Chapter 5

Applications of Multivariable Functions

5.1 The Chain Rule

We apply the chain rule to related rate problems which are unable to get solved by using derivatives of one variable functions.

Example 1 A cone pile of sand increases by 2 inches/second in height and by 1 inch/second in base radius. Find the rate of volume's change of this sand pile when its cone is 30 inches in height and has base radius 20 inches.

Example 2: An airplane flies from the west to the east of an observer on the ground. Suppose the plane flies with horizontal speed 440 feet/second and vertical speed 10 feet/second. What is the rate of distance's change between the plane and the observer when the plane is 12000 feet above ground and 16000 feet to the west of the observer?

5.2 **Total Differential**

In the case of one variable functions, we have the differential:

$$df = f'(x)\Delta x.$$

Similarly, in the case of two variable functions, this df is called the total differential of f and is defined as follows.

Definition: The total differential of $f(x, y)$, denoted by df , is defined to be

$$df = f_x(x, y)dx + f_y(x, y)dy$$

Example Find dz where $z = \ln(x^3y^2)$.

Definition: The difference of function between two points

$$\Delta f = f(x+\Delta x, y+\Delta y) - f(x, y).$$

Application of total difference/differential

We know that $\Delta f \approx df$ when $\Delta x, \Delta y \to 0$.

Thus, $f(x+\Delta x, y+\Delta y) - f(x,y) \approx f_x(x,y)dx + f_y(x,y)dy$.

i.e. $f(x+\Delta x, y+\Delta y) \approx f(x,y) + f_x(x,y)dx + f_y(x,y)dy$.

This relation can be used to approximate the value of the function f at some points $(x+\Delta x, y+\Delta y)$ near the point (x,y) when the value of $f(x,y)$ is much easier to compute. It is called the *linear approximation* of two variable functions.

Example Estimate $\sqrt{(5.98)^2 + (8.01)^2}$.

5.3 Max and Min of multivariable functions

Definition 5.3.1

If a function f of two variable has a point $(a,b) \in D_f$ such that $f(x,y) \leq f(a,b)$ for all points (x,y) in an open disk centered at (a,b), then the point $(a,b,f(a,b))$ is called a *local maximum point* of f, and the value $f(a,b)$ is called a *local maximum value* of f.

If $f(x,y) \leq f(a,b)$ for all points (x,y) in domain of f, we call the point $(a,b,f(a,b))$ as an *absolute maximum point* of f and call the value $f(a,b)$ as an *absolute maximum value* of f.

If a function f of two variables has a point $(c,d) \in D_f$ such that $f(x,y) \geq f(c,d)$ for all points (x,y) in an open disk centered at (c,d), then the point $(c,d,f(c,d))$ is called as a *local minimum point* of f and the value $f(c,d)$ is called a *local minimum value* of f.

If $f(x,y) \geq f(c,d)$ for all points (x,y) in domain of f, we call the point $(c,d,f(c,d))$ as an *absolute minimum point* of f and called the value $f(c,d)$ as an *absolute minimum value*.

How to find local maximum and minimum

Theorem: If f has a local max or local min at (a,b) and its first partial derivatives exist at (a,b), we would have both

$$f_x(a,b)=0 \text{ and } f_y(a,b)=0.$$

Definition: A point (a,b) in domain of f is called a critical point of f if either both $f_x(a,b)=0$ and $f_y(a,b)=0$ or at least one partial derivative of f does not exist at (a,b).

<u>Remark</u>: A critical point may not be a local max or local min; it could be a saddle point.

Sometimes we may use a graph of f to locate a local max and a local min.

Example: Find all critical points and locate local max and local min of $f(x,y)=x^2-6x+y^2-4y$.

The test of local max and local min

Theorem: Let $z = f(x,y)$. Suppose f has continuous second partial derivatives in an open disk around (x_0, y_0) such that both $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$. There are three possibilities:

1. If $f_{xx}(x_0, y_0) \cdot f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2 > 0$,
   - and $f_{xx}(x_0, y_0) > 0$, then f has a local min at (x_0, y_0).
   - and $f_{xx}(x_0, y_0) < 0$, then f has a local max at (x_0, y_0).
2. If $f_{xx}(x_0, y_0) \cdot f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2 < 0$, then the point (x_0, y_0) is a saddle point of f.
3. If $f_{xx}(x_0, y_0) \cdot f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2 = 0$, then the test fails. We have no conclusion about the point (x_0, y_0). It may be the local max, the local min, the saddle point or none of these.

Example Suppose $f(x,y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$. Find the local max and local min of f.

5.3.2 Maximun and Minimum of a Function in a Closed Domain.

Theorem: Suppose $f : D \to R$ where D is a closed plane. If the function f is a continuous function and bounded on D, then f always has the absolute maximum and absolute minimum points in D.

Example Find the absolute maximum and absolute minimum points of

$$f(x, y) = x^2 - 6x + y^2 - 4y$$

on the area bounded by x-axis, y-axis and the line $x + y = 7$.

Problem Find the size of a rectangular box whose volume $V = 1000$ cubic feet and has the least surface area A.

5.4 Maximum and minimum of a function with boundary conditions

Example Find the size and the volume of a rectangular box with the maximum volume where the box is located in the first octant so that one boxes' corner is at the origin and the opposite corner is on the paraboloid $z = 4 - x^2 - 4y^2$.

Most of the time, the given conditions are not easy to put in the functions directly. For example, we want to find a local max and a local min of $f(x,y) = xy$ with the condition: $g(x,y) = \frac{x^2}{8} + \frac{y^2}{2} - 1 = 0$. Hence, we may need the following method:

Method of Lagrange Multipliers

This method can be used to find the local max or local min of two variable functions (in general, multivariable functions) f with the given condition g. This can be done by following these steps.

1. Form a function F with 3 variables: x, y, λ such that $F(x,y,\lambda) = f(x,y) - \lambda g(x,y)$ where f and g have continuous first partial derivatives.
2. Find all critical points of F. That is, we find x, y, λ such that $F_x(x,y,\lambda) = 0$, $F_y(x,y,\lambda) = 0$, $F_\lambda(x,y,\lambda) = 0$.
3. Calculate the value of f at all critical points (x,y) found in 2.

Example Find maximum and minimum of $f(x,y)=xy$ with the condition: $g(x,y)=\dfrac{x^2}{8}+\dfrac{y^2}{2}-1=0$.

Exercises

1. Estimate the value of $\sqrt{8.99}+\cos\left(0.02\right)$.

2. Find all local extreme (maximum and/or minimum), and saddle points of $f(x,y)=x^2y+2y^2-2xy-15y$.

3. Find the maximum temperature defined by

$$T(x,y)=x^2+y^2+4x-4y+3$$

on the disk circumference $x^2+y^2=2$.

Answer 1. $4-\frac{1}{600}=3.99833...$

2. Saddle points at $(-3,0)$ and $(5,0)$

   Local minimum at $(1,4)$

3. Maximum temperature = 13