MTH 101

Mathematics I

Module 3

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# INTEGRATION

## 1. Antiderivative

This lesson concerns the reverse process of taking derivative of a function as we learned last section. In particular, if $y' = f'(x)$, we want to find $y = f(x)$. Consider the following:

If $f(x) = x^2$, then $f'(x) = 2x$

If $f(x) = x^2 + 1$, then $f'(x) = 2x$

If $f(x) = x^2 + 2$, then $f'(x) = 2x$

………

If $f(x) = x^2 + C$, then $f'(x) = 2x$

Thus $f'(x) = 2x$ may have $f(x) = x^2$ or in general, $f(x) = x^2 + C$, where $C$ is some constant.

We call $x^2 + C$ an **antiderivative** of $2x$.

**Definition 1.1** Function $F(x)$ such that $F'(x) = f(x)$ is called an "an antiderivative of $f(x)$"

For examples, for any constant $C$

1. $F(x) = x^2 + \frac{1}{x} + C$ is an antiderivative of $f(x) = 2x - \frac{1}{x^2}$ since $F'(x) = 2x - \frac{1}{x^2}$.
2. $F(x) = \sin x + C$ is an antiderivative of $f(x) = \cos x$ since $F'(x) = \cos x$.
3. $F(x) = e^x + \tan^{-1} x + C$ is an antiderivative of $f(x) = e^x + \frac{1}{1+x^2}$ since $F'(x) = e^x + \frac{1}{1+x^2}$.

**Properties of an antiderivative of $f(x)$**

1. Every continuous function $f(x)$ has infinitely many antiderivatives of $f(x)$.
2. If $F_1(x), F_2(x)$ are both antiderivatives of $f(x)$, then the difference $F_1(x) - F_2(x) =$ constant.
3. If $F(x)$ is an antiderivative $f(x)$, then $F(x) + C$ where $C$ is some constant is also the antiderivative of $f(x)$. Thus we say that all antiderivatives of $f(x)$ are in the form of $F(x) + C$.

**Definition 1.2** The process of finding an antiderivative of $f(x)$ is called an integration

$$f(x) \to F(x) \text{ if } F'(x) = f(x)$$

**Notion**: $\int f(x)dx$ is called an "integral of $f(x)$ with respect to $x$"

$\int$ is an integration notation, $dx$ refers to the independent variable $x$ and $f(x)$ is called an integrand.

There are two types of integrations: indefinite and definite Integrals.

## 2 Indefinite Integral

Since $\frac{d}{dx}F(x) = f(x)$ or $dF(x) = f(x)dx$, then $\int dF(x) = \int f(x)dx = F(x) + C$ where $C$ is some constant.

**Note** that the notation $\int$ is the reverse operation of the derivative notation and we call this process an indefinite integral. Examples:

| | |
|---|---|
| $\frac{d}{dx}(x) = 1$ | $\int 1dx = \int dx = x + C$ |
| $\frac{d}{dx}(ax) = a$ | $\int adx = ax + C$ |
| $\frac{d}{dx}(x^{n+1}) = (n+1)x^n$ | $\int x^n dx = \frac{x^{n+1}}{n+1} + C,\ n \neq 1$ |
| $\frac{d}{dx}(\ln x) = \frac{1}{x}$ | $\int \frac{1}{x}dx = \ln\lvert x\rvert + C$ |
| $\frac{d}{dx}(e^x) = e^x$ | $\int e^x dx = e^x + C$ |

$$\frac{d}{dx}(\sin x) = \cos x \qquad \int \cos x dx = \sin x + C$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \int \sin x dx = -\cos x + C$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \int \sec^2 x dx = \tan x + C$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x \qquad \int \csc^2 x dx = -\cot x + C$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \int \sec x \tan x dx = \sec x + C$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x \qquad \int \csc x \cot x dx = -\csc x + C$$

**Rule of Algebra for Antiderivative**

1. Constant multiplication

   $\int af(x)dx = a\int f(x)dx,\ a$ is some constant

2. Addition and subtraction

$$\int \left[ f(x) \pm g(x) \right] dx = \int f(x)dx \pm \int g(x)dx$$

**Example 1** Evaluate $\int (5x - x^2 + 2)dx$

<u>Solution</u> $\int (5x - x^2 + 2)dx = \int 5xdx - \int x^2 dx + \int 2dx$

$$= 5\int xdx - \int x^2 dx + 2\int dx$$

$$= 5\left(\frac{x^2}{2} + c_1\right) - \left(\frac{x^3}{3} + c_2\right) + 2\left(x + c_3\right)$$

$$= \frac{5x^2}{2} + 5c_1 - \frac{x^3}{3} - c_2 + 2x + 2c_3$$

$$= \frac{5x^2}{2} - \frac{x^3}{3} + 2x + C$$

where $C = 5c_1 - c_2 + 2c_3$

**Example 2** Evaluate $\int (8x^3 + 4x - 6\sqrt{x} - \frac{2}{\sqrt[3]{x}} + \frac{5}{x^2})dx$

<u>Solution</u>

**Example 3** Evaluate $\int (3e^x - 7\sin x + \frac{5}{x})dx$

Solution

**Example 4** Evaluate $\int \frac{\cos x}{\sin^2 x} dx$

Solution

## 3. Definite Integral

A definite integral of $f(x)$ from $a$ to $b$ is written as

$$\int_a^b f(x)dx$$

$a$ and $b$ are called limits of integration, where $a$ is the lower limit and $b$ is the upper limit.

**Definition 3.1** A definite integral of $f(x)$ is a continuous function on $a \le x \le b$ such that

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$$

**Definition 3.2** If $a < b$ and $f(x)$ is integrable on $a \le x \le b$

1. $\int_a^a f(x)dx = 0$
2. $\int_a^b f(x)dx = -\int_b^a f(x)dx$
3. $\int_a^b f(x)dx > 0$ where $f(x) > 0$, and

   $\int_a^b f(x)dx < 0$ where $f(x) < 0$

**Evaluation process of an definite integral**

Step 1 Find antiderivative of $F(x)$

Step 2 Calculate $F(b)-F(a)$ by plugging $x=b$ and $x=a$ into $F(x)$ we found in step 1

**Properties of a definite integral**

Let $f(x)$ and $g(x)$ be integrable functions on $a \le x \le b$ and $C$ be some constant.

1. $\int_a^b C dx = C(b-a)$
2. $\int_a^b Cf(x)dx = C\int_a^b f(x)dx$
3. $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

**Example 5** Evaluate $\int_1^2 \left[ 5x^2 + 3x - 1 - \dfrac{6}{x} \right] dx$

<u>Solution</u>

$$\int_1^2 \left[ 5x^2 + 3x - 1 - \frac{6}{x} \right] dx = \int_1^2 5x^2 dx + \int_1^2 3x dx - \int_1^2 dx - \int_1^2 \frac{6}{x} dx$$

$$= 5\int_1^2 x^2 dx + 3\int_1^2 x dx - \int_1^2 dx - 6\int_1^2 \frac{1}{x} dx$$

$$= 5\left.\frac{x^3}{3}\right|_1^2 + 3\left.\frac{x^2}{2}\right|_1^2 - x\Big|_1^2 - 6\ln|x|\Big|_1^2$$

$$= 5\left(\frac{8-1}{3}\right) + 3\left(\frac{4-1}{2}\right) - (2-1)$$

$$-6(\ln 2 - \ln 1)$$

$$= 5\left(\frac{7}{3}\right) + 3\left(\frac{3}{2}\right) - (1) - 6(\ln 2 - 0)$$

$$= \frac{35}{3} + \frac{9}{2} - 1 - 6\ln 2$$

$$= \frac{70 + 27 - 6}{6} - 6\ln 2$$

$$= \frac{91}{6} - 6\ln 2$$

Thus $$\int_1^2 \left[ 5x^2 + 3x - 1 - \frac{6}{x} \right] dx = \frac{91}{6} - 6\ln 2$$

**Example 6** Evaluate $\int_{\pi}^{\pi}\left[e^{x}+4\sin x\right]dx$

<u>Solution</u>

**Example 7** Evaluate $\int_0^3 |x-2|dx$

<u>Solution</u> From $f(x) = |x-2|$

We can write $f(x) = \begin{cases} x-2; & x \geq 2 \\ -(x-2); & x < 2 \end{cases}$

Thus
$$\int_0^3 |x-2|dx = \int_0^2 |x-2|dx + \int_2^3 |x-2|dx$$
$$= \int_0^2 (-x+2)dx + \int_2^3 (x-2)dx$$
$$= -\int_0^2 xdx + \int_0^2 2dx + \int_2^3 xdx - \int_2^3 2dx$$
$$= -\frac{x^2}{2}\Bigg|_0^2 + 2x\Big|_0^2 + \frac{x^2}{2}\Bigg|_2^3 - 2x\Big|_2^3$$
$$= -\frac{1}{2}[4-0] + 2[2-0]$$
$$+\frac{1}{2}[9-4] - 2[3-2]$$
$$= -2 + 4 + \frac{5}{2} - 2$$
$$= \frac{5}{2}$$

Hence $\int_0^3 |x-2|dx = \frac{5}{2}$

**Example 8** Evaluate

$$\int_{-2}^{1} f(x)dx \text{ where } f(x) = \begin{cases} 2 - x^2; & x \geq 0 \\ x + 2; & x < 0 \end{cases}$$

<u>Solution</u>

# 4. Techniques of Integration

## 4.1 Integration by Substitution

We change the integrand by substitution.

**Example 9** Evaluate $\int (3x-5)^{20}\,dx$

Solution Let $u = 3x-5$. Then $du = 3dx$ or $dx = \dfrac{du}{3}$

Thus
$$\int (3x-5)^{20}\,dx = \int u^{20}\,\frac{du}{3}$$
$$= \frac{1}{3}\int u^{20}du$$
$$= \frac{1}{3}\cdot\frac{u^{21}}{21} + C$$
$$= \frac{(3x-5)^{21}}{63} + C$$

Hence
$$\int (3x-5)^{20}\,dx = \frac{(3x-5)^{21}}{63} + C$$

**Example 10** Evaluate $\int \frac{(\ln x)^2}{x \ln 9} dx$

<u>Solution</u>

**Example 11** Evaluate $\int (x+3)\sqrt{x+1}dx$

Solution

**The procedure of integration by substitution**

1. Define $u = g(x)$ and find $du = g'(x)dx$
2. Rewrite $\int f(x)dx$ in terms of new variable $u$ to get $\int h(u)du$
3. Find the integral $\int h(u)du = H(u) + C$
4. Plug $u = g(x)$ back into the resulting function in step 3.

$$\int f(x)dx = H(u) + C = H(g(x)) + C = F(x) + C$$

**Example 12** Evaluate $\int x^2(1-x)^{100}\,dx$

Solution Let $u = 1 - x$ or $x = 1 - u$

Then $x^2 = (1-u)^2$ and $du = -dx$

Thus $\int x^2(1-x)^{100}\,dx = \int (1-u)^2 u^{100}(-du)$

$$= \int (1 - 2u + u^2)(-u^{100})du$$

$$= \int -u^{100}du + \int 2u^{101}du - \int u^{102}du$$

$$= -\frac{u^{101}}{101} + 2\frac{u^{102}}{102} - \frac{u^{103}}{103} + C$$

$$= \frac{2(1-x)^{102}}{102} - \frac{(1-x)^{101}}{101} - \frac{(1-x)^{103}}{103} + C$$

Hence $\int x^2(1-x)^{100}\,dx = \dfrac{2(1-x)^{102}}{102} - \dfrac{(1-x)^{101}}{101}$

$$-\frac{(1-x)^{103}}{103} + C$$

**Example 13** Evaluate $\int \frac{\sec^2 2x dx}{1+\tan 2x}$

Solution

**Example 14** Evaluate $\int \frac{(x^2+1)dx}{2x-3}$

<u>Solution</u> Let $u = 2x-3$. Then $du = 2dx$ or $\frac{du}{2} = dx$

and $x = \frac{u+3}{2}$, $x^2 = \left(\frac{u+3}{2}\right)^2$, $x^2 = \frac{1}{4}\left(u^2+6u+9\right)$

then $x^2+1 = \frac{1}{4}\left(u^2+6u+9+4\right)$

Substitution:

$$\begin{aligned}
\int \frac{(x^2+1)dx}{2x-3} &= \int \frac{1}{4}\cdot\frac{(u^2+6u+9+4)}{u}\cdot\frac{du}{2} \\
&= \frac{1}{8}\int \frac{(u^2+6u+13)}{u}du \\
&= \frac{1}{8}\int (u+6+\frac{13}{u})du \\
&= \frac{1}{8}\left[\frac{u^2}{2}+6u+13\ln|u|\right]+C \\
&= \frac{1}{8}\left\{\frac{(2x-3)^2}{2}+(2x-3)\right. \\
&\qquad \left.+13\ln|2x-3|\right\}+C
\end{aligned}$$

Thus

$$\int \frac{(x^2+1)dx}{2x-3} = \frac{1}{8}\left[\frac{(2x-3)^2}{2}+(2x-3)+13\ln|2x-3|\right]+C$$

**Remark**

Once we change the variable in the definite integral by substitution technique, we also need to change the limits of integration.

