MTH101

Mathematics I (International)

Module 1

Functions

Definition 1

A **function** is a rule that takes certain numbers as inputs and assigns to exactly one output number. The set of all input numbers is called the **domain** of the function and the set of resulting output numbers is called the **range** of the function.

Note: A function can be considered as a set of ordered pairs (x, y).

Notations:

Let f be a function from A to B ($f : A \to B$)

- D_f represents domain of function f
- R_f represents range of function f
- Image of x is y since $f(x) = y$

$f : A \to B$ is called a function from A **onto** B if $R_f = B$

Normally, we may present a function via four common ways:

1) Description (words)
2) Numeric (tables)
3) Visual (graphs)
4) Algebra (formulas)

Example 1

Consider a set $\{(-3,1),(0,2),(3,-1),(5,4)\}$. Is it a function?

Domain:

Range:

Example 2

Let $f = \{(x, y) : x, y \in \mathbb{R} \text{ and } y = x^2 - 2\}$.

So $D_f = \mathbb{R}$ and $R_f = [-2, +\infty)$. We usually write $f(x) = x^2 - 2$.

The values of f at some points are as follow.

$$f(0) = (0)^2 - 2 = -2$$

$$f(-1) = (-1)^2 - 2 = -1$$

$$f(\sqrt{3}) = (\sqrt{3})^2 - 2 = 1$$

$$f(c) = c^2 - 2$$

$$f(x+h) = (x+h)^2 - 2 = x^2 + 2hx + h^2 - 2$$

$$f(x+h) - f(x) = (x^2 + 2hx + h^2 - 2) - (x^2 - 2) = 2hx + h^2$$

and $\dfrac{f(x+h) - f(x)}{h} = 2x + h, \quad h \neq 0$

Example 3

Let $f = \left\{(x, y) : x^2 + y^2 = 1^2\right\}$. Is f a function?

Example 4 Find the domain of the following functions.

(1) $f(x)=\dfrac{4}{x-1}$

(2) $f(x)=\dfrac{x}{x^2-9}$

(3) $f(x)=\dfrac{\sqrt{4-x}}{x}$

(4) $f(x)=\sqrt{4-x^2}$

Example 5

1) $y=\sin x$ has the set of all real numbers as its domain and the interval $[-1,1]$ as its range.

2) $y=\sqrt{x^2+4}$ has the set of all real numbers as a domain and the interval $[2,+\infty)$ as its range.

Example 6

$$h(x)=\begin{cases}\dfrac{2x^2-9x+4}{x-4} & ,x\neq 4\\ 5 & ,x=4\end{cases} \quad \text{or} \quad h(x)=\begin{cases}2x-1 & ,x\neq 4\\ 5 & ,x=4\end{cases}$$

$D_f =$

$R_f =$

Definition 2 The function f equals to the function g if and only if

1. $D_f = D_g$
2. $f(x)=g(x)$ for all $x\in D_f$.

Example 7 Check if the following functions are equal.

1) Let $f(x)=\dfrac{\sqrt{2+x}-\sqrt{2}}{x}$ and $g(x)=\dfrac{1}{\sqrt{2+x}+\sqrt{2}}$

2) Let $f(x)=x+3$ and $g(x)=\begin{cases} \dfrac{2x^2+7x+3}{2x+1} & , x\neq -\dfrac{1}{2} \\ \dfrac{5}{2} & , x=-\dfrac{1}{2} \end{cases}$

Definition 3

Let f and g be functions and $R_g \cap D_f \neq \varnothing$.

A composite function of f and g (denoted by $f \circ g$) is a function $(f \circ g)(x) = f(g(x))$ whose domain is $\{x : x \in D_g \text{ and } g(x) \in D_f\}$.

Example 8 Let $f(x)=\sqrt{x-3}$ and $g(x)=2x-1$

a) Let $F = f \circ g$ Find $F(x)$ and domain of F

b) Let $G = g \circ f$ Find $G(x)$ and domain of G

c) Let $H = f \circ f$ Find $H(x)$ and domain of H

Solutions

a) The domain of g is $(-\infty, \infty)$ and domain of f is $[3, \infty)$.

To find the domain of $F = f \circ g$, we consider only x where $g(x)$ is in domain of f. That is, $2x - 1 \geq 3$.

Thus domain of F is a set of x where $x \geq 2$ i.e. $[2, \infty)$.

Then, the function $F = f \circ g$ can be found by

$$F(x) = f \circ g(x) = f(g(x)) = f(2x-1) = \sqrt{(2x-1)-3} = \sqrt{2x-4}.$$

Symmetry

Definition 4 Let f be a function.

a. If $f(-x) = -f(x)$, f is called an **odd function** whose graph is symmetric about the origin.

b. If $f(-x) = f(x)$, f is called an **even function** whose graph is symmetric about the y-axis.

Example 9

a) Let $f(x) = x^3$.

Consider $f(-x) = (-x)^3 = -x^3 = -f(x)$.

Thus f is an odd function and it graph is shown in figure 1 below.

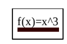


Figure 1

b) Let $f(x) = 3x^2 - 1$.

Consider $f(-x) = 3(-x)^2 - 1 = 3x^2 - 1 = f(x)$.

Thus f is an even function whose graph shown in Figure 2.

