Functions

Definition 1

A **function** is a rule that takes certain numbers as inputs and assigns to exactly one output number. The set of all input numbers is called the **domain** of the function and the set of resulting output numbers is called the **range** of the function.

Note: A function can be considered as a set of ordered pairs (x, y).

Notations:

Let f be a function from A to B ($f : A \to B$)

- D_f represents domain of function f
- R_f represents range of function f
- Image of x is y since $f(x) = y$

$f : A \to B$ is called a function from A **onto** B if $R_f = B$

Normally, we may present a function via four common ways:

1) Description (words)
2) Numeric (tables)
3) Visual (graphs)
4) Algebra (formulas)

Example 1

Consider a set $\{(-3,1),(0,2),(3,-1),(5,4)\}$. Is it a function?

Domain:

Range:

Example 2

Let $f=\{(x,y): x,y\in\mathbb{R}$ and $y=x^2-2\}$.

So $D_f=\mathbb{R}$ and $R_f=[-2,+\infty)$. We usually write $f(x)=x^2-2$.

The values of f at some points are as follow.

$f(0)=(0)^2-2=-2$

$f(-1)=(-1)^2-2=-1$

$f(\sqrt{3})=(\sqrt{3})^2-2=1$

$f(c)=c^2-2$

$f(x+h)=(x+h)^2-2=x^2+2hx+h^2-2$

$f(x+h)-f(x)=(x^2+2hx+h^2-2)-(x^2-2)=2hx+h^2$

and $\dfrac{f(x+h)-f(x)}{h}=2x+h, \quad h\neq 0$

Example 3

Let $f=\left\{(x,y): x^2+y^2=1^2\right\}$. Is f a function?

Example 4 Find the domain of the following functions.

(1) $f(x)=\dfrac{4}{x-1}$

(2) $f(x)=\dfrac{x}{x^2-9}$

(3) $f(x)=\dfrac{\sqrt{4-x}}{x}$

(4) $f(x)=\sqrt{4-x^2}$

Example 5

1) $y=\sin x$ has the set of all real numbers as its domain and the interval $[-1,1]$ as its range.

2) $y=\sqrt{x^2+4}$ has the set of all real numbers as a domain and the interval $[2,+\infty)$ as its range.

Example 6

$$h(x)=\begin{cases}\dfrac{2x^2-9x+4}{x-4} & ,x\neq 4\\ 5 & ,x=4\end{cases} \quad \text{or} \quad h(x)=\begin{cases}2x-1 & ,x\neq 4\\ 5 & ,x=4\end{cases}$$

$D_f =$

$R_f =$

Definition 2 The function f equals to the function g if and only if

1. $D_f = D_g$
2. $f(x)=g(x)$ for all $x\in D_f$.

Example 7 Check if the following functions are equal.

1) Let $f(x)=\dfrac{\sqrt{2+x}-\sqrt{2}}{x}$ and $g(x)=\dfrac{1}{\sqrt{2+x}+\sqrt{2}}$

2) Let $f(x)=x+3$ and $g(x)=\begin{cases}\dfrac{2x^2+7x+3}{2x+1} & ,x\neq -\dfrac{1}{2}\\ \dfrac{5}{2} & ,x=-\dfrac{1}{2}\end{cases}$

Definition 3

Let f and g be functions and $R_g \cap D_f \neq \varnothing$.

A composite function of f and g (denoted by $f \circ g$) is a function $(f \circ g)(x) = f(g(x))$ whose domain is $\{x : x \in D_g \text{ and } g(x) \in D_f\}$.

Example 8 Let $f(x) = \sqrt{x-3}$ and $g(x) = 2x-1$

a) Let $F = f \circ g$ Find $F(x)$ and domain of F

b) Let $G = g \circ f$ Find $G(x)$ and domain of G

c) Let $H = f \circ f$ Find $H(x)$ and domain of H

Solutions

a) The domain of g is $(-\infty, \infty)$ and domain of f is $[3, \infty)$.

To find the domain of $F = f \circ g$, we consider only x where $g(x)$ is in domain of f. That is, $2x - 1 \geq 3$.

Thus domain of F is a set of x where $x \geq 2$ i.e. $[2, \infty)$.

Then, the function $F = f \circ g$ can be found by

$$F(x) = f \circ g(x) = f(g(x)) = f(2x-1) = \sqrt{(2x-1)-3} = \sqrt{2x-4}.$$

Symmetry

Definition 4 Let f be a function.

a. If $f(-x) = -f(x)$, f is called an **odd function** whose graph is symmetric about the origin.

b. If $f(-x) = f(x)$, f is called an **even function** whose graph is symmetric about the y-axis.

Example 9

a) Let $f(x) = x^3$.

