MTH 101

Mathematics I

Module 2

The Derivative

Definition of Derivative

Let L be a line connecting points P and Q on the curve of $y = f(x)$ as shown in Figure 1 here.

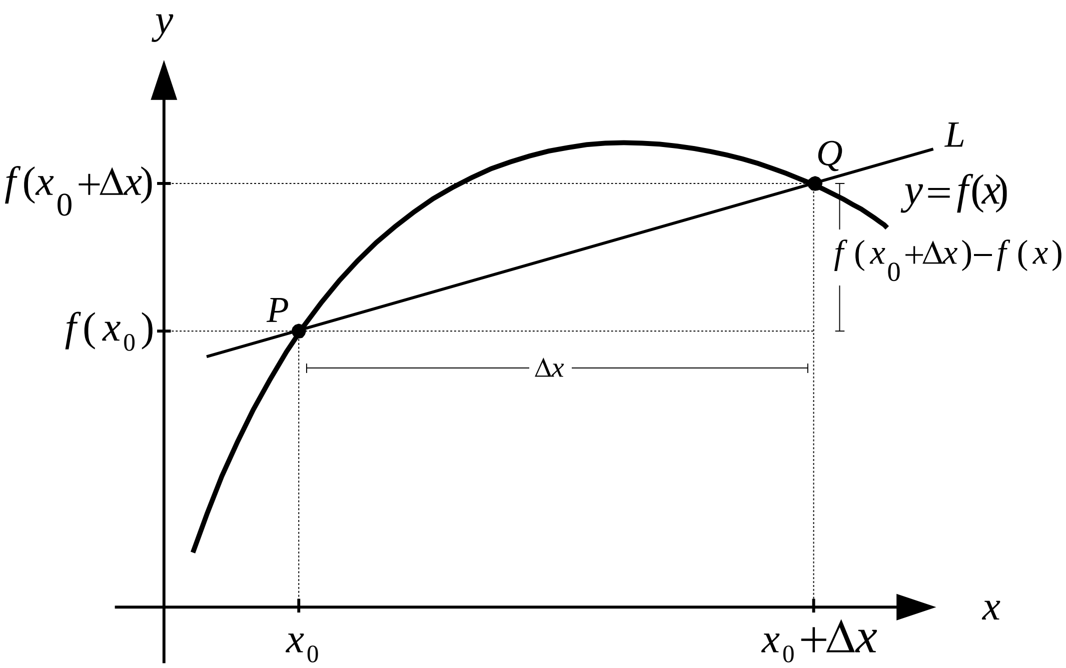


Figure 1

From Figure 1, consider the slope of line L:

$$\text{Slope of the line } L = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \frac{\text{Changed values of } f}{\text{Changed values of } x} \qquad [\text{From } x \text{ to } x+\Delta x]$$

$$= \text{Average rate of change of } f \text{ from } x \text{ to } x+\Delta x\,.$$

Next, consider the slope of line L when point Q is moved closer and closer to point P along the curve of $y = f(x)$ as in Figure 2.

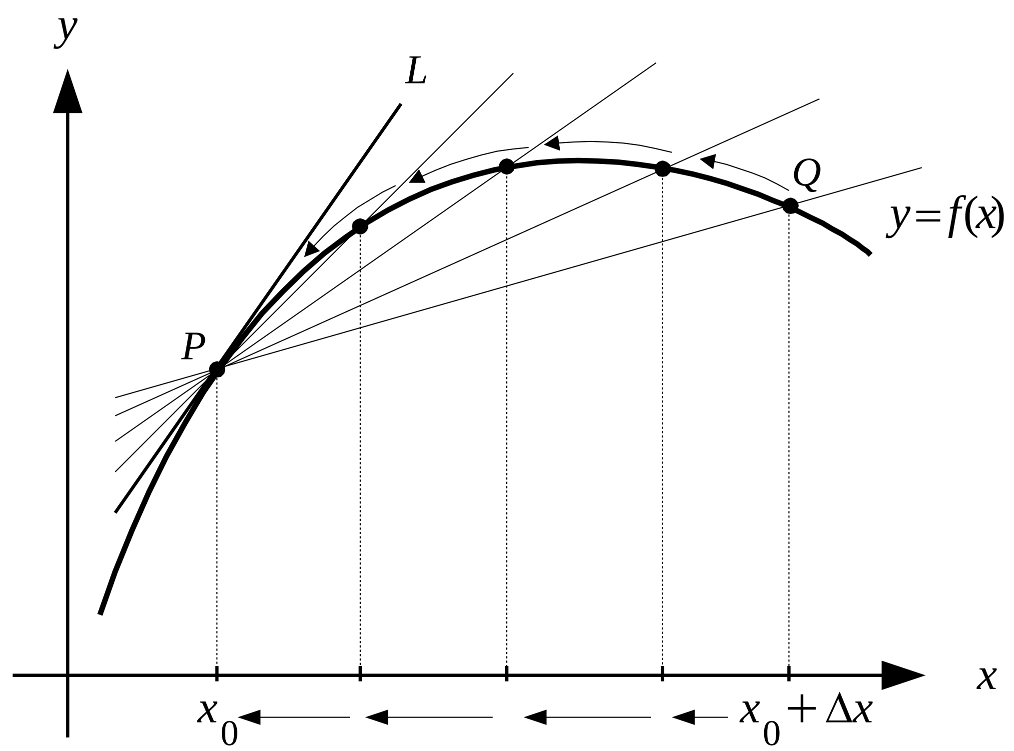


Figure 2

As Q is moved closer to P or as Δx gets smaller to 0, the slope of line L gets closer to the slope of a tangent line of the curve $y = f(x)$ at the point P.

Slope of the tangent line of $y = f(x)$ at the point P

$$= \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

= Instantaneous rate of change of f at $x = x_0$.

Definition1:

Let f be a function defined on an open interval containing x. Then the **derivative of** f **at** x**,** denoted by $f'(x)$, is defined by

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

Remark

- Beside the notation $f'(x)$, the following notations are also used to denote the derivative of f at x:

$$\frac{dy}{dx}, \ \frac{d}{dx} f(x) \ \text{ or just } y'.$$

For the derivative of f at $x = a$, we may write it as

$$f'(a) \ \text{ or } \ \left.\frac{dy}{dx}\right|_{x=a}.$$

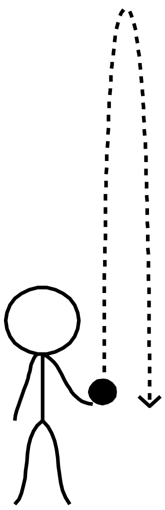
- If f is continuous at $x = a$, then

$$f'(a) = \text{The slope of tangent line of } y = f(x) \text{ at } x = a$$

$$= \lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x}$$

$$= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} .$$

Throw an object vertically. At time t seconds, the object is at the position $s(t) = -4.9t^2 + 49t$.



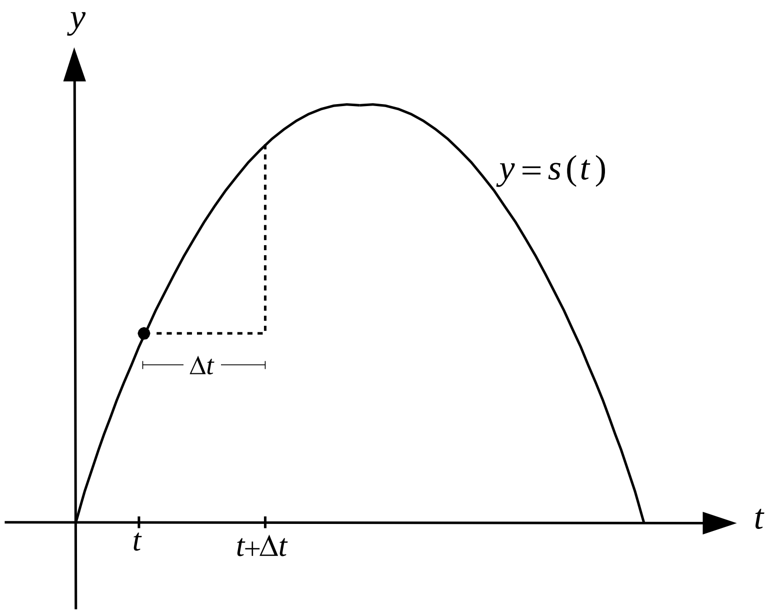


Figure 3 shows the object's position at time t

- **Average velocity of the object from time t to $t+\Delta t$** is the average rate of distance change from time t to $t+\Delta t$ seconds.

$$\text{Average velocity from time } t \textbf{ to } t+\Delta t = \frac{s(t+\Delta t)-s(t)}{\Delta t}.$$

- **Instantaneous velocity of the object at time t** is the instantaneous rate of distance change at time t seconds as $\Delta t \to 0$

$$\text{Instantaneous velocity at time } t = \lim_{\Delta t\to 0}\frac{s(t+\Delta t)-s(t)}{\Delta t} = \frac{ds}{dt}.$$

Thus, at time t seconds, the object is thrown from the ground with the velocity $v(t) = \dfrac{ds}{dt} = -9.8t+49$.

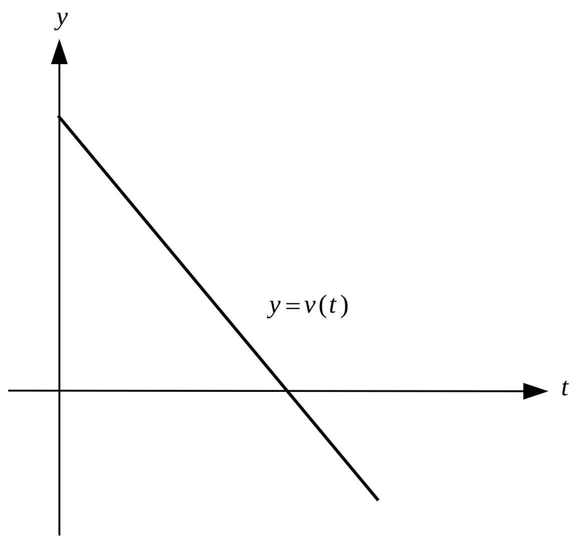


Figure 4 shows the velocity of the object at time t seconds

Example Let $f(x) = x^2 + 4$.

(a) Find derivative of f.

(b) Find derivative of f at $x = 5$.

Example Find the slope of the tangent line of $y = \frac{1}{x^2}$ at the point $(1,1)$.

Example An object moves horizontally. At time t seconds, the object has distance $s = 5 - 2t + t^2$ meters. Compute

a. the average velocity of the object from 1 to 3 seconds,

b. the instantaneous velocity of the object at t seconds,

c. the instantaneous velocity of the object at $t = 1$ second.

Theorem

If a function f has derivative (or say "is differentiable") at $x = a$ ($f'(a)$ exists as a real number), then f is continuous at $x = a$.

Proof Consider

$$\begin{aligned}
\lim_{x\to a} f(x) - f(a) &= \lim_{x\to a} f(x) - \lim_{x\to a} f(a) \\
&= \lim_{x\to a}\left[f(x) - f(a)\right] \\
&= \lim_{x\to a} \frac{(x-a)\left[f(x) - f(a)\right]}{x-a} \\
&= \lim_{x\to a}(x-a) \cdot \lim_{x\to a} \frac{f(x) - f(a)}{x-a} \\
&= 0 \cdot f'(a) \\
&= 0 .
\end{aligned}$$

That is, $\lim_{x\to a} f(x) = f(a)$.

Thus, f is continuous at $x = a$.

$\square$

Remark: The converse may not be true. That is, if f is continuous at $x=a$, then f **may or may not** be differentiable at $x=a$.

Example Find the derivative of $f(x)=|x|$ at $x=0$.

<u>Example</u> Let $f(x) = \sqrt{x}$. Compute $f'(x)$.

Derivative Formulas

Finding a derivative by using the definition is quite complicated and time consuming. However, there are several theorems and formulas to help us calculating derivatives easier and faster. Consider the following formulas.

Formulas

Let u, v be functions of x and c, n are some constants.

1. $\dfrac{dc}{dx} = 0$ and $\dfrac{dx}{dx} = 1$

2. $\dfrac{d}{dx}(cu) = c\dfrac{du}{dx}$

3. $\dfrac{d}{dx}(u \pm v) = \dfrac{du}{dx} \pm \dfrac{dv}{dx}$

4. $\dfrac{du^n}{dx} = nu^{n-1} \cdot \dfrac{du}{dx}$

$\dfrac{dx^n}{dx} = nx^{n-1} \cdot \dfrac{dx}{dx} = nx^{n-1}$

5. $\dfrac{d}{dx}(uv) = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

6. $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

<u>Example</u> Find the derivatives of the following functions .

(a) $f(x) = 2x^9 - \dfrac{5}{x} + 7x - 1$

(b) $g(x) = \sqrt[3]{x^4} + \dfrac{1}{\sqrt[3]{x^4 + 2}}$

(c) $p(x) = \left(2x^7 - x^{-1}\right)\left(5x^9 - 10\right)$

Example Find the derivatives of the following functions.

