Module # 1.

Limit and Continuity of Function

2.1 Limit of function

Let f be a function. The limit of $f(x)$ when x approaches to a is not the value of $f\left(a\right)$ but it is a value that $f(x)$ is approaching to (as x approaches to a). There are two types of the limit.

2.1.1 Limit of function as $x \to a$ (a is a **real number.)**

Suppose that $f(x) = 5x - 1$ and $g(x) = [\![x]\!]$ defined by the largest integer which is less than or equal to x. For example,

$$g(4) = [\![4]\!] = 4\ ,\ g(3.8) = [\![3.8]\!] = 3,\ g(-1.2) = [\![-1.2]\!] = -2$$

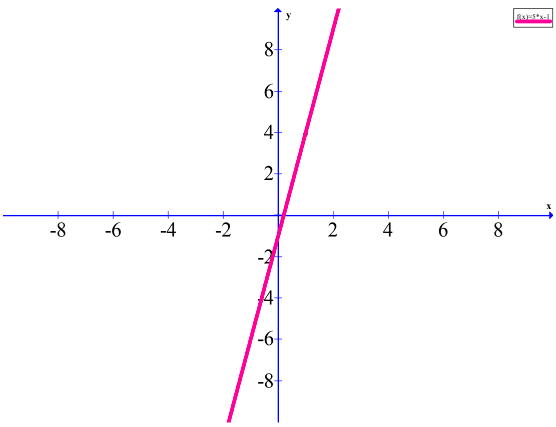
For some values of x which approaches to $a = 1$, the value $f(x)$ and $g(x)$ are shown in Table 1

x	0.5	0.9	0.99	0.999	...	1.001	1.01	1.1
$f(x)$	1.5	3.5	3.95	3.995	...	4.005	4.05	4.5
$g(x)$	0	0	0	0	...	1	1	1

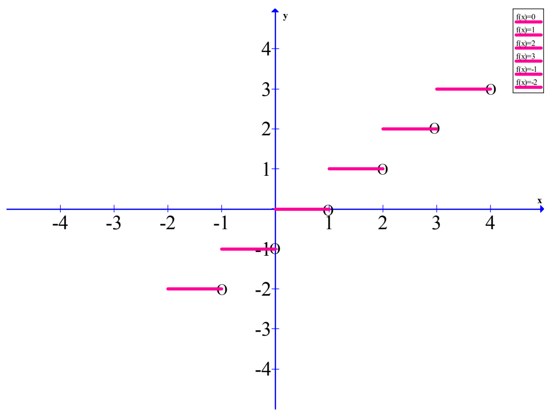
Table 1

We can see that when x approaches to $a=1$, $f(x)$ gets closer and closer to the value 4. However, $g(x)=1$ when $x \geq 1$ and $g(x)=0$ when $x<1$. Thus $g(x)$ does not approach to one number. Therefore, we say that $f(x)$ has the limit equal to 4 as x approaches to 1 and $g(x)$ does not have a limit when x approaches to 1. We may write them as

$$\lim_{x \to 1} f(x) = 4 \quad \text{and} \quad \lim_{x \to 1} g(x) \text{ does not exist.}$$



$f(x)=5x-1$



$g(x)=[\![x]\!]$

The graph of the function f shows that the value of $f(x)$ gets closer to 4 when x approaches to 1. But the graph of the function g jumps from $y=0$ to $y=1$ at $x=1$. Thus $g(x)=[\![x]\!]$ has no limit at $x=1$.

Using this concept, one can define the limit as follows:

Definition If $f(x)$ gets closer to L when x approaches to a, we say that L is the limit of $f(x)$ when x approaches to a, denoted by $\lim\limits_{x\to a} f(x) = L$.

The values of x approaches to a from two sides:

- x **approaches to** a **from the right side** is denoted by

$x \to a^{+}$. In this case, we focus on x when $x > a$.

- x **approaches to** a **from the left side** is denoted by

$x \to a^{-}$. In this case, we focus on x when $x < a$.

From the above example, we have $\lim\limits_{x\to 1^{+}} \llbracket x \rrbracket = 1$ but $\lim\limits_{x\to 1^{-}} \llbracket x \rrbracket = 0$ and $\lim\limits_{x\to 1^{+}} 5x-1 = \lim\limits_{x\to 1^{-}} 5x-1 = 4$.

We see that the function f has the same limit from both sides when x approaches to 1 and

(Right limit) $\lim\limits_{x\to a^{+}} f(x) =$ (Left limit) $\lim\limits_{x\to a^{-}} f(x) = \lim\limits_{x\to a} f(x)$.