Consider $f(-x) = (-x)^3 = -x^3 = -f(x)$.

Thus f is an odd function and it graph is shown in figure 1 below.

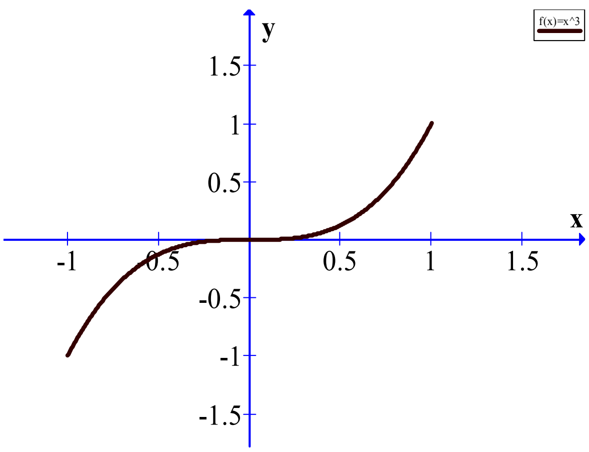


Figure 1

b) Let $f(x)=3x^2-1$

Consider $f(-x)=3(-x)^2-1=3x^2-1=f(x)$.

Thus f is an even function whose graph shown in Figure 2.

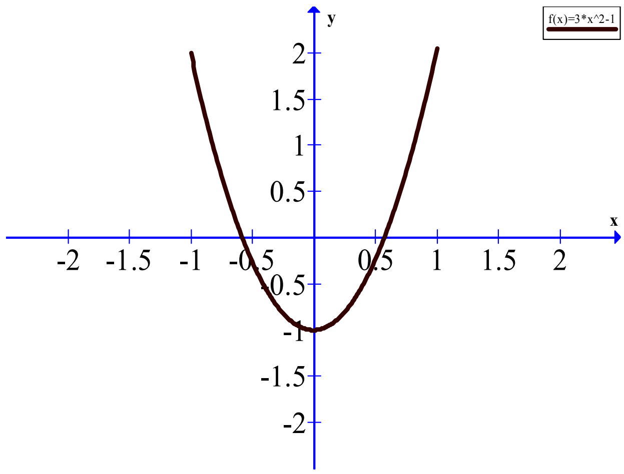


Figure 2

Inverse function

Definition 5 The function f is called a **one-to-one** function if and only if for all x, y, z if (x, y) and $(z, y) \in f$ then $x = z$.

Definition 6 Let f be a one-to-one function from A onto B.

An inverse function of f is defined by $f^{-1}=\{(b,a) \mid (a,b)\in f\}$ which is also a one-to-one function from B to A.

Remark Graphs of f and f^{-1} are symmetric about the line $y = x$ as shown in Figure 3 below.

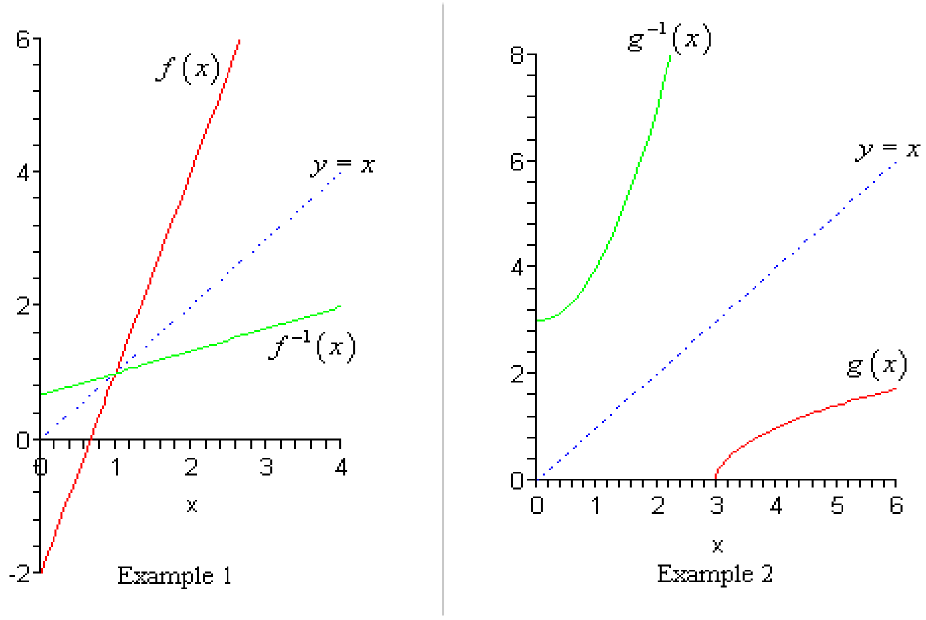


Figure 3

Example10 Find an inverse of f where $f(x) = x^3 - 1$.