(a) $y = \dfrac{(2x-1)^2}{x^2+7}$

Solution $$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{(2x-1)^2}{x^2+7}\right]$$

$$= \frac{(x^2+7)\dfrac{d}{dx}(2x-1)^2 - (2x-1)^2\dfrac{d}{dx}(x^2+7)}{(x^2+7)^2}$$

$$= \frac{(x^2+7)\cdot 2(2x-1)\dfrac{d}{dx}(2x-1) - (2x-1)^2\cdot(2x)}{(x^2+7)^2}$$

$$= \frac{(x^2+7)\cdot 2(2x-1)\cdot 2 - (2x-1)^2\cdot(2x)}{(x^2+7)^2}$$

$$= \frac{4(x^2+7)(2x-1) - (2x-1)^2\cdot(2x)}{(x^2+7)^2}$$

(b) $f(x)=\dfrac{x^7-4\sqrt{x}-2}{x^2}$

Solution $f'(x)=\dfrac{d}{dx}\left[\dfrac{x^7-4\sqrt{x}-2}{x^2}\right]$

$$=\frac{d}{dx}\left(x^5-4x^{-\frac{3}{2}}-2x^{-2}\right)$$

$$=\frac{dx^5}{dx}-4\frac{dx^{-\frac{3}{2}}}{dx}-2\frac{dx^{-2}}{dx}$$

$$=5x^4-4\left(-\frac{3}{2}\right)x^{-\frac{5}{2}}-2(-2)x^{-3}$$

$$=5x^4+6x^{-\frac{5}{2}}+4x^{-3}$$

(c) $r(t)=\left(4t^3-7t^{-6}\right)^{12}$

Solution

$$r'(t)=\frac{d}{dt}\left(4t^3-7t^{-6}\right)^{12}$$

$$=12\left(4t^3-7t^{-6}\right)^{11}\cdot\frac{d}{dt}\left(4t^3-7t^{-6}\right)$$

$$=12\left(4t^3-7t^{-6}\right)^{11}\cdot\frac{d}{dt}\left(4t^3-7t^{-6}\right)$$

$$=12\left(4t^3-7t^{-6}\right)^{11}\left(4\frac{dt^3}{dt}-7\frac{dt^{-6}}{dt}\right)$$

$$=12\left(4t^3-7t^{-6}\right)^{11}\left(12t^2+42t^{-7}\right)$$

$$=72\left(4t^3-7t^{-6}\right)^{11}\left(2t^2+7t^{-7}\right)$$

(d) $y=\left(5x^{10}-8\right)\cdot\sqrt[4]{x^2+9}$

Solution

$$\frac{dy}{dx}=\frac{d}{dx}\left(5x^{10}-8\right)\cdot\sqrt[4]{x^2+9}$$

$$=\left(5x^{10}-8\right)\frac{d}{dx}\left(x^2+9\right)^{\frac{1}{4}}+\left(x^2+9\right)^{\frac{1}{4}}\frac{d}{dx}\left(5x^{10}-8\right)$$

$$=\left(5x^{10}-8\right)\left(\frac{1}{4}\left(x^2+9\right)^{-\frac{3}{4}}\right)\frac{d}{dx}\left(x^2+9\right)+\left(x^2+9\right)^{\frac{1}{4}}\left(50x^9\right)$$

$$=\left(5x^{10}-8\right)\left(\frac{1}{4}\left(x^2+9\right)^{-\frac{3}{4}}\right)\left(2x\right)+\left(x^2+9\right)^{\frac{1}{4}}\left(50x^9\right)$$

$$=\frac{1}{2}x\left(5x^{10}-8\right)\left(x^2+9\right)^{-\frac{3}{4}}+50x^9\left(x^2+9\right)^{\frac{1}{4}}$$

(e) $y=\dfrac{4x^2+5}{3x-1}$

Solution

$$\frac{dy}{dx}=\frac{d}{dx}\left[\frac{4x^2+5}{3x-1}\right]$$

$$=\frac{(3x-1)\frac{d}{dx}\left(4x^2+5\right)-(4x^2+5)\frac{d}{dx}\left(3x-1\right)}{(3x-1)^2}$$

$$=\frac{(3x-1)\left(8x\right)-(4x^2+5)\left(3\right)}{(3x-1)^2}$$

$$=\frac{12x^2-8x-15}{(3x-1)^2}$$

(f) $g(x)=\sqrt{\dfrac{8x^4}{2-x^7}}$

Solution

<u>Example</u> Find an equation of the tangent line of $y = \dfrac{1}{\sqrt{x^4 + 8x}}$

at the point $x = 1$.

The Chain Rule

If functions f and g are differentiable and $F = f \circ g$ is a composite function given by $F(x) = f(g(x))$, then F is also differentiable at x and

$$\boxed{F'(x) = f'(g(x)) \cdot g'(x)}$$

In other words, if $y = f(u)$ and $u = g(x)$ are differentiable, then

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$$

Remark:

In case of more than two functions composed, we can extend the chain rule as follows.

Let $y = f(u)$, $u = g(x)$ and $x = h(t)$ be differentiable, then

$$\boxed{\frac{dy}{dt} = \frac{dy}{du} \cdot \frac{du}{dx} \cdot \frac{dx}{dt}}$$

Example Find $\dfrac{dy}{dx}$ where $y = \sqrt{x^{\frac{2}{3}} + x^{\frac{4}{3}} - 1}$

Solution From $y = \sqrt{x^{\frac{2}{3}} + x^{\frac{4}{3}} - 1} = \sqrt{x^{\frac{2}{3}} + (x^{\frac{2}{3}})^2 - 1}$,

we consider $y = \sqrt{u}$, $u = v + v^2 - 1$ and $v = x^{\frac{2}{3}}$.

Apply the following chain rule: $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dv} \cdot \dfrac{dv}{dx}$

where
$$\frac{dy}{du} = \frac{du^{\frac{1}{2}}}{du} = \frac{1}{2}u^{-\frac{1}{2}},$$
$$\frac{du}{dv} = \frac{d}{dv}(v + v^2 - 1) = 1 + 2v,$$

and
$$\frac{dv}{dx} = \frac{dx^{\frac{2}{3}}}{dx} = \frac{2}{3}x^{-\frac{1}{3}}.$$

Thus,

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{2}u^{-\frac{1}{2}} \cdot (1 + 2v) \cdot \frac{2}{3}x^{-\frac{1}{3}} \\
&= \frac{1}{3} \cdot (x^{\frac{2}{3}} + x^{\frac{4}{3}} - 1)^{-\frac{1}{2}} \cdot (1 + 2x^{\frac{2}{3}}) \cdot x^{-\frac{1}{3}} \\
&= \frac{1}{3} \cdot (x^{\frac{2}{3}} + x^{\frac{4}{3}} - 1)^{-\frac{1}{2}} \cdot (x^{-\frac{1}{3}} + 2x^{\frac{1}{3}}) .
\end{aligned}$$

Example Let $y = \dfrac{u^2}{u^3 - 16}$, $u = 3x^2 - 8$ and $x = \sqrt[4]{t+5}$.

Find $\dfrac{dy}{dt}$ at $t = 11$.

Derivatives of Trigonometric Functions

Derivative Formulas

Let u be a function of x and differentiable.

1. $$\frac{d}{dx}\sin u = \cos u \cdot \frac{du}{dx}$$

2. $$\frac{d}{dx}\cos u = -\sin u \cdot \frac{du}{dx}$$

3. $$\frac{d}{dx}\tan u = \sec^2 u \cdot \frac{du}{dx}$$

4. $$\frac{d}{dx}\cot u = -\csc^2 u \cdot \frac{du}{dx}$$

5. $$\frac{d}{dx}\sec u = \sec u \tan u \cdot \frac{du}{dx}$$

6. $$\frac{d}{dx}\csc u = -\csc u \cot u \cdot \frac{du}{dx}$$

<u>Example</u> Find $\dfrac{dy}{dx}$ where

(a) $y = (x^4 + 1)\tan x$,

<u>Solution</u>

$$\frac{dy}{dx} = \frac{d}{dx}(x^4 + 1)\tan x$$

$$= (x^4 + 1)\frac{d}{dx}\tan x + \tan x\frac{d}{dx}(x^4 + 1)$$

$$= (x^4 + 1)\sec^2 x + \tan x(4x^3)$$

$$= (x^4 + 1)\sec^2 x + 4x^3\tan x$$

(b) $y = \dfrac{\sin(2x)}{7 - \cos(3x)}$,

(c) $y = \cot \sqrt[3]{4 - x^3}$,

(d) $\dfrac{d}{dx}\left[\sec(2x) + \tan(2x)\right]^3$,

(e) $y = 8 + 3\cos(x^4)\sec(7x)$.

<u>Example</u> Find the tangent line equation of $y = \cos(x)$ at $x = \dfrac{3\pi}{2}$.

<u>Solution</u> At $x = \dfrac{3\pi}{2}$, we have $y = 0$.

Consider $\dfrac{dy}{dx} = \dfrac{d}{dx}\cos(x) = -\sin(x)$.

Thus, the slope of the tangent line at $x = \dfrac{3\pi}{2}$ equals to

$$\left.\frac{dy}{dx}\right|_{x=\frac{3\pi}{2}} = -\sin\left(\frac{3\pi}{2}\right) = 1.$$

Therefore, the equation of the tangent line at $x = \dfrac{3\pi}{2}$ is

$$(y-0) = 1\left(x - \frac{3\pi}{2}\right) \quad \text{or}$$

$$x - y = \frac{3\pi}{2}.$$

Derivatives of Logarithmic Functions

Derivative Formulas

Let u be a function of x and differentiable.

1. $$\frac{d}{dx}\log_a u = \frac{1}{u\ln a}\cdot\frac{du}{dx}$$
2. $$\frac{d}{dx}(\ln u) = \frac{1}{u}\cdot\frac{du}{dx}$$

<u>Example</u> Find $\dfrac{dy}{dx}$ where

(a) $y = \log_3(7x^4 + 1)$,

(b) $y = [\ln(3 - x^2)]^4$,

(c) $y = \ln\left[\dfrac{(8x-9)^4}{\sqrt{1+x^6}}\right]$.

Derivatives of Exponential Functions

Derivative formulas

Let u be a function of x and differentiable.

1. $$\frac{d}{dx}a^u = a^u \ln a \cdot \frac{du}{dx}$$

2. $$\frac{d}{dx}e^u = e^u \cdot \frac{du}{dx}$$

<u>Example</u> Find $f'(x)$ where

(a) $f(x) = 10^{\sin(4x)}$,

(b) $f(x) = e^{5x}\sin(\ln x)$,

(c) $f(x) = e^{(x^2-3)\tan x}$.

<u>Example</u> Let $y=\sqrt{e^{8x}+3e^{-8x}}$. Find $\dfrac{dy}{dx}$ at $x=0$.

Derivatives of Inverse Trigonometric Functions

Derivative formulas

Let u be a function of x and differentiable.

1. $$\frac{d}{dx}\sin^{-1}u = \frac{1}{\sqrt{1-u^2}}\cdot\frac{du}{dx}$$

2. $$\frac{d}{dx}\cos^{-1}u = \frac{-1}{\sqrt{1-u^2}}\cdot\frac{du}{dx}$$

3. $$\frac{d}{dx}\tan^{-1}u = \frac{1}{1+u^2}\cdot\frac{du}{dx}$$

4. $$\frac{d}{dx}\cot^{-1}u = \frac{-1}{1+u^2}\cdot\frac{du}{dx}$$

5. $$\frac{d}{dx}\sec^{-1}u = \frac{1}{u\sqrt{u^2-1}}\cdot\frac{du}{dx}$$

6. $$\frac{d}{dx}\csc^{-1}u = \frac{-1}{u\sqrt{u^2-1}}\cdot\frac{du}{dx}$$

Proof 1. We want to show that $\dfrac{d}{dx}\sin^{-1}u = \dfrac{1}{\sqrt{1-u^2}}\cdot\dfrac{du}{dx}$.

Let $\quad y = \sin^{-1}u$.

That is, $\quad u = \sin y$

$$\frac{du}{dx} = \frac{d}{dx}\sin y$$

$$\frac{du}{dx} = \cos y\cdot\frac{dy}{dx}$$

$$\frac{du}{dx} = \cos y\cdot\frac{dy}{dx} \; .$$

Thus,

$$\frac{dy}{dx} = \frac{1}{\cos y}\cdot\frac{du}{dx}$$

$$= \frac{1}{\sqrt{1-\sin^2 y}}\cdot\frac{du}{dx} \; .$$

Therefore, $\quad \dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-u^2}}\cdot\dfrac{du}{dx}$.

* Formulas 2-6 can be proven analogously to the formula 1 above.

<u>Example</u> Find $\dfrac{dy}{dx}$ where

(a) $y = \sin^{-1}(1 + x^2)$,

(b) $y = [1 + \cos^{-1}(\sqrt{x})]^6$,

(c) $y = e^{\sec^{-1}(\sqrt{x})}$,

$$\frac{dy}{dx} = \frac{d}{dx} e^{\sec^{-1}(\sqrt{x})}$$

$$= e^{\sec^{-1}(\sqrt{x})} \frac{d}{dx} \sec^{-1}(\sqrt{x})$$

$$= e^{\sec^{-1}(\sqrt{x})} \cdot \frac{1}{\sqrt{x}\,\sqrt{(\sqrt{x})^2 - 1}} \cdot \frac{d}{dx}\sqrt{x}$$

$$= e^{\sec^{-1}(\sqrt{x})} \cdot \frac{1}{\sqrt{x}\,\sqrt{x-1}} \cdot \frac{dx^{\frac{1}{2}}}{dx}$$

$$= e^{\sec^{-1}(\sqrt{x})} \cdot \frac{1}{\sqrt{x}\,\sqrt{x-1}} \cdot \frac{1}{2\sqrt{x}}$$

$$= \frac{e^{\sec^{-1}(\sqrt{x})}}{2x\,\sqrt{x-1}}$$

(d) $y = \tan^{-1}\left[\dfrac{1-x}{2+x}\right]$.

Solution

Derivatives of Hyperbolic Functions

Derivative formulas

Let u be a function of x and differentiable.

1. $$\frac{d}{dx}\sinh u = \cosh u \cdot \frac{du}{dx}$$
2. $$\frac{d}{dx}\cosh u = \sinh u \cdot \frac{du}{dx}$$
3. $$\frac{d}{dx}\tanh u = \operatorname{sech}^2 u \cdot \frac{du}{dx}$$
4. $$\frac{d}{dx}\coth u = -\operatorname{csch}^2 u \cdot \frac{du}{dx}$$
5. $$\frac{d}{dx}\operatorname{sech} u = -\operatorname{sech} u \tanh u \cdot \frac{du}{dx}$$
6. $$\frac{d}{dx}\operatorname{csch} u = -\operatorname{csch} u \coth u \cdot \frac{du}{dx}$$

Example Find the following derivatives.

(a) $\dfrac{d}{dx}\cosh(9x^2 - 2) =$

(b) $\dfrac{d}{dx}\ln(\tanh(x^3)) =$

(c) $\dfrac{d}{dx}\left(e^{7x}\sinh^3(5x)\right) =$

(d) $\dfrac{d}{dx}\left[\dfrac{\sinh x}{\cosh x-1}\right]$

$$= \frac{(\cosh x-1)\dfrac{d}{dx}\sinh x-\sinh x\dfrac{d}{dx}(\cosh x-1)}{(\cosh x-1)^2}$$

$$= \frac{(\cosh x-1)\cdot\cosh x-\sinh x(\sinh x)}{(\cosh x-1)^2}$$

$$= \frac{\cosh^2 x-\cosh x-\sinh^2 x}{(\cosh x-1)^2}$$

$$= \frac{1-\cosh x}{(\cosh x-1)^2}$$

$$= \frac{1}{1-\cosh x}$$

(e) $\dfrac{d}{dx}\sinh^5(e^x+1)$

Solution

(f) $\dfrac{d}{dx}\left(x^9+\sin(\coth 2x)\right)$.

