IMPROPER INTEGRALS

Introduction

Previously, we learned about the definite integral $\int_a^b f(x)\,dx$ where a and b are constants when the integrand f is continuous and bounded on the interval $[a,b]$.

In this chapter, we are interested in $\int_a^b f(x)\,dx$ where the limits of integration a and b may be infinity or the integrand function is unbounded at some points in the domain of integration. This type of integral is called an "***Improper Integral***". There are three cases:

Case 1 The limit of integration is infinite (Infinite interval).

$$[a,+\infty), \qquad (-\infty,b] \qquad \text{or} \qquad (-\infty,+\infty)$$

For examples,

$$\int_1^{+\infty} \frac{dx}{x^2}, \qquad \int_{-\infty}^{0} e^x\,dx \qquad \text{and} \qquad \int_{-\infty}^{+\infty} \frac{dx}{1+x^2}$$

Case 2 The integrand f is unbounded at some point $x=c$ in $[a,b]$. That is,

$$\lim_{x\to c} f(x) = \pm\infty .$$

For examples,

$$\int_{-3}^{3} \frac{dx}{x^2}, \qquad \int_1^2 \frac{dx}{x-1} \qquad \text{and} \qquad \int_0^{\pi} \tan x\,dx$$

Case 3 The combination of both case 1 and case 2.

For examples,

$$\int_0^{+\infty} \frac{dx}{\sqrt{x}}, \qquad \int_{-\infty}^{+\infty} \frac{dx}{x^2-9} \qquad \text{and} \qquad \int_1^{+\infty} \sec x\,dx$$

1. Evaluation of improper integral of case 1

Definition Let a be a real number and f a bounded and integrable function on $[a,t]$ for all t such that $t > a$. The improper integral of f on $\left[a, +\infty\right)$, denoted by $\int_a^{+\infty} f(x)\,dx$, is defined by

$$\int_a^{+\infty} f(x)\,dx = \lim_{t\to+\infty} \int_a^t f(x)\,dx.$$

Remark

- If $\lim_{t\to+\infty} \int_a^t f(x)\,dx$ exists, then $\int_a^{+\infty} f(x)\,dx$ converges.
- If $\lim_{t\to+\infty} \int_a^t f(x)\,dx$ does not exist, then $\int_a^{+\infty} f(x)\,dx$ diverges.

Definition Let b be a real number and f a bounded and integrable function on $[t,b]$ for all t such that $t < b$. The improper integral of f on $\left(-\infty, b\right]$, denoted by $\int_{-\infty}^{b} f(x)\,dx$, is defined by

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx.$$

Remark

- If $\lim_{t\to-\infty} \int_t^b f(x)\,dx$ exists, then $\int_{-\infty}^{b} f(x)\,dx$ converges.
- If $\lim_{t\to-\infty} \int_t^b f(x)\,dx$ does not exist, then $\int_{-\infty}^{b} f(x)\,dx$ diverges.

Definition Let f be a bounded and integrable function. The improper integral of f on $\left(-\infty, +\infty\right)$, denoted by $\int_{-\infty}^{+\infty} f(x)\,dx$, is defined by

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{+\infty} f(x)\,dx \quad \text{where } c \in \mathbb{R}.$$

Remark

- $\int_{-\infty}^{+\infty} f(x)\,dx$ converges if both integrals on the right converge.
- $\int_{-\infty}^{+\infty} f(x)\,dx$ diverges if at least one integral on the right diverges.

Example Evaluate $\int_1^{+\infty} \frac{1}{x^3}\,dx$.

Solution

$$\int_1^{+\infty} \frac{1}{x^3}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{1}{x^3}\,dx$$

$$= \lim_{t\to\infty}\left[-\frac{1}{2x^2}\right]_1^t$$

$$= \lim_{t\to\infty}\left(\frac{1}{2}-\frac{1}{2t^2}\right) = \frac{1}{2}.$$

Thus, we conclude that the given integral converges to $\frac{1}{2}$. ■

Example Evaluate $\int_1^{+\infty} \frac{1}{x}\,dx$.

Solution

Example Identify p such that $\int_1^{+\infty} \frac{1}{x^p}\,dx$ converges or diverges.

Solution $$\int_1^{+\infty} \frac{1}{x^p}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{1}{x^p}\,dx$$

$$= \lim_{t\to\infty}\left[\frac{x^{1-p}}{1-p}\right]_1^t$$

$$= \lim_{t\to\infty}\left(\frac{t^{1-p}}{1-p} - \frac{1}{1-p}\right).$$

If $p>1$, then $1-p<0$ and $t^{1-p}\to 0$ as $t\to+\infty$.