**Example 15** Evaluate $\int_0^1 xe^{4x^2+1}dx$

Solution Let $u = 4x^2 + 1$. Then $du = 8xdx$ or $xdx = \dfrac{du}{8}$

When $x = 0$, then $u = 1$. And when $x = 1$, then $u = 5$

Substitution: $$\int_0^1 xe^{4x^2+1}dx = \int_0^1 e^{4x^2+1}xdx$$

$$= \int_1^5 \frac{e^u du}{8}$$

$$= \frac{1}{8}\int_1^5 e^u du$$

$$= \frac{1}{8}e^u\Big|_1^5$$

$$= \frac{1}{8}\left[e^5 - e^1\right]$$

Thus $$\int_0^1 xe^{4x^2+1}dx = \frac{1}{8}\left(e^5 - e\right)$$

**Example 16** Evaluate $\int_0^3 x(1+x)^{\frac{1}{2}}\,dx$

<u>Solution</u>

## 4.2 Integration by Parts

We use this technique when integration by substitution doesn't work. We consider the integral as $\int udv$ where $dv$ is a part of the function consisting of $dx$ and $f(x)$ or $g(x)$.

Formula used to find the integration by parts:

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

or $$d(uv) = udv + vdu$$

$$udv = d(uv) - vdu$$

and $$\int udv = \int d(uv) - \int vdu$$

$$\int udv = uv - \int vdu$$

**Remark**

This technique is to express $\int udv$ in terms of $uv$ and $\int vdu$ which is easier to be integrated. Thus choosing appropriate $u$ and $v$ is a crucial step for doing integration by parts.

**Summary**

Let $\int f(x)g(x)dx = \int h(x)dx = \int udv = uv - \int vdu$

To pick $u$ and $v$, we consider

1. $dv$ is easy to get integrated so that we have $v$
2. $\int vdu$ exists

In case of, definite integral:

$$\int_a^b udv = uv\Big|_a^b - \int_a^b vdu$$

**Example 17** Evaluate $\int x\ln x dx$

Solution Let $u = \ln x$ and $dv = xdx$

$$du = \frac{dx}{x} \text{ and } \int dv = \int xdx \quad \text{or} \quad v = \frac{x^2}{2}$$

From $\int udv = uv - \int vdu$

Then $\int x\ln x dx = \int \ln x(xdx)$

$$= \ln(x)\left(\frac{x^2}{2}\right) - \int\left(\frac{x^2}{2}\right)\left(\frac{dx}{x}\right)$$

$$= \ln(x)\left(\frac{x^2}{2}\right) - \frac{1}{2}\int xdx$$

$$= \ln(x)\left(\frac{x^2}{2}\right) - \frac{1}{2}\cdot\frac{x^2}{2} + C$$

$$= \frac{x^2\ln x}{2} - \frac{x^2}{4} + C$$

Thus $$\int x\ln x dx = \frac{x^2\ln x}{2} - \frac{x^2}{4} + C$$

**Example 18** Evaluate $\int_1^2 \ln x dx$

<u>Solution</u>

**Example 19** Evaluate $\int \tan^{-1} x \, dx$

<u>Solution</u>

Note: Some integrals may need several integrations by parts.

**Example 20** Evaluate $\int e^{2x} \sin x \ dx$

<u>Solution</u> Let $u = e^{2x}$ and $dv = \sin x dx$

$du = 2e^{2x} dx$ and $v = -\cos x$

Then

$$\int e^{2x} \sin x \ dx = e^{2x}(-\cos x) - \int -\cos x (2e^{2x} dx)$$
$$= -e^{2x} \cos x + 2\int e^{2x} \cos x dx$$

Next, consider $2\int e^{2x} \cos x dx$

Let $u = e^{2x}$ and $dv = \cos x dx$

$du = 2e^{2x} dx$ and $v = \sin x$

$$2\int e^{2x} \cos x dx = 2\left[ e^{2x} \sin x - \int \sin x (2e^{2x} dx) \right]$$
$$= 2e^{2x} \sin x - 4\int e^{2x} \cos x dx$$

Then

$$\int e^{2x} \sin x \ dx = -e^{2x} \cos x + 2\int e^{2x} \cos x dx$$
$$= -e^{2x} \cos x + 2e^{2x} \sin x - 4\int e^{2x} \sin x \ dx + C$$
$$5\int e^{2x} \sin x \ dx = -e^{2x} \cos x + 2e^{2x} \sin x + C$$

Hence, $\int e^{2x} \sin x \ dx = \dfrac{1}{5}\left[ -e^{2x} \cos x + 2e^{2x} \sin x \right] + C$

**Rules to pick $u$ and $dv$**

1. $u$ should have a simple derivative.
2. $dv$ may be complicated but easy to get integrated.
3. $\int vdu$ is easier to evaluate than $\int udv$

Examples of $u$ and $dv$

1. $\int x^n e^{ax} dx, \int x^n \cos axdx, \int x^n \sin axdx$

   Then $u = x^n$ and $dv$ is the rest of the integrand

2. $\int x^n \sin^{-1} xdx, \int x^n \cos^{-1} xdx, \int x^n \tan^{-1} xdx$

   Then $u = \sin^{-1} x$ or $u = \cos^{-1} x$ or $u = \tan^{-1} x$, respectively and $dv$ is the rest

3. $\int x^m [\ln x]^n dx$ where $m \neq -1$

   Then $u = [\ln x]^n$ and $dv$ is the rest

### 4.3 Integration of Rational Function by Partial Fraction

It is used when the integrand is in a form of rational function $\dfrac{f(x)}{g(x)}$

$$\frac{f(x)}{g(x)} = \frac{a_0 + a_1 x + a_2 x^2 + \ldots\ldots\ldots + a_{n-1}x^{n-1} + a_n x^n}{b_0 + b_1 x + b_2 x^2 + \ldots\ldots\ldots + b_{m-1}x^{m-1} + b_m x^m}; n < m$$

Express $\dfrac{f(x)}{g(x)}$ as a partial fraction: ex $\dfrac{5x-3}{x^2-2x-3} = \dfrac{2}{x+1} + \dfrac{3}{x-3}$

which is found by

$$\frac{5x-3}{x^2-2x-3} = \frac{5x-3}{(x+1)(x-3)} = \frac{A}{x+1} + \frac{B}{x-3}$$

and $\quad 5x-3 = A(x-3) + B(x+1)$

$$5x-3 = (A+B)x + (-3A+B)$$

Calculating $A$ and $B$ by comparing coefficients of $x$

$$A+B=5 \qquad \text{and} \qquad -3A+B=-3$$

Solve to get $A=2$ and $B=3.$ We call $A$ and $B$ constants calculated by undeterminated coefficients.

**Conditions on partial fractions:**

$\dfrac{f(x)}{g(x)}$ **can be expressed as a partial fraction if**

1. Power of $f(x)$ is higher than or equal to power of $g(x)$ ($n \geq m$), we first have to divide $g(x)$ by $f(x)$ to get

$$\frac{f(x)}{g(x)} = \phi(x) + \frac{h(x)}{g(x)}$$

where $h(x), g(x)$ are both polynomials and power of $h(x)$ is less than power of $g(x)$.

2. $g(x)$ can be factor out as linear or quadratic factors

2.1 Types of factors

a. Linear factor is in a form of $(ax+b)$ where $a, b$ are real.

b. Irreducible quadratic factor is in a form of $(ax^2+bx+c)$ where $a, b, c$ are real

*Procedure of Integration by Partial Fraction*

***Consider a rational function*** $\dfrac{f(x)}{g(x)}$

***Case 1*** $g(x)$ has only non-repeated linear factors

$$g(x) = (a_1x+b_1)(a_2x+b_2)........(a_nx+b_n)$$

where $\dfrac{b_1}{a_1} \neq \dfrac{b_2}{a_2} \neq ........ \neq \dfrac{b_n}{a_n}$ and $a_1, a_2, ......., a_n \neq 0$

Then

$$\frac{f(x)}{g(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + ..... + \frac{A_n}{a_nx+b_n}$$

where $A_1, A_2, ......, A_n$ are all constants we need to find.

**Example 37** Evaluate $\int \dfrac{2x^2+5x-1}{x^3+x^2-2x}dx$

<u>Solution</u> Consider $x^3+x^2-2x$

$$x^3+x^2-2x = x(x-1)(x+2)$$

Thus $\dfrac{2x^2+5x-1}{x^3+x^2-2x} = \dfrac{A_1}{x} + \dfrac{A_2}{x-1} + \dfrac{A_3}{x+2}$

$$2x^2+5x-1 = A_1(x-1)(x+2) + A_2(x)(x+2) + A_3x(x-1)$$

$$= A_1(x^2+x-2) + A_2(x^2+2x) + A_3(x^2-x)$$

Compare coefficients:

$$A_1 + A_2 + A_3 = 2$$

$$A_1 + 2A_2 - A_3 = 5$$

$$-2A_1 = -1$$

Solve to get $\quad A_1 = \dfrac{1}{2},\ A_2 = 2,\ A_3 = -\dfrac{1}{2}$

Thus $\quad \dfrac{2x^2+5x-1}{x^3+x^2-2x} = \dfrac{1}{2x} + \dfrac{2}{x-1} - \dfrac{1}{2(x+2)}$

Plug it back into the integral:

$$\int \frac{2x^2+5x-1}{x^3+x^2-2x}\,dx = \int \frac{1}{2x}\,dx + \int \frac{2}{x-1}\,dx - \int \frac{1}{2(x+2)}\,dx$$

$$= \frac{1}{2}\ln|x| + 2\ln|x-1| - \frac{1}{2}\ln|x+2| + C$$

Hence

$$\int \frac{2x^2+5x-1}{x^3+x^2-2x}dx = \frac{1}{2}\ln|x| + 2\ln|x-1| - \frac{1}{2}\ln|x+2| + C$$

***Case 2*** $g(x)$ has only repeated linear factors.

$$g(x) = (ax+b)^n$$

Then

$$\frac{f(x)}{g(x)} = \frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \ldots\ldots + \frac{A_n}{(ax+b)^n}$$

where $A_1, A_2, \ldots\ldots, A_n$ are all constant we need to find.