Solution From $y = f(x) = x^3 - 1$ (i.e. $x = \sqrt[3]{y+1}$), we have that

$$f^{-1} = \left\{(y,x) \middle|\ y = x^3 - 1\right\} \text{ or } f^{-1} = \left\{(x,y) \middle|\ y = \sqrt[3]{x+1}\right\}$$

We normally write $f^{-1}(x) = \sqrt[3]{x+1}$ so that we can easily draw graphs of both functions f and f^{-1} as follows

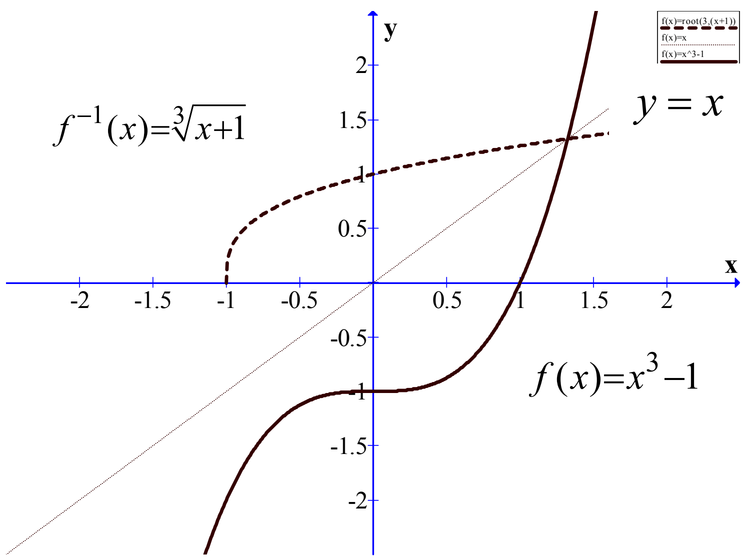


Figure 4

Other Interesting Functions

All functions here will be useful in the next sections.

Algebraic Function

a. Polynomial Functions are functions of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

where a_i is a real number for each $i = 0,1,2,\ldots,n$ and

n is a non-negative integer.

If n is the largest number such that $a_n \neq 0$, we call f a polynomial function of degree n such as $f(x) = 3x^3 - 5x^2 + x + 4$ is a polynomial function of degree 3.

Normally, if there is nothing specific, the domain of a polynomial function is the set of all real numbers.

b. Rational Functions are functions formed by a ratio between two polynomial functions.

$$f(x) = \frac{a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0}{b_m x^m + b_{m-1} x^{m-1} + \ldots + b_0},$$

Note that, if there is nothing specific, the domain of this rational function is $\left\{ x \in \mathbb{R} \;\middle|\; b_m x^m + b_{m-1} x^{m-1} + \ldots + b_0 \neq 0 \right\}$

Example 11 Let $y = f(x) = \dfrac{x^2 + x}{x}$

Rewrite function f : $f(x) = x + 1$ where $x \neq 0$

Thus graph of $f(x)$ is the graph of $y = x+1$, but undefined at $x = 0$

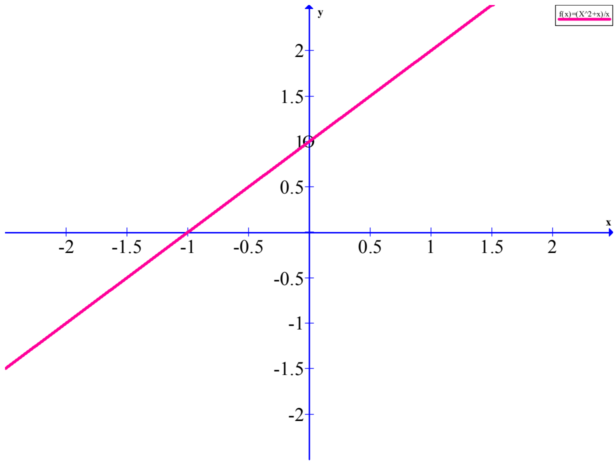


Figure 5

c. Functions of the form $\sqrt[n]{f(x)}$; $n \in \mathbb{N}$ where the function $f(x)$ is either a polynomial or a rational function.

The domain of this type of functions can be considered as follows

<u>Case1</u> n is odd

The domain of $\sqrt[n]{f(x)}$ is exactly the domain D_f of $f(x)$

<u>Case2</u> n is even

The domain of $\sqrt[n]{f(x)}$ is $D_f \cap \{x \mid f(x) \geq 0\}$

d. Functions formed by summation, multiplication and division of functions in part a. to c.

Below are some examples of functions in part c. and d.

