MTH 101

Mathematics I

Module 4

Multivariable Functions

(Functions of several variables)

1. Functions of several variables

For examples:

The area of a triangle has the formula: $A=\frac{1}{2}bh$

where b = length of the base and h = height .

The volume of a rectangular box: $V = Lwh$

where L = length , w = width , h = height .

The arithmetic mean: $\bar{x} = \frac{1}{n}\left(x_1+x_2+\ldots+x_n\right)$

where $x_1, x_2, \ldots, x_n$ are n real numbers.

From the above example , we have that

A is a function of 2 variables: b and h,

V is a function of 3 variables: L, w, h,

$\bar{x}$ is a function of n variables: $x_1, x_2, \ldots, x_n$.

Notation: $z = f(x,y)$

It means that z is a function of x and y. The variables x and y are called *independent variables* or inputs while the variable z is called a *dependent variable* or output. Analogously, $w = f(x,y,z)$ means w is a function of $x,\ y$ and z. Also, $u = f(x_1, x_2, ..., x_n)$ refers to u as a function in terms of variables $x_1, x_2, ..., x_n$.

Definition 1: A function f of two variables x and y is the assignment of each point (x,y) in its domain $D \subseteq \mathbb{R}^2$ to some real number $f(x,y)$.

Definition 2: A function f of three variables x, y and z is the assignment of a point (x,y,z) in its domain $D \subseteq \mathbb{R}^3$ to some real number $f(x,y,z)$.

Example 1 Find the domain and draw the graph of the domain of the functions below:

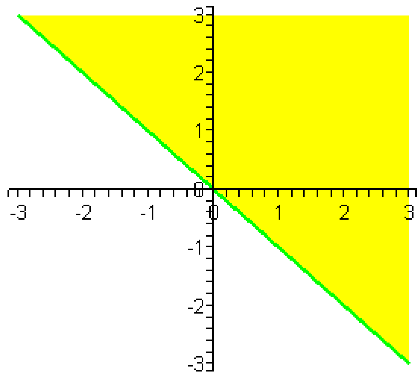
1.1 $f(x,y) = \sqrt{x+y}$.

1.2 $f(x,y) = \sqrt{x}+\sqrt{y}$.

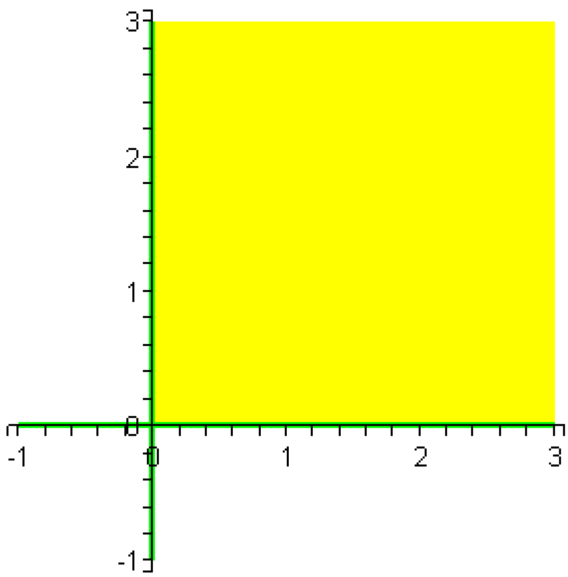
1.3 $f(x,y) = \ln(9-x^2-9y^2)$.

Solution

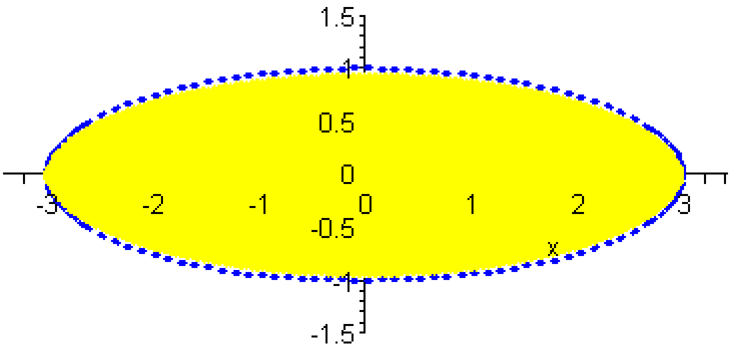
1.1



1.2



1.3



Example 2 Let $f(x,y,z) = \dfrac{1}{\sqrt{x^2+y^2+z^2-16}}$. Find its domain.

Solution

Three Dimensional Spaces

1. Rectangular Coordinate System in Three-dimensional Space

It consists of three orthogonal axes called x-axis, y-axis and z-axis. The intersection point of all axes is called the *origin*.

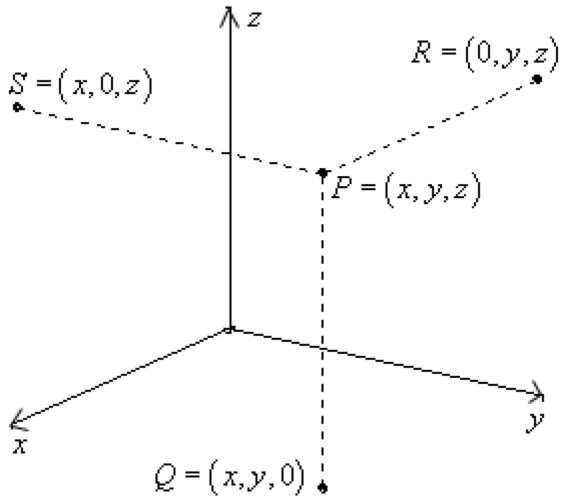


Figure 1 Three dimensional space

Every pair of (x, y, z) forms a coordinate plane: xy-plane, xz-plane and yz-plane. These three planes divide the space into eight parts. Each of which is called *octant*. The first octant is the octant that all x, y, z are positive. The other octants show different signs of x, y, z.

The rectangular coordinate system (x, y, z) refers to the distance from the point to each plane. In particular, the x-coordinate measures the distance from the point to yz-plane while the y-coordinate and z

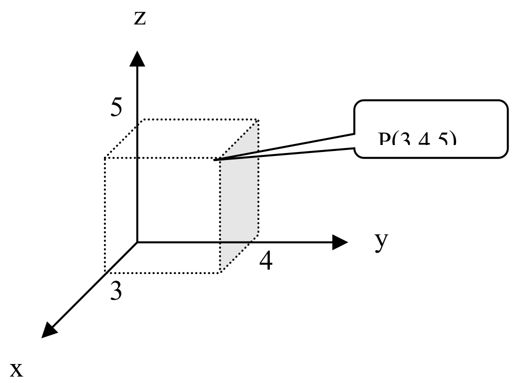
-coordinate indicate the distance from the point to the xz-plane and the xy-plane, respectively.

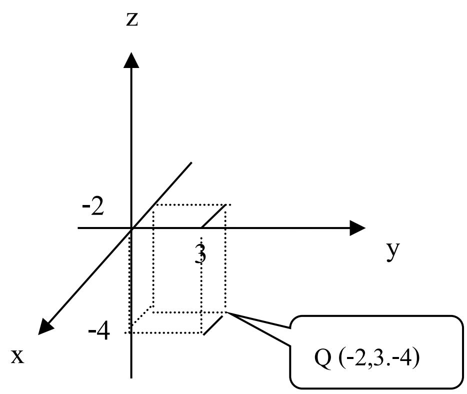
Example1 Locate the following points in the three-dim space.

a. $P(3,4,5)$

b. $Q(-2,3,-4)$

Solution

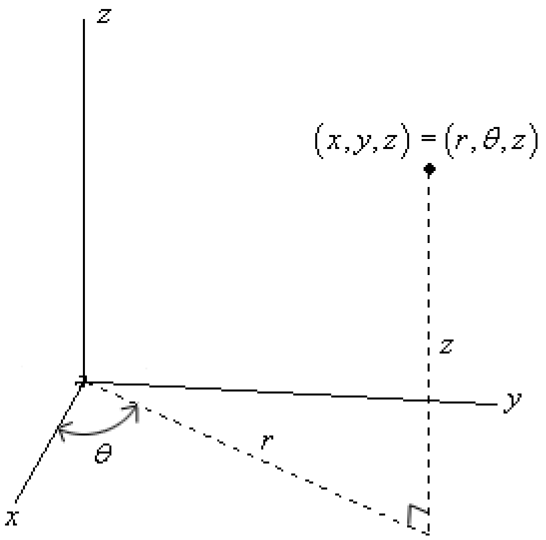




2. Cylindrical and Spherical Coordinates

Beside the rectangular coordinate system, we can locate a point in three dimensional space by cylindrical and/or spherical coordinates as follow:

- Cylindrical coordinate, we use the coordinate (r, θ, z).
- Spherical coordinate, we use the coordinate (ρ, θ, ϕ).



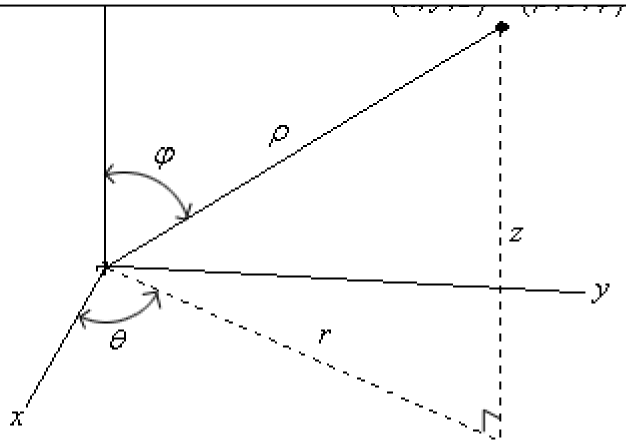
Cylindrical coordinate

Relationship between rectangular and cylindrical coordinates:

$$x = r\cos\theta,\ \ y = r\sin\theta,\ \ z = z$$

$$x^2 + y^2 = r^2\ ,\ \tan\theta = \frac{y}{x}$$

where $\quad r \geq 0,\ \ 0 \leq \theta \leq 2\pi$.



Spherical coordinate

The relationship between rectangular and spherical coordinates:

$$x=\rho\sin\phi\cos\theta,\ \ y=\rho\sin\phi\sin\theta,\ \ z=\rho\cos\phi$$

$$x^2+y^2+z^2=\rho^2\ ,\ \tan\theta=\frac{y}{x},\ \cos\phi=\frac{z}{\sqrt{x^2+y^2+z^2}}$$

where $\rho\geq 0,\quad 0\leq\theta\leq 2\pi,\quad 0\leq\phi\leq\pi$.

Example 1 Find the rectangular coordinates of these two points.

a. $(r,\theta,z)=\left(4,\frac{\pi}{3},-3\right)$ b. $(\rho,\theta,\phi)=\left(4,\frac{\pi}{3},\frac{\pi}{4}\right)$

Solution a.

$$x=r\cos\theta,\ \ y=r\sin\theta,\ \ z=z$$

$$x^2+y^2=r^2\ ,\ \tan\theta=\frac{y}{x}$$

b.

$$\boxed{\begin{gathered} x=\rho\sin\phi\cos\theta,\ \ y=\rho\sin\phi\sin\theta,\ \ z=\rho\cos\phi \\ x^2+y^2+z^2=\rho^2\ ,\ \tan\theta=\frac{y}{x},\ \cos\phi=\frac{z}{\sqrt{x^2+y^2+z^2}} \end{gathered}}$$

Example 2 Covert the following equations:

a. $z = x^2 + y^2 - 2x + y$ to cylindrical coordinate system.

b. $z = x^2 + y^2$ to spherical coordinate system.

Solution: a.

$$\boxed{\begin{gathered} x = r\cos\theta,\ \ y = r\sin\theta,\ \ z = z \\ x^2 + y^2 = r^2\ ,\ \tan\theta = \frac{y}{x} \end{gathered}}$$

b.

