The Derivative

Definition of Derivative

Let L be a line connecting points P and Q on the curve of $y = f(x)$ as shown in Figure 1 here.

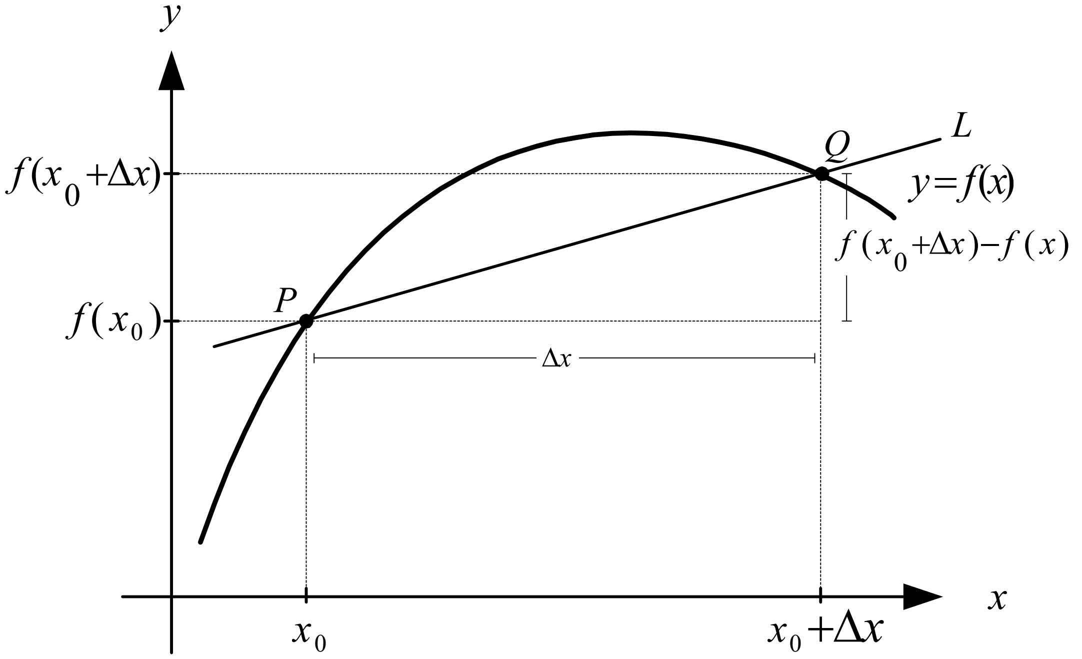


Figure 1

From Figure 1, consider the slope of line L:

$$\text{Slope of the line } L = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \frac{\text{Changed values of } f}{\text{Changed values of } x} \qquad [\text{From } x \text{ to } x+\Delta x]$$

$$= \text{Average rate of change of } f \text{ from } x \text{ to } x+\Delta x.$$

Next, consider the slope of line L when point Q is moved closer and closer to point P along the curve of $y = f(x)$ as in Figure 2.

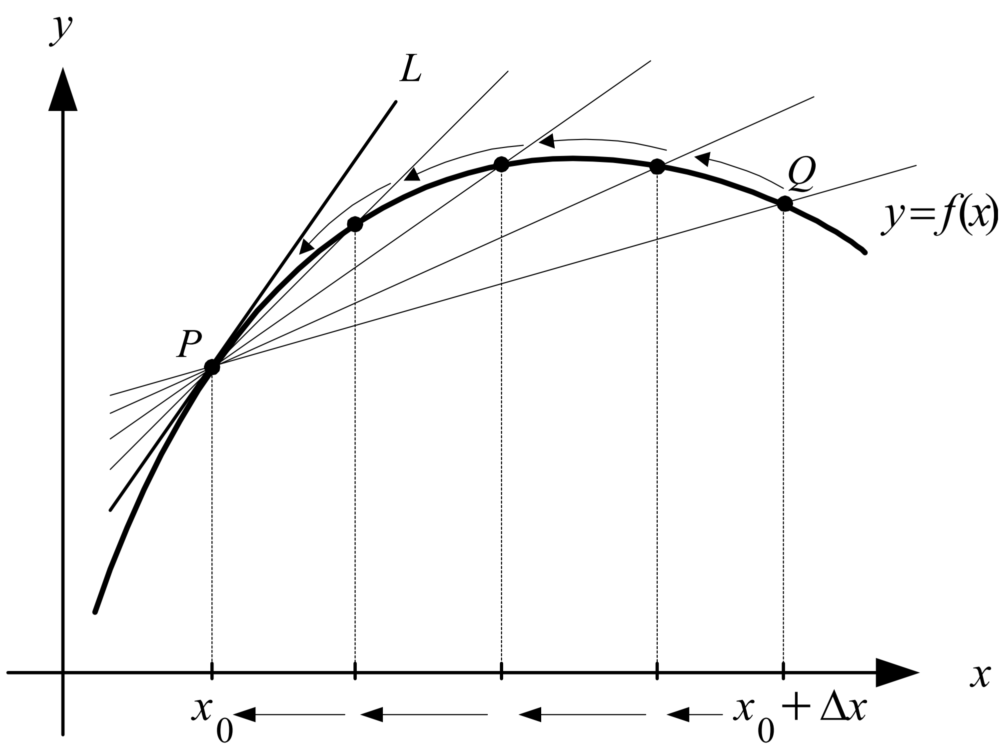


Figure 2

As Q is moved closer to P or as Δx gets smaller to 0, the slope of line L gets closer to the slope of a tangent line of the curve $y = f(x)$ at the point P.

Slope of the tangent line of $y = f(x)$ at the point P

$$= \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

= Instantaneous rate of change of f at $x = x_0$.

Definition 1:

Let f be a function defined on an open interval containing x. Then the **derivative of** f **at** x, denoted by $f'(x)$, is defined by

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

Remark

- Beside the notation $f'(x)$, the following notations are also used to denote the derivative of f at x:

$$\frac{dy}{dx}, \ \frac{d}{dx} f(x) \ \text{or just } y'.$$

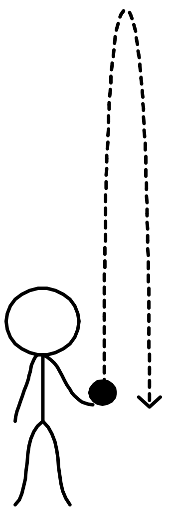
For the derivative of f at $x = a$, we may write it as

$$f'(a) \ \text{or} \ \left.\frac{dy}{dx}\right|_{x=a}.$$

- If f is continuous at $x = a$, then

$$\begin{aligned} f'(a) &= \text{The slope of tangent line of } y = f(x) \text{ at } x = a \\ &= \lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x} \\ &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a}. \end{aligned}$$

◇ Throw an object vertically. At time t seconds, the object is at the position $s(t) = -4.9t^2 + 49t$.