1) $f(x) = x^{\frac{2}{3}}$ 2) $f(x) = \sqrt[4]{\dfrac{x}{x+1}}$

3) $f(x) = \dfrac{\sqrt{x}}{\sqrt{x}+1}$

<u>Transcendental Functions</u>

a. Exponential Functions are functions of the form

$$y = a^x, \quad \text{where} \quad a > 0 \text{ and } a \neq 1$$

When $a > 1$, its graph can be shown in Figure 6 below.

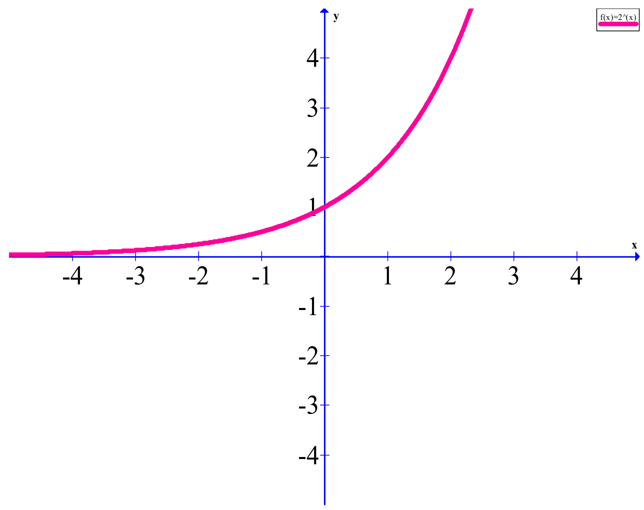


Figure 6

When $0 < a < 1$, its graph can be shown in figure 7 below

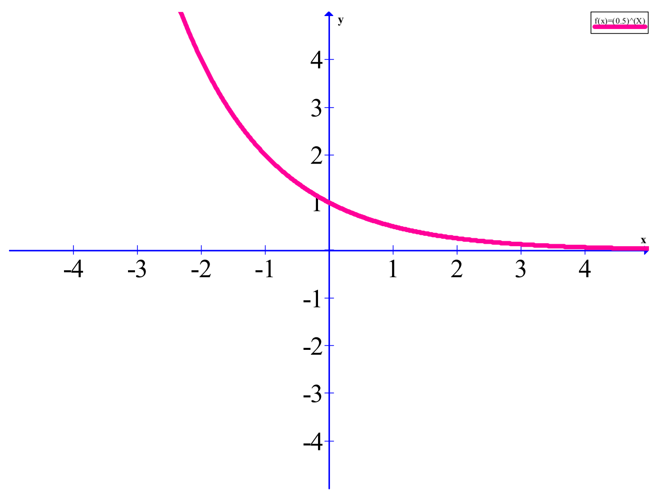


Figure 7

b. Logarithmic Function

Logarithmic function is an inverse of exponential function. Given an exponential function $y = a^x$. Then its inverse function is $x = a^y$ or we can rewrite it as $y = \log_a x$.

If $y = \log_a x$, $a > 1$, then its graph is shown in Figure 8.

If $y = \log_a x$, $0 < a < 1$, then its graph is shown in Figure 9.

