive integers less than 10:

  $O = \{1,3,5,7,9\}$

- Set of all positive integers less than 100:

  $S = \{1,2,3,\ldots,99\}$

- Set of all integers less than 0:

  $S = \{\ldots, -3,-2,-1\}$

Some Important Sets

N = *natural numbers* = {0,1,2,3…}

Z = *integers* = {…,-3,-2,-1,0,1,2,3,…}

$\mathbf{Z}^+$ = *positive integers* = {1,2,3,…}

R = set of *real numbers*

$\mathbf{R}^+$ = set of *positive real numbers*

C = set of *complex numbers*.

Q = set of rational numbers

Set-Builder Notation

- Specify the property or properties that all members must satisfy:

  $S = \{x \mid x \text{ is a positive integer less than } 100\}$

  $O = \{x \mid x \text{ is an odd positive integer less than } 10\}$

  $O = \{x \in \mathbf{Z}^+ \mid x \text{ is odd and } x < 10\}$

- A predicate may be used:

  $$S = \{x \mid \mathrm{P}(x)\}$$

- Example: $S = \{x \mid \mathrm{Prime}(x)\}$
- Positive rational numbers:

  $\mathbf{Q}^+ = \{x \in \mathbf{R} \mid x = p/q, \text{ for some positive integers } p, q\}$

Interval Notation

$$[a, b] = \{x \mid a \le x \le b\}$$
$$[a, b) = \{x \mid a \le x < b\}$$
$$(a, b] = \{x \mid a < x \le b\}$$
$$(a, b) = \{x \mid a < x < b\}$$

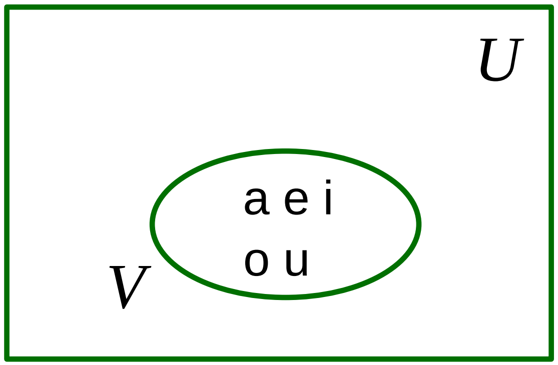
closed interval [a, b]

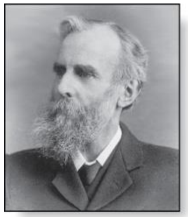
open interval (a, b)

Universal Set and Empty Set

- The ***universal set U*** is the set containing everything currently under consideration.
  - Sometimes implicit
  - Sometimes explicitly stated.
  - Contents depend on the context.
- The **empty set** is the set with no elements. Symbolized ∅, but { } also used.

Venn Diagram





John Venn (1834-1923)
Cambridge, UK

Some things to remember

- Sets can be elements of sets.

  $\{\{1, 2, 3\}, a, \{b, c\}\}$

  $\{\mathbf{N}, \mathbf{Z}, \mathbf{Q}, \mathbf{R}\}$

- The empty set is different from a set containing the empty set.

  $\emptyset = \{\ \} \neq \{\emptyset\}$

Set Equality

Definition: Two sets are **equal** if and only if they have the same elements.

- Therefore if A and B are sets, then A and B are equal if and only if

  .

$$\forall x(x \in A \leftrightarrow x \in B)$$

- We write *A* = *B* if *A* and *B* are equal sets.

$$\{1,3,5\} = \{3, 5, 1\}$$

$$\{1,5,5,5,3,3,1\} = \{1,3,5\}$$

Subsets

Definition: The set A is a *subset* of B, if and only if every element of A is also an element of B.

- The notation $A \subseteq B$ is used to indicate that A is a subset of the set B.
- $A \subseteq B$ holds if and only if $\forall x(x \in A \to x \in B)$ is true.
  - Because $a \in \emptyset$ is always false, $\emptyset \subseteq S$,for every set S.
  - Because $a \in S \to a \in S$, $S \subseteq S$, for every set S.

Equality of Sets

- Recall that two sets *A* and *B* are **equal**, denoted by *A* = *B*, iff

$$\forall x(x \in A \leftrightarrow x \in B)$$

- Using logical equivalences we have that *A* = *B* iff

$$\forall x[(x \in A \rightarrow x \in B) \wedge (x \in B \rightarrow x \in A)]$$

- This is equivalent to

$$A \subseteq B \quad \text{and} \quad B \subseteq A$$

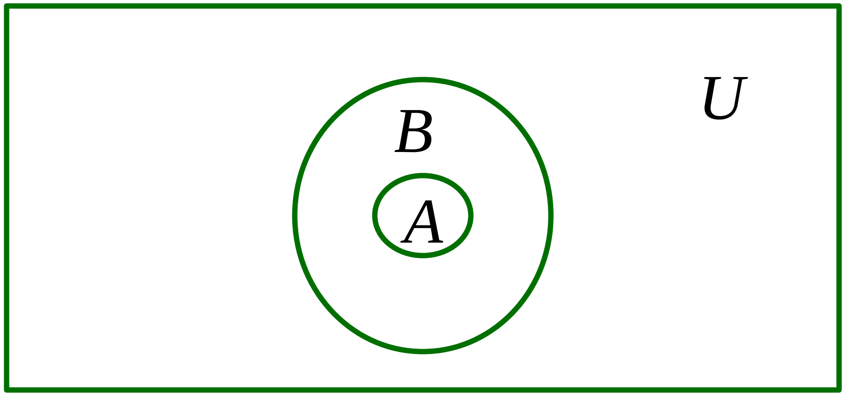
Proper Subsets

Definition: If $A \subseteq B$, but $A \neq B$, then we say A is a proper subset of B, denoted by $A \subset B$. If $A \subset B$, then

$$\forall x(x \in A \rightarrow x \in B) \wedge \exists x(x \in B \wedge x \notin A)$$

is true.

Venn Diagram



Set Cardinality

Definition: If there are exactly n distinct elements in *S* where *n* is a nonnegative integer, we say that *S* is *finite*. Otherwise it is *infinite*.

Definition: The *cardinality* of a finite set *A,* denoted by |*A*|, is the number of (distinct) elements of *A*.

Examples:

- |ø| = 0
- Let S be the letters of the English alphabet. Then |*S*| = 26
- |{1,2,3}| = 3
- |{ø}| = 1
- The set of integers is infinite.

Power Sets

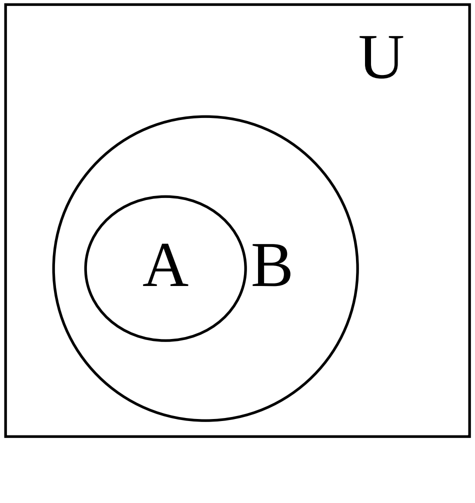
Definition: The set of all subsets of a set *A*, denoted P**(***A***)**, is called the **power set** of *A*.

