Limit and Continuity of Function

2.1 Limit of function

Let f be a function. The limit of $f(x)$ when x approaches to a is not the value of $f\left(a\right)$ but it is a value that $f(x)$ is approaching to (as x approaches to a). There are two types of the limit.

2.1.1 Limit of function as $x \to a$ (a is a real number.)

Suppose that $f(x) = 5x - 1$ and $g(x) = [\![x]\!]$ defined by the largest integer which is less than or equal to x. For example,

$$g(4) = [\![4]\!] = 4\ ,\ g(3.8) = [\![3.8]\!] = 3,\ g(-1.2) = [\![-1.2]\!] = -2$$

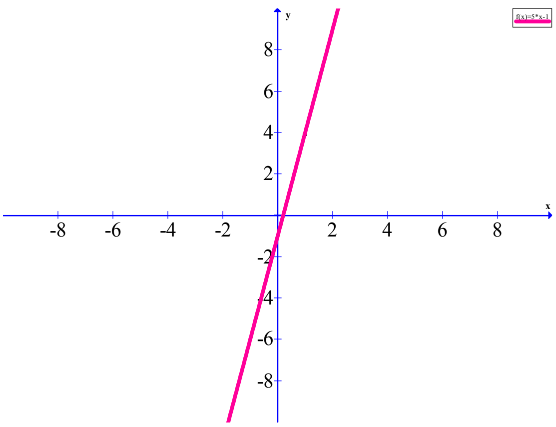
For some values of x which approaches to $a = 1$, the value $f(x)$ and $g(x)$ are shown in Table 1

x	0.5	0.9	0.99	0.999	...	1.001	1.01	1.1
$f(x)$	1.5	3.5	3.95	3.995	...	4.005	4.05	4.5
$g(x)$	0	0	0	0	...	1	1	1

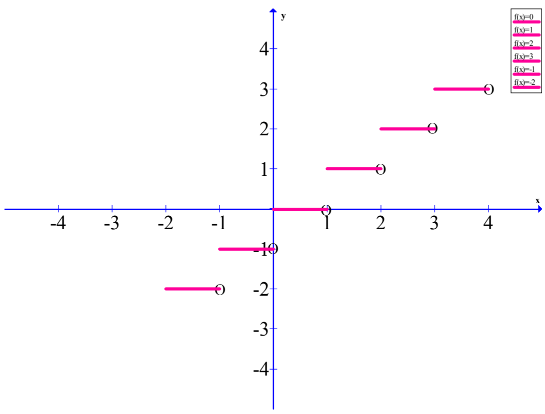
Table 1

We can see that when x approaches to $a = 1$, $f(x)$ gets closer and closer to the value 4. However, $g(x) = 1$ when $x \geq 1$ and $g(x) = 0$ when $x < 1$. Thus $g(x)$ does not approach to one number. Therefore, we say that $f(x)$ has the limit equal to 4 as x approaches to 1 and $g(x)$ does not have a limit when x approaches to 1. We may write them as

$$\lim_{x \to 1} f(x) = 4 \quad \text{and} \quad \lim_{x \to 1} g(x) \text{ does not exist.}$$



$f(x) = 5x - 1$



$g(x) = [\![x]\!]$

The graph of the function f shows that the value of $f(x)$ gets closer to 4 when x approaches to 1. But the graph of the function g jumps from $y = 0$ to $y = 1$ at $x = 1$. Thus $g(x) = [\![x]\!]$ has no limit at $x = 1$.

Using this concept, one can define the limit as follows:

Definition If $f(x)$ gets closer to L when x approaches to a, we say that L is the limit of $f(x)$ when x approaches to a, denoted by $\lim\limits_{x\to a} f(x) = L$.

The values of x approaches to a from two sides:

- x **approaches to** a **from the right side** is denoted by

$x \to a^+$. In this case, we focus on x when $x > a$.

- x **approaches to** a **from the left side** is denoted by

$x \to a^-$. In this case, we focus on x when $x < a$.

From the above example, we have $\lim\limits_{x\to 1^+} [\![x]\!] = 1$ but $\lim\limits_{x\to 1^-} [\![x]\!] = 0$ and $\lim\limits_{x\to 1^+} 5x-1 = \lim\limits_{x\to 1^-} 5x-1 = 4$.

