Applications of the Definite Integral

If $y = f(x)$ is a continuous function on $a \le x \le b$ and $F(x)$ is an antiderivative of $f(x)$ and may be denoted by

$$\int f(x)dx = F(x) + C \text{ , where } C \text{ is some constant.} \quad (1)$$

The definite integral of $f(x)$ on the interval (a,b) is

$$\int_a^b f(x)dx = F(b) - F(a) \quad (2)$$

The definite integrals have a lot of applications in geometry and physics such as area under a curve, area between curves, volume, arc length, surface area, moment and work.

1. Area Under a Curve

If $y = f(x)$ is a non-negative and continuous function on $a \le x \le b$ as shown in figure 1

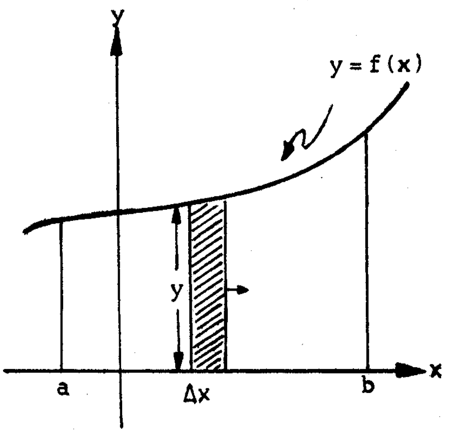


Figure 1

Area under the curve of $y = f(x)$ from $x = a$ to $x = b$ as shown here is

$$A = \int_a^b y\,dx \text{ .}$$

If we want to find the area covered by the curves of $x = g(y)$ where $g(y) \geq 0$, $y-$axis, $y = c$ and $y = d$ as shown in Figure 2, we partition the area into n small parts, all parts' widths are denoted by $\Delta y_1, \Delta y_2, \ldots, \Delta y_n$ and each length is $x = g(y)$.

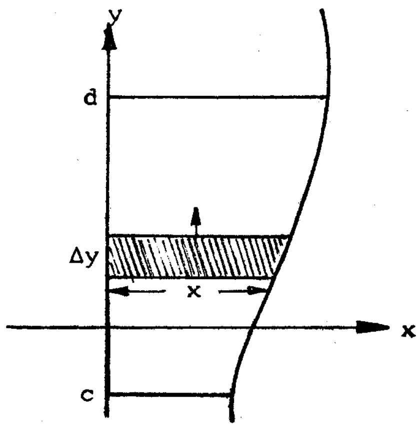


Figure 2

Consider the i^{th} partition.

Area $\Delta A_i \approx x \cdot \Delta y_i$ = width × length.

Then, $A \approx \sum_{i=1}^{n} \Delta A_i = \sum_{i=1}^{n} (x \cdot \Delta y_i)$.

If $n \to \infty$ (or $\Delta y_i \to 0$), we have

$$A = \lim_{\Delta y_i \to 0} \sum_{i=1}^{n} (x \cdot \Delta y_i) = \int_c^d x\,dy.$$

Summary Area under a curve

1. Area covered by the curves of $y = f(x)$ where $f(x) \geq 0$, x-axis, $x = a$, and $x = b$ is

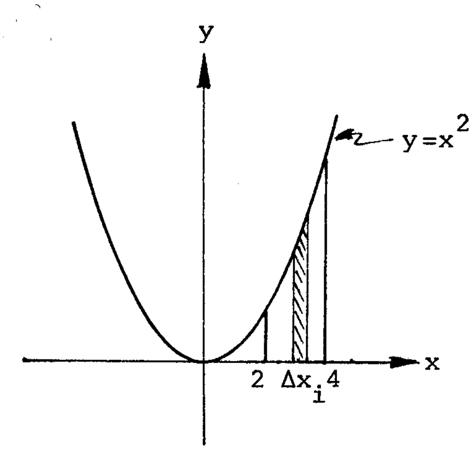
$$A = \int_a^b y\,dx \,. \tag{3}$$

2. Area covered by the curves of $x = g(y)$ where $g(y) \geq 0$, $y-$axis, $y = c$ and $y = d$ is

$$A = \int_c^d x\,dy. \tag{4}$$

Example 1 Compute the area covered by $y=x^2$, the $x-$axis, $x=2$ and $x=4$.