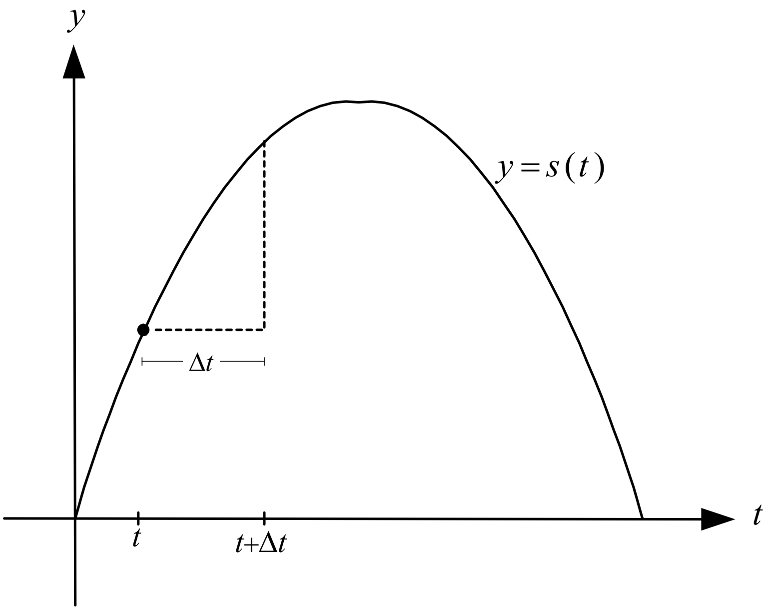


Figure 3 shows the object's position at time t

- **Average velocity of the object from time t to $t+\Delta t$** is the average rate of distance change from time t to $t+\Delta t$ seconds.

$$\text{Average velocity from time } t \textbf{ to } t+\Delta t = \frac{s(t+\Delta t)-s(t)}{\Delta t}.$$

- **Instantaneous velocity of the object at time t** is the instantaneous rate of distance change at time t seconds as $\Delta t \to 0$

$$\text{Instantaneous velocity at time } t = \lim_{\Delta t \to 0} \frac{s(t+\Delta t)-s(t)}{\Delta t} = \frac{ds}{dt}.$$

Thus, at time t seconds, the object is thrown from the ground with the velocity $v(t) = \dfrac{ds}{dt} = -9.8t + 49$.

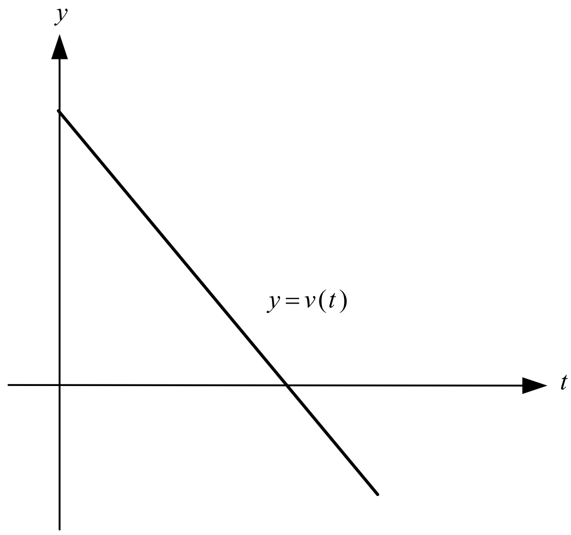


Figure 4 shows the velocity of the object at time t seconds

<u>Example</u> Let $f(x) = x^2 + 4$.

(a) Find derivative of f.

(b) Find derivative of f at $x = 5$.

<u>Example</u> Find the slope of the tangent line of $y = \dfrac{1}{x^2}$ at the point $(1,1)$.

Example An object moves horizontally. At time t seconds, the object has distance $s = 5 - 2t + t^2$ meters. Compute

a. the average velocity of the object from 1 to 3 seconds,

b. the instantaneous velocity of the object at t seconds,

c. the instantaneous velocity of the object at $t = 1$ second.

Theorem

If a function f has derivative (or say "is differentiable") at $x = a$ ($f'(a)$ exists as a real number), then f is continuous at $x = a$.

Proof Consider

$$
\begin{aligned}
\lim_{x\to a} f(x) - f(a) &= \lim_{x\to a} f(x) - \lim_{x\to a} f(a) \\
&= \lim_{x\to a}\left[f(x) - f(a)\right] \\
&= \lim_{x\to a} \frac{(x-a)\left[f(x) - f(a)\right]}{x-a} \\
&= \lim_{x\to a}(x-a) \cdot \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \\
&= 0 \cdot f'(a) \\
&= 0\ .
\end{aligned}
$$

That is, $\lim_{x\to a} f(x) = f(a)$.

Thus, f is continuous at $x = a$.

□

Remark: The converse may not be true. That is, if f is continuous at $x = a$, then f **may or may not** be differentiable at $x = a$.

Example Find the derivative of $f(x) = |x|$ at $x = 0$.

Example Let $f(x) = \sqrt{x}$. Compute $f'(x)$.

Derivative Formulas

Finding a derivative by using the definition is quite complicated and time consuming. However, there are several theorems and formulas to help us calculating derivatives easier and faster. Consider the following formulas.

Formulas

Let u, v be functions of x and c, n are some constants.

1. $\dfrac{dc}{dx} = 0$ and $\dfrac{dx}{dx} = 1$

2. $\dfrac{d}{dx}(cu) = c\dfrac{du}{dx}$

3. $\dfrac{d}{dx}(u \pm v) = \dfrac{du}{dx} \pm \dfrac{dv}{dx}$

4. $\dfrac{du^n}{dx} = nu^{n-1} \cdot \dfrac{du}{dx}$

$$\frac{dx^n}{dx} = nx^{n-1} \cdot \frac{dx}{dx} = nx^{n-1}$$

5. $\dfrac{d}{dx}(uv) = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

6. $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

<u>Example</u> Find the derivatives of the following functions .

(a) $f(x) = 2x^9 - \dfrac{5}{x} + 7x - 1$

(b) $g(x) = \sqrt[3]{x^4} + \dfrac{1}{\sqrt[3]{x^4 + 2}}$

(c) $p(x) = \left(2x^7 - x^{-1}\right)\left(5x^9 - 10\right)$

Example Find the derivatives of the following functions.

(a) $y = \dfrac{(2x-1)^2}{x^2+7}$

Solution $$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{(2x-1)^2}{x^2+7}\right]$$

$$= \frac{(x^2+7)\dfrac{d}{dx}(2x-1)^2 - (2x-1)^2\dfrac{d}{dx}(x^2+7)}{(x^2+7)^2}$$

$$= \frac{(x^2+7)\cdot 2(2x-1)\dfrac{d}{dx}(2x-1) - (2x-1)^2\cdot(2x)}{(x^2+7)^2}$$

$$= \frac{(x^2+7)\cdot 2(2x-1)\cdot 2 - (2x-1)^2\cdot(2x)}{(x^2+7)^2}$$

$$= \frac{4(x^2+7)(2x-1) - (2x-1)^2\cdot(2x)}{(x^2+7)^2}$$

(b) $f(x) = \dfrac{x^7 - 4\sqrt{x} - 2}{x^2}$

Solution $f'(x) = \dfrac{d}{dx}\left[\dfrac{x^7 - 4\sqrt{x} - 2}{x^2}\right]$

$$= \frac{d}{dx}\left(x^5 - 4x^{-\frac{3}{2}} - 2x^{-2}\right)$$

$$= \frac{dx^5}{dx} - 4\frac{dx^{-\frac{3}{2}}}{dx} - 2\frac{dx^{-2}}{dx}$$

$$= 5x^4 - 4\left(-\frac{3}{2}\right)x^{-\frac{5}{2}} - 2(-2)x^{-3}$$

$$= 5x^4 + 6x^{-\frac{5}{2}} + 4x^{-3}$$

(c) $r(t) = \left(4t^3 - 7t^{-6}\right)^{12}$

Solution

$$r'(t) = \frac{d}{dt}\left(4t^3 - 7t^{-6}\right)^{12}$$

$$= 12\left(4t^3 - 7t^{-6}\right)^{11} \cdot \frac{d}{dt}\left(4t^3 - 7t^{-6}\right)$$

