Persistent Unobservables

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Grad IO

Persistent Unobserved Heterogeneity

Suppose we think about a model with a friction such as a switching cost.

- If $y_{it} \neq y_{i,t-1}$ you pay a switching cost F_i.
- How do we use data to tell apart large switching costs $F_i \gg 0$ from persistent tastes $Cov(\epsilon_{i,t}, \epsilon_{i,t-1}) > 0$?
- The conditional independence assumption tells us it has to be the switching cost not the autocorrelated unobservables.
- This is probably why people don't like this assumption.

Discrete Unobserved Types

- Up until now we consider models satisfying Rust's conditional independence assumption on the ε's. This rules out persistence in unobservables which are economically meaningful.
- Suppose there are two types of buses good $(s_i = g)$ and bad $(s_i = b)$.
- Assume that this is known to HZ but not the econometrician.
- Single period utility now depends on s_i so $u(x_{it}, s_i, d_{it}; \theta)$ unobserved state variable.
- In case of the nested fixed point algorithm, this unobserved persistent heterogeneity is not a big problem as we can solve for the value function (and expected policy functions) given the state variables and integrate it out in the likelihood

Unobserved State Variables: What happened?

$$\Pr(d_{i1}, \dots, d_{iT} | x_{i1}, \dots, x_{iT}) = \sum_{s} \prod_{t=1}^{T} \Pr(d_{it} | x_{it}) p(s_i)$$

- Conditional on s_i replacement decisions are independent across t given x_{it}.
- The resulting likelihood is just a finite mixture model.
- These can be hard to solve when both s_i and its distribution $p(s_i)$ are unknown.
- Arcidiacono and Miller (2011) provide theoretical results for these types of problems.

Pakes (1986): Patents as Options

How much are patents worth? Valuable for optimal patent length and design? Sufficient incentive for innovation?

- Q_A: value of patent at age A
- Goal of paper is to estimate Q_A using data on their renewal. Q_A is inferred from patent renewal process via revealed preference for patent renewal behavior.
- Treat renewal systems as exogenous (in Europe)

Timing

- For $a = 1, \dots, L$ a patent can be renewed by paying the fee c_a.
- At age $a = 1$ patent holder gets r_1 from patent
- Decide whether or not to renew (pay c_1 and go to a_2).
- At age $a = 2$ get r_2 from patent
- and so on...

Gives us the value function

$$
\begin{aligned}
V &\equiv \max_{t \in [a,L]} \sum_{a'=1}^{L-a} \beta^{a'} R(a+a') \\
R(a) &= \begin{cases} r_a - c_a, & \text{if } t \geq a \text{ when you hold patent} \\ 0 & \text{if } t < a \text{ after patent expires} \end{cases}
\end{aligned}
$$

- t above denotes the age which allows the patent to expire and is the choice variable. Another optimal stopping problem.
- $R(a)$ are the profits from year a. This is a controlled stochastic process. It is random but affected by the actions of the agent.

Pakes (1986)

- The maximum age L is finite so it is finite-horizon DP.
- The single period revenue r_a is the state variable.
- We can solve the problem with *backward recursion*.

$$V_a(r_a) = \max \{0, Q_a \equiv r_a + \beta E[V_{a+1}(r_{a+1})|\Omega_a] - c_a\}$$

- Renew iff $Q_a - c_a > 0$.
- Ω_a: history up to age $a = \{r_1, r_2, \ldots, r_a\}$.
- Expectation is over $r_{a+1}|\Omega_a$. The sequence of conditional distributions $G_a \equiv F(r_{a+1}|\Omega_a)$, $a = 1, 2, \ldots$ is an important component of model specification.

$$r_{a+1} = \begin{cases} 0 & \text{w. prob } \exp(-\theta r_a) \\ \max(\delta r_a, z) & \text{w. prob } 1 - \exp(-\theta r_a) \end{cases}$$

Model has the following parameters

- density of z $q_a = \frac{1}{\sigma_a}$ exp$[-(\gamma + z)/\sigma_a]$ and $\sigma_a = \phi^{a-1}\sigma$, for $a = 1, \ldots, L-1$.
- $(\delta, \theta, \gamma, \phi, \sigma)$ are the structural parameters of the model
- Break down the model period by period and decide whether or not to renew if $Q_a = r_a +$ "option value".
- Option value is about keeping the patent alive in case it pays off in the future.