**Example 38** Evaluate $\int \frac{x^2+2x+3}{(x-1)(x+1)^2}dx$

<u>Solution</u> Consider

$$\frac{x^2+2x+3}{(x-1)(x+1)^2} = \frac{A_1}{x-1} + \frac{A_2}{x+1} + \frac{A_3}{(x+1)^2}$$

$$x^2+2x+3 = A_1(x+1)^2 + A_2(x-1)(x+1) + A_3(x-1)$$
$$= A_1(x^2+2x+1) + A_2(x^2-1) + A_3(x-1)$$

Compare the coefficients:

$$A_1 + A_2 = 1$$
$$2A_1 + A_3 = 2$$
$$A_1 - A_2 - A_3 = 3$$

Solve to get $A_1 = \frac{3}{2},\ A_2 = -\frac{1}{2},\ A_3 = -1$

Thus $\frac{x^2+2x+3}{(x-1)(x+1)^2} = \frac{3}{2(x-1)} - \frac{1}{2(x+1)} - \frac{1}{(x+1)^2}$

Plug it back to the integral:

$$\int \frac{(x^2+2x+3)dx}{(x-1)(x+1)^2} = \frac{3}{2}\int \frac{dx}{x-1} - \frac{1}{2}\int \frac{dx}{x+1} - \int \frac{dx}{(x+1)^2}$$
$$= \frac{3}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + \frac{1}{(x+1)} + C$$

Hence,

$$\int \frac{(x^2+2x+3)dx}{(x-1)(x+1)^2} = \frac{3}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + \frac{1}{(x+1)} + C$$

***Case 3*** $g(x)$ has only non repeated irreducible quadratic factors $ax^2+bx+c$:

$$\frac{f(x)}{g(x)}=\frac{Ax+B}{ax^2+bx+c}$$

where $A, B$ are constants we need to find.

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**Example 39** Evaluate $\int \frac{5x^2+3x-2}{x^3-1}\,dx$

<u>Solution</u> Consider

$$\frac{5x^2+3x-2}{x^3-1}=\frac{5x^2+3x-2}{(x-1)(x^2+x+1)}$$

$$\frac{5x^2+3x-2}{x^3-1}=\frac{A}{x-1}+\frac{Bx+C}{x^2+x+1}$$

$$5x^2+3x-2=A(x^2+x+1)+(Bx+C)(x-1)$$

$$=A(x^2+x+1)+Bx^2-Bx+Cx-C$$

Compare the coefficients:

$$A+B=5$$

$$A-B+C=3$$

$$A-C=-2$$

Solve to get $A=2,\ B=3,\ C=4$

Thus $$\frac{5x^2+3x-2}{x^3-1}=\frac{2}{x-1}+\frac{3x+4}{x^2+x+1}$$

Plug it back into the integral:

$$\int \frac{(5x^2+3x-2)dx}{x^3-1} = \int \frac{2dx}{x-1} + \int \frac{(3x+4)dx}{x^2+x+1}$$
$$= \int \frac{2dx}{x-1} + \int \frac{(3x+4)dx}{x^2+x+1}$$
$$= 2\ln|x-1| + \int \frac{(3x+4)dx}{x^2+x+1}$$

Next consider $\int \frac{(3x+4)dx}{x^2+x+1} = \int \frac{(3x+4)dx}{\left[x+\frac{1}{2}\right]^2+\frac{3}{4}}$

Let $u = x+\frac{1}{2}$ and $du = dx$

Thus

$$\int \frac{(3x+4)dx}{\left[x+\frac{1}{2}\right]^2+\frac{3}{4}} = \int \frac{3\left[u-\frac{1}{2}\right]+4}{u^2+\frac{3}{4}}du$$
$$= \int \frac{3u+\frac{5}{2}}{u^2+\frac{3}{4}}du$$
$$= 3\int \frac{udu}{u^2+\frac{3}{4}} + \frac{5}{2}\int \frac{du}{u^2+\frac{3}{4}}$$
$$= \frac{3}{2}\ln(u^2+\frac{3}{4}) + \frac{5(2)}{2(\sqrt{3})}\tan^{-1}\frac{2}{\sqrt{3}}u + C$$
$$= \frac{3}{2}\ln(x^2+x+1) + \frac{5}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}} + C$$

Hence

$$\int \frac{(5x^2+3x-2)dx}{x^3-1} = 2\ln\left|x-1\right| + \frac{3}{2}\ln(x^2+x+1) + \frac{5}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}} + C$$

***Case 4*** $g(x)$ has only repeated irreducible quadratic factors: $(ax^2+bx+c)^n, n \geq 2$:

$$\frac{f(x)}{g(x)} = \frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \ldots\ldots + \frac{A_nx+B_n}{(ax^2+bx+c)^n}$$

where $A_1, \ldots\ldots A_n, B_1, \ldots\ldots, B_n$ are all constants we need to find.

**Example 40** Evaluate $\int \frac{(x^3+1)dx}{(x^2+4)^2}$

<u>Solution</u> Consider

$$\frac{x^3+1}{(x^2+4)^2} = \frac{Ax+B}{x^2+4} + \frac{Cx+D}{(x^2+4)^2}$$

$$x^3+1 = (Ax+B)(x^2+4) + (Cx+D)$$

$$= Ax^3 + 4Ax + Bx^2 + 4B + Cx + D$$

Compare coefficients:

$$A = 1$$

$$B = 0$$

$$4A + C = 0$$

$$4B + D = 1$$

Solve to get $A = 1,\ B = 0,\ C = -4, D = 1$

Thus $$\frac{x^3+1}{(x^2+4)^2} = \frac{1x+0}{x^2+4} + \frac{-4x+1}{(x^2+4)^2}$$

Plug it back into the integral:

$$\int \frac{(x^3+1)dx}{(x^2+4)^2} = \int \frac{xdx}{x^2+4} - 4\int \frac{xdx}{(x^2+4)^2} + \int \frac{dx}{(x^2+4)^2}$$

$$= \frac{1}{2}\ln(x^2+4) - 4\int \frac{xdx}{(x^2+4)^2} + \int \frac{dx}{(x^2+4)^2}$$ Next

consider $$-4\int \frac{xdx}{(x^2+4)^2}$$

Let $u = x^2 + 4$ and $du = 2xdx$

So we have $$-4\int \frac{xdx}{(x^2+4)^2} = -2\int \frac{du}{u^2}$$

$$= 2u^{-1} + C$$

$$= \frac{2}{x^2+4} + C$$

And for $\int \frac{dx}{(x^2+4)^2}$

We let $x = 2\tan\theta$ and $dx = 2\sec^2\theta d\theta$

We then have $$\int \frac{dx}{(x^2+4)^2} = \int \frac{2\sec^2\theta d\theta}{(4\tan^2\theta+4)^2}$$

$$= \frac{1}{8}\int \frac{\sec^2\theta d\theta}{(\tan^2\theta+1)^2}$$

$$= \frac{1}{8}\int \frac{d\theta}{\sec^2\theta}$$

$$= \frac{1}{8}\int \cos^2\theta d\theta$$

$$= \frac{1}{16}\int (1+\cos 2\theta)d\theta$$

$$= \frac{1}{16}\left[\theta + \frac{\sin 2\theta}{2}\right] + C$$

$$= \frac{1}{16}\left[\tan^{-1}\frac{x}{2} + \frac{2x}{(x^2+4)}\right] + C$$

Hence

$$\int \frac{(x^3+1)dx}{(x^2+4)^2} = \frac{1}{2}\ln(x^2+4) + \frac{2}{x^2+4} + \frac{1}{16}\tan^{-1}\frac{x}{2}$$

$$+ \frac{x}{8(x^2+4)} + C$$

**Example 41** Evaluate $\int \frac{(x^5 - x^4 - 3x + 5)}{x^4 - 2x^3 + 2x^2 - 2x + 1} dx$

Solution

**Exercise 1**

Evaluate the following integrals

1. $\int 3x^2(x^3+2)^2\,dx$

2. $\int x^2\sqrt{x^3+2}\,dx$

3. $\int \dfrac{8x^2}{(x^3+2)}\,dx$

4. $\int \dfrac{x^2}{\sqrt{x^3+2}}\,dx$

5. $\int 3x\sqrt{1-2x^2}\,dx$

6. $\int \dfrac{x+3}{\sqrt[3]{x^2+6x}}\,dx$

7. $\int (3x^2-2)(x^3-2x)\,dx$

8. $\int \dfrac{x+1}{x^2+2x+5}\,dx$

9. $\int x^2\sqrt{1+x}\,dx$

10. $\int \dfrac{x^2}{1-2x^3}\,dx$

11. $\int (e^x+1)^3\,dx$

12. $\int \cos^3 2x \sin 2x\,dx$

13. $\int e^{\cos x}\sin x\,dx$

14. $\int \dfrac{\cos x\,dx}{\sqrt{4-\sin^2 x}}$

15. $\int \dfrac{e^{\sqrt{1+x}}}{\sqrt{1+x}}\,dx$

16. $\int \cos 2x\sqrt{1-\sin 2x}\,dx$

17. $\int 3^{2x+1}\,dx$

18. $\int \dfrac{e^{\tan^{-1}2x}}{1+4x^2}\,dx$

19. $\int \left[\dfrac{\ln x}{x}\right]^3 dx$

20. $\int \dfrac{1}{x\ln x}\,dx$

Evaluate the following definite integrals

21. $\int_1^5 \dfrac{x+3}{\sqrt{2x-1}}\,dx$

22. $\int_0^1 \dfrac{x}{x^2+4}\,dx$

23. $\int_{1}^{8} \sqrt{1+3x}\,dx$

24. $\int_{4}^{8} \frac{x\,dx}{\sqrt{x^2-15}}$

25. $\int_{0}^{2\pi} \sin\frac{x}{2}\,dx$

**Answers to exercise 1**

1. $\left[\frac{x^3+2}{3}\right]^3 + C$

2. $\frac{2}{9}(x^3+2)^{\frac{3}{2}} + C$

3. $\frac{-4}{3(x^3+2)^2} + C$

4. $\frac{2}{3}\sqrt{x^3+2} + C$

5. $-\frac{1}{2}(1-2x^2)^{\frac{3}{2}} + C$

6. $\frac{3}{4}(x^2+6x)^{\frac{2}{3}} + C$

7. $\frac{1}{6}(x^3-2x)^6 + C$

8. $\frac{1}{2}\ln\left|x^2+2x+5\right| + C$

9. $\frac{2}{7}(1+x)^{\frac{7}{2}} - \frac{4}{5}(1+x)^{\frac{5}{2}} + \frac{2}{3}(1+x)^{\frac{3}{2}} + C$

10. $-\frac{1}{6}\ln\left|1-2x^3\right| + C$

11. $\frac{1}{4}(e^x+1) + C$

12. $-\frac{\cos^4 2x}{8} + C$

13. $-e^{\cos x} + C$

14. $\sin^{-1}\left(\frac{\sin x}{2}\right) + C$

15. $2e^{\sqrt{1+x}} + C$

16. $-\frac{1}{3}(1-\sin 2x)^{\frac{3}{2}} + C$

17. $\frac{3^{2x+1}}{2\ln 3} + C$

18. $\frac{1}{2}e^{\tan^{-1} 2x} + C$

19. $\frac{1}{4}\left[\ln x\right]^4 + C$

20. $\ln\left|\ln x\right| + C$

21. 20

22. $\frac{1}{2}\ln\frac{5}{4}$

23. 26

24. 6

25. 4

**Exercise 2**

Evaluate the following integrals

1. $\int x\sin x dx$
2. $\int xe^{x} dx$
3. $\int x^{2}\ln x dx$
4. $\int x\sqrt{1+x} dx$
5. $\int \sec^{3} x dx$
6. $\int x^{2}\sin x dx$
7. $\int x^{2}e^{2x} dx$
8. $\int x\cos x dx$
9. $\int x\sec^{2} 3x dx$
10. $\int \cos^{-1} 2x dx$
11. $\int \tan^{-1} x dx$
12. $\int \frac{xe^{x}}{(1+x)^{2}} dx$
13. $\int x\tan^{-1} x dx$
14. $\int x^{2}e^{-3x} dx$
15. $\int x^{3}\sin x dx$
16. $\int x\sin^{-1} x^{2} dx$
17. $\int \sin x\sin 3x dx$
18. $\int \sin(\ln x) dx$
19. $\int e^{ax}\cos bx dx$
20. $\int e^{ax}\sin bx dx$

Show how to use reduction formula to the following integrals.