$$= 12\left(4t^3 - 7t^{-6}\right)^{11} \cdot \frac{d}{dt}\left(4t^3 - 7t^{-6}\right)$$

$$= 12\left(4t^3 - 7t^{-6}\right)^{11}\left(4\frac{dt^3}{dt} - 7\frac{dt^{-6}}{dt}\right)$$

$$= 12\left(4t^3 - 7t^{-6}\right)^{11}\left(12t^2 + 42t^{-7}\right)$$

$$= 72\left(4t^3 - 7t^{-6}\right)^{11}\left(2t^2 + 7t^{-7}\right)$$

(d) $y=\left(5x^{10}-8\right)\cdot\sqrt[4]{x^2+9}$

Solution

$$\frac{dy}{dx}=\frac{d}{dx}\left(5x^{10}-8\right)\cdot\sqrt[4]{x^2+9}$$

$$=\left(5x^{10}-8\right)\frac{d}{dx}\left(x^2+9\right)^{\frac{1}{4}}+\left(x^2+9\right)^{\frac{1}{4}}\frac{d}{dx}\left(5x^{10}-8\right)$$

$$=\left(5x^{10}-8\right)\left(\frac{1}{4}\left(x^2+9\right)^{-\frac{3}{4}}\right)\frac{d}{dx}\left(x^2+9\right)+\left(x^2+9\right)^{\frac{1}{4}}\left(50x^9\right)$$

$$=\left(5x^{10}-8\right)\left(\frac{1}{4}\left(x^2+9\right)^{-\frac{3}{4}}\right)\left(2x\right)+\left(x^2+9\right)^{\frac{1}{4}}\left(50x^9\right)$$

$$=\frac{1}{2}x\left(5x^{10}-8\right)\left(x^2+9\right)^{-\frac{3}{4}}+50x^9\left(x^2+9\right)^{\frac{1}{4}}$$

(e) $y=\dfrac{4x^2+5}{3x-1}$

Solution

$$\frac{dy}{dx}=\frac{d}{dx}\left[\frac{4x^2+5}{3x-1}\right]$$

$$=\frac{(3x-1)\frac{d}{dx}\left(4x^2+5\right)-(4x^2+5)\frac{d}{dx}\left(3x-1\right)}{(3x-1)^2}$$

$$=\frac{(3x-1)\left(8x\right)-(4x^2+5)\left(3\right)}{(3x-1)^2}$$

$$=\frac{12x^2-8x-15}{(3x-1)^2}$$

(f) $g(x)=\sqrt{\dfrac{8x^4}{2-x^7}}$

Solution

<u>Example</u> Find an equation of the tangent line of $y = \dfrac{1}{\sqrt{x^4 + 8x}}$ at the point $x = 1$.

The Chain Rule

If functions f and g are differentiable and $F = f \circ g$ is a composite function given by $F(x) = f(g(x))$, then F is also differentiable at x and

$$\boxed{F'(x) = f'(g(x)) \cdot g'(x)}$$

In other words, if $y = f(u)$ and $u = g(x)$ are differentiable, then

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$$

Remark:

In case of more than two functions composed, we can extend the chain rule as follows.

Let $y = f(u)$, $u = g(x)$ and $x = h(t)$ be differentiable, then

$$\boxed{\frac{dy}{dt} = \frac{dy}{du} \cdot \frac{du}{dx} \cdot \frac{dx}{dt}}$$

Example Find $\dfrac{dy}{dx}$ where $y=\sqrt{x^{\frac{2}{3}}+x^{\frac{4}{3}}-1}$

Solution From $y = \sqrt{x^{\frac{2}{3}}+x^{\frac{4}{3}}-1} = \sqrt{x^{\frac{2}{3}}+(x^{\frac{2}{3}})^2-1}$,

we consider $y=\sqrt{u}$, $u=v+v^2-1$ and $v=x^{\frac{2}{3}}$.

Apply the following chain rule: $\dfrac{dy}{dx}=\dfrac{dy}{du}\cdot\dfrac{du}{dv}\cdot\dfrac{dv}{dx}$

where
$$\frac{dy}{du} = \frac{du^{\frac{1}{2}}}{du} = \frac{1}{2}u^{-\frac{1}{2}},$$
$$\frac{du}{dv} = \frac{d}{dv}(v+v^2-1) = 1+2v,$$

and
$$\frac{dv}{dx} = \frac{dx^{\frac{2}{3}}}{dx} = \frac{2}{3}x^{-\frac{1}{3}}.$$

Thus,

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{2}u^{-\frac{1}{2}}\cdot(1+2v)\cdot\frac{2}{3}x^{-\frac{1}{3}} \\ &= \frac{1}{3}\cdot(x^{\frac{2}{3}}+x^{\frac{4}{3}}-1)^{-\frac{1}{2}}\cdot(1+2x^{\frac{2}{3}})\cdot x^{-\frac{1}{3}} \\ &= \frac{1}{3}\cdot(x^{\frac{2}{3}}+x^{\frac{4}{3}}-1)^{-\frac{1}{2}}\cdot(x^{-\frac{1}{3}}+2x^{\frac{1}{3}})\,.\end{aligned}$$

Example Let $y = \dfrac{u^2}{u^3 - 16}$, $u = 3x^2 - 8$ and $x = \sqrt[4]{t+5}$.

Find $\dfrac{dy}{dt}$ at $t = 11$.

Derivatives of Trigonometric Functions

Derivative Formulas

Let u be a function of x and differentiable.

1. $$\frac{d}{dx}\sin u = \cos u \cdot \frac{du}{dx}$$

2. $$\frac{d}{dx}\cos u = -\sin u \cdot \frac{du}{dx}$$

3. $$\frac{d}{dx}\tan u = \sec^2 u \cdot \frac{du}{dx}$$

4. $$\frac{d}{dx}\cot u = -\csc^2 u \cdot \frac{du}{dx}$$

5. $$\frac{d}{dx}\sec u = \sec u \tan u \cdot \frac{du}{dx}$$

6. $$\frac{d}{dx}\csc u = -\csc u \cot u \cdot \frac{du}{dx}$$

<u>Example</u> Find $\dfrac{dy}{dx}$ where

(a) $y = (x^4 + 1)\tan x$,

<u>Solution</u>

$$\frac{dy}{dx} = \frac{d}{dx}(x^4 + 1)\tan x$$

$$= (x^4 + 1)\frac{d}{dx}\tan x + \tan x\frac{d}{dx}(x^4 + 1)$$

$$= (x^4 + 1)\sec^2 x + \tan x(4x^3)$$

$$= (x^4 + 1)\sec^2 x + 4x^3 \tan x$$

(b) $y = \dfrac{\sin(2x)}{7 - \cos(3x)}$,

(c) $y = \cot\sqrt[3]{4 - x^3}$,

(d) $\dfrac{d}{dx}\left[\sec(2x) + \tan(2x)\right]^3$,

(e) $y = 8 + 3\cos(x^4)\sec(7x)$.

