INTEGRATION

1. Antiderivative

This lesson concerns the reverse process of taking derivative of a function as we learned last section. In particular, if $y' = f'(x)$, we want to find $y = f(x)$. Consider the following:

If $f(x) = x^2$, then $f'(x) = 2x$

If $f(x) = x^2 + 1$, then $f'(x) = 2x$

If $f(x) = x^2 + 2$, then $f'(x) = 2x$

………

If $f(x) = x^2 + C$, then $f'(x) = 2x$

Thus $f'(x) = 2x$ may have $f(x) = x^2$ or in general, $f(x) = x^2 + C$, where C is some constant.

We call $x^2 + C$ an **<u>antiderivative</u>** of $2x$.

Definition 1.1 Function $F(x)$ such that $F'(x) = f(x)$ is called an "an antiderivative of $f(x)$"

For examples, for any constant C

1. $F(x) = x^2 + \dfrac{1}{x} + C$ is an antiderivative of $f(x) = 2x - \dfrac{1}{x^2}$ since $F'(x) = 2x - \dfrac{1}{x^2}$.
2. $F(x) = \sin x + C$ is an antiderivative of $f(x) = \cos x$ since $F'(x) = \cos x$.

3. $F(x) = e^x + \tan^{-1} x + C$ is an antiderivative of $f(x) = e^x + \dfrac{1}{1+x^2}$ since $F'(x) = e^x + \dfrac{1}{1+x^2}$.

Properties of an antiderivative of $f(x)$

1. Every continuous function $f(x)$ has infinitely many antiderivatives of $f(x)$.
2. If $F_1(x), F_2(x)$ are both antiderivatives of $f(x)$, then the difference $F_1(x) - F_2(x) =$ constant.
3. If $F(x)$ is an antiderivative $f(x)$, then $F(x)+C$ where C is some constant is also the antiderivative of $f(x)$. Thus we say that all antiderivatives of $f(x)$ are in the form of $F(x)+C$.

Definition 1.2 The process of finding an antiderivative of $f(x)$ is called an integration

$$f(x) \to F(x) \text{ if } F'(x) = f(x)$$

Notion: $\int f(x)dx$ is called an "integral of $f(x)$ with respect to x"

$\int$ is an integration notation, dx refers to the independent variable x and $f(x)$ is called an integrand.

There are two types of integrations: indefinite and definite Integrals.

2 Indefinite Integral

Since $\dfrac{d}{dx}F(x) = f(x)$ or $dF(x) = f(x)dx$, then $\int dF(x) = \int f(x)dx = F(x) + C$ where C is some constant.

Note that the notation $\int$ is the reverse operation of the derivative notation and we call this process an indefinite integral. Examples:

$\dfrac{d}{dx}(x) = 1$	$\int 1dx = \int dx = x + C$
$\dfrac{d}{dx}(ax) = a$	$\int adx = ax + C$
$\dfrac{d}{dx}(x^{n+1}) = (n+1)x^n$	$\int x^n dx = \dfrac{x^{n+1}}{n+1} + C,\ n \neq 1$
$\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$	$\int \dfrac{1}{x}dx = \ln\lvert x\rvert + C$
$\dfrac{d}{dx}(e^x) = e^x$	$\int e^x dx = e^x + C$
$\dfrac{d}{dx}(\sin x) = \cos x$	$\int \cos xdx = \sin x + C$
$\dfrac{d}{dx}(\cos x) = -\sin x$	$\int \sin xdx = -\cos x + C$
$\dfrac{d}{dx}(\tan x) = \sec^2 x$	$\int \sec^2 xdx = \tan x + C$
$\dfrac{d}{dx}(\cot x) = -\csc^2 x$	$\int \csc^2 xdx = -\cot x + C$
$\dfrac{d}{dx}(\sec x) = \sec x \tan x$	$\int \sec x \tan xdx = \sec x + C$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x \qquad \int \csc x \cot x dx = -\csc x + C$$

Rule of Algebra for Antiderivative

1. Constant multiplication

$\int af(x)dx = a\int f(x)dx,\ a$ is some constant

2. Addition and subtraction

$$\int \left[f(x) \pm g(x) \right] dx = \int f(x)dx \pm \int g(x)dx$$

Example 1 Evaluate $\int (5x - x^2 + 2)dx$

<u>Solution</u>
$$\begin{aligned}\int (5x - x^2 + 2)dx &= \int 5xdx - \int x^2 dx + \int 2dx \\ &= 5\int xdx - \int x^2 dx + 2\int dx \\ &= 5\left(\frac{x^2}{2} + c_1\right) - \left(\frac{x^3}{3} + c_2\right) + 2\left(x + c_3\right) \\ &= \frac{5x^2}{2} + 5c_1 - \frac{x^3}{3} - c_2 + 2x + 2c_3 \\ &= \frac{5x^2}{2} - \frac{x^3}{3} + 2x + C\end{aligned}$$

where $C = 5c_1 - c_2 + 2c_3$

Example 2 Evaluate $\int (8x^3 + 4x - 6\sqrt{x} - \frac{2}{\sqrt[3]{x}} + \frac{5}{x^2})dx$

<u>Solution</u>

Example 3 Evaluate $\int (3e^x - 7\sin x + \frac{5}{x})dx$

Solution

Example 4 Evaluate $\int \frac{\cos x}{\sin^2 x} dx$

Solution

3. Definite Integral

A definite integral of $f(x)$ from a to b is written as

$$\int_a^b f(x)dx$$

a and b are called limits of integration, where a is the lower limit and b is the upper limit.