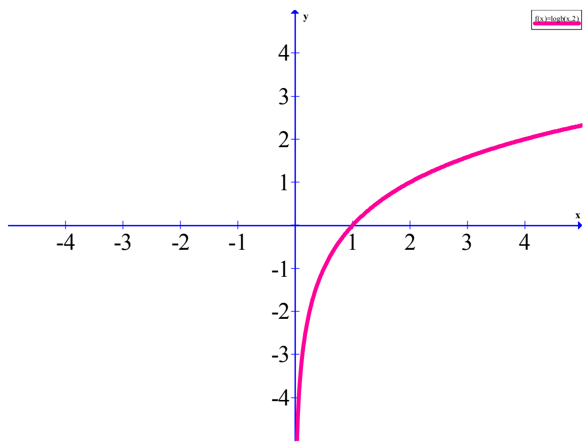


Figure 8

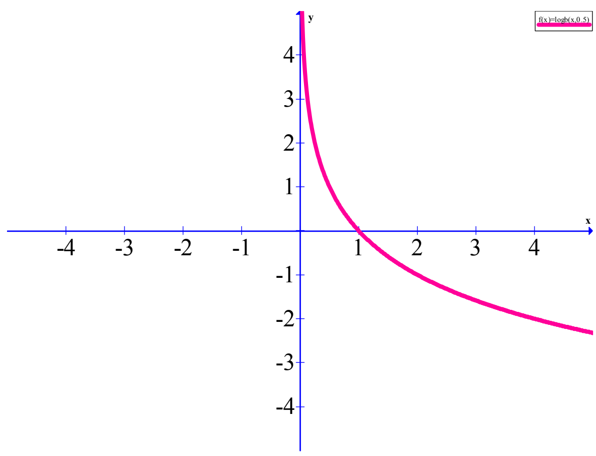


Figure 9

Some facts about logarithmic functions

1. Domain of a logarithmic function is $\{x : x > 0\}$

   and its range is $\{y : y \in \mathbb{R}\}$

2. A logarithmic function is a one-to-one function.

3. $\log_a 1 = 0$

4. Graph of $y = \log_a x$ is a reflection of the graph $y = a^x$ across the line $y = x$.

Remark: When $a = e$ (where $e = 2.71818...$= natural number) $y = e^x$ has the inverse $y = \log_e x$ which is normally written as $y = \ln x$ and it is called a natural logarithm.

The properties of $y = e^x$ and $y = \ln x$ are the same as of the following properties of $y = a^x$ and $y = \log_a x$ $(a > 0)$, respectively

Properties of logarithmic and exponential functions

Given positive numbers a, b where $a \neq 1, b \neq 1$ and $x, y \in R$

1. $a^x \cdot a^y = a^{x+y}$

2. $\dfrac{a^x}{a^y} = a^{x-y}$

3. $a^x \cdot b^x = (ab)^x$ and $\dfrac{a^x}{b^x} = \left[\dfrac{a}{b}\right]^x$

4. $\left(a^x\right)^y = a^{xy}$

5. $a^{-x} = \dfrac{1}{a^x}$

6. If $x > 0, y > 0$, then $\log_a(xy) = \log_a x + \log_a y$

$$\log_a(\frac{x}{y}) = \log_a x - \log_a y$$

7. $\log_a x^r = r\log_a x$

8. $\log_a x = \dfrac{\log_b x}{\log_b a}$

9. $\log_a a = 1$

10. $\ln e^x = x$ and $e^{\ln x} = x \quad , x > 0$

11. $a^x = y$ and $x = \log_a y \quad , y > 0$

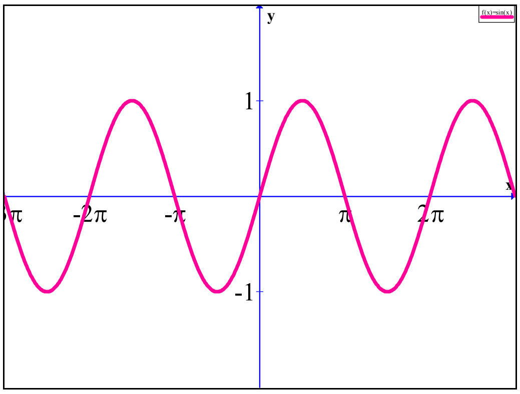
Example 12 Find the values of x

(a) $4\cdot 3^x = 8\cdot 6^x$ (b) $7^{x+2} = e^{17x}$

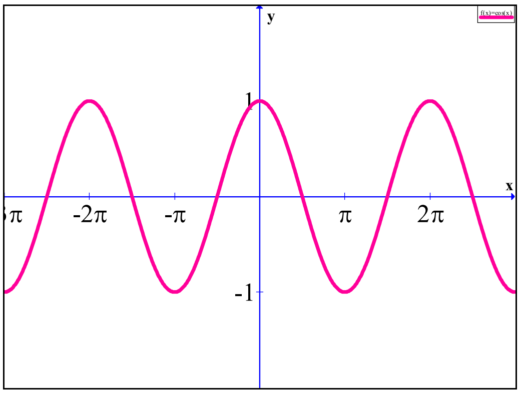
c. Trigonometric Function

$y = \sin x$ $\qquad$ $y = \cos x$ $\qquad$ $y = \tan x = \dfrac{\sin x}{\cos x}$

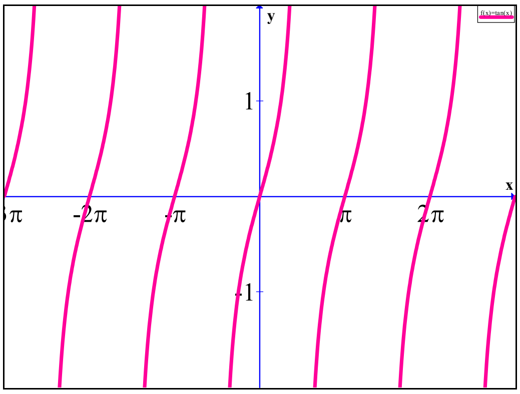
$y = \csc x = \dfrac{1}{\sin x}$ $\qquad$ $y = \sec x = \dfrac{1}{\cos x}$ $\qquad$ $y = \cot x = \dfrac{\cos x}{sinx}$