Example: If *A* = {a,b} then

$$P(A) = \{ø, \{a\},\{b\},\{a,b\}\}$$

- If a set has *n* elements, then the cardinality of the power set is 2^n.

Venn Diagram



U = Universal Set

Empty Set or Null Set $= \varnothing = \{\ \}$

Subset

$A \subset B$ iff $\forall x\ (x \in A \rightarrow x \in B)$

$A \subseteq B$; is possible A=B

|S| = Cardinality of S

EX Let A be a set of odd positive integer less than 10.00

So, $|A| = |\{1,3,5,7,9\}| = 5$

EX $|\varnothing| = 0$

Finite Set & Infinite Set

EX The set of positive integers is **infinite**.

Power set

Giving a set S , the power set of S is the set of all subsets of set S, denoted by ***P(S)***

EX : P({ 0,1,2 })

$$=\{ \varnothing,\{0\},\{1\},\{2\},\{0,1\},\{0,2\},\{1,2\},\{0,1,2\} \}$$

$$P(S) = 2^n \quad \text{when} \quad | S | = n$$

Cartesian Products

DEFINITION 8: Let A, B be sets.
Cartesian product of A and B is $\boldsymbol{A \times B}$

$$= \{(a,b) \mid a \in A \wedge b \in B\}$$

$$\mathbf{A \times B \neq B \times A}$$

EX: Let A = {1, 2}, B = {a, b, c}

A x B = {(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)}
B x A = {(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)}

Cartesian Product

René Descartes
(1596-1650)

Definition: The *Cartesian Product* of two sets *A* and *B*, denoted by $A \times B$ is the set of ordered pairs (a, b) where $a \in A$ and $b \in B$.

$$A \times B = \{(a, b) | a \in A \wedge b \in B\}$$

Example:

$A = \{a, b\}$ $B = \{1,2,3\}$

$A \times B = \{(a,1),(a,2),(a,3),(b,1),(b,2),(b,3)\}$

- **Definition**: A subset *R* of the Cartesian product $A \times B$ is called a *relation* from the set A to the set B. (Relations will be covered in depth in Chapter 9.)

Cartesian Product

Definition: The cartesian products of the sets $A_1, A_2, \ldots\ldots, A_n$, denoted by $A_1 \times A_2 \times \ldots\ldots \times A_n$, is the set of ordered n-tuples $(a_1, a_2, \ldots\ldots, a_n)$ where a_i belongs to A_i for $i = 1, \ldots n$.

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) | a_i \in A_i \text{ for } i = 1, 2, \ldots n\}$$

Example: What is $A \times B \times C$ where $A = \{0,1\}$, $B = \{1,2\}$ and $C = \{0,1,2\}$

Solution: $A \times B \times C$ **=** {(0,1,0), (0,1,1), (0,1,2),(0,2,0), (0,2,1), (0,2,2),(1,1,0), (1,1,1), (1,1,2), (1,2,0), (1,2,1), (1,1,2)}

2.2 Set Operations

Let A and B be sets.

1)	Union	$A \cup B$
2)	Intersection	$A \cap B$
3)	Disjoint	$A \cap B = \varnothing$
4)	Difference	$A - B$
5)	Complement	$\bar{A} = U - A = \{x \mid x \notin A\}$

Union

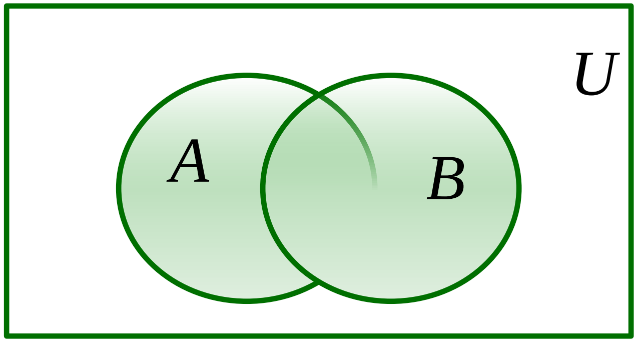
- **Definition**: Let A and B be sets. The *union* of the sets A and B, denoted by $A \cup B$, is the set:

$$\{x|x \in A \vee x \in B\}$$

- **Example**: What is $\{1,2,3\} \cup \{3, 4, 5\}$?

Solution: $\{1, 2, 3, 4, 5\}$

Venn Diagram for $A \cup B$



Intersection

- **Definition**: The *intersection* of sets *A* and *B*, denoted by $A \cap B$, is
$$\{x|x \in A \wedge x \in B\}$$

- Note if the intersection is empty, then *A* and *B* are said to be *disjoint*.

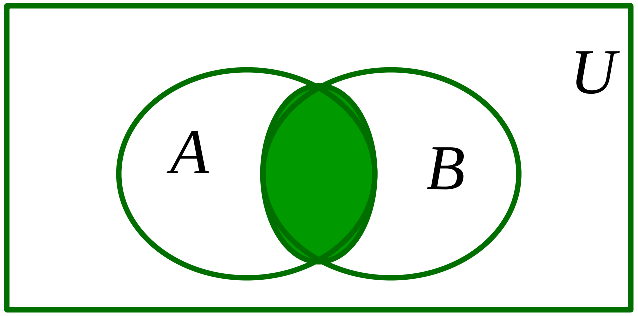
- **Example**: What is $\{1,2,3\} \cap \{3,4,5\}$?

  Solution: $\{3\}$

- **Example:** What is $\{1,2,3\} \cap \{4,5,6\}$?

  Solution: $\emptyset$

Venn Diagram for $A \cap B$



Complement

Definition: If A is a set, then the complement of the A (with respect to U), denoted by $\bar{A}$ is the set $U - A$

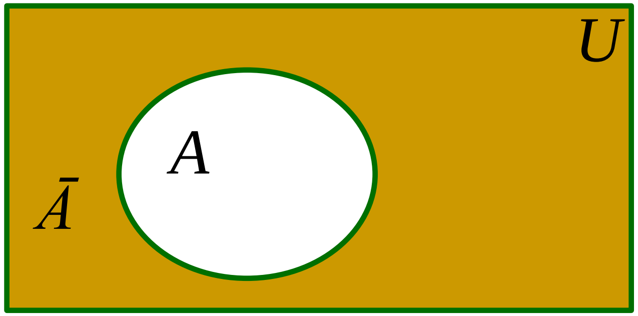
$$\bar{A} = \{x \in U \mid x \notin A\}$$

(The complement of A is sometimes denoted by A^c.)