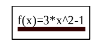


Figure 2

Inverse function

Definition 5 The function f is called a **one-to-one** function if and only if for all x, y, z if (x, y) and $(z, y) \in f$ then $x = z$.

Definition 6 Let f be a one-to-one function from A onto B.

An inverse function of f is defined by $f^{-1} = \{(b,a) \mid (a,b) \in f\}$

which is also a one-to-one function from B to A.

Remark Graphs of f and f^{-1} are symmetric about the line $y = x$ as shown in Figure 3 below.

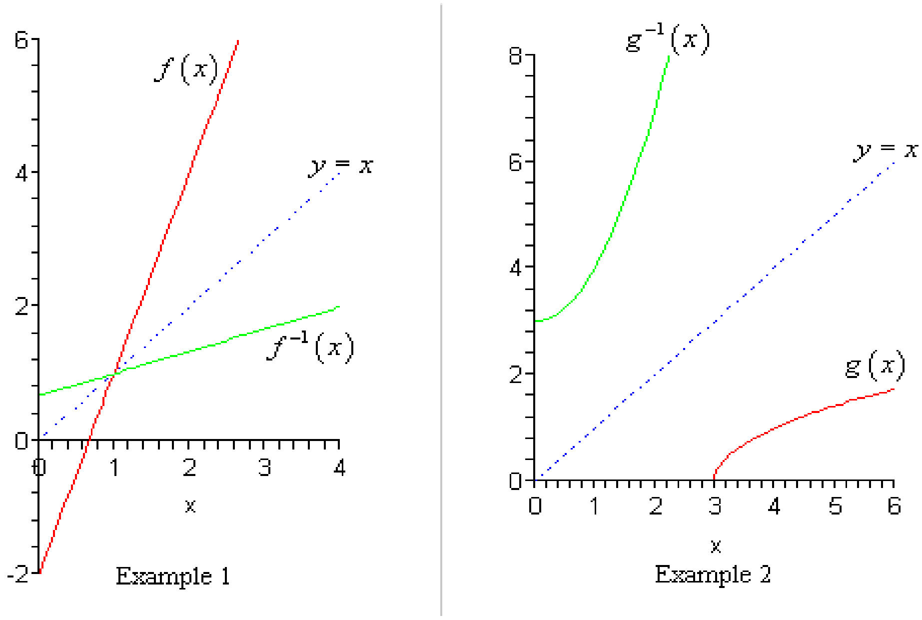


Figure 3

Example10 Find an inverse of f where $f(x) = x^3 - 1$.

Solution From $y = f(x) = x^3 - 1$ (i.e. $x = \sqrt[3]{y+1}$), we have that $f^{-1} = \left\{(y,x) \middle| \; y = x^3 - 1\right\}$ or $f^{-1} = \left\{(x,y) \middle| \; y = \sqrt[3]{x+1}\right\}$

We normally write $f^{-1}(x) = \sqrt[3]{x+1}$ so that we can easily draw graphs of both functions f and f^{-1} as follows

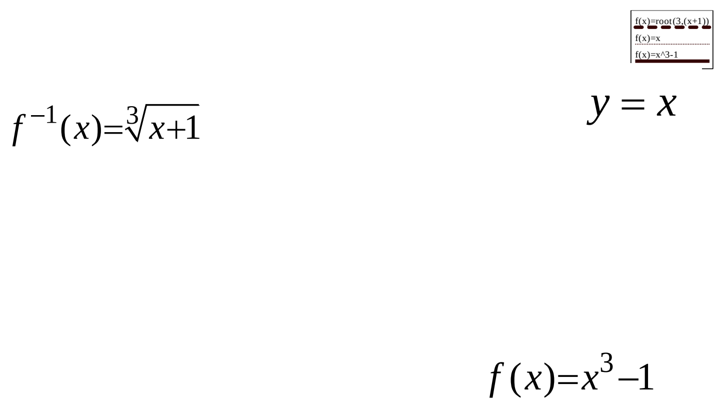


Figure 4

Other Interesting Functions

All functions here will be useful in the next sections.

Algebraic Function

a. Polynomial Functions are functions of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

where a_i is a real number for each $i = 0, 1, 2, \ldots, n$ and

n is a non-negative integer.

If n is the largest number such that $a_n \neq 0$, we call f a polynomial function of degree n such as $f(x) = 3x^3 - 5x^2 + x + 4$ is a polynomial function of degree 3.

Normally, if there is nothing specific, the domain of a polynomial function is the set of all real numbers.

b. Rational Functions are functions formed by a ratio between two polynomial functions.

$$f(x) = \frac{a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0}{b_m x^m + b_{m-1} x^{m-1} + \ldots + b_0},$$

Note that, if there is nothing specific, the domain of this rational function is $\left\{ x \in \mathbb{R} \;\middle|\; b_m x^m + b_{m-1} x^{m-1} + \ldots + b_0 \neq 0 \right\}$

Example 11 Let $y = f(x) = \dfrac{x^2 + x}{x}$

Rewrite function f : $f(x) = x + 1$ where $x \neq 0$

Thus graph of $f(x)$ is the graph of $y = x + 1$, but undefined at $x = 0$

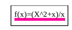


Figure 5

c. Functions of the form $\sqrt[n]{f(x)}$; $n \in \mathbb{N}$ where the function $f(x)$ is either a polynomial or a rational function.