Example Find the tangent line equation of $y = \cos(x)$ at $x = \dfrac{3\pi}{2}$.

Solution At $x = \dfrac{3\pi}{2}$, we have $y = 0$.

Consider $\dfrac{dy}{dx} = \dfrac{d}{dx}\cos(x) = -\sin(x)$.

Thus, the slope of the tangent line at $x = \dfrac{3\pi}{2}$ equals to

$$\left.\frac{dy}{dx}\right|_{x=\frac{3\pi}{2}} = -\sin(\frac{3\pi}{2}) = 1.$$

Therefore, the equation of the tangent line at $x = \dfrac{3\pi}{2}$ is

$$(y-0) = 1(x - \frac{3\pi}{2}) \quad \text{or}$$

$$x - y = \frac{3\pi}{2}.$$

Derivatives of Logarithmic Functions

Derivative Formulas

Let u be a function of x and differentiable.

1. $$\frac{d}{dx}\log_a u = \frac{1}{u\ln a}\cdot\frac{du}{dx}$$

2. $$\frac{d}{dx}(\ln u) = \frac{1}{u}\cdot\frac{du}{dx}$$

<u>Example</u> Find $\dfrac{dy}{dx}$ where

(a) $y = \log_3(7x^4+1)$,

(b) $y = [\ln(3 - x^2)]^4$,

(c) $y = \ln\left[\dfrac{(8x-9)^4}{\sqrt{1+x^6}}\right]$.

Derivatives of Exponential Functions

Derivative formulas

Let u be a function of x and differentiable.

1. $$\frac{d}{dx}a^u = a^u \ln a \cdot \frac{du}{dx}$$

2. $$\frac{d}{dx}e^u = e^u \cdot \frac{du}{dx}$$

<u>Example</u> Find $f'(x)$ where

(a) $f(x) = 10^{\sin(4x)}$,

(b) $f(x) = e^{5x}\sin(\ln x)$,

(c) $f(x) = e^{(x^2-3)\tan x}$.

<u>Example</u> Let $y = \sqrt{e^{8x} + 3e^{-8x}}$. Find $\dfrac{dy}{dx}$ at $x = 0$.

Derivatives of Inverse Trigonometric Functions

Derivative formulas

Let u be a function of x and differentiable.

1. $$\frac{d}{dx}\sin^{-1}u = \frac{1}{\sqrt{1-u^2}}\cdot\frac{du}{dx}$$

2. $$\frac{d}{dx}\cos^{-1}u = \frac{-1}{\sqrt{1-u^2}}\cdot\frac{du}{dx}$$

3. $$\frac{d}{dx}\tan^{-1}u = \frac{1}{1+u^2}\cdot\frac{du}{dx}$$

4. $$\frac{d}{dx}\cot^{-1}u = \frac{-1}{1+u^2}\cdot\frac{du}{dx}$$

5. $$\frac{d}{dx}\sec^{-1}u = \frac{1}{u\sqrt{u^2-1}}\cdot\frac{du}{dx}$$

6. $$\frac{d}{dx}\csc^{-1}u = \frac{-1}{u\sqrt{u^2-1}}\cdot\frac{du}{dx}$$

Proof 1. We want to show that $\dfrac{d}{dx}\sin^{-1} u = \dfrac{1}{\sqrt{1-u^2}} \cdot \dfrac{du}{dx}$.

Let $\quad y = \sin^{-1} u$.

That is, $\quad u = \sin y$

$$\frac{du}{dx} = \frac{d}{dx}\sin y$$

$$\frac{du}{dx} = \cos y \cdot \frac{dy}{dx}$$

$$\frac{du}{dx} = \cos y \cdot \frac{dy}{dx} \ .$$

Thus,

$$\frac{dy}{dx} = \frac{1}{\cos y} \cdot \frac{du}{dx}$$

$$= \frac{1}{\sqrt{1-\sin^2 y}} \cdot \frac{du}{dx} \ .$$

Therefore, $\quad \dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-u^2}} \cdot \dfrac{du}{dx}$.

* Formulas 2-6 can be proven analogously to the formula 1 above.

Example Find $\dfrac{dy}{dx}$ where

(a) $y = \sin^{-1}(1 + x^2)$,

(b) $y = [1 + \cos^{-1}(\sqrt{x})]^6$,

(c) $y = e^{\sec^{-1}(\sqrt{x})}$,

$$\frac{dy}{dx} = \frac{d}{dx} e^{\sec^{-1}(\sqrt{x})}$$

$$= e^{\sec^{-1}(\sqrt{x})} \frac{d}{dx} \sec^{-1}(\sqrt{x})$$

$$= e^{\sec^{-1}(\sqrt{x})} \cdot \frac{1}{\sqrt{x}\,\sqrt{(\sqrt{x})^2 - 1}} \cdot \frac{d}{dx}\sqrt{x}$$

$$= e^{\sec^{-1}(\sqrt{x})} \cdot \frac{1}{\sqrt{x}\,\sqrt{x-1}} \cdot \frac{dx^{\frac{1}{2}}}{dx}$$

$$= e^{\sec^{-1}(\sqrt{x})} \cdot \frac{1}{\sqrt{x}\,\sqrt{x-1}} \cdot \frac{1}{2\sqrt{x}}$$

$$= \frac{e^{\sec^{-1}(\sqrt{x})}}{2x\,\sqrt{x-1}}$$

(d) $y = \tan^{-1}\left[\dfrac{1-x}{2+x}\right]$.

Solution

Derivatives of Hyperbolic Functions

Derivative formulas

Let u be a function of x and differentiable.

1. $$\frac{d}{dx}\sinh u = \cosh u \cdot \frac{du}{dx}$$

2. $$\frac{d}{dx}\cosh u = \sinh u \cdot \frac{du}{dx}$$

3. $$\frac{d}{dx}\tanh u = \operatorname{sech}^2 u \cdot \frac{du}{dx}$$

4. $$\frac{d}{dx}\coth u = -\operatorname{csch}^2 u \cdot \frac{du}{dx}$$

5. $$\frac{d}{dx}\operatorname{sech} u = -\operatorname{sech} u \tanh u \cdot \frac{du}{dx}$$

6. $$\frac{d}{dx}\operatorname{csch} u = -\operatorname{csch} u \coth u \cdot \frac{du}{dx}$$

<u>Example</u> Find the following derivatives.

(a) $\frac{d}{dx}\cosh(9x^2-2) =$

(b) $\frac{d}{dx}\ln(\tanh(x^3)) =$

(c) $\frac{d}{dx}\left(e^{7x}\sinh^3(5x)\right) =$

(d) $\dfrac{d}{dx}\left[\dfrac{\sinh x}{\cosh x - 1}\right]$

$$= \frac{(\cosh x - 1)\dfrac{d}{dx}\sinh x - \sinh x \dfrac{d}{dx}(\cosh x - 1)}{(\cosh x - 1)^2}$$

$$= \frac{(\cosh x - 1)\cdot \cosh x - \sinh x(\sinh x)}{(\cosh x - 1)^2}$$

$$= \frac{\cosh^2 x - \cosh x - \sinh^2 x}{(\cosh x - 1)^2}$$

$$= \frac{1 - \cosh x}{(\cosh x - 1)^2}$$

$$= \frac{1}{1 - \cosh x}$$

(e) $\dfrac{d}{dx}\sinh^5(e^x + 1)$

Solution

(f) $\dfrac{d}{dx}\left(x^9 + \sin(\coth 2x)\right)$.