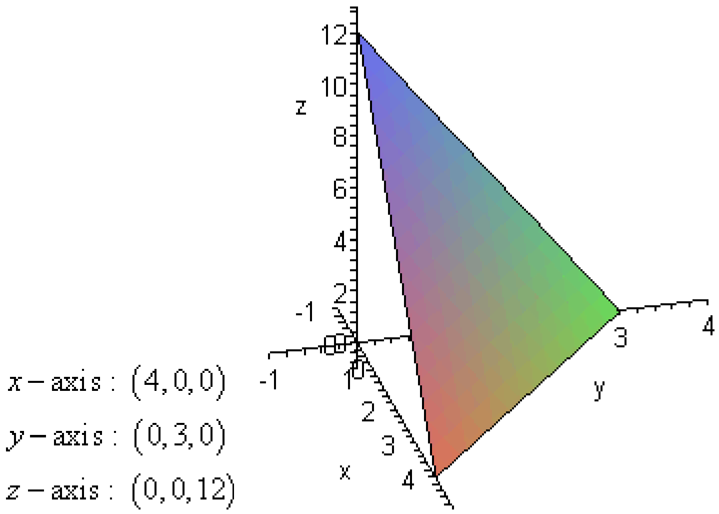
$$\boxed{\begin{gathered} x = \rho\sin\phi\cos\theta,\ \ y = \rho\sin\phi\sin\theta,\ \ z = \rho\cos\phi \\ x^2 + y^2 + z^2 = \rho^2\ ,\ \tan\theta = \frac{y}{x},\ \ \cos\phi = \frac{z}{\sqrt{x^2 + y^2 + z^2}} \end{gathered}}$$

2. Graphs of multivariable functions

A graph of two variable functions $z = f(x,y)$ on three dimensional space is just a surface.

Example 3 Draw a graph of $f(x,y) = 12-3x-4y$ on XYZ-space.

Solution



2.1 Level Curves

By using the same method, we are not able to draw a graph of $w = f(x, y, z)$ since its graph will be in four dimensions. How can we solve this problem?

Let us go back to a function of two variables. Consider a geological map which is a picture of area in 2 and 3 dimensions. Figures A and C below show the three dimensional pictures of mountain area with contour lines (traces). Figure B shows a 2-dimensional picture with different heights indicated.

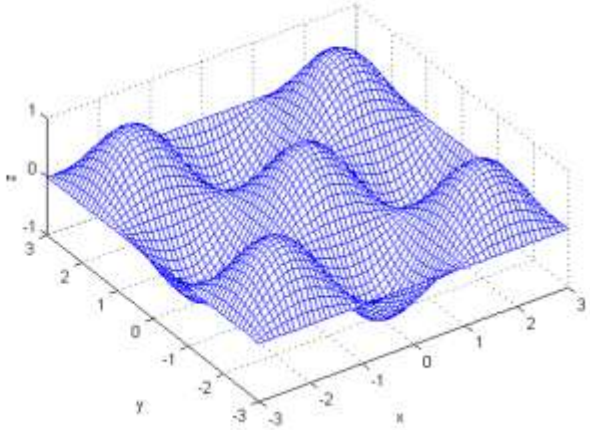


Figure A

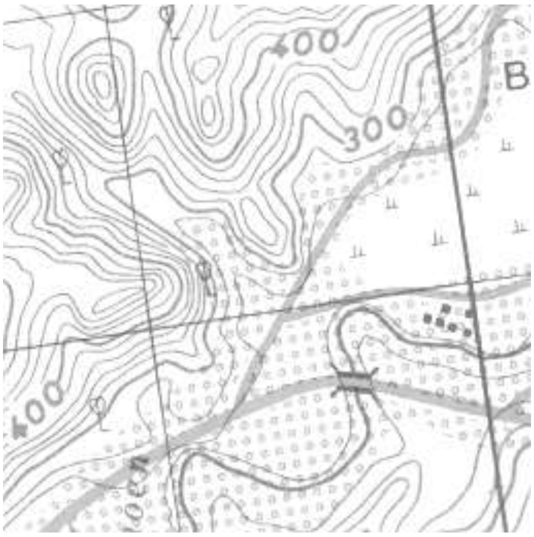


Figure B

Figure D shows several curves of area at different heights in 3 dimensions. Figure E is the projection of curves in figure D onto the xy-plane. This shows how we obtain a map as in figure B.

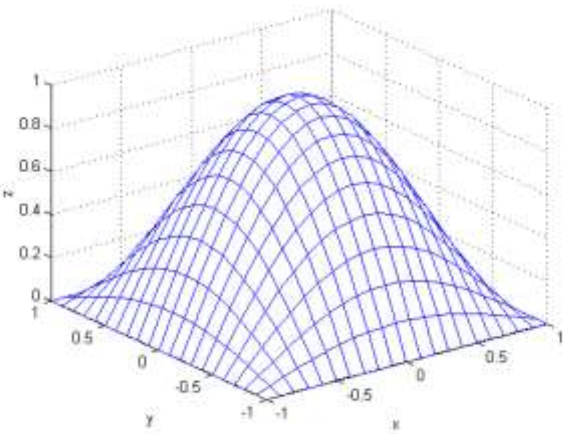


Figure C

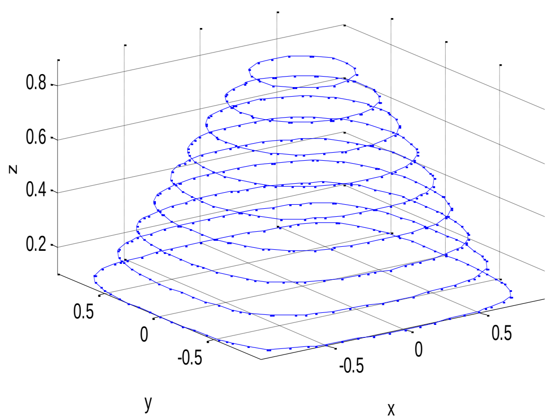


Figure D

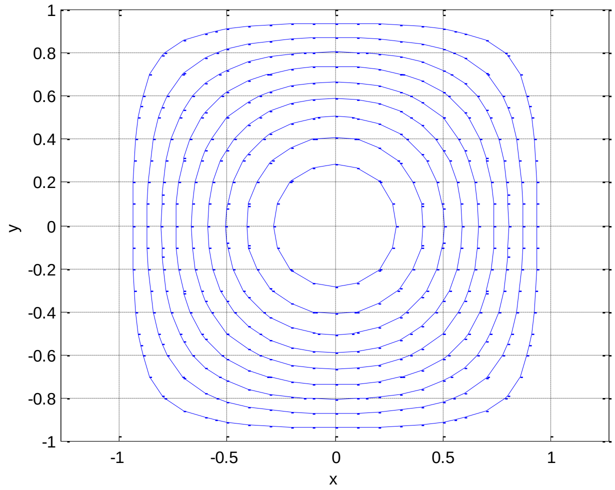
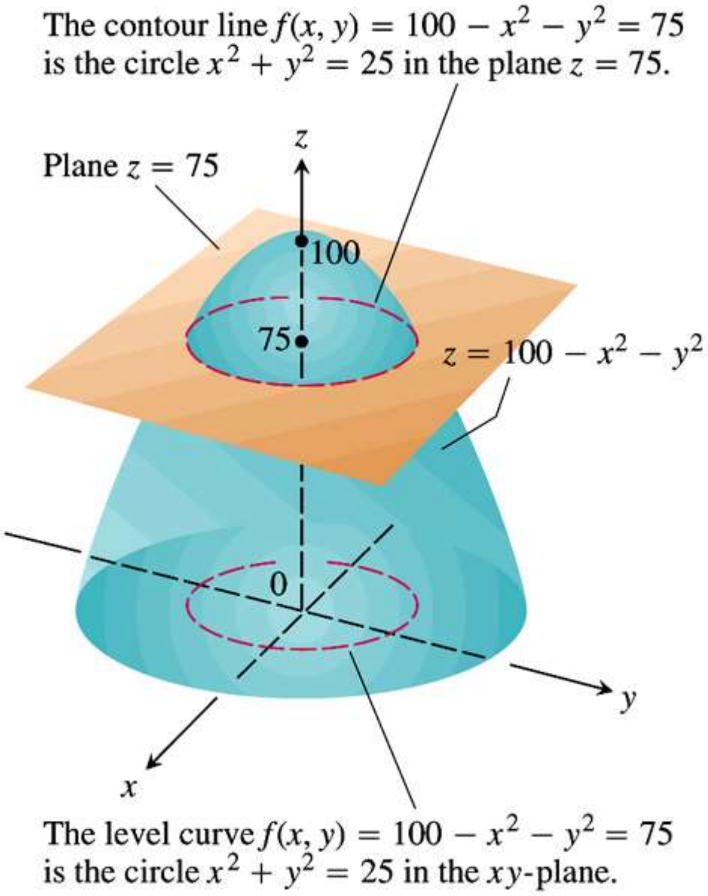


Figure E

The curves in figure E are called *level curves*. Each curve represents $f(x, y) = k$ on the xy-plan.

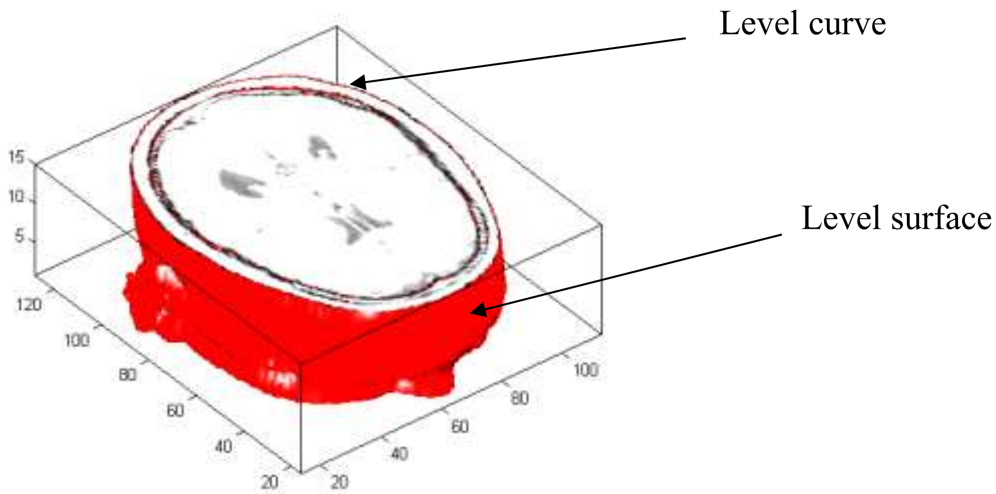
Level curves have been widely used in the atmospheric map to indicate the area with some fixed condition such as constant pressure level and constant temperature. We call the level curve for a constant pressure "isobar," and call the level curve for a constant temperature "isotherm."

Example 4 Draw several level curves of $f(x,y) = 100 - x^2 - y^2$ when $k = 100$, $k = 75$, $k = 0$.



2.2 Level Surface

In the case of 4 variables function $w = f(x, y, z)$, we are not able to draw a graph in 3 dimensional space. For example, we take the MRI picture of someone's brain at a given time. Here, the time becomes another variable. We can see the change by looking at the brain at several different times. At time c, we have $f(x, y, z) = c$. This given equation forms the area called "level surface," as shown here.



Exercise

1. Let $f(x,y) = 3x + y^2$. Evaluate the following:

   (a) $f(2,3)$ (b) $f(2,\sqrt{2})$ (c) $f(0,0)$.