The domain of this type of functions can be considered as follows

<u>Case1</u> n is odd

The domain of $\sqrt[n]{f(x)}$ is exactly the domain D_f of $f(x)$

<u>Case2</u> n is even

The domain of $\sqrt[n]{f(x)}$ is $D_f \cap \{x \mid f(x) \geq 0\}$

d. Functions formed by summation, multiplication and division of functions in part a. to c.

Below are some examples of functions in part c. and d.

1) $f(x) = x^{\frac{2}{3}}$ 2) $f(x) = \sqrt[4]{\dfrac{x}{x+1}}$

3) $f(x) = \dfrac{\sqrt{x}}{\sqrt{x}+1}$

<u>**Transcendental Functions**</u>

a. Exponential Functions are functions of the form

$$y = a^x, \text{ where } a > 0 \text{ and } a \neq 1$$

When $a > 1$, its graph can be shown in Figure 6 below.

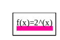


Figure 6

When $0 < a < 1$, its graph can be shown in figure 7 below

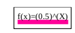


Figure 7

b. Logarithmic Function

Logarithmic function is an inverse of exponential function. Given an exponential function $y = a^x$. Then its inverse function is $x = a^y$ or we can rewrite it as $y = \log_a x$.

If $y = \log_a x$, $a > 1$, then its graph is shown in Figure 8.

If $y = \log_a x$, $0 < a < 1$, then its graph is shown in Figure 9.

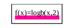


Figure 8

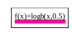


Figure 9

Some facts about logarithmic functions

1. Domain of a logarithmic function is $\{x : x > 0\}$ and its range is $\{y : y \in \mathbb{R}\}$

2. A logarithmic function is a one-to-one function.

3. $\log_a 1 = 0$

4. Graph of $y = \log_a x$ is a reflection of the graph $y = a^x$ across the line $y = x$.

Remark: When $a = e$ (where $e = 2.71818...$= natural number) $y = e^x$ has the inverse $y = \log_e x$ which is normally written as $y = \ln x$ and it is called a natural logarithm.

The properties of $y = e^x$ and $y = \ln x$ are the same as of the following properties of $y = a^x$ and $y = \log_a x$ $(a > 0)$, respectively

Properties of logarithmic and exponential functions

Given positive numbers a, b where $a \neq 1, b \neq 1$ and $x, y \in R$

1. $a^x \cdot a^y = a^{x+y}$

2. $\dfrac{a^x}{a^y} = a^{x-y}$

3. $a^x \cdot b^x = (ab)^x$ and $\dfrac{a^x}{b^x} = \left[\dfrac{a}{b}\right]^x$

4. $\left(a^x\right)^y = a^{xy}$

5. $a^{-x} = \dfrac{1}{a^x}$

6. If $x > 0, y > 0$, then $\log_a(xy) = \log_a x + \log_a y$

$$\log_a(\frac{x}{y}) = \log_a x - \log_a y$$

7. $\log_a x^r = r \log_a x$

8. $\log_a x = \dfrac{\log_b x}{\log_b a}$

9. $\log_a a = 1$

10. $\ln e^x = x$ and $e^{\ln x} = x \quad , x > 0$

11. $a^x = y$ and $x = \log_a y \quad , y > 0$

Example 12 Find the values of x

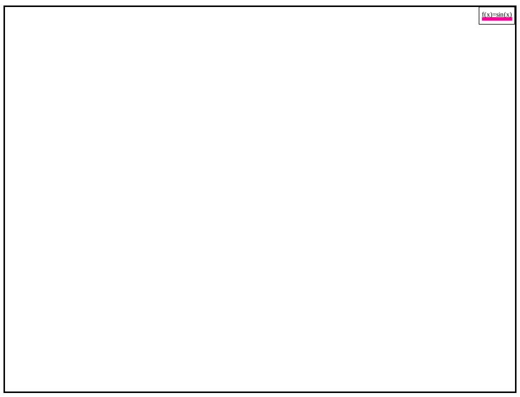
(a) $4 \cdot 3^x = 8 \cdot 6^x$

(b) $7^{x+2} = e^{17x}$

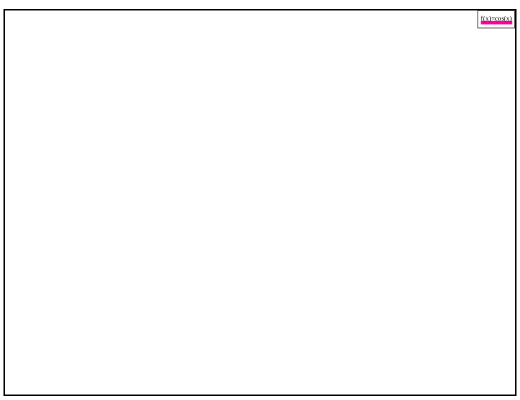
c. Trigonometric Function

$y = \sin x$ $\qquad$ $y = \cos x$ $\qquad$ $y = \tan x = \dfrac{\sin x}{\cos x}$