Solution



Partition along the $x-$axis

$$\Delta A_i = y\cdot\Delta x_i$$

$$A=\int_2^4 y\,dx$$

$$=\int_2^4 x^2\,dx = \left[\frac{x^3}{3}\right]_2^4$$

$$=\frac{4^3}{3}-\frac{2^3}{3} = \frac{64}{3}-\frac{8}{3} = \frac{56}{3}$$

$$= 18\frac{2}{3} \quad \text{unit}^3.$$

Example 2 Find the area covered by $y=x^3$, $x=-1$, $x=2$ and the $x-$axis.

Solution

Remark The area is a non-negative value, but the definite integral may be negative. So, we may write the area as $A = \left| \int_a^b f(x)\, dx \right|$.

If we integrate along the x-axis, the definite integral is positive when the graph is above x-axis and negative when the graph is below x-axis. For example, as in example 2,

$$\int_{-1}^{2} y\, dx = \int_{-1}^{2} x^3\, dx = \left[\frac{x^4}{4} \right]_{x=-1}^{x=2} = \left[\frac{2^4}{4} - \frac{(-1)^4}{4} \right] = 4 - \frac{1}{4} = 3\frac{3}{4}.$$

It is the area under the curve above x-axis from 0 to 2 minus the area below the x-axis from -1 to 0.

To find the total area of under the curve of $y = f(x),\ a \le x \le b$ as shown here

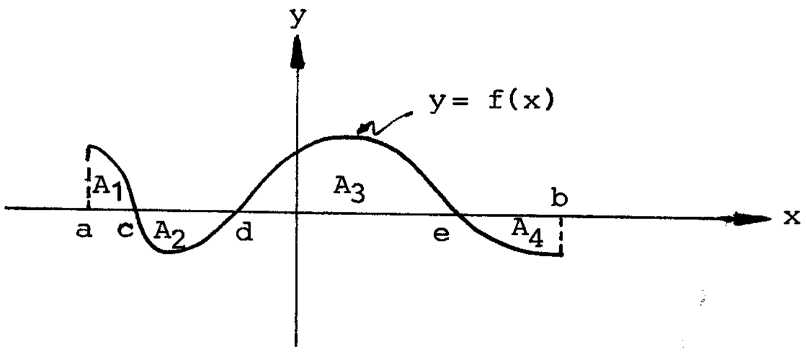


Figure 5

Total area

$$A = |A_1| + |A_2| + |A_3| + |A_4|$$

$$= \left| \int_a^c f(x)dx \right| + \left| \int_c^d f(x)dx \right| + \left| \int_d^e f(x)dx \right| + \left| \int_e^b f(x)dx \right|.$$

1. Analogously, if we integrate along the y-axis, the definite integral is positive when the graph is on the right and negative when the graph is on the left of the y-axis.
2. If a graph is symmetric, we can integrate just one part and multiply by number of symmetries as shown in example 3.

Example 3 Compute the area covered by $|x|+|y|=a$.

Solution By definition of absolute value

$$|x| = \begin{cases} x & x\geq 0 \\ -x & x<0 \end{cases},$$ we have

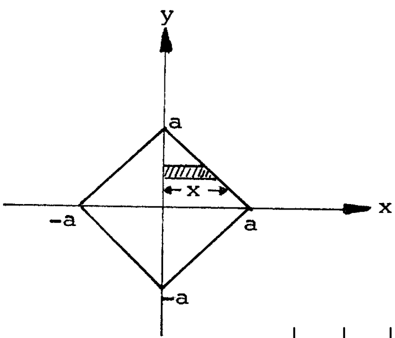


Figure 6

$$|x|+|y|=a \rightarrow \begin{cases} x+y=a \text{ when } x\geq 0 \text{ and } y\geq 0 \\ x-y=a \text{ when } x\geq 0 \text{ and } y<0 \\ -x+y=a \text{ when } x<0 \text{ and } y\geq 0 \\ -x-y=a \text{ when } x<0 \text{ and } y<0\,. \end{cases}$$

As we can see, this graph is symmetric about the origin. So we can just find the area in the first Quadrant, called it A_1. The total area is then four times A_1.