Definition 3.1 A definite integral of $f(x)$ is a continuous function on $a \le x \le b$ such that

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$$

Definition 3.2 If $a < b$ and $f(x)$ is integrable on $a \le x \le b$

1. $\int_a^a f(x)dx = 0$
2. $\int_a^b f(x)dx = -\int_b^a f(x)dx$
3. $\int_a^b f(x)dx > 0$ where $f(x) > 0$, and

   $\int_a^b f(x)dx < 0$ where $f(x) < 0$

Evaluation process of an definite integral

Step 1 Find antiderivative of $F(x)$

Step 2 Calculate $F(b)-F(a)$ by plugging $x=b$ and $x=a$ into $F(x)$ we found in step 1

Properties of a definite integral

Let $f(x)$ and $g(x)$ be integrable functions on $a \le x \le b$ and C be some constant.

1. $\int_a^b C dx = C(b-a)$
2. $\int_a^b Cf(x)dx = C\int_a^b f(x)dx$
3. $\int_a^b \left[f(x) \pm g(x) \right] dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

Example 5 Evaluate $\int_1^2 \left[5x^2 + 3x - 1 - \frac{6}{x}\right] dx$

<u>Solution</u>

$$\int_1^2 \left[5x^2 + 3x - 1 - \frac{6}{x}\right] dx = \int_1^2 5x^2 dx + \int_1^2 3x dx - \int_1^2 dx - \int_1^2 \frac{6}{x} dx$$

$$= 5\int_1^2 x^2 dx + 3\int_1^2 x dx - \int_1^2 dx - 6\int_1^2 \frac{1}{x} dx$$

$$= 5\frac{x^3}{3}\Big|_1^2 + 3\frac{x^2}{2}\Big|_1^2 - x\Big|_1^2 - 6\ln|x|\Big|_1^2$$

$$= 5\left(\frac{8-1}{3}\right) + 3\left(\frac{4-1}{2}\right) - (2-1)$$

$$-6(\ln 2 - \ln 1)$$

$$= 5\left(\frac{7}{3}\right) + 3\left(\frac{3}{2}\right) - (1) - 6(\ln 2 - 0)$$

$$= \frac{35}{3} + \frac{9}{2} - 1 - 6\ln 2$$

$$= \frac{70 + 27 - 6}{6} - 6\ln 2$$

$$= \frac{91}{6} - 6\ln 2$$

Thus $$\int_1^2 \left[5x^2 + 3x - 1 - \frac{6}{x}\right] dx = \frac{91}{6} - 6\ln 2$$

Example 6 Evaluate $\int_{\pi}^{\pi} \left[e^x + 4\sin x \right] dx$

<u>Solution</u>

Example 7 Evaluate $\int_0^3 |x-2|dx$

<u>Solution</u> From $f(x)=|x-2|$

We can write $f(x)=\begin{cases} x-2; & x \geq 2 \\ -(x-2); & x<2 \end{cases}$

Thus
$$\int_0^3 |x-2|dx = \int_0^2 |x-2|dx + \int_2^3 |x-2|dx$$
$$= \int_0^2 (-x+2)dx + \int_2^3 (x-2)dx$$
$$= -\int_0^2 xdx + \int_0^2 2dx + \int_2^3 xdx - \int_2^3 2dx$$
$$= -\frac{x^2}{2}\bigg|_0^2 + 2x\Big|_0^2 + \frac{x^2}{2}\bigg|_2^3 - 2x\Big|_2^3$$
$$= -\frac{1}{2}[4-0]+2[2-0]$$
$$+\frac{1}{2}[9-4]-2[3-2]$$
$$= -2+4+\frac{5}{2}-2$$
$$= \frac{5}{2}$$

Hence $\int_0^3 |x-2|dx = \frac{5}{2}$

Example 8 Evaluate

$$\int_{-2}^{1} f(x)dx \quad \text{where} \quad f(x) = \begin{cases} 2 - x^2; & x \geq 0 \\ x + 2; & x < 0 \end{cases}$$

Solution

4. Techniques of Integration

4.1 Integration by Substitution

We change the integrand by substitution.

Example 9 Evaluate $\int (3x-5)^{20}\,dx$

Solution Let $u = 3x-5$. Then $du = 3dx$ or $dx = \dfrac{du}{3}$

Thus
$$\int (3x-5)^{20}\,dx = \int u^{20}\,\frac{du}{3}$$
$$= \frac{1}{3}\int u^{20}du$$
$$= \frac{1}{3}\cdot\frac{u^{21}}{21} + C$$
$$= \frac{(3x-5)^{21}}{63} + C$$

Hence
$$\int (3x-5)^{20}\,dx = \frac{(3x-5)^{21}}{63} + C$$

Example 10 Evaluate $\int \frac{(\ln x)^2}{x \ln 9} dx$

<u>Solution</u>

Example 11 Evaluate $\int (x+3)\sqrt{x+1}dx$

Solution

The procedure of integration by substitution

1. Define $u = g(x)$ and find $du = g'(x)dx$
2. Rewrite $\int f(x)dx$ in terms of new variable u to get $\int h(u)du$
3. Find the integral $\int h(u)du = H(u) + C$
4. Plug $u = g(x)$ back into the resulting function in step 3.