The following theorem guarantees the above remark.

Theorem 1 $\lim\limits_{x\to a} f(x)$ exists and equals to L if

(1) both $\lim\limits_{x\to a^{+}} f(x)$ and $\lim\limits_{x\to a^{-}} f(x)$ exist and

(2) $\lim\limits_{x\to a^{+}} f(x) = \lim\limits_{x\to a^{+}} f(x) = L$

Example 1 Compare $\lim\limits_{x\to 0}\dfrac{x}{|x|}$ and $\lim\limits_{x\to 0}\dfrac{x^2}{|x|}$

Solution

Sol $|x| = \begin{cases} x\ , & x \geq 0 \quad \text{right of zero} \\ -x\ , & x < 0 \quad \text{left of zero} \end{cases}$

1.1 $\lim\limits_{x\to 0^-}\dfrac{x}{|x|} = \lim\limits_{x\to 0^-}\dfrac{x}{-x} = \lim\limits_{x\to 0^-}(-1) = -1$

$\lim\limits_{x\to 0^+}\dfrac{x}{|x|} = \lim\limits_{x\to 0^+}\dfrac{x}{x} = \lim\limits_{x\to 0^+}(1) = 1$

$\therefore \lim\limits_{x\to 0}\dfrac{x}{|x|}$ does not exist. #

1.2 $\lim\limits_{x\to 0^-}\dfrac{x^2}{|x|} = \lim\limits_{x\to 0^-}\left(\dfrac{x^2}{-x}\right) = \lim\limits_{x\to 0^-}(-x) = 0$

$\lim\limits_{x\to 0^+}\dfrac{x^2}{|x|} = \lim\limits_{x\to 0^+}\left(\dfrac{x^2}{x}\right) = \lim\limits_{x\to 0^+}(x) = 0$

$\therefore \lim\limits_{x\to 0}\dfrac{x^2}{|x|}$ exists. #

Properties of limits

Let a, k, L and M be real numbers. Suppose that $\lim\limits_{x\to a} f(x) = L$ and $\lim\limits_{x\to a} g(x) = M$. Then,

1. $\lim\limits_{x\to a} kf(x) = kL$,
2. $\lim\limits_{x\to a} [f(x) \pm g(x)] = L \pm M$,
3. $\lim\limits_{x\to a} f(x)g(x) = LM$,
4. $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}, M \neq 0$,
5. If f is a polynomial function, then for any number a

   $\lim\limits_{x\to a} f(x) = f(a)$,
6. $\lim\limits_{x\to a} \sqrt[n]{g(x)} = \sqrt[n]{\lim\limits_{x\to a} g(x)}$ where n is a natural number.

Example 2 Evaluate $\lim\limits_{x\to 0}\dfrac{x^3-3x^2+4}{\cos x}$

Solution

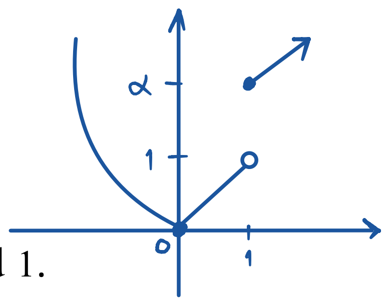
Sol $\lim\limits_{x\to 0}\dfrac{x^3-3x^2+4}{\cos x} = \dfrac{0^3-3(0)^2+4}{\cos(0)}$

$= \dfrac{4}{1}$

$= 4$ #

Example 3 Let f be a function defined by

$$f(x)=\begin{cases}2x^2 & ,x<0,\\ x & ,0\le x<1,\\ x+1 & ,x\ge 1.\end{cases}$$



Find the limits of $f(x)$ when x approaches 0 and 1.

Solution

$\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} 2x^2 = 0$

$\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} x = 0$

$\therefore \lim\limits_{x\to 0} f(x) = 0$ [exists]

$\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} x = 1$

$\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} (x+1) = 2$

$\therefore \lim\limits_{x\to 1} f(x)$ does not exist

Example 4 Evaluate $\lim_{x\to 9}\left(2x^{\frac{3}{2}}-9\sqrt{x}\right)^{\frac{1}{3}}\sin 2x.$

Solution

$$\underline{\text{Sol}}\quad \left(\lim_{x\to 9}\left(2x^{3/2}-9\sqrt{x}\right)\right)^{\frac{1}{3}}\lim_{x\to 9}(\sin 2x)$$

$$=\left(2(9)^{3/2}-9\sqrt{9}\right)^{\frac{1}{3}}\sin 18$$

$$=(54-27)^{\frac{1}{3}}\sin 18 = 3\sin(18)$$

Sometimes, we find the limit by replacing x by a and may get the result in the form of $\frac{0}{0}$. So, we can use these two techniques to find the limit.