21. $\int u^{n}e^{au} du$
22. $\int u^{n}\cos bu du$

Evaluate the following definite integrals

23. $\int_1^e \ln x dx$

24. $\int_0^{\frac{\pi}{3}} x^2 \sin 3x dx$

25. $\int_0^{\sqrt{2}} x^3 e^{x^2} dx$

**Answers to exercise 2**

1. $-x\sin x + \sin x + C$
2. $xe^x - e^x + C$
3. $\frac{x^3 \ln x}{3} - \frac{1}{9}x^3 + C$
4. $\frac{2}{3}x(1+x)^{\frac{3}{2}} - \frac{4}{15}(1+x)^{\frac{5}{2}} + C$
5. $\frac{1}{2}(\sec x \tan x + \ln\left|\sec x \tan x\right|) + C$
6. $-x^2 \cos x + 2x\sin x + 2\cos x + C$
7. $\frac{1}{2}x^3 e^{2x} - \frac{3}{4}x^2 e^{2x} + \frac{3}{4}xe^{2x} - \frac{3}{8}e^{2x} + C$
8. $x\sin x + \cos x + C$
9. $\frac{1}{3}x\tan 3x - \frac{1}{9}\ln\left|\sec 3x\right| + C$
10. $x\cos^{-1} 2x - \frac{1}{2}\sqrt{1-4x^2} + C$
11. $x\tan^{-1} x - \ln\sqrt{1+x^2} + C$

12. $\dfrac{e^x}{1+x}+C$

13. $\dfrac{1}{2}(x^2+1)\tan^{-1}x-\dfrac{1}{2}x+C$

14. $-\dfrac{1}{3}e^{-3x}(x^2+\dfrac{2}{9}x+\dfrac{2}{9})+C$

15. $-x^3\cos x+3x^2\sin x+6x\cos x-6\sin x+C$

16. $\dfrac{1}{2}x^2\sin^{-1}x^2+\dfrac{1}{2}\sqrt{1-x^4}+C$

17. $\dfrac{1}{8}\sin 3x\cos x-\dfrac{3}{8}\cos 3x\sin x+C$

18. $\dfrac{1}{2}\left[x\sin(\ln x)-x\cos(\ln x)\right]+C$

19. $\dfrac{e^{ax}(b\sin bx+a\cos bx)}{a^2+b^2}+C$

20. $\dfrac{e^{ax}(a\sin bx-b\cos bx)}{a^2+b^2}+C$

21. $\dfrac{1}{a}u^n e^{au}-\dfrac{n}{a}\int u^{n-1}e^{au}du$

22. $\dfrac{1}{b}u^n\sin bu-\dfrac{n}{b}\int u^{n-1}\sin bu du$

23. 1

24. $\dfrac{1}{27}(\pi^2-4)$

25. $\dfrac{1}{2}(e^2+1)$

## Exercise 3

Evaluate the following integrals

1. $\int \frac{1}{x^2-4}dx$

2. $\int \frac{x+1}{x^3+x^2-6x}dx$

3. $\int \frac{1}{x^2+7x+6}dx$

4. $\int \frac{x}{x^2-3x-4}dx$

5. $\int \frac{x^2-3x-1}{x^3+x^2-2x}dx$

6. $\int \frac{x^2+3x-4}{x^2-2x-8}dx$

7. $\int \frac{x}{(x-2)^2}dx$

8. $\int \frac{3x+5}{x^3-x^2-x+1}dx$

9. $\int \frac{x^4-x^3-x-1}{x^3-x^2}dx$

10. $\int \frac{x^3+x^2+x+2}{x^4+3x^2+2}dx$

11. $\int \frac{x^2}{a^4-x^4}dx$

12. $\int \frac{2x^2+3}{(x^2+1)^2}dx$

13. $\int \frac{1}{x^3+x}dx$

14. $\int \frac{x^3+x^2+x+3}{(x^2+1)(x^2+3)}dx$

15. $\int \frac{2x^3}{(x^2+1)^2}dx$

16. $\int \frac{2x^3+x^2+4}{(x^2+4)^2}dx$

17. $\int \frac{x^3+x-1}{(x^2+1)^2}dx$

18. $\int \frac{x^4}{(1-x)^3}dx$

19. $\int \frac{x^5-x^4+4x^3-4x^2+8x-4}{(x^2+2)^3}dx$

20. $\int \frac{1}{e^{2x}-3e^x}dx$

21. $\int \frac{\sin x}{\cos x(1+\cos^2 x)}dx$

22. $\int \frac{(2+\tan^2\theta)\sec^2\theta}{1+\tan^3\theta}d\theta$

Evaluate the following definite integrals

23. $\int_{-1}^{2} \frac{1}{x^2-9}dx$

24. $\int_{-8}^{-3} \frac{x+2}{x(x-2)^2}dx$

25. $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{\sin x dx}{\cos^2 x - 5\cos x + 4}$

**Answers to exercise 3**

1. $\frac{1}{4}\ln\left|\frac{x-2}{x+2}\right| + C$

2. $\frac{1}{30}\ln\left|\frac{(x-2)^9}{(x)^5(x+3)^4}\right| + C$

3. $\frac{1}{5}\ln\left|\frac{x+1}{x+6}\right| + C$

4. $\frac{1}{5}\ln\left|(x-4)(x+1)^4\right| + C$

5. $\frac{1}{2}\ln\left|\frac{x(x+2)^3}{(x-1)^2}\right| + C$

6. $x + \ln\left|(x-4)^4(x+2)\right| + C$

7. $\ln|x-2| - \frac{2}{x-2} + C$

8. $-\frac{4}{x-1} + \frac{1}{2}\ln\left|\frac{x+1}{x-1}\right| + C$

9. $\frac{x^2}{2} - \frac{1}{x} - 2\ln\left|\frac{x-1}{x}\right| + C$

10. $\tan^{-1} x + \frac{1}{2}\ln\left|x^2+2\right| + C$

11. $\frac{1}{4a}\ln\left|\frac{a+x}{a-x}\right| - \frac{1}{2a}\tan^{-1}\frac{x}{a} + C$

12. $\frac{5}{2}\tan^{-1} x + \frac{x}{2(x^2+1)} + C$

13. $\ln\left|\dfrac{x}{\sqrt{x^2+1}}\right|+C$

14. $\ln\left|\sqrt{x^2+3}\right|+\tan^{-1}x+C$

15. $\ln\left|x^2+1\right|+\dfrac{1}{x^2+1}+C$

16. $\ln\left|x^2+4\right|+\dfrac{1}{2}\tan^{-1}\dfrac{x}{2}+\dfrac{4}{x^2+4}+C$

17. $\dfrac{1}{2}\ln\left|x^2+1\right|-\dfrac{1}{2}\tan^{-1}x-\dfrac{x}{2(x^2+1)}+C$

18. $-\dfrac{x^2}{2}-3x-6\ln|1-x|-\dfrac{4}{1-x}+\dfrac{1}{2(1-x)^2}+C$

19. $\dfrac{1}{2}\ln\left|x^2+2\right|-\dfrac{\sqrt{2}}{2}\tan^{-1}\dfrac{x}{\sqrt{2}}-\dfrac{x}{(x^2+2)^2}+C$

20. $\dfrac{1}{3e^x}+\dfrac{1}{9}\ln\left|\dfrac{e^x-3}{e^x}\right|+C$

21. $\ln\left|\dfrac{\sqrt{1+\cos^2 x}}{\cos x}\right|+C$

22. $\ln|1+\tan\theta|+\dfrac{2}{\sqrt{3}}\tan^{-1}\dfrac{2\tan\theta-1}{\sqrt{3}}+C$

23. $-\dfrac{1}{6}\ln 10$

24. $\dfrac{1}{2}\ln\dfrac{3}{4}+\dfrac{1}{5}$

25. $\dfrac{1}{3}\ln\left|\dfrac{-\dfrac{\sqrt{2}}{2}-1}{-\dfrac{\sqrt{2}}{2}-4}\right|-\dfrac{1}{3}\ln\left|\dfrac{-\dfrac{\sqrt{2}}{2}-1}{\dfrac{\sqrt{2}}{2}-4}\right|$

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# Applications of the Definite Integral

If $y = f(x)$ is a continuous function on $a \le x \le b$ and $F(x)$ is an antiderivative of $f(x)$ and may be denoted by

$$\int f(x)dx = F(x) + C \text{ , where } C \text{ is some constant.} \tag{1}$$

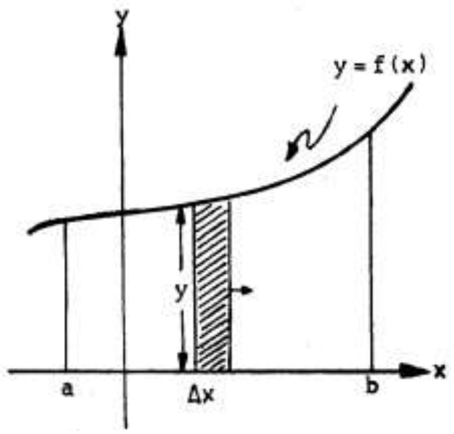
The definite integral of $f(x)$ on the interval $(a,b)$ is

$$\int_a^b f(x)dx = F(b) - F(a) \tag{2}$$

The definite integrals have a lot of applications in geometry and physics such as area under a curve, area between curves, volume, arc length, surface area, moment and work.

## 1. Area Under a Curve

If $y = f(x)$ is a non-negative and continuous function on $a \le x \le b$ as shown in figure 1




Area under the curve of $y = f(x)$ from $x = a$ to $x = b$ as shown here is

$$A = \int_a^b y\,dx \text{ .}$$

**Figure 1**

If we want to find the area covered by the curves of $x = g(y)$ where $g(y) \geq 0$, $y-$axis, $y = c$ and $y = d$ as shown in Figure 2, we partition the area into $n$ small parts, all parts' widths are denoted by $\Delta y_1$, $\Delta y_2$, ..., $\Delta y_n$ and each length is $x = g(y)$.

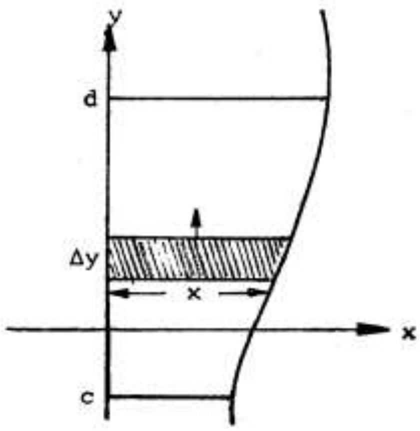



Figure 2

**Consider the $i^{th}$ partition.**

Area $\Delta A_i \approx x \cdot \Delta y_i$ = width × length.

Then, $A \approx \sum_{i=1}^{n} \Delta A_i = \sum_{i=1}^{n} (x \cdot \Delta y_i)$.

If $n \to \infty$ ( or $\Delta y_i \to 0$ ), we have

$$A = \lim_{\Delta y_i \to 0} \sum_{i=1}^{n} (x \cdot \Delta y_i) = \int_c^d x\, dy.$$

<u>**Summary**</u> Area under a curve

1. Area covered by the curves of $y = f(x)$ where $f(x) \geq 0$, $x$-axis, $x = a$, and $x = b$ is

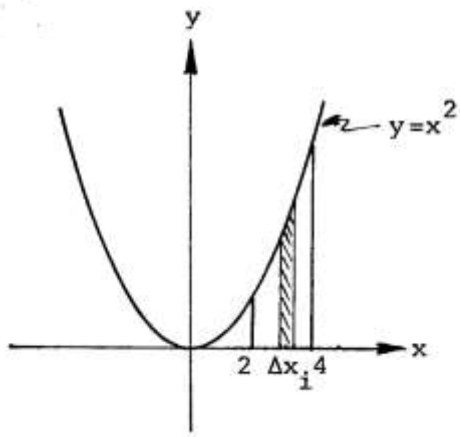
$$A = \int_a^b y\, dx \,. \qquad (3)$$

2. Area covered by the curves of $x = g(y)$ where $g(y) \geq 0$, $y-$axis, $y = c$ and $y = d$ is

$$A = \int_c^d x\, dy. \qquad (4)$$

**Example 1** Compute the area covered by $y = x^2$, the $x-$axis, $x = 2$ and $x = 4$.

**Solution**




Partition along the $x-$axis

$$\Delta A_i = y \cdot \Delta x_i$$

$$A = \int_2^4 y\,dx$$

$$= \int_2^4 x^2\,dx = \left[\frac{x^3}{3}\right]_2^4$$

$$= \frac{4^3}{3} - \frac{2^3}{3} = \frac{64}{3} - \frac{8}{3} = \frac{56}{3}$$

$$= 18\frac{2}{3} \quad \text{unit}^3.$$

**Example 2** Find the area covered by $y = x^3$, $x = -1$, $x = 2$ and the $x-$axis.

**Solution**

**Remark** The area is a non-negative value, but the definite integral may be negative. So, we may write the area as $A = \left| \int_a^b f(x)\,dx \right|$.