Example: If U is the positive integers less than 100, what is the complement of $\{x \mid x > 70\}$

Solution: $\{x \mid x \leq 70\}$

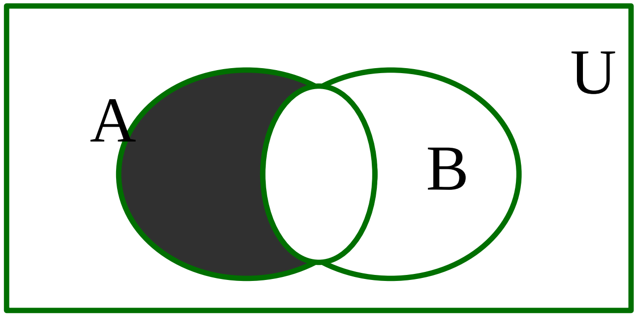
Venn Diagram for Complement



Difference

- **Definition**: Let A and B be sets. The *difference* of A and B, denoted by $A - B$, is the set containing the elements of A that are not in B. The difference of A and B is also called the complement of B with respect to A.

$$A - B = \{x \mid x \in A \wedge x \notin B\} = A \cap \overline{B}$$

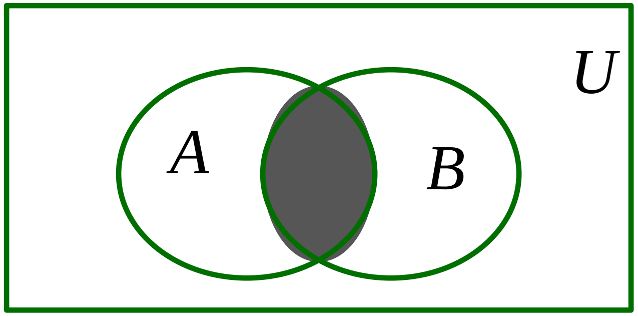


Venn Diagram for $A - B$

The Cardinality of the Union of Two Sets

Inclusion-Exclusion

$|A \cup B| = |A| + |B| - |A \cap B|$



Venn Diagram for $A \cap B$

Example: Let *A* be number of students with math majors in your class and *B* be the CS majors. To count the number of students who are math majors or CS majors, we add the number of math majors and the number of CS majors, and subtract the number of joint CS/math majors.

EX 1: The union of the sets {1, 3, 5} and {1, 2, 3}

$$\{1, 3, 5\} \cup \{1, 2, 3\} = \{1, 2, 3, 5\};$$

EX 2: The intersection of the sets {1, 3, 5} and {1, 2, 3}

$$\{1, 3, 5\} \cap \{1, 2, 3\} = \{ 1, 3\}.$$

EX 3: The difference of {1,3,5} and {1,2,3} is the set {5}

$$\{1,3,5\} - \{1,2,3\} = \{ 5\}$$

while $\{1,2,3\} - \{1,3,5\} = \{2\}$

EX 4: Let A be the set of positive integers greater than 10 (with universal set the set of all positive integers). Then

$$A' = \{1, 2, 3, 4, 5, \ldots, 9, 10\}$$

PROPERTY OF SET

Identity laws

$$A \cup \emptyset = A$$
$$A \cap U = A$$

Domination laws

$$A \cup U = U$$
$$A \cap \emptyset = \emptyset$$

Idempotent laws

$$A \cup A = A$$
$$A \cap A = A$$

Complementary laws

$$\overline{\overline{A}} = A$$

Property of Set (continue)

Commutative laws

$$A \cup B = B \cup A$$

$$A \cap B = B \cap A$$

Associative laws

$$A \cup (B \cup C) = (A \cup B) \cup C$$

$$A \cap (B \cap C) = (A \cap B) \cap C$$

Distributive laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

De Morgan's laws

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

Absorption laws

$$A \cup (A \cap B) = A$$

$$A \cap (A \cup B) = A$$

<u>EX13:</u> Use a truth table to show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Solution:

A B C	B ∪ C	A∩ (B∪ C)	A∩B	A∩C	(A∩B) ∪ (A∩C)
1 1 1	1	**1**	1	1	**1**
1 1 0	1	**1**	1	0	**1**
1 0 1	1	**1**	0	1	**1**
1 0 0	0	**0**	0	0	**0**
0 1 1	1	**0**	0	0	**0**
0 1 0	1	**0**	0	0	**0**
0 0 1	1	**0**	0	0	**0**
0 0 0	0	**0**	0	0	**0**

EX14: Let A, B, and C be sets.

Show that $(A \cup (B \cap C))' = (C' \cup B') \cap A'$

Solution:

$(A \cup (B \cap C))' = A' \cap (B \cap C)'$ by the first De Morgan law

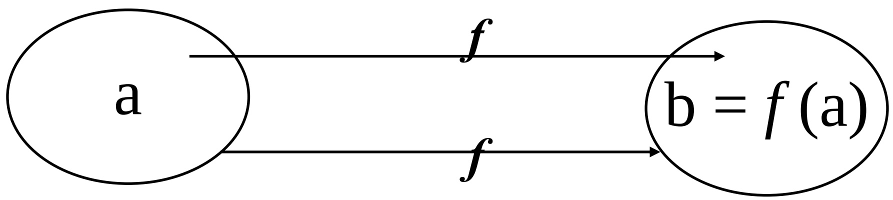
$= A' \cap (B' \cup C')$ by the second De Morgan law

$= (B' \cup C') \cap A'$ by the commutative law for intersections

$= (C' \cup B')' \cap A'$ by the commutative law for unions.

2.3 Functions

DEFINITIONS: Let A and B be sets. A function *f* from A to B is an assignment of exactly one element of B to each element of A, we write ***f* : A→B**