Solution

> ◈ Let a, b be constants, and u, v functions of x.
>
> Consider the following derivatives:
>
> 1. $\frac{d}{dx}(a^b) \quad = \quad 0$
> 2. $\frac{d}{dx}[u]^a \quad = \quad a\,u^{a-1} \cdot \frac{du}{dx}$
> 3. $\frac{d}{dx}[a^u] \quad = \quad a^u \ln a \cdot \frac{du}{dx}$
> 4. $\frac{d}{dx}[u]^v \quad = \quad \ldots \qquad .$

Logarithmic Derivative

It is used to find derivative of a function in a form of $[u(x)]^{v(x)}$ and when the function consists of several products or quotients of functions. The process is as follows.

1. Take natural log both sides.
2. Apply properties of logarithm.
3. Find derivatives of both sides.
4. Solve equation for $\frac{dy}{dx}$.

Example Find $\dfrac{dy}{dx}$ where $y = x^{\sin(3x)}$, $x > 0$.

Example Find $\dfrac{dy}{dx}$ where $y = (\sin^2 x + 4)^x$.

Solution

Take $\ln$ both sides of the equation

$$\begin{aligned} \ln y &= \ln(\sin^2 x + 4)^x \\ \ln y &= x \ln(\sin^2 x + 4) \end{aligned}$$

Take derivative with respect to x both sides

$$\frac{d}{dx}\ln y = \frac{d}{dx} x \ln(\sin^2 x + 4)$$

$$\frac{1}{y}\frac{dy}{dx} = x\frac{d}{dx}\ln(\sin^2 x + 4) + \ln(\sin^2 x + 4)\frac{dx}{dx}$$

$$\frac{1}{y}\frac{dy}{dx} = x \cdot \frac{1}{\sin^2 x + 4} \cdot \frac{d}{dx}(\sin^2 x + 4) + \ln(\sin^2 x + 4)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{x}{\sin^2 x + 4} \cdot 2\sin x \cos x + \ln(\sin^2 x + 4)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{2x \sin x \cos x}{\sin^2 x + 4} + \ln(\sin^2 x + 4)$$

Thus,

$$\begin{aligned} \frac{dy}{dx} &= y\left[\frac{2x \sin x \cos x}{\sin^2 x + 4} + \ln(\sin^2 x + 4)\right] \\ &= (\sin^2 x + 4)^x \left[\frac{2x \sin x \cos x}{\sin^2 x + 4} + \ln(\sin^2 x + 4)\right]. \end{aligned}$$

<u>Example</u> Find derivative of $y = x^{3x}\sqrt{\dfrac{(x^2+3)(6+x^4)}{3x+4}}$.

Solution

Exercise 1

1. Use definition to find $f'(x)$ where

(1.1) $f(x) = \pi$ (1.2) $f(x) = 4x - 3$

(1.3) $f(x) = 2 - x^2$ (1.4) $f(x) = (2 - x)^2$

(1.5) $f(x) = x^3 - 9$ (1.6) $f(x) = \dfrac{1}{(5x-1)^2}$

(1.7) $f(x) = \sqrt{x}$ (1.8) $f(x) = \dfrac{1}{\sqrt{x}}$

(1.9) $f(x) = \dfrac{2x-1}{2x+1}$ (1.10) $f(x) = \dfrac{1}{\sqrt{1+2x}}$.

Answers

(1.1) 0 (1.5) $3x^2$ (1.8) $-\dfrac{1}{2x\sqrt{x}}$

(1.2) 4 (1.6) $-10(5x-1)^{-3}$ (1.9) $\dfrac{4}{(2x+1)^2}$

(1.3) $-2x$ (1.7) $\dfrac{1}{2\sqrt{x}}$ (1.10) $-\dfrac{1}{2(2+x)^{\frac{3}{2}}}$

(1.4) $2x - 4$

2. Find $f'(a)$ if it exists where

(2.1) $f(x)=\left|x^2-9\right| \;\;;\;\; a=3$

(2.2) $f(x)=\dfrac{x-4}{|x-4|} \;\;;\;\; a=4$

(2.3) $f(x)=\begin{cases} x^2\,, & x\geq 0 \\ 2x\,, & x<0 \end{cases} \;\;;\;\; a=0$

(2.4) $f(x)=\begin{cases} 1-x^2\,, & x\leq 1 \\ 2-2x\,, & x>1 \end{cases} \;\;;\;\; a=1.$

Answers (2.1) doesn't exist (2.3) doesn't exist

(2.2) 0 (2.4) -2

3. A ball is being inflated. Let V be the volume of the ball in cm^3 and r be the ball's radius in cm such that $V=\dfrac{4}{3}\pi r^3$. Find

(3.1) The average rate of volume's change with respect to radius when the radius changes from 6 cm to 9 cm.

(3.2) The instantaneous rate of volume's change with respect to radius when the radius is 9 cm.

Answers (3.1) 228π (3.2) 324π

Exercise 2

1. Find the derivatives of the following functions.

(1.1) $y = 7 + 9x - 7x^3 + 4x^7$

(1.2) $y = \dfrac{1}{x} + \dfrac{3}{x^2} + \dfrac{2}{x^3}$

(1.3) $y = 2\sqrt{x} + 6\sqrt[3]{x} - 2\sqrt{x^3}$

(1.4) $y = \sqrt[3]{3x^2} - \dfrac{1}{\sqrt{5x}}$

(1.5) $y = (x^5 - 4x)^{43}$

(1.6) $y = \dfrac{3}{(a^2 - x^2)^2}$

(1.7) $y = \sqrt{x^2 + 6x + 3}$

(1.8) $y = \dfrac{3 - 2x}{3 + 2x}$

(1.9) $y = (x^2 + 4)^2 (2x^3 - 1)^3$

(1.10) $y = \dfrac{x^2}{\sqrt{4 - x^2}}$

(1.11) $y = \dfrac{x^3 - 2x\sqrt{x}}{x}$

(1.12) $y = \dfrac{1}{x^4 + x^2 + 1}$

(1.13) $y = \dfrac{x}{x + \dfrac{a}{x}}$

Answers

(1.1) $9 - 21x^2 + 28x^6$

(1.2) $-\dfrac{1}{x^2} - \dfrac{6}{x^3} - \dfrac{6}{x^4}$

(1.3) $\dfrac{1}{\sqrt{x}} + \dfrac{2}{\sqrt[3]{x^2}} - \dfrac{3}{\sqrt{x}}$

(1.4) $\dfrac{2}{\sqrt[3]{9x}} + \dfrac{1}{2x\sqrt{5x}}$

(1.5) $43(5x^4 - 4)(x^5 - 4x)^{42}$

(1.6) $\dfrac{12x}{(a^2 - x^2)^3}$

(1.7) $\dfrac{x + 3}{\sqrt{x^2 + 6x + 3}}$

(1.8) $\dfrac{-12x}{(3 + 2x)^2}$

(1.9) $2x(x^2 + 4)(2x^3 - 1)^2 (13x^3 + 36x - 2)$

(1.10) $\dfrac{8x-x^3}{(4-x^2)^{\frac{3}{2}}}$ (1.11) $2x-\dfrac{1}{\sqrt{x}}$

(1.12) $\dfrac{-(4x^3+2x)}{(x^4+x^2+1)^2}$ (1.13) $\dfrac{2ax}{(x^2+a)^2}$

2. Find the equation of a tangent line of $y=\dfrac{2x}{x+1}$ at $(1,1)$.

Answer $y=\dfrac{x+1}{2}$

3. Find the equation of a tangent line of $y=x+\sqrt{x}$ at $(1,2)$.

Answer $y=\dfrac{3}{2}x+\dfrac{1}{2}$

Exercise 3

1. Find the derivatives of the following functions.

(1.1) $f(x)=x-3\sin x$ (1.2) $y=\sin x+10\tan x$

(1.3) $g(t)=t^3\cos t$ (1.4) $h(\theta)=\theta\csc\theta-\cot\theta$

(1.5) $y=\dfrac{x}{\cos x}$ (1.6) $f(\theta)=\dfrac{\sec\theta}{1+\sec\theta}$

(1.7) $y=\sqrt{\sin x}$ (1.8) $y=\cos(a^3+x^3)$

(1.9) $y=\cot(\dfrac{x}{2})$ (1.10) $y=\sin(x\cos x)$

(1.11) $y=\tan(\cos x)$ (1.12) $y=(1+\cos^2 x)^6$

(1.13) $y = \sec^2 x + \tan^2 x$ (1.14) $y = \cot^2(\sin\theta)$

(1.15) $y = \sin(\tan\sqrt{\sin x})$

Answer

(1.1) $f'(x) = 1 - 3\cos x$ (1.2) $y' = \cos x + 10\sec^2 x$

(1.3) $g'(t) = 3t^2\cos t - t^3\sin t$

(1.4) $h'(\theta) = \csc\theta - \theta\csc\theta\cot\theta + \csc^2\theta$

(1.5) $y' = \dfrac{\cos x + x\sin x}{\cos^2 x}$ (1.6) $f(\theta) = \dfrac{\sec\theta\tan\theta}{(1+\sec\theta)^2}$

(1.7) $y' = \dfrac{\cos x}{2\sqrt{\sin x}}$ (1.8) $y' = 3x^2\sin(a^3 + x^3)$

(1.9) $y' = \dfrac{1}{2}\csc^2(\dfrac{x}{2})$

(1.10) $y' = (\cos x - x\sin x)\cos(x\cos x)$

(1.11) $y' = -\sin x\sec^2(\cos x)$

(1.12) $y' = -12\cos x\sin x(1+\cos^2 x)^5$

(1.13) $y' = 4\sec^2 x\tan x$

(1.14) $y' = -2\cos\theta\cot(\sin\theta)\csc^2(\sin\theta)$

(1.15) $y' = \cos(\tan\sqrt{\sin x})(\sec^2\sqrt{\sin x})(2\sqrt{\sin x})(\cos x)$

2. Find the equation of a tangent line of each function at a given point.

(2.1) $y = \tan x$ at $(\frac{\pi}{4},1)$

(2.2) $y = x + \cos x$ at $(0,1)$

(2.3) $y = x\cos x$ at $(\pi,-\pi)$

(2.4) $y = \sin(\sin x)$ at $(\pi,0)$

(2.5) $y = \tan(\frac{\pi x^2}{4})$ at $(1,1)$

Answer

(2.1) $y = 2x + 1 - \frac{\pi}{2}$ (2.2) $y = x + 1$ (2.3) $y = -x$

(2.4) $y = -x + \pi$ (2.5) $y = \pi x - \pi + 1$

3. Find the value of x such that there exists a tangent line of $f(x) = x + 2\sin x$ to be a horizontal line.

Answer $(2n+1)\pi \pm \frac{\pi}{3}$ where n is an integerๅ

Exercise 4

1. Find the derivatives of the following functions.

(1.1) $y=\log_a(3x^2-5)$

(1.2) $y=\ln(x+3)^2$

(1.3) $y=\ln^2(x+3)$

(1.4) $y=\ln(x^3+2)(x^2+3)$

(1.5) $y=\ln\dfrac{x^4}{(3x-4)^2}$

(1.6) $y=\ln(\sin 3x)$

(1.7) $y=\ln(x+\sqrt{1+x^2})$

(1.8) $y=x\ln x-x$

(1.9) $y=\ln(\sec x+\tan x)$

(1.10) $y=\ln(\ln\tan x)$

(1.11) $y=\dfrac{(\ln x^2)}{x^2}$

(1.12) $y=\dfrac{1}{5}x^5(\ln x-1)$

(1.13) $y=x(\sin\ln x-\cos\ln x)$

(1.14) $y=\dfrac{\ln x}{1+\ln(2x)}$

(1.15) $y=\ln(e^{-x}+xe^{-x})$

(1.16) $y=\ln(\dfrac{1}{x})+\dfrac{1}{\ln x}$

Answer

(1.1) $\dfrac{6x}{(3x^2-5)\ln a}$

(1.2) $\dfrac{2}{x+3}$

(1.3) $\dfrac{2\ln(x+3)}{x+3}$

(1.4) $\dfrac{3x^2}{x^3+2}+\dfrac{2x}{x^2+3}$

(1.5) $\dfrac{4}{x}-\dfrac{6}{3x-4}$

(1.6) $3\cot 3x$

(1.7) $\dfrac{1}{\sqrt{1+x^2}}$

(1.8) $\ln x$

(1.9) $\sec x$

(1.10) $\dfrac{2}{\sin(2x)\cdot\ln(\tan x)}$

(1.11) $\dfrac{2-4\ln x}{x^3}$ (1.12) $x^4 \ln x$

(1.13) $2\sin \ln x$ (1.14) $\dfrac{1+\ln 2}{x[1+\ln(2x)]^2}$

(1.15) $\dfrac{-x}{1+x}$ (1.16) $-\dfrac{1}{x}[1+\dfrac{1}{(\ln x)^2}]$

2. Let $f(x)=\dfrac{x}{\ln x}$. Find $f'(e)$.

Answer 0

3. Find the equation of a tangent line of $y=\ln(\ln x)$ at $(e,0)$.

Answer $x-ey=e$

Exercise 5

1. Find the derivatives of the following functions

(1.1) $f(x)=x^2e^x$ (1.2) $y=3^{ax^3}$

(1.3) $f(u)=e^{1/u}$ (1.4) $f(t)=e^{t\sin 2t}$

(1.5) $y=\sqrt{1+2e^{3x}}$ (1.6) $y=e^{e^x}$

(1.7) $y=\dfrac{ae^x+b}{ce^x+d}$ (1.8) $f(t)=\cos(e^{-t\ln t})$

(1.9) $y=\sqrt{\cos x}\cdot a^{\sqrt{\cos x}}$ (1.10) $y=7^{x^3+8}$

(1.11) $y = 7^{x^3+8}\ (x^4 - x)$ (1.12) $h(t) = (\ln t + 1)10^{\ln t}$

(1.13) $g(x) = \dfrac{\ln x}{e^{x^2} - e^x}$ (1.14) $y = \tan^2(e^{3x})$

(1.15) $f(x) = e^{\sin^3(\ln(x^2+1))}$

Answer

(1.1) $f'(x) = x(x+2)e^x$

(1.2) $y' = 3^{ax^3}\ \ln 3 \cdot (3ax^2)$

(1.3) $f(u) = (-1/u^2)e^{1/u}$

(1.4) $f'(t) = e^{t\sin 2t}\ (2t\cos 2t + \sin 2t)$

(1.5) $y' = 3e^{3x} / \sqrt{1+2e^{3x}}$

(1.6) $y' = e^{e^x}\ e^x$

(1.7) $y' = \dfrac{(ad-bc)e^x}{(ce^x+d)^2}$

(1.8) $f'(t) = \sin(e^{-t\ln t}) \cdot e^{-t\ln t}(1 + \ln t)$

(1.9) $y' = -\dfrac{1}{2} a^{\sqrt{\cos x}} \cdot \sin x\ (\ln a - \dfrac{1}{\sqrt{\cos x}})$