Solution

Let a, b be constants, and u, v functions of x.

Consider the following derivatives:

1. $\dfrac{d}{dx}(a^b) \quad = \quad 0$

2. $\dfrac{d}{dx}[u]^a \quad = \quad a\, u^{a-1} \cdot \dfrac{du}{dx}$

3. $\dfrac{d}{dx}[a^u] \quad = \quad a^u \ln a \cdot \dfrac{du}{dx}$

4. $\dfrac{d}{dx}[u]^v \quad = \quad \ldots \qquad .$

Logarithmic Derivative

It is used to find derivative of a function in a form of $[u(x)]^{v(x)}$ and when the function consists of several products or quotients of functions. The process is as follows.

1. Take natural log both sides.
2. Apply properties of logarithm.
3. Find derivatives of both sides.
4. Solve equation for $\dfrac{dy}{dx}$.

Example Find $\dfrac{dy}{dx}$ where $y = x^{\sin(3x)}$, $\quad x > 0$.

Example Find $\dfrac{dy}{dx}$ where $y = (\sin^2 x + 4)^x$.

Solution

Take $\ln$ both sides of the equation

$$\ln y = \ln(\sin^2 x + 4)^x$$

$$\ln y = x\ln(\sin^2 x + 4)$$

Take derivative with respect to x both sides

$$\frac{d}{dx}\ln y = \frac{d}{dx}x\ \ln(\sin^2 x + 4)$$

$$\frac{1}{y}\frac{dy}{dx} = x\ \frac{d}{dx}\ln(\sin^2 x + 4) + \ln(\sin^2 x + 4)\frac{dx}{dx}$$

$$\frac{1}{y}\frac{dy}{dx} = x \cdot \frac{1}{\sin^2 x + 4} \cdot \frac{d}{dx}(\sin^2 x + 4) + \ln(\sin^2 x + 4)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{x}{\sin^2 x + 4} \cdot 2\sin x\cos x + \ln(\sin^2 x + 4)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{2x\sin x\cos x}{\sin^2 x + 4} + \ln(\sin^2 x + 4)$$

Thus,

$$\frac{dy}{dx} = y\left[\frac{2x\sin x\cos x}{\sin^2 x + 4} + \ln(\sin^2 x + 4)\right]$$

$$= (\sin^2 x + 4)^x\left[\frac{2x\sin x\cos x}{\sin^2 x + 4} + \ln(\sin^2 x + 4)\right].$$

<u>Example</u> Find derivative of $y = x^{3x}\sqrt{\dfrac{(x^2+3)(6+x^4)}{3x+4}}$.

Solution

Exercise 1

1. Use definition to find $f'(x)$ where

(1.1) $f(x)=\pi$ (1.2) $f(x)=4x-3$

(1.3) $f(x)=2-x^2$ (1.4) $f(x)=(2-x)^2$

(1.5) $f(x)=x^3-9$ (1.6) $f(x)=\dfrac{1}{(5x-1)^2}$

(1.7) $f(x)=\sqrt{x}$ (1.8) $f(x)=\dfrac{1}{\sqrt{x}}$

(1.9) $f(x)=\dfrac{2x-1}{2x+1}$ (1.10) $f(x)=\dfrac{1}{\sqrt{1+2x}}$.

Answers

(1.1) 0 (1.5) $3x^2$ (1.8) $-\dfrac{1}{2x\sqrt{x}}$

(1.2) 4 (1.6) $-10(5x-1)^{-3}$ (1.9) $\dfrac{4}{(2x+1)^2}$

(1.3) $-2x$ (1.7) $\dfrac{1}{2\sqrt{x}}$ (1.10) $-\dfrac{1}{2(2+x)^{\frac{3}{2}}}$

(1.4) $2x-4$

2. Find $f'(a)$ if it exists where

(2.1) $f(x)=\left|x^2-9\right|$; $a=3$

(2.2) $f(x)=\dfrac{x-4}{|x-4|}$; $a=4$

(2.3) $f(x)=\begin{cases} x^2\ , & x\geq 0 \\ 2x\ , & x<0 \end{cases}$; $a=0$

(2.4) $f(x)=\begin{cases} 1-x^2\ , & x\leq 1 \\ 2-2x\ , & x>1 \end{cases}$; $a=1$.

Answers (2.1) doesn't exist (2.3) doesn't exist

(2.2) 0 (2.4) -2

3. A ball is being inflated. Let V be the volume of the ball in cm^3 and r be the ball's radius in cm such that $V=\dfrac{4}{3}\pi r^3$. Find

(3.1) The average rate of volume's change with respect to radius when the radius changes from 6 cm to 9 cm.

(3.2) The instantaneous rate of volume's change with respect to radius when the radius is 9 cm.