Implications

- Drop out at age a if $c_a > Q_a$
- Optimal decision is characterized by cutoff points $Q_a > c_a \Leftrightarrow r_a > \overline{r}_a$ (Key assumptions is Q_a increasing /single crossing)
- Cutoff points are increasing sequence $\overline{r}_a < \overline{r}_{a+1} < \ldots < \overline{r}_{L-1}$.

Instead of using Pakes' notation r_t for the patent revenue. We will use the generic Rust notation of ϵ_t the unobserved state variable, and i_t to denote the choice (renewal).

- For a single patent $\tilde{T}$ denotes the age at which it is allowed to expire. Let $T = \min(L-1, \tilde{T})$ denote the period sins which the agent makes a renewal decision where we model the agent's choice.
- ϵ follows a first-order Markov process $F(\epsilon'|\epsilon)$
- Age-specific policy function by $i_t^*(\epsilon)$.

Likelihood function is

$$\ell(i_1, \ldots, i_T|\epsilon_0, i_0, \theta) = \prod_{t=1}^{T} \Pr(i_t|i_0, \ldots, x_{t-1}, i_{t-1}; \epsilon_0, \theta)$$

Serial correlation in ϵ means there is dependence among i_t, i_{t-2} even after conditioning on x_{t-1}, i_{t-1}.

- It might seem like we were stuck since it no longer has a closed form. However, we can simulate the "outer loop" of the nested fixed point routine given a guess of $i_t^*(\epsilon, \theta)$.
- Because ϵ is serially correlated we need to start with an initial ϵ_0 (or distribution) and assume that it is known. This is the initial conditions problem of finite MDPs.
- Note that simulation is part of the "outer loop" of nested fixed point estimation routine. So at the point when we simulate, we already know the policy functions $i_t^*(\epsilon, \theta)$ (How would you compute this?)

Naive Frequency Simulator (Don't do this...)

Go back to the full likelihood function (condition on initial ϵ_0 for serial correlation):

$$\ell(i_1, \ldots, i_T | i_0, \epsilon_0, \theta) = \mathsf{Pr}(i_t^*(\epsilon_t, \theta) = i_t, \quad \forall t = 1, \ldots, T)$$

Need to take probability over distribution of $(\epsilon_1, \ldots, \epsilon_T | \epsilon_0)$. Let $F(\epsilon_{t+1} | \epsilon_t, \theta)$ then the above probability can be expressed as the integral:

$$\int \cdots \int \prod_t \mathbf{1}(i_t^*(\epsilon_t, \theta) = i_t) \prod_t dF(\epsilon_t | \epsilon_{t-1}; \theta)$$

Simulate by drawing sequences of (ϵ_t).

Naive Frequency Simulator (Don't do this...)

Simulate by drawing sequences of (ϵ_t) and for each draw $s = 1, \ldots, S$ we take as initial values (x_0, i_0, ϵ_0) then

- Generate (ϵ_1^s, i_1^s)
  1. Generate $\epsilon_1^s \sim F(\epsilon_1 | \epsilon_0)$
  2. Compute $i_1^s = i_1^*(\epsilon_1^s; \theta)$
- Generate (ϵ_2^s, i_2^s)
  1. Generate $\epsilon_2^s \sim F(\epsilon_2 | \epsilon_1^s)$
  2. Subsequently compute $i_2^s = i_2^*(\epsilon_2^s; \theta)$
- And so on, up to (ϵ_T^s, i_T^s).

Naive Frequency Simulator (Don't do this...)