(1.10) $y' = 3x^2 \ln 7 \cdot 7^{x^3+8}$

(1.11) $y' = 7^{x^3+8}[(4x^3-1)+(3x^6-3x^3)\ln 7]$

(1.12) $h'(t) = \dfrac{10^{\ln t}}{t}[\ln 10(\ln t+1)+1]$

(1.13) $g'(x) = \dfrac{1}{x(e^{x^2}-e^x)} - \dfrac{\ln x(2xe^{x^2}-e^x)}{(e^{x^2}-e^x)^2}$

(1.14) $y' = 6e^{3x}\tan(e^{3x})\sec^2(e^{3x})$

(1.15) $f'(x) = \dfrac{6x}{x^2+1}\sin^2(\ln(x^2+1))\cdot\cos(\ln(x^2+1))\cdot e^{\sin^3(\ln(x^2+1))}$

Exercise 6

1. Find the derivatives of the following functions.

(1.1) $y = \arcsin(2x-3)$ (1.2) $y = \arccos(x^2)$

(1.3) $y = \arctan 3x^2$ (1.4) $y = \operatorname{arc}\cot\dfrac{1+x}{1-x}$

(1.5) $f(x) = x\csc^{-1}(\dfrac{1}{x})+\sqrt{1+x^2}$

(1.6) $y = \dfrac{1}{ab}\arctan(\dfrac{b}{a}\tan x)$

(1.7) $y = x\ln(4+x^2)+4\arctan\dfrac{x}{2}-2x$

(1.8) $h(t) = \cot^{-1}(t)+\cot^{-1}(\dfrac{1}{t})$

(1.9) $y = \cos^{-1}\left[\dfrac{b + a\cos x}{a + b\cos x}\right]$

Answer

(1.1) $y' = \dfrac{1}{\sqrt{3x - x^2 - 2}}$

(1.2) $y' = -\dfrac{2x}{\sqrt{1 - x^4}}$

(1.3) $y' = \dfrac{6x}{1 + 9x^4}$

(1.4) $y' = -\dfrac{1}{1 + x^2}$

(1.5) $f'(x) = \csc^{-1}(\dfrac{1}{x})$

(1.6) $y' = \dfrac{1}{a^2\cos^2 x + b^2\sin^2 x}$

(1.7) $y' = \ln(4 + x^2)$

(1.8) $h'(t) = 0$

(1.9) $y' = \dfrac{\sqrt{a^2 - b^2}}{a + b\cos x}$

2. Show that

$$\frac{d}{dx}\left[\frac{1}{2}\tan^{-1} x + \frac{1}{4}\ln\frac{(x+1)^2}{x^2+1}\right] = \frac{1}{(1+x)(1+x^2)}\ .$$

Exercise 7

1. Evaluate $\frac{dy}{dx}$ of the following functions.

(1.1) $y = \sinh 3x$

(1.2) $y = \tanh(1 + x^2)$

(1.3) $y = \coth(\frac{1}{x})$

(1.4) $y = x \operatorname{sech} x^2$

(1.5) $y = \operatorname{csch}^2(x^2 + 1)$

(1.6) $y = \ln(\tanh(2x))$

(1.7) $y = \sinh(\tan^{-1} e^{3x})$

Answer

(1.1) $3\cosh 3x$

(1.2) $2x \operatorname{sech}^2(1 + x^2)$

(1.3) $\frac{1}{x^2} \operatorname{csch}^2(\frac{1}{x})$

(1.4) $-2x^2 \operatorname{sech} x^2 \tanh x^2 + \operatorname{sech} x^2$

(1.5) $-4x \operatorname{csch}^2(x^2 + 1) \coth(x^2 + 1)$

(1.6) $4 \operatorname{csch} 4x$

(1.7) $\frac{3e^{3x} \cosh(\tan^{-1} e^{3x})}{1 + e^{6x}}$

Exercise 8

1. Use logarithmic derivative to find $\dfrac{dy}{dx}$ where

(1.1) $y=(x^2+2)(1-x^3)^4$

(1.2) $y=\dfrac{x(1-x^2)^2}{\sqrt{1+x^2}}$

(1.3) $y=\dfrac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}$

(1.4) $y=(2x+1)^5(x^4-3)^6$

(1.5) $y=\dfrac{\sin^2 x\tan^4 x}{(x^2+1)^2}$

(1.6) $y=x^2e^{2x}\cos 3x$

(1.7) $y=x^x$

(1.8) $y=x^{\ln x}$

(1.9) $y=x^{\sin x}$.

Answer

(1.1) $y'=6x(x^2+2)^2(1-x^3)^3(1-4x-3x^3)$

(1.2) $y'=\dfrac{(1-5x^2-4x^4)(1-x^2)}{(1+x^2)^{3/2}}$

(1.3) $y'=\dfrac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}\left[\dfrac{3}{4x}+\dfrac{x}{x^2+1}-\dfrac{15}{3x+2}\right]$

(1.4) $y'=(2x+1)^5(x^4-3)^6\left(\dfrac{10}{2x+1}+\dfrac{24x^3}{x^2+1}\right)$

(1.5) $y' = \dfrac{\sin^2 x \tan^4 x}{(x^2+1)^2}\left(2\cot x + \dfrac{4\sec^2 x}{\tan x} - \dfrac{4x}{x^2+1}\right)$

(1.6) $y' = x^2 e^{2x}\cos 3x[2/x + 2 - 3\tan 3x]$

(1.7) $y' = x^x(1+\ln x)$

(1.8) $y' = 2x^{\ln x - 1}\ln x$

(1.9) $y' = x^{\sin x}[(\sin x)/x + \ln x \cos x]$

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Implicit Differentiation

Definition Implicit function

Implicit function of one independent variable function is a function written in a form of $F(x,y)=0$. For example, $x^2+3xy^2+2y-5=0$.

Implicit Differentiation

Since we start with $F(x,y)=0$, we can find $\frac{dy}{dx}$ by taking derivatives with respect to x both sides separately. Then we solve the equation for $\frac{dy}{dx}$. This method is called "implicit differentiation."

Example 1 Find $\frac{dy}{dx}$ where y is a function of x and is implicitly defined by $y^2+xy-6x=0$.

Example 2 Given that $a > 0$, find $\frac{dy}{dx}$ where y is a function of x and is implicitly defined by

$$x = a\sin^{-1}\left(\frac{y}{a}\right) - \sqrt{a^2 - y^2}\,.$$

Exercise 9

Find $f'(x)$ of each function defined by

1. $2x^4 - 3x^2y^2 + y^4 = 0$

2. $(x+y)^2 - (x-y)^2 = x^4 + y^4$

3. $x^2 + xy + y^2 - 3 = 0$

4. $x\sec 5x = 4y - y\tan 8x$

5. $\sqrt{x} + \sqrt{\sqrt{x} + \cos y} = 1$

Answer

1. $\dfrac{3xy^2 - 4x^3}{2y^3 - 3x^2y}$ **2.** $\dfrac{x^3 - y}{x - y^3}$ **3.** $\dfrac{2x + y}{x^2 + xy - x}$

4. $\dfrac{8y\sec^2 8x + \sec 5x + 5x\sec 5x\tan 5x}{4 - \tan 8x}$

5. $\dfrac{\sqrt{\sqrt{x} + \cos y}}{\sin y\sqrt{x}} + \dfrac{1}{2\sin y\sqrt{x}}$

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Derivatives of Higher Order

Definition Derivatives of Higher Order

Derivatives of higher order refer to finding derivatives several times. So, we call them second order and third order derivatives depending on the number of times taking derivatives.

For example, let $y = f(x)$. By taking derivative one time, we obtain $\dfrac{dy}{dx} = f'(x)$ and it is called the *first order derivative of* f.

If we find the derivative of $\dfrac{dy}{dx} = f'(x)$ one more time, the derivative of $\dfrac{dy}{dx} = f'(x)$ is called the *second (order) derivative of* f and is denoted by $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d}{dx} f'(x) = f''(x)$.

Similarly, if we find the derivative of $\dfrac{d^2y}{dx^2} = f''(x)$ one more time, we will obtain the *third (order) derivative of* f and is denoted by

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^3}\right) = \frac{d}{dx} f''(x) = f'''(x).$$

Continue to do these steps, we may define the n^{th} *(order) derivative* for any positive integers n as follows :

$$\frac{d^ny}{dx^n} = \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) = \frac{d}{dx} f^{(n-1)}(x) = f^{(n)}(x).$$

<u>Remark</u>: The notation $f^{(n)}(x)$ is used when $n \geq 4$.

Example 3 Find $f^{(n)}(x)$ where $y = f(x) = \dfrac{1}{(x-1)}$.

Example 4 Find $f^{(n)}(x)$ where $y = \ln(1-x)$.

Exercise 10

1. Find $\dfrac{d^5y}{dx^5}$ **of** $y = 3^x$.
2. Find $f'''(x)$ **of** $f(x) = e^{3x+1}$.
3. Find $f^{(n)}(x)$ **of** $f(x) = (ax+b)^n$.

Answer

1. $3^x \ln^5 3$ **2.** $27e^{3x+1}$ **3.** $a^n n!$

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<u>Indeterminate forms</u>

Suppose $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist such that $\lim_{x\to a} g(x) \neq 0$. We have that $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ exist.

However, if $\lim_{x\to a} g(x) = 0$, we are not able to say anything about $\lim_{x\to a} \frac{f(x)}{g(x)}$. There are two cases as follow:

1. If $\lim_{x\to a} f(x)$ equals some nonzero number and $\lim_{x\to a} g(x) = 0$, then we can conclude that $\lim_{x\to a} \frac{f(x)}{g(x)}$ does not exist (DNE).

2. If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, then we say that $\lim_{x\to a} \frac{f(x)}{g(x)}$ has the indeterminate form, namely $\frac{0}{0}$.

Besides the indeterminate form $\frac{0}{0}$, the indeterminate forms also include the forms $\frac{\pm\infty}{\pm\infty}$, $0 \cdot \pm\infty$, $\pm\infty \pm \infty$, 0^0, $\pm\infty^0$, and $1^{\pm\infty}$.

To find the limits of these indeterminate forms, a French mathematician named *L'Hopital* made the following rule:

For any real number a and two functions $f(x)$ and $g(x)$ which are differentiable on the interval $0<|x-a|<\delta$ for some $\delta>0$, if $\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=0$, then $\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$.

If $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ still has the indeterminate form $\frac{0}{0}$, then we may apply *L'Hopital rule* until $\lim_{x\to a} f^{(n)}(x)$ and $\lim_{x\to a} g^{(n)}(x)$ are not zero simultaneously.

Example 1: Evaluate $\lim_{x\to\frac{\pi}{2}}\frac{\sqrt{4+\cos\theta}-2}{\theta-\frac{\pi}{2}}$.

Example 2: Evaluate $\lim_{t \to 0} \dfrac{\sqrt{1+t} - \left(1 + \dfrac{t}{2}\right)}{t^2}$.

Example 3: Evaluate $\lim\limits_{x\to 0}\dfrac{e^x(1-e^x)}{(1+x)\ln(1-x)}$.

In the case of $\frac{\pm\infty}{\pm\infty}$, we can do one of the following:

(1) Eliminate the terms of ∞ by dividing every term by the highest term as we do for polynomial functions in the chapter 2.

(2) Apply the *L'Hopital's* rule by rewriting the functions:

Suppose $\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}\ \left(=\frac{\infty}{\infty}\right)$. Then, we rewrite as $\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}=\frac{\frac{1}{\lim_{x\to a} g(x)}}{\frac{1}{\lim_{x\to a} f(x)}}\ \left(=\frac{0}{0}\right)$ before applying the *L'Hopital'*s rule.

Theorem1: Suppose $\lim_{x\to a} f(x)=\infty$ and $\lim_{x\to a} g(x)=\infty$, and both functions $f(x)$, $g(x)$ are differentiable. Then, we also have $\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$.

If $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ still has the indeterminate form $\frac{\infty}{\infty}$, then we can apply *L'Hopital rule* until $\lim_{x\to a} f^{(n)}(x)$ and $\lim_{x\to a} g^{(n)}(x)$ do not approach infinity simultaneously.

Example 4: Evaluate $\lim\limits_{x\to\infty} \dfrac{5x+2\ln x}{x+3\ln x}$.

Example 5: Evaluate $\lim\limits_{x\to 0} \dfrac{\cot x}{\cot 2x}$.

Example 6: Evaluate $\lim\limits_{x\to 0^+} \dfrac{e^{-3/x}}{x^2}$.

For those indeterminate forms $0 \cdot \pm\infty, \pm\infty \pm \infty$, we have to convert them to either $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$ before applying L'Hoptial's rule.

Example 7: Evaluate $\lim\limits_{x\to 0^+} x^3 \ln x$.

Example 8: Evaluate $\lim\limits_{x\to 0^+} \left(\csc x - \dfrac{1}{x} \right)$.

Example 9: Evaluate $\lim_{x\to\infty}\left(x-\sqrt{x^2+x}\right)$.

For the rest of indeterminate forms: $0^0, \pm\infty^0, 1^{\pm\infty}$, we take the natural log (ln) to the function so that it has the form $0\cdot\pm\infty$. Then, we continue just like what we do in the last section.

<u>**Note:**</u> 1^∞ is not always equal to 1. For example, $\lim_{x\to 0}1^{1/x}=1$, but $\lim_{x\to 0}(1+x)^{1/x}=e$.

Example 10: Evaluate $\lim_{x \to 0} \left(\sec^3 2x \right)^{\cot^2 3x}$.

Example 11: Evaluate $\lim_{x \to 0^+} \left(1 + \dfrac{5}{x} \right)^{2x}$.

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Applications of Derivative

1. Differentials

Definition (Differential)

If f is a differentiable function with $\dfrac{dy}{dx} = f'(x)$, then $dy = f'(x)dx$ is called a **differential** of y (dependent variable) at some point x.

The following are differential formulas of some basic functions.

1. $da = 0$; a is a constant.
2. $dx^n = nx^{n-1}dx$
3. $d\sin x = \cos x dx$
4. $d\cos x = -\sin x dx$
5. $da^x = a^x \ln a \; dx$ where a is a positive constant.
6. $de^x = e^x dx$
7. $d\ln x = \dfrac{1}{x}dx$
8. $d\log_a x = \dfrac{1}{x\ln a}dx$ where a is a positive constant with $a \neq 1$.