1) Factoring
2) Conjugating

Example 5 Calculate $\lim_{x\to 3}\dfrac{x^3-x^2-9x+9}{x^2-x-6}.$

Solution $\lim_{x\to 3}\dfrac{x^3-x^2-9x+9}{x^2-x-6} = \lim_{x\to 3}\dfrac{x^2(x-1)-9(x-1)}{(x-3)(x+2)}$

$$=\lim_{x\to 3}\frac{(x-1)(x^2-9)}{(x-3)(x+2)}$$

$$=\lim_{x\to 3}\frac{(x-1)(x+3)\cancel{(x-3)}}{\cancel{(x-3)}(x+2)} = \frac{12}{5} \quad \#$$

Example 6 Calculate $\lim_{x\to 0^+} \dfrac{2\sqrt{x}}{\sqrt{16+2\sqrt{x}}-4}$

Solution $\lim_{x\to 0^+} \dfrac{2\sqrt{x}}{\sqrt{16+2\sqrt{x}}-4} \times \dfrac{\sqrt{16+2\sqrt{x}}+4}{\sqrt{16+2\sqrt{x}}+4}$

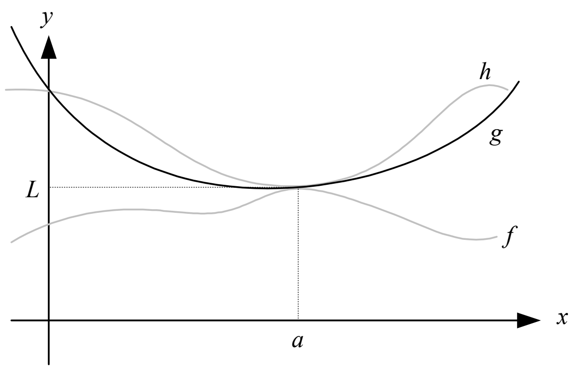
$= \lim_{x\to 0^+} \dfrac{(2\sqrt{x})(\sqrt{16+2\sqrt{x}}+4)}{16+2\sqrt{x}-16}$

$= \sqrt{16}+4 = 4+4 = 8$ #

The following theorem is one of an important theorem that helps us to find the limit. It is typically used to confirm the limit of a function via comparison with two other functions whose limits are known or easily computed.

Squeeze Theorem

If $f(x) \le g(x) \le h(x)$ for all values of x, $x \ne a$ at some points a and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$



Example 7 Use **the** Squeeze Theorem to show that

$$\lim_{x\to 0} \frac{x^2}{1+\left(1+x^4\right)^{\frac{5}{2}}} = 0.$$

Sol. $0 \leqslant \dfrac{x^2}{1+(1+x^4)^{\frac{5}{2}}} \leqslant x^2$

$\lim\limits_{x\to 0} 0 = \lim\limits_{x\to 0} x^2 = 0$

$\Bigg\}$ ทำให้ $\dfrac{x^2}{1+(1+x^4)^{5/2}} = 0$ โดยอ้างทฤษฎี Squeeze Theorem.

Example 8

1. If $3x \le f(x) \le x^3 + 2$ for $0 \le x \le 2$, evaluate $\lim\limits_{x\to 1} f(x)$
2. Calculate $\lim\limits_{x\to 0} x^2 \sin\dfrac{2}{x}$

Solution

1) $\lim\limits_{x\to 1} 3x = 3$

$\lim\limits_{x\to 1} (x^3+2) = 3$

$\Big\}$ สามารถด่วนสรุปโดยการใช้ Squeeze Theorem

$\therefore \lim\limits_{x\to 1} f(x) = 3$ #

2) $-1 \leqslant \sin\left(\frac{2}{x}\right) \leqslant 1.$

$-x^2 \leqslant x^2 \sin\left(\frac{2}{x}\right) \leqslant x^2$

$\lim\limits_{x\to 0} (-x^2) = \lim\limits_{x\to 0} x^2 = 0$

$\therefore \lim\limits_{x\to 0} x^2 \sin\left(\frac{2}{x}\right) = 0$; by squeeze theorem.

Theorem

1. $\lim_{x\to 0}\dfrac{\sin x}{x}=1$ 2. $\lim_{x\to 0}\dfrac{\cos x-1}{x}=0$

Example 9 Use $\lim_{x\to 0}\dfrac{\sin x}{x}=1$ to show that $\lim_{x\to 0}\dfrac{\cos x-1}{x}=0$.