And for the case where (i, x) are both discrete (Rust) we can approximate:

$$\ell(i_t, \dots, i_T | \epsilon_0, i_0; \theta) \approx \frac{1}{S} \sum_s \prod_{t=1}^{T} \mathbf{1}(i_t^s = i_t)$$

Frequency of simulated sequences which match observed sequence. T long or S small you're in trouble (non-smooth).

Importance Sampling: Particle Filtering

- We can use importance sampling to simulate the likelihood function.
- This is not straightforward given time dependence in (i_t, ϵ_t)
- Consider particle filtering approach from Fernandez-Villaverde and Rubio-Ramirez (2007) or Flury and Shehard (2008) (non-Gaussian Kalman filtering).
- A more up to date take: Blevins (2016) : Sequential Monte Carlo Methods for Estimating Dynamic Microeconomic Models

- Evolution of utility shocks $\epsilon_t | \epsilon_{t-1} \sim f(\epsilon' | \epsilon)$. Ignore dependence of distribution of ϵ on age t for convenience.
- As before, the policy function is $i_t = i^*(\epsilon_t)$
- Let $\epsilon^t \equiv \{\epsilon_1, \dots, \epsilon_t\}$.
- The initial values of y_0 and ϵ_0 are known

Go back to the factorized likelihood

$$\begin{aligned} \ell(y^T | y_0, \epsilon_0) &= \prod_{t=1}^{T} \ell(y_t | y^{t-1}, y_0, \epsilon_0) = \prod_{t=1} \int \ell(y_t | \epsilon^t, y^{t-1}) p(\epsilon^t | y^{t-1}) d\epsilon^t \\ &\approx \frac{1}{S} \sum_s \ell(y_t | \epsilon^{t|t-1,s}, y^{t-1}) \end{aligned}$$

We omit conditioning on (ϵ_0, y_0) for convenience, and $\epsilon^{t|t-1,s}$ is a simulated draw of $\epsilon^t \sim p(\epsilon^t | y^{t-1})$.

Let's look more closely at the last line:

- first term: $\ell(y_t, |\epsilon^t, y^{t-1})$ we can calculate for a value of ϵ_t

$$\ell(y_t|\epsilon^t, y^{t-1}) = p(i_t|\epsilon^t, y^{t-1}) = p(i_t|\epsilon_t) = \mathbf{1}(i(\epsilon_t) = i_t)$$

- the second term $p(\epsilon^t|y^{t-1})$ is generally not obtainable in closed form. So numerical integration is not feasible. Particle filtering let's us draw ϵ^t from this distribution for every period t.

Particle filtering proposes a recursive approach to draw sequences $p(\epsilon^t|y^{t-1})$ for every t

First period: $t = 1$ In order to simulate the integral corresponding to the first period we need to draw from $p(\epsilon^1|y^0, \epsilon_0)$ (easy).

- We draw $\{\epsilon^{1|0,s}\}_{s=1}^{S}$ according to $f(\epsilon'|\epsilon_0)$.
- The notation $\epsilon^{1|0,s}$ makes it explicit that the ϵ is a draw from $p(\epsilon^1|y^0, \epsilon_0)$
- Use the S draws we can evaluate the period $t = 1$ likelihood.

Second period: $t = 2$. We need to draw from $p(\epsilon^2|y^1)$ factorize as:

$$p(\epsilon^2|y^1) = p(\epsilon^1|y^1) \cdot p(\epsilon_2|\epsilon^1) \text{ recall } \epsilon^2 \equiv \{\epsilon_1, \epsilon_2\}$$