Thus, $$\lim_{t\to\infty}\left(\frac{t^{1-p}}{1-p} - \frac{1}{1-p}\right) = \frac{1}{p-1}.$$

If $p<1$, then $1-p>0$ and $t^{1-p}\to+\infty$ as $t\to+\infty$.

Thus, $$\lim_{t\to\infty}\left(\frac{t^{1-p}}{1-p} - \frac{1}{1-p}\right) = +\infty.$$

If $p=1$, then $\int_1^{+\infty} \frac{1}{x}\,dx$ diverges. From all three cases, we have

$$\int_1^{+\infty} \frac{1}{x^p}\,dx = \begin{cases} \dfrac{1}{p-1} & , \ p>1, \\ \text{diverge} & , \ p\le 1. \end{cases}$$

■

Example Determine if the following improper integrals converges or diverges.

1) $\displaystyle\int_{1}^{+\infty} \frac{1}{x^{2/3}}\,dx$

2) $\displaystyle\int_{1}^{+\infty} \frac{1}{x^{5}}\,dx$

3) $\displaystyle\int_{1}^{+\infty} \frac{1}{x^{3/2}}\,dx$

Answer 1) $\displaystyle\int_{1}^{+\infty} \frac{1}{x^{2/3}}\,dx$ diverges.

2) $\displaystyle\int_{1}^{+\infty} \frac{1}{x^{5}}\,dx$ converges to $\displaystyle\frac{1}{5-1}=\frac{1}{4}$.

3) $\displaystyle\int_{1}^{+\infty} \frac{1}{x^{3/2}}\,dx$ converges to $\displaystyle\frac{1}{(3/2)-1}=2$. ■

Example Evaluate $\displaystyle\int_{0}^{+\infty} (1-x)e^{-x}\,dx$.

Solution

Example Determine if $\int_{-\infty}^{1} xe^{-x^2}\,dx$ diverges or converges to which value.

Solution

$$\begin{aligned}
\int_{-\infty}^{1} xe^{-x^2}\,dx &= \lim_{t\to-\infty}\int_{t}^{1} xe^{-x^2}\,dx \\
&= \lim_{t\to-\infty}\left[\frac{-e^{-x^2}}{2}\right]_t^1 \\
&= \lim_{t\to-\infty}\left[-\frac{1}{2e}+\frac{e^{-t^2}}{2}\right] \\
&= -\frac{1}{2e}+0 \\
&= -\frac{1}{2e}.
\end{aligned}$$

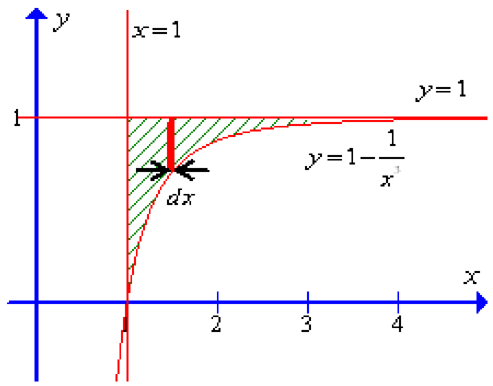
Hence, $\int_{-\infty}^{1} xe^{-x^2}\,dx$ converges to $-\frac{1}{2e}$. ■

Example Evaluate $\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\,dx$.

Solution

Example Find the area bounded by curve $y = 1 - \frac{1}{x^3}$, line $x = 1$ and line $y = 1$.

Solution Area A as shown below can be computed by



$$
\begin{aligned}
A &= \int_1^\infty (y_2 - y_1)dx \\
&= \int_1^\infty \left[1 - \left(1 - \frac{1}{x^3}\right)\right] dx = \int_1^\infty \frac{1}{x^3} dx \\
&= \lim_{t\to\infty} \int_1^t \frac{1}{x^3} dx \\
&= \lim_{t\to\infty} \left[-\frac{1}{2x^2}\right]_1^t \\
&= \lim_{t\to\infty} \left[-\frac{1}{2t^2} + \frac{1}{2}\right] \\
A &= 1/2 \quad \text{unit}^2.
\end{aligned}
$$

■

2. Evaluation of improper integral of case 2

Definition Let a and b be real numbers such that $a < b$. Suppose that f is a bounded and integrable function on $[t,b]$ for all t such that $a < t < b$, but f goes to infinity at $x = a$, i.e.,

$$\lim_{x\to a^+} f(x) = \pm\infty.$$

The improper integral of f on $[a,b]$ is defined by

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx.$$

Remark If the limit exists, then the integral $\int_a^b f(x)\,dx$ converges. Otherwise, $\int_a^b f(x)\,dx$ diverges.

