

# Multinomial Discrete Choice: IIA Logit

---

Chris Conlon

Fall 2025

Grad IO

Most decisions agents make are not necessarily binary:

- ▶ Choosing a level of schooling (or a major).
- ▶ Choosing an occupation.
- ▶ Choosing a partner.
- ▶ Choosing where to live.
- ▶ Choosing a brand of (yogurt, laundry detergent, orange juice, cars, etc.).

# Nonparametric Setup

We consider a **multinomial discrete choice**:

- ▶ in period  $t$
- ▶ with  $\mathcal{J}_t$  alternatives.
- ▶ subscript individual agents by  $i$ .
- ▶ agents choose  $j \in J_t$  with probability  $s_{ijt}$ .
- ▶ Agent  $i$  receives utility  $U_{ijt}$  for choosing  $j$ .
- ▶ Choice is exhaustive and mutually exclusive.

Consider the simple example ( $t = 1$ ):

$$s_{ij} = \mathbb{P}(U_{ij} > U_{ik} \quad \forall k \neq j)$$

# Nonparametric Setup

We consider a **multinomial discrete choice**:

- ▶ in period  $t$
- ▶ with  $\mathcal{J}_t$  alternatives.
- ▶ subscript individual agents by  $i$ .
- ▶ agents choose  $j \in J_t$  with probability  $s_{ijt}$ .
- ▶ Agent  $i$  receives utility  $U_{ijt}$  for choosing  $j$ .
- ▶ Choice is exhaustive and mutually exclusive.

Consider the simple example ( $t = 1$ ):

$$s_{ij} = \mathbb{P}(U_{ij} > U_{ik} \quad \forall k \neq j)$$

# Nonparametric Setup

Now consider separating the utility into the **observed**  $V_{ij}$  and **unobserved** components  $\varepsilon_{ij}$ .

$$\begin{aligned}s_{ij} &= \mathbb{P}(U_{ij} > U_{ik} \quad \forall k \neq j) \\ &= \mathbb{P}(V_{ij} + \varepsilon_{ij} > V_{ik} + \varepsilon_{ik} \quad \forall k \neq j) \\ &= \mathbb{P}(\varepsilon_{ij} - \varepsilon_{ik} > V_{ik} - V_{ij} \quad \forall k \neq j)\end{aligned}$$

It is helpful to define  $f(\varepsilon_i)$  as the  $J$  vector of individual  $i$ 's unobserved utility.

$$\begin{aligned}s_{ij} &= \mathbb{P}(\varepsilon_{ij} - \varepsilon_{ik} > V_{ik} - V_{ij} \quad \forall k \neq j) \\ &= \int I(\varepsilon_{ij} - \varepsilon_{ik} > V_{ik} - V_{ij}) f(\varepsilon_i) d\varepsilon_i\end{aligned}$$

# Nonparametric Setup

Now consider separating the utility into the **observed**  $V_{ij}$  and **unobserved** components  $\varepsilon_{ij}$ .

$$\begin{aligned}s_{ij} &= \mathbb{P}(U_{ij} > U_{ik} \quad \forall k \neq j) \\ &= \mathbb{P}(V_{ij} + \varepsilon_{ij} > V_{ik} + \varepsilon_{ik} \quad \forall k \neq j) \\ &= \mathbb{P}(\varepsilon_{ij} - \varepsilon_{ik} > V_{ik} - V_{ij} \quad \forall k \neq j)\end{aligned}$$

It is helpful to define  $f(\varepsilon_i)$  as the  $J$  vector of individual  $i$ 's unobserved utility.

$$\begin{aligned}s_{ij} &= \mathbb{P}(\varepsilon_{ij} - \varepsilon_{ik} > V_{ik} - V_{ij} \quad \forall k \neq j) \\ &= \int I(\varepsilon_{ij} - \varepsilon_{ik} > V_{ik} - V_{ij}) f(\varepsilon_i) d\varepsilon_i\end{aligned}$$

# Nonparametric Setup

In order to compute the choice probabilities, we must perform a  $J$  dimensional integral over  $f(\varepsilon_i)$ .