Example 1 Find dy where $y = x^2(x+1)$.

Solution

Remark

For parametric functions, we may use

$$\frac{\text{differential of } y}{\text{differential of } x} = \frac{dy}{dx}$$

where $y = f(x)$ and x, y are parametric functions in term of t.

Example 2 Evaluate $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ where $y = f(x)$ is defined by $y = t^2 + 1$ and $x = t^3 + 3$.

Solution

Exercise of Differential

Find differentials of the following functions.

1. $y = \dfrac{x}{\sin x}$

2. $x^2 y - y^2 x + 2 = 0$

3. $x + xy\sin x + \dfrac{y^2 \cos x}{x} = 1$

Answers

1. $\dfrac{\sin x - x\cos x}{\sin^2 x}dx$ **2.** $\dfrac{y^2 - 2xy}{x^2 - 2xy}dx$

3. $\left(\dfrac{\dfrac{x\sin x + \cos x}{x^2}y^2 - xy\cos x - y\sin x - 1}{x\sin x + 2y\dfrac{\cos x}{x}} \right)dx$

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2. Linear Approximation

From the definition of differential: $dy = f'(x)dx$ and the change of dependent variable: $\Delta y = y_2 - y_1$, $\Delta x = x_2 - x_1$, we may consider the geometric meaning as follows.

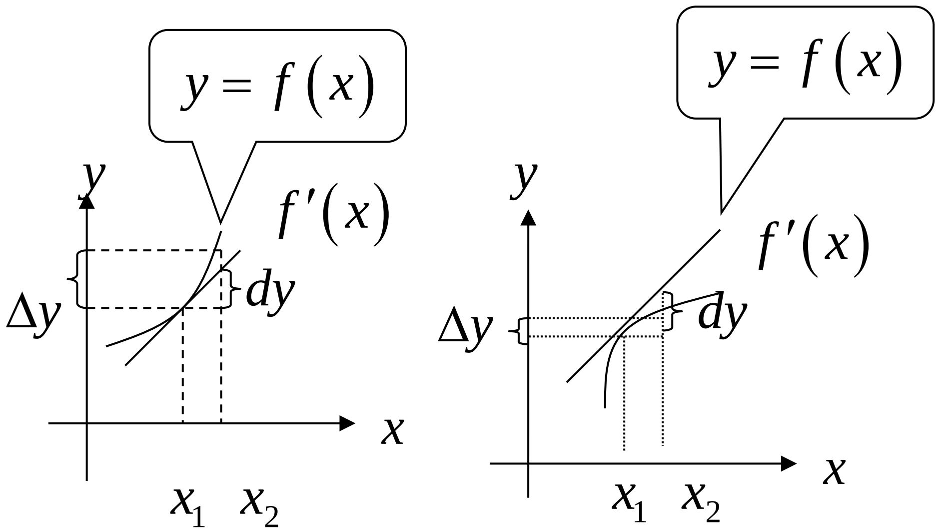

Figure shows $dy, \Delta y$.

The differential of y (dy) at some small interval of x is calculated from the different values on the tangent line to the curve, while the difference of y (Δy) is computed from the real values of the curve. So we may say $dy \approx \Delta y$. However, $dx = \Delta x$ and when $\Delta x \to 0$ we have $dy = \Delta y$.

To find the linear approximation at some point x, we use the following approximation: $\Delta y \approx dy$

$$y_2 - y_1 \approx f'(x)dx$$

$$f(x+\Delta x) - f(x) \approx f'(x)dx$$

$$f(x+\Delta x) \approx f(x) + f'(x)dx.$$

The above equation shows that we may approximate $f(x+\Delta x)$ by adding function $f(x)$ to its differential where $\Delta x = dx$.

Example 3 Use the differential to find a linear approximation of $f(x) = \sqrt{1+x}$ at $x = 0$ when $\Delta x = a$.

Solution

Example 4 Approximate $\sqrt{1.1}$ linearly.

Solution

Example 5 Linearly approximate $\cos 62^{\circ}$.

solution

Exercise of Linear Approximation

Use differential to find linear approximations of the following:

1. $\sqrt{63.999}$ 2. $(31)^{\frac{3}{5}}$ 3. $\cos(44^\circ)$

4. $(1.01)^5 + 3(1.01)^{\frac{3}{2}} - 1.$

Answer

1. 7.9999 2. 7.85 3. 0.7194 4. 3.095

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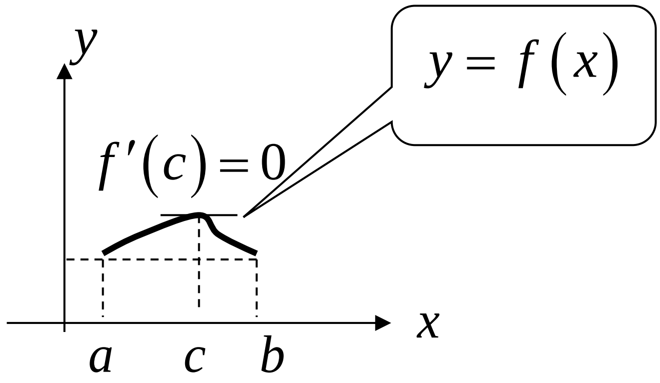
3. Some Useful Theorems

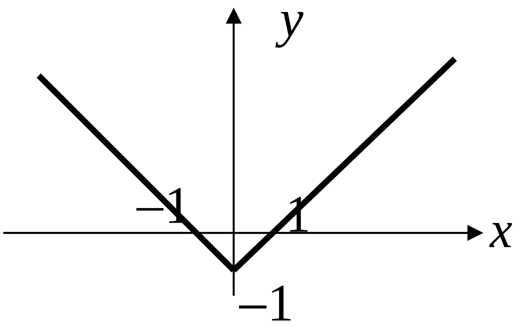
Roll's Theorem

If $f(x)$ is a continuous function on the interval $[a,b]$, differentiable on (a,b) and $f(a)=f(b)$, then there exist at least one point $x=c$ in the interval (a,b) such that $f'(c)=0$.



Remark

Absolute functions such as $y=|x|-1$ is continuous everywhere but not differentiable at $x=0$. Thus, the Roll's theorem does not apply to this function, i.e., there is no c such that $f'(c)=0$.



Graph of $y=|x|-1$

Mean - Value Theorem

If f is a continuous function on the interval $[a,b]$, and differentiable on (a,b), then there exists at least one point $x=c$ in (a,b) such that $f'(c)=\dfrac{f(b)-f(a)}{b-a}$.

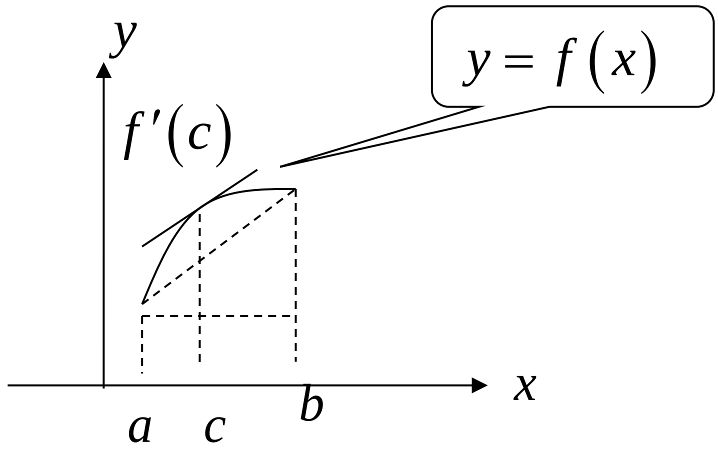


Figure shows the Mean – Value Theorem

$$f'(c)=\frac{f(b)-f(a)}{b-a}=\frac{\Delta y}{\Delta x}.$$

Example 6 Verify the Roll's Theorem to the given function: $f(x)=x^2-6x+8$ on the interval $[2,4]$.

Solution

Example 7 Let $f(x)=2x^3-6x^2+6x-3$. Find x such that $f'(x)$ equals to the average rate of change of f over $0\le x\le 2$.

Solution

Exercise of Mean-Value Function

Find a point c on $[a,b]$ such that $f'(c)=\dfrac{f(b)-f(a)}{b-a}$ where the functions and intervals are given as follow.

1. $f(x)=x^6$; $x\in[-3,3]$
2. $f(x)=\sin x$; $x\in[0,2\pi]$
3. $f(x)=x^3-2x^2+x+1$; $x\in[0,1]$

Answers

1. 0
2. $\dfrac{\pi}{2}$ or $\dfrac{3\pi}{2}$
3. $\dfrac{1}{3}$

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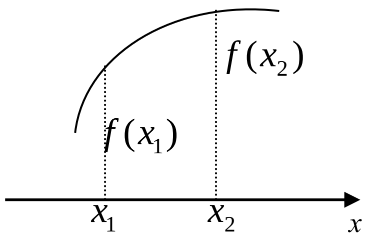
4. Increasing and Decreasing Functions

<u>Definition</u>: (Increasing, decreasing and constant functions)

Let f be a function defined on the interval I, and x_1, x_2 are two points in I.

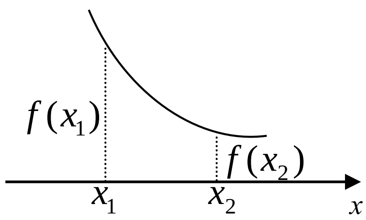
1. The function f is an increasing function on the interval I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$.
2. The function f is a decreasing function on the interval I if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.
3. The function f is a constant function on the interval I if $f(x_1) = f(x_2)$ for any x_1, x_2 in I.



<u>Increasing</u>

$f(x_1) < f(x_2)$

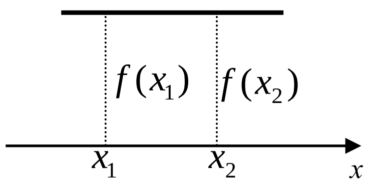
when $x_1 < x_2$.



<u>Decreasing</u>

$f(x_1) > f(x_2)$

when $x_1 < x_2$.



<u>Constant</u>

$f(x_1) = f(x_2)$

for all x_1, x_2.

<u>Theorem</u>

Let f be a continuous function on $[a,b]$ and differentiable on (a,b).

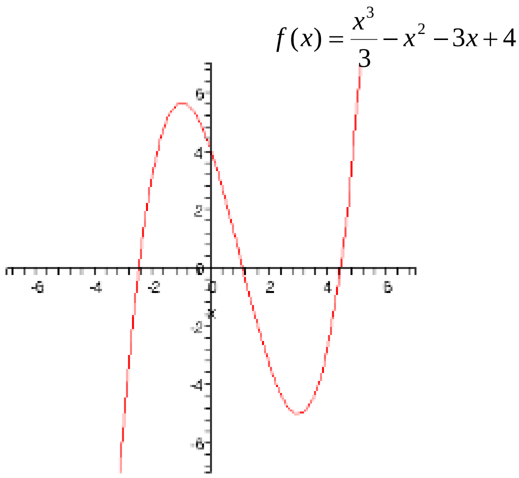
1. If $f'(x) > 0$ for all $x \in (a,b)$, then f increases on $[a,b]$.
2. If $f'(x) < 0$ for all $x \in (a,b)$, then f decreases on $[a,b]$.
3. If $f'(x) = 0$ for all $x \in (a,b)$, then f is constant on $[a,b]$.

Example 8 Let f be a function defined by

$$f(x) = \frac{x^3}{3} - x^2 - 3x + 4 .$$

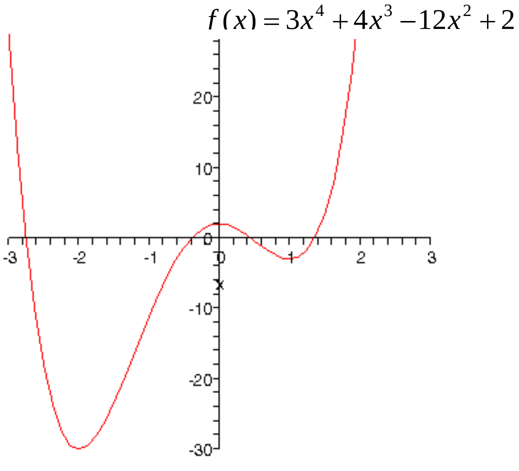
Find the intervals of x for which f is increasing and is decreasing.

<u>Solution</u>



Example 9 Identify the intervals of x where the given function $f(x) = 3x^4 + 4x^3 - 12x^2 + 2$ is increasing and decreasing.

<u>Solution</u>

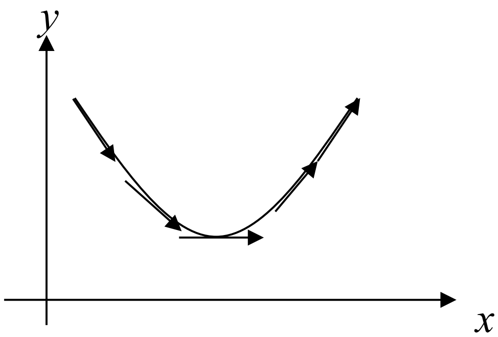


5. Concavity and Point of Inflection

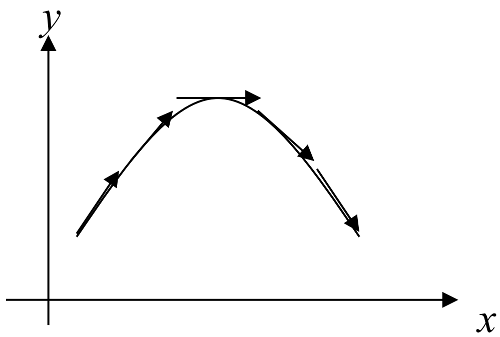
<u>Definition</u> (Concave Up and Concave Down)

Let f be a differentiable function on an open interval I. Then

1. The function f is called concave up on I if f' increases on I.
2. The function f is called concave down on I if f' decreases on I.



concave up



concave down

<u>Theorem</u>

Let f be a function such that $f''(x)$ exists on an open interval I.

1. If $f''(x) > 0$ for all $x \in I$, then f is concave up on I.
2. If $f''(x) < 0$ for all $x \in I$, then f is concave down on I.

Example 10 Identify the intervals where the following functions are concave up and where they are concave down.