If we integrate along the $x$-axis, the definite integral is positive when the graph is above $x$-axis and negative when the graph is below $x$-axis. For example, as in example 2,

$$\int_{-1}^{2} y\,dx = \int_{-1}^{2} x^3\,dx = \left[ \frac{x^4}{4} \right]_{x=-1}^{x=2} = \left[ \frac{2^4}{4} - \frac{(-1)^4}{4} \right] = 4 - \frac{1}{4} = 3\frac{3}{4}.$$

It is the area under the curve above $x$-axis from 0 to 2 minus the area below the $x$-axis from -1 to 0.

To find the total area of under the curve of $y = f(x),\ a \le x \le b$ as shown here

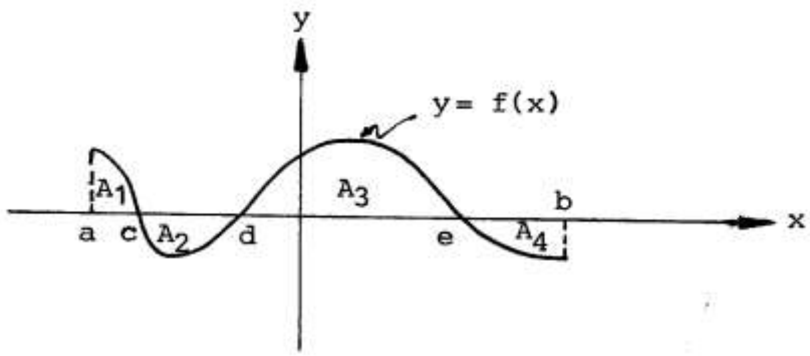



**Figure 5**

**Total area**

$$A = |A_1| + |A_2| + |A_3| + |A_4|$$

$$= \left|\int_a^c f(x)\,dx\right| + \left|\int_c^d f(x)\,dx\right| + \left|\int_d^e f(x)\,dx\right| + \left|\int_e^b f(x)\,dx\right|.$$

1. Analogously, if we integrate along the $y$-axis, the definite integral is positive when the graph is on the right and negative when the graph is on the left of the $y$-axis.
2. If a graph is symmetric, we can integrate just one part and multiply by number of symmetries as shown in example 3.

<u>**Example 3**</u> Compute the area covered by $|x|+|y|=a$.

<u>**Solution**</u> By definition of absolute value

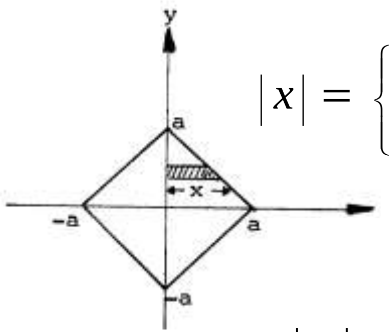



**Figure 6**

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases},$$ we have

$$|x|+|y|=a \rightarrow \begin{cases} x+y=a \text{ when } x \geq 0 \text{ and } y \geq 0 \\ x-y=a \text{ when } x \geq 0 \text{ and } y < 0 \\ -x+y=a \text{ when } x < 0 \text{ and } y \geq 0 \\ -x-y=a \text{ when } x < 0 \text{ and } y < 0\,. \end{cases}$$

As we can see, this graph is symmetric about the origin. So we can just find the area in the first Quadrant, called it $A_1$. The total area is then four times $A_1$.

**Consider** $A_1$ If partition along the $y$-axis,

$$\Delta A_1 = x \cdot \Delta y \qquad \text{where} \quad x = a - y\,.$$

$$A_1 = \int_0^a (a-y)\,dy = \left[ay - \frac{y^2}{2}\right]_{y=0}^{y=a}$$

$$= a^2 - \frac{a^2}{2} = \frac{a^2}{2}.$$

Finally, we obtain

$$A = 4A_1 = \frac{4a^2}{2} = 2a^2 \text{ unit}^3.$$

## 2. Area Between Curves

### 2.1 Rectangular Form

If $y_1 = f(x)$ and $y_2 = g(x)$ are continuous functions such that $y_2 \geq y_1$ for $a \leq x \leq b$, we may compute the areas between these two curves $y_1, y_2$ from $x = a$ and $x = b$ as shown below.

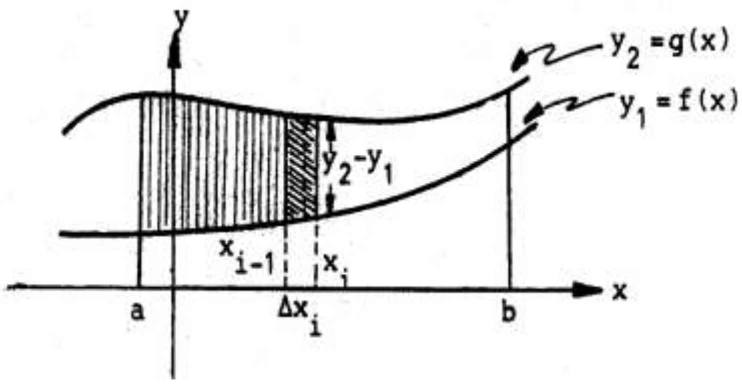



**Figure 7**

Partition the area into small $n$ parts with widths $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$.

Let $\Delta A_i$ = the area of the $i^{th}$ partition.

Then $\Delta A_i \approx (y_2 - y_1)\cdot\Delta x_i =$ width × length

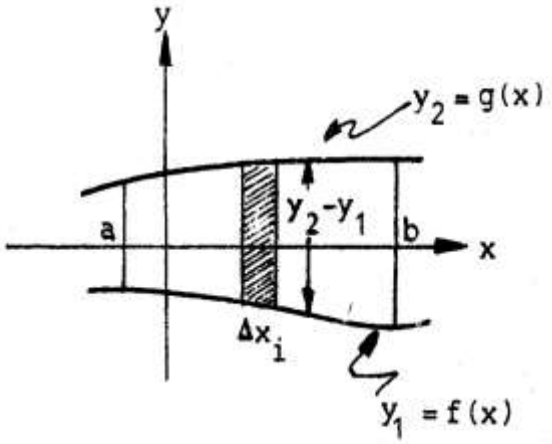
Thus, the total area

$$A \approx \sum_{i=1}^{n} \Delta A_i \approx \sum_{i=1}^{n} (y_2 - y_1) \cdot \Delta x_i \, .$$

If $\Delta x_i \to 0$, the length $(y_2 - y_1)$ of the interval $(x_{i-1}, x_i)$ will approach $(y_2 - y_1)$ at $x_{i-1}$ and $x_i$. Thus, the approximation is closer and closer to the exact area. Therefore,

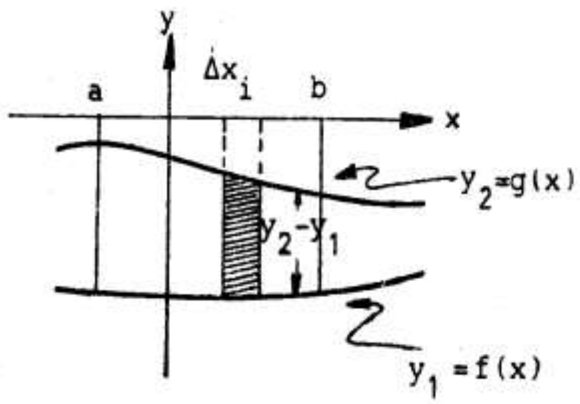
$$A = \lim_{\substack{\Delta x_i \to 0 \\ n \to \infty}} \sum_{i=1}^{n} \Delta A_i = \lim_{\substack{\Delta x_i \to 0 \\ n \to \infty}} \sum_{i=1}^{n} \left(y_2 - y_1\right) \Delta x_i = \int_a^b \left(y_2 - y_1\right) dx \, .$$

This formula is always valid if $y_2 > y_1$. The above or below $x$-axis locations do not matter. Here are some examples.




$$\Delta A_i = (y_2 - y_1) \cdot \Delta x_i$$

**Figure 8**




$$\Delta A_i = (y_2 - y_1) \cdot \Delta x_i$$

**Figure 9**

**Note** If $y_2 \geq y_1$, $y_2$ is always above $y_1$.

**Summary** If $y_1 = f(x)$ and $y_2 = g(x)$ are continuous functions such that $y_2 \geq y_1$ for $a \leq x \leq b$, then the area covered by the curves $y_1$ and $y_2$ from $x = a$ to $x = b$ is

$$A = \int_a^b \left(y_2 - y_1\right) dx = \int_a^b \left(g(x) - f(x)\right) dx \,. \tag{5}$$

Analogously if $g_1(y)$ and $g_2(y)$ are continuous function such that $g_2(y) \geq g_1(y)$ for $c \leq y \leq d$ , we may compute the area covered by $x_1 = g_1(y)$, $x_2 = g_2(y)$ from $y = c$ to $y = d$. For $x_2 > x_1$ as shown in three figures below, we partition along the $y$-axis to $n$ parts with widths $\Delta y_1, \Delta y_2, \ldots, \Delta y_n$ .

Area of the $i^{th}$ partition is

$$\Delta A_i \approx (x_2 - x_1) \cdot \Delta y_i = \text{length} \times \text{width} \,.$$

Thus, the total area

$$A = \lim_{\substack{\Delta y_i \to 0 \\ (n\to\infty)}} \sum_{i=1}^{n} \Delta A_i = \lim_{\substack{\Delta y_i \to 0 \\ (n\to\infty)}} \sum_{i=1}^{n} \left(x_2 - x_1\right)\Delta y_i = \int_c^d \left(x_2 - x_1\right) dy .$$

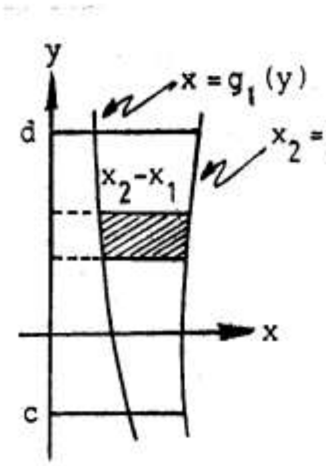



**Figure 10**

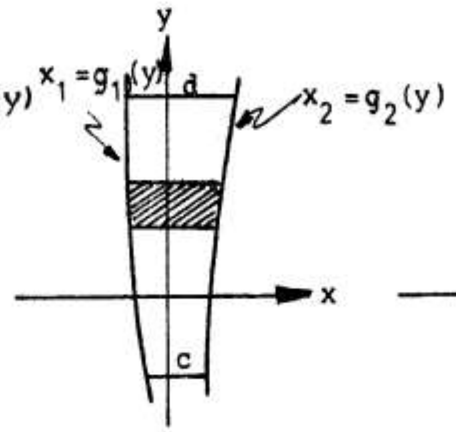



**Figure 11**

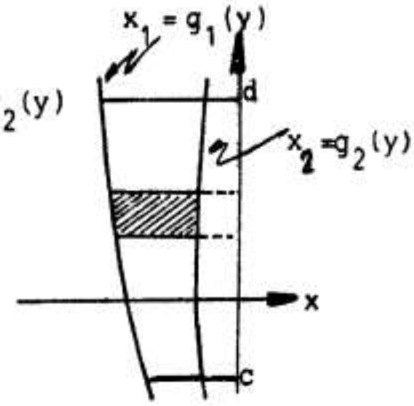



**Figure 12**

<u>**Summary**</u> If $x_1 = g_1(y)$ and $x_2 = g_2(y)$ are continuous functions such that $x_2 \geq x_1$ for $c \leq y \leq d$, then the area covered by $x_1$, $x_2$, $y = c$ and $y = d$ is

$$A = \int_c^d \left(x_2 - x_1\right) dy = \int_c^d \left(g_2(y) - g_1(y)\right) dy . \qquad (6)$$

**Example 4** Compute the area covered by $x^2 = y$, $x^2 = 4y$ and the line $x = 2$.

**Approach 1** Partition along the $x$-axis.

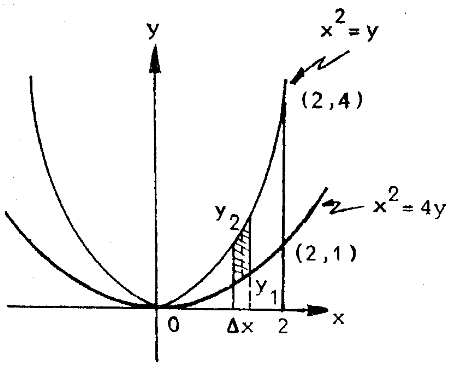



**Figure 13**

Then $\Delta A = (y_2 - y_1)\cdot \Delta x$

where $y_2 = x^2$ and $y_1 = \dfrac{x^2}{4}$ .