$$s_{ij} = \int I(\varepsilon_{ij} - \varepsilon_{ik} > V_{ik} - V_{ij}) f(\varepsilon_i) d\varepsilon_i$$

There are some choices that make our life easier

- ▶ Multivariate normal:  $\varepsilon_i \sim N(0, \Omega)$ .  $\longrightarrow$  **multinomial probit**.
- ▶ Gumbel/Type 1 EV:  $f(\varepsilon_i) = e^{-\varepsilon_{ij}} e^{-e^{-\varepsilon_{ij}}}$  and  $F(\varepsilon_i) = 1 - e^{-e^{-\varepsilon_{ij}}}$   $\longrightarrow$  **multinomial logit**
- ▶ There are also heteroskedastic variants of the Type I EV/ Logit framework.

Allowing for full support  $(-\infty, \infty)$  errors provide two key features:

- ▶ Smoothness:  $s_{ij}$  is everywhere continuously differentiable in  $V_{ij}$ .
- ▶ Bound  $s_{ij} \in (0, 1)$  so that we can rationalize any observed pattern in the data.
  - ▶ Caveat: zero and one (interpretation).
- ▶ What does  $\varepsilon_{ij}$  really mean? (unobserved utility, idiosyncratic tastes, etc.)



# Basic Identification

- ▶ Only differences in utility matter:  $\mathbb{P}(\varepsilon_{ij} - \varepsilon_{ik} > V_{ik} - V_{ij} \quad \forall k \neq j)$
- ▶ Adding constants is irrelevant: if  $U_{ij} > U_{ik}$  then  $U_{ij} + a > U_{ik} + a$ .
- ▶ Only differences in alternative specific constants can be identified

$$U_b = v_{ib} + k_b + \varepsilon_{ib}$$

$$U_c = v_{ic} + k_c + \varepsilon_{ic}$$

only  $d = k_b - k_c$  is identified.

- ▶ This means that we can only include  $J - 1$  such  $k$ 's and need to normalize one to zero. (Much like fixed effects).
- ▶ We cannot have individual specific factors that enter the utility of all options such as income  $\theta Y_i$ . We can allow for interactions between individual and choice characteristics  $\theta p_j / Y_i$ .

$$U_b = v_b + \theta y_i + \varepsilon_b$$

$$U_c = v_c + \theta y_i + \varepsilon_c$$

## Basic Identification: Location

- ▶ Technically we can't really fully specify  $f(\varepsilon_i)$  since we can always re-normalize:  
 $\widetilde{\varepsilon}_{ijk} = \varepsilon_{ij} - \varepsilon_{ik}$  and write  $g(\widetilde{\varepsilon}_{ik})$ . Thus any  $g(\widetilde{\varepsilon}_{ik})$  is consistent with infinitely many  $f(\varepsilon_i)$ .
- ▶ Logit pins down  $f(\varepsilon_i)$  sufficiently with parametric restrictions.
- ▶ Probit does not. We must generally normalize one dimension of  $f(\varepsilon_i)$  in the probit model. Usually a diagonal term of  $\Omega$  so that  $\omega_{11} = 1$  for example. (Actually we need to do more!).

## Basic Identification: Scale

- ▶ Consider:  $U_{ij}^0 = V_{ij} + \varepsilon_{ij}$  and  $U_{ij}^1 = \lambda V_{ij} + \lambda \varepsilon_{ij}$  with  $\lambda > 0$ . Multiplying by constant  $\lambda$  factor doesn't change any statements about  $U_{ij} > U_{ik}$ .
- ▶ We normalize this by fixing the variance of  $\varepsilon_{ij}$  since  $Var(\lambda \varepsilon_{ij}) = \sigma_e^2 \lambda^2$ .
- ▶ Normalizing this variance normalizes the scale of utility.
- ▶ For the logit case the variance is normalized to  $\pi^2/6$ . (this emerges as a constant of integration to guarantee a proper density).