$$\int f(x)dx = H(u) + C = H(g(x)) + C = F(x) + C$$

Example 12 Evaluate $\int x^2(1-x)^{100}\,dx$

<u>Solution</u> Let $u = 1 - x$ or $x = 1 - u$

Then $x^2 = (1-u)^2$ and $du = -dx$

Thus
$$\begin{aligned}\int x^2(1-x)^{100}\,dx &= \int (1-u)^2 u^{100}(-du)\\
&= \int (1-2u+u^2)(-u^{100})du\\
&= \int -u^{100}du + \int 2u^{101}du - \int u^{102}du\\
&= -\frac{u^{101}}{101} + 2\frac{u^{102}}{102} - \frac{u^{103}}{103} + C\\
&= \frac{2(1-x)^{102}}{102} - \frac{(1-x)^{101}}{101} - \frac{(1-x)^{103}}{103} + C\end{aligned}$$

Hence
$$\int x^2(1-x)^{100}\,dx = \frac{2(1-x)^{102}}{102} - \frac{(1-x)^{101}}{101} - \frac{(1-x)^{103}}{103} + C$$

Example 13 Evaluate $\int \frac{\sec^2 2x dx}{1+\tan 2x}$

Solution

Example 14 Evaluate $\int \frac{(x^2+1)dx}{2x-3}$

Solution Let $u = 2x-3$. Then $du = 2dx$ or $\frac{du}{2} = dx$

and $x = \frac{u+3}{2}$, $x^2 = \left(\frac{u+3}{2}\right)^2$, $x^2 = \frac{1}{4}\left(u^2+6u+9\right)$

then $x^2+1 = \frac{1}{4}\left(u^2+6u+9+4\right)$

Substitution:

$$\begin{aligned}
\int \frac{(x^2+1)dx}{2x-3} &= \int \frac{1}{4}\cdot\frac{(u^2+6u+9+4)}{u}\cdot\frac{du}{2} \\
&= \frac{1}{8}\int \frac{(u^2+6u+13)}{u}du \\
&= \frac{1}{8}\int (u+6+\frac{13}{u})du \\
&= \frac{1}{8}\left[\frac{u^2}{2}+6u+13\ln|u|\right]+C \\
&= \frac{1}{8}\left\{\frac{(2x-3)^2}{2}+(2x-3)\right. \\
&\quad \left. +13\ln|2x-3|\right\}+C
\end{aligned}$$

Thus

$$\int \frac{(x^2+1)dx}{2x-3} = \frac{1}{8}\left[\frac{(2x-3)^2}{2}+(2x-3)+13\ln|2x-3|\right]+C$$

Remark

Once we change the variable in the definite integral by substitution technique, we also need to change the limits of integration.

Example 15 Evaluate $\int_0^1 xe^{4x^2+1}dx$

<u>Solution</u> Let $u = 4x^2 + 1$. Then $du = 8xdx$ or $xdx = \dfrac{du}{8}$

When $x = 0$, then $u = 1$. And when $x = 1$, then $u = 5$

Substitution: $$\int_0^1 xe^{4x^2+1}dx = \int_0^1 e^{4x^2+1}xdx$$

$$= \int_1^5 \frac{e^u du}{8}$$

$$= \frac{1}{8}\int_1^5 e^u du$$

$$= \frac{1}{8}e^u\Big|_1^5$$

$$= \frac{1}{8}\left[e^5 - e^1\right]$$

Thus $$\int_0^1 xe^{4x^2+1}dx = \frac{1}{8}\left(e^5 - e\right)$$

Example 16 Evaluate $\int_0^3 x(1+x)^{\frac{1}{2}}\,dx$

<u>Solution</u>

4.2 Integration by Parts

We use this technique when integration by substitution doesn't work. We consider the integral as $\int udv$ where dv is a part of the function consisting of dx and $f(x)$ or $g(x)$.

Formula used to find the integration by parts:

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

or $$d(uv) = udv + vdu$$

$$udv = d(uv) - vdu$$

and $$\int udv = \int d(uv) - \int vdu$$

$$\int udv = uv - \int vdu$$

Remark

This technique is to express $\int udv$ in terms of uv and $\int vdu$ which is easier to be integrated. Thus choosing appropriate u and v is a crucial step for doing integration by parts.

Summary

Let $\int f(x)g(x)dx = \int h(x)dx = \int udv = uv - \int vdu$

To pick u and v, we consider

1. dv is easy to get integrated so that we have v
2. $\int vdu$ exists

In case of, definite integral:

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

Example 17 Evaluate $\int x \ln x dx$

Solution Let $u = \ln x$ and $dv = xdx$

$$du = \frac{dx}{x} \text{ and } \int dv = \int xdx \quad \text{or} \quad v = \frac{x^2}{2}$$

From $\int udv = uv - \int vdu$

Then $\int x \ln x dx = \int \ln x (xdx)$

$$= \ln(x)\left(\frac{x^2}{2}\right) - \int \left(\frac{x^2}{2}\right)\left(\frac{dx}{x}\right)$$

$$= \ln(x)\left(\frac{x^2}{2}\right) - \frac{1}{2}\int xdx$$

$$= \ln(x)\left(\frac{x^2}{2}\right) - \frac{1}{2}\cdot\frac{x^2}{2} + C$$

$$= \frac{x^2 \ln x}{2} - \frac{x^2}{4} + C$$

Thus $$\int x \ln x dx = \frac{x^2 \ln x}{2} - \frac{x^2}{4} + C$$

Example 18 Evaluate $\int_{1}^{2} \ln x dx$

<u>Solution</u>

Example 19 Evaluate $\int \tan^{-1} x \, dx$

<u>Solution</u>

Note: Some integrals may need several integrations by parts.