2. Find the domain of the following functions and draw a graph of its domain.

   (a) $f(x,y) = \dfrac{x^2 + y^2 + 8}{(x-4)(y-3)}$

   (b) $g(x,y) = e^{\sqrt{x}} + \ln y$

Blank Page

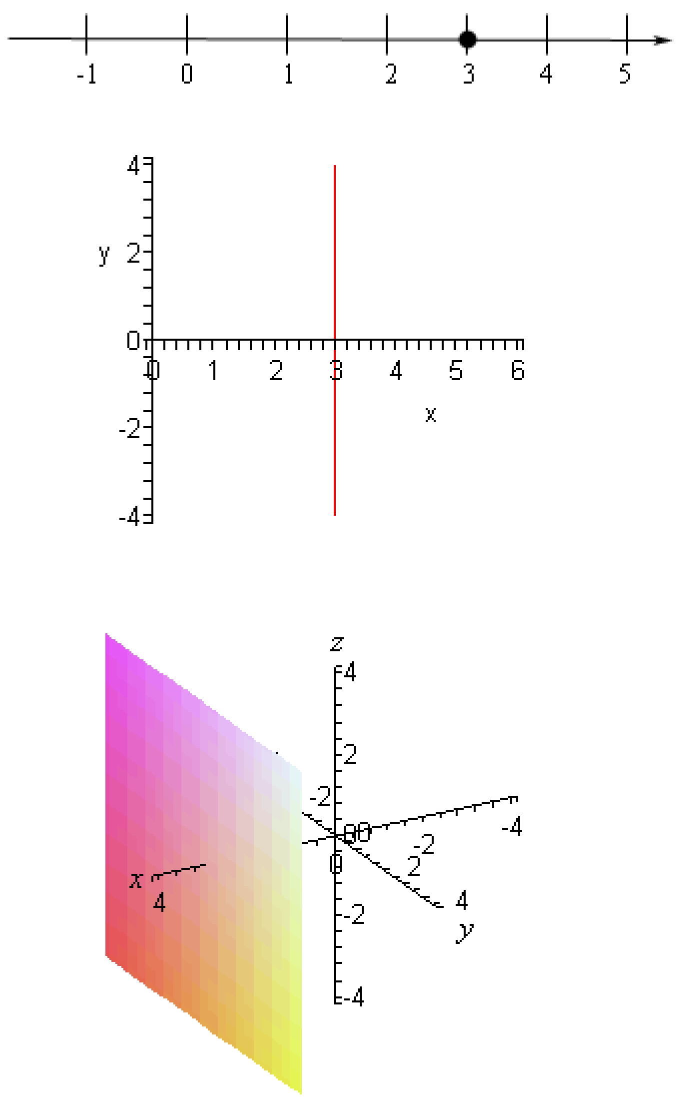
3. Graph of Planar Surface

Let A, B, C, D be some constants.

Definition: A plane is a set of all points in the three dimensional space satisfying the equation $Ax + By + Cz + D = 0$.

Example 1 Draw the graphs of $x = 3$ in one-, two- and three-dimensional spaces.

Solution:



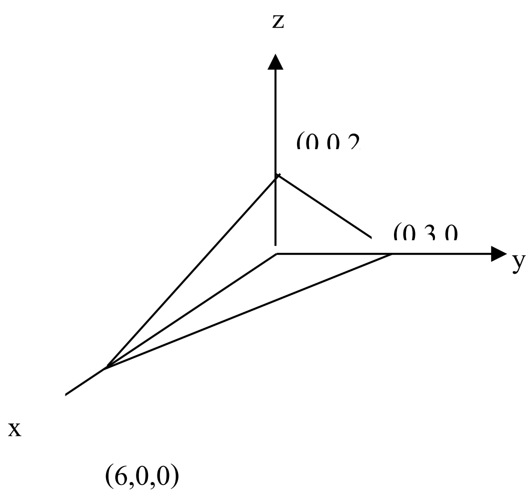
To sketch a planar graph of $Ax + By + Cz + D = 0$, we generally draw a plane only on the octant where the plane lies on by finding the intercepts of the plane and each axis. Then connect all intercepted points with lines.

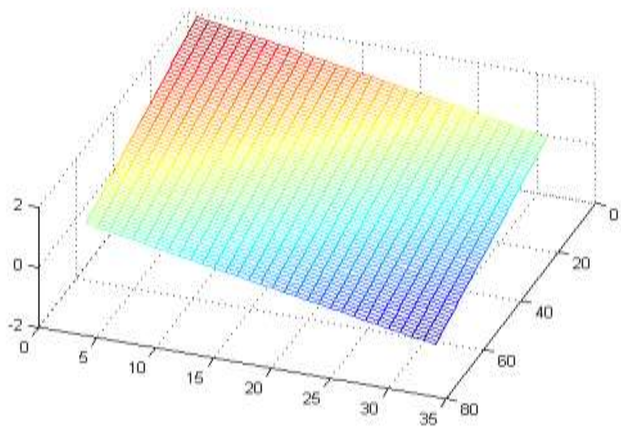
Three intercepted points:

1. If $x = 0,\ y = 0$, the point $(0,0,z)$ is the z-intercept.
2. If $x = 0,\ z = 0$, the point $(0,y,0)$ is the y-intercept.
3. If $y = 0,\ z = 0$, the point $(x,0,0)$ is the x-intercept.

Example 2 Draw a graph of $x + 2y + 3z = 6$.

Solution:





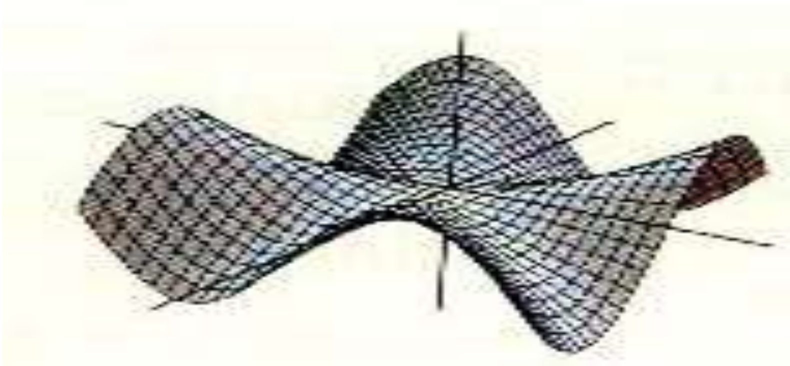
4. Graphs of Quadratic Surfaces

A graph of quadratic surface is a set of all points in three dimensional space satisfying the following equation:

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$$

where $A, B, C, \ldots, I, J$ are constants.

To draw a graph of quadratic surface, we cut the surface by the planes parallel to xy-, yz- and xz-planes making several traces on the surface. These traces form the graph of a surface.



The figure above shows traces of function: $z = \dfrac{xy\left(x^2 - y^2\right)}{x^2 + y^2}$.

The important quadratic surfaces you should know are the following:

1. Sphere
2. Ellipsoid
3. Hyperboloid of one sheet

4. Hyperboloid of two sheets

5. Elliptic cone

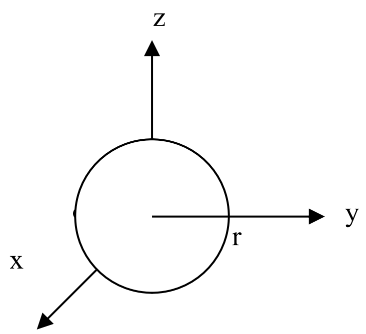
6. Elliptic paraboloid

7. Hyperbolic paraboloid

8. Cylinder

4.1 Sphere

The equation of a sphere centered at (0,0,0) with radius r has the form: $x^2 + y^2 + z^2 = r^2$.

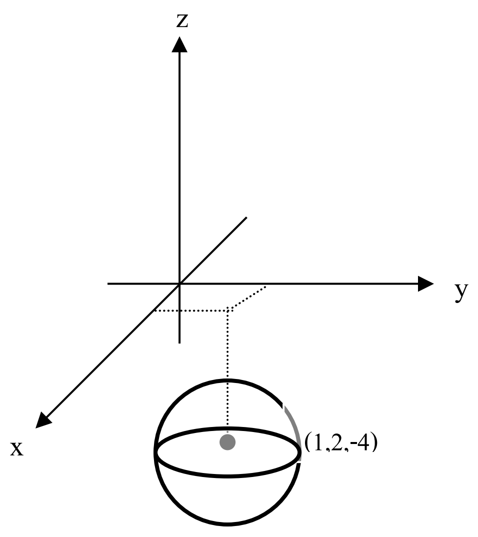


Similarly, the equation of a sphere centered at (x_0, y_0, z_0) with radius r has the form:

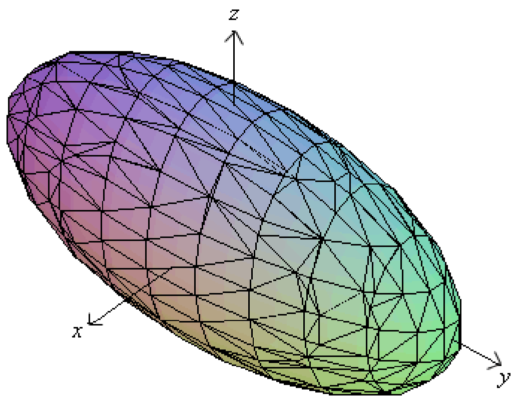
$$(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = r^2.$$

Example 1: Draw a graph of

$$x^2 + y^2 + z^2 - 2x - 4y + 8z + 17 = 0.$$



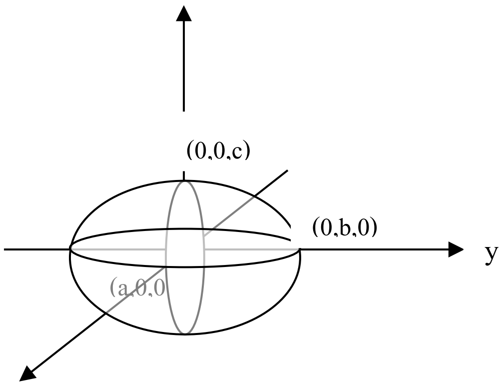
4.2 Ellipsoid



The equation of an ellipsoid centered at $(0,0,0)$ has the form:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

where a, b, c are some constants.



Similarly, the equation of an ellipsoid centered at (x_0, y_0, z_0) has the form:

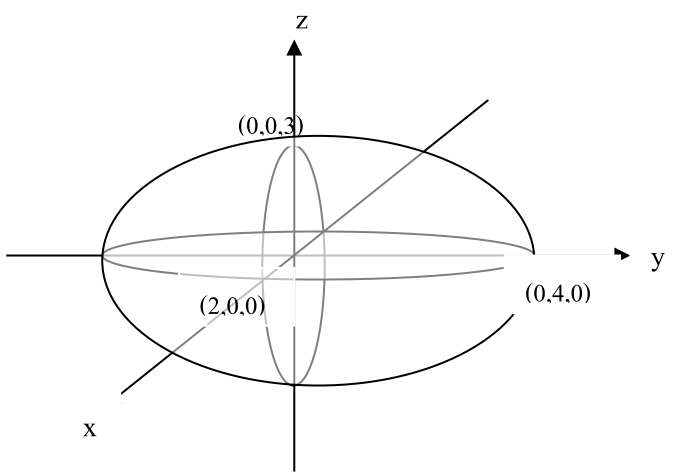
$$\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}+\frac{(z-z_0)^2}{c^2}=1.$$

Example Draw a graph of quadratic surface of

$$\frac{x^2}{4}+\frac{y^2}{16}+\frac{z^2}{9}=1$$

and find the equation of a graph after cutting the surface by the plane $x=k$, where k is some constant.

Solution



The plane $x=k$ is parallel to the yz-plane, after cutting the surface by this plane, we get the equation: $\dfrac{y^2}{16}+\dfrac{z^2}{9}=1-\dfrac{k^2}{4}$ which is an ellipse on the plane $x=k$; $-2\le k\le 2$.