Answers (3.1) 228π (3.2) 324π

Exercise 2

1. Find the derivatives of the following functions.

(1.1) $y = 7 + 9x - 7x^3 + 4x^7$

(1.2) $y = \dfrac{1}{x} + \dfrac{3}{x^2} + \dfrac{2}{x^3}$

(1.3) $y = 2\sqrt{x} + 6\sqrt[3]{x} - 2\sqrt{x^3}$

(1.4) $y = \sqrt[3]{3x^2} - \dfrac{1}{\sqrt{5x}}$

(1.5) $y = (x^5 - 4x)^{43}$

(1.6) $y = \dfrac{3}{(a^2 - x^2)^2}$

(1.7) $y = \sqrt{x^2 + 6x + 3}$

(1.8) $y = \dfrac{3 - 2x}{3 + 2x}$

(1.9) $y = (x^2 + 4)^2 (2x^3 - 1)^3$

(1.10) $y = \dfrac{x^2}{\sqrt{4 - x^2}}$

(1.11) $y = \dfrac{x^3 - 2x\sqrt{x}}{x}$

(1.12) $y = \dfrac{1}{x^4 + x^2 + 1}$

(1.13) $y = \dfrac{x}{x + \dfrac{a}{x}}$

Answers

(1.1) $9 - 21x^2 + 28x^6$

(1.2) $-\dfrac{1}{x^2} - \dfrac{6}{x^3} - \dfrac{6}{x^4}$

(1.3) $\dfrac{1}{\sqrt{x}} + \dfrac{2}{\sqrt[3]{x^2}} - \dfrac{3}{\sqrt{x}}$

(1.4) $\dfrac{2}{\sqrt[3]{9x}} + \dfrac{1}{2x\sqrt{5x}}$

(1.5) $43(5x^4 - 4)(x^5 - 4x)^{42}$

(1.6) $\dfrac{12x}{(a^2 - x^2)^3}$

(1.7) $\dfrac{x + 3}{\sqrt{x^2 + 6x + 3}}$

(1.8) $\dfrac{-12x}{(3 + 2x)^2}$

(1.9) $2x(x^2 + 4)(2x^3 - 1)^2 (13x^3 + 36x - 2)$

(1.10) $\dfrac{8x-x^3}{(4-x^2)^{\frac{3}{2}}}$ (1.11) $2x-\dfrac{1}{\sqrt{x}}$

(1.12) $\dfrac{-(4x^3+2x)}{(x^4+x^2+1)^2}$ (1.13) $\dfrac{2ax}{(x^2+a)^2}$

2. Find the equation of a tangent line of $y=\dfrac{2x}{x+1}$ at $(1,1)$.

Answer $y=\dfrac{x+1}{2}$

3. Find the equation of a tangent line of $y=x+\sqrt{x}$ at $(1,2)$.

Answer $y=\dfrac{3}{2}x+\dfrac{1}{2}$

Exercise 3

1. Find the derivatives of the following functions.

(1.1) $f(x)=x-3\sin x$ (1.2) $y=\sin x+10\tan x$

(1.3) $g(t)=t^3\cos t$ (1.4) $h(\theta)=\theta\csc\theta-\cot\theta$

(1.5) $y=\dfrac{x}{\cos x}$ (1.6) $f(\theta)=\dfrac{\sec\theta}{1+\sec\theta}$

(1.7) $y=\sqrt{\sin x}$ (1.8) $y=\cos(a^3+x^3)$

(1.9) $y=\cot(\dfrac{x}{2})$ (1.10) $y=\sin(x\cos x)$

(1.11) $y=\tan(\cos x)$ (1.12) $y=(1+\cos^2 x)^6$

(1.13) $y = \sec^2 x + \tan^2 x$ (1.14) $y = \cot^2(\sin\theta)$

(1.15) $y = \sin(\tan\sqrt{\sin x})$

Answer

(1.1) $f'(x) = 1 - 3\cos x$ (1.2) $y' = \cos x + 10\sec^2 x$

(1.3) $g'(t) = 3t^2\cos t - t^3\sin t$

(1.4) $h'(\theta) = \csc\theta - \theta\csc\theta\cot\theta + \csc^2\theta$

(1.5) $y' = \dfrac{\cos x + x\sin x}{\cos^2 x}$ (1.6) $f(\theta) = \dfrac{\sec\theta\tan\theta}{\left(1+\sec\theta\right)^2}$

(1.7) $y' = \dfrac{\cos x}{2\sqrt{\sin x}}$ (1.8) $y' = 3x^2\sin(a^3 + x^3)$

(1.9) $y' = \dfrac{1}{2}\csc^2(\dfrac{x}{2})$

(1.10) $y' = (\cos x - x\sin x)\cos(x\cos x)$

(1.11) $y' = -\sin x\sec^2(\cos x)$

(1.12) $y' = -12\cos x\sin x(1+\cos^2 x)^5$

(1.13) $y' = 4\sec^2 x\tan x$

(1.14) $y' = -2\cos\theta\cot(\sin\theta)\csc^2(\sin\theta)$

(1.15) $y' = \cos(\tan\sqrt{\sin x})(\sec^2\sqrt{\sin x})(2\sqrt{\sin x})(\cos x)$

2. Find the equation of a tangent line of each function at a given point.

(2.1) $y = \tan x$ at $(\frac{\pi}{4}, 1)$

(2.2) $y = x + \cos x$ at $(0,1)$

(2.3) $y = x\cos x$ at $(\pi, -\pi)$

(2.4) $y = \sin(\sin x)$ at $(\pi, 0)$

(2.5) $y = \tan(\frac{\pi x^2}{4})$ at $(1,1)$

Answer

(2.1) $y = 2x + 1 - \frac{\pi}{2}$ (2.2) $y = x + 1$ (2.3) $y = -x$

(2.4) $y = -x + \pi$ (2.5) $y = \pi x - \pi + 1$

3. Find the value of x such that there exists a tangent line of $f(x) = x + 2\sin x$ to be a horizontal line.

Answer $(2n+1)\pi \pm \frac{\pi}{3}$ where n is an integer

Exercise 4

1. Find the derivatives of the following functions.

(1.1) $y=\log_a(3x^2-5)$

(1.2) $y=\ln(x+3)^2$

(1.3) $y=\ln^2(x+3)$

(1.4) $y=\ln(x^3+2)(x^2+3)$

(1.5) $y=\ln\dfrac{x^4}{(3x-4)^2}$

(1.6) $y=\ln(\sin 3x)$

(1.7) $y=\ln(x+\sqrt{1+x^2})$

(1.8) $y=x\ln x-x$

(1.9) $y=\ln(\sec x+\tan x)$

(1.10) $y=\ln(\ln\tan x)$

(1.11) $y=\dfrac{(\ln x^2)}{x^2}$

(1.12) $y=\dfrac{1}{5}x^5(\ln x-1)$

(1.13) $y=x(\sin\ln x-\cos\ln x)$

(1.14) $y=\dfrac{\ln x}{1+\ln(2x)}$

(1.15) $y=\ln(e^{-x}+xe^{-x})$

(1.16) $y=\ln(\dfrac{1}{x})+\dfrac{1}{\ln x}$

Answer

(1.1) $\dfrac{6x}{(3x^2-5)\ln a}$

(1.2) $\dfrac{2}{x+3}$

(1.3) $\dfrac{2\ln(x+3)}{x+3}$

(1.4) $\dfrac{3x^2}{x^3+2}+\dfrac{2x}{x^2+3}$

(1.5) $\dfrac{4}{x}-\dfrac{6}{3x-4}$

(1.6) $3\cot 3x$

(1.7) $\dfrac{1}{\sqrt{1+x^2}}$

(1.8) $\ln x$

(1.9) $\sec x$

(1.10) $\dfrac{2}{\sin(2x)\cdot\ln(\tan x)}$

(1.11) $\dfrac{2-4\ln x}{x^3}$ (1.12) $x^4 \ln x$

(1.13) $2\sin\ln x$ (1.14) $\dfrac{1+\ln 2}{x[1+\ln(2x)]^2}$

(1.15) $\dfrac{-x}{1+x}$ (1.16) $-\dfrac{1}{x}[1+\dfrac{1}{(\ln x)^2}]$

2. Let $f(x)=\dfrac{x}{\ln x}$. Find $f'(e)$.

Answer 0

3. Find the equation of a tangent line of $y=\ln(\ln x)$ at $(e,0)$.

Answer $x-ey=e$

Exercise 5

1. Find the derivatives of the following functions

(1.1) $f(x)=x^2e^x$ (1.2) $y=3^{ax^3}$

(1.3) $f(u)=e^{1/u}$ (1.4) $f(t)=e^{t\sin 2t}$

(1.5) $y=\sqrt{1+2e^{3x}}$ (1.6) $y=e^{e^x}$

(1.7) $y=\dfrac{ae^x+b}{ce^x+d}$ (1.8) $f(t)=\cos(e^{-t\ln t})$

(1.9) $y=\sqrt{\cos x}\cdot a^{\sqrt{\cos x}}$ (1.10) $y=7^{x^3+8}$

(1.11) $y = 7^{x^3+8}\ (x^4 - x)$ (1.12) $h(t) = (\ln t + 1)10^{\ln t}$

(1.13) $g(x) = \dfrac{\ln x}{e^{x^2} - e^x}$ (1.14) $y = \tan^2(e^{3x})$

(1.15) $f(x) = e^{\sin^3(\ln(x^2+1))}$

Answer

(1.1) $f'(x) = x(x+2)e^x$

(1.2) $y' = 3^{ax^3}\ \ln 3 \cdot (3ax^2)$

(1.3) $f(u) = (-1/u^2)e^{1/u}$

(1.4) $f'(t) = e^{t\sin 2t}\ (2t\cos 2t + \sin 2t)$

(1.5) $y' = 3e^{3x} / \sqrt{1+2e^{3x}}$

(1.6) $y' = e^{e^x}\ e^x$

(1.7) $y' = \dfrac{(ad-bc)e^x}{(ce^x + d)^2}$

(1.8) $f'(t) = \sin(e^{-t\ln t}) \cdot e^{-t\ln t}(1 + \ln t)$

(1.9) $y' = -\dfrac{1}{2}a^{\sqrt{\cos x}} \cdot \sin x\ (\ln a - \dfrac{1}{\sqrt{\cos x}})$