Example 20 Evaluate $\int e^{2x} \sin x \, dx$

<u>Solution</u> Let $u = e^{2x}$ and $dv = \sin x dx$

$du = 2e^{2x} dx$ and $v = -\cos x$

Then

$$\int e^{2x} \sin x \, dx = e^{2x}(-\cos x) - \int -\cos x(2e^{2x} dx)$$
$$= -e^{2x} \cos x + 2\int e^{2x} \cos x dx$$

Next, consider $2\int e^{2x} \cos x dx$

Let $u = e^{2x}$ and $dv = \cos x dx$

$du = 2e^{2x} dx$ and $v = \sin x$

$$2\int e^{2x} \cos x dx = 2\left[e^{2x} \sin x - \int \sin x(2e^{2x} dx)\right]$$
$$= 2e^{2x} \sin x - 4\int e^{2x} \cos x dx$$

Then

$$\int e^{2x} \sin x \, dx = -e^{2x} \cos x + 2\int e^{2x} \cos x dx$$
$$= -e^{2x} \cos x + 2e^{2x} \sin x - 4\int e^{2x} \sin x \, dx + C$$
$$5\int e^{2x} \sin x \, dx = -e^{2x} \cos x + 2e^{2x} \sin x + C$$

Hence, $\int e^{2x} \sin x \, dx = \frac{1}{5}\left[-e^{2x} \cos x + 2e^{2x} \sin x\right] + C$

Rules to pick u and dv

1. u should have a simple derivative.
2. dv may be complicated but easy to get integrated.
3. $\int vdu$ is easier to evaluate than $\int udv$

Examples of u and dv

1. $\int x^n e^{ax} dx,\ \int x^n \cos axdx,\ \int x^n \sin axdx$

   Then $u = x^n$ and dv is the rest of the integrand

2. $\int x^n \sin^{-1} xdx,\ \int x^n \cos^{-1} xdx,\ \int x^n \tan^{-1} xdx$

   Then $u = \sin^{-1} x$ or $u = \cos^{-1} x$ or $u = \tan^{-1} x$, respectively and dv is the rest

3. $\int x^m [\ln x]^n dx$ where $m \neq -1$

   Then $u = [\ln x]^n$ and dv is the rest

4.3 Integration of Rational Function by Partial Fraction

It is used when the integrand is in a form of rational function $\dfrac{f(x)}{g(x)}$

$$\frac{f(x)}{g(x)}=\frac{a_0+a_1x+a_2x^2+........+a_{n-1}x^{n-1}+a_nx^n}{b_0+b_1x+b_2x^2+........+b_{m-1}x^{m-1}+b_mx^m};n<m$$

Express $\dfrac{f(x)}{g(x)}$ as a partial fraction: ex $\dfrac{5x-3}{x^2-2x-3}=\dfrac{2}{x+1}+\dfrac{3}{x-3}$

which is found by

$$\frac{5x-3}{x^2-2x-3}=\frac{5x-3}{(x+1)(x-3)}=\frac{A}{x+1}+\frac{B}{x-3}$$

and $5x-3=A(x-3)+B(x+1)$

$$5x-3=(A+B)x+(-3A+B)$$

Calculating A and B by comparing coefficients of x

$$A+B=5 \quad \text{and} \quad -3A+B=-3$$

Solve to get $A=2$ and $B=3$. We call A and B constants calculated by undeterminated coefficients.

Conditions on partial fractions:

$\dfrac{f(x)}{g(x)}$ **can be expressed as a partial fraction if**

1. Power of $f(x)$ is higher than or equal to power of $g(x)$ ($n \geq m$), we first have to divide $g(x)$ by $f(x)$ to get

$$\frac{f(x)}{g(x)}=\phi(x)+\frac{h(x)}{g(x)}$$

where $h(x), g(x)$ are both polynomials and power of $h(x)$ is less than power of $g(x)$.

2. $g(x)$ can be factor out as linear or quadratic factors

   2.1 Types of factors

   a. Linear factor is in a form of $(ax+b)$ where a,b are real.

   b. Irreducible quadratic factor is in a form of (ax^2+bx+c) where a,b,c are real

Procedure of Integration by Partial Fraction

Consider a rational function $\dfrac{f(x)}{g(x)}$

Case 1 $g(x)$ has only non repeated linear factors

$$g(x)=(a_1x+b_1)(a_2x+b_2)........(a_nx+b_n)$$

where $\dfrac{b_1}{a_1} \neq \dfrac{b_2}{a_2} \neq .... \neq \dfrac{b_n}{a_n}$ and $a_1, a_2, ..., a_n \neq 0$

Then

$$\frac{f(x)}{g(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + ... + \frac{A_n}{a_nx+b_n}$$

where $A_1, A_2, ..., A_n$ are all constants we need to find.

Example 37 Evaluate $\int \frac{2x^2+5x-1}{x^3+x^2-2x}dx$

<u>Solution</u> Consider x^3+x^2-2x

$$x^3+x^2-2x = x(x-1)(x+2)$$

Thus $$\frac{2x^2+5x-1}{x^3+x^2-2x} = \frac{A_1}{x} + \frac{A_2}{x-1} + \frac{A_3}{x+2}$$

$$2x^2+5x-1 = A_1(x-1)(x+2) + A_2(x)(x+2) + A_3x(x-1)$$
$$= A_1(x^2+x-2) + A_2(x^2+2x) + A_3(x^2-x)$$

Compare coefficients:

$$A_1 + A_2 + A_3 = 2$$
$$A_1 + 2A_2 - A_3 = 5$$
$$-2A_1 = -1$$

Solve to get $A_1 = \frac{1}{2},\ A_2 = 2,\ A_3 = -\frac{1}{2}$

Thus $$\frac{2x^2+5x-1}{x^3+x^2-2x} = \frac{1}{2x} + \frac{2}{x-1} - \frac{1}{2(x+2)}$$