4.3 Hyperboloid of one sheet

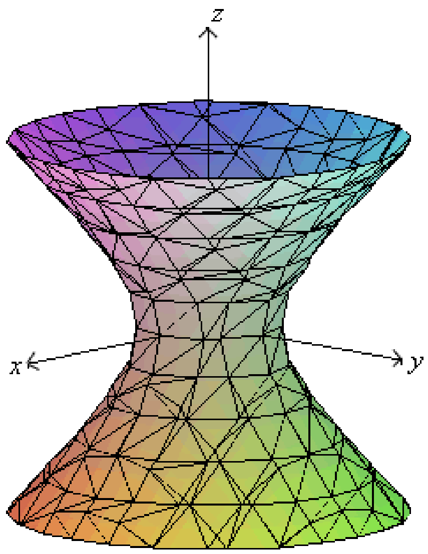
The equation of a hyperboloid of one sheet centered at $(0,0,0)$ has the following forms:

Lie along z-axis: $$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

Lie along y-axis: $$\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

Lie along x-axis: $$-\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

where a,b,c are some constants.



The graph of quadratic surface of $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$

Remark: If the hyperboloid of one sheet lies along the line parallel to the z-axis and centered at (x_0,y_0,z_0), its equation is of the form:

$$\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}-\frac{(z-z_0)^2}{c^2}=1.$$

Similarly, if it lies along the line parallel to the y-axis, its equation has the form: $\dfrac{(x-x_0)^2}{a^2} - \dfrac{(y-y_0)^2}{b^2} + \dfrac{(z-z_0)^2}{c^2} = 1.$

If it lies along the line parallel to the x-axis, its equation has the form: $-\dfrac{(x-x_0)^2}{a^2} + \dfrac{(y-y_0)^2}{b^2} + \dfrac{(z-z_0)^2}{c^2} = 1.$

4.4 Hyperboloid of two sheets

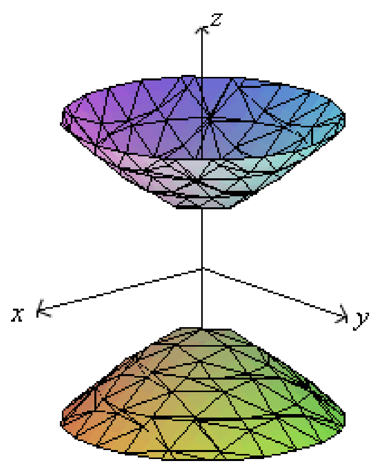
The equation of a hyperboloid of two sheets centered at $(0,0,0)$ has the following forms:

Lie along z-axis: $$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Lie along y-axis: $$-\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

Lie along x-axis: $$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

where a, b, c are some constants.



The graph of quadratic surface $-\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

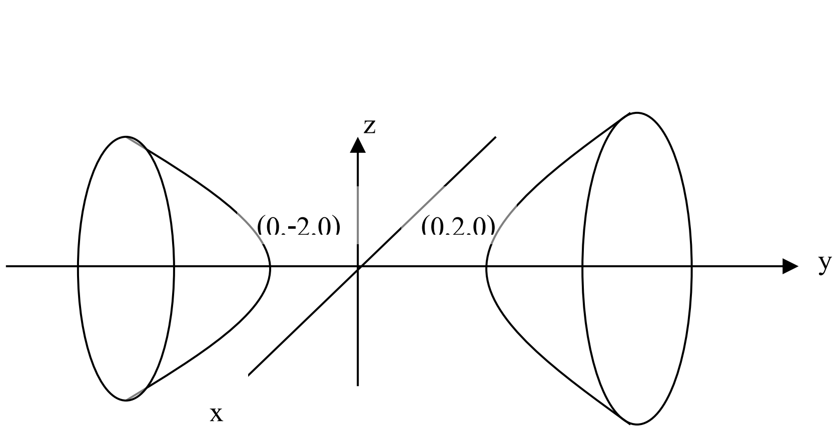
Remark:

If the hyperboloid of two sheets lies along the line parallel to the z-axis and centered at (x_0, y_0, z_0), its equation is of the form:

$$-\frac{(x-x_0)^2}{a^2} - \frac{(y-y_0)^2}{b^2} + \frac{(z-z_0)^2}{c^2} = 1.$$

Similarly, in the case of hyperboloid of two sheets lying along the y-axis or x-axis, we get the similar form of the equation.

Example: Draw a surface of $4x^2 - y^2 + 2z^2 + 4 = 0$ and find the equations of the traces after cutting the surface by each plane.



4.5 Elliptic cones

The equation of an elliptic cone centered at $(0,0,0)$ has the following forms:

Lie along z-axis: $$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$$

Lie along y-axis: $$\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=0$$

Lie along x-axis: $$-\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=0$$

where a,b,c are some constants.

Remark:

If the elliptic cone lies along the line parallel to the z-axis and centered at (x_0,y_0,z_0), its equation is of the form:

$$\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}-\frac{(z-z_0)^2}{c^2}=0.$$

Similarly, in the case of elliptic cone lying along the y-axis or the x-axis, we get the similar form of the equation.

Example: Draw a graph of the quadratic surface

$$x^2 + y^2 - z^2 - 2x + 6z - 8 = 0.$$

4.6 Paraboloid

The equation of a paraboloid centered at $(0,0,0)$ has the following forms:

Lie along z-axis and open on positive z: $$z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$$

open on negative z: $$z=-\frac{x^2}{a^2}-\frac{y^2}{b^2}$$

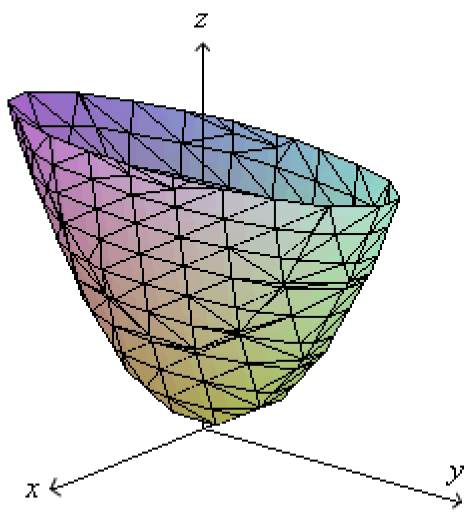
Lie along y-axis and open on positive y: $$y=\frac{x^2}{a^2}+\frac{z^2}{c^2}$$

open on negative y: $$y=-\frac{x^2}{a^2}-\frac{z^2}{c^2}$$

Lie along x-axis and open on positive x: $$x=\frac{y^2}{b^2}+\frac{z^2}{c^2}$$

open on negative x: $$x=-\frac{y^2}{b^2}-\frac{z^2}{c^2}$$

where a,b,c are some constants.



The graph of quadratic surface $z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$

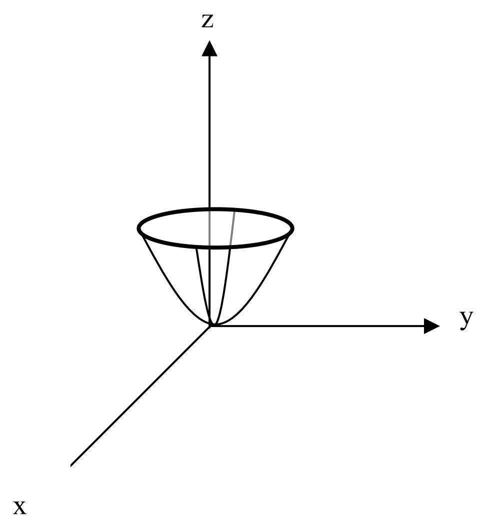
Remark:

If the paraboloid lies along the line parallel to z-axis, centered at (x_0, y_0, z_0) and opens on positive z, its equation is of the form:

$$(z - z_0) = \frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2}$$

Similarly, in the case of paraboloid lying along y-axis or x-axis, we get the similar form of the equation.

Example Draw a graph of $z = \dfrac{x^2}{4} + \dfrac{y^2}{9}$.



4.7 Hyperbolic Paraboloid

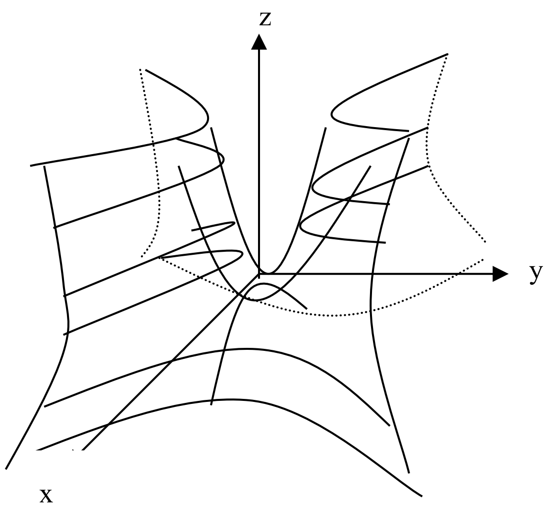
The equation of a hyperbolic paraboloid centered at $(0,0,0)$ has the following forms:

Lie along z-axis : $z=-\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}$, $z=\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}$

Lie along y-axis : $y=-\dfrac{x^2}{a^2}+\dfrac{z^2}{c^2}$, $y=\dfrac{x^2}{a^2}-\dfrac{z^2}{c^2}$

Lie along x-axis : $x=-\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}$, $x=\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}$

where a,b,c are some constants.



The graph of quadratic surface $z=-\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}$

Remark: If the hyperbolic paraboloid lies along the line parallel to z-axis and centered at (x_0, y_0, z_0), its equation is of the form:

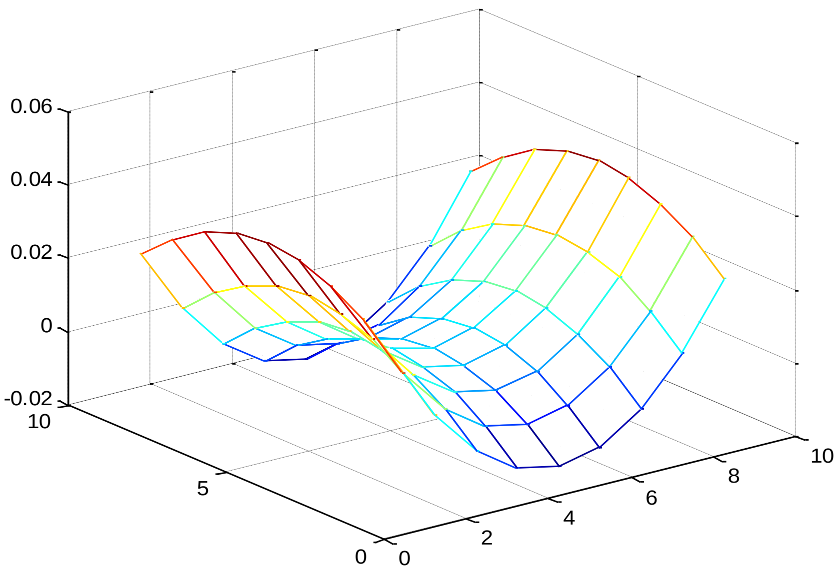
$$(z-z_0) = -\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2}$$

or $$(z-z_0) = \frac{(x-x_0)^2}{a^2} - \frac{(y-y_0)^2}{b^2}$$

Analogously, in the case of hyperbolic paraboloid lying along y-axis or x-axis, we get the similar form of the equation.

Note that in the case of a hyperbolic paraboloid, its center can also be called as a *saddle point*.

Example: Draw the quadratic surface of $z = \frac{x^2}{4} - \frac{y^2}{9}$.