Consider A_1 If partition along the y-axis,

$$\Delta A_1 = x\cdot\Delta y \qquad \text{where } x=a-y.$$

$$A_1 = \int_0^a (a-y)\,dy = \left[ay-\frac{y^2}{2}\right]_{y=0}^{y=a}$$

$$= a^2-\frac{a^2}{2} = \frac{a^2}{2}\,.$$

Finally, we obtain

$$A = 4A_1 = \frac{4a^2}{2} = 2a^2 \text{ unit}^3.$$

2. Area Between Curves

2.1 Rectangular Form

If $y_1 = f(x)$ and $y_2 = g(x)$ are continuous functions such that $y_2 \geq y_1$ for $a \leq x \leq b$, we may compute the areas between these two curves y_1, y_2 from $x = a$ and $x = b$ as shown below.

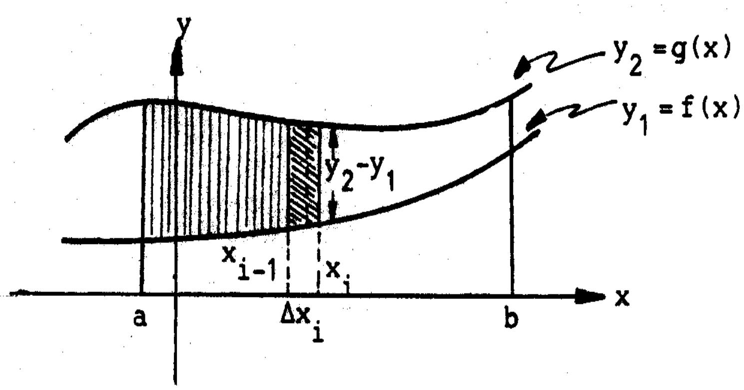


Figure 7

Partition the area into small n parts with widths $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$.

Let ΔA_i = the area of the i^{th} partition.

Then $\Delta A_i \approx (y_2 - y_1) \cdot \Delta x_i =$ width × length

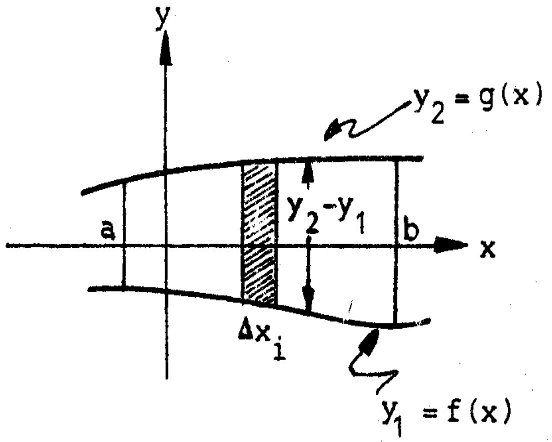
Thus, the total area

$$A \approx \sum_{i=1}^{n} \Delta A_i \approx \sum_{i=1}^{n} (y_2 - y_1) \cdot \Delta x_i .$$

If $\Delta x_i \to 0$, the length $(y_2 - y_1)$ of the interval (x_{i-1}, x_i) will approach $(y_2 - y_1)$ at x_{i-1} and x_i. Thus, the approximation is closer and closer to the exact area. Therefore,