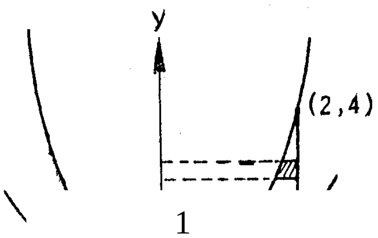
$$A = \int_0^2 (y_2 - y_1)dx = \int_0^2 \left( x^2 - \frac{x^2}{4} \right) dx$$

$$= \frac{3}{4}\int_0^2 x^2\,dx = \frac{3}{4}\left[ \frac{x^3}{3} \right]_{x=0}^{x=2} = 2.$$

**Approach 2** Partition along the $y$-axis: there are 2 parts.

$A_1$: $y$ from $0 \to 1$, we have

$$\Delta A_1 = (x_2 - x_1)\Delta y$$




where $x_2 = \sqrt{4y}$ and $x_1 = \sqrt{y}$ .

(Do not forget that $x_2$ is on the right of $x_1$)

$$A_1 = \int_0^1 (x_2 - x_1)dy = \int_0^1 \left( \sqrt{4y} - \sqrt{y} \right) dy$$

$$= \int_0^1 \sqrt{y}\,dy = \left[ \frac{2}{3} y^{3/2} \right]_{y=0}^{y=1} = \frac{2}{3}.$$

$A_2$: $y$ from $1 \to 4$, we have

$$\Delta A_1 = (x_2 - x_1)\Delta y \quad \text{where} \quad x_2 = 2 \text{ and } x_1 = \sqrt{y} \ .$$

$$A_2 = \int_1^4 (x_2 - x_1)\, dy = \int_1^4 \left(2 - \sqrt{y}\right) dy$$

$$= \left[2y - \frac{2}{3}y^{3/2}\right]_{y=1}^{y=4} = \frac{4}{3}.$$

Therefore,

$$A = A_1 + A_2 = \frac{2}{3} + \frac{4}{3} = 2.$$

No matter which approach you choose, the correct answer is always the same.

**Example5** Compute the area covered by $y^2 = 2x$ and $x - y = 4$.

**Solution**

**Example 6** Compute the area covered by $y = -x^2 - 2x + 3$, its tangent line at (2,-5) and the $y$-axis.

**Solution** Consider $y = -x^2 - 2x + 3 = -\left(x^2 + 2x + 1\right) + 4$.

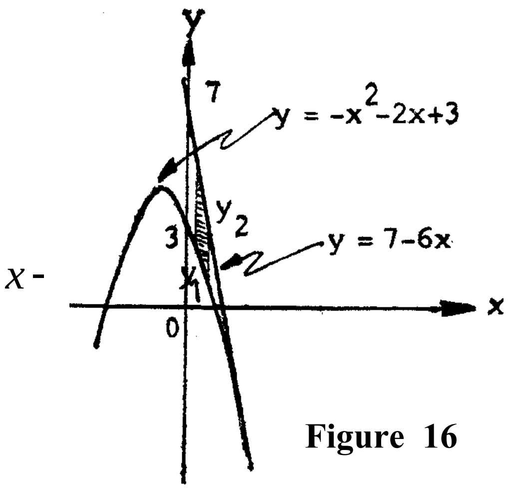



Figure 16

$y - 4 = -\left(x+1\right)^2$ is a parabolic curve having a vertex at (-1, 4). This curve has the $y$-intercept as (0, 3) and has the intercept at $x = -3$, and $x = 1$.

Consider $\dfrac{dy}{dx} = -2x - 2$.

The slope of the tangent line at (2, -5) is $-2(2) - 2 = -6$.

The equation of this tangent line can be found by $y - y_1 = m\left(x - x_1\right)$.

Here, we have $y_1 = -5,\ x_1 = 2,\ m = -6$.

So, $y - (-5) = -6\left(x - 2\right)$. That is, we have the equation of the tangent line of this parabola at (2, -5) is $y = 7 - 6x$.

If partition on $x$-axis,

$$\Delta A = \left(y_2 - y_1\right)\Delta x$$

where $y_2 = 7 - 6x$ and $y_1 = -x^2 - 2x + 3$.

Then, $$A = \int_0^2 \left[\left(7 - 6x\right) - \left(-x^2 - 2x + 3\right)\right] dx$$

$$= \int_0^2 \left[4 - 4x + x^2\right] dx = \frac{8}{3}.$$

**Exercise 1**

Compute each area covered by the following graphs

1. $x-$axis, $y=2x-x^2$
2. $y-$axis, $x=y^2-y^3$
3. $y^2=x$, $x=4$
4. $y=2x-x^2$, $y=-3$
5. $y=x^2$, $y=x$
6. $x=3y-y^2$, $x+y=3$
7. $y=x^4-2x^2$, $y=2x^2$
8. First part of $y=\sin x$
9. $y-$axis, $y^2-4x-4=0$
10. Ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$
11. $x=y^2$, $x=y$
12. $y^2=8x$, $x^2=4y$
13. $x^2-5x+y=0$, $y=x$
14. $y^2=9x$, $y^2=x^3$
15. $y=x^2$, $y=x$, $y=2x$
16. $y^2=4x$, $2x-y-4=0$
17. $y=x^3-4x$, $x-$axis
18. $x+2y=2$, $y-x=1$, $2x+y=7$
19. $x^2y=x^2-4$, $x-$axis, $x=2$ and $x=4$

20. $y = 6x + x^2 - x^3,\ \ x-$axis

21. $f(x) = \begin{cases} x^2 &, \ x \le 2 \\ -x+6, & x > 2 \end{cases}$ from $x = 0$ and $x = 3$

22. $y = x(x-3)(x+3),\ \ y = -5x$

23. $y = x^2,\ \ y = 8 - x^2$ and $y = 4x + 12$

24. $x = 0,\ x = 2,\ y = 2^x$ and $y = 2x - x^2$

25. $x = -2y^2,\ \ x = 1 - 3y^2$

26. $y = x + 1,\ \ y = \cos x$ and the $x-$axis (largest region)

27. One loop of $y^2 = (x-1)(x-2)^2$

28. $y = x^2 - 2x + 2$, its tangent line at the point $M(3, 5)$, the $y-$axis

29. $\sqrt{x} + \sqrt{y} = 1$ and $x + y = 1$

30. $y = x^2,\ y = 4$ This area is divided into 2 equal parts by the line $y = c$. Evaluate the value of $c$.

31. $x^2 = 4y,\ \ y = \dfrac{8}{x^2+4}$

32. One loop of $y^2 = (x-1)^2$

33. $y^2 = 4x,\ \ x^2 = 4y$ and $x^2 + y^2 = 5$ where $x \ge 0,\ y \ge 0$

34. Hypocycloid: $x^{2/3} + y^{2/3} = a^{2/3}$

35. $y^3 = x^2$ the cord connecting (-1, 1) and (8, 4)

36. $y^2 = x^2\left(1 - x^2\right)$

37. $xy = 4,\ \ y = x,\ \ x = 5$ and $x = \sqrt{-y}$

**<u>Answer 1</u>**

| | | |
|---|---|---|
| 1. $\dfrac{4}{3}$ | 2. $\dfrac{1}{12}$ | 3. $\dfrac{32}{3}$ |

4. $\frac{32}{3}$ 5. $\frac{1}{6}$ 6. $\frac{4}{3}$

7. $\frac{128}{15}$ 8. $2$ 9. $\frac{8}{3}$

10. $\pi ab$ 11. $\frac{1}{6}$ 12. $\frac{16}{3}$

13. $\frac{32}{3}$ 14. $\frac{24\sqrt{3}}{5}$ 15. $\frac{7}{6}$

16. 9 17. 8 18. 6

19. 1 (Figure 17) 20. $\frac{253}{12}$ (Figure 18)

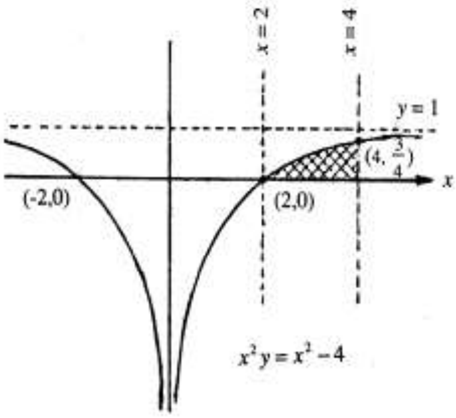



**Figure 17**

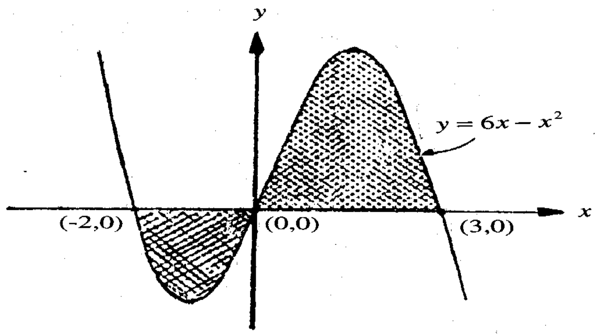



**Figure 18**

21. $\frac{37}{6}$ 22. 8 23. 64

24. $\frac{3}{\ln 2} - \frac{4}{3}$ 25. $\frac{4}{3}$ 26. $\frac{3}{2}$

27. $\frac{8}{15}$ 28. 9 29. $\frac{1}{3}$

30. $\frac{32}{3}$, $c = \sqrt[3]{16}$ 31. $2\pi - \frac{4}{3}$ (Figure 19)

32. $\frac{8}{15}$ (Figure 20)

**Figure 19**

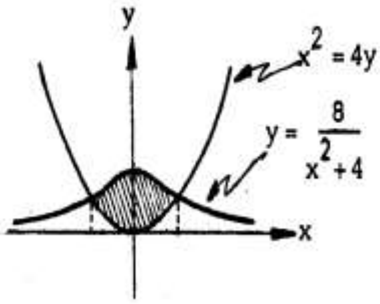
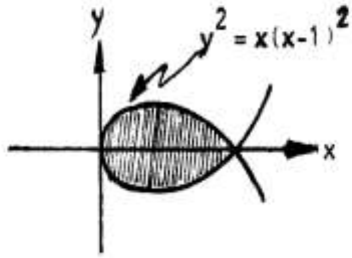



**Figure 20**




33. $\frac{2}{3}+\frac{5}{2}\sin^{-1}\frac{3}{5}$

34. $\frac{3}{8}\pi a^2$

35. 2.7

36. $\frac{4}{3}$

37. $4(\ln 5-\ln 2)+\frac{131}{3}$

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# IMPROPER INTEGRALS

## Introduction

Previously we learned about the definite integral $\int_a^b f(x)\,dx$ where $a$ and $b$ are constants. The integrand $f(x)$ is continuous and bounded on the interval $[a,b]$.