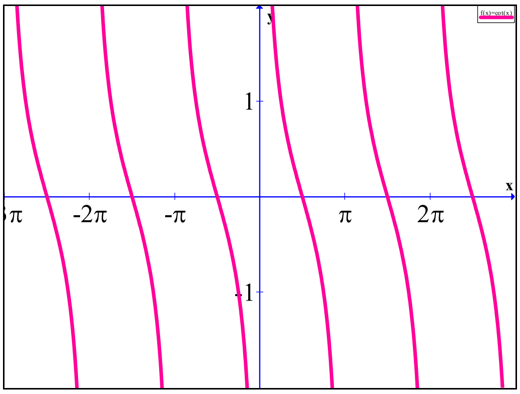
Graph of $y = \sin x$



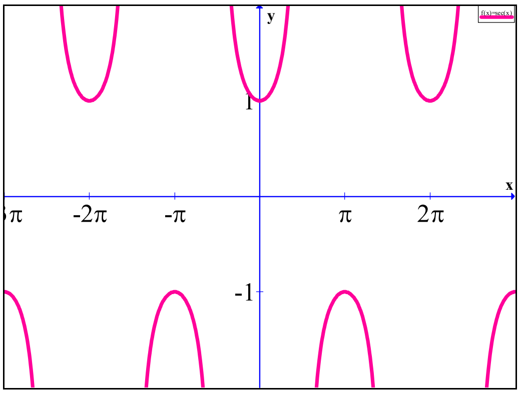
Graph of $y = \cos x$



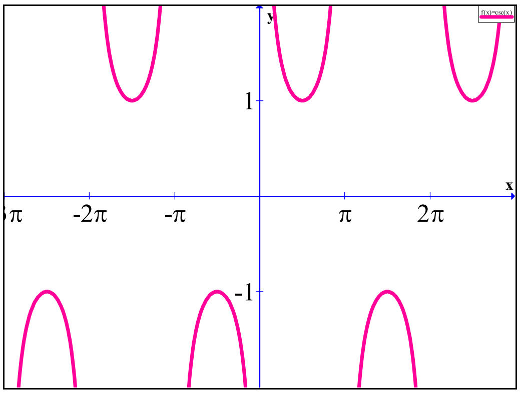
Graph of $y = \tan x$



Graph of $y = \cot x$



Graph of $y = \sec x$

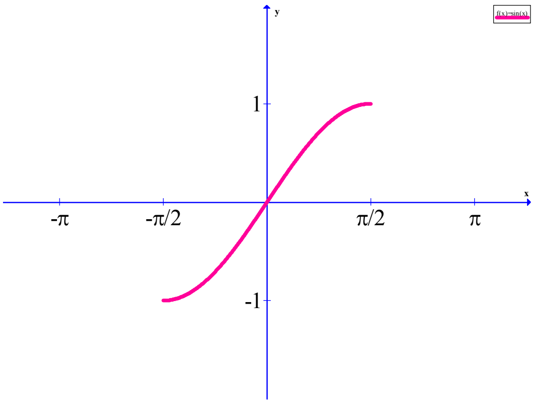


Graph of $y = \csc x$

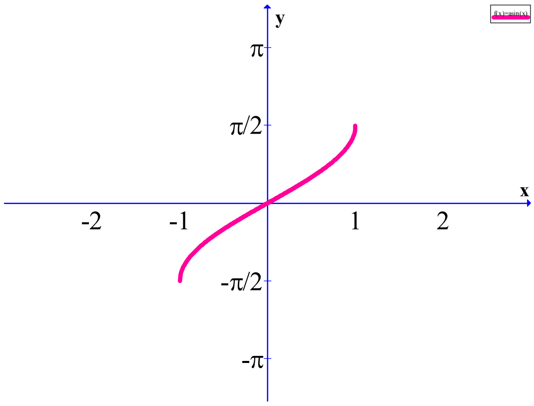
Normally, the inverse of a trigonometric function is not a function since each trigonometric function is not one-to-one. However, if we restrict the domain, we can make a one-to-one trigonometric function and define an inverse function as follows.

1) Restrict the domain of $y = \sin x$ to $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

Its inverse function is $y = \arcsin x$.