$y = \csc x = \dfrac{1}{\sin x}$ $\qquad$ $y = \sec x = \dfrac{1}{\cos x}$ $\qquad$ $y = \cot x = \dfrac{\cos x}{sinx}$

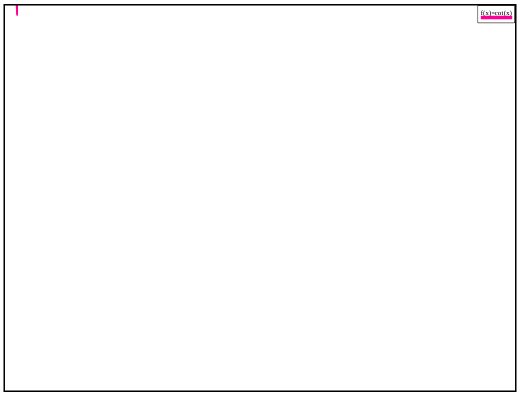


Graph of $y = \sin x$

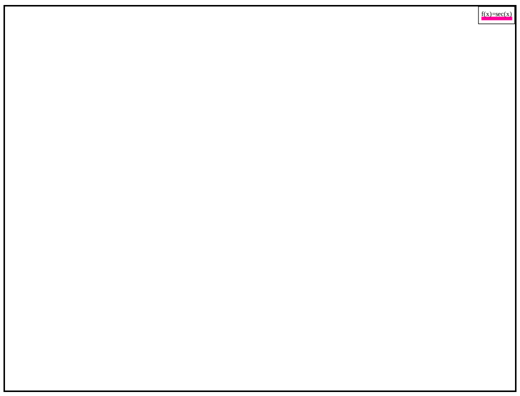


Graph of $y = \cos x$

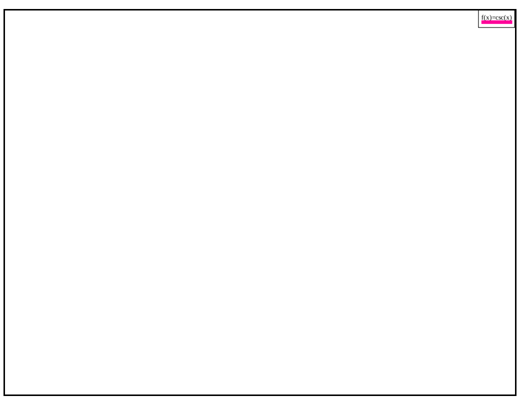
Graph of $y = \tan x$



Graph of $y = \cot x$



Graph of $y = \sec x$



Graph of $y = \csc x$

Normally, the inverse of a trigonometric function is not a function since each trigonometric function is not one-to-one. However, if we restrict the domain, we can make a one-to-one trigonometric function and define an inverse function as follows.

1) Restrict the domain of $y = \sin x$ to $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

Its inverse function is $y = \arcsin x$.

f(x)=sin(x) f(x)=asin(x)

$y = \sin x$ $y = \arcsin x$

2) Restrict domain of $y = \cos x$ to $[0, \pi]$

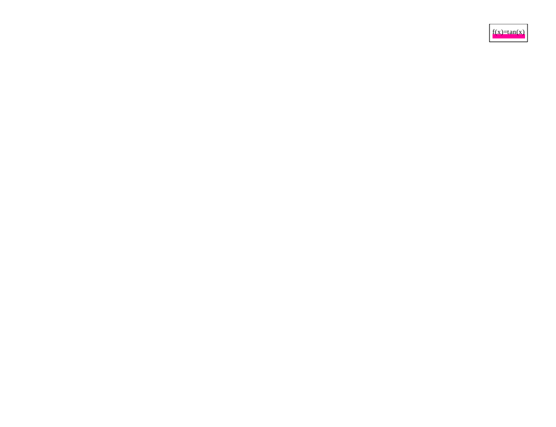
Its inverse function is $y = \arccos x$.

f(x)=cos(x) f(x)=acos(x)

$y = \cos x$ $y = \arccos x$

3) Restrict domain of $y = \tan x$ to $\left(\frac{-\pi}{2}, \frac{\pi}{2} \right)$

Its inverse function is $y = \arctan x$.



$y = \tan x$ $\qquad\qquad$ $y = \arctan x$

4) Restrict domain of $y = \cot x$ to $(0, \pi)$

Its inverse function is $y = \text{arccot}\, x$.

$y = \cot x$ $\qquad\qquad$ $y = \text{arccot}\, x$

5) Restrict domain of $y = \sec x$ to $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$

Its inverse function is $y = \operatorname{arcsec} x$.

$y = \sec x$ $\qquad\qquad$ $y = \operatorname{arcsec} x$

6) Restrict domain of $y = \csc x$ to $[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$

Its inverse function is $y = \operatorname{arccsc} x$.