A is the domain of ***f***
B is the codomain of ***f***

DEFINITION 3: Let f_1 and f_2 be functions from A to **R**, then

$$(f_1+f_2)(x) = f_1(x) + f_2(x)$$
$$(f_1f_2)(x) = f_1(x)f_2(x)$$

<u>Ex:</u> Let $\mathbf{f_1}$, $\mathbf{f_2}$ be functions from **R** to **R**

$f_1(x) = x^2$, $f_2(x) = x-x^2$

$(f_1 + f_2)(x) = f_1(x) + f_2(x) = x^2 + (x - x^2) = x$

$(f_1f_2)(x) = f_1(x)f_2(x) = x^2(x - x^2) = x^3 - x^4$

<u>DEFINITION 5:</u> **One-to-One** or **Injective**

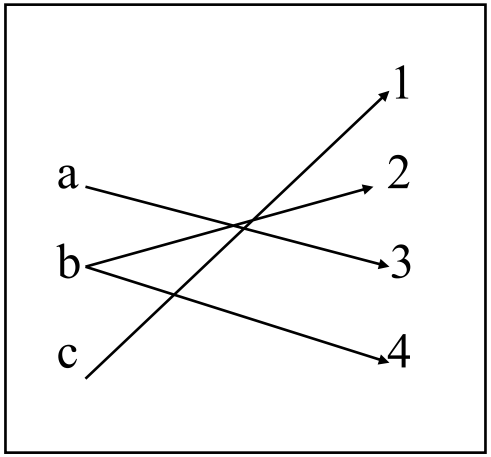
If f(x) = f(y) then x = y

<u>DEFINITION 7:</u>

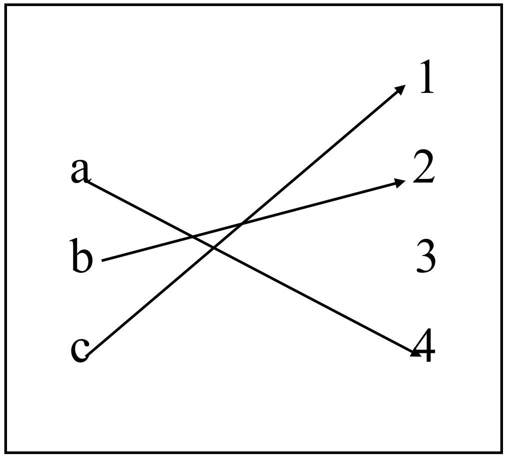
A function ***f*** from A to B is called ***onto***, or ***surjective*** if an only if for every element **b** is member of B, there is an element **a** is member of A with f(a) = b.

<u>DEFINITION 8:</u>

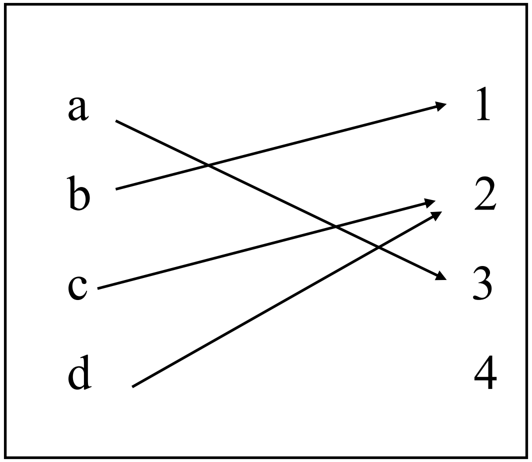
The function ***f*** is a ***one-to-one correspondence***, or a ***bijection***, if it is both **one-to-one** and **onto**.



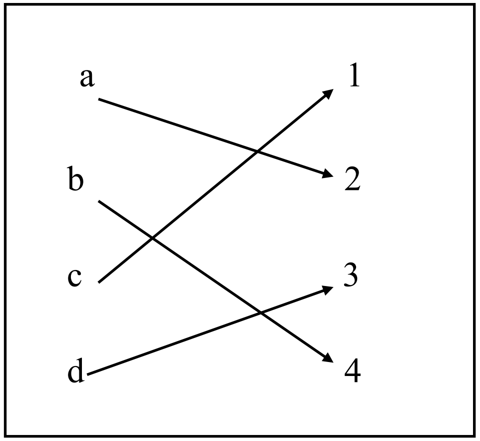
not a function



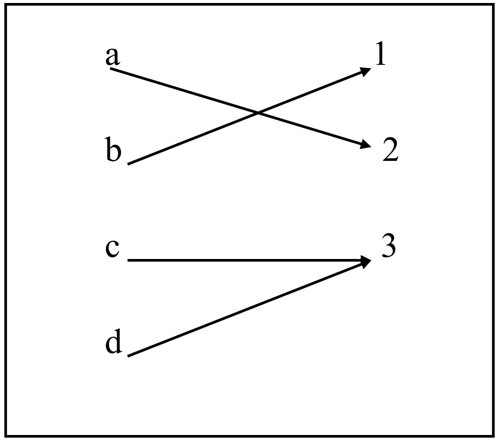
one-to-one, not onto



not one-to-one, not onto



one-to-one, onto



not one-to-one, onto

Example: Consider function f(x) from the set of integers to the set of integers

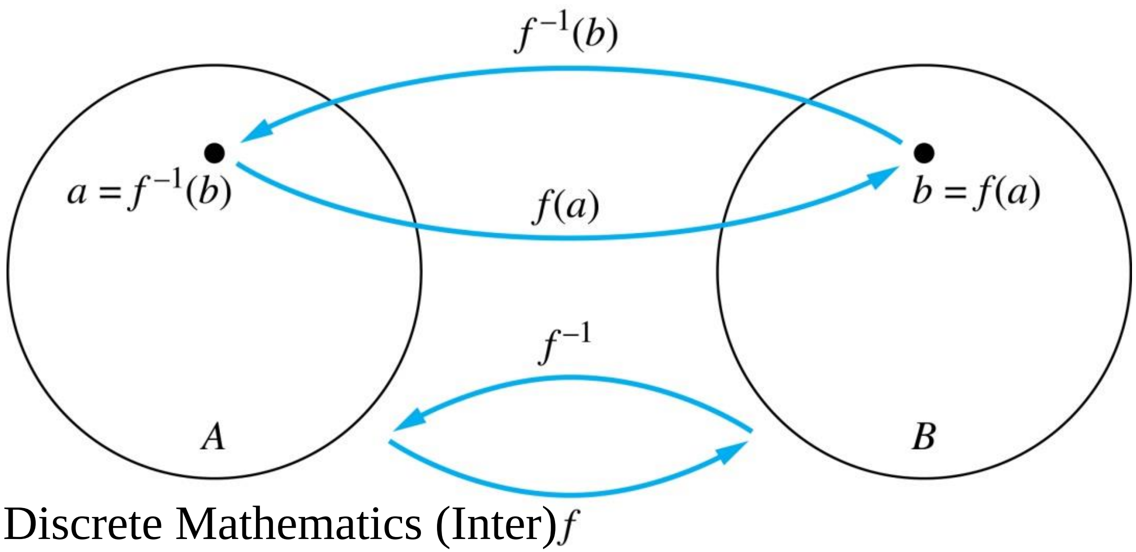
1. $f(x) = +x, -x$; not a function
2. $f(x) = |x|$; not one to one
3. $f(x) = x^2$; not one to one
4. $f(x) = x^3$; one to one
5. $f(x) = x+3$; one to one