Definition Let a and b be real numbers such that $a<b$. Suppose that f is a bounded and integrable function on $[a,t]$ for all t such that $a<t<b$, but f goes to infinity at $x=b$, i.e.,

$$\lim_{x\to b^-} f(x) = \pm\infty.$$

The improper integral of f on $[a,b]$ is defined by

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx.$$

Remark If the limit exists, then $\int_a^b f(x)\,dx$ converges. Otherwise, $\int_a^b f(x)\,dx$ diverges.

Definition Let a and b be real numbers such that $a<b$. Suppose that f is a bounded and integrable function on $[a,b]$, but f goes to infinity at $x=c$ in (a,b). Tthe improper integral of f on $[a,b]$ is defined by

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Remark

- $\int_a^b f(x)\,dx$ converges if both integrals on the right converge.
- $\int_a^b f(x)\,dx$ diverges if at least one integral on the right diverges.

Example Evaluate $\int_0^1 \frac{1}{\sqrt{x}}\,dx$.

Solution
$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^+} \int_t^1 \frac{1}{\sqrt{x}}\,dx$$

$$= \lim_{t\to 0^+} \left[2\sqrt{x}\right]_t^1$$

$$= \lim_{t\to 0^+} \left(2-2\sqrt{t}\right) = 2.$$

Hence, $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ converges to 2. ■

Example Evaluate $\displaystyle\int_1^2 \frac{dx}{1-x}$.

Solution

$$\int_1^2 \frac{dx}{1-x} = \lim_{t\to 1^+} \int_t^2 \frac{dx}{1-x}$$

$$= \lim_{t\to 1^+} \left[-\ln|1-x|\right]_t^2$$

$$= \lim_{t\to 1^+} \left(-\ln|-1| + \ln|1-t|\right)$$

$$= 0 + \lim_{t\to 1^+} \ln|1-t| = -\infty.$$

Hence, $\displaystyle\int_1^2 \frac{dx}{1-x}$ diverges. ■

Example Evaluate $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x}}$.

Solution

Example Determine if $\displaystyle\int_1^4 \frac{dx}{(x-2)^{2/3}}$ converges or diverges.

Solution

3. Evaluation of improper integral of case 3

Example Evaluate $\int_0^\infty \frac{dx}{\sqrt{x}(x+1)}$.

Solution

$$\begin{aligned}
\int_0^\infty \frac{dx}{\sqrt{x}(x+1)} &= \int_0^1 \frac{dx}{\sqrt{x}(x+1)} + \int_1^\infty \frac{dx}{\sqrt{x}(x+1)} \\
&= \lim_{t\to 0^+} \int_t^1 \frac{dx}{\sqrt{x}(x+1)} + \lim_{t\to\infty} \int_1^t \frac{dx}{\sqrt{x}(x+1)} \\
&= \lim_{t\to 0^+} \left[2\tan^{-1}\sqrt{x}\right]_t^1 + \lim_{t\to\infty} \left[2\tan^{-1}\sqrt{x}\right]_1^t \\
&= 2\left[\frac{\pi}{4} - 0\right] + 2\left[\frac{\pi}{2} - \frac{\pi}{4}\right] \\
&= \pi.
\end{aligned}$$

Hence, $\int_0^\infty \frac{dx}{\sqrt{x}(x+1)}$ converges to π. ■

Exercise Separate the following integrals into several parts according to their improperity.

1) $\int_{-3}^\infty \frac{dx}{x+2}$

Solution

2) $$\int_{-\infty}^{0} \frac{dx}{(x+3)^2}$$

Solution

3) $$\int_{-\infty}^{+\infty} \frac{dx}{x^3}$$

Solution

Exercise 7.1

1. Determine if the following improper integral converges or diverges and find its value.

1.1 $\int_{2}^{\infty} \frac{1}{(x+1)^2}\,dx$ **1.2** $\int_{0}^{\infty} \cos x\,dx$

1.3 $\int_{1}^{\infty} \frac{\ln x}{x}\,dx$ **1.4** $\int_{e}^{\infty} \frac{1}{x\ln^3 x}\,dx$

1.5 $\int_{0}^{\infty} \frac{1}{1+2^x}\,dx$ **1.6** $\int_{-1}^{\infty} \frac{x}{1+x^2}\,dx$