# Observed Heteroskedasticity

Consider the case where  $Var(\varepsilon_{ib}) = \sigma^2$  and  $Var(\varepsilon_{ic}) = k^2\sigma^2$  :

- ▶ We can estimate

$$U_{ib} = v_{ib} + \varepsilon_{ib}$$

$$U_{ic} = v_{ic} + \varepsilon_{ic}$$

becomes:

$$U_{ib} = v_{ib} + \varepsilon_{ib}$$

$$U_{ic} = v_{ic} + \varepsilon_{ic}$$

- ▶ Some interpret this as saying that in segment  $C$  the unobserved factors are  $\hat{k}$  times larger.

Different ways to look at identification

- ▶ Are we interested in non-parametric identification of  $V_{ij}$ , specifying  $f(\varepsilon_i)$ ?
- ▶ Or are we interested in non-parametric identification of  $U_{ij}$ . (Generally hard).
  - ▶ Generally we require a large support (special-regressor) or “completeness” condition.
  - ▶ Lewbel (2000) does random utility with additively separable but nonparametric error.
  - ▶ Berry and Haile (2015) with non-separable error (and endogeneity).

# Multinomial Logit

- ▶ Multinomial Logit (Gumbel/Type I EV) has closed form choice probabilities

$$s_{ij} = \frac{e^{V_{ij}}}{\sum_k e^{V_{ik}}}$$

- ▶ Often we approximate  $V_{ij} \approx X_{ik}\beta$  with something linear in parameters.

# Logit Inclusive Value

Expected maximum also has closed form:

$$\mathbb{E}[\max_j U_{ij}] = \log \left( \sum_j \exp[V_{ij}] \right) + C$$

Logit Inclusive Value is helpful for several reasons

- ▶ Expected utility of best option (without knowledge of  $\epsilon_i$ ) does not depend on  $\epsilon_{ij}$ .
- ▶ This is a globally concave function in  $V_{ij}$  (more on that later).
- ▶ Allows simple computation of  $\Delta CS$  for consumer welfare (but not  $CS$  itself).

# Multinomial Logit

Multinomial Logit goes by a lot of names in various literatures

- ▶ The problem of multiple choice is often called **multiclass classification** or **softmax regression** in other literatures.
- ▶ In general these models assume you have individual level data

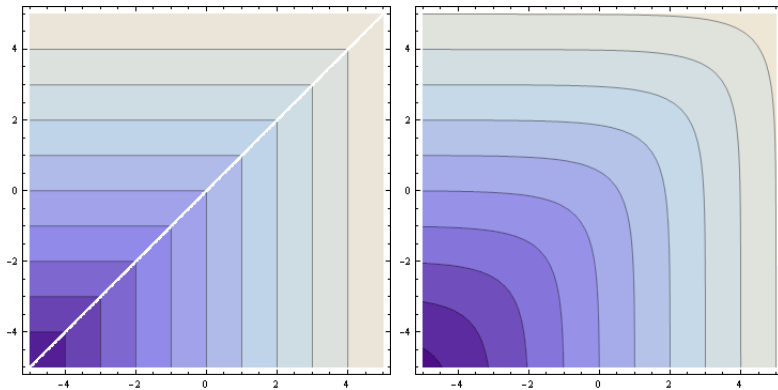


# Alternative Interpretation

Statistics/Computer Science offer an alternative interpretation

- ▶ Sometimes this is called **softmax** regression.
- ▶ Think of this as a continuous/concave approximation to the maximum.
- ▶ Consider  $\max\{x, y\}$  vs  $\log(\exp(x) + \exp(y))$ . The exp exaggerates the differences between  $x$  and  $y$  so that the larger term dominates.
- ▶ We can accomplish this by rescaling  $k$ :  $\log(\exp(kx) + \exp(ky))/k$  as  $k$  becomes large the derivatives become infinite and this approximates the “hard” maximum.
- ▶  $g(1, 2) = 2.31$ , but  $g(10, 20) = 20.00004$ .

# Alternative Interpretation



# Multinomial Logit: Identification

What is actually identified here?