Inverse Functions

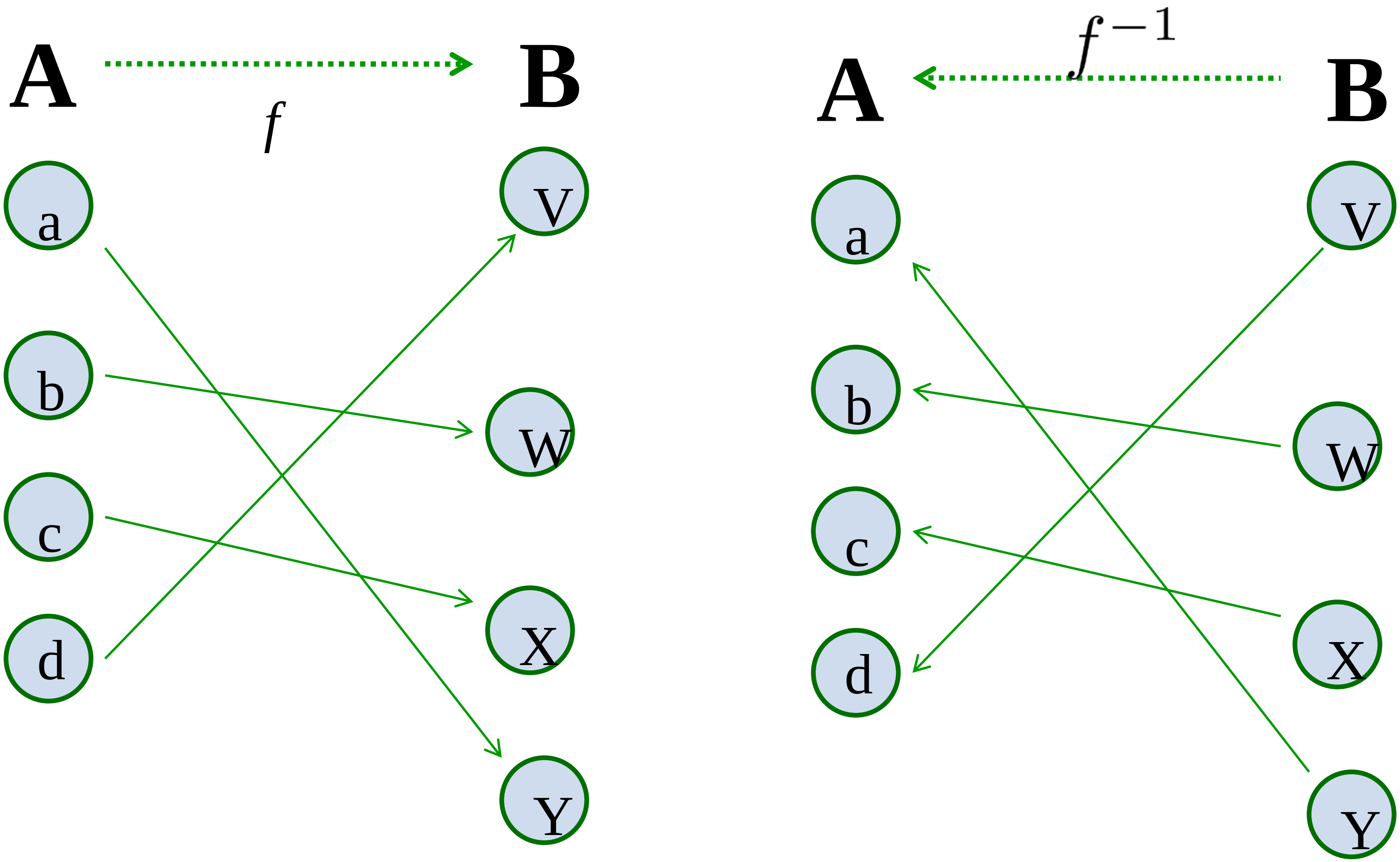
Definition: Let *f* be a bijection from *A* to *B*. Then the *inverse* of *f*, denoted f^{-1}, is the function from *B* to *A* defined as

$$f^{-1}(y) = x \text{ iff } f(x) = y$$

No inverse exists unless *f* is a bijection. Why?