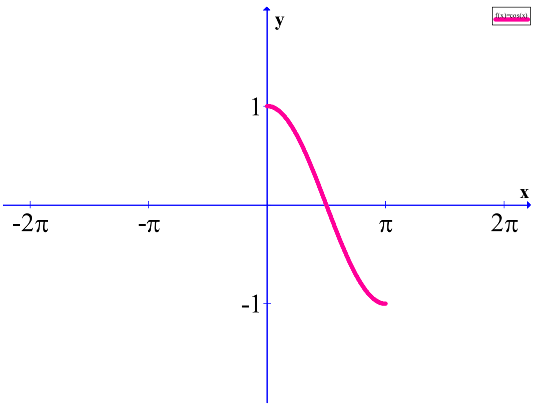
$y = \sin x$



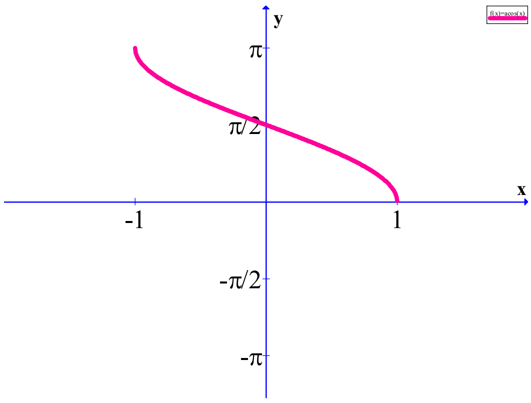
$y = \arcsin x$

2) Restrict domain of $y = \cos x$ to $[0, \pi]$

Its inverse function is $y = \arccos x$.



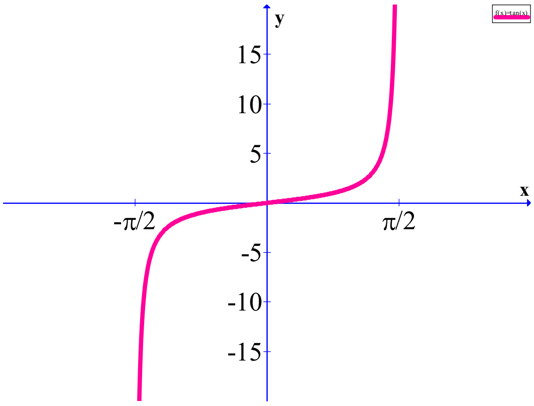
$y = \cos x$



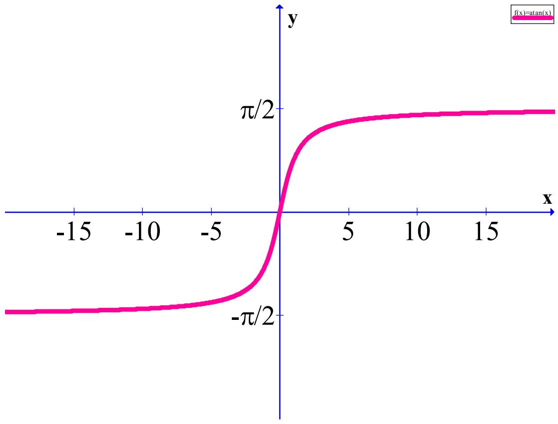
$y = \arccos x$

3) Restrict domain of $y = \tan x$ to $\left(\frac{-\pi}{2}, \frac{\pi}{2} \right)$

Its inverse function is $y = \arctan x$.



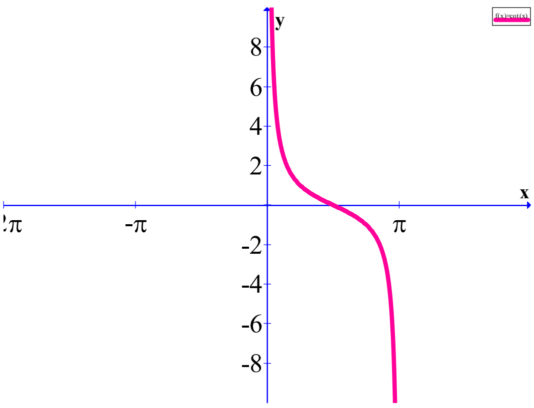
$y = \tan x$



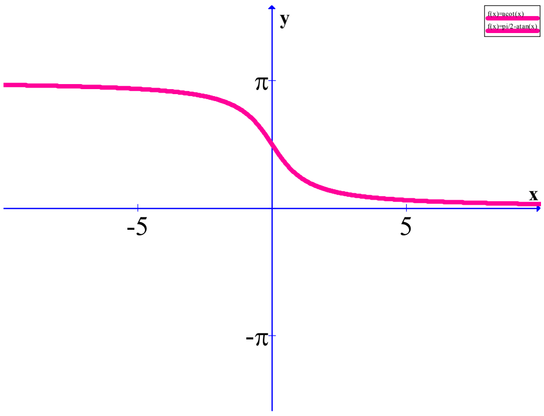
$y = \arctan x$

4) Restrict domain of $y = \cot x$ to $(0, \pi)$

Its inverse function is $y = \operatorname{arccot} x$.