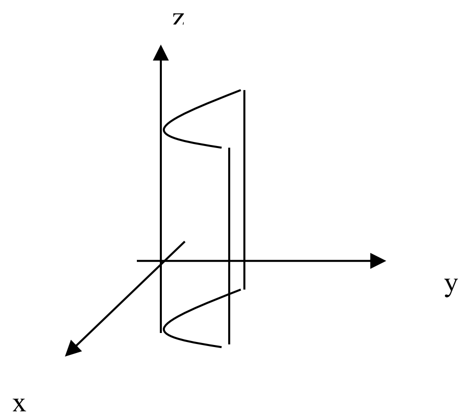


0.06
0.04
0.02
0
-0.02
10
5
0
0
2
4
6
8
10

4.8 Cylinders

There are several types of cylinders we need to learn such as cyclic cylinders, elliptic cylinders, hyperbolic cylinders, and parabolic cylinders.

The equation forms of each cylindrical surface are in 2-variables. Since we consider a graph in 3-dimensional space, the missing variable has values in $(-\infty, \infty)$. Thus, the cylinder will lie along the axis of missing variable. For example, the function $y = ax^2$ where a is a positive constant. This equation forms a parabolic cylinder lying along z-axis as shown in the figure below:



Parabolic cylinder of $y = ax^2$ when $a > 0$.

Problem: Draw the following graphs in the 3-dimensional space.

a. $z = \sqrt{y^2 + 1}$ b. $y - z^2 = 0$

c. $y^2 - x^2 = 1$ d. $25x^2 + 9z^2 = 1$

Solution:

Partial Derivatives of a Two Variable Function

In case of one variable function $y = f(x)$, we have

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

But it does not apply to the case of two variable function $z = f(x, y)$ since there are two independent variables.

To find the derivatives of $f(x, y)$, we need to calculate the derivatives with respect to each independent variable separately. We call it ***"partial derivative"***.

Partial Derivatives

How to calculate the partial derivative?

To find a partial derivative of $f(x, y)$ with respect to one variable, we consider another input variable as a constant. For example, to find the partial derivative of $f(x, y)$ with respect to x, we considers y as a constant. Then, we take an ordinary derivative of $f(x, y)$ with respect to x as in the case of one variable function.

Notations for Partial Derivatives

Let $z = f(x, y)$.

The partial derivative of $f(x, y)$ with respect to x is denoted by

$$\frac{\partial z}{\partial x} \quad \text{or} \quad \frac{\partial f}{\partial x} \quad \text{or} \quad f_x.$$

If we want to evaluate the partial derivative at (x_0, y_0), we use the notations:

$$\left.\frac{\partial z}{\partial x}\right|_{(x_0, y_0)} \quad \text{or} \quad \frac{\partial f}{\partial x}(x_0, y_0) \quad \text{or} \quad f_x(x_0, y_0).$$

Similarly, the partial derivative of $f(x, y)$ with respect to y is denoted by

$$\frac{\partial z}{\partial y} \quad \text{or} \quad \frac{\partial f}{\partial y} \quad \text{or} \quad f_y.$$

Partial Derivative with Respect to x

Definition The partial derivative of $f(x, y)$ with respect to x at (x_0, y_0) is defined as

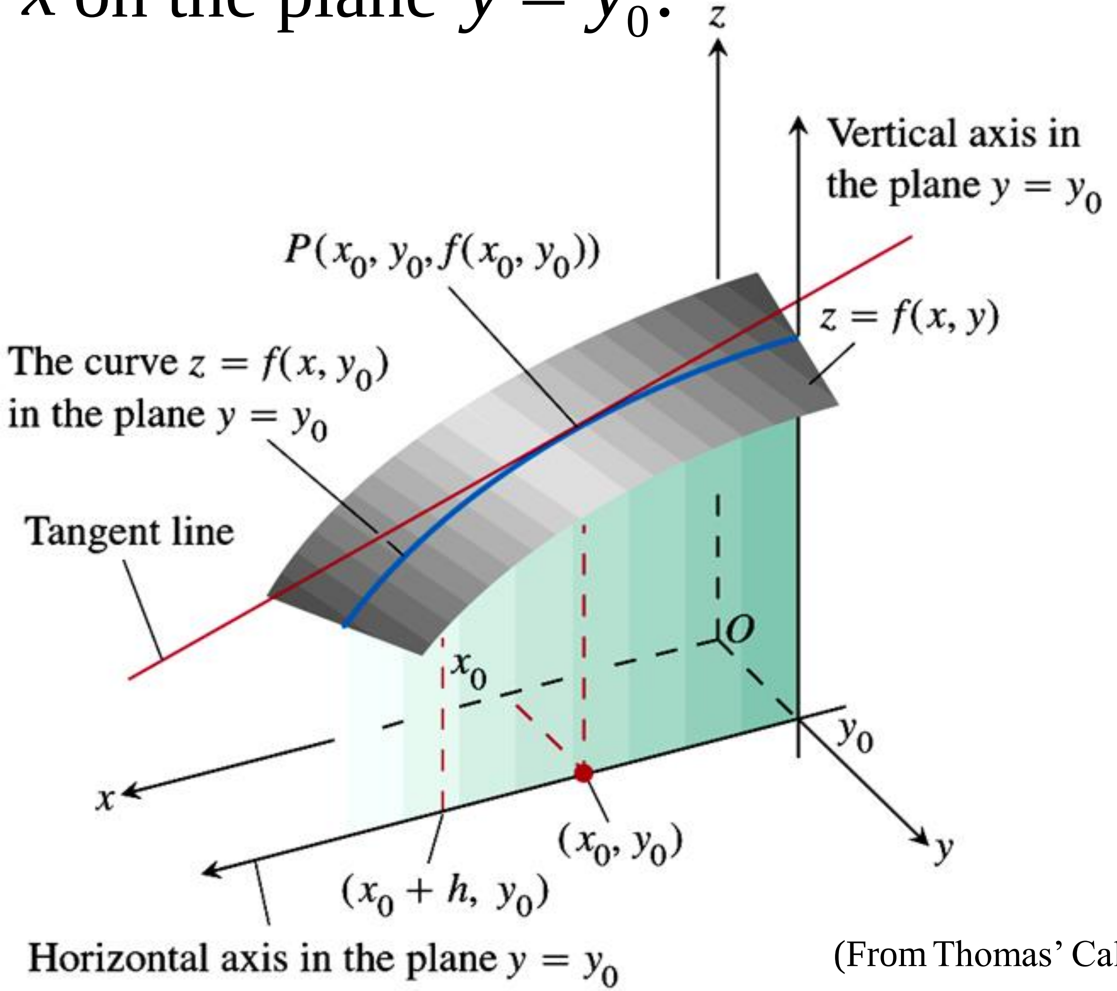
$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \left.\frac{d}{dx} f(x, y_0)\right|_{x=x_0} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}.$$

Example Let $f(x, y) = x^3 - 3x^2 y + 3xy^2 - y^3$. Find $f_x(1, 0)$.

Geometrical Interpretation of f_x

Let y_0 be a constant. The curve $(x, y_0, f(x, y_0))$ is the intersection between the surface $z = f(x, y)$ and the plane $y = y_0$. Note that curve $(x, y_0, f(x, y_0))$ is a function of only one variable x on the plane $y = y_0$.

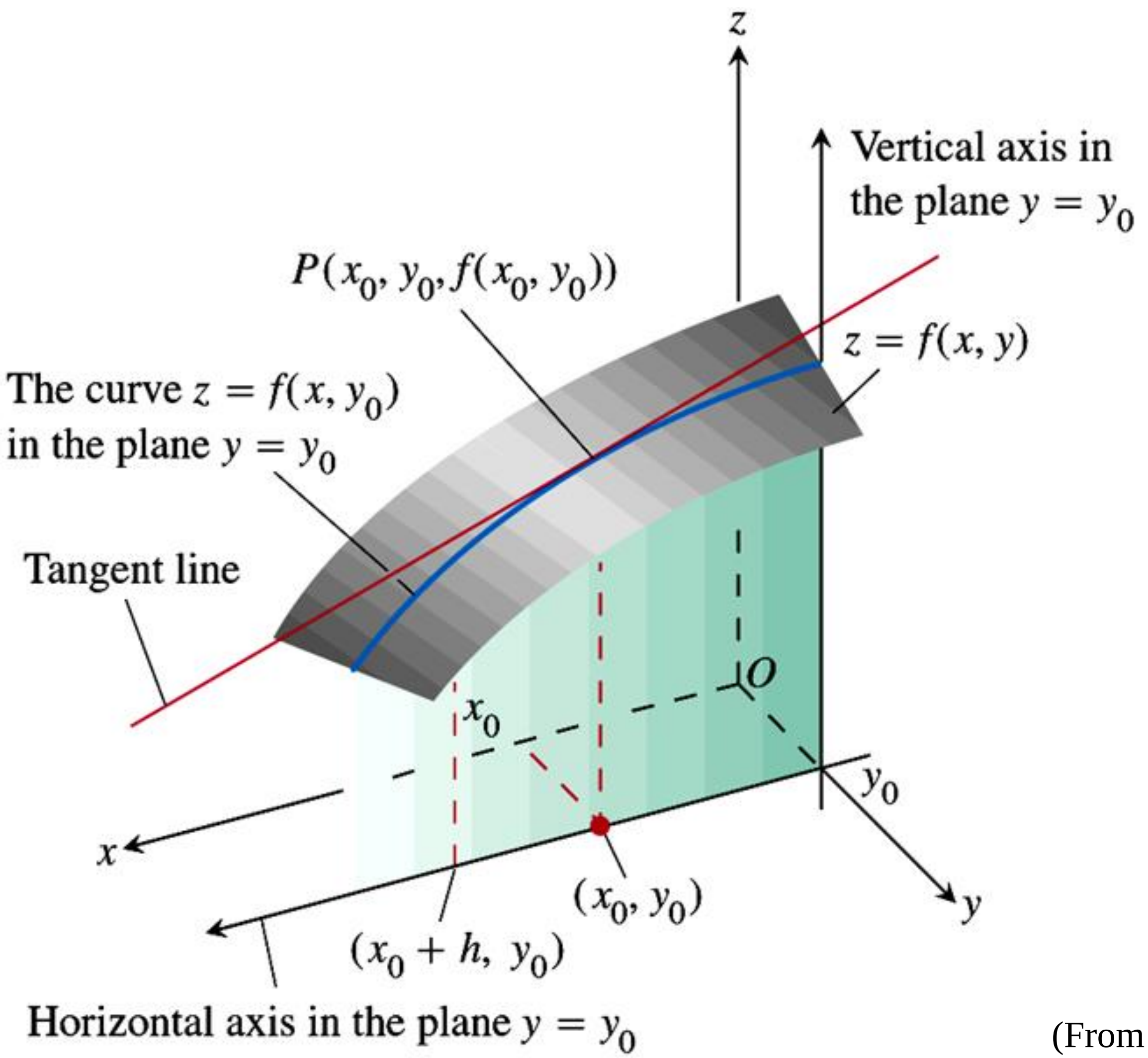
Set $y = y_0$ to be a constant.



(From Thomas' Calculus)

Geometrical Interpretation of f_x

$f_x(x_0, y_0)$ means the slope of the tangent line to the curve $(x, y_0, f(x, y_0))$ at (x_0, y_0).