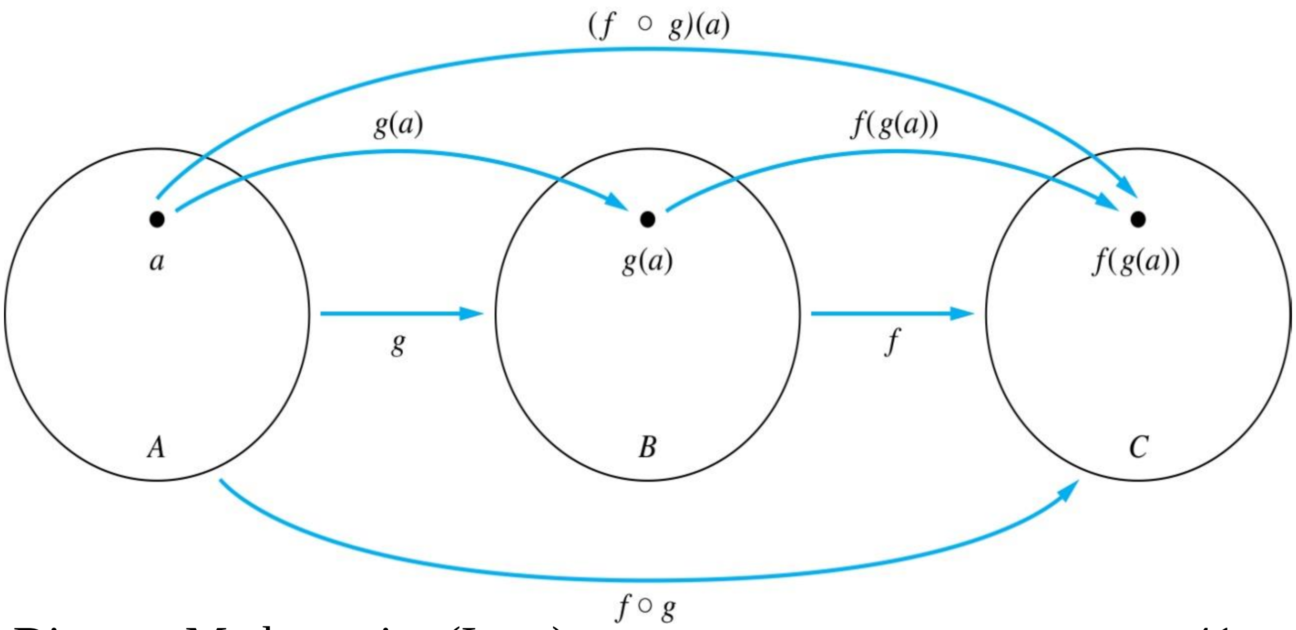


Inverse Functions

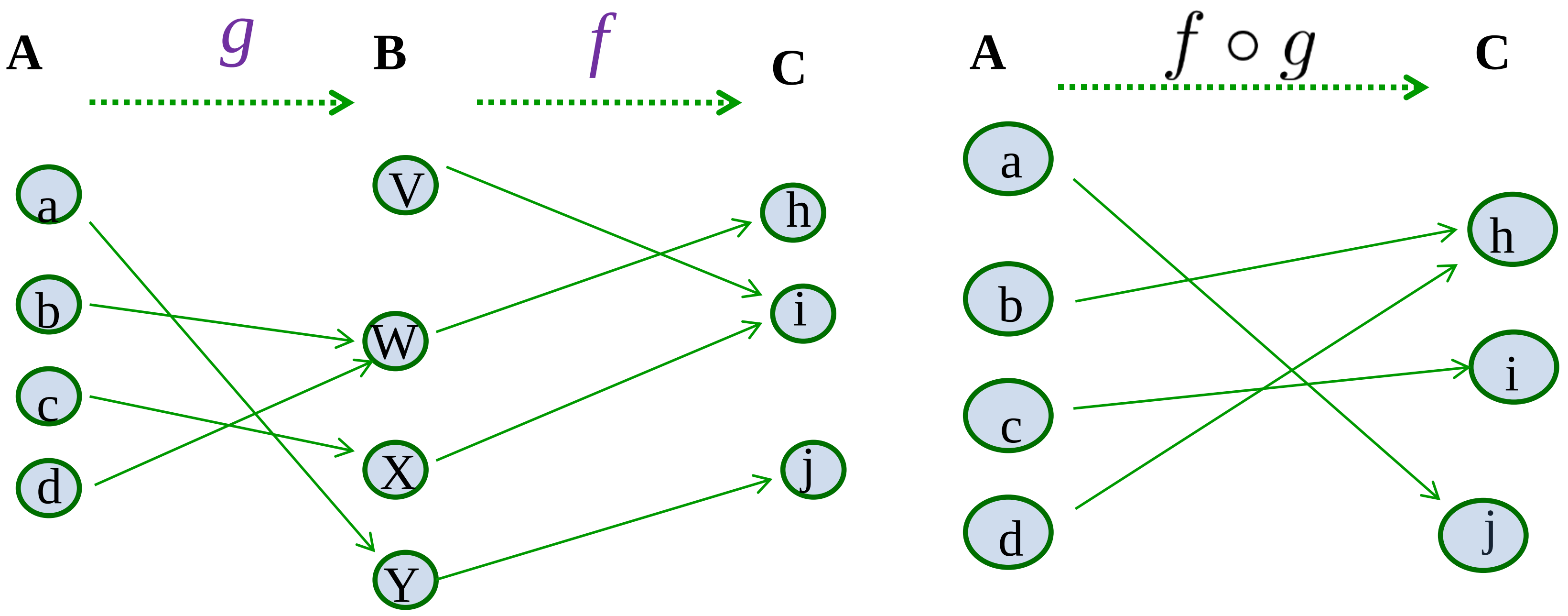


Composition

- **Definition**: Let *f*: *B* → *C*, *g*: *A* → *B*. The *composition of f with g*, denoted $f \circ g$ is the function from *A* to *C* defined by $$f \circ g(x) = f(g(x))$$



Composition



Composition

Example 1: If $f(x) = x^2$ and $g(x) = 2x + 1$ then

$$f(g(x)) = (2x + 1)^2$$

and

$$g(f(x)) = 2x^2 + 1$$

Some Important Functions

- The *floor* function, denoted $f(x) = \lfloor x \rfloor$

is the largest integer less than or equal to *x*.

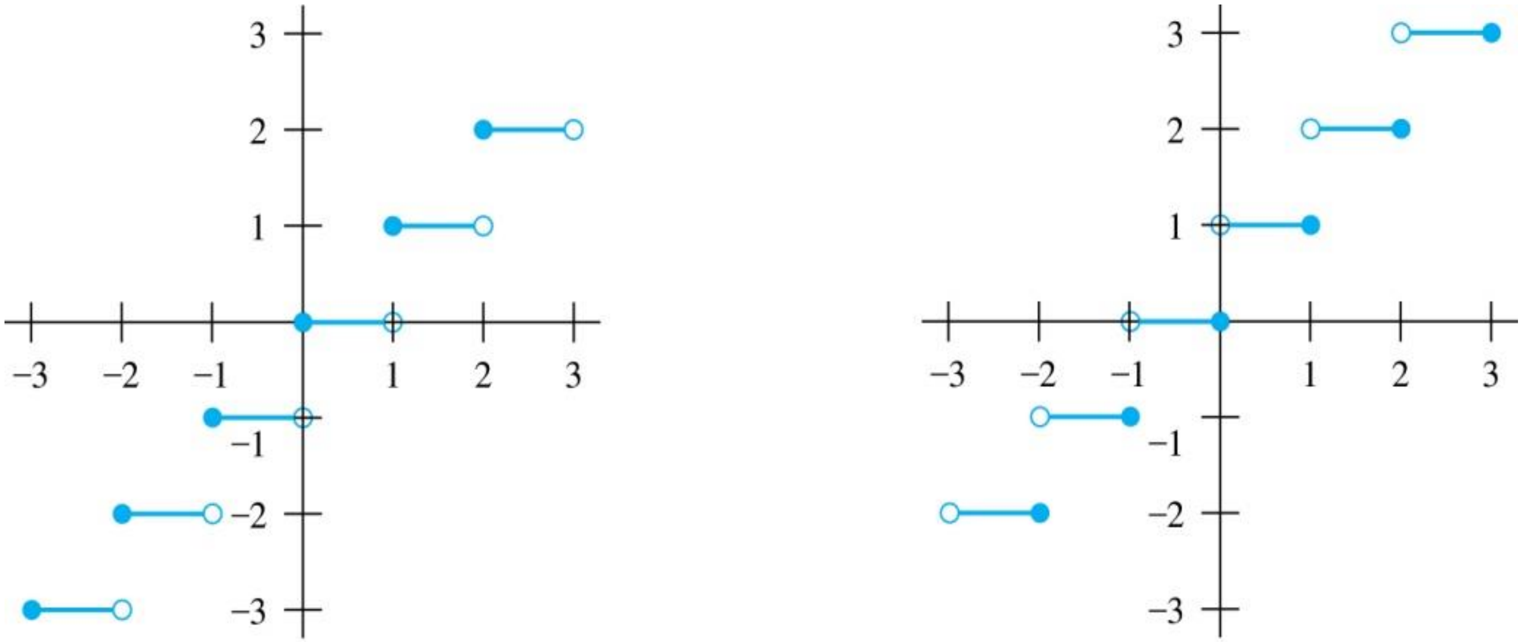
- The *ceiling* function, denoted $f(x) = \lceil x \rceil$

is the smallest integer greater than or equal to *x*

Example: $\lceil 3.5 \rceil = 4$ $\lfloor 3.5 \rfloor = 3$

$\lceil -1.5 \rceil = -1$ $\lfloor -1.5 \rfloor = -2$

Floor and Ceiling Functions



(a) $y = \lfloor x \rfloor$ (b) $y = \lceil x \rceil$

Graph of (a) Floor and (b) Ceiling Functions

Floor and Ceiling Functions

TABLE 1 Useful Properties of the Floor and Ceiling Functions.
(n is an integer, x is a real number)

(1a)	$\lfloor x \rfloor = n$ if and only if $n \le x < n+1$
(1b)	$\lceil x \rceil = n$ if and only if $n-1 < x \le n$
(1c)	$\lfloor x \rfloor = n$ if and only if $x-1 < n \le x$
(1d)	$\lceil x \rceil = n$ if and only if $x \le n < x+1$
(2)	$x-1 < \lfloor x \rfloor \le x \le \lceil x \rceil < x+1$
(3a)	$\lfloor -x \rfloor = -\lceil x \rceil$
(3b)	$\lceil -x \rceil = -\lfloor x \rfloor$
(4a)	$\lfloor x+n \rfloor = \lfloor x \rfloor + n$
(4b)	$\lceil x+n \rceil = \lceil x \rceil + n$

2.4 Sequences and Summations

Def 1: A sequence is a function from a subset of the set of integers to a set S. a_n is a term of the sequence

Ex1: Consider the sequence $\{a_n\}$, where $a_n = 1/n$
The list of the terms of this sequence is

$$1, 1/2, 1/3, 1/4, 1/5, \ldots$$

Ex2: $c_n = 4^n$

The list of the terms of this sequence is $c_0, c_1, c_2, \ldots$

$$= 1, 4, 16, 64, 256, \ldots$$

SOME USEFUL SEQUENCES : $n^2, n^3, n^4, 2^n, 3^n, n!$

Def 2. A Geometric progression is a sequence of the form $a, ar, ar^2, \ldots, ar^n, \ldots$ where the initial term a and the common ratio r are real numbers.