$y = \csc x$ $\qquad\qquad$ $y = \operatorname{arccsc} x$

Exercises on Functions

1. Determine if the following are functions. Locate domain and range.

(a) $\{(1,3),(2,3),(3,4),(4,5)\}$

(b) $\{(x,y): y > 4x-1\}$

(c) $y = x^4 - 1$

(d) Let

x	y
15	2
2	13
13	13
5	3

2. Determine if each following function is either even or odd or neither.

(a) $f(x) = x^3 + 2x$

(b) $g(x) = \dfrac{8}{x^2 - 2}$

(c) $h(x) = 3x|x|$

(d) $k(x) = x + |x|$

3. What is the difference of $\sin x^2$, $\sin^2 x$ and $\sin(\sin x)$? Show in terms of composite functions.

Answers to Function Exercises

1. (a) yes $D = \{1, 2, 3, 4\}$ and $R = \{3, 4, 5\}$

   (b) no $D = R =$ all real numbers

   (c) yes $D = \mathbb{R}$ and $R = \{y : y \geq -1\}$

   (d) yes $D = \{2, 5, 13, 15\}$ and $R = \{2, 3, 13\}$

2. (a) odd (b) even

   (c) odd (d) neither

3. Let $f(x) = \sin x$ and $g(x) = x^2$

$\sin x^2 = f(g(x))$, $\sin^2 x = g(f(x))$, while

$\sin(\sin x) = f(f(x))$.

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Limit and Continuity of Function

2.1 Limit of function

Let f be a function. The limit of $f(x)$ when x approaches to a is not the value of $f\left(a\right)$ but it is a value that $f(x)$ is approaching to (as x approaches to a). There are two types of the limit.

2.1.1 Limit of function as $x \to a$ (a is a real number.)

Suppose that $f(x)=5x-1$ and $g(x)=\ x\ $ defined by the largest integer which is less than or equal to x. For example,

$$g(4)=\ 4\ =4,\ g(3.8)=\ 3.8\ =3,\ g(-1.2)=\ -1.2\ =-2.$$

For some values of x which approaches to $a=1$, the value $f(x)$ and $g(x)$ are shown in Table 1.

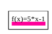
x	0.5	0.9	0.99	0.999	...	1.001	1.01	1.1
$f(x)$	1.5	3.5	3.95	3.995	...	4.005	4.05	4.5
$g(x)$	0	0	0	0	...	1	1	1

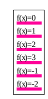
Table 1

We can see that when x approaches to $a=1$, $f(x)$ gets closer and closer to the value 4. However, $g(x)=1$ when $x \geq 1$ and $g(x)=0$ when $x<1$. Thus $g(x)$ does not approach to one number.

Therefore, we say that $f(x)$ has the limit equal to 4 as x approaches to 1 and $g(x)$ does not have a limit when x approaches to 1. We may write them as

$$\lim_{x \to 1} f(x) = 4 \quad \text{and} \quad \lim_{x \to 1} g(x) \text{ does not exist.}$$





$f(x) = 5x - 1$

$g(x) = \ x$

The graph of the function f shows that the value of $f(x)$ gets closer to 4 when x approaches to 1. But the graph of the function g jumps from $y=0$ to $y=1$ at $x=1$. Thus $g(x) = \ x$ has no limit at $x=1$.

Using this concept, one can define the limit as follows:

Definition If $f(x)$ gets closer to L when x approaches to a, we say that L is the limit of $f(x)$ when x approaches to a, denoted by $\lim\limits_{x \to a} f(x) = L$.

The values of x approaches to a from two sides:

- x **approaches to** a **from the right side** is denoted by

$x \to a^+$. In this case, we focus on x when $x > a$.

- x **approaches to** a **from the left side** is denoted by

$x \to a^-$. In this case, we focus on x when $x < a$.

From the above example, we have $\lim\limits_{x \to 1^+} \lfloor x \rfloor = 1$ but $\lim\limits_{x \to 1^-} \lfloor x \rfloor = 0$ and $\lim\limits_{x \to 1^+} 5x - 1 = \lim\limits_{x \to 1^-} 5x - 1 = 4$.

We see that the function f has the same limit from both sides when x approaches to 1 and

(Right limit) $\lim\limits_{x \to a^+} f(x) =$ (Left limit) $\lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a} f(x)$.

The following theorem guarantees the above remark.

Theorem 1 $\lim_{x \to a} f(x)$ exists and equals to L if

(1) both $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ exist and

(2) $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$

Example 1 Compare $\lim_{x \to 0} \frac{x}{|x|}$ and $\lim_{x \to 0} \frac{x^2}{|x|}$.

Solution

Properties of limits

Let a, k, L and M be real numbers. Suppose that $\lim\limits_{x\to a} f(x) = L$ and $\lim\limits_{x\to a} g(x) = M$. Then,

1. $\lim\limits_{x\to a} kf(x) = kL$,
2. $\lim\limits_{x\to a} [f(x) \pm g(x)] = L \pm M$,
3. $\lim\limits_{x\to a} f(x)g(x) = LM$,
4. $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$, $M \neq 0$,
5. If f is a polynomial function, then for any number a

   $\lim\limits_{x\to a} f(x) = f(a)$,
6. $\lim\limits_{x\to a} \sqrt[n]{g(x)} = \sqrt[n]{\lim\limits_{x\to a} g(x)}$ where n is a natural number.

Example 2 Evaluate $\lim\limits_{x\to 0} \dfrac{x^3 - 3x^2 + 4}{\cos x}$.

Solution

Example 3 Let f be a function defined by $f(x) = \begin{cases} 2x^2 & , x < 0, \\ x & , 0 \le x < 1, \\ x + 1 & , x \ge 1. \end{cases}$

Find the limits of $f(x)$ when x approaches 0 and 1.