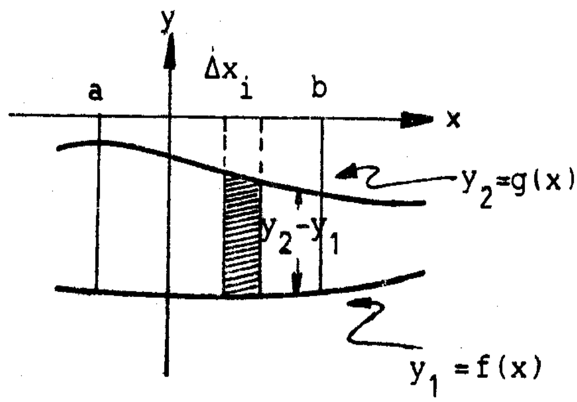
$$A = \lim_{\substack{\Delta x_i \to 0 \\ n \to \infty}} \sum_{i=1}^{n} \Delta A_i = \lim_{\substack{\Delta x_i \to 0 \\ n \to \infty}} \sum_{i=1}^{n} \left(y_2 - y_1\right)\Delta x_i = \int_a^b \left(y_2 - y_1\right) dx .$$

This formula is always valid if $y_2 > y_1$. The above or below x-axis locations do not matter. Here are some examples.



$\Delta A_i = (y_2 - y_1) \cdot \Delta x_i$

Figure 8



$\Delta A_i = (y_2 - y_1) \cdot \Delta x_i$

Figure 9

<u>**Note**</u> If $y_2 \ge y_1$, y_2 is always above y_1.

Summary If $y_1 = f(x)$ and $y_2 = g(x)$ are continuous functions such that $y_2 \geq y_1$ for $a \leq x \leq b$, then the area covered by the curves y_1 and y_2 from $x = a$ to $x = b$ is

$$A = \int_a^b \left(y_2 - y_1\right) dx = \int_a^b \left(g(x) - f(x)\right) dx \,. \tag{5}$$

Analogously if $g_1(y)$ and $g_2(y)$ are continuous function such that $g_2(y) \geq g_1(y)$ for $c \leq y \leq d$, we may compute the area covered by $x_1 = g_1(y)$, $x_2 = g_2(y)$ from $y = c$ to $y = d$. For $x_2 > x_1$ as shown in three figures below, we partition along the y-axis to n parts with widths $\Delta y_1, \Delta y_2, \ldots, \Delta y_n$.

Area of the i^{th} partition is

$$\Delta A_i \approx (x_2 - x_1) \cdot \Delta y_i = \text{length} \times \text{width} \,.$$

Thus, the total area

$$A = \lim_{\substack{\Delta y_i \to 0 \\ (n \to \infty)}} \sum_{i=1}^{n} \Delta A_i = \lim_{\substack{\Delta y_i \to 0 \\ (n \to \infty)}} \sum_{i=1}^{n} \left(x_2 - x_1\right) \Delta y_i = \int_c^d \left(x_2 - x_1\right) dy \,.$$

