Mastermind

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Mastermind

i. Codemaker vs. Codebreaker

ii. Queries: Guess a vector from $[k]^n$. Get two-color feedback.

Knuth Paper – 1976

i. Original Mastermind: Four spots and six colors
ii. Deterministic strategy, wins in at most five turns
iii. Minimax algorithm

Minimax Example

i. Hidden Vector: $(3, 5, 0, 0)$
ii. What to guess? Eliminate as many as possible.
iii. Simulation [Format: Number Remaining $\rightarrow$ Guess $\rightarrow$ (Black Hits, White Hits)]

$$1296 \rightarrow$$

Minimax Example

i. Hidden Vector: $(3, 5, 0, 0)$
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$$1296 \rightarrow (0, 0, 1, 1) \rightarrow$$

Minimax Example

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$$1296 \rightarrow (0, 0, 1, 1) \rightarrow (0, 2)$$

Minimax Example

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ii. What to guess? Eliminate as many as possible.
iii. Simulation [Format: Number Remaining $\rightarrow$ Guess $\rightarrow$ (Black Hits, White Hits)]

$$
\begin{aligned}
1296 &\rightarrow (0, 0, 1, 1) \rightarrow (0, 2) \\
96 &
\end{aligned}
$$

Minimax Example

i. Hidden Vector: $(3, 5, 0, 0)$

ii. What to guess? Eliminate as many as possible.

iii. Simulation [Format: Number Remaining $\rightarrow$ Guess $\rightarrow$ (Black Hits, White Hits)]

$$\begin{aligned} 1296 &\rightarrow (0, 0, 1, 1) \rightarrow (0, 2) \\ 96 &\rightarrow (1, 2, 3, 3) \rightarrow (0, 1) \\ 14 &\rightarrow (4, 1, 0, 4) \rightarrow (1, 0) \\ 3 &\rightarrow (2, 5, 0, 0) \rightarrow (3, 0) \\ 1 &\rightarrow (3, 5, 0, 0) \rightarrow (4, 0) \end{aligned}$$

Generalizations

i. Basic Extension: n spots, k colors
ii. Repeats vs. no repeats
iii. Non-adaptive vs. adaptive strategies

Known Bounds

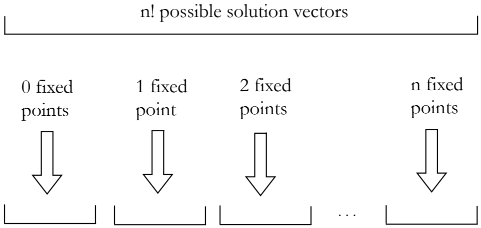
1. *Mastermind with Repeats:*
   i. When $n \leq k \leq n^2 \log\log(n)$, [Chvatal] and [Doerr, et. al.]
   $$\frac{n\log(k)}{\log\binom{n+2}{2}} \leq f(n,k) \leq O(n\log\log(n))$$
   ii. When $k = n^\alpha$ for $0 < \alpha < 1$, [Doerr, et. al.]
   $$f(n,k) = \Theta(n)$$

Known Bounds (cont.)

2. *Mastermind without Repeats:*
   i. When $n = k$, we have the permutation game: [Our Result] and [Ouali et. al]

$$n - \log\log(n) \leq f(n, k) \leq n\log(n) + O(n)$$

Deriving $n - \log\log n$ Lower Bound



i. Uses more information than previous proofs, using "buckets."

ii. Bucket $i \iff i$ fixed points

iii. Size of buckets:

$$\begin{aligned} |B_i| &= \binom{n}{i} \cdot D(n-i) \\ &= \binom{n}{i} \cdot (n-i)! \cdot \sum_{j=0}^{n-i} \frac{(-1)^j}{j!} \end{aligned}$$

Deriving $n \log \log n$ Upper Bound when $k = n$

[Doerr et. al., 2013]

i. Split hidden vector into "coins" (subvectors).

ii. Use coin weighing problem to eliminate colors.

Basics of Entropy

Definition: Let X be a random variable with domain D.

$$H(X) = -\sum_{x \in D} \mathbb{P}[X = x] \cdot \log_2 (\mathbb{P}[X = x])$$

Key Properties:

i. If random variables X, Y uniquely determine each other's outcomes, then $H(X) = H(Y)$.

ii. *Subadditivity:* A vector $X = (X_1, X_2, \ldots, X_n)$ of random variables has

$$H(X) \leq \sum_{i=1}^{n} H(X_i)$$

Use these properties to give lower bound on non-adaptive strategies.

Future Topics to Explore

i. Alternative Queries: Can query any subgraph
ii. Does our lower bound extend to general n and k?
iii. Non-adaptive strategies: Submit all guesses at beginning