(1.10) $y' = 3x^2 \ln 7 \cdot 7^{x^3+8}$

(1.11) $y' = 7^{x^3+8}[(4x^3-1)+(3x^6-3x^3)\ln 7]$

(1.12) $h'(t) = \dfrac{10^{\ln t}}{t}[\ln 10(\ln t+1)+1]$

(1.13) $g'(x) = \dfrac{1}{x(e^{x^2}-e^x)} - \dfrac{\ln x(2xe^{x^2}-e^x)}{(e^{x^2}-e^x)^2}$

(1.14) $y' = 6e^{3x}\tan(e^{3x})\sec^2(e^{3x})$

(1.15) $f'(x) = \dfrac{6x}{x^2+1}\sin^2(\ln(x^2+1))\cdot\cos(\ln(x^2+1))\cdot e^{\sin^3(\ln(x^2+1))}$

Exercise 6

1. Find the derivatives of the following functions.

(1.1) $y = \arcsin(2x-3)$

(1.2) $y = \arccos(x^2)$

(1.3) $y = \arctan 3x^2$

(1.4) $y = \operatorname{arc}\cot\dfrac{1+x}{1-x}$

(1.5) $f(x) = x\csc^{-1}(\dfrac{1}{x}) + \sqrt{1+x^2}$

(1.6) $y = \dfrac{1}{ab}\arctan(\dfrac{b}{a}\tan x)$

(1.7) $y = x\ln(4+x^2) + 4\arctan\dfrac{x}{2} - 2x$

(1.8) $h(t) = \cot^{-1}(t) + \cot^{-1}(\dfrac{1}{t})$

(1.9) $y = \cos^{-1}\left[\dfrac{b + a\cos x}{a + b\cos x}\right]$

Answer

(1.1) $y' = \dfrac{1}{\sqrt{3x - x^2 - 2}}$

(1.2) $y' = -\dfrac{2x}{\sqrt{1 - x^4}}$

(1.3) $y' = \dfrac{6x}{1 + 9x^4}$

(1.4) $y' = -\dfrac{1}{1 + x^2}$

(1.5) $f'(x) = \csc^{-1}(\dfrac{1}{x})$

(1.6) $y' = \dfrac{1}{a^2\cos^2 x + b^2\sin^2 x}$

(1.7) $y' = \ln(4 + x^2)$

(1.8) $h'(t) = 0$

(1.9) $y' = \dfrac{\sqrt{a^2 - b^2}}{a + b\cos x}$

2. Show that

$$\frac{d}{dx}\left[\frac{1}{2}\tan^{-1} x + \frac{1}{4}\ln\frac{(x+1)^2}{x^2+1}\right] = \frac{1}{(1+x)(1+x^2)}\ .$$

Exercise 7

1. Evaluate $\dfrac{dy}{dx}$ of the following functions.

(1.1) $y = \sinh 3x$

(1.2) $y = \tanh(1+x^2)$

(1.3) $y = \coth(\frac{1}{x})$

(1.4) $y = x\,\mathrm{sech}\,x^2$

(1.5) $y = \mathrm{csch}^2(x^2+1)$

(1.6) $y = \ln(\tanh(2x))$

(1.7) $y = \sinh(\tan^{-1} e^{3x})$

Answer

(1.1) $3\cosh 3x$

(1.2) $2x\,\mathrm{sech}^2(1+x^2)$

(1.3) $\dfrac{1}{x^2}\mathrm{csch}^2(\frac{1}{x})$

(1.4) $-2x^2\,\mathrm{sech}\,x^2\tanh x^2 + \mathrm{sech}\,x^2$

(1.5) $-4x\,\mathrm{csch}^2(x^2+1)\coth(x^2+1)$

(1.6) $4\,\mathrm{csch}\,4x$

(1.7) $\dfrac{3e^{3x}\cosh(\tan^{-1} e^{3x})}{1+e^{6x}}$

Exercise 8

1. Use logarithmic derivative to find $\dfrac{dy}{dx}$ where

(1.1) $y=(x^2+2)(1-x^3)^4$

(1.2) $y=\dfrac{x(1-x^2)^2}{\sqrt{1+x^2}}$

(1.3) $y=\dfrac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}$

(1.4) $y=(2x+1)^5(x^4-3)^6$

(1.5) $y=\dfrac{\sin^2 x\tan^4 x}{(x^2+1)^2}$

(1.6) $y=x^2e^{2x}\cos 3x$

(1.7) $y=x^x$

(1.8) $y=x^{\ln x}$

(1.9) $y=x^{\sin x}$.

Answer

(1.1) $y'=6x(x^2+2)^2(1-x^3)^3(1-4x-3x^3)$

(1.2) $y'=\dfrac{(1-5x^2-4x^4)(1-x^2)}{(1+x^2)^{3/2}}$

(1.3) $y'=\dfrac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}\left[\dfrac{3}{4x}+\dfrac{x}{x^2+1}-\dfrac{15}{3x+2}\right]$

(1.4) $y'=(2x+1)^5(x^4-3)^6\left(\dfrac{10}{2x+1}+\dfrac{24x^3}{x^2+1}\right)$

(1.5) $y' = \dfrac{\sin^2 x \tan^4 x}{(x^2+1)^2}\left(2\cot x + \dfrac{4\sec^2 x}{\tan x} - \dfrac{4x}{x^2+1}\right)$

(1.6) $y' = x^2 e^{2x} \cos 3x[2/x + 2 - 3\tan 3x]$

(1.7) $y' = x^x(1+\ln x)$

(1.8) $y' = 2x^{\ln x - 1}\ln x$

(1.9) $y' = x^{\sin x}[(\sin x)/x + \ln x \cos x]$

Implicit Differentiation

Definition Implicit function

Implicit function of one independent variable function is a function written in a form of $F(x, y) = 0$. For example, $x^2 + 3xy^2 + 2y - 5 = 0$.

Implicit Differentiation

Since we start with $F(x, y) = 0$, we can find $\frac{dy}{dx}$ by taking derivatives with respect to x both sides separately. Then we solve the equation for $\frac{dy}{dx}$. This method is called "implicit differentiation."

Example 1 Find $\frac{dy}{dx}$ where y is a function of x and is implicitly defined by $y^2 + xy - 6x = 0$.