Getting a draw from $p(\epsilon^1|y^1)$, given that we already have draws $\{\epsilon^{1|0,s}\}$ from $p(\epsilon^1|y_0)$, from the previous period $t = 1$, is the heart of particle filtering. We use the principle of importance sampling: by Bayes' Rule

$$p(\epsilon^1|y^1) \propto p(y_1|\epsilon^1, y^0) \cdot p(\epsilon^1|y^0)$$

Hence, if our desired sampling density is $p(\epsilon^1|y^1)$, but we actually have draws $\{\epsilon^{1|0,s}\}$ from $p(\epsilon^1|y^0)$, then the importance sampling weight for the draw $\epsilon^{1|0,s}$ is proportional to

$$\tau_1^s \equiv p(y_1|\epsilon^{1|0,s}, y^0)$$

Note that this coincides with the likelihood contribution for period 1, evaluated at the shock $\epsilon^{1|0,s}$.The SIR algorithm in Rubin (1988) proposes that making S draws with replacement from samples $\{\epsilon^{1|0,s}\}_{s=1}^S$, using weights proportional τ_1^s yields draws from the desired density $p(\epsilon^1|y^1)$ which we denote $\{\epsilon^{1|0,s}\}_{s=1}^S$.

Prediction Step

For the second term in the equation: we simply draw one ϵ_2^s from $f(\epsilon'|\epsilon^{1,s})$, for each draw $\epsilon^{1,s}$ from the filtering step. This is the **prediction** step.

By combining the draws from these two terms, we have $\{\epsilon^{2|1,s}\}_{s=1}^{S}$.which is S drawn sequences from $p(\epsilon^2|y^1)$. Using these S draws, we can evaluate the simulated likelihood for period 2

Third period, $t = 3$: start again by factoring

$$p(\epsilon^3|y^2) = p(\epsilon^2|y^2) \cdot p(\epsilon^3|\epsilon^2)$$

As above, drawing from requires filtering the draws $\{\epsilon^{2|1,s}\}_{s=1}^S$, from the previous period $t = 2$, to obtain draws $\{\epsilon^{2,s}\}_{s=1}^S$. Given these draws, draw $\epsilon_3^s \sim f(\epsilon'|\epsilon^{2,s})$ for each s.

And so on. By the last period $t = T$, you have

$$\left\{\{\epsilon^{t|t-1,s}\}_{s=1}^S\right\}_{t=1}^T$$

Prediction Step (continued)

Hence the factorized likelihood can be approximated by simulation as:

$$\prod_t \frac{1}{S} \sum_s \ell(y_t | \epsilon^{t|t-1,s}, y^{t-1})$$

As noted above, the likelihood term $\ell(y_t|\epsilon^{t|t-1,s}, y^{t-1})$ coincides with the simulation weight τ_t^s. Hence the simulated likelihood can also be constructed as:

$$\log \ell(y^T | y_0, \epsilon_0) = \sum_t \log \left\{ \frac{1}{S} \sum_s \tau_t^s \right\}$$

Particle Filtering (Summary)

- Start by drawing $\{\epsilon^{1|0,s}\}_{s=1}^{S}$ from $p(\epsilon^1|y^0, \epsilon_0)$.
- In period t, we start with $\{\epsilon^{t-1|t-2,s}\}_{s=1}^{S}$ draws from $p(\epsilon^{t-1}|y^{t-2}, \epsilon_0)$.
    1. **Filter step:** Calculate proportion weights $\tau_{t-1}^{s} \equiv p(y_{t-1}|\epsilon^{t-1|t-2,s}, y^{t-2})$ using $p(i_t|\epsilon_t)$.
       Draw$\{\epsilon^{t-1|t-1,s}\}_{s=1}^{S}$ by resampling from $\{\epsilon^{t-1|t-2,s}\}_{s=1}^{S}$ with weights τ_{t-1}^{s}.
    2. **Prediction step:** Draw ϵ_t^s from $p(\epsilon_t|\epsilon^{t-1|t-1,s})$, for $s = 1, \dots, S$. Combine to get $\{\epsilon^{t|t-1,s}\}_{s=1}^{S}$.
- Set $t = t + 1$ and go back to step 2. Stop when $t = T + 1$.

The difference is that the crude simulator draws S sequences and puts zero weight on those which don't match the observed sequence. In each period t we just keep sequences where predicted choices match observed choice of *that period*. This is more accurate likelihood as long as S is large enough that we don't have all the weight on a single sequence in period t.

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