We see that the function f has the same limit from both sides when x approaches to 1 and

(Right limit) $\lim\limits_{x\to a^+} f(x) =$ (Left limit) $\lim\limits_{x\to a^-} f(x) = \lim\limits_{x\to a} f(x)$.

The following theorem guarantees the above remark.

Theorem 1 $\lim\limits_{x\to a} f(x)$ exists and equals to L if

(1) both $\lim\limits_{x\to a^+} f(x)$ and $\lim\limits_{x\to a^-} f(x)$ exist and

(2) $\lim\limits_{x\to a^+} f(x) = \lim\limits_{x\to a^+} f(x) = L$

Example 1 Compare $\lim_{x\to 0} \frac{x}{|x|}$ and $\lim_{x\to 0} \frac{x^2}{|x|}$

Solution

Properties of limits

Let a, k, L and M be real numbers. Suppose that $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$. Then,

1. $\lim_{x\to a} kf(x) = kL$,
2. $\lim_{x\to a} [f(x) \pm g(x)] = L \pm M$,
3. $\lim_{x\to a} f(x)g(x) = LM$,
4. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}, M \neq 0$,
5. If f is a polynomial function, then for any number a

   $\lim_{x\to a} f(x) = f(a)$,
6. $\lim_{x\to a} \sqrt[n]{g(x)} = \sqrt[n]{\lim_{x\to a} g(x)}$ where n is a natural number.

Example 2 Evaluate $\lim\limits_{x\to 0} \dfrac{x^3 - 3x^2 + 4}{\cos x}$

Solution

Example 3 Let f be a function defined by

$$f(x) = \begin{cases} 2x^2 & ,x < 0, \\ x & ,0 \le x < 1, \\ x + 1 & ,x \ge 1. \end{cases}$$

Find the limits of $f(x)$ when x approaches 0 and 1.

Solution

Example 4 Evaluate $\lim_{x \to 9} \left(2x^{\frac{3}{2}} - 9\sqrt{x}\right)^{\frac{1}{3}} \sin 2x.$

Solution

Sometimes, we find the limit by replacing x by a and may get the result in the form of $\frac{0}{0}$. So, we can use these two techniques to find the limit.

1) Factoring
2) Conjugating

Example 5 Calculate $\lim_{x \to 3} \frac{x^3 - x^2 - 9x + 9}{x^2 - x - 6}$.

Solution

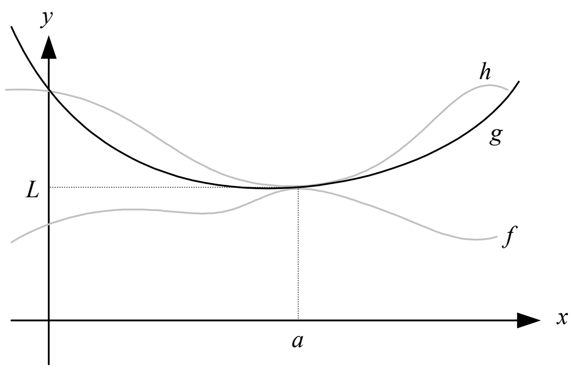
Example 6 Calculate $\lim\limits_{x\to 0^+} \dfrac{2\sqrt{x}}{\sqrt{16+2\sqrt{x}}-4}$

Solution

The following theorem is one of an important theorem that helps us to find the limit. It is typically used to confirm the limit of a function via comparison with two other functions whose limits are known or easily computed.