Proof
$$\lim_{x\to 0}\frac{\cos x-1}{x}=\lim_{x\to 0}\frac{\cos x-1}{x}\left(\frac{\cos x+1}{\cos x+1}\right)$$
$$=\lim_{x\to 0}\frac{\cos^2 x-1}{x(\cos x+1)}=\lim_{x\to 0}\frac{-\sin^2 x}{x(\cos x+1)}$$
$$=\lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{-\sin x}{\cos x+1}$$
$$=\lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{-\sin x}{\cos x+1}=1\cdot 0=0.$$

Example 10 Evaluate $\lim_{x\to 1}\dfrac{\sin(x-1)}{x^2+x-2}$.

Solution

Sol'n
$$\lim_{x\to 1}\frac{\sin(x-1)}{x^2+x-2}=\lim_{x\to 1}\frac{\sin(x-1)}{(x-1)(x+2)}$$
$$=\lim_{x\to 1}\frac{\sin(x-1)}{x-1}\cdot\lim_{x\to 1}\frac{1}{(x+2)}$$

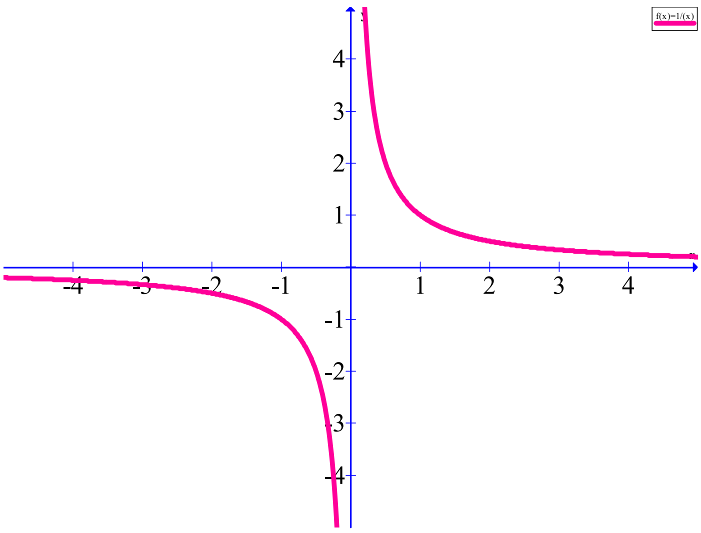
Let $y=x-1$; $x=1\Rightarrow y=0$

$$=\lim_{y\to 0}\frac{\sin y}{y}\cdot\lim_{x\to 1}\frac{1}{(x+2)}$$
$$=\sin 0^{\circ}\cdot\frac{1}{3}$$
$$=1\cdot\frac{1}{3}=\frac{1}{3}\ \#$$

2.1.2 Limit of function as $x \to \infty$ (infinity)

When the domain of a function f is unbounded, the values of $f(x)$ may get closer to one value when x increases unboundedly (written as $x \to +\infty$) or x decreases unboundedly (written as $x \to -\infty$).

Let $f(x) = \dfrac{1}{x}$. Its graph can be shown here.



Consider the value of $f(x)$ in the following table.

x	100	1000	10000	Increases unboundedly
$f(x) = \dfrac{1}{x}$	0.01	0.001	0.0001	$\ldots \to 0$
x	-100	-1000	-10000	Decreases unboundedly
$f(x) = \dfrac{1}{x}$	-0.01	-0.001	-0.0001	$\ldots \to 0$

Table 2

We see that, when $x \to +\infty$, the values of $f(x)$ get closer to 0 and $f(x) > 0$. So, we say that limit of $f(x)$ equals 0 as $x \to +\infty$, denoted by $\lim\limits_{x \to +\infty} \frac{1}{x} = 0$. Also, when $x \to -\infty$, the values of $f(x)$ get closer to 0 as well, but $f(x) < 0$. We say that limit of $f(x)$ equals 0 as $x \to -\infty$ and denote it by $\lim\limits_{x \to -\infty} \frac{1}{x} = 0$.

The above graph shows that $f(x) = \frac{1}{x}$ gets closer to x-axis as x increasing to infinity and decreasing to negative infinity, but it never hit the x-axis. We call a line that the graph gets closer to as an **asymptote** of function.

Properties of infinite limits

Many properties of infinite limits are the same as those of limits at a finite number a.