Def 3. An Arithmetic progression is a sequence of the form $a, a+d, a+2d, \ldots, a+nd, \ldots$ where the initial term a and the common difference d are real numbers.

Def 4. A recurrence relation for the sequence $\{a_n\}$ is an equation that expresses an in terms of one or more of the previous terms of the sequence, namely $a_0, a_1, a_2, \ldots, a_{n-1}$, for all integer n with $n \geq n_0$, where a_0 is a non-negative integer.

A sequence is called a solution of a recurrence relation if its term satisfies the recurrence relation.

Ex1: Suppose that ***f*** is defined Recursively by

$$f(0) = 3 \quad \text{and} \quad f(n + 1) = 2f(n) + 3$$

Find f(1), f(2), f(3) and f(4)

$f(1) = 2f(0)+3 = 9$

Def 5. The **Fibonacci sequence**, $f_0, f_1, f_2, \ldots$ is defined by the initial conditions $f_0 = 0$, $f_1 = 1$, and the recurrence relation

$$f_n = f_{n-1} + f_{n-2} \quad \text{for } n = 2, 3, 4, \ldots$$

Ex2: Find the Fibonacci numbers f_2, f_3, f_4, f_5, and f_6

$f_2 = f_1 + f_0 = 1 + 0 = 1$

$f_3 = f_2 + f_1 =$

$f_4 =$

Useful Sequences

TABLE 1 Some Useful Sequences.

nth Term	*First 10 Terms*
n^2	1, 4, 9, 16, 25, 36, 49, 64, 81, 100, . . .
n^3	1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, . . .
n^4	1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, . . .
2^n	2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, . . .
3^n	3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, . . .
$n!$	1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, . . .
f_n	1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, . . .

Summations

$$\sum_{j=m}^{n} a_j = a_m + a_{m+1} + \ldots + a_n$$

EX: $$\sum_{j=1}^{4} j^2 = ?$$

EX: $$\sum_{j=1}^{5} j^2 = \sum_{k=0}^{4} (k+1)^2 = 55$$

EX: $$\sum_{i=1}^{4} \sum_{j=1}^{3} i\,j = ?$$

Summation Formulae

Summation	Closed Form
$\sum_{k=1}^{n} a r^{k}$	$\frac{ar^{n+1} - a}{r - 1}$, $r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^{2}$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^{3}$	$\frac{n^{2}(n+1)^{2}}{4}$

Some Useful Summation Formulae

TABLE 2 Some Useful Summation Formulae.

Sum	*Closed Form*
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\dfrac{ar^{n+1} - a}{r - 1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\dfrac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\dfrac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\dfrac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x\rvert < 1$	$\dfrac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \lvert x\rvert < 1$	$\dfrac{1}{(1-x)^2}$

2.5 Cardinality

Definition: The ***cardinality*** of a set A is equal to the cardinality of a set B, denoted

$$|A| = |B|,$$

if and only if there is a one-to-one correspondence (*i.e.*, a bijection) from A to B.

- If there is a one-to-one function (*i.e.*, an injection) from A to B, the cardinality of A is less than or the same as the cardinality of B and we write $|A| \leq |B|$.
- When $|A| \leq |B|$ and A and B have different cardinality, we say that the cardinality of A is less than the cardinality of B and write $|A| < |B|$.

Cardinality

- **Definition**: A set that is either finite or has the same cardinality as the set of positive integers ($\mathbf{Z}^+$) is called *countable*. A set that is not countable is *uncountable*.

- The set of real numbers **R** is an uncountable set.

Showing that a Set is Countable

- An infinite set is countable if and only if it is possible to list the elements of the set in a sequence (indexed by the positive integers).

- The reason for this is that a one-to-one correspondence f from the set of positive integers to a set S can be expressed in terms of a sequence $a_1, a_2, \ldots, a_n, \ldots$ where

$$a_1 = f(1),\ a_2 = f(2), \ldots, a_n = f(n), \ldots$$

2.6 Matrices

- Definition of a Matrix
- Matrix Arithmetic
- Transposes and Powers of Arithmetic
- Zero-One matrices

Matrices

- Matrices are useful discrete structures that can be used in many ways. In later chapters, we will see matrices used to build models of
  - Transportation systems
  - Communication networks

Matrix

Definition: A *matrix* is a rectangular array of numbers. A matrix with *m* rows and *n* columns is called an *m x n* matrix.

- The plural of matrix is *matrices*.
- A matrix with the same number of rows as columns is called *square*.
- Two matrices are *equal* if they have the same number of rows and the same number of columns and the corresponding entries in every position are equal.

$$3 \times 2 \text{ matrix} \qquad \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 3 \end{bmatrix}$$

Notation

- Let *m* and *n* be positive integers and let

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Matrix Arithmetic: Addition

Defintion: Let $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ be $m \quad n$ matrices. The sum of **A** and **B** = **A** + **B**, is the $m \; x \; n$ matrix that has $a_{ij} + b_{ij}$ as its $(i, j)^{\text{th}}$ element.

$$\mathbf{A} + \mathbf{B} = [a_{ij} + b_{ij}].$$

Example:

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & -3 \\ 3 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 4 & -1 \\ 1 & -3 & 0 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 4 & -2 \\ 3 & -1 & -3 \\ 2 & 5 & 2 \end{bmatrix}$$

matrices of different sizes can not be added.

Matrix Multiplication

Definition: Let **A** be an $n \ x \ k$ matrix and **B** be a $k \ x \ n$ matrix. The *product* of **A** and **B** = **AB**, is the $m \ x \ n$ matrix

if **AB** = [c_{ij}] then $c_{ij} = a_{i1}b_{1\mathrm{j}} + a_{i2}b_{2j} + \ldots + a_{kj}b_{2j}$.