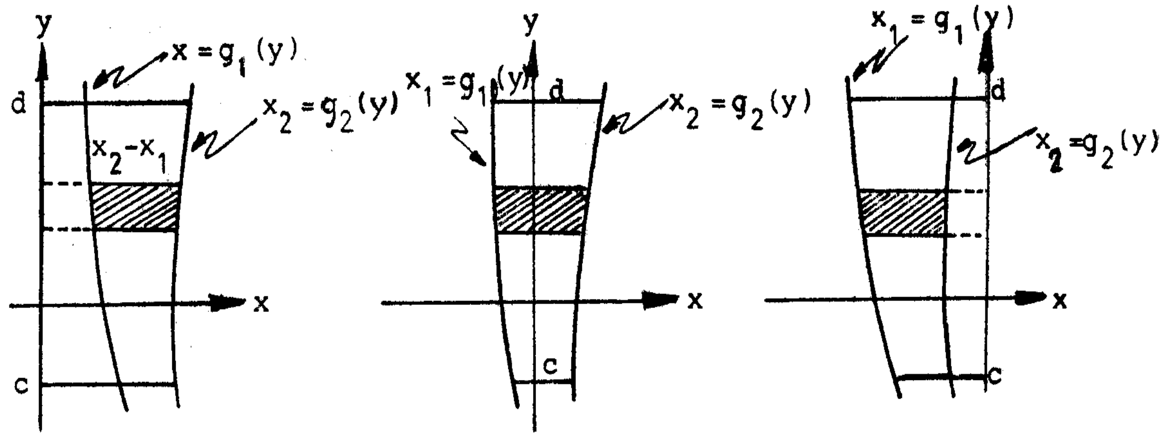


Figure 10 **Figure 11** **Figure 12**

Summary If $x_1 = g_1(y)$ and $x_2 = g_2(y)$ are continuous functions such that $x_2 \ge x_1$ for $c \le y \le d$, then the area covered by x_1, x_2, $y = c$ and $y = d$ is $$A = \int_c^d \left(x_2 - x_1\right) dy = \int_c^d \left(g_2(y) - g_1(y)\right) dy . \qquad (6)$$

Example 4 Compute the area covered by $x^2 = y,\ x^2 = 4y$ and the line $x = 2$.

Approach 1 Partition along the x-axis.

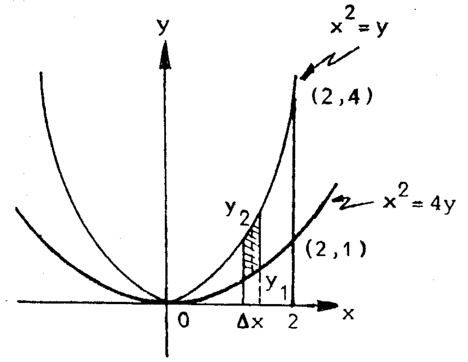


Figure 13

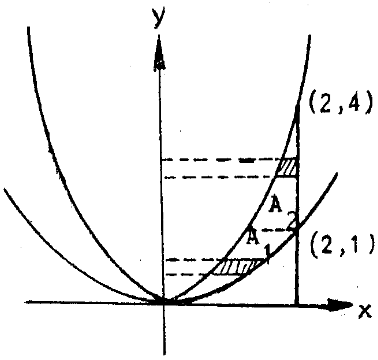
Then $\Delta A = (y_2 - y_1) \cdot \Delta x$

where $y_2 = x^2$ and $y_1 = \dfrac{x^2}{4}$.

$$A = \int_0^2 \left(y_2 - y_1\right) dx = \int_0^2 \left(x^2 - \frac{x^2}{4}\right) dx$$

$$= \frac{3}{4} \int_0^2 x^2\, dx = \frac{3}{4}\left[\frac{x^3}{3}\right]_{x=0}^{x=2} = 2.$$

Approach 2 Partition along the y-axis: there are 2 parts.



A_1: y from $0 \to 1$, we have

$$\Delta A_1 = (x_2 - x_1)\Delta y$$

where $x_2 = \sqrt{4y}$ and $x_1 = \sqrt{y}$.

(Do not forget that x_2 is on the right of x_1)

$$A_1 = \int_0^1 (x_2 - x_1)\,dy = \int_0^1 \left(\sqrt{4y} - \sqrt{y}\right) dy$$

$$= \int_0^1 \sqrt{y}\ dy = \left[\frac{2}{3} y^{3/2}\right]_{y=0}^{y=1} = \frac{2}{3}.$$

A_2: y from $1 \to 4$, we have

$$\Delta A_1 = (x_2 - x_1)\Delta y \quad \text{where} \quad x_2 = 2 \text{ and } x_1 = \sqrt{y}\ .$$

$$A_2 = \int_1^4 (x_2 - x_1)\,dy = \int_1^4 \left(2 - \sqrt{y}\right) dy$$

$$= \left[2y - \frac{2}{3} y^{3/2}\right]_{y=1}^{y=4} = \frac{4}{3}.$$

Therefore,

$$A = A_1 + A_2 = \frac{2}{3} + \frac{4}{3} = 2.$$

No matter which approach you choose, the correct answer is always the same.

Example5 Compute the area covered by $y^2 = 2x$ and $x - y = 4$.

Solution

Example 6 Compute the area covered by $y=-x^2-2x+3$, its tangent line at (2,-5) and the y-axis.

Solution Consider $y=-x^2-2x+3=-\left(x^2+2x+1\right)+4$.