Solution

Example 4 Evaluate $\lim_{x \to 9} \left(2x^{\frac{3}{2}} - 9\sqrt{x} \right)^{\frac{1}{3}} \sin 2x.$

Solution

Sometimes, we find the limit by replacing x by a and may get the result in the form of $\frac{0}{0}$. So, we can use these two techniques to find the limit.

1) Factoring
2) Conjugating

Example 5 Calculate $\lim\limits_{x \to 3} \dfrac{x^3 - x^2 - 9x + 9}{x^2 - x - 6}$.

Solution

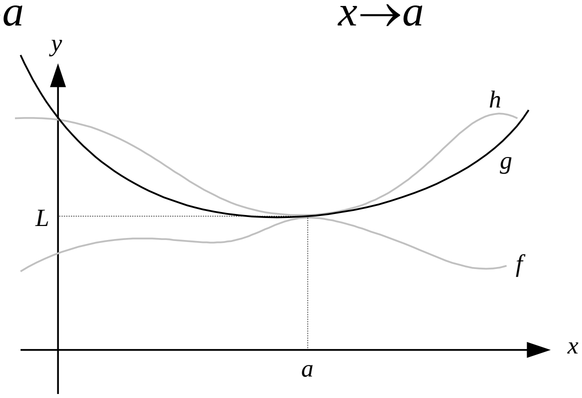
Example 6 Calculate $\lim\limits_{x \to 0^+} \dfrac{2\sqrt{x}}{\sqrt{16 + 2\sqrt{x}} - 4}$.

Solution

The following theorem is one of an important theorem that helps us to find the limit. It is typically used to confirm the limit of a function via comparison with two other functions whose limits are known or easily computed.

Squeeze Theorem

If $f(x) \le g(x) \le h(x)$ for all values of x, $x \ne a$ at some points a and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$.



Example 7 Use the Squeeze Theorem to show that

$$\lim_{x\to 0} \frac{x^2}{1+\left(1+x^4\right)^{\frac{5}{2}}} = 0.$$

Example 8

1. If $3x \le f(x) \le x^3 + 2$ for $0 \le x \le 2$, evaluate $\lim_{x \to 1} f(x)$.

2. Calculate $\lim_{x \to 0} x^2 \sin \dfrac{2}{x}$.

Solution

Theorem

1. $\lim_{x\to 0}\frac{\sin x}{x}=1$

2. $\lim_{x\to 0}\frac{\cos x-1}{x}=0$

Example 9 Use $\lim_{x\to 0}\frac{\sin x}{x} = 1$ to show that $\lim_{x\to 0}\frac{\cos x-1}{x}=0.$

Proof $\lim_{x\to 0}\frac{\cos x-1}{x}=\lim_{x\to 0}\frac{\cos x-1}{x}\left(\frac{\cos x+1}{\cos x+1}\right)$

$$=\lim_{x\to 0}\frac{\cos^2 x-1}{x(\cos x+1)} =\lim_{x\to 0}\frac{-\sin^2 x}{x(\cos x+1)}$$

$$=\lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{-\sin x}{\cos x+1}$$

$$=\lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{-\sin x}{\cos x+1}=1\cdot 0=0.$$

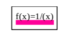
Example 10 Evaluate $\lim_{x\to 1}\frac{\sin(x-1)}{x^2+x-2}$.

Solution

2.1.2 Limit of function as $x \to \infty$ (infinity)

When the domain of a function f is unbounded, the values of $f(x)$ may get closer to one value when x increases unboundedly (written as $x \to +\infty$) or x decreases unboundedly (written as $x \to -\infty$).

Let $f(x) = \frac{1}{x}$. Its graph can be shown here.



Consider the value of $f(x)$ in the following table.

x	100	1000	10000	Increases unboundedly
$f(x) = \frac{1}{x}$	0.01	0.001	0.0001	$\ldots \to 0$
x	-100	-1000	-10000	Decreases unboundedly
$f(x) = \frac{1}{x}$	-0.01	-0.001	-0.0001	$\ldots \to 0$

Table 2

We see that, when $x \to +\infty$, the values of $f(x)$ get closer to 0 and $f(x) > 0$. So, we say that limit of $f(x)$ equals 0 as $x \to +\infty$, denoted by $\lim\limits_{x \to +\infty} \frac{1}{x} = 0$. Also, when $x \to -\infty$, the values of $f(x)$ get closer to 0 as well, but $f(x) < 0$. We say that limit of $f(x)$ equals 0 as $x \to -\infty$ and denote it by $\lim\limits_{x \to -\infty} \frac{1}{x} = 0$.

The above graph shows that $f(x) = \frac{1}{x}$ gets closer to x-axis as x increases to infinity and decreases to negative infinity, but it never hit the x-axis. We call a line that the graph gets closer to as an **asymptote** of the function.

Properties of infinite limits

Many properties of infinite limits are the same as those of limits at a finite number a.