Let k, L and M be real numbers. Suppose that $\lim\limits_{x \to +\infty} f(x) = L$ and $\lim\limits_{x \to +\infty} g(x) = M$. Then,

1. $\lim\limits_{x \to +\infty} k = k$,
2. $\lim\limits_{x \to +\infty} [f(x) \pm g(x)] = L \pm M$,
3. $\lim\limits_{x \to +\infty} f(x)g(x) = LM$,
4. $\lim\limits_{x \to +\infty} \frac{f(x)}{g(x)} = \frac{L}{M}, M \neq 0,$

5. $\lim\limits_{x\to+\infty} [f(x)]^{\frac{1}{n}} = L^{\frac{1}{n}}$ where n is positive and $L \geq 0$,

6. $\lim\limits_{x\to+\infty} \dfrac{1}{x^n} = 0$ where n is a positive integer.

All 6 properties are the same when we replace $x \to +\infty$ by $x \to -\infty$

Example 1 Calculate

a) $\lim\limits_{x\to+\infty} \dfrac{5}{x^3}$, b) $\lim\limits_{x\to-\infty} \dfrac{-3}{x^{\frac{2}{3}}}$, c) $\lim\limits_{x\to\infty} \dfrac{4^x - 4^{-x}}{4^x + 4^{-x}}$.

Solution

a) $\lim\limits_{x\to+\infty} \dfrac{5}{x^3} = 5 \lim\limits_{x\to+\infty} \dfrac{1}{x^3} = 5 \cdot 0 = 0$ #

b) $\lim\limits_{x\to-\infty} \dfrac{-3}{x^{2/3}} = -3 \lim\limits_{x\to-\infty} \dfrac{1}{x^{2/3}} = -3 \cdot 0 = 0$ #

c) $\lim\limits_{x\to\infty} \dfrac{4^x - 4^{-x}}{4^x + 4^{-x}} = \lim\limits_{x\to\infty} \dfrac{(4^x - \frac{1}{4^x})}{(4^x + \frac{1}{4^x})}$

$= \lim\limits_{x\to\infty} \dfrac{\cancel{4^x}(1 - \frac{1}{4^{2x}})}{\cancel{4^x}(1 + \frac{1}{4^{2x}})} = \dfrac{1 - \lim\limits_{x\to\infty} \frac{1}{4^{2x}}}{1 + \lim\limits_{x\to\infty} \frac{1}{4^{2x}}} = \dfrac{1-0}{1+0} = 1$ #

Example 2 Evaluate $\displaystyle\lim_{x\to+\infty}\frac{\sqrt{3x^4+7x^2+6}}{4x^2-3x-6}$.

Solution

$$\lim_{x\to+\infty}\frac{\sqrt{x^4(3+7/x^2+6/x^4)}}{x^2(4-\frac{3}{x}-\frac{6}{x^2})}=\lim_{x\to+\infty}\frac{\cancel{|x^2|}\sqrt{3+7/x^2+\frac{6}{x^4}}}{\cancel{x^2}(4-3/x-6/x^2)}$$

$$=\frac{\sqrt{3}}{4}\ \#$$

Example 3 Evaluate $\displaystyle\lim_{x\to-\infty}\frac{\sqrt{x^2+3}}{x+3}$.

Solution

$$\lim_{x\to-\infty}\frac{\sqrt{x^2(1+3/x^2)}}{x(1+3/x)}=\lim_{x\to-\infty}\frac{\sqrt{x^2}\cdot\sqrt{1+3/x^2}}{x(1+\frac{3}{x})}$$

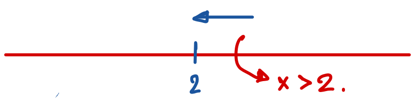
$$=\lim_{x\to-\infty}\frac{|x|\cdot\sqrt{1+3/x^2}}{x(1+\frac{3}{x})}$$

$$=\lim_{x\to-\infty}\frac{\overset{-1}{\cancel{(-x)}}\cdot\sqrt{1+3/x^2}}{\cancel{x}(1+\frac{3}{x})}$$

$$=-1\ \#$$

$-\frac{1}{0} \to \infty,\ -\infty,\ \text{D.N.E}$

Example 4 Calculate $\lim\limits_{x\to 2^+} \dfrac{x-3}{x-2}$. $\begin{matrix}-\\+\end{matrix}$



Solution

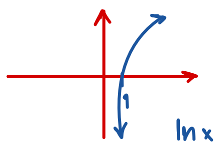
$\lim\limits_{x\to 2^+} x-3 = -1 < 0$

$\lim\limits_{x\to 2^+} x-2 = 0$ where $x-2>0$

$\therefore \lim\limits_{x\to 2^+} \dfrac{x-3}{x-2} = -\infty$

Example 5 Calculate $\lim\limits_{x\to 0^+} (x-1)\ln x$.