Plug it back into the integral:

$$\int \frac{2x^2+5x-1}{x^3+x^2-2x}dx = \int \frac{1}{2x}dx + \int \frac{2}{x-1}dx - \int \frac{1}{2(x+2)}dx$$
$$= \frac{1}{2}\ln|x| + 2\ln|x-1| - \frac{1}{2}\ln|x+2| + C$$

Hence

$$\int \frac{2x^2+5x-1}{x^3+x^2-2x}dx = \frac{1}{2}\ln|x| + 2\ln|x-1| - \frac{1}{2}\ln|x+2| + C$$

Case 2 $g(x)$ has only repeated linear factors.

$$g(x) = (ax+b)^n$$

Then

$$\frac{f(x)}{g(x)} = \frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \ldots\ldots + \frac{A_n}{(ax+b)^n}$$

where $A_1, A_2, \ldots\ldots, A_n$ are all constant we need to find.

Example 38 Evaluate $\int \frac{x^2+2x+3}{(x-1)(x+1)^2}\,dx$

<u>Solution</u> Consider

$$\frac{x^2+2x+3}{(x-1)(x+1)^2} = \frac{A_1}{x-1} + \frac{A_2}{x+1} + \frac{A_3}{(x+1)^2}$$

$$x^2+2x+3 = A_1(x+1)^2 + A_2(x-1)(x+1) + A_3(x-1)$$

$$= A_1(x^2+2x+1) + A_2(x^2-1) + A_3(x-1)$$

Compare the coefficients:

$$A_1 + A_2 = 1$$

$$2A_1 + A_3 = 2$$

$$A_1 - A_2 - A_3 = 3$$

Solve to get $A_1 = \frac{3}{2},\ A_2 = -\frac{1}{2},\ A_3 = -1$

Thus $$\frac{x^2+2x+3}{(x-1)(x+1)^2} = \frac{3}{2(x-1)} - \frac{1}{2(x+1)} - \frac{1}{(x+1)^2}$$

Plug it back to the integral:

$$\int \frac{(x^2+2x+3)dx}{(x-1)(x+1)^2} = \frac{3}{2}\int \frac{dx}{x-1} - \frac{1}{2}\int \frac{dx}{x+1} - \int \frac{dx}{(x+1)^2}$$

$$= \frac{3}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + \frac{1}{(x+1)} + C$$

Hence,

$$\int \frac{(x^2+2x+3)dx}{(x-1)(x+1)^2} = \frac{3}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + \frac{1}{(x+1)} + C$$

Case 3 $g(x)$ has only non repeated irreducible quadratic factors ax^2+bx+c:

$$\frac{f(x)}{g(x)} = \frac{Ax+B}{ax^2+bx+c}$$

where A, B are constants we need to find.

Example 39 Evaluate $\int \frac{5x^2+3x-2}{x^3-1}dx$

<u>Solution</u> Consider

$$\frac{5x^2+3x-2}{x^3-1} = \frac{5x^2+3x-2}{(x-1)(x^2+x+1)}$$

$$\frac{5x^2+3x-2}{x^3-1} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$$

$$\begin{aligned} 5x^2+3x-2 &= A(x^2+x+1)+(Bx+C)(x-1) \\ &= A(x^2+x+1)+Bx^2-Bx+Cx-C \end{aligned}$$

Compare the coefficients:

$$A+B=5$$

$$A-B+C=3$$

$$A-C=-2$$

Solve to get $A=2,\ B=3,\ C=4$

Thus $$\frac{5x^2+3x-2}{x^3-1} = \frac{2}{x-1} + \frac{3x+4}{x^2+x+1}$$

Plug it back into the integral:

$$\begin{aligned} \int \frac{(5x^2+3x-2)dx}{x^3-1} &= \int \frac{2dx}{x-1} + \int \frac{(3x+4)dx}{x^2+x+1} \\ &= \int \frac{2dx}{x-1} + \int \frac{(3x+4)dx}{x^2+x+1} \\ &= 2\ln|x-1| + \int \frac{(3x+4)dx}{x^2+x+1} \end{aligned}$$

Next consider $\int \frac{(3x+4)dx}{x^2+x+1} = \int \frac{(3x+4)dx}{\left[x+\frac{1}{2}\right]^2+\frac{3}{4}}$

Let $u = x+\frac{1}{2}$ and $du = dx$

Thus

$$\int \frac{(3x+4)dx}{\left[x+\frac{1}{2}\right]^2+\frac{3}{4}} = \int \frac{3\left[u-\frac{1}{2}\right]+4}{u^2+\frac{3}{4}}du$$

$$= \int \frac{3u+\frac{5}{2}}{u^2+\frac{3}{4}}du$$

$$= 3\int \frac{udu}{u^2+\frac{3}{4}} + \frac{5}{2}\int \frac{du}{u^2+\frac{3}{4}}$$

$$= \frac{3}{2}\ln(u^2+\frac{3}{4}) + \frac{5(2)}{2(\sqrt{3})}\tan^{-1}\frac{2}{\sqrt{3}}u + C$$

$$= \frac{3}{2}\ln(x^2+x+1) + \frac{5}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}} + C$$

Hence

$$\int \frac{(5x^2+3x-2)dx}{x^3-1} = 2\ln|x-1| + \frac{3}{2}\ln(x^2+x+1)$$

$$+\frac{5}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}} + C$$

Case 4 $g(x)$ has only repeated irreducible quadratic factors: $(ax^2+bx+c)^n, n \geq 2$:

$$\frac{f(x)}{g(x)} = \frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \ldots\ldots.$$

$$+ \frac{A_nx+B_n}{(ax^2+bx+c)^n}$$

where $A_1, \ldots\ldots A_n, B_1, \ldots\ldots, B_n$ are all constants we need to find.