In this chapter we are interested in $\int_a^b f(x)\,dx$ where limits of integration $a$ and $b$ may be infinity or the integrand function is unbounded at some points in the interval $[a,b]$. This type of integral called "***Improper Integral***." There are three cases:

**Case 1** Limit of integration is infinity (Infinite interval)

$$[a,+\infty) \qquad (-\infty,b] \qquad (-\infty,+\infty)$$

For examples,

$$\int_1^{+\infty} \frac{dx}{x^2} \qquad \int_{-\infty}^{0} e^x dx \qquad \int_{-\infty}^{+\infty} \frac{dx}{1+x^2}$$

**Case 2** Integrand $f(x)$ is unbounded at some point $x=c$ in $[a,b]$

$$\lim_{x\to c} f(x) = \pm\infty$$

For examples,

$$\int_{-3}^{3} \frac{dx}{x^2} \qquad \int_1^2 \frac{dx}{x-1} \qquad \int_0^{\pi} \tan x\,dx$$

**Case 3** Combination of both case 1 and case 2

For examples,

$$\int_0^{+\infty} \frac{dx}{\sqrt{x}} \qquad \int_{-\infty}^{+\infty} \frac{dx}{x^2-9} \qquad \int_1^{+\infty} \sec x\,dx$$

**1. Evaluation of improper integral case 1**

**Definition** Let $a$ are a real number and $f$ be bounded and integrable on $[a,t]$ for all $t$ such that $t > a$. Thus the improper integral of $f(x)$ on $[a,+\infty)$ denoted by $\int_a^{+\infty} f(x)\,dx$ is defined by

$$\int_a^{+\infty} f(x)\,dx = \lim_{t\to+\infty} \int_a^t f(x)\,dx$$

**Remark**

- If $\lim_{t\to+\infty} \int_a^t f(x)\,dx$ exists, then $\int_a^{+\infty} f(x)\,dx$ converges.
- If $\lim_{t\to+\infty} \int_a^t f(x)\,dx$ does not exist, then $\int_a^{+\infty} f(x)\,dx$ diverges.

**Definition** Let $b$ be a real number and $f$ be bounded and integrable on $[t,b]$ for all $t$ such that $t < b$. Thus the improper integral of $f(x)$ on $(-\infty,b]$ denoted by $\int_{-\infty}^{b} f(x)\,dx$ is defined by

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

**Remark**

- If $\lim_{t\to-\infty} \int_t^b f(x)\,dx$ exists, then $\int_{-\infty}^{b} f(x)\,dx$ converges.
- If $\lim_{t\to-\infty} \int_t^b f(x)\,dx$ does not exist, then $\int_{-\infty}^{b} f(x)\,dx$ diverges.

**Definition** Let $f$ be a bounded and integrable function on $[a,b]$ for constants $a$ and $b$, where $a < b$. Thus the improper integral of $f(x)$ on $(-\infty,+\infty)$ denoted by $\int_{-\infty}^{+\infty} f(x)\,dx$ is defined by

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{+\infty} f(x)\,dx \quad \text{where } c \in \mathbb{R}$$

**Remark**

- $\int_{-\infty}^{+\infty} f(x)\,dx$ converges if both integral on the right converge.
- $\int_{-\infty}^{+\infty} f(x)\,dx$ diverges if at least one integral on the right diverges.

**Example** Evaluate $\int_1^{+\infty} \frac{1}{x^3}\,dx$

**Solution** $$\int_1^{+\infty} \frac{1}{x^3}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{1}{x^3}\,dx$$

$$= \lim_{t\to\infty}\left[-\frac{1}{2x^2}\right]_1^t$$

$$= \lim_{t\to\infty}\left(\frac{1}{2}-\frac{1}{2t^2}\right) = 1/2$$

Thus we conclude that the given integral converges to $1/2$. ■

**Example** Evaluate $\int_1^{+\infty} \frac{1}{x}\,dx$

**Solution**

**Example** Identify $p$ such that $\int_1^{+\infty} \frac{1}{x^p}\,dx$ converges or diverges

**Solution**

$$\int_1^{+\infty} \frac{1}{x^p}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{1}{x^p}\,dx$$

$$= \lim_{t\to\infty} \left[ \frac{x^{1-p}}{1-p} \right]_1^t$$

$$= \lim_{t\to\infty} \left( \frac{t^{1-p}}{1-p} - \frac{1}{1-p} \right)$$

If $p > 1$, then $1-p < 0$ and $t^{1-p} \to 0$ as $t \to +\infty$.

Thus

$$\lim_{t\to\infty} \left( \frac{t^{1-p}}{1-p} - \frac{1}{1-p} \right) = \frac{1}{p-1}$$

If $p < 1$, then $1-p > 0$ and $t^{1-p} \to +\infty$ as $t \to +\infty$

Thus

$$\lim_{t\to\infty} \left( \frac{t^{1-p}}{1-p} - \frac{1}{1-p} \right) = +\infty$$

If $p = 1$, then $\int_1^{+\infty} \frac{1}{x}\,dx$ diverges. From all three cases, we have

$$\int_1^{+\infty} \frac{1}{x^p}\,dx = \begin{cases} \dfrac{1}{p-1} & , \ p > 1 \\ \text{diverge} & , \ p \le 1 \end{cases}$$

■

**Example** Determine if the following improper integrals converges or diverges.

1) $\int_1^{+\infty} \frac{1}{x^{2/3}}\,dx$

2) $\int_1^{+\infty} \frac{1}{x^5}\,dx$

3) $\int_1^{+\infty} \frac{1}{x^{3/2}}\,dx$

**Answer**

1) $\int_1^{+\infty} \frac{1}{x^{2/3}}\,dx$ diverges

2) $\int_1^{+\infty} \frac{1}{x^5}\,dx$ converges to $\frac{1}{5-1} = \frac{1}{4}$

3) $\int_1^{+\infty} \frac{1}{x^{3/2}}\,dx$ converges to $\frac{1}{(3/2)-1} = 2$

■

**Example** Evaluate $\int_{0}^{+\infty} (1-x)e^{-x}dx$

**Solution**

**Example** Determine if $\int_{-\infty}^{1} xe^{-x^2}\,dx$ converges or diverges to which value.

**Solution**

$$\int_{-\infty}^{1} xe^{-x^2}\,dx = \lim_{t\to-\infty}\int_{t}^{1} xe^{-x^2}\,dx$$

$$= \lim_{t\to-\infty}\left[\frac{-e^{-x^2}}{2}\right]_t^1$$

$$= \lim_{t\to-\infty}\left[-\frac{1}{2e}+\frac{e^{-t^2}}{2}\right]$$

$$= -\frac{1}{2e}+0$$

$$= -\frac{1}{2e}$$

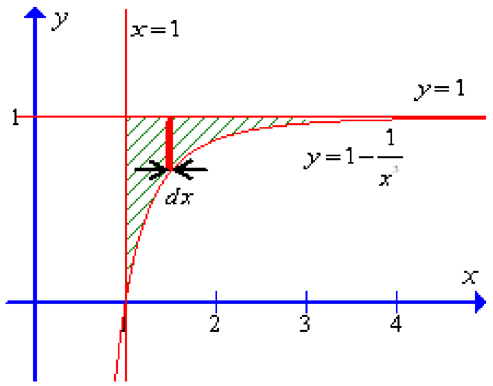
Hence, $\int_{-\infty}^{1} xe^{-x^2}\,dx$ converges to $-1/2e$ ■

**Example** Evaluate $\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\,dx$

**Solution**

**Example** Find the area bounded by curve $y=1-\frac{1}{x^3}$, line $x=1$ and line $y=1$

**Solution** Area $A$ as shown below can be computed by




$$
\begin{aligned}
A &= \int_1^{\infty} (y_2 - y_1)dx \\
&= \int_1^{\infty} \left[1-\left(1-\frac{1}{x^3}\right)\right]dx = \int_1^{\infty} \frac{1}{x^3}dx \\
&= \lim_{t\to\infty} \int_1^{t} \frac{1}{x^3}dx \\
&= \lim_{t\to\infty} \left[-\frac{1}{2x^2}\right]_1^t \\
&= \lim_{t\to\infty} \left[-\frac{1}{2t^2}+\frac{1}{2}\right] \\
A &= 1/2 \quad \text{unit}^2
\end{aligned}
$$

■

## 2. Evaluation of improper integral case 2

**Definition** Let $a$ and $b$ be real numbers such that $a<b$. Suppose $f$ is a bounded and integrable function on $[t,b]$ for all $t$ such that $a<t<b$, but $f$ goes to infinity at $x=a$, i.e.

$$\lim_{x\to a^+} f(x) = \pm\infty$$

Thus the improper integral of $f(x)$ on $[a,b]$ is defined by

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$$

**Remark**

- If the limit exists, then the integral $\int_a^b f(x)\,dx$ converges
- Otherwise $\int_a^b f(x)\,dx$ diverges.

**Definition** Let $a$ and $b$ be real numbers such that $a < b$. Suppose $f$ is a bounded and integrable function on $[a,t]$ for all $t$ such that $a < t < b$, but $f$ goes to infinity at $x = b$, i.e.

$$\lim_{x \to b^-} f(x) = \pm\infty$$

Thus the improper integral of $f(x)$ on $[a,b]$ is defined by

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$

**Remark**

- If the limit exists, then $\int_a^b f(x)\,dx$ converges
- Otherwise $\int_a^b f(x)\,dx$ diverges

**Definition** Let $a$ and $b$ be real numbers such that $a < b$. Suppose $f$ is a bounded and integrable function on $[a,b]$, but $f$ goes to infinity at $x = c$ in $(a,b)$. Thus the improper integral of $f$ on $[a,b]$ is defined by

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

**Remark**

- $\int_a^b f(x)\,dx$ converges if both integral on the right converge.
- $\int_a^b f(x)\,dx$ diverges if at least one integral on the right diverge.