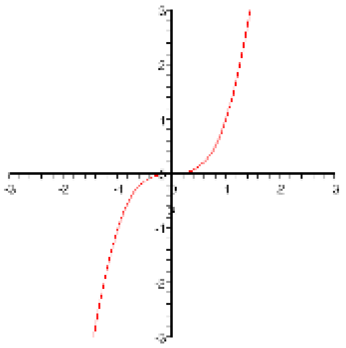
a. $f(x) = x^3$

b. $f(x) = x^3 - 3x^2 + 1$

<u>Solution</u>

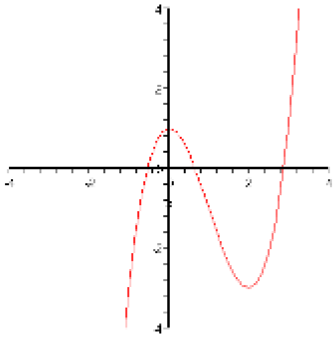
a.

$f(x) = x^3$



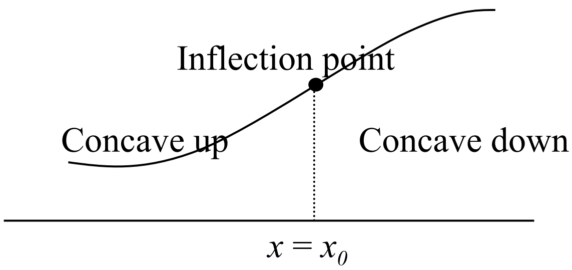
b.

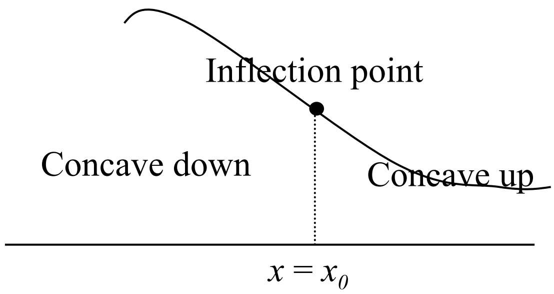
$f(x) = x^3 - 3x^2 + 1$



<u>Definition</u> (Inflection Point)

A point $(x_0, f(x_0))$ is called an inflection point if the graph of *f* changes the concavity at $x = x_0$.





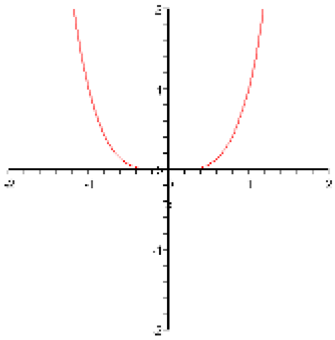
<u>Theorem</u>

If the point $(x_0, f(x_0))$ is an inflection point, then either $f''(x_0) = 0$ or $f''(x_0)$ does not exist.

<u>Remark</u> The converse of this theorem is not true.

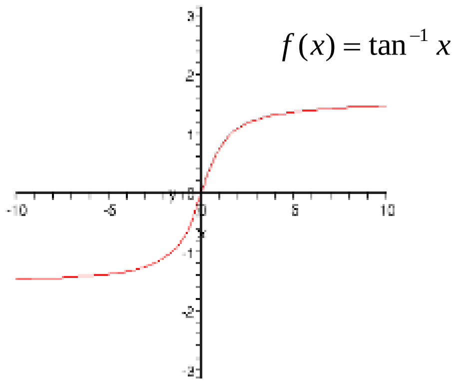
That is, if $f''(x_0) = 0$ or $f''(x_0)$ does not exist, the point $(x_0, f(x_0))$ may or may not be an inflection point.

For example, $f(x) = x^4$ has $f''(x) = 12x^2$ and $f''(0) = 0$, but $x = 0$ is not an inflection point.

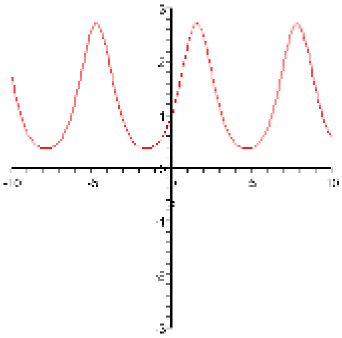


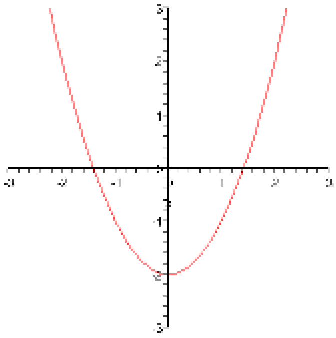
Example 11 Find all inflection points of $f(x) = \tan^{-1} x$.

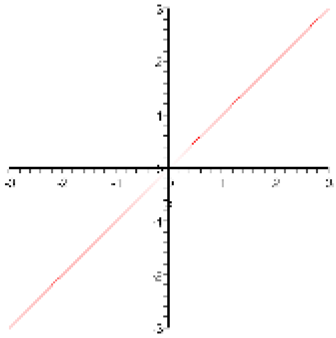
<u>Solution</u>



6. Maximum Value and Minimum Value of Function







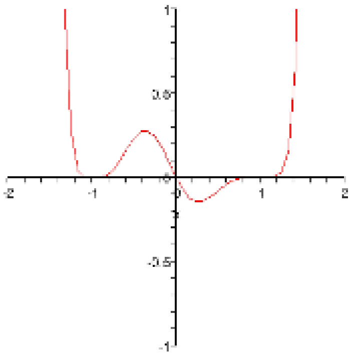
<u>Definition</u> (Relative Maximum and Relative Minimum)

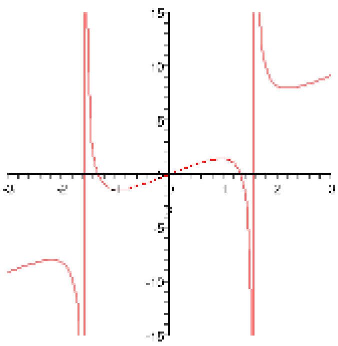
Let f be a real valued function. We say that

1. f has a relative maximum at $x = x_0$ if $f(x) \le f(x_0)$ for all x in an open interval containing x_0. The point $(x_0, f(x_0))$ is called a relative maximum point of f.

2. f has a relative minimum at $x = x_0$ if $f(x) \ge f(x_0)$ for all x in an open interval containing x_0. The point $(x_0, f(x_0))$ is called a relative minimum point of f.

We may refer to a relative maximum and a relative minimum of a function as its relative extrema.





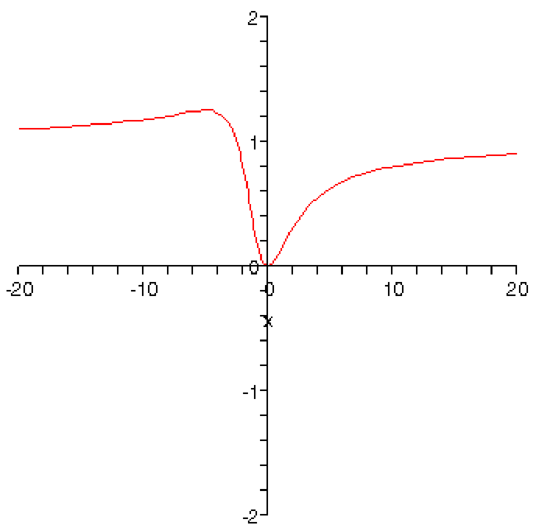
Definition (Absolute Maximum and Absolute Minimum)

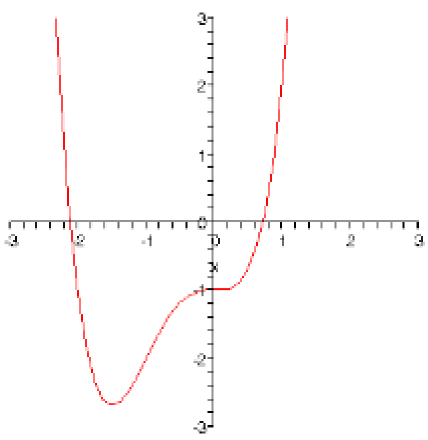
Let f be a real valued function. Then

1. f has an absolute maximum at $x = x_0$ if $f(x_0) \geq f(x)$ for all $x \in D_f$. We call $(x_0, f(x_0))$ an absolute maximum point of f.

2. f has an absolute minimum at $x = x_0$ if $f(x_0) \leq f(x)$ for all $x \in D_f$. We call $(x_0, f(x_0))$ an absolute minimum point of f.

Also, an absolute max and an absolute min of a function can be referred as its absolute extrema.



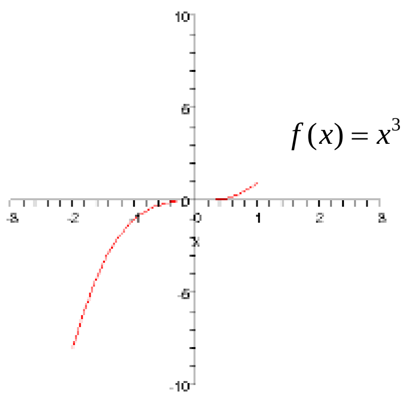


<u>Extreme Value Theorem</u>

If f is a continuous function on $[a,b]$, then f has both absolute minimum and absolute maximum values on $[a,b]$.

Example 12 Find all absolute extreme values of $f(x) = x^3$ where $x \in [-2,1]$.

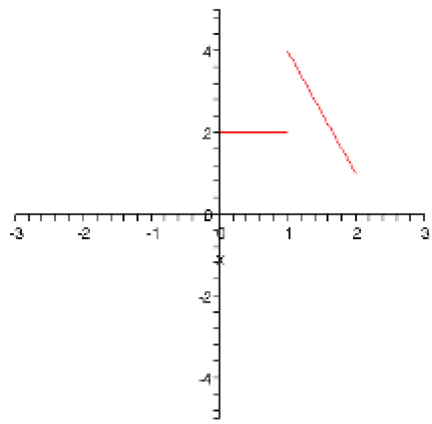
<u>Solution</u>



Example 13 Find all absolute extreme values of the function

$$f(x) = \begin{cases} 2 & , \quad 0 \le x \le 1 \\ -3x+7 & , \quad 1 < x \le 2 \, . \end{cases}$$

<u>Solution</u>



From the above example, we can see that if a given function is not continuous on a closed interval $[a,b]$, that function may or may not have absolute extreme values.

<u>Theorem</u>

If f has a relative extreme value at $x = x_0$, then either

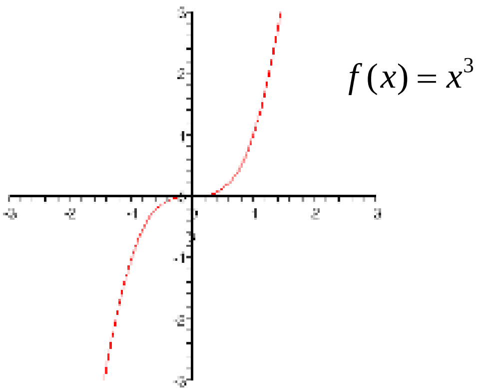
$f'(x_0) = 0$ or $f'(x_0)$ does not exist.

Note:

1. If $f'(x_0) = k \neq 0$, then $x = x_0$ is not a relative extreme point.

2. If $f'(x_0) = 0$ or undefined, then $x = x_0$ is not necessarily a relative extreme point.

In example 5, we see that $f'(x) = 0$ at $x = 0$, but f does not have a relative max and relative min at $x = 0$.



<u>Procedure of finding maximum and minimum values</u>

Definition

A point $x = x_0$ is called a **critical point** of a function f if either

1. $f'(x_0) = 0$ or
2. $f'(x_0)$ does not exist.

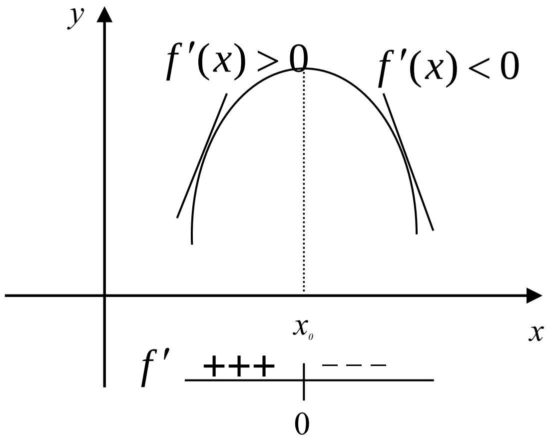
We call $f(x_0)$ a critical value of f.

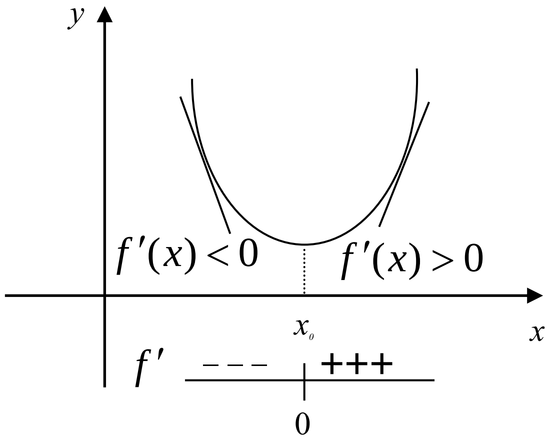
Theorem (The First Derivative Test for Relative Extreme Points.)

Suppose $x = x_0$ is a critical point of a continuous function f.

Consider the sign of the derivative of f around x_0.

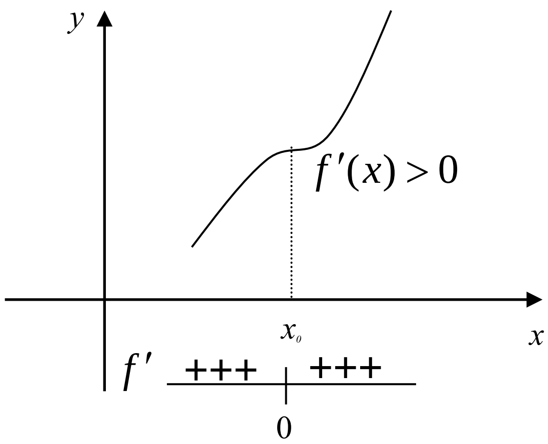
1. If the sign of $f'(x)$ changes from positive to negative at x_0, then f has a relative maximum at $x = x_0$.
2. If the sign of $f'(x)$ changes from negative to positive, then f has a relative minimum at $x = x_0$.