Example 2 Given that $a > 0$, find $\frac{dy}{dx}$ where y is a function of x and is implicitly defined by

$$x = a\sin^{-1}\left(\frac{y}{a}\right) - \sqrt{a^2 - y^2}\,.$$

Exercise 9

Find $f'(x)$ of each function defined by

1. $2x^4 - 3x^2y^2 + y^4 = 0$

2. $(x+y)^2 - (x-y)^2 = x^4 + y^4$

3. $x^2 + xy + y^2 - 3 = 0$

4. $x\sec 5x = 4y - y\tan 8x$

5. $\sqrt{x} + \sqrt{\sqrt{x} + \cos y} = 1$

Answer

1. $\dfrac{3xy^2 - 4x^3}{2y^3 - 3x^2y}$ **2.** $\dfrac{x^3 - y}{x - y^3}$ **3.** $\dfrac{2x+y}{x^2 + xy - x}$

4. $\dfrac{8y\sec^2 8x + \sec 5x + 5x\sec 5x\tan 5x}{4 - \tan 8x}$

5. $\dfrac{\sqrt{\sqrt{x} + \cos y}}{\sin y\sqrt{x}} + \dfrac{1}{2\sin y\sqrt{x}}$

Derivatives of Higher Order

Definition Derivatives of Higher Order

Derivatives of higher order refer to finding derivatives several times. So, we call them second order and third order derivatives depending on the number of times taking derivatives.

For example, let $y = f(x)$. By taking derivative one time, we obtain $\dfrac{dy}{dx} = f'(x)$ and it is called the *first order derivative of* f .

If we find the derivative of $\dfrac{dy}{dx} = f'(x)$ one more time, the derivative of $\dfrac{dy}{dx} = f'(x)$ is called the *second (order) derivative of* f and is denoted by $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d}{dx} f'(x) = f''(x)$.

Similarly, if we find the derivative of $\dfrac{d^2y}{dx^2} = f''(x)$ one more time, we will obtain the *third (order) derivative of* f and is denoted by

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^3}\right) = \frac{d}{dx} f''(x) = f'''(x).$$

Continue to do these steps, we may define the n^{th} *(order) derivative* for any positive integers n as follows :

$$\frac{d^ny}{dx^n} = \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) = \frac{d}{dx} f^{(n-1)}(x) = f^{(n)}(x).$$

<u>Remark</u>: The notation $f^{(n)}(x)$ is used when $n \geq 4$.

Example 3 Find $f^{(n)}(x)$ where $y = f(x) = \dfrac{1}{(x-1)}$.

Example 4 Find $f^{(n)}(x)$ where $y = \ln(1-x)$.

Exercise 10

1. Find $\dfrac{d^5y}{dx^5}$ **of** $y = 3^x$.

2. Find $f'''(x)$ **of** $f(x) = e^{3x+1}$.

3. Find $f^{(n)}(x)$ **of** $f(x) = (ax+b)^n$.

Answer

1. $3^x \ln^5 3$ **2.** $27e^{3x+1}$ **3.** $a^n n!$

<u>Indeterminate forms</u>

Suppose $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist such that $\lim_{x\to a} g(x) \neq 0$. We have that $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ exist.

However, if $\lim_{x\to a} g(x) = 0$, we are not able to say anything about $\lim_{x\to a} \frac{f(x)}{g(x)}$. There are two cases as follow:

1. If $\lim_{x\to a} f(x)$ equals some nonzero number and $\lim_{x\to a} g(x) = 0$, then we can conclude that $\lim_{x\to a} \frac{f(x)}{g(x)}$ does not exist (DNE).

2. If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, then we say that $\lim_{x\to a} \frac{f(x)}{g(x)}$ has the indeterminate form, namely $\frac{0}{0}$.

Besides the indeterminate form $\frac{0}{0}$, the indeterminate forms also include the forms $\frac{\pm\infty}{\pm\infty}$, $0\cdot\pm\infty$, $\pm\infty\pm\infty$, 0^0, $\pm\infty^0$, and $1^{\pm\infty}$.

To find the limits of these indeterminate forms, a French mathematician named *L'Hopital* made the following rule:

For any real number a and two functions $f(x)$ and $g(x)$ which are differentiable on the interval $0<|x-a|<\delta$ for some $\delta>0$, if $\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=0$, then $\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$.

If $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ still has the indeterminate form $\frac{0}{0}$, then we may apply *L'Hopital rule* until $\lim_{x\to a} f^{(n)}(x)$ and $\lim_{x\to a} g^{(n)}(x)$ are not zero simultaneously.

Example 1: Evaluate $\lim_{x\to\frac{\pi}{2}}\frac{\sqrt{4+\cos\theta}-2}{\theta-\frac{\pi}{2}}$.

Example 2: Evaluate $\lim\limits_{t\to 0}\dfrac{\sqrt{1+t}-\left(1+\dfrac{t}{2}\right)}{t^2}$.

Example 3: Evaluate $\lim\limits_{x\to 0}\dfrac{e^x(1-e^x)}{(1+x)\ln(1-x)}$.

In the case of $\frac{\pm\infty}{\pm\infty}$, we can do one of the following:

(1) Eliminate the terms of ∞ by dividing every term by the highest term as we do for polynomial functions in the chapter 2.

(2) Apply the *L'Hopital's* rule by rewriting the functions:

Suppose $\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{x\to a}f(x)}{\lim_{x\to a}g(x)}\quad\left(=\frac{\infty}{\infty}\right)$. Then, we rewrite as $\frac{\lim_{x\to a}f(x)}{\lim_{x\to a}g(x)}=\frac{\frac{1}{\lim_{x\to a}g(x)}}{\frac{1}{\lim_{x\to a}f(x)}}\quad\left(=\frac{0}{0}\right)$ before applying the *L'Hopital'*s rule.

Theorem1: Suppose $\lim_{x\to a}f(x)=\infty$ and $\lim_{x\to a}g(x)=\infty$, and both functions $f(x)$, $g(x)$ are differentiable. Then, we also have $\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$.

If $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ still has the indeterminate form $\frac{\infty}{\infty}$, then we can apply *L'Hopital rule* until $\lim_{x\to a}f^{(n)}(x)$ and $\lim_{x\to a}g^{(n)}(x)$ do not approach infinity simultaneously.

Example 4: Evaluate $\lim_{x\to\infty} \frac{5x + 2\ln x}{x + 3\ln x}$.

Example 5: Evaluate $\lim_{x\to 0} \frac{\cot x}{\cot 2x}$.

Example 6: Evaluate $\lim_{x\to 0^+} \dfrac{e^{-3/x}}{x^2}$.

For those indeterminate forms $0\cdot\pm\infty, \pm\infty\pm\infty$, we have to convert them to either $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$ before applying L'Hoptial's rule.

Example 7: Evaluate $\lim_{x \to 0^+} x^3 \ln x$.

Example 8: Evaluate $\lim_{x \to 0^+} \left(\csc x - \dfrac{1}{x} \right)$.

Example 9: Evaluate $\lim_{x \to \infty} \left(x - \sqrt{x^2 + x} \right)$.

For the rest of indeterminate forms: $0^0, \pm\infty^0, 1^{\pm\infty}$, we take the natural log (ln) to the function so that it has the form $0 \cdot \pm\infty$. Then, we continue just like what we do in the last section.

<u>**Note:**</u> 1^∞ is not always equal to 1. For example, $\lim_{x \to 0} 1^{1/x} = 1$, but $\lim_{x \to 0} (1 + x)^{1/x} = e$.

Example 10: Evaluate $\lim\limits_{x\to 0}\left(\sec^3 2x\right)^{\cot^2 3x}$.

Example 11: Evaluate $\lim\limits_{x\to 0^+}\left(1+\dfrac{5}{x}\right)^{2x}$.