Squeeze Theorem

If $f(x) \le g(x) \le h(x)$ for all values of x, $x \ne a$ at some points a and $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} h(x) = L$, then $\lim\limits_{x\to a} g(x) = L$



Example 7 Use **the** Squeeze Theorem to show that

$$\lim_{x\to 0} \frac{x^2}{1+\left(1+x^4\right)^{\frac{5}{2}}} = 0.$$

Example 8

1. If $3x \le f(x) \le x^3 + 2$ for $0 \le x \le 2$, evaluate $\lim\limits_{x\to 1} f(x)$
2. Calculate $\lim\limits_{x\to 0} x^2 \sin\dfrac{2}{x}$

Solution

Theorem

1. $\displaystyle\lim_{x\to 0}\frac{\sin x}{x}=1$ 2. $\displaystyle\lim_{x\to 0}\frac{\cos x-1}{x}=0$

Example 9 Use $\displaystyle\lim_{x\to 0}\frac{\sin x}{x} = 1$ to show that $\displaystyle\lim_{x\to 0}\frac{\cos x-1}{x}=0.$

Proof $$\lim_{x\to 0}\frac{\cos x-1}{x}=\lim_{x\to 0}\frac{\cos x-1}{x}\left(\frac{\cos x+1}{\cos x+1}\right)$$

$$=\lim_{x\to 0}\frac{\cos^2 x-1}{x(\cos x+1)} =\lim_{x\to 0}\frac{-\sin^2 x}{x(\cos x+1)}$$

$$=\lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{-\sin x}{\cos x+1}$$

$$=\lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{-\sin x}{\cos x+1}=1\cdot 0=0.$$

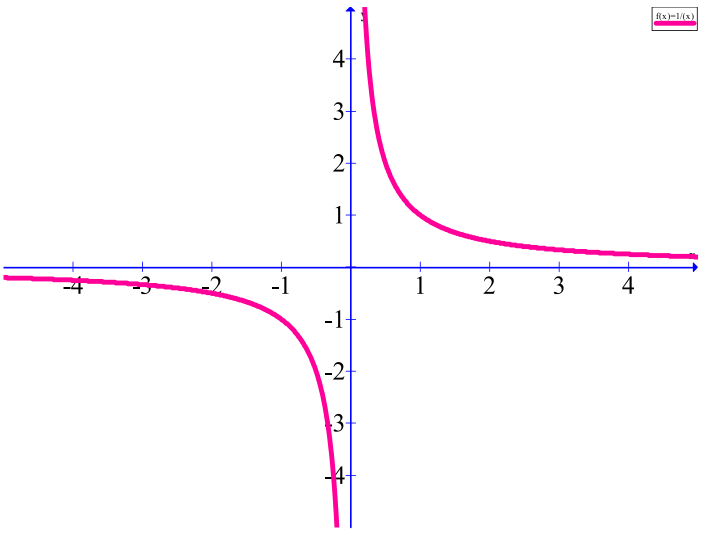
Example 10 Evaluate $\displaystyle\lim_{x\to 1}\frac{\sin(x-1)}{x^2+x-2}.$

Solution

2.1.2 Limit of function as $x \to \infty$ (infinity)

When the domain of a function f is unbounded, the values of $f(x)$ may get closer to one value when x increases unboundedly (written as $x \to +\infty$) or x decreases unboundedly (written as $x \to -\infty$).

Let $f(x) = \frac{1}{x}$. Its graph can be shown here.



Consider the value of $f(x)$ in the following table.

x	100	1000	10000	Increases unboundedly
$f(x) = \frac{1}{x}$	0.01	0.001	0.0001	$\ldots \to 0$
x	-100	-1000	-10000	Decreases unboundedly
$f(x) = \frac{1}{x}$	-0.01	-0.001	-0.0001	$\ldots \to 0$

Table 2

We see that, when $x \to +\infty$, the values of $f(x)$ get closer to 0 and $f(x) > 0$. So, we say that limit of $f(x)$ equals 0 as $x \to +\infty$, denoted by $\lim_{x \to +\infty} \frac{1}{x} = 0$. Also, when $x \to -\infty$, the values of $f(x)$ get closer to 0 as well, but $f(x) < 0$. We say that limit of $f(x)$ equals 0 as $x \to -\infty$ and denote it by $\lim_{x \to -\infty} \frac{1}{x} = 0$.

The above graph shows that $f(x) = \frac{1}{x}$ gets closer to x-axis as x increasing to infinity and decreasing to negative infinity, but it never hit the x-axis. We call a line that the graph gets closer to as an **asymptote** of function.

Properties of infinite limits

Many properties of infinite limits are the same as those of limits at a finite number a.