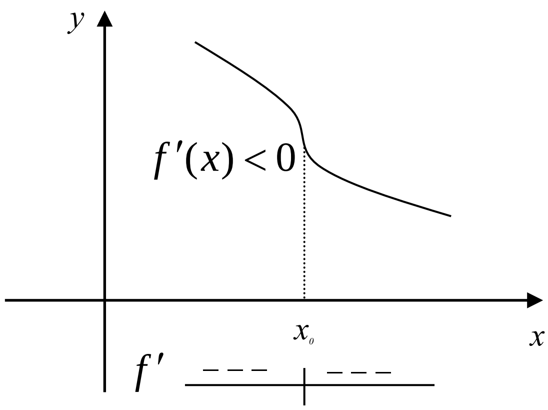




Relative Maximum

Relative Minimum



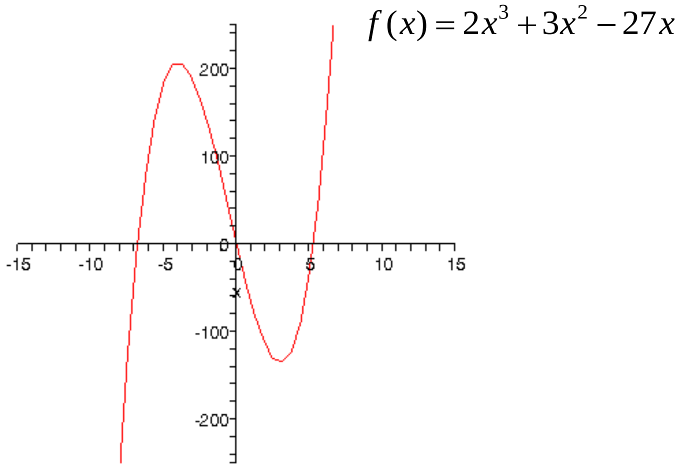


Neither Relative Maximum nor Minimum

Example 14 Find relative max and relative min values of

$$f(x) = 2x^3 + 3x^2 - 27x \ .$$

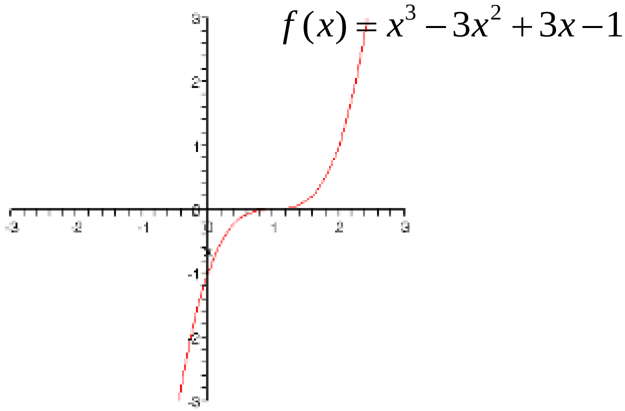
<u>Solution</u>



Example 15 Find relative max and relative min values of

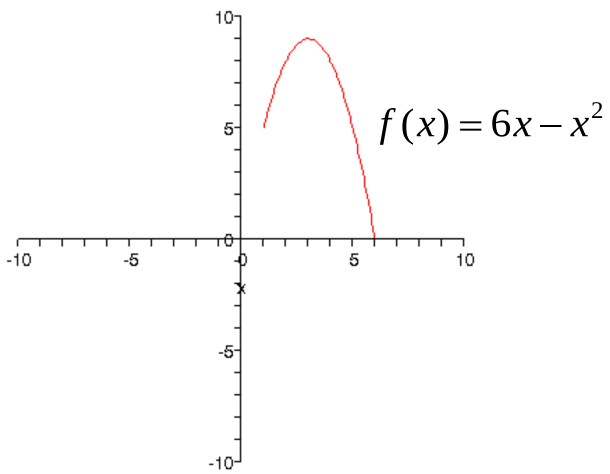
$$f(x) = x^3 - 3x^2 + 3x - 1 .$$

<u>Solution</u>

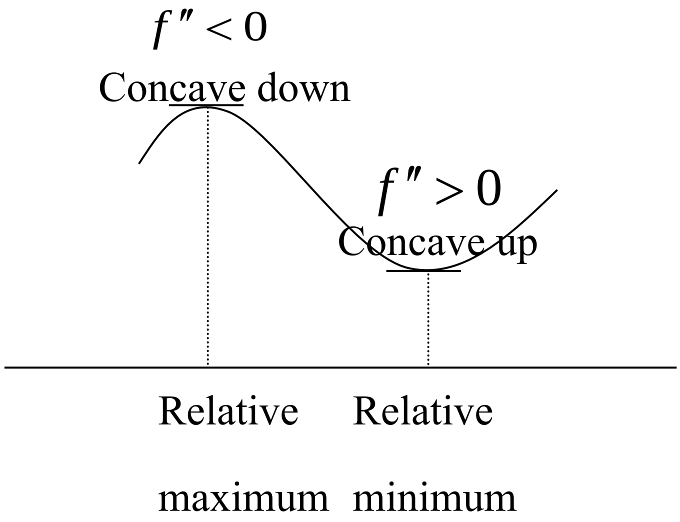


Example 16 Find all relative extrema and absolute extrema of the function $f(x) = 6x - x^2$ on the interval [1,6].

<u>Solution</u>



We may use the second derivative to identify relative max and relative min points of a function via the concavity concepts as follows:



Theorem (The Second Derivative Test for Relative Extremum)

Let f be a differentiable function such that $f''(x_0)$ exists and $f'(x_0) = 0$.

1. If $f''(x_0) > 0$, then f has a relative minimum at x_0.
2. If $f''(x_0) < 0$, then f has a relative maximum at x_0.
3. If $f''(x_0) = 0$, the test fails. We have no conclusions.

   (That is, x_0 may or may not be a relative extreme point.)

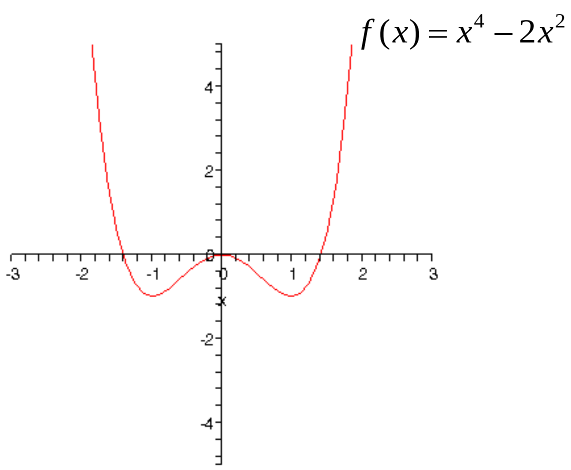
Example 17 Let $f(x) = x^4 - 2x^2$.

(a) Identify the intervals of x for which f is concave up and concave down.

(b) Find all inflection points of f.

(c) and both relative extreme points.

<u>Solution</u>



Example18 Find a and b so that $f(x) = x^3 + ax^2 + bx$ has a relative maximum at $x = -1$ and a relative minimum at $x = 3$.

Solution

7. Sketching a graph of rational function

Let f be a rational function. To sketch the graph of f, we do the following procedure:

1. Find basic properties of function f such as
   a. domain and range
   b. x-intercept and y-intercept
   c. symmetry
   d. asymptotes.
2. Apply the first derivative f' to find
   a. critical points
   b. intervals of x for which f is increasing and decreasing
   c. relative extreme points.
3. Apply the second derivative f'' to find
   a. inflection points
   b. intervals of x for which f is concave up and concave down.

Asymptotes

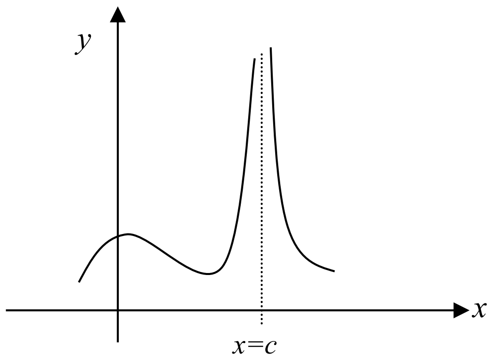
An asymptote is a line which a graph of a function gets arbitrarily close to as x or y or both increases (or decreases) unboudedly. In general, there are 3 types of asymptotes.

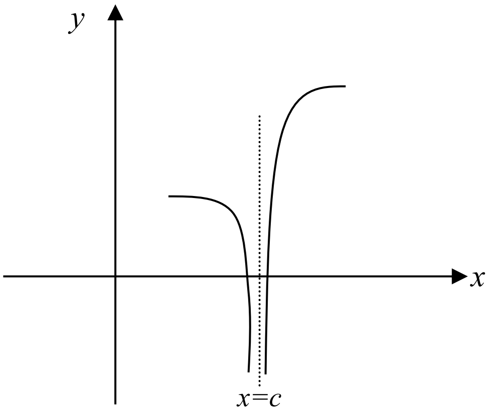
1. **Vertical Asymptote**: $x = c$

A line $x = c$ is called a vertical asymptote of a function f if

$$\lim_{x \to c^-} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to c^+} f(x) = \pm\infty.$$

In case of a rational function, its vertical asymptote can be easily identified by only considering all the points $x = c$ where the function is undefined ($x = c$ which make the denominator of the function equal zero). Then the lines $x = c$ will automatically be its asymptotes.



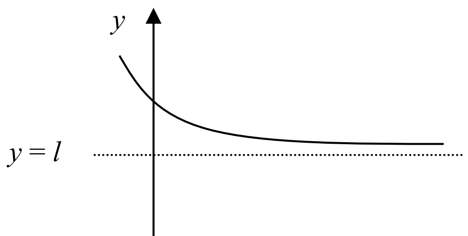


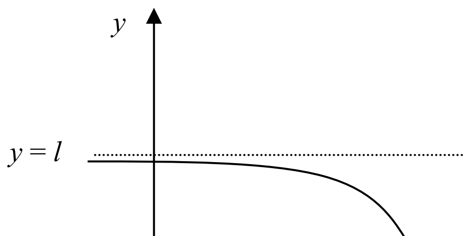
2. **Horizontal Asymptote**: $y = l$

A line $y = l$ is called a horizontal asymptote of a function f if

$$\lim_{x \to +\infty} f(x) = l \quad \text{or} \quad \lim_{x \to -\infty} f(x) = l.$$

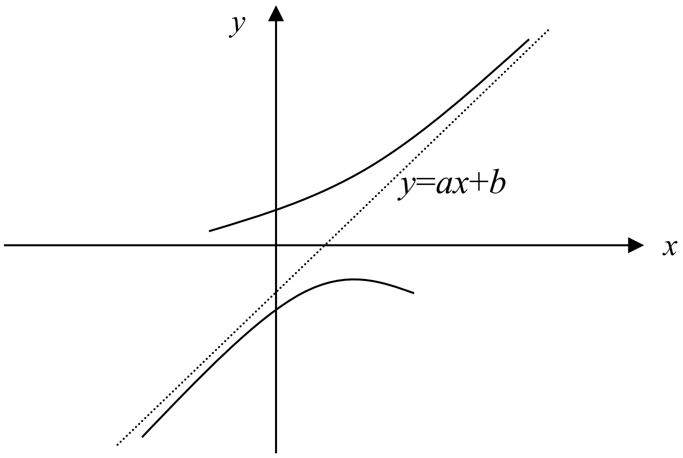
From the above definition, a horizontal asymptote can be found by evaluating the limits of function as $x \to \pm\infty$.





3. **Oblique Asymptote**: $y = ax + b$

A line $y = ax + b$ is called an oblique asymptote of a function f if both $a = \lim_{x\to\infty} \frac{f(x)}{x}$ and $b = \lim_{x\to\infty}[f(x) - ax]$ exist.

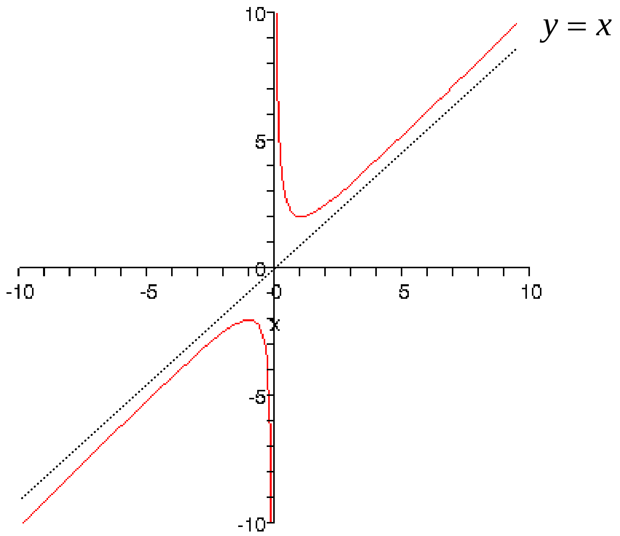


Example19 Find all asymptotes of the following functions.