Solution $\lim\limits_{x\to 0^+} (x-1) = -1 < 0$



$\lim\limits_{x\to 0^+} \ln x = -\infty$

$\ln x \to -\infty$ as $x \to 0^+$

$\therefore \lim\limits_{x\to 0^+} (x-1)\ln = +\infty$

Limit of a function associating with the number e

For any constant a,

$$\lim_{x\to 0}(1+ax)^{1/x} = e^a \quad \text{and} \quad \lim_{x\to\infty}(1+\frac{a}{x})^x = e^a.$$

Example 6 Calculate $\lim\limits_{x\to\infty}\left(\dfrac{x+4}{x+1}\right)^{x+1}$

Solution $\underset{x\to\infty}{\text{limit}} \left(\frac{x+4}{x+1}\right)^{x+1} = \lim\limits_{x\to\infty}\left(\frac{x+4}{x+1}\right)\left(\frac{x+4}{x+1}\right)^x$

$= 1 \cdot \lim\limits_{x\to\infty}\left(\frac{x+4}{x+1}\right)^x$

$= \lim\limits_{x\to\infty}\left(\frac{\cancel{x}(1+\frac{4}{x})}{\cancel{x}(1+\frac{1}{x})}\right)^x$

$= \lim\limits_{x\to\infty}\frac{(1+\frac{4}{x})^x}{(1+\frac{1}{x})^x} = \frac{e^4}{e} = e^3$

2.2 Continuity of Function

Definition Function f is continuous at $x = a$ if all of the three following conditions are satisfied:

1. $f(a)$ exists,
2. $\lim_{x\to a} f(x)$ exists, (That is, $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$.)
3. $\lim_{x\to a} f(x) = f(a)$.

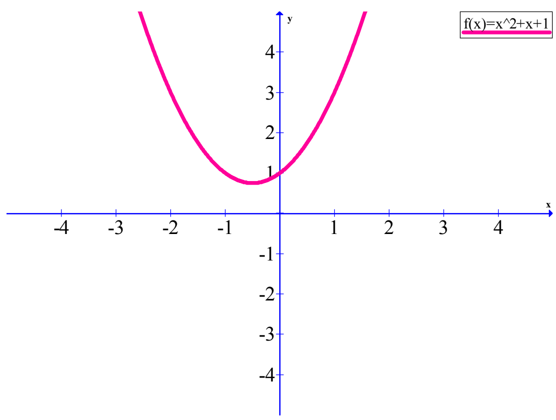
Remark: If at least one of the above conditions is not satisfied, then the given function is discontinuous at $x = a$.

Example 1 Let $f(x) = x^2 + 2x + 1$

Consider the continuity of this function at $x = 0$:

1. $f(0) = 1$ exists,
2. $\lim_{x\to 0} f(x) = 1$ exists, and
3. $\lim_{x\to 0} f(0) = f(0) = 1$.

Thus, $f(x)$ is continuous at $x = 0$. Its graph is here.



Example 2 Let f be a function defined by

$$f(x)=\begin{cases}\dfrac{1-x^2}{1-x} & ,x\neq 1,\\ 3 & ,x=1.\end{cases}$$

Determine if this function is continuous at $x=1$.

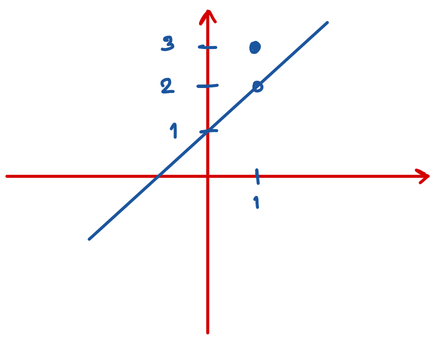
Solution

1) $f(1) = 3$ [exists]

2) $\lim\limits_{x\to 1} f(x) = \lim\limits_{x\to 1} \dfrac{1-x^2}{1-x} = \lim\limits_{x\to 1} \dfrac{(1-\cancel{x})(1+x)}{(1-\cancel{x})} = 2$ (exists)

3) $f(1) \neq \lim\limits_{x\to 1} f(x)$

$\therefore$ f is discontinuous at $x=1$



Example 3 Let f be a function defined by

$$f(x)=\begin{cases} bx^2+1 & ,x<-2, \\ x & ,x\geq -2. \end{cases}$$

Find b that makes this function continuous at $x=-2$.