Example:

$$\begin{bmatrix} \boxed{1 \quad 0 \quad 4} \\ 2 \quad 1 \quad 1 \\ 3 \quad 1 \quad 0 \\ 0 \quad 2 \quad 2 \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} 2 \\ 1 \\ 3 \end{matrix}} & \begin{matrix} 4 \\ 1 \\ 0 \end{matrix} \end{bmatrix} = \begin{bmatrix} 14 & 4 \\ 8 & 9 \\ 7 & 13 \\ 8 & 2 \end{bmatrix}$$

The product of two matrices is <u>undefined</u> when the number of columns in the first matrix is <u>not</u> the same as the umber of rows in the second**.**

Illustration of Matrix Multiplication

- The Product of **A** = [a_{ij}] and **B** = [b_{ij}]

$$
\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2k} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{i1} & a_{i2} & \dots & a_{ik} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{m1} & a_{m2} & \dots & a_{mk} \end{bmatrix}
$$

$$
\mathbf{B} = \begin{bmatrix} b_{11} & a_{12} & \dots & b_{1j} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2j} & \dots & b_{2n} \\ \cdot & \cdot & & \cdot & & \\ \cdot & \cdot & & \cdot & & \\ b_{k1} & b_{k2} & \dots & b_{kj} & \dots & b_{kn} \end{bmatrix}
$$

$$
\mathbf{AB} = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & \boxed{c_{ij}} & \cdot \\ \cdot & \cdot & & \cdot \\ c_{m1} & c_{m2} & \dots & c_{mn} \end{bmatrix}
$$

$$
c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj}
$$

Matrix Multiplication is not Commutative

Example: Let

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

Does ***AB*** = ***BA***?

Solution:

$$AB = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix} \qquad BA = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}$$

AB $\neq$ ***BA***

Identity Matrix and Powers of Matrices

Definition: The *identity matrix of order n* is the *m x n* matrix $\mathbf{I}_n = [\delta_{ij}]$, where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

$$\mathbf{I_n} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

$$\mathbf{A}\mathbf{I}_n = \mathbf{I}_m\mathbf{A} = \mathbf{A}$$

when **A** is an *m* x *n* matrix

Powers of square matrices can be defined. When A is an $n \times n$ matrix, we have:

$$\mathbf{A}^0 = \mathbf{I}_n \qquad \mathbf{A}^r = \underbrace{\mathbf{AAA}\ldots\mathbf{A}}_{\text{r times}}$$

Transposes of Matrices

Definition: Let **A** = [a_{ij}] be an m x n matrix. The transpose of **A**, denoted by $\mathbf{A}^t$,is the n x m matrix obtained by interchanging the rows and columns of **A**.

If $\mathbf{A}^t = [b_{ij}]$, then $b_{ij} = a_{ji}$ for i =1, 2,…, n

and j = 1, 2, ..., m

The transpose of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ is the matrix $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$.

Transposes of Matrices

Definition: A square matrix **A** is called **symmetric** if **A** = **A**^{t}. Thus **A** = [a_{ij}] is symmetric if $a_{ij} = a_{ji}$ for i and j with $1 \leq i \leq n$ and $1 \leq j \leq n$.

$$\text{The matrix } \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \text{ is square.}$$

Square matrices do not change when their rows and columns are interchanged.

Zero-One Matrices

Definition: A matrix all of whose entries are either 0 or 1 is called a ***zero-one matrix***. These will be used in Chapters 9 and 10. (Relations and Graphs)

Algorithms operating on discrete structures represented by zero-one matrices are based on Boolean arithmetic defined by the following Boolean operations:

$$b_1 \wedge b_2 = \begin{cases} 1 & \text{if } b_1 = b_2 = 1 \\ 0 & \text{otherwise} \end{cases} \qquad b_1 \vee b_2 = \begin{cases} 1 & \text{if } b_1 = 1 \text{ or } b_2 = 1 \\ 0 & \text{otherwise} \end{cases}$$

Zero-One Matrices

Definition: Let $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ be an $m \times n$ zero-one matrices.

The *join* of **A** and **B** is the zero-one matrix with (i,j)th entry $a_{ij} \vee b_{ij}$. The join of **A** and **B** is denoted by **A** ∨ **B**.

Example: Find the join of the zero-one matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

Solution: The **join** of **A** and **B** is

$$\mathbf{A} \vee \mathbf{B} = \begin{bmatrix} 1 \vee 0 & 0 \vee 1 & 1 \vee 0 \\ 0 \vee 1 & 1 \vee 1 & 0 \vee 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Joins and Meets of Zero-One Matrices

The meet of **A** and **B** is the zero-one matrix with (i,j)th entry $a_{ij} \wedge b_{ij}$. The **meet** of **A** and **B** is denoted by **A ∧ B**.

Example: Find the meet of the zero-one matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

Solution: The **meet** of **A** and **B** is

$$\mathbf{A} \wedge \mathbf{B} = \begin{bmatrix} 1 \wedge 0 & 0 \wedge 1 & 1 \wedge 0 \\ 0 \wedge 1 & 1 \wedge 1 & 0 \wedge 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Boolean Product of Zero-One Matrices

Definition: Let $\mathbf{A} = [a_{ij}]$ be an $m \times k$ zero-one matrix and $\mathbf{B} = [b_{ij}]$ be a $k \times n$ zero-one matrix. The *Boolean product* of **A** and **B**, denoted by $\mathbf{A} \odot \mathbf{B}$, is the $m \times n$ zero-one matrix with (i, j)th entry

$$c_{ij} = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2j}) \vee \ldots \vee (a_{ik} \wedge b_{kj})$$

Example: Find the Boolean product of **A** and **B**, where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Boolean Product of Zero-One Matrices

$$\mathbf{A} = \begin{bmatrix} \boxed{1 \quad 0} \\ 0 \quad 1 \\ 1 \quad 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Solution: The Boolean product **A** ⊙ **B** is given by

$$\mathbf{A} \odot \mathbf{B} = \begin{bmatrix} (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \\ (0 \wedge 1) \vee (1 \wedge 0) & (0 \wedge 1) \vee (1 \wedge 1) & (0 \wedge 0) \vee (1 \wedge 1) \\ (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \\ 0 \vee 0 & 0 \vee 1 & 0 \vee 1 \\ 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Boolean Powers of Zero-One Matrices

Definition: Let **A** be a square zero-one matrix and let *r* be a positive integer. The r^{th} Boolean power of **A** is the Boolean product of *r* factors of **A**, denoted by $\mathbf{A}^{[r]}$.

$$\mathbf{A}^{[r]} = \underbrace{\mathbf{A} \odot \mathbf{A} \odot \ldots \odot \mathbf{A}}_{r \text{ times}}.$$

Boolean Powers of Zero-One Matrices

Example: Let

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

Find $\mathbf{A}^n$ for all positive integers *n*.

Solution:

$$\mathbf{A}^{[2]} = \mathbf{A} \odot \mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \qquad \mathbf{A}^{[3]} = \mathbf{A}^{[2]} \odot \mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\mathbf{A}^{[4]} = \mathbf{A}^{[3]} \odot \mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\mathbf{A}^{[5]} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \mathbf{A}^{[\mathbf{n}]} = \mathbf{A}^{\mathbf{5}} \quad \text{for all positive integers } n \text{ with } n \geq 5.$$