(From Thomas' Calculus)

Partial Derivative with Respect to y

Definition The partial derivative of $f(x, y)$ with respect to y at (x_0, y_0) is defined as

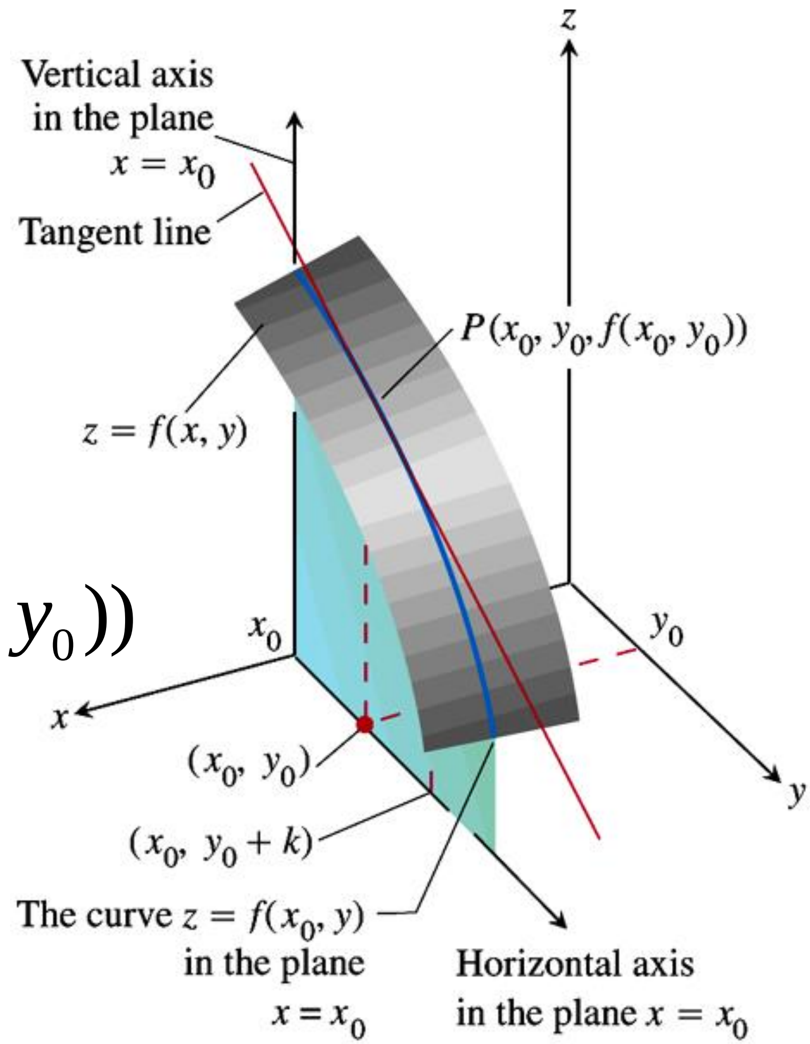
$$\left.\frac{\partial f}{\partial y}\right|_{(x_0, y_0)} = \left.\frac{d}{dy} f(x_0, y)\right|_{y=y_0} = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}.$$

Example Let $f(x, y) = x^3 - 3x^2 y + 3xy^2 - y^3$. Find $f_y(1, 0)$.

Geometrical Interpretation of f_y

$f_y(x_0, y_0)$ means the slope of the tangent line to the curve $(x, y_0, f(x, y_0))$ at (x_0, y_0).

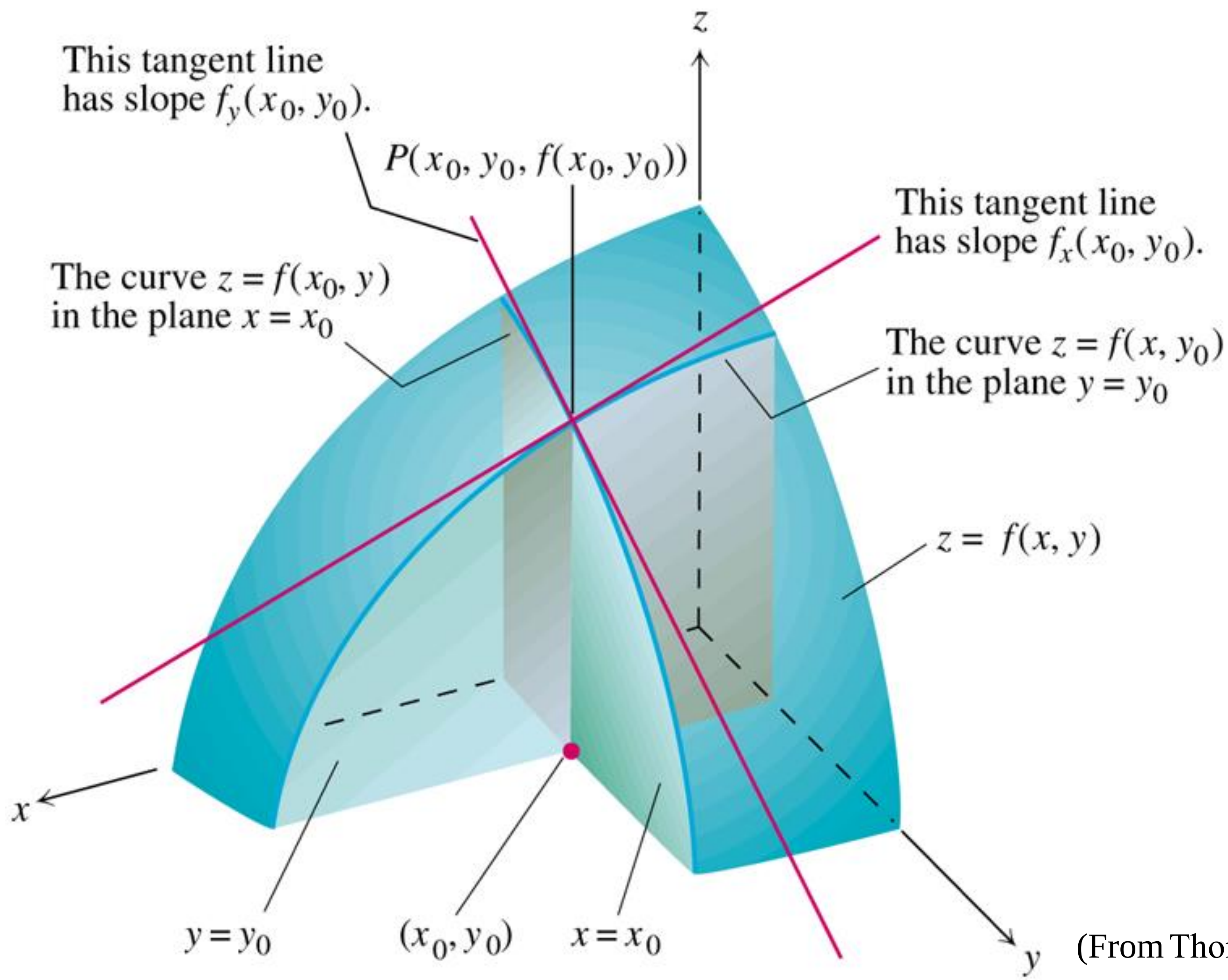
The curve $(x, y_0, f(x, y_0))$ is the intersection between the surface $z = f(x, y)$ and the plane $x = x_0$.
Note that the curve $(x, y_0, f(x, y_0))$ is a function of one variable y.



(From Thomas' Calculus)

Partial Derivatives

There are many tangent lines to the surface $z = f(x, y)$ at $(x_0, y_0, f(x_0, y_0))$ depending on the directions.



(From Thomas' Calculus)

Example: Partial Derivatives

Let $f(x, y) = \dfrac{x}{x^2 + y^2}$. Then

$$\frac{\partial f}{\partial x} =$$

$$\frac{\partial f}{\partial y} =$$

Example: Partial Derivatives

Let $f(x, y) = \cos(3x - y^2)$. Then

$$\frac{\partial f}{\partial x} =$$

$$\frac{\partial f}{\partial y} =$$

Implicit Partial Differentiation

In some case, for example $z^2+2z+1=x^2+2xy+y^2$, it's not easy to write $z=f(x,y)$ before calculating the derivatives. In this situation, we can find the partial derivative by taking partial derivative operator $\dfrac{\partial}{\partial x}$ on both sides of the equation:

$$\frac{\partial}{\partial x}(z^2+2z+1)=\frac{\partial}{\partial x}(x^2+2xy+y^2).$$

$$\left.\begin{aligned}
&\frac{\partial}{\partial x}(z^2)=\frac{\partial(z^2)}{\partial z}\frac{\partial z}{\partial x}=2z\frac{\partial z}{\partial x}\\
&\frac{\partial}{\partial x}(2z)=2\frac{\partial z}{\partial x}\\
&\frac{\partial}{\partial x}(x^2+2xy+y^2)=2x+2y
\end{aligned}\right\}\quad (2z+2)\frac{\partial z}{\partial x}=2x+2y$$

Thus, $\dfrac{\partial z}{\partial x}=\dfrac{x+y}{z+1}$.

Example: Implicit Partial Differentiation

Let $xz^2 + e^z = \sin(x^2 + y^2)$. Find $\dfrac{\partial z}{\partial x}$.

Functions of More Than 2 Variables

To find partial derivatives of multivariable functions, we use the same method as in the case of two-variable functions. For example, the case of the partial derivative of $f(x, y, z)$ with respect to x, we consider y and z as constants. Then, we take a usual derivative of $f(x, y, z)$ with respect to x as in the case of one variable function.

Example Let $f(x, y, z) = x\cos(3y - z^2)$. Then

$$\frac{\partial f}{\partial z} =$$

Second Order Partial Derivatives

We just take partial derivative twice which consists of four possibilities.

$$\frac{\partial^2 f}{\partial x^2} \quad \text{or} \quad f_{xx} \qquad\qquad \frac{\partial^2 f}{\partial y^2} \quad \text{or} \quad f_{yy}$$

$$\frac{\partial^2 f}{\partial x \partial y} \quad \text{or} \quad f_{yx} \qquad\qquad \frac{\partial^2 f}{\partial y \partial x} \quad \text{or} \quad f_{xy}$$

Second Order Partial Derivatives

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

Example Let $f(x,y) = x^3 - 3x^2y + 3xy^2 - y^3$. Then

$$\frac{\partial^2 f}{\partial x^2} =$$

$$\frac{\partial^2 f}{\partial y^2} =$$

$$\frac{\partial^2 f}{\partial x \partial y} = \qquad \frac{\partial^2 f}{\partial y \partial x} =$$

Example: Second Order Partial Derivatives

Let $f(x,y)=\dfrac{x}{x^2+y^2}$. Then

$$\frac{\partial f}{\partial x}=\frac{y^2-x^2}{\left(x^2+y^2\right)^2},\qquad \frac{\partial f}{\partial y}=\frac{-2xy}{\left(x^2+y^2\right)^2},$$

$$\frac{\partial^2 f}{\partial x^2}=\frac{-2x\left(x^2+y^2\right)^2-2\left(x^2+y^2\right)(2x)(y^2-x^2)}{\left(x^2+y^2\right)^4},$$

$$\frac{\partial^2 f}{\partial y^2}=\frac{-2x\left(x^2+y^2\right)^2-2\left(x^2+y^2\right)(2y)(-2xy)}{\left(x^2+y^2\right)^4},$$

$$\frac{\partial^2 f}{\partial x\partial y}=\frac{-2y\left(x^2+y^2\right)^2-2\left(x^2+y^2\right)(2x)(-2xy)}{\left(x^2+y^2\right)^4}\quad \text{and}$$

$$\frac{\partial^2 f}{\partial y\partial x}=\frac{2y\left(x^2+y^2\right)^2-2\left(x^2+y^2\right)(2y)(y^2-x^2)}{\left(x^2+y^2\right)^4}.$$

The Mixed Derivative Theorem

The 2nd partial derivatives $\dfrac{\partial^2 f}{\partial x \partial y}$ and $\dfrac{\partial^2 f}{\partial y \partial x}$ contain $\dfrac{\partial}{\partial x}$ and $\dfrac{\partial}{\partial y}$.

They are called "Mixed partial derivatives".