- ▶ Helpful to look at the ratio of two choice probabilities

$$\frac{s_{ij}(\theta)}{s_{ik}(\theta)} = \frac{e^{V_{ij}}}{e^{V_{ik}}} = e^{V_{ij} - V_{ik}}$$

- ▶ We only identify the **difference in indirect utilities** not the levels.
- ▶ The ratio of choice probabilities for  $j$  and  $k$  depends only on  $j$  and  $k$  and not on any alternative  $l$ , this is known as **independence of irrelevant alternatives**.
- ▶ For some (Luce (1959)) IIA was an attractive property for axiomatizing choice. (A feature or a bug?)
- ▶ In fact the logit was derived in the search for a statistical model that satisfied various axioms.

# Multinomial Logit: Identification

As another idea suppose we add a constant  $C$  to each  $\beta_j$ .

$$s_{ij} = \frac{\exp[\mathbf{x}_i(\beta_j + C)]}{\sum_k \exp[\mathbf{x}_i(\beta_k + C)]} = \frac{\exp[\mathbf{x}_i C] \exp[\mathbf{x}_i \beta_j]}{\exp[\mathbf{x}_i C] \sum_k \exp[\mathbf{x}_i \beta_k]}$$

This has no effect. That means we need to fix a normalization  $C$ .

The most convenient is generally that  $C = -\beta_K$ .

- ▶ We normalize one of the choices to provide a utility of zero.
- ▶ We actually already made another normalization. Does anyone know which?

# Multinomial Logit: Identification

The most sensible normalization in demand settings is to allow for an **outside option** which produces no utility in expectation so that  $e^{V_{i0}} = e^0 = 1$ :

$$s_{ij} = \frac{e^{V_{ij}}}{1 + \sum_k e^{V_{ik}}}$$

- ▶ Hopefully the choice of outside option is well defined: not buying a yogurt, buying some other used car, etc.
- ▶ Now this resembles the binomial logit model more closely.

## Back to Scale of Utility

- ▶ Consider  $U_{ij}^* = V_{ij} + \varepsilon_{ij}^*$  with  $Var(\varepsilon^*) = \sigma^2 \pi^2 / 6$ .
- ▶ Without changing behavior we can divide by  $\sigma$  so that  $U_{ij} = V_{ij}/\sigma + \varepsilon_{ij}$  and  $Var(\varepsilon^*/\sigma) = Var(\varepsilon) = \pi^2/6$

$$s_{ij} = \frac{e^{V_{ij}/\sigma}}{\sum_k e^{V_{ik}/\sigma}} \approx \frac{e^{\beta^*/\sigma \cdot x_{ij}}}{\sum_k e^{\beta^*/\sigma \cdot x_{ik}}}$$

- ▶ Every coefficient  $\beta$  is rescaled by  $\sigma$ . This implies that only the ratio  $\beta^*/\sigma$  is identified.
- ▶ Coefficients are relative to variance of unobserved factors. More unobserved variance  $\longrightarrow$  smaller  $\beta$ .
- ▶ Ratio  $\beta_1/\beta_2$  is invariant to the scale parameter  $\sigma$ . (**marginal rate of substitution**).

# IIA Property

The well known critique:

- ▶ You can choose to go to work on a car  $c$  or blue bus  $bb$ .  $S_c = S_{bb} = \frac{1}{2}$  so that  $\frac{S_c}{S_{bb}} = 1$ .
- ▶ Now we introduce a red bus  $rb$  that is identical to  $bb$ . Then  $\frac{S_{rb}}{S_{bb}} = 1$  and  $S_c = S_{bb} = S_{rb} = \frac{1}{3}$  as the logit model predicts.
- ▶ In reality we don't expect painting a bus red would change the number of individuals who drive a car so we would anticipate  $S_c = \frac{1}{2}$  and  $S_{bb} = S_{rb} = \frac{1}{4}$ .
- ▶ We may not encounter too many cases where  $\rho_{\varepsilon_{ik}, \varepsilon_{ij}} \approx 1$ , but we have many cases where this  $\rho_{\varepsilon_{ik}, \varepsilon_{ij}} \neq 0$
- ▶ What we need is the ratio of probabilities to change when we introduce a third option!