Example 40 Evaluate $\int \frac{(x^3+1)dx}{(x^2+4)^2}$

<u>Solution</u> Consider

$$\frac{x^3+1}{(x^2+4)^2} = \frac{Ax+B}{x^2+4} + \frac{Cx+D}{(x^2+4)^2}$$

$$x^3+1 = (Ax+B)(x^2+4) + (Cx+D)$$

$$= Ax^3 + 4Ax + Bx^2 + 4B + Cx + D$$

Compare coefficients:

$$A = 1$$

$$B = 0$$

$$4A + C = 0$$

$$4B + D = 1$$

Solve to get $A = 1,\ B = 0,\ C = -4, D = 1$

Thus $$\frac{x^3+1}{(x^2+4)^2} = \frac{1x+0}{x^2+4} + \frac{-4x+1}{(x^2+4)^2}$$

Plug it back into the integral:

$$\int \frac{(x^3+1)dx}{(x^2+4)^2} = \int \frac{xdx}{x^2+4} - 4\int \frac{xdx}{(x^2+4)^2} + \int \frac{dx}{(x^2+4)^2}$$

$$= \frac{1}{2}\ln(x^2+4) - 4\int \frac{xdx}{(x^2+4)^2} + \int \frac{dx}{(x^2+4)^2}$$

Next consider $-4\int \frac{xdx}{(x^2+4)^2}$

Let $u = x^2+4$ and $du = 2xdx$

So we have $-4\int \frac{xdx}{(x^2+4)^2} = -2\int \frac{du}{u^2}$

$$= 2u^{-1} + C$$

$$= \frac{2}{x^2+4} + C$$

And for $\int \frac{dx}{(x^2+4)^2}$

We let $x = 2\tan\theta$ and $dx = 2\sec^2\theta d\theta$

We then have $\int \frac{dx}{(x^2+4)^2} = \int \frac{2\sec^2\theta d\theta}{(4\tan^2\theta+4)^2}$

$$= \frac{1}{8}\int \frac{\sec^2\theta d\theta}{(\tan^2\theta+1)^2}$$

$$= \frac{1}{8}\int \frac{d\theta}{\sec^2\theta}$$

$$= \frac{1}{8}\int \cos^2\theta d\theta$$

$$= \frac{1}{16}\int (1+\cos 2\theta)d\theta$$

$$= \frac{1}{16}\left[\theta + \frac{\sin 2\theta}{2}\right] + C$$

$$= \frac{1}{16}\left[\tan^{-1}\frac{x}{2} + \frac{2x}{(x^2+4)}\right] + C$$

Hence

$$\int \frac{(x^3+1)dx}{(x^2+4)^2} = \frac{1}{2}\ln(x^2+4) + \frac{2}{x^2+4} + \frac{1}{16}\tan^{-1}\frac{x}{2}$$

$$+ \frac{x}{8(x^2+4)} + C$$

Example 41 Evaluate $\int \frac{(x^5 - x^4 - 3x + 5)}{x^4 - 2x^3 + 2x^2 - 2x + 1} dx$

Solution

Exercise 1

Evaluate the following integrals

1. $\int 3x^2(x^3+2)^2\,dx$
2. $\int x^2\sqrt{x^3+2}\,dx$
3. $\int \dfrac{8x^2}{(x^3+2)}\,dx$
4. $\int \dfrac{x^2}{\sqrt{x^3+2}}\,dx$
5. $\int 3x\sqrt{1-2x^2}\,dx$
6. $\int \dfrac{x+3}{\sqrt[3]{x^2+6x}}\,dx$
7. $\int (3x^2-2)(x^3-2x)\,dx$
8. $\int \dfrac{x+1}{x^2+2x+5}\,dx$
9. $\int x^2\sqrt{1+x}\,dx$
10. $\int \dfrac{x^2}{1-2x^3}\,dx$
11. $\int (e^x+1)^3\,dx$
12. $\int \cos^3 2x \sin 2x\,dx$
13. $\int e^{\cos x}\sin x\,dx$
14. $\int \dfrac{\cos x\,dx}{\sqrt{4-\sin^2 x}}$
15. $\int \dfrac{e^{\sqrt{1+x}}}{\sqrt{1+x}}\,dx$
16. $\int \cos 2x\sqrt{1-\sin 2x}\,dx$
17. $\int 3^{2x+1}\,dx$
18. $\int \dfrac{e^{\tan^{-1}2x}}{1+4x^2}\,dx$
19. $\int \left[\dfrac{\ln x}{x}\right]^3 dx$
20. $\int \dfrac{1}{x\ln x}\,dx$