**Example** Evaluate $\int_0^1 \frac{1}{\sqrt{x}}\,dx$

**Solution**
$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t \to 0^+} \int_t^1 \frac{1}{\sqrt{x}}\,dx$$

$$= \lim_{t \to 0^+} \left[2\sqrt{x}\right]_t^1$$

$$= \lim_{t \to 0^+} \left(2 - 2\sqrt{t}\right) = 2$$

Hence, $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ converges to $2$. ■

**Example** Evaluate $\int_1^2 \frac{dx}{1-x}$

**Solution** $$\int_1^2 \frac{dx}{1-x} = \lim_{t\to 1^+} \int_t^2 \frac{dx}{1-x}$$
$$= \lim_{t\to 1^+} \left[-\ln|1-x|\right]_t^2$$
$$= \lim_{t\to 1^+} \left(-\ln|-1| + \ln|1-t|\right)$$
$$= 0 + \lim_{t\to 1^+} \ln|1-t| = -\infty$$

Hence, we have $\int_1^2 \frac{dx}{1-x}$ diverges. ■

**Example** Evaluate $\int_0^1 \frac{dx}{\sqrt{1-x}}$

**Solution**

**Example** Determine if $\int_1^4 \frac{dx}{(x-2)^{2/3}}$ converges or diverges

**Solution**

**3. Evaluation of improper integral case 3**

**Example** Evaluate $\int_0^{\infty} \frac{dx}{\sqrt{x}(x+1)}$

**Solution**

$$\begin{aligned}
\int_0^{\infty} \frac{dx}{\sqrt{x}(x+1)} &= \int_0^{1} \frac{dx}{\sqrt{x}(x+1)} + \int_1^{\infty} \frac{dx}{\sqrt{x}(x+1)} \\
&= \lim_{t\to 0^+} \int_t^{1} \frac{dx}{\sqrt{x}(x+1)} + \lim_{t\to\infty} \int_1^{t} \frac{dx}{\sqrt{x}(x+1)} \\
&= \lim_{t\to 0^+} \left[2\tan^{-1}\sqrt{x}\right]_t^1 + \lim_{t\to\infty} \left[2\tan^{-1}\sqrt{x}\right]_1^t \\
&= 2\left[\frac{\pi}{4} - 0\right] + 2\left[\frac{\pi}{2} - \frac{\pi}{4}\right] \\
&= \pi
\end{aligned}$$

Hence, $\int_0^{\infty} \frac{dx}{\sqrt{x}(x+1)}$ diverges to $\pi$. ■

**Exercise** Separate the following integrals into several parts according to their impropriety.

1) $\int_{-3}^{\infty} \frac{dx}{x+2}$

**Solution**

2) $\int_{-\infty}^{0} \frac{dx}{(x+3)^2}$

**Solution**

3) $\int_{-\infty}^{+\infty} \frac{dx}{x^3}$

**Solution**

**Exercise 7.1**

**1** Determine if the following improper integral converges or diverges and find its value.

**1.1** $\displaystyle\int_{2}^{\infty} \frac{1}{(x+1)^2}\,dx$

**1.2** $\displaystyle\int_{0}^{\infty} \cos x\,dx$

**1.3** $\displaystyle\int_{1}^{\infty} \frac{\ln x}{x}\,dx$

**1.4** $\displaystyle\int_{e}^{\infty} \frac{1}{x\ln^3 x}\,dx$

**1.5** $\displaystyle\int_{0}^{\infty} \frac{1}{1+2^x}\,dx$

**1.6** $\displaystyle\int_{-1}^{\infty} \frac{x}{1+x^2}\,dx$

**1.7** $\displaystyle\int_{2}^{\infty} \frac{1}{x^2+4}\,dx$

**1.8** $\displaystyle\int_{0}^{\infty} \frac{1}{\sqrt{e^x}}\,dx$

**1.9** $\displaystyle\int_{0}^{\infty} xe^{-x}\,dx$

**1.10** $\displaystyle\int_{0}^{\infty} e^{-x}\cos x\,dx$

**1.11** $\displaystyle\int_{0}^{\infty} \frac{1}{e^{2x}+e^x}\,dx$

**1.12** $\displaystyle\int_{1}^{\infty} \frac{1}{\sqrt{x}(1+e^{\sqrt{x}})^2}\,dx$

**1.13** $\displaystyle\int_{-\infty}^{1} \frac{1}{3-2x}\,dx$

**1.14** $\displaystyle\int_{-\infty}^{0} e^{3x}\,dx$

**1.15** $\displaystyle\int_{-\infty}^{-1} \frac{x}{\sqrt{1+x^2}}\,dx$

**1.16** $\displaystyle\int_{-\infty}^{0} \frac{1}{(1-x)^{5/2}}\,dx$

**1.17** $\displaystyle\int_{-\infty}^{0} \frac{e^x}{3-2e^x}\,dx$

**1.18** $\displaystyle\int_{-\infty}^{0} \frac{1}{(x-8)^{2/3}}\,dx$

**1.19** $\displaystyle\int_{-\infty}^{0} \frac{1}{2x^2+2x+1}\,dx$

**1.20** $\displaystyle\int_{-\infty}^{\infty} \frac{|x+1|}{x^2+1}\,dx$

**1.21** $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{x^2+1}\,dx$

**1.22** $\displaystyle\int_{-\infty}^{\infty} \frac{x}{(x^2+3)^2}\,dx$

**1.23** $\displaystyle\int_{-\infty}^{\infty} \frac{1}{e^x+e^{-x}}\,dx$

**1.24** $\displaystyle\int_{-\infty}^{\infty} xe^{-x^2}\,dx$

**2** Find value of $a$ such that $\displaystyle\int_{0}^{\infty} e^{-ax}\,dx = 5$

**3** Show that $\displaystyle\int_{1}^{\infty} \frac{1}{x^p}\,dx$ converges if $p>1$ and diverges if $p\le 1$

**4** Find the area between the curve $y=\dfrac{8}{x^2-4}$ and $x$-axis where $x\ge 3$

**5** Find the area between the curves $y=\dfrac{1}{x}$ and $y=\dfrac{1}{x^2}$ where $x\in[1,\infty)$

**6** Let $R=\{(x,y)\mid x\ge 4$ และ $0\le y\le x^{-3/2}\}$. Find

**6.1** area of region $R$

**6.2** volume of a solid generated when the region $R$ is revolved about the $x$-axis

**7** Let $R$ be the region between the curve $y=\dfrac{4}{x^2+1}$ and $x$-axis where $x \ge 0$. Find

**7.1** area of region $R$

**7.2** volume of a solid generated when the region $R$ is revolved about the $x$-axis

## Answer 7.1

**1**

| | | | |
|---|---|---|---|
| **1.1** | $1/3$ | **1.2** | diverges |
| **1.3** | diverges | **1.4** | $1/2$ |
| **1.5** | $1$ | **1.6** | diverges |
| **1.7** | $\pi/8$ | **1.8** | $2$ |
| **1.9** | $1$ | **1.10** | $1/2$ |
| **1.11** | $1-\ln 2$ | **1.12** | $2\left(\ln(1+e)-1-\dfrac{1}{1+e}\right)$ |
| **1.13** | diverges | **1.14** | $1/3$ |
| **1.15** | diverges | **1.16** | $2/3$ |
| **1.17** | $\dfrac{1}{2}\ln 3$ | **1.18** | diverges |
| **1.19** | $3\pi/4$ | **1.20** | diverges |
| **1.21** | diverges | **1.22** | $0$ |
| **1.23** | $\pi/2$ | **1.24** | $0$ |

**2** $1/5$

**4** $2\ln 5$

**5** undefined

**6**

**6.1** $1$ **6.2** $\pi/32$

**7**

**7.1** $2\pi$ **7.2** $4\pi^2$

## Exercise 7.2

**1** Determine if the following improper integral converges or diverges and find its value.

**1.1** $\displaystyle\int_0^9 \frac{1}{\sqrt{x}}\,dx$ **1.2** $\displaystyle\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$

**1.3** $\int_3^4 \frac{1}{(x-3)^2}\,dx$

**1.4** $\int_0^4 \frac{1}{(4-x)^{3/2}}\,dx$

**1.5** $\int_1^2 \frac{x}{\sqrt{x-1}}\,dx$

**1.6** $\int_0^1 x\ln x\,dx$

**1.7** $\int_0^{\pi/6} \frac{\cos x}{\sqrt{1-2\sin x}}\,dx$

**1.8** $\int_0^{\pi/2} \sec^2 x\,dx$

**1.9** $\int_0^2 \frac{2x+1}{x^2+x-6}\,dx$

**1.10** $\int_0^1 \ln x\,dx$

**1.11** $\int_0^4 \frac{\ln\sqrt{x}}{\sqrt{x}}\,dx$

**1.12** $\int_0^1 \frac{1}{\sqrt{1-\sqrt{x}}}\,dx$

**1.13** $\int_2^4 \frac{x}{\sqrt[3]{x-2}}\,dx$

**1.14** $\int_0^2 \frac{x}{(x^2-1)^2}\,dx$

**1.15** $\int_{-1}^8 \frac{1}{\sqrt[3]{x}}\,dx$

**1.16** $\int_{-2}^7 \frac{1}{(x+1)^{2/3}}\,dx$

**1.17** $\int_{-1}^1 \frac{1}{\sqrt{|x|}}\,dx$

**1.18** $\int_2^4 \frac{1}{(x-3)^7}\,dx$

**1.19** $\int_0^3 \frac{1}{x^2+2x-3}\,dx$

**1.20** $\int_1^3 \frac{x}{(x^2-4)^3}\,dx$

**1.21** $\int_{-1}^2 \frac{1}{x^2}\cos\frac{1}{x}\,dx$

**1.22** $\int_0^2 \frac{1}{\sqrt{2x-x^2}}\,dx$

**1.23** $\int_{-1}^2 \frac{1}{x^2-x-2}\,dx$

**1.24** $\int_0^1 \frac{1}{x(\ln x)^{1/5}}\,dx$

**2** Show that $\int_0^1 \frac{1}{x^p}\,dx$ converges if $p<1$ and diverges if $p\ge 1$

**3** Find the area between the curve $y=\frac{1}{(1-x)^2}$ and $x$-axis where $x\in[0,4]$

**4** Find (a) Area of region $R$, and

(b) Volume of a solid generated by revolving region $R$ about the $x$-axis, when R is given as follows.

**4.1** $R=\{(x,y)\,|\,-4\le x\le 4$ และ $0\le y\le 1/(x+4)\}$

**4.2** $R=\{(x,y)\,|\,0\le x\le 1$ และ $0\le y\le 1/\sqrt{x}\}$

**5** Find the area between the curves $y=\frac{1}{x}$ and $y=\frac{1}{x(x^2+1)}$ where $x\in[0,1]$

**Answer 7.2**

**1**

**1.1** 6 **1.2** $\pi/2$

**1.3** diverges **1.4** diverges

| | | | |
|---|---|---|---|
| **1.5** | $8/3$ | **1.6** | $-1/4$ |
| **1.7** | $1$ | **1.8** | diverges |
| **1.9** | diverges | **1.10** | $-1$ |
| **1.11** | $4(\ln 2-1)$ | **1.12** | $8/3$ |
| **1.13** | $21\sqrt[3]{4}/5$ | **1.14** | diverges |
| **1.15** | $9/2$ | **1.16** | $3$ |
| **1.17** | $4$ | **1.18** | diverges |
| **1.19** | diverges | **1.20** | diverges |
| **1.21** | diverges | **1.22** | $\pi$ |
| **1.23** | diverges | **1.24** | diverges |

**3** Undefined

**4**

**4.1** (a) DNE (b) DNE

**4.2** (a) 2 (b) DNE

**5** $\frac{1}{2}\ln 2$

## Exercise 7.3

**1.** Determine if the following improper integral converges or diverges and find its value.

**1.1** $\int_0^{\infty} \frac{1}{(x-1)^{2/3}}\,dx$ **1.2** $\int_1^{\infty} \frac{1}{x\sqrt{x^2-1}}\,dx$

**1.3** $\int_{-1}^{\infty} \frac{1}{x^2-1}\,dx$ **1.4** $\int_1^{\infty} \frac{1}{x\ln x}\,dx$

**1.5** $\int_0^{\infty} x^{-0.1}dx$ **1.6** $\int_0^{\infty} \frac{1}{\sqrt{x}(x+4)}\,dx$

**1.7** $\int_1^{\infty} \frac{1}{x^2-6x+8}\,dx$ **1.8** $\int_0^{\infty} \frac{\sqrt{x}}{1-\sqrt{x}}\,dx$

**1.9** $\int_2^{\infty} \frac{1}{(x+7)\sqrt{x-2}}\,dx$ **1.10** $\int_{-\infty}^{\infty} \frac{1}{x^2+2x+1}\,dx$

**1.11** $\int_{-\infty}^{\infty} \frac{1}{x^2 - 3x + 2} dx$

**1.12** $\int_{-\infty}^{\infty} \frac{e^x}{e^x - 1} dx$

**Answer 7.3**

| | | | |
|---|---|---|---|
| **1.1** | diverges | **1.2** | $\pi/2$ |
| **1.3** | diverges | **1.4** | diverges |
| **1.5** | diverges | **1.6** | $\pi/2$ |
| **1.7** | diverges | **1.8** | diverges |
| **1.9** | $\pi/3$ | **1.10** | diverges |
| **1.11** | diverges | **1.12** | diverges |

**Numerical Integration**

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