# IIA Property

- ▶ IIA implies that we can obtain consistent estimates for  $\beta$  on any subset of alternatives.
- ▶ This means instead of using all  $\mathcal{J}$  alternatives in the choice set, we could estimate on some subset  $\mathcal{S} \subset \mathcal{J}$ .
- ▶ This used to be a way to reduce the computational burden of estimation (not clear this is an issue in 21st century).
- ▶ Sometimes we have **choice based samples** where we oversample people who choose a particular alternative. Manski and Lerman (1977) show we can get consistent estimates for all but the ASC. This requires knowledge of the difference between the true rate  $A_j$  and the choice-based sample rate  $S_j$ .
- ▶ Hausman proposes a specification test of the logit model: estimate on the full dataset to get  $\hat{\beta}$ , construct a smaller subsample  $\mathcal{S}^k \subset \mathcal{J}$  and  $\hat{\beta}^k$  for one or more subsets  $k$ . If  $|\hat{\beta}^k - \hat{\beta}|$  is small enough.



## IIA Property

For the linear  $V_{ij}$  case we have that  $\frac{\partial V_{ij}}{\partial z_{ij}} = \beta_z$ .

$$\frac{\partial s_{ij}}{\partial z_{ij}} = s_{ij}(1 - s_{ij}) \frac{\partial V_{ij}}{\partial z_{ij}}$$

And Elasticity: 
$$\frac{\partial \log s_{ij}}{\partial \log z_{ij}} = s_{ij}(1 - s_{ij}) \frac{\partial V_{ij}}{\partial z_{ij}} \frac{z_{ij}}{s_{ij}} = (1 - s_{ij}) z_{ij} \frac{\partial V_{ij}}{\partial z_{ij}}$$

With cross effects: 
$$\frac{\partial s_{ij}}{\partial z_{ik}} = -s_{ij}s_{ik} \frac{\partial V_{ik}}{\partial z_{ik}}$$

and elasticity : 
$$\frac{\partial \log s_{ij}}{\partial \log z_{ik}} = -s_{ik} z_{ik} \frac{\partial V_{ik}}{\partial z_{ik}}$$

# Own and Cross Elasticity

An important output from a demand system are elasticities

- ▶ This implies that  $\eta_{jj} = \frac{\partial s_{ij}}{\partial p_j} \frac{p_j}{s_{ij}} = \beta_p \cdot p_j \cdot (1 - s_{ij})$ .
- ▶ The price elasticity is increasing in own price! (Why is this a bad idea?)
- ▶ Also mechanical relationship between elasticity and **share** so that popular products necessarily have higher markups (holding fixed prices).

# Proportional Substitution

Cross elasticity doesn't really depend on  $j$ .

$$\frac{\partial \log s_{ij}}{\partial \log z_{ik}} = -s_{ik} z_{ik} \underbrace{\frac{\partial V_{ik}}{\partial z_{ik}}}_{\beta_z}.$$

- ▶ This leads to the idea of proportional substitution. As option  $k$  gets better it proportionally reduces the shares of the all other choices.
- ▶ This might be a desirable property but probably not.

# Diversion Ratios

Recall the diversion ratio:

$$D_{jk} = \frac{\frac{\partial s_{ik}}{\partial p_j}}{\left| \frac{\partial s_{ij}}{\partial p_j} \right|} = \frac{\beta_p s_{ik} s_{ij}}{\beta_p s_{ij} (1 - s_{ij})} = \frac{s_{ik}}{1 - s_{ij}}$$

- ▶ Again proportional substitution. As price of  $j$  goes up we proportionally inflate choice probabilities of substitutes.
- ▶ Likewise removing an option  $j$  means that  $\tilde{s}_{ik}(\mathcal{J} \setminus j) = \frac{s_{ik}}{1 - s_{ij}}$  for all other  $k$ .
- ▶ IIA/Logit means **constant diversion ratios**.

Thanks!

---