Solution

Since f is continuous at $x=-2$

$$\lim_{x\to -2^-} f(x) = \lim_{x\to -2^+} f(x)$$

$$\lim_{x\to -2^-} bx^2+1 = \lim_{x\to -2^+} x$$

$$b(-2)^2+1 = -2$$

$$4b+1 = -2$$

$$b = \frac{-3}{4} \quad \#$$

Three Types of Discontinuities

Consider the continuity of $f(x)$ at $x = a$,

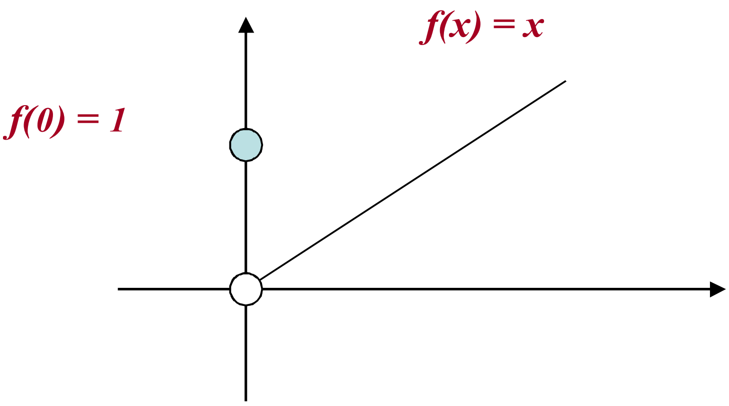
1. Removable discontinuity

It occurs when

(i) $\lim\limits_{x \to a} f(x)$ exists, but not equal to $f(a)$ or

(ii) $f(a)$ is undefined.

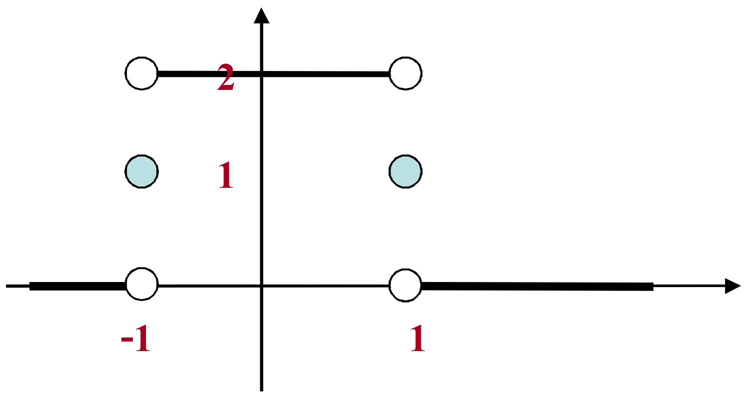
For example, $f(x) = \begin{cases} 1 & , x = 0 \\ x & , x \neq 0 \end{cases}$ has a removable discontinuity at $x = 0$ as show in the Figure below.



2. Jump discontinuity or Ordinary discontinuity

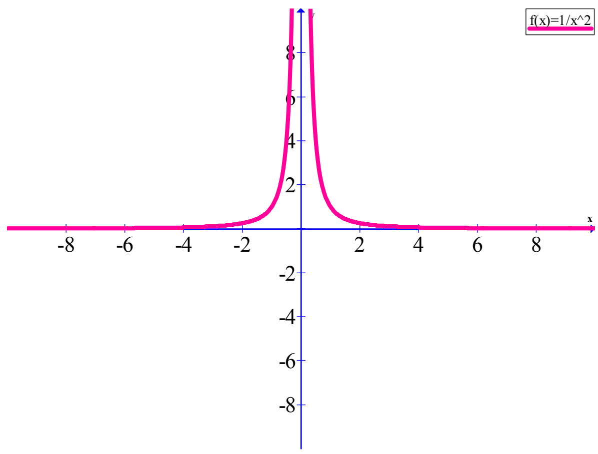
It occurs when $\lim\limits_{x \to a} f(x)$ does not exist due to the **unequal** existence of $\lim\limits_{x \to a^-} f(x)$ and $\lim\limits_{x \to a^+} f(x)$. For example, the function

$f(x) = \begin{cases} 2 & , |x| < 1 \\ 1 & , |x| = 1 \\ 0 & , |x| > 1 \end{cases}$ has a jump discontinuity at $x = 1, -1$.



3. Infinite discontinuity

It occurs when at least one of the left limit or the right limit does not exist. For example, $f(x)=\dfrac{1}{x^2}$ has an infinite discontinuity at $x=0$ as shown here.