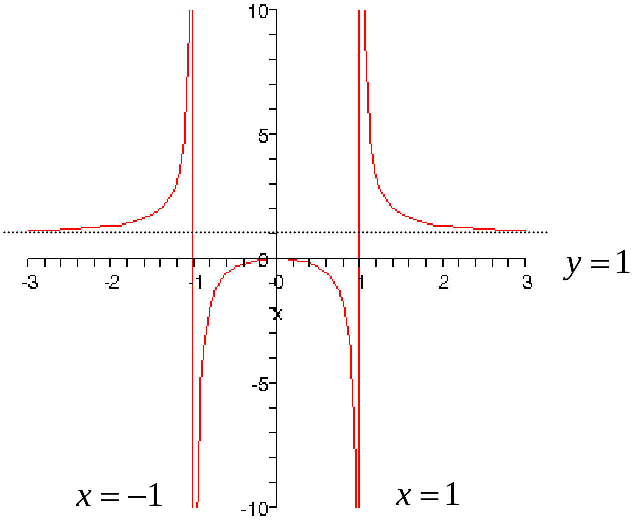
a. $f(x) = \dfrac{x^2+1}{x}$ b. $f(x) = \dfrac{x^2}{x^2-1}$

<u>Solution</u>

a. $f(x) = \dfrac{x^2+1}{x}$



b. $f(x)=\dfrac{x^2}{x^2-1}$

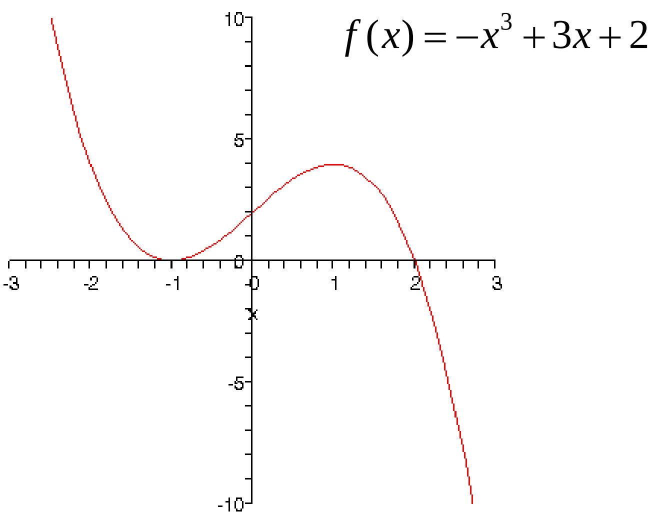


Example20 Analyze and sketch graphs of the following functions.

a. $f(x) = -x^3 + 3x + 2$ b. $f(x) = \dfrac{2x^2 - 8}{x^2 - 16}$

<u>Solution</u>

a.



b. Domain and range: $D_f = \mathbb{R} - \{-4, 4\}$,

$$R_f = \{y \mid y \le \frac{1}{2} \cup y > 2\}$$

x-intercept: $(-2, 0)$ and $(2, 0)$

y-intercept: $(0, \frac{1}{2})$

Symmetry: symmetric about the y-axis

Asymptote:

Vertical asymptote: $x = -4$ and $x = 4$

Horizontal asymptote: $y = 2$

Oblique asymptote: none

$$f'(x) = \frac{4x}{(x^2 - 16)} - \frac{2x(2x^2 - 8)}{(x^2 - 16)^2} = -\frac{48x}{(x^2 - 16)^2}$$

$$f''(x) = -\frac{48}{(x^2 - 16)^2} + \frac{192x^2}{(x^2 - 16)^3}$$

Critical point: $x = 0$

$f'(x)$ (+) (+) (-) (-)

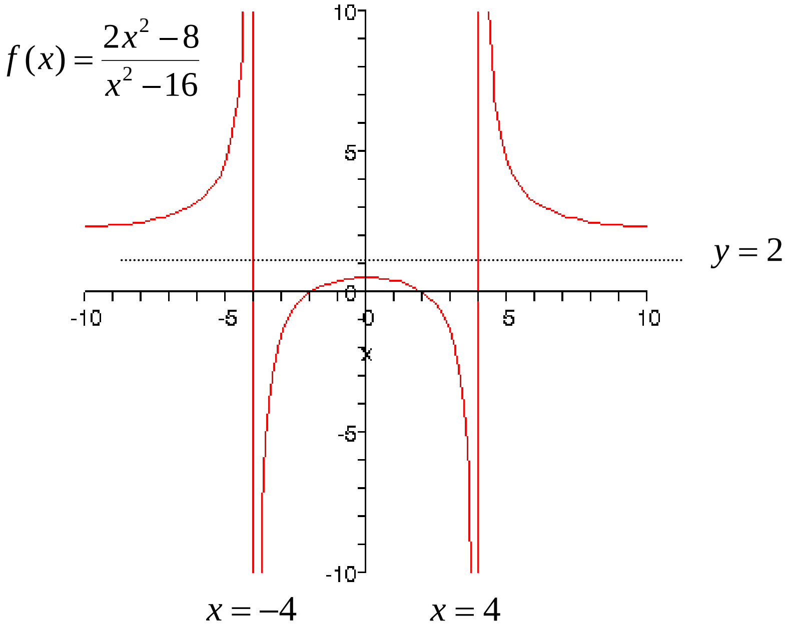
-4 0 4 x

Thus, f is decreasing on $[0,4)\cup(4,\infty)$ and

increasing on $(-\infty,-4)\cup(-4,0]$.

Also, f is concave up on $(-\infty,-4)\cup(4,\infty)$ and

concave down on $(-4,4)$.



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8. Applications of Maxima and Minima

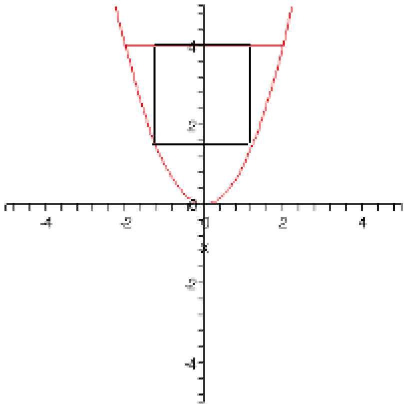
Here is the procedure on how to solve the maximum and minimum problems.

1. Draw a picture (if possible).
2. Define relevant variables.
3. Write down a function we want to optimize.
4. If the function has more than one independent variable, reduce number of variables by applying the conditions given in the problem.
5. Use the derivative to find the extreme points.

Example 21 We want to make a rectangular box with open top from a square paper. Each side of this square paper is 12 cm long. We cut out all of its four corners as shown below. To obtain the highest volume of the box, how long should we cut at each corner?

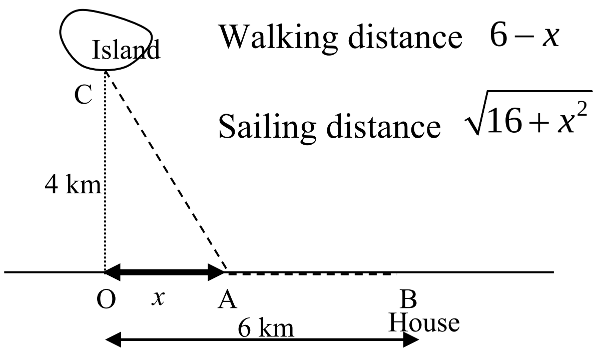
Example 22 Find the dimension of a rectangle which occupies the largest area between a parabola of $y = x^2$ and a line $y = 4$.

<u>Solution</u>



Example 23 A man is on a deserted island which is 4 km vertically far from the seashore. He wants to sail back to his house 6 km away from the given point O as shown below. If he can sail with speed 4 km per hour and walk by 5 km per hour. How should he travel so that he reaches his house fastest?

Solution Let A be a point where the man reaches seashore, and x be the distance from point O to A.



Let T = total travel time

= sailing time + walking time

$$T = \frac{\sqrt{16+x^2}}{4} + \frac{6-x}{5}, \; 0 \le x \le 6$$

9. Related Rates

Related rate is the rate of change of some quantity compared to time. It can be found by computing the derivative with respect to time. The following is the procedure.

1. Draw a picture (if possible).
2. Define relevant variables.
3. Write down a function and a rate of change we want to optimize in terms of time.
4. Find the derivative of the function with respect to time.
5. Evaluate the rate of change by using the given quantities and their rates of changes.

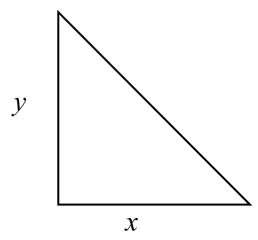
Note: The rate of change has a sign.

If t increases and the value of x also increases, we have $\dfrac{dx}{dt} = +$.

If t increases but the value of x decreases, we then have $\dfrac{dx}{dt} = -$.

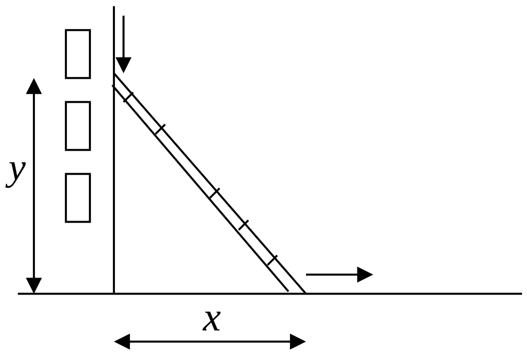
Example 24 Assume that the area of a right triangle is fixed to be 6 square inch. Suppose that it has 4 inches long base at the beginning. If its height increases by 0.5 inches per minute, what is the rate of change of the base of this triangle?

Solution

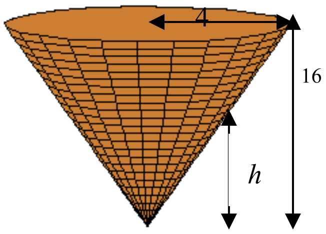


Example 25 A 13 meters long ladder is put up against the wall. Its top end of the ladder is moving down along the wall by 5 meters per minute. This makes the other end of the ladder on the ground moves horizontally away from the wall. Calculate the rate of change of the ground distance between the wall and the ladder when the ladder end is 5 meters away from the wall.

<u>Solution</u>



Example 26 A circular cone has top radius 4 cm. and height 16 cm. Water is pouring into this cone by the rate of 10 cm^3 per minute. Find the rate of change of the water's height in the cone when water is 6 cm high from the bottom.



Let

More Exercises on Applications of Derivatives

1. Locate the intervals of x where each function is increasing and where it is decreasing.

   1.1. $f(x) = 6x^2 - 2x^3 - 3$

   1.2. $f(x) = x^3 - 6x^2 + 9x - 5$

   1.3 $f(x) = \ln(1 + x^2)$

2. Find all the critical points of the following functions.

   2.1 $y = x^3 - 2x^2$

   2.2 $y = x^2 + \frac{2}{x}$

   2.3 $y = \frac{x - 1}{x^2}$

3. Show that these functions have no absolute extreme points.

   3.1 $y = 2x^3 - 9x^2 + 12x$

   3.2 $y = x + \sin x$

4. Locate the intervals of x where each function's graph is concave up and where it is concave down. Identify the inflection points and calculate relative extreme values.

   4.1 $f(x) = x^4 - 4x^3 + 8x - 2$

   4.2 $f(x) = 5 + 12x - x^3$

   4.3 $f(x) = 2x^3 - 9x^2 + 12x$

5. Find all extreme values of the following functions.

5.1 $f(x) = \tan^2 3x$

5.2 $f(x) = 2x^3 + 3x^2 - 72x \ , \ x \in [-10, 5]$

5.3 $f(x) = \dfrac{ax}{x^2 + a^2}$

6. Analyze and sketch a graph of each function.

6.1 $y = x^4 - 4x^3 + 8x - 2$

6.2 $y = \dfrac{8}{4 - x^2}$

6.3 $y = \dfrac{3x^2 - 4x - 4}{x^2}$

7. Find the maximum volume of a cylinder inscribed in a sphere whose radius is r.

8. An area of 14,4000 m^2 is required to construct one 7-Eleven shop in Bangkok. Its floor plan has a rectangular shape. The shop has three brick walls and one glass wall in the front. The cost of the material is calculated by the length. Suppose glass wall costs 1.88 times as much as the brick wall costs. Find the dimension of this shop so that the material cost is minimized.

9. Identify the point on the curve of $xy^2 = 128$ which is closest to the origin.

10. A rectangular bucket has the dimension: width$\times$length$\times$height = $x \times y \times x$ inch^3. It is made of a piece of tin with area 1350 inch^2. Calculate the possible maximum volume of this bucket.

11. A six-foot tall man walks along the road toward the lamp pole with speed 5 feet per second. The lamp is 16 feet above ground. Find the velocity of his shadow's tip and how the shadow's length changes when he is 10 feet away from the lamp pole.

12. Suppose the volume of a symmetric cube increases by 4 cm^3 per second. Find the rate of change of the cube's surface area when the surface area 24 cm^2.

13. A boy is flying a kite 300 feet high above the ground. If the wind pushes the kite away from the boy by horizontal speed of 25 feet per second, then how fast does the boy release the kite's rope when the kite is 500 feet far from him?

14. Two sailors sails two ships from the same position. The first ship starts sailing at noon and sail toward the east by 20 miles per hour. The second one starts sailing at 1 pm and sail toward the south by 25 miles per hour. Find the rate of change of the distance between these two ships at 2 pm.

Answers

1.1 decrease $(-\infty,0]\cup[2,\infty)$ / increase $[0,2]$

1.2 decrease $[1,3]$ / increase $(-\infty,1]\cup[3,\infty)$

1.3 decrease $(-\infty,0]$ / increase $[0,\infty)$

2.1 $(0,0)$, $(\frac{4}{3},-\frac{37}{27})$

2.2 $(1,3)$

2.3 $(2,\frac{1}{4})$

4.1 concave down $(0,2)$ / concave up $(-\infty,0)\cup(2,\infty)$ /

inflection point $(0,-2)$ / relative max 3 / relative min -6

4.2 concave down $(0,\infty)$ / concave up $(-\infty,0)$ /

inflection point $(0,5)$ / relative max 21 / relative min -11

4.3 concave down $(0,\frac{3}{2})$ / concave up $(\frac{3}{2},\infty)$ /

Inflection point $(\frac{3}{2},\frac{9}{2})$ / relative max 5 / relative min 4

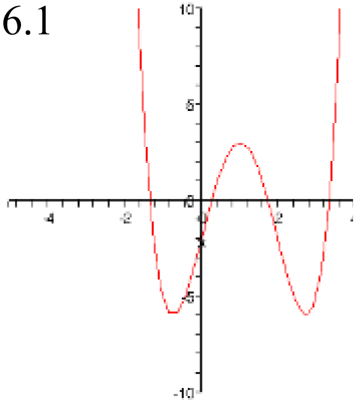
5.1 relative min 0

5.2 absolute max $f(-4)=208$/ relative min $f(3)=-135$

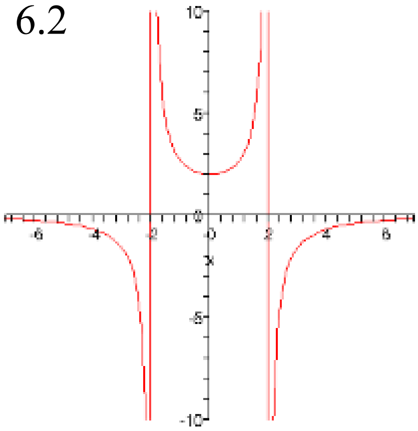
absolute min $f(-10)=-980$

5.3 relative max $\frac{1}{2}$ / relative min $-\frac{1}{2}$

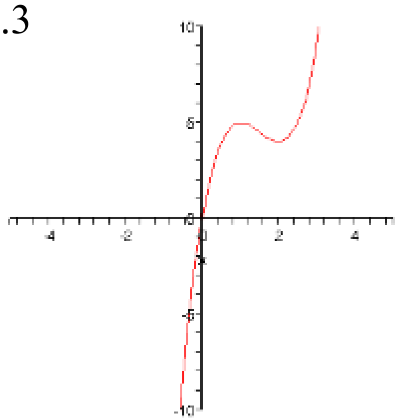
6.1



6.2



6.3



7. $\frac{4}{9}r^3\sqrt{3}$

8. 144 meters wide, 100 meters long

9. $(4, \pm 4\sqrt{2})$

10. 4500 inch^3

11. 8 ft/sec, decrease by 3 ft/sec

12. 8 cm^2/sec

13. 20 ft/sec

14. $\frac{285}{\sqrt{89}}$ mph (miles per hour)