Theorem:
If f and its partial derivatives f_x, f_y, f_{xy} and f_{yx} are defined in open region containing (a,b) and all are continuous at (a,b), then

$$f_{xy}(a,b) = f_{yx}(a,b).$$

This theorem implies that if f, f_x, f_y, f_{xy} and f_{yx} are all continuous, then the order of $\dfrac{\partial}{\partial x}$ and $\dfrac{\partial}{\partial y}$ in mixed partial derivative does not matter.

Example: The Mixed Derivative Theorem

Let $f(x, y) = \cos(3x - y^2)$. Then

$$\frac{\partial f}{\partial x} = -3\sin\left(3x - y^2\right),$$

$$\frac{\partial^2 f}{\partial y \partial x} = -3\cos\left(3x - y^2\right)(-2y) = 6y\cos\left(3x - y^2\right),$$

$$\frac{\partial f}{\partial y} = 2y\sin\left(3x - y^2\right) \quad \text{and}$$

$$\frac{\partial^2 f}{\partial x \partial y} = 2y\cos\left(3x - y^2\right)(3) = 6y\cos\left(3x - y^2\right).$$

In this case, $\boxed{\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}.}$

Partial Derivatives of Higher Order

Partial derivative operators $\dfrac{\partial}{\partial x}$ and $\dfrac{\partial}{\partial y}$ may be applied several times such as

$$\frac{\partial^3 f}{\partial y \partial x^2} \quad \text{or} \quad f_{xxy}, \qquad\qquad \frac{\partial^4 f}{\partial y^2 \partial x^2} \quad \text{or} \quad f_{xxyy}.$$

Example Let $f(x, y) = \cos(3x - y^2)$. Then

$$\frac{\partial f}{\partial x} = -3\sin\left(3x - y^2\right),$$

$$\frac{\partial^2 f}{\partial x^2} = -9\cos\left(3x - y^2\right) \quad \text{and}$$

$$\frac{\partial^3 f}{\partial y \partial x^2} = 9\sin\left(3x - y^2\right)\left(-2y\right) = -18y\sin\left(3x - y^2\right).$$

Differentiability of a Functions of Two Variables

Definition A function f is said to be *differentiable* at (x_0, y_0) if $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist and it is called a *differentiable function* if it is differentiable everywhere.

Continuity of Partial Derivatives

If f_x and f_y are both continuous in open region R, then f is differentiable everywhere in R.

Differentiability Implies Continuity

If f is differentiable at (x_0, y_0), then f is continuous at (x_0, y_0).

Composite Functions in Higher Dimensions

Let $w = f(x, y)$, $y = h(t)$ and $x = g(t)$.

Thus,

$$w = f(g(t), h(t))$$

We say that w is a composite function in terms of t,
x and y are called "intermediate variables",
w is called a "dependent variable",
and t is called an "independent variable".

Chain Rule for Functions of Two Independent Variables

Derivative of a composite function can be calculated by applying the chain rule. In case of one-variable functions $y = f(x)$ and $x = g(t)$, we have

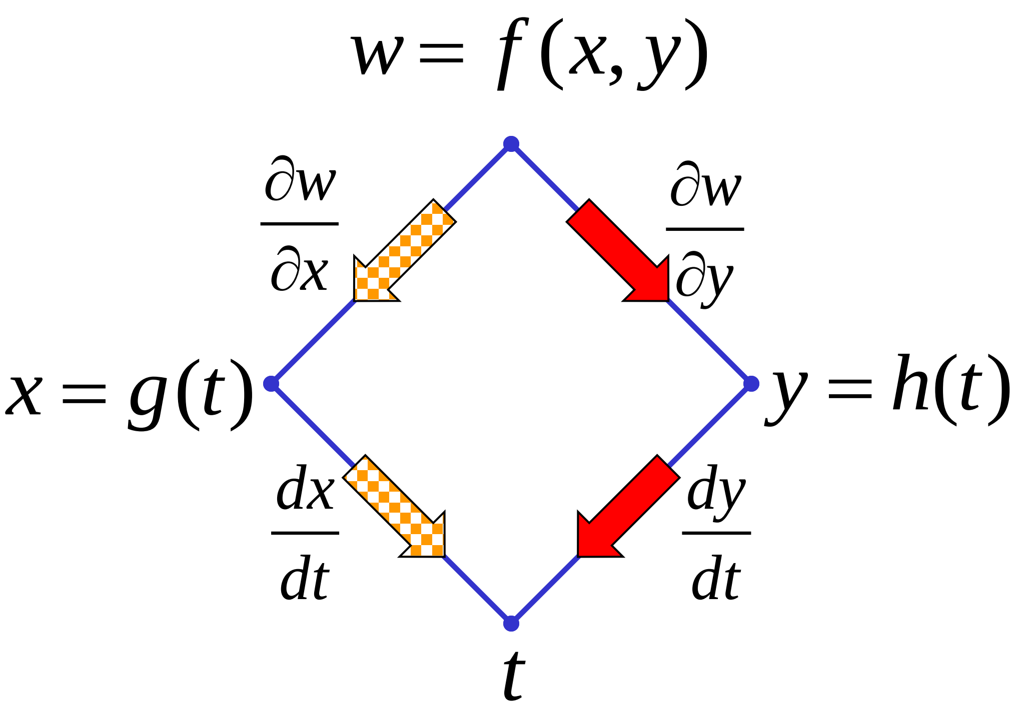
$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}.$$

In case of two-variable functions, let $w = f(x, y)$ and $y = h(t)$ and $x = g(t)$ where f, g and h are differentiable. We have

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Tree Diagram

This diagram shows how to find the derivatives of a composite function.



1. Each node represents each variable.
2. Arrow is the derivative of beginning node with respect to the ending node.
3. Start with the dependent variable w and then walk along all branches to t.
4. Add them all up.

$$\left.\begin{array}{l} \text{Get } \dfrac{\partial w}{\partial x}, \dfrac{dx}{dt} \\[2ex] \text{Get } \dfrac{\partial w}{\partial y}, \dfrac{dy}{dt} \end{array}\right\} \frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}.$$

Example: Chain Rule for Functions of 2 Variables

Let $w = x^2 + 2xy + y^2$, $x = \cos(t)$ and $y = \sin(t)$. Find $\dfrac{dw}{dt}$.

Chain Rule for Functions of 3 Independent Variables

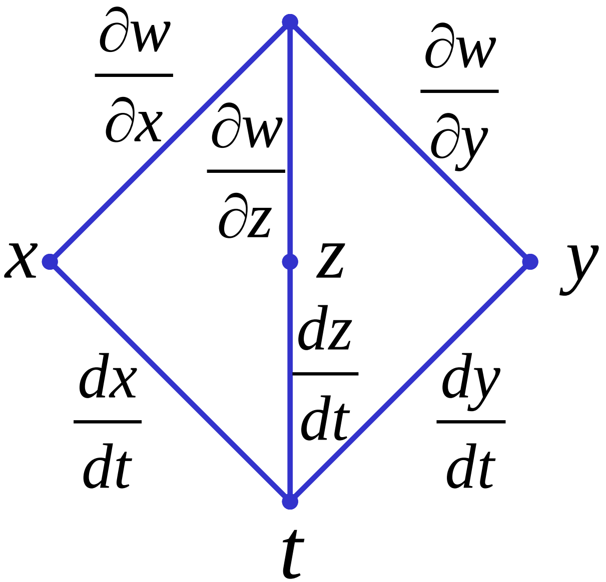
In case of three-variable functions

$$w = f(x, y, z),\ x = g(t),\ y = h(t) \quad \text{and} \quad z = k(t)$$

where f, g, h and k are differentiable. We have

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}.$$

$$w = f(x, y, z)$$



$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}.$$

Example: Chain Rule for Functions of 3 Variables

Let $w = xy + yz + zx,\ x = \cos(t),\ y = \sin(t)$ and $z = t^2.$ Then

$$\frac{\partial w}{\partial x} = y + z, \qquad \frac{\partial w}{\partial y} = x + z, \qquad \frac{\partial w}{\partial z} = y + x,$$

$$\frac{dx}{dt} = -\sin(t), \qquad \frac{dy}{dt} = \cos(t), \qquad \frac{dz}{dt} = 2t,$$

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

$$= (y+z)\big(-\sin(t)\big) + (x+z)\big(\cos(t)\big) + (y+x)(2t)$$

$$= (\sin t + t^2)\big(-\sin t\big) + (\cos t + t^2)\big(\cos t\big) + (\sin t + \cos t)(2t).$$

Functions Defined on Surfaces

Suppose that we have several two-variable functions as intermediate variables

$$w = f(x, y, z),\ x = g(r, s),\ y = h(r, s),\ z = k(r, s).$$

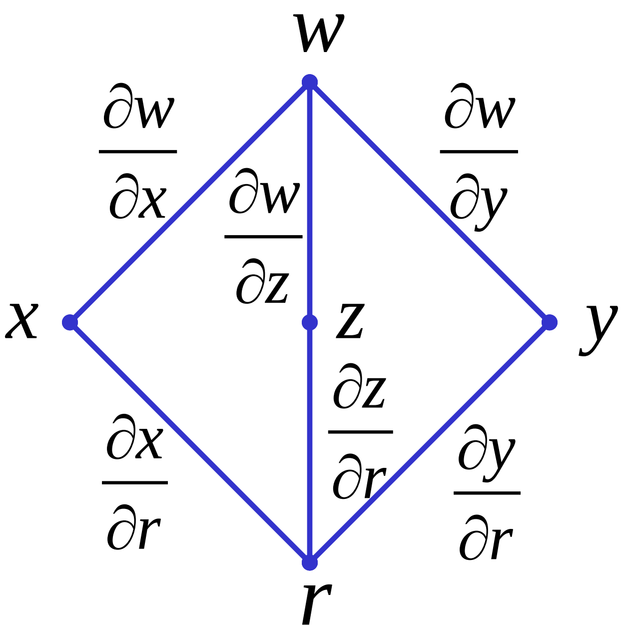
All x, y, z are considered as surfaces while w is a function of all three surfaces. Its partial derivatives are

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

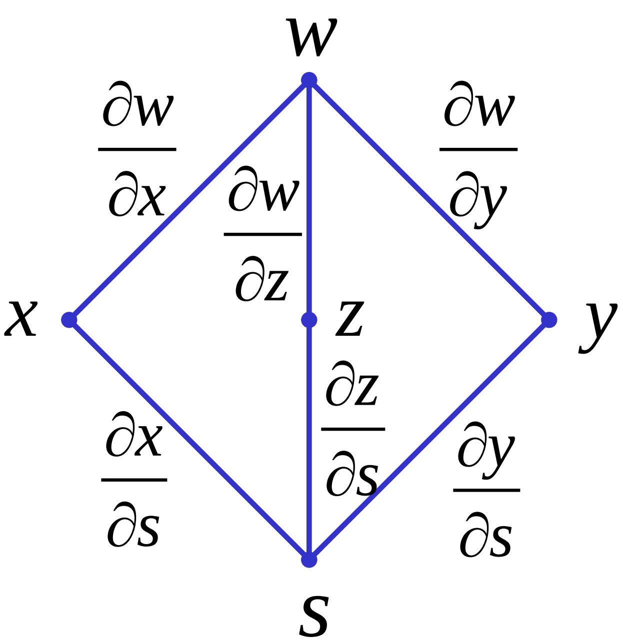
and

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}.$$

Tree Diagram for $f(g(r,s),h(r,s),k(r,s))$



$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}.$$



$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}.$$

Example

Let $w = xz + y^2$, $x = \dfrac{r}{s}$, $y = r^2 + \ln(s)$ and $z = r^2$. Then

$$\frac{\partial w}{\partial x} = z, \qquad \frac{\partial w}{\partial y} = 2y, \qquad \frac{\partial w}{\partial z} = x,$$

$$\frac{\partial x}{\partial r} = \frac{1}{s}, \qquad \frac{\partial y}{\partial r} = 2r, \qquad \frac{\partial z}{\partial r} = 2r,$$

$$\frac{\partial x}{\partial s} = \frac{-r}{s^2}, \qquad \frac{\partial y}{\partial s} = \frac{1}{s}, \qquad \frac{\partial z}{\partial s} = 0,$$