1.7 $\int_{2}^{\infty} \frac{1}{x^2+4}\,dx$ **1.8** $\int_{0}^{\infty} \frac{1}{\sqrt{e^x}}\,dx$

1.9 $\int_{0}^{\infty} xe^{-x}\,dx$ **1.10** $\int_{0}^{\infty} e^{-x}\cos x\,dx$

1.11 $\int_{0}^{\infty} \frac{1}{e^{2x}+e^x}\,dx$ **1.12** $\int_{1}^{\infty} \frac{1}{\sqrt{x}(1+e^{\sqrt{x}})^2}\,dx$

1.13 $\int_{-\infty}^{1} \frac{1}{3-2x}\,dx$ **1.14** $\int_{-\infty}^{0} e^{3x}\,dx$

1.15 $\int_{-\infty}^{-1} \frac{x}{\sqrt{1+x^2}}\,dx$ **1.16** $\int_{-\infty}^{0} \frac{1}{(1-x)^{5/2}}\,dx$

1.17 $\int_{-\infty}^{0} \frac{e^x}{3-2e^x}\,dx$ **1.18** $\int_{-\infty}^{0} \frac{1}{(x-8)^{2/3}}\,dx$

1.19 $\int_{-\infty}^{0} \frac{1}{2x^2+2x+1}\,dx$ **1.20** $\int_{-\infty}^{\infty} \frac{|x+1|}{x^2+1}\,dx$

1.21 $\int_{-\infty}^{\infty} \frac{x^2}{x^2+1}\,dx$ **1.22** $\int_{-\infty}^{\infty} \frac{x}{(x^2+3)^2}\,dx$

1.23 $\int_{-\infty}^{\infty} \frac{1}{e^x+e^{-x}}\,dx$ **1.24** $\int_{-\infty}^{\infty} xe^{-x^2}\,dx$

2. Find value of a such that $\int_{0}^{\infty} e^{-ax}\,dx = 5$.

3. Show that $\int_{1}^{\infty} \frac{1}{x^p}\,dx$ converges if $p>1$ and diverges if $p\le 1$.

4. Find the area between the curve $y=\frac{8}{x^2-4}$ and x-axis where $x\ge 3$.

5. Find the area between the curves $y=\frac{1}{x}$ and $y=\frac{1}{x^2}$ where $x\in[1,\infty)$.

6. Let $R=\{(x,y)\,|\,x\geq 4 \text{ and } 0\leq y\leq x^{-3/2}\}$. Find the area of region R.

7. Let R be the region between the curve $y=\dfrac{4}{x^2+1}$ and x-axis where $x\geq 0$. Find the area of region R.

Answer 7.1

1

1.1	$\frac{1}{3}$	**1.2**	diverges
1.3	diverges	**1.4**	$\frac{1}{2}$
1.5	1	**1.6**	diverges
1.7	$\frac{\pi}{8}$	**1.8**	2
1.9	1	**1.10**	$\frac{1}{2}$
1.11	$1-\ln 2$	**1.12**	$2\left(\ln(1+e)-1-\frac{1}{1+e}\right)$
1.13	diverges	**1.14**	$\frac{1}{3}$
1.15	diverges	**1.16**	$\frac{2}{3}$
1.17	$\frac{1}{2}\ln 3$	**1.18**	diverges
1.19	$\frac{3\pi}{4}$	**1.20**	diverges
1.21	diverges	**1.22**	0
1.23	$\frac{\pi}{2}$	**1.24**	0

2 $\frac{1}{5}$

4 $2\ln 5$

5 undefined

6 1

7 2π

Exercise 7.2

1 Determine if the following improper integral converges or diverges and find its value.

1.1 $\int_0^9 \frac{1}{\sqrt{x}}\,dx$ **1.2** $\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$

1.3 $\int_3^4 \frac{1}{(x-3)^2}\,dx$ **1.4** $\int_0^4 \frac{1}{(4-x)^{3/2}}\,dx$

1.5 $\int_1^2 \frac{x}{\sqrt{x-1}}\,dx$ **1.6** $\int_0^1 x\ln x\,dx$

1.7 $\int_0^{\pi/6} \frac{\cos x}{\sqrt{1-2\sin x}}\,dx$ **1.8** $\int_0^{\pi/2} \sec^2 x\,dx$