Let k, L and M be real numbers. Suppose that $\lim_{x \to +\infty} f(x) = L$ and $\lim_{x \to +\infty} g(x) = M$. Then,

1. $\lim_{x \to +\infty} k = k$,
2. $\lim_{x \to +\infty} [f(x) \pm g(x)] = L \pm M$,
3. $\lim_{x \to +\infty} f(x)g(x) = LM$,
4. $\lim_{x \to +\infty} \frac{f(x)}{g(x)} = \frac{L}{M}, M \neq 0,$

5. $\lim\limits_{x\to+\infty} [f(x)]^{\frac{1}{n}} = L^{\frac{1}{n}}$ where n is positive and $L \geq 0$,

6. $\lim\limits_{x\to+\infty} \dfrac{1}{x^n} = 0$ where n is a positive integer.

All 6 properties are the same when we replace $x \to +\infty$ by

$x \to -\infty$

Example 1 Calculate

a) $\lim\limits_{x\to+\infty} \dfrac{5}{x^3}$, b) $\lim\limits_{x\to-\infty} \dfrac{-3}{x^{\frac{2}{3}}}$, c) $\lim\limits_{x\to\infty} \dfrac{4^x - 4^{-x}}{4^x + 4^{-x}}$.

Solution

Example 2 Evaluate $\lim\limits_{x\to+\infty}\dfrac{\sqrt{3x^4+7x^2+6}}{4x^2-3x-6}$.

Solution

Example 3 Evaluate $\lim\limits_{x\to-\infty}\dfrac{\sqrt{x^2+3}}{x+3}$.

Solution

Example 4 Calculate $\lim_{x\to 2^+} \dfrac{x-3}{x-2}$.

Solution

Example 5 Calculate $\lim_{x\to 0^+} (x-1)\ln x$.

Solution

Limit of a function associating with the number e

For any constant a,

$$\lim_{x\to 0}(1+ax)^{1/x} = e^a \quad \text{and} \quad \lim_{x\to\infty}(1+\frac{a}{x})^x = e^a.$$

Example 6 Calculate $\lim_{x\to\infty}\left(\dfrac{x+4}{x+1}\right)^{x+1}$

Solution

2.2 Continuity of Function

Definition Function f is continuous at $x = a$ if all of the three following conditions are satisfied:

1. $f(a)$ exists,
2. $\lim_{x \to a} f(x)$ exists, (That is, $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$.)
3. $\lim_{x \to a} f(x) = f(a)$.

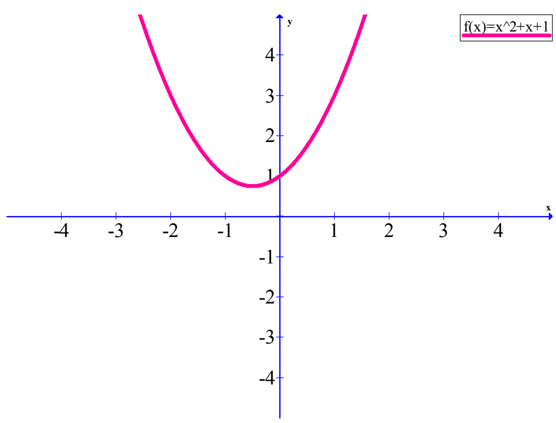
Remark: If at least one of the above conditions is not satisfied, then the given function is discontinuous at $x = a$.

Example 1 Let $f(x) = x^2 + 2x + 1$

Consider the continuity of this function at $x = 0$:

1. $f(0) = 1$ exists,
2. $\lim_{x \to 0} f(x) = 1$ exists, and
3. $\lim_{x \to 0} f(0) = f(0) = 1$.

Thus, $f(x)$ is continuous at $x = 0$. Its graph is here.



Example 2 Let f be a function defined by

$$f(x)=\begin{cases}\dfrac{1-x^2}{1-x} & ,x\neq 1,\\ 3 & ,x=1.\end{cases}$$

Determine if this function is continuous at $x=1$.

Solution

Example 3 Let f be a function defined by

$$f(x)=\begin{cases} bx^2+1 & ,x<-2, \\ x & ,x\geq -2. \end{cases}$$

Find b that makes this function continuous at $x=-2$.