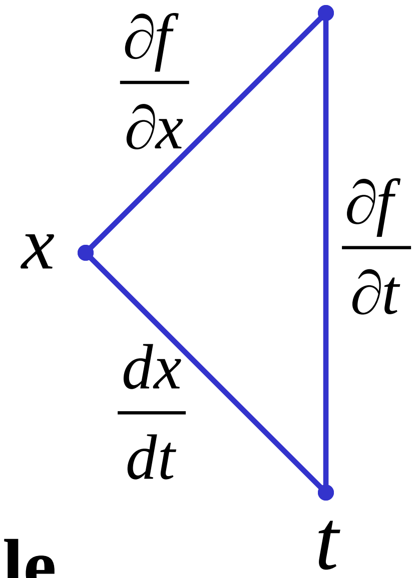
$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} = \frac{z}{r} + 2y(2r) + x(2r)$$

$$= r + 4r(r^2 + \ln s) + 2\frac{r^2}{s},$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} = \frac{-zr}{s^2} + 2\frac{y}{s} + x(0) = \frac{-r^3}{s^2} + 2\frac{r^2 + \ln s}{s}.$$

Functions in a Form of $w = f(t, x(t))$

Suppose that w is a function of one independent and one intermediate variables: $w = f(t, x)$



$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial t}$$

Example

Let $w = xt + x^2$ where $x = (t-2)^2$. Find $\frac{dw}{dt}$.

$$\frac{\partial f}{\partial x} = t + 2x, \qquad \frac{\partial f}{\partial t} = 2(t-2) \quad \text{and}$$

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial t} = (t+2x)2(t-2) = 2\left(t + 2(t-2)^2\right)(t-2).$$

Implicit Functions

Explicit function is a function in a form $y = h(x)$. Then $\frac{dy}{dx}$ can be calculated easily. **Implicit function** is a function in a form $F(x, y) = 0$. We may compute $\frac{dy}{dx}$ by 2 methods:

<u>Method 1</u>

Rewrite $F(x, y) = 0$ into $y = h(x)$ before computing $\frac{dy}{dx}$.

This is sometimes difficult.

<u>Method 2</u>

Use the chain rule for multivariable functions.

Implicit Differentiation

Let y be a function defined implicitly in term of x.

We can find $\dfrac{dy}{dx}$ by following this procedure.

1. Set up $F(x, y) = 0.$

2. Differentiate $F(x, y) = 0$ with respect to x on both sides:

$$\frac{d}{dx} F(x, y) = \frac{d}{dx} 0.$$

Then, we get

$$0 = F_x \frac{dx}{dx} + F_y \frac{dy}{dx} = F_x + F_y \frac{dy}{dx}.$$

Therefore,

$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$

Example: Implicit Differentiation

Suppose that $x^2 + 2xy + y^2 = \sin(xy)$. Find $\dfrac{dy}{dx}$.

Implicit differentiations of a system of equations

- **<u>Example</u>** Let u and v be functions x and y such that

$$uv = x + y \quad \text{and} \quad u - v^2 = x - y.$$

Find $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}.$

<u>Definition</u> Let F and G be functions of u, v. We call

$$\frac{\partial(F,G)}{\partial(u,v)} = \begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix} = F_u G_v - F_v G_u$$

as a Jacobian determinant of $F(u,v)$ and $G(u,v)$.

Jacobian formulas:

Now, we have $F(u,v,x,y)=0$ and $G(u,v,x,y)=0.$

By Cramer's rule: If $F_uG_v - F_vG_u \neq 0,$ then

$$u_x = -\frac{\begin{vmatrix} F_x & F_v \\ G_x & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = -\frac{\dfrac{\partial(F,G)}{\partial(x,v)}}{\dfrac{\partial(F,G)}{\partial(u,v)}}, \qquad u_y = -\frac{\begin{vmatrix} F_y & F_v \\ G_y & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = -\frac{\dfrac{\partial(F,G)}{\partial(y,v)}}{\dfrac{\partial(F,G)}{\partial(u,v)}},$$

$$v_x = -\frac{\begin{vmatrix} F_u & F_x \\ G_u & G_x \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = -\frac{\dfrac{\partial(F,G)}{\partial(u,x)}}{\dfrac{\partial(F,G)}{\partial(u,v)}} \quad \text{and} \quad v_y = -\frac{\begin{vmatrix} F_u & F_y \\ G_u & G_y \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = -\frac{\dfrac{\partial(F,G)}{\partial(u,y)}}{\dfrac{\partial(F,G)}{\partial(u,v)}}.$$

Example Let u and v be functions x and y such that

$$uv^2 + xy = x^2 + y \quad \text{and} \quad u^2 - 3v = x^2 + y^2.$$

Find $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$.

Applications of Multivariable Functions

1. The Chain Rule

We apply the chain rule to related rate problems which are unable to get solved by using derivatives of one variable functions.

Example 1 A cone pile of sand increases by 2 inches/second in height and by 1 inch/second in base radius. Find the rate of volume's change of this sand pile when its cone is 30 inches in height and has base radius 20 inches.

Example 2: An airplane flies from the west to the east of an observer on the ground. Suppose the plane flies with horizontal speed 440 feet/second and vertical speed 10 feet/second. What is the rate of distance's change between the plane and the observer when the plane is 12000 feet above ground and 16000 feet to the west of the observer?

2. **Total Differential**

In the case of one variable functions, we have the differential:

$$df = f'(x)\Delta x.$$

Similarly, in the case of two variable functions, this df is called the total differential of f and is defined as follows.

Definition: The total differential of $f(x, y)$, denoted by df , is defined to be

$$df = f_x(x, y)dx + f_y(x, y)dy$$

Example Find dz where $z = \ln(x^3 y^2)$.

Definition: The difference of function between two points

$$\Delta f = f(x+\Delta x, y+\Delta y) - f(x,y).$$

Application of total difference/differential

We know that $\Delta f \approx df$ when $\Delta x, \Delta y \to 0$.

Thus, $f(x+\Delta x, y+\Delta y) - f(x,y) \approx f_x(x,y)dx + f_y(x,y)dy$.

i.e. $f(x+\Delta x, y+\Delta y) \approx f(x,y) + f_x(x,y)dx + f_y(x,y)dy$.

This relation can be used to approximate the value of the function f at some points $(x+\Delta x, y+\Delta y)$ near the point (x,y) when the value of $f(x,y)$ is much easier to compute. It is called the *linear approximation* of two variable functions.

Example Estimate $\sqrt{(5.98)^2+(8.01)^2}$.

3. Max and Min of multivariable functions

Definition

If a function f of two variable has a point $(a,b) \in D_f$ such that $f(x,y) \le f(a,b)$ for all points (x,y) in an open disk centered at (a,b), then the point $(a,b,f(a,b))$ is called a *local maximum point* of f, and the value $f(a,b)$ is called a *local maximum value* of f.

If $f(x,y) \le f(a,b)$ for all points (x,y) in domain of f, we call the point $(a,b,f(a,b))$ as an *absolute maximum point* of f and call the value $f(a,b)$ as an *absolute maximum value* of f.

If a function f of two variables has a point $(c,d) \in D_f$ such that $f(x,y) \ge f(c,d)$ for all points (x,y) in an open disk centered at (c,d), then the point $(c,d,f(c,d))$ is called as a *local minimum point* of f and the value $f(c,d)$ is called a *local minimum value* of f.

If $f(x,y) \ge f(c,d)$ for all points (x,y) in domain of f, we call the point $(c,d,f(c,d))$ as an *absolute minimum point* of f and called the value $f(c,d)$ as an *absolute minimum value*.

How to find local maximum and minimum

Theorem: If f has a local max or local min at (a,b) and its first partial derivatives exist at (a,b), we would have both

$$f_x(a,b)=0 \text{ and } f_y(a,b)=0.$$

Definition: A point (a,b) in domain of f is called a critical point of f if either both $f_x(a,b)=0$ and $f_y(a,b)=0$ or at least one partial derivative of f does not exist at (a,b).

Remark: A critical point may not be a local max or local min; it could be a saddle point.

Sometimes we may use a graph of f to locate a local max and a local min.

Example: Find all critical points and locate local max and local min of $f(x,y)=x^2-6x+y^2-4y$.

The test of local max and local min

Theorem: Let $z = f(x, y)$. Suppose f has continuous second partial derivatives in an open disk around (x_0, y_0) such that both $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$. There are three possibilities:

1. If $f_{xx}(x_0, y_0) \cdot f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2 > 0$,
   - and $f_{xx}(x_0, y_0) > 0$, then f has a local min at (x_0, y_0).
   - and $f_{xx}(x_0, y_0) < 0$, then f has a local max at (x_0, y_0).
2. If $f_{xx}(x_0, y_0) \cdot f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2 < 0$, then the point (x_0, y_0) is a saddle point of f.
3. If $f_{xx}(x_0, y_0) \cdot f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2 = 0$, then the test fails. We have no conclusion about the point (x_0, y_0). It may be the local max, the local min, the saddle point or none of these.

Example Suppose $f(x, y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$. Find the local max and local min of f.

Maximum and Minimum of a Function in a Closed Domain.

Theorem: Suppose $f : D \rightarrow R$ where D is a closed plane. If the function f is a continuous function and bounded on D, then f always has the absolute maximum and absolute minimum points in D.

Example Find the absolute maximum and absolute minimum points of

$$f(x, y) = x^2 - 6x + y^2 - 4y$$

on the area bounded by x-axis, y-axis and the line $x + y = 7$.

Problem Find the size of a rectangular box whose volume $V = 1000$ cubic feet and has the least surface area A.

Maximum and minimum of a function with boundary conditions

Example Find the size and the volume of a rectangular box with the maximum volume where the box is located in the first octant so that one boxes' corner is at the origin and the opposite corner is on the paraboloid $z = 4 - x^2 - 4y^2$.

Most of the time, the given conditions are not easy to put in the functions directly. For example, we want to find a local max and a local min of $f(x,y)=xy$ with the condition: $g(x,y)=\frac{x^2}{8}+\frac{y^2}{2}-1=0$. Hence, we may need the following method:

Method of Lagrange Multipliers

This method can be used to find the local max or local min of two variable functions (in general, multivariable functions) f with the given condition g . This can be done by following these steps.

1. Form a function F with 3 variables: x,y,λ such that $F(x,y,\lambda)=f(x,y)-\lambda g(x,y)$ where f and g have continuous first partial derivatives.
2. Find all critical points of F . That is, we find x,y,λ such that $F_x(x,y,\lambda)=0$, $F_y(x,y,\lambda)=0$, $F_\lambda(x,y,\lambda)=0$.
3. Calculate the value of f at all critical points (x,y) found in 2.

Example Find maximum and minimum of $f(x,y)=xy$ with the condition: $g(x,y)=\frac{x^2}{8}+\frac{y^2}{2}-1=0$.

Exercises

1. Estimate the value of $\sqrt{8.99}+\cos\left(0.02\right)$.
2. Find all local extreme (maximum and/or minimum), and saddle points of $f(x,y)=x^2y+2y^2-2xy-15y$.
3. Find the maximum temperature defined by

$$T(x,y)=x^2+y^2+4x-4y+3$$

on the disk circumference $x^2+y^2=2$.

Answer 1. $4-\frac{1}{600}=3.99833...$

2. Saddle points at $(-3,0)$ and $(5,0)$

   Local minimum at $(1,4)$

3. Maximum temperature = 13