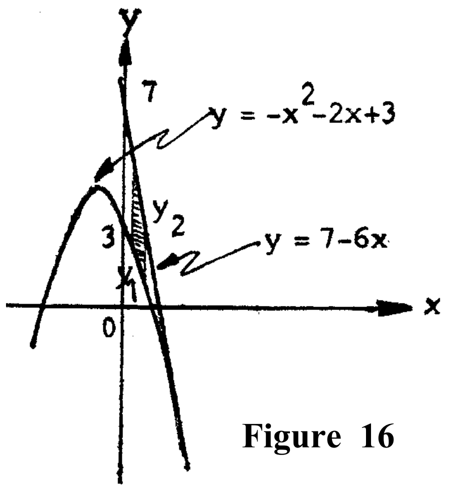


Figure 16

$y-4=-\left(x+1\right)^2$ is a parabolic curve having a vertex at (-1, 4). This curve has the y-intercept as (0, 3) and has the x-intercept at $x=-3$, and $x=1$.

Consider $\frac{dy}{dx}=-2x-2$.

The slope of the tangent line at (2, -5) is $-2(2)-2=-6$.

The equation of this tangent line can be found by $y-y_1=m\left(x-x_1\right)$.

Here, we have $y_1=-5,\ x_1=2,\ m=-6$.

So, $y-\left(-5\right)=-6\left(x-2\right)$. That is, we have the equation of the tangent line of this parabola at (2, -5) is $y=7-6x$.

If partition on x-axis,

$$\Delta A=\left(y_2-y_1\right)\Delta x$$

where $y_2=7-6x$ and $y_1=-x^2-2x+3$.

Then,
$$A=\int_0^2\left[\left(7-6x\right)-\left(-x^2-2x+3\right)\right]dx$$
$$=\int_0^2\left[4-4x+x^2\right]dx=\frac{8}{3}.$$

Exercise 1

Compute each area covered by the following graphs

1. $x-$axis, $y=2x-x^2$
2. $y-$axis, $x=y^2-y^3$
3. $y^2=x,\ x=4$
4. $y=2x-x^2,\ y=-3$
5. $y=x^2,\ y=x$
6. $x=3y-y^2,\ x+y=3$
7. $y=x^4-2x^2,\ y=2x^2$
8. First part of $y=\sin x$
9. $y-$axis, $y^2-4x-4=0$
10. Ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$
11. $x=y^2,\ x=y$
12. $y^2=8x,\ x^2=4y$
13. $x^2-5x+y=0,\ y=x$
14. $y^2=9x,\ y^2=x^3$
15. $y=x^2,\ y=x,\ y=2x$
16. $y^2=4x,\ 2x-y-4=0$
17. $y=x^3-4x,\ x-$axis
18. $x+2y=2,\ y-x=1,\ 2x+y=7$

19. $x^2y = x^2-4$, $x-$axis, $x=2$ and $x = 4$

20. $y=6x+x^2-x^3$, $x-$axis

21. $f(x)=\begin{cases} x^2 & , x\le 2 \\ -x+6, & x>2 \end{cases}$ from $x=0$ and $x=3$

22. $y=x(x-3)(x+3)$, $y=-5x$

23. $y=x^2$, $y=8-x^2$ and $y = 4x+12$

24. $x=0$, $x=2$, $y=2^x$ and $y=2x-x^2$

25. $x=-2y^2$, $x=1-3y^2$

26. $y=x+1$, $y=\cos x$ and the $x-$axis (largest region)

27. One loop of $y^2=(x-1)(x-2)^2$

28. $y=x^2-2x+2$, its tangent line at the point $M(3, 5)$, the $y-$axis

29. $\sqrt{x}+\sqrt{y}=1$ and $x+y=1$

30. $y=x^2$, $y=4$ This area is divided into 2 equal parts by the line $y=c$. Evaluate the value of c.