Evaluate the following definite integrals

21. $\int_1^5 \dfrac{x+3}{\sqrt{2x-1}}\,dx$
22. $\int_0^1 \dfrac{x}{x^2+4}\,dx$

23. $\int_1^8 \sqrt{1+3x}\,dx$

24. $\int_4^8 \frac{x\,dx}{\sqrt{x^2-15}}$

25. $\int_0^{2\pi} \sin\frac{x}{2}\,dx$

Answers to exercise 1

1. $\left[\frac{x^3+2}{3}\right]^3 + C$

2. $\frac{2}{9}(x^3+2)^{\frac{3}{2}} + C$

3. $\frac{-4}{3(x^3+2)^2} + C$

4. $\frac{2}{3}\sqrt{x^3+2} + C$

5. $-\frac{1}{2}(1-2x^2)^{\frac{3}{2}} + C$

6. $\frac{3}{4}(x^2+6x)^{\frac{2}{3}} + C$

7. $\frac{1}{6}(x^3-2x)^6 + C$

8. $\frac{1}{2}\ln\left|x^2+2x+5\right| + C$

9. $\frac{2}{7}(1+x)^{\frac{7}{2}} - \frac{4}{5}(1+x)^{\frac{5}{2}} + \frac{2}{3}(1+x)^{\frac{3}{2}} + C$

10. $-\frac{1}{6}\ln\left|1-2x^3\right| + C$

11. $\frac{1}{4}(e^x+1) + C$

12. $-\frac{\cos^4 2x}{8} + C$

13. $-e^{\cos x} + C$

14. $\sin^{-1}\left(\frac{\sin x}{2}\right) + C$

15. $2e^{\sqrt{1+x}} + C$

16. $-\frac{1}{3}(1-\sin 2x)^{\frac{3}{2}} + C$

17. $\frac{3^{2x+1}}{2\ln 3} + C$

18. $\frac{1}{2}e^{\tan^{-1} 2x} + C$

19. $\frac{1}{4}[\ln x]^4 + C$

20. $\ln|\ln x| + C$

21. 20

22. $\frac{1}{2}\ln\frac{5}{4}$

23. 26

24. 6

25. 4

Exercise 2

Evaluate the following integrals

1. $\int x \sin x dx$
2. $\int xe^x dx$
3. $\int x^2 \ln x dx$
4. $\int x\sqrt{1+x} dx$
5. $\int \sec^3 x dx$
6. $\int x^2 \sin x dx$
7. $\int x^2 e^{2x} dx$
8. $\int x \cos x dx$
9. $\int x \sec^2 3x dx$
10. $\int \cos^{-1} 2x dx$
11. $\int \tan^{-1} x dx$
12. $\int \frac{xe^x}{(1+x)^2} dx$
13. $\int x \tan^{-1} x dx$
14. $\int x^2 e^{-3x} dx$
15. $\int x^3 \sin x dx$
16. $\int x \sin^{-1} x^2 dx$
17. $\int \sin x \sin 3x dx$
18. $\int \sin(\ln x) dx$
19. $\int e^{ax} \cos bx dx$
20. $\int e^{ax} \sin bx dx$

Show how to use reduction formula to the following integrals.

21. $\int u^n e^{au} du$
22. $\int u^n \cos bu du$

Evaluate the following definite integrals

23. $\int_{1}^{e} \ln x dx$

24. $\int_{0}^{\frac{\pi}{3}} x^2 \sin 3x dx$

25. $\int_{0}^{\sqrt{2}} x^3 e^{x^2} dx$

Answers to exercise 2

1. $-x \sin x + \sin x + C$
2. $xe^x - e^x + C$
3. $\dfrac{x^3 \ln x}{3} - \dfrac{1}{9} x^3 + C$
4. $\dfrac{2}{3} x(1+x)^{\frac{3}{2}} - \dfrac{4}{15}(1+x)^{\frac{5}{2}} + C$
5. $\dfrac{1}{2}(\sec x \tan x + \ln\left|\sec x \tan x\right|) + C$
6. $-x^2 \cos x + 2x \sin x + 2\cos x + C$
7. $\dfrac{1}{2} x^3 e^{2x} - \dfrac{3}{4} x^2 e^{2x} + \dfrac{3}{4} x e^{2x} - \dfrac{3}{8} e^{2x} + C$
8. $x \sin x + \cos x + C$
9. $\dfrac{1}{3} x \tan 3x - \dfrac{1}{9} \ln\left|\sec 3x\right| + C$
10. $x \cos^{-1} 2x - \dfrac{1}{2}\sqrt{1-4x^2} + C$
11. $x \tan^{-1} x - \ln\sqrt{1+x^2} + C$