1.9 $\int_0^2 \frac{2x+1}{x^2+x-6}\,dx$ **1.10** $\int_0^1 \ln x\,dx$

1.11 $\int_0^4 \frac{\ln\sqrt{x}}{\sqrt{x}}\,dx$ **1.12** $\int_0^1 \frac{1}{\sqrt{1-\sqrt{x}}}\,dx$

1.13 $\int_2^4 \frac{x}{\sqrt[3]{x-2}}\,dx$ **1.14** $\int_0^2 \frac{x}{(x^2-1)^2}\,dx$

1.15 $\int_{-1}^8 \frac{1}{\sqrt[3]{x}}\,dx$ **1.16** $\int_{-2}^7 \frac{1}{(x+1)^{2/3}}\,dx$

1.17 $\int_{-1}^1 \frac{1}{\sqrt{|x|}}\,dx$ **1.18** $\int_2^4 \frac{1}{(x-3)^7}\,dx$

1.19 $\int_0^3 \frac{1}{x^2+2x-3}\,dx$ **1.20** $\int_1^3 \frac{x}{(x^2-4)^3}\,dx$

1.21 $\int_{-1}^2 \frac{1}{x^2}\cos\frac{1}{x}\,dx$ **1.22** $\int_0^2 \frac{1}{\sqrt{2x-x^2}}\,dx$

1.23 $\int_{-1}^2 \frac{1}{x^2-x-2}\,dx$ **1.24** $\int_0^1 \frac{1}{x(\ln x)^{1/5}}\,dx$

2 Show that $\int_0^1 \frac{1}{x^p}\,dx$ converges if $p<1$ and diverges if $p\ge 1$.

3 Find the area between the curve $y=\frac{1}{(1-x)^2}$ and x-axis where $x\in[0,4]$.

4 Find the area of region R when R is given as follows.

4.1 $R=\{(x,y)\,|\,-4\le x\le 4 \text{ and } 0\le y\le 1/(x+4)\}$,

4.2 $R=\{(x,y)\,|\,0\le x\le 1 \text{ and } 0\le y\le 1/\sqrt{x}\}$.

5 Find the area between the curves $y=\dfrac{1}{x}$ and $y=\dfrac{1}{x(x^2+1)}$ where $x\in[0,1]$.

Answer 7.2

1

1.1	6	**1.2**	$\dfrac{\pi}{2}$
1.3	diverges	**1.4**	diverges
1.5	$\dfrac{8}{3}$	**1.6**	$-\dfrac{1}{4}$
1.7	1	**1.8**	diverges
1.9	diverges	**1.10**	-1
1.11	$4(\ln 2-1)$	**1.12**	$\dfrac{8}{3}$
1.13	$\dfrac{21\sqrt[3]{4}}{5}$	**1.14**	diverges
1.15	$\dfrac{9}{2}$	**1.16**	3
1.17	4	**1.18**	diverges
1.19	diverges	**1.20**	diverges
1.21	diverges	**1.22**	π
1.23	diverges	**1.24**	diverges

3 Undefined

4 **4.1** DNE **4.1** 2

5 $\dfrac{1}{2}\ln 2$

Exercise 7.3

1. Determine if the following improper integral converges or diverges and find its value.

1.1 $\int_0^{\infty} \frac{1}{(x-1)^{2/3}}dx$

1.2 $\int_1^{\infty} \frac{1}{x\sqrt{x^2-1}}dx$

1.3 $\int_{-1}^{\infty} \frac{1}{x^2-1}dx$

1.4 $\int_1^{\infty} \frac{1}{x\ln x}dx$

1.5 $\int_0^{\infty} x^{-0.1}dx$

1.6 $\int_0^{\infty} \frac{1}{\sqrt{x}(x+4)}dx$

1.7 $\int_1^{\infty} \frac{1}{x^2-6x+8}dx$

1.8 $\int_0^{\infty} \frac{\sqrt{x}}{1-\sqrt{x}}dx$

1.9 $\int_2^{\infty} \frac{1}{(x+7)\sqrt{x-2}}dx$

1.10 $\int_{-\infty}^{\infty} \frac{1}{x^2+2x+1}dx$

1.11 $\int_{-\infty}^{\infty} \frac{1}{x^2-3x+2}dx$

1.12 $\int_{-\infty}^{\infty} \frac{e^x}{e^x-1}dx$

Answer 7.3

1.1 diverges

1.2 $\frac{\pi}{2}$

1.3 diverges

1.4 diverges

1.5 diverges

1.6 $\frac{\pi}{2}$

1.7 diverges

1.8 diverges

1.9 $\frac{\pi}{3}$

1.10 diverges

1.11 diverges

1.12 diverges

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