Let k, L and M be real numbers. Suppose that $\lim_{x \to +\infty} f(x) = L$ and $\lim_{x \to +\infty} g(x) = M$. Then,

1. $\lim_{x \to +\infty} k = k,$
2. $\lim_{x \to +\infty} [f(x) \pm g(x)] = L \pm M$,
3. $\lim_{x \to +\infty} f(x)g(x) = LM$,
4. $\lim_{x \to +\infty} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}, M \neq 0,$
5. $\lim_{x \to +\infty} [f(x)]^{\frac{1}{n}} = L^{\frac{1}{n}}$ where n is positive and $L \geq 0,$
6. $\lim_{x \to +\infty} \dfrac{1}{x^n} = 0$ where n is a positive integer.

All 6 properties are the same when we replace $x \to +\infty$ by $x \to -\infty$

Example 1 Calculate

a) $\lim\limits_{x \to +\infty} \dfrac{5}{x^3}$, b) $\lim\limits_{x \to -\infty} \dfrac{-3}{x^{\frac{2}{3}}}$, c) $\lim\limits_{x \to \infty} \dfrac{4^x - 4^{-x}}{4^x + 4^{-x}}$.

Solution

Example 2 Evaluate $\lim\limits_{x \to +\infty} \dfrac{\sqrt{3x^4 + 7x^2 + 6}}{4x^2 - 3x - 6}$.

Solution

Example 3 Evaluate $\lim\limits_{x \to -\infty} \dfrac{\sqrt{x^2 + 3}}{x + 3}$.

Solution

Example 4 Calculate $\lim_{x \to 2^+} \dfrac{x-3}{x-2}$.

Solution

Example 5 Calculate $\lim_{x \to 0^+} (x-1)\ln x$.

Solution

Limit of a function associating with the number e

For any constant a,

$$\lim_{x\to 0}(1+ax)^{1/x} = e^a \quad \text{and} \quad \lim_{x\to\infty}(1+\frac{a}{x})^x = e^a.$$

Example 6 Calculate $\lim_{x\to\infty}\left(\frac{x+4}{x+1}\right)^{x+1}$.

Solution

2.2 Continuity of Function

Definition A function f is continuous at $x=a$ if all of the three following conditions are satisfied:

1. $f(a)$ exists,
2. $\lim_{x\to a} f(x)$ exists, (That is, $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$.)
3. $\lim_{x\to a} f(x) = f(a)$.

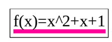
Remark: If at least one of the above conditions is not satisfied, then the given function is discontinuous at $x=a$.

Example 1 Let $f(x) = x^2 + 2x + 1$

Consider the continuity of this function at $x = 0$:

1. $f(0) = 1$ exists,
2. $\lim\limits_{x \to 0} f(x) = 1$ exists, and
3. $\lim\limits_{x \to 0} f(0) = f(0) = 1$.

Thus, $f(x)$ is continuous at $x = 0$. Its graph is here.



Example 2 Let f be a function defined by

$$f(x)=\begin{cases} \dfrac{1-x^2}{1-x} & ,x\neq 1, \\ 3 & ,x=1. \end{cases}$$

Determine if this function is continuous at $x=1$.

Solution

Example 3 Let f be a function defined by

$$f(x)=\begin{cases} bx^2+1 & ,x<-2, \\ x & ,x\geq -2. \end{cases}$$

Find b that makes this function continuous at $x=-2$.

Solution

Three Types of Discontinuities

Consider the continuity of $f(x)$ at $x = a$.

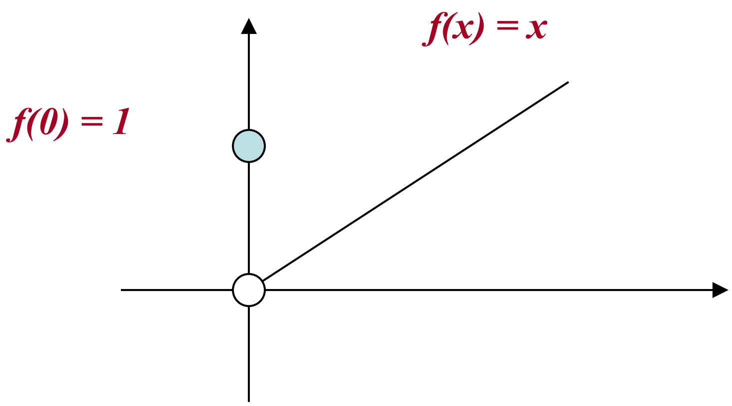
1. Removable discontinuity

It occurs when

(i) $\lim_{x \to a} f(x)$ exists, but not equal to $f(a)$ or

(ii) $f(a)$ is undefined.

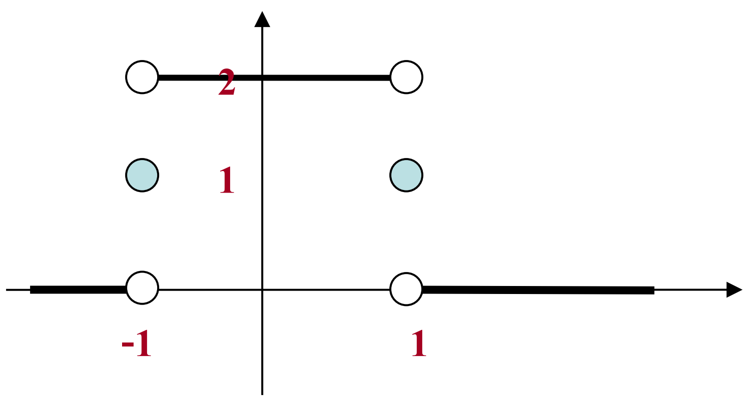
For example, $f(x) = \begin{cases} 1 & , x = 0 \\ x & , x \neq 0 \end{cases}$ has a removable discontinuity at $x = 0$ as show in the Figure below.