31. $x^2=4y$, $y=\dfrac{8}{x^2+4}$

32. One loop of $y^2=(x-1)^2$

33. $y^2=4x$, $x^2=4y$ and $x^2+y^2=5$ where $x\ge 0$, $y\ge 0$

34. Hypocycloid: $x^{2/3}+y^{2/3}=a^{2/3}$

35. $y^3=x^2$ the cord connecting (-1, 1) and (8, 4)

36. $y^2 = x^2(1-x^2)$

37. $xy=4$, $y=x$, $x=5$ and $x=\sqrt{-y}$

Answer 1

1. $\frac{4}{3}$	2. $\frac{1}{12}$	3. $\frac{32}{3}$
4. $\frac{32}{3}$	5. $\frac{1}{6}$	6. $\frac{4}{3}$
7. $\frac{128}{15}$	8. 2	9. $\frac{8}{3}$
10. πab	11. $\frac{1}{6}$	12. $\frac{16}{3}$
13. $\frac{32}{3}$	14. $\frac{24\sqrt{3}}{5}$	15. $\frac{7}{6}$
16. 9	17. 8	18. 6
19. 1 (Figure 17)	20. $\frac{253}{12}$ (Figure 18)	

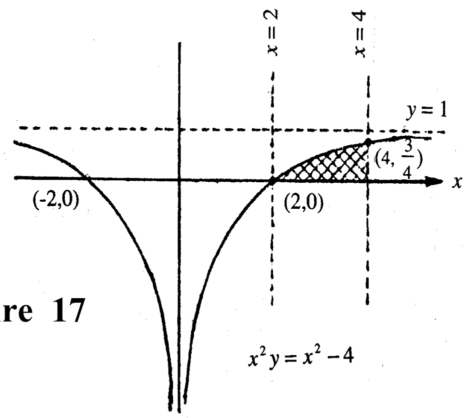


Figure 17

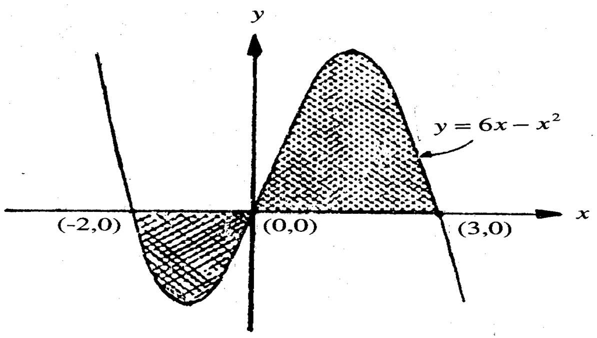
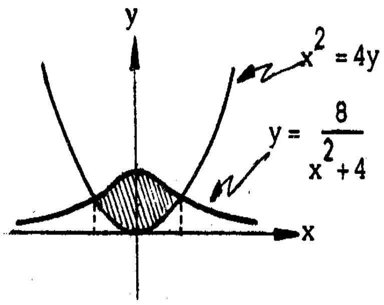


Figure 18

21. $\frac{37}{6}$	22. 8	23. 64
24. $\frac{3}{\ln 2}-\frac{4}{3}$	25. $\frac{4}{3}$	26. $\frac{3}{2}$
27. $\frac{8}{15}$	28. 9	29. $\frac{1}{3}$
30. $\frac{32}{3}$, $c=\sqrt[3]{16}$	31. $2\pi-\frac{4}{3}$ (Figure 19)	

32. $\dfrac{8}{15}$ (Figure 20)

Figure 19



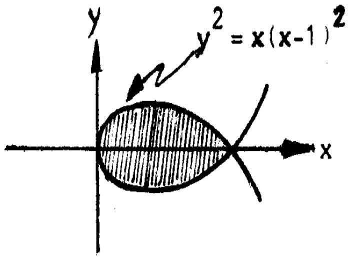


Figure 20

33. $\dfrac{2}{3}+\dfrac{5}{2}\sin^{-1}\dfrac{3}{5}$

34. $\dfrac{3}{8}\pi a^2$

35. 2.7

36. $\dfrac{4}{3}$

37. $4(\ln 5-\ln 2)+\dfrac{131}{3}$