Algebraic properties of functions on the continuity

1. If f and g are continuous at $x=a$, then $f \pm g$, $f.g$, $\dfrac{f}{g}$ ($g(a)\neq 0$) and kf (k is a constant) are also continuous at $x=a$.

2. If f is continuous at $x=b$ and $\lim\limits_{x\to a} g(x)=b$, then $\lim\limits_{x\to a}(f\circ g)(x)=f(b)$.

3. If g is continuous at $x=a$ and f is continuous at $g(a)$, then the composite function $f\circ g$ is continuous at $x=a$.

Example 4 Let f be a function defined by

$$f(x) = \frac{2(x^2+4x+2)}{(x^2-9)(x-1)}.$$

Locate where this function is continuous.

Everywhere except when

$x^2 - 9 = 0$ and $x - 1 = 0$

$x = \pm 3$ $x = 1.$ —— ans. #

Definition Let f be a function. If f is continuous everywhere in the interval (a,b), we say that f is continuous on (a,b).

Definition A function f is continuous in $[a,b]$ where $a < b$ if

1. $f(x)$ is continuous on (a,b),
2. $\lim_{x \to a^+} f(x) = f(a)$ and
3. $\lim_{x \to b^-} f(x) = f(b)$

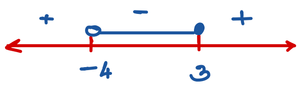
Example 5 Let g be a function defined by $g(x)=\sqrt{\dfrac{3-x}{4+x}}$.

Locate where this function is continuous.

Solution

$$\frac{3-x}{4+x} \geqslant 0 \quad \text{and} \quad x \neq -4$$

$$(4+x)\left(\frac{3-x}{4-x}\right) \geqslant 0$$



$$(4+x)(x-3) \leqslant 0$$

$(-4, 3]$ #

Limit and Continuity Exercises

1. Find the limits of the following functions.

(a) $f(x)=\dfrac{x^3}{|x-1|}$ Find $\lim\limits_{x\to 1} f(x)$

(b) $\lim\limits_{x\to 1} 3x[\![x]\!]$

(c) $g(x)=\begin{cases} x^2-2; & x>0 \\ -2-x; & x<0 \end{cases}$ Calculate $\lim\limits_{x\to 0} g(x)$

(d) $\lim\limits_{x\to\infty}\dfrac{6\sqrt{x^2-3}}{2x-1}$

(e) $\lim\limits_{x\to-\infty}\dfrac{\sqrt{x^2+7}}{2x-4}$

2. Make the following functions continuous at $x=a$

(a) $f(x)=\dfrac{\sqrt{3x^2}}{2|x|}$, $a=0$

(b) $g(x)=\dfrac{x^n-1}{x-1}$, $n\in\mathbb{Z}^+$, $a=1$

3. Locate domain that makes the following function continuous

(a) $h(x)=\dfrac{2}{x^2+3x-28}$

(b) $k(x)=\sqrt[3]{(x-a)(x-b)}$

4. Find k that makes $f(x)=\begin{cases} \dfrac{x^2-5x+6}{x-2}; & x\neq 2 \\ kx-3; & x=2 \end{cases}$ continuous everywhere.

5. Find k that makes each following limit exists

(a) $\lim_{x\to 1}\dfrac{x^2-kx+4}{x-1}$

(b) $\lim_{x\to\infty}\dfrac{x^4+3x-5}{2x^2-1+x^k}$

(c) $\lim_{x\to-\infty}\dfrac{e^{2x}-5}{e^{kx}+4}$

6. Compute the following limits

(a) $\lim_{h\to 0}\dfrac{(3+h)^2-9}{h}$

(b) $\lim_{h\to 0}\dfrac{1/(1+h)-1}{h}$

(c) $\lim_{h\to 0}\dfrac{\sqrt{4+h}-2}{h}$

7. Compute the following limits

(a) $\lim_{x\to 0}\dfrac{\cos x-1}{\sin x}$

(b) $\lim_{x\to 0}\dfrac{\sin 3x}{\sin 5x}$

(c) $\lim_{x\to\infty} x\sin\dfrac{\pi}{x}$

Answers to limit and continuity exercises

1. (a) $+\infty$

   (b) Does not exist

   (c) -2

   (d) 3

   (e) $-1/2$

2. (a) add $f(0)=\dfrac{\sqrt{3}}{2}$

   (b) add $g(1)=n$

3. (a) $x\neq -7,4$

   (b) $(-\infty,\infty)$

4. 1

5. (a) 5

   (b) greater than or equal to 4

   (c) less than or equal to 2

6. (a) 6

   (b) -1

   (c) $-1/16$

7. (a) 0

   (b) 3/5

   (c) π