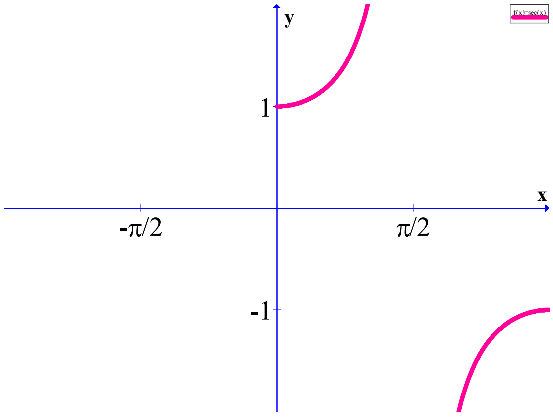
$y = \cot x$



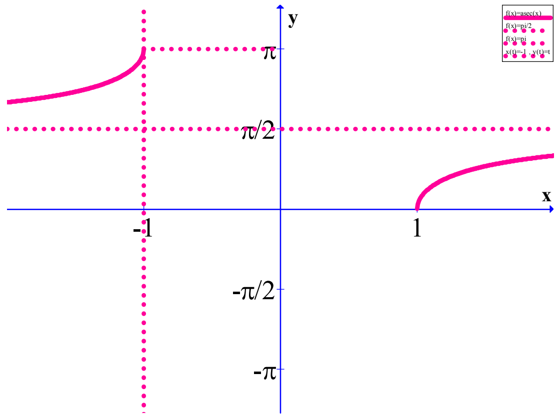
$y = \operatorname{arccot} x$

5) Restrict domain of $y=\sec x$ to $[0,\frac{\pi}{2})\cup(\frac{\pi}{2},\pi]$

Its inverse function is $y=\operatorname{arcsec} x$.



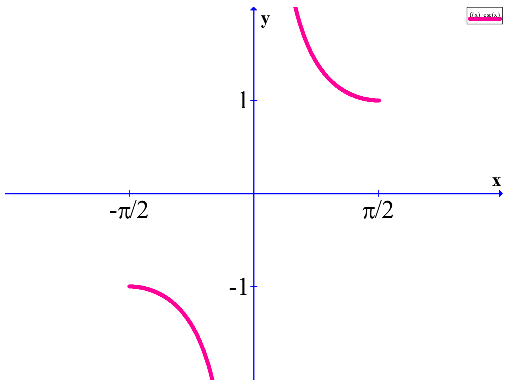
$y=\sec x$



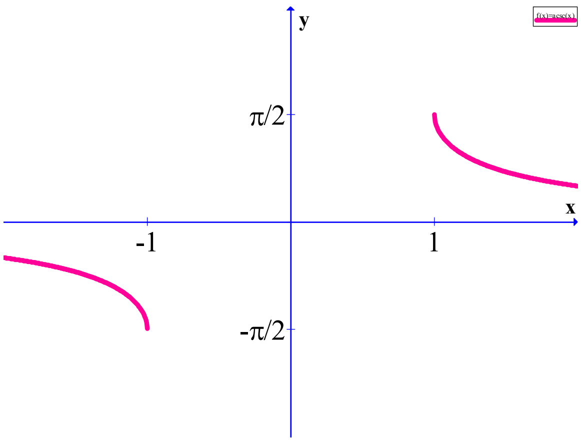
$y=\operatorname{arcsec} x$

6) Restrict domain of $y=\csc x$ to $[-\frac{\pi}{2},0)\cup(0,\frac{\pi}{2}]$

Its inverse function is $y=\operatorname{arccsc} x$.



$y=\csc x$



$y=\operatorname{arccsc} x$

Exercises on Functions

1. Determine if the following are functions. Locate domain and range.

(a) $\{(1,3),(2,3),(3,4),(4,5)\}$

(b) $\{(x,y): y > 4x-1\}$

(c) $y = x^4 - 1$

(d) Let

x	y
15	2
2	13
13	13
5	3

2. Determine if each following function is either even or odd or neither.

(a) $f(x) = x^3 + 2x$

(b) $g(x) = \dfrac{8}{x^2 - 2}$

(c) $h(x) = 3x|x|$

(d) $k(x) = x + |x|$

3. What is the difference of $\sin x^2$, $\sin^2 x$ and $\sin(\sin x)$? Show in terms of composite functions.

Answers to Function Exercises

1. (a) yes $D = \{1,2,3,4\}$ and $R = \{3,4,5\}$

   (b) no $D = R =$ all real numbers

   (c) yes $D = \mathbb{R}$ and $R = \{y : y \geq -1\}$

   (d) yes $D = \{2,5,13,15\}$ and $R = \{2,3,13\}$

2. (a) odd (b) even

   (c) odd (d) neither

3. Let $f(x) = \sin x$ and $g(x) = x^2$

$\sin x^2 = f(g(x))$, $\sin^2 x = g(f(x))$, while

$\sin(\sin x) = f(f(x))$