12. $\dfrac{e^x}{1+x}+C$

13. $\dfrac{1}{2}(x^2+1)\tan^{-1}x-\dfrac{1}{2}x+C$

14. $-\dfrac{1}{3}e^{-3x}(x^2+\dfrac{2}{9}x+\dfrac{2}{9})+C$

15. $-x^3\cos x+3x^2\sin x+6x\cos x-6\sin x+C$

16. $\dfrac{1}{2}x^2\sin^{-1}x^2+\dfrac{1}{2}\sqrt{1-x^4}+C$

17. $\dfrac{1}{8}\sin 3x\cos x-\dfrac{3}{8}\cos 3x\sin x+C$

18. $\dfrac{1}{2}\left[x\sin(\ln x)-x\cos(\ln x)\right]+C$

19. $\dfrac{e^{ax}(b\sin bx+a\cos bx)}{a^2+b^2}+C$

20. $\dfrac{e^{ax}(a\sin bx-b\cos bx)}{a^2+b^2}+C$

21. $\dfrac{1}{a}u^n e^{au}-\dfrac{n}{a}\int u^{n-1}e^{au}du$

22. $\dfrac{1}{b}u^n\sin bu-\dfrac{n}{b}\int u^{n-1}\sin bu du$

23. 1

24. $\dfrac{1}{27}(\pi^2-4)$

25. $\dfrac{1}{2}(e^2+1)$

Exercise 3

Evaluate the following integrals

1. $\int \frac{1}{x^2-4}\,dx$

2. $\int \frac{x+1}{x^3+x^2-6x}\,dx$

3. $\int \frac{1}{x^2+7x+6}\,dx$

4. $\int \frac{x}{x^2-3x-4}\,dx$

5. $\int \frac{x^2-3x-1}{x^3+x^2-2x}\,dx$

6. $\int \frac{x^2+3x-4}{x^2-2x-8}\,dx$

7. $\int \frac{x}{(x-2)^2}\,dx$

8. $\int \frac{3x+5}{x^3-x^2-x+1}\,dx$

9. $\int \frac{x^4-x^3-x-1}{x^3-x^2}\,dx$

10. $\int \frac{x^3+x^2+x+2}{x^4+3x^2+2}\,dx$

11. $\int \frac{x^2}{a^4-x^4}\,dx$

12. $\int \frac{2x^2+3}{(x^2+1)^2}\,dx$

13. $\int \frac{1}{x^3+x}\,dx$

14. $\int \frac{x^3+x^2+x+3}{(x^2+1)(x^2+3)}\,dx$

15. $\int \frac{2x^3}{(x^2+1)^2}\,dx$

16. $\int \frac{2x^3+x^2+4}{(x^2+4)^2}\,dx$

17. $\int \frac{x^3+x-1}{(x^2+1)^2}\,dx$

18. $\int \frac{x^4}{(1-x)^3}\,dx$

19. $\int \frac{x^5-x^4+4x^3-4x^2+8x-4}{(x^2+2)^3}\,dx$

20. $\int \frac{1}{e^{2x}-3e^x}\,dx$

21. $\int \frac{\sin x}{\cos x(1+\cos^2 x)}\,dx$

22. $\int \frac{(2+\tan^2\theta)\sec^2\theta}{1+\tan^3\theta}\,d\theta$

Evaluate the following definite integrals

23. $\int_{-1}^{2} \frac{1}{x^2-9}dx$

24. $\int_{-8}^{-3} \frac{x+2}{x(x-2)^2}dx$

25. $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{\sin x dx}{\cos^2 x - 5\cos x + 4}$

Answers to exercise 3

1. $\frac{1}{4}\ln\left|\frac{x-2}{x+2}\right| + C$

2. $\frac{1}{30}\ln\left|\frac{(x-2)^9}{(x)^5(x+3)^4}\right| + C$

3. $\frac{1}{5}\ln\left|\frac{x+1}{x+6}\right| + C$

4. $\frac{1}{5}\ln\left|(x-4)(x+1)^4\right| + C$

5. $\frac{1}{2}\ln\left|\frac{x(x+2)^3}{(x-1)^2}\right| + C$

6. $x + \ln\left|(x-4)^4(x+2)\right| + C$

7. $\ln|x-2| - \frac{2}{x-2} + C$

8. $-\frac{4}{x-1} + \frac{1}{2}\ln\left|\frac{x+1}{x-1}\right| + C$

9. $\frac{x^2}{2} - \frac{1}{x} - 2\ln\left|\frac{x-1}{x}\right| + C$

10. $\tan^{-1} x + \frac{1}{2}\ln\left|x^2+2\right| + C$

11. $\frac{1}{4a}\ln\left|\frac{a+x}{a-x}\right| - \frac{1}{2a}\tan^{-1}\frac{x}{a} + C$

12. $\frac{5}{2}\tan^{-1} x + \frac{x}{2(x^2+1)} + C$

13. $\ln\left|\dfrac{x}{\sqrt{x^2+1}}\right|+C$

14. $\ln\left|\sqrt{x^2+3}\right|+\tan^{-1}x+C$

15. $\ln\left|x^2+1\right|+\dfrac{1}{x^2+1}+C$

16. $\ln\left|x^2+4\right|+\dfrac{1}{2}\tan^{-1}\dfrac{x}{2}+\dfrac{4}{x^2+4}+C$

17. $\dfrac{1}{2}\ln\left|x^2+1\right|-\dfrac{1}{2}\tan^{-1}x-\dfrac{x}{2(x^2+1)}+C$

18. $-\dfrac{x^2}{2}-3x-6\ln\left|1-x\right|-\dfrac{4}{1-x}+\dfrac{1}{2(1-x)^2}+C$

19. $\dfrac{1}{2}\ln\left|x^2+2\right|-\dfrac{\sqrt{2}}{2}\tan^{-1}\dfrac{x}{\sqrt{2}}-\dfrac{x}{(x^2+2)^2}+C$

20. $\dfrac{1}{3e^x}+\dfrac{1}{9}\ln\left|\dfrac{e^x-3}{e^x}\right|+C$

21. $\ln\left|\dfrac{\sqrt{1+\cos^2 x}}{\cos x}\right|+C$

22. $\ln\left|1+\tan\theta\right|+\dfrac{2}{\sqrt{3}}\tan^{-1}\dfrac{2\tan\theta-1}{\sqrt{3}}+C$

23. $-\dfrac{1}{6}\ln 10$

24. $\dfrac{1}{2}\ln\dfrac{3}{4}+\dfrac{1}{5}$

25. $\dfrac{1}{3}\ln\left|\dfrac{-\dfrac{\sqrt{2}}{2}-1}{-\dfrac{\sqrt{2}}{2}-4}\right|-\dfrac{1}{3}\ln\left|\dfrac{-\dfrac{\sqrt{2}}{2}-1}{\dfrac{\sqrt{2}}{2}-4}\right|$