Solution

Three Types of Discontinuities

Consider the continuity of $f(x)$ at $x = a$,

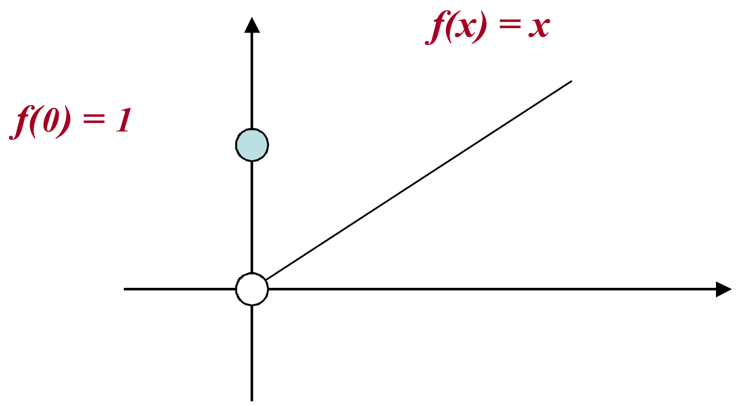
1. Removable discontinuity

It occurs when

(i) $\lim\limits_{x \to a} f(x)$ exists, but not equal to $f(a)$ or

(ii) $f(a)$ is undefined.

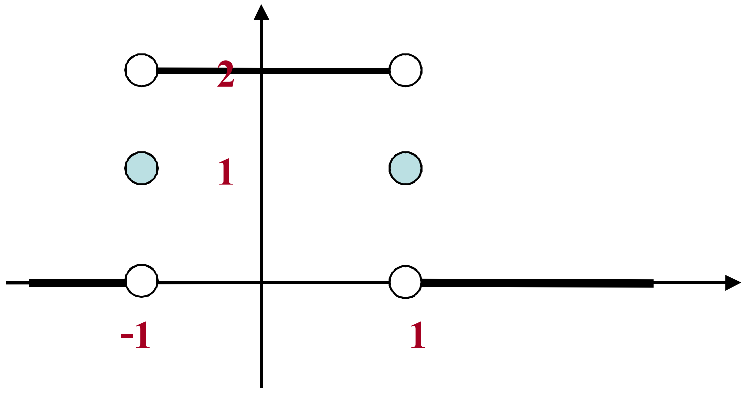
For example, $f(x) = \begin{cases} 1 & , x = 0 \\ x & , x \neq 0 \end{cases}$ has a removable discontinuity at $x = 0$ as show in the Figure below.



2. Jump discontinuity or Ordinary discontinuity

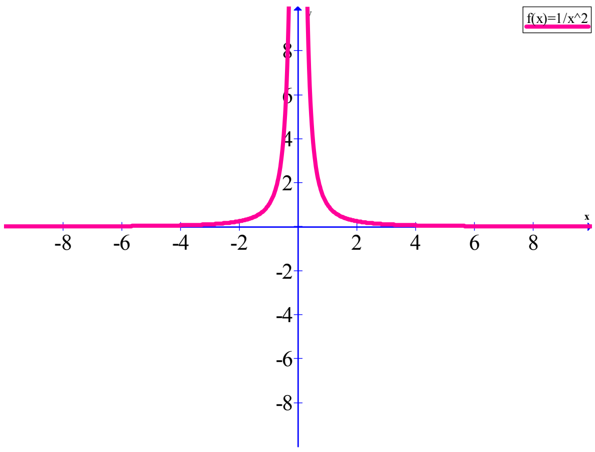
It occurs when $\lim\limits_{x \to a} f(x)$ does not exist due to the **unequal** existence of $\lim\limits_{x \to a^-} f(x)$ and $\lim\limits_{x \to a^+} f(x)$. For example, the function

$f(x) = \begin{cases} 2 & , |x| < 1 \\ 1 & , |x| = 1 \\ 0 & , |x| > 1 \end{cases}$ has a jump discontinuity at $x = 1, -1$.



3. Infinite discontinuity

It occurs when at least one of the left limit or the right limit does not exist. For example, $f(x)=\dfrac{1}{x^2}$ has an infinite discontinuity at $x=0$ as shown here.