2. Jump discontinuity or Ordinary discontinuity

It occurs when $\lim_{x\to a} f(x)$ does not exist due to the **unequal** existence of $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$. For example, the function

$$f(x)=\begin{cases} 2 & ,|x|<1 \\ 1 & ,|x|=1 \\ 0 & ,|x|>1 \end{cases}$$ has a jump discontinuity at $x=1,-1$.



3. Infinite discontinuity

It occurs when at least one of the left limit or the right limit does not exist. For example, $f(x)=\dfrac{1}{x^2}$ has an infinite discontinuity at $x=0$ as shown here.

f(x)=1/x^2

Algebraic properties of functions on the continuity

1. If f and g are continuous at $x=a$, then $f\pm g,\ f.g,\ \dfrac{f}{g}$ ($g(a)\neq 0$) and kf (k is a constant) are also continuous at $x=a$.

2. If f is continuous at $x=b$ and $\lim\limits_{x\to a} g(x)=b$, then $\lim\limits_{x\to a}(f\circ g)(x)=f(b)$.

3. If g is continuous at $x=a$ and f is continuous at $g(a)$, then the composite function $f\circ g$ is continuous at $x=a$.

Example 4 Let f be a function defined by

$$f(x) = \frac{2(x^2 + 4x + 2)}{(x^2 - 9)(x - 1)}.$$

Locate where this function is continuous.

Definition If the function f is continuous everywhere in the interval (a,b), we say that f is continuous on (a,b).

Definition A function f is continuous in $[a,b]$ where $a<b$ if

1. $f(x)$ is continuous on (a,b),
2. $\lim\limits_{x\to a^+} f(x)=f(a)$ and
3. $\lim\limits_{x\to b^-} f(x)=f(b)$.

Example 5 Let g be a function defined by $g(x)=\sqrt{\dfrac{3-x}{4+x}}$.

Locate where this function is continuous.

Solution

Limit and Continuity Exercises

1. Find the limits of the following functions.

(a) Let $f(x) = \dfrac{x^3}{|x-1|}$. Find $\lim_{x\to1} f(x)$.

(b) $\lim_{x\to1} 3x \quad x$

(c) Let $g(x) = \begin{cases} x^2 - 2; & x > 0 \\ -2 - x; & x < 0 \end{cases}$. Calculate $\lim_{x\to0} g(x)$.

(d) $\lim_{x\to\infty} \dfrac{6\sqrt{x^2 - 3}}{2x-1}$

(e) $\lim_{x\to-\infty} \dfrac{\sqrt{x^2 + 7}}{2x-4}$

2. Make the following functions continuous at $x = a$.

(a) $f(x) = \dfrac{\sqrt{3x^2}}{2|x|}$, $a = 0$

(b) $g(x) = \dfrac{x^n - 1}{x-1}, \quad n \in \mathbb{Z}^+, \quad a = 1$

3. Locate domain that makes the following function continuous.

(a) $h(x) = \dfrac{2}{x^2 + 3x - 28}$

(b) $k(x) = \sqrt[3]{(x-a)(x-b)}$

4. Find k that makes $f(x)=\begin{cases} \dfrac{x^2-5x+6}{x-2}; & x\neq 2 \\ kx-3 \quad ; & x=2 \end{cases}$ continuous everywhere.

5. Find k that makes each following limit exists.

(a) $\lim\limits_{x\to 1}\dfrac{x^2-kx+4}{x-1}$

(b) $\lim\limits_{x\to \infty}\dfrac{x^4+3x-5}{2x^2-1+x^k}$

(c) $\lim\limits_{x\to -\infty}\dfrac{e^{2x}-5}{e^{kx}+4}$

6. Compute the following limits.

(a) $\lim\limits_{h\to 0}\dfrac{(3+h)^2-9}{h}$

(b) $\lim\limits_{h\to 0}\dfrac{1/(1+h)-1}{h}$

(c) $\lim\limits_{h\to 0}\dfrac{\sqrt{4+h}-2}{h}$

7. Compute the following limits.

(a) $\lim\limits_{x\to 0}\dfrac{\cos x-1}{\sin x}$

(b) $\lim\limits_{x\to 0}\dfrac{\sin 3x}{\sin 5x}$

(c) $\lim\limits_{x\to \infty} x\sin\dfrac{\pi}{x}$

Answers to limit and continuity exercises

1. (a) $+\infty$

   (b) Does not exist

   (c) –2

   (d) 3

   (e) –1 / 2

2. (a) add $f(0)=\dfrac{\sqrt{3}}{2}$

   (b) add $g(1)=n$

3. (a) $x \neq -7, 4$

   (b) $(-\infty,\infty)$

4. 1

5. (a) 5

   (b) greater than or equal to 4

   (c) less than or equal to 2

6. (a) 6

   (b) –1

   (c) –1/16

7. (a) 0

   (b) 3/5

   (c) π