Algebraic properties of functions on the continuity

1. If f and g are continuous at $x=a$, then $f \pm g$, $f.g$, $\dfrac{f}{g}$ ($g(a)\neq 0$) and kf (k is a constant) are also continuous at $x=a$.

2. If f is continuous at $x=b$ and $\lim\limits_{x\to a} g(x)=b$, then $\lim\limits_{x\to a}(f\circ g)(x)=f(b)$.

3. If g is continuous at $x=a$ and f is continuous at $g(a)$, then the composite function $f\circ g$ is continuous at $x=a$.

Example 4 Let f be a function defined by

$$f(x)=\frac{2(x^2+4x+2)}{(x^2-9)(x-1)}.$$

Locate where this function is continuous.

Definition Let f be a function. If f is continuous everywhere in the interval (a,b), we say that f is continuous on (a,b).

Definition A function f is continuous in $[a,b]$ where $a<b$ if

1. $f(x)$ is continuous on (a,b),
2. $\lim\limits_{x\to a^+} f(x)=f(a)$ and
3. $\lim\limits_{x\to b^-} f(x)=f(b)$

Example 5 Let g be a function defined by $g(x)=\sqrt{\dfrac{3-x}{4+x}}$.

Locate where this function is continuous.

Solution

Limit and Continuity Exercises

1. Find the limits of the following functions.

(a) $f(x)=\dfrac{x^3}{|x-1|}$ Find $\lim\limits_{x\to 1} f(x)$

(b) $\lim\limits_{x\to 1} 3x[\![x]\!]$

(c) $g(x)=\begin{cases} x^2-2; & x>0 \\ -2-x; & x<0 \end{cases}$ Calculate $\lim\limits_{x\to 0} g(x)$

(d) $\lim\limits_{x\to\infty}\dfrac{6\sqrt{x^2-3}}{2x-1}$

(e) $\lim\limits_{x\to-\infty}\dfrac{\sqrt{x^2+7}}{2x-4}$

2. Make the following functions continuous at $x=a$

(a) $f(x)=\dfrac{\sqrt{3x^2}}{2|x|}$, $a=0$

(b) $g(x)=\dfrac{x^n-1}{x-1},\quad n\in\mathbb{Z}^+,\quad a=1$

3. Locate domain that makes the following function continuous

(a) $h(x)=\dfrac{2}{x^2+3x-28}$

(b) $k(x)=\sqrt[3]{(x-a)(x-b)}$

4. Find k that makes $f(x)=\begin{cases} \dfrac{x^2-5x+6}{x-2}; & x\neq 2 \\ kx-3; & x=2 \end{cases}$ continuous

everywhere.

5. Find k that makes each following limit exists

(a) $\lim\limits_{x\to 1}\dfrac{x^2-kx+4}{x-1}$

(b) $\lim\limits_{x\to \infty}\dfrac{x^4+3x-5}{2x^2-1+x^k}$

(c) $\lim\limits_{x\to -\infty}\dfrac{e^{2x}-5}{e^{kx}+4}$

6. Compute the following limits

(a) $\lim\limits_{h\to 0}\dfrac{(3+h)^2-9}{h}$

(b) $\lim\limits_{h\to 0}\dfrac{1/(1+h)-1}{h}$

(c) $\lim\limits_{h\to 0}\dfrac{\sqrt{4+h}-2}{h}$

7. Compute the following limits

(a) $\lim\limits_{x\to 0}\dfrac{\cos x-1}{\sin x}$

(b) $\lim\limits_{x\to 0}\dfrac{\sin 3x}{\sin 5x}$

(c) $\lim\limits_{x\to \infty} x\sin\dfrac{\pi}{x}$

Answers to limit and continuity exercises

1. (a) $+\infty$

   (b) Does not exist

   (c) -2

   (d) 3

   (e) $-1/2$

2. (a) add $f(0)=\dfrac{\sqrt{3}}{2}$

   (b) add $g(1)=n$

3. (a) $x\neq -7,4$

   (b) $(-\infty,\infty)$

4. 1

5. (a) 5

   (b) greater than or equal to 4

   (c) less than or equal to 2

6. (a) 6

   (b) –1

   (c) –1/16

7. (a) 0

   (b) 3/5

   (c) π