# Week 2: Demand Estimation (Part two) [OLD]

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Grad IO

# Reading for Today

These notes are based on Ken Train's book Discrete Choice Methods with Simulation

https://eml.berkeley.edu/books/choice2.html

That book summarizes about 20+ years of work by Daniel McFadden and other scholars.

# Statistical Models of Product Differentiation

# Motivation

Most decisions agents make are not necessarily binary:

- Choosing a level of schooling (or a major).
- Choosing an occupation.
- Choosing a partner.
- Choosing where to live.
- Choosing a brand of (yogurt, laundry detergent, orange juice, cars, etc.).

# Nonparametric Setup

We consider a multinomial discrete choice:

- in period $t$
- with $J_t$ alternatives.
- subscript individual agents by $i$.
- agents choose $j \in J_t$ with probability $P_{ijt}$.
- Agent $i$ receives utility $U_{ij}$ for choosing $j$.
- Choice is exhaustive and mutually exclusive.

Consider the simple example ($t = 1$):

$$P_{ij} = Prob(U_{ij} > U_{ik} \quad \forall j \neq k)$$

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$$P_{ij} = Prob(U_{ij} > U_{ik} \quad \forall j \neq k)$$

# Nonparametric Setup

Now consider separating the utility into the observed $V_{ij}$ and unobserved components $\varepsilon_{ij}$.

$$\begin{aligned} P_{ij} &= Prob(U_{ij} > U_{ik} \quad \forall j \neq k) \\ &= Prob(V_{ij} + \varepsilon_{ij} > V_{ik} + \varepsilon_{ik} \quad \forall j \neq k) \\ &= Prob(\varepsilon_{ij} - \varepsilon_{ik} > V_{ik} - V_{ij} \quad \forall j \neq k) \end{aligned}$$

It is helpful to define $f(\varepsilon_i)$ as the $J$ vector of individual $i$'s unobserved utility.

$$\begin{aligned} P_{ij} &= Prob(\varepsilon_{ij} - \varepsilon_{ik} > V_{ik} - V_{ij} \quad \forall j \neq k) \\ &= \int I(\varepsilon_{ij} - \varepsilon_{ik} > V_{ik} - V_{ij}) f(\varepsilon_i) \partial \varepsilon_i \end{aligned}$$

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# Nonparametric Setup

In order to compute the choice probabilities, we must perform a $J$ dimensional integral over $f(\varepsilon_i)$.

$$P_{ij} \quad = \quad \int I(\varepsilon_{ij} - \varepsilon_{ik} > V_{ik} - V_{ij}) f(\varepsilon_i) \partial \varepsilon_i$$

There are some choices that make our life easier

- Multivariate normal: $\varepsilon_i \sim N(0, \Omega)$. $\longrightarrow$ multinomial probit.
- Gumbel/Type 1 EV: $f(\varepsilon_i) = e^{-\varepsilon_{ij}} e^{-e^{-\varepsilon_{ij}}}$ and $F(\varepsilon_i) = 1 - e^{-e^{-\varepsilon_{ij}}}$ $\longrightarrow$ multinomial logit
- There are also heteroskedastic variants of the Type I EV/ Logit framework.

# Errors

Allowing for full support $[-\infty, \infty]$ errors provide two key features:

- Smoothness: $P_{ij}$ is everywhere continuously differentiable in $V_{ij}$.
- Bound $P_{ij} \in (0, 1)$ so that we can rationalize any observed pattern in the data.
- What does $\varepsilon_{ij}$ really mean? (unobserved utility, idiosyncratic tastes, etc.)

# Basic Identification

- Only differences in utility matter: $Prob(\varepsilon_{ij} - \varepsilon_{ik} > V_{ik} - V_{ij} \quad \forall j \neq k)$
- Adding constants is irrelevant: if $U_{ij} > U_{ik}$ then $U_{ij} + a > U_{ik} + a$.
- Only differences in alternative specific constants can be identified

  $$\begin{aligned} U_b &= X_b\beta + k_b + \varepsilon_b \\ U_c &= X_c\beta + k_c + \varepsilon_c \end{aligned}$$

  only $d = k_b - k_c$ is identified.
- This means that we can only include $J - 1$ such $k$'s and need to normalize one to zero. (Much like fixed effects).
- We cannot have individual specific factors that enter the utility of all options such as income $\theta Y_i$. We can allow for interactions between individual and choice characteristics $\theta p_j / Y_i$.

# Basic Identification

## Location

- Technically we can't really fully specify $f(\varepsilon_i)$ since we can always re-normalize: $\widetilde{\varepsilon_{ijk}} = \varepsilon_{ij} - \varepsilon_{ik}$ and write $g(\widetilde{\varepsilon_{ik}})$. Thus any $g(\widetilde{\varepsilon_{ik}})$ is consistent with infinitely many $f(\varepsilon_i)$.
- Logit pins down $f(\varepsilon_i)$ sufficiently with parametric restrictions.
- Probit does not. We must generally normalize one dimension of $f(\varepsilon_i)$ in the probit model. Usually a diagonal term of $\Omega$ so that $\omega_{11} = 1$ for example. (Actually we need to do more!).

## Scale

- Consider: $U_{ij}^0 = V_{ij} + \varepsilon_{ij}$ and $U_{ij}^1 = \lambda V_{ij} + \lambda\varepsilon_{ij}$ with $\lambda > 0$. Multiplying by constant $\lambda$ factor doesn't change any statements about $U_{ij} > U_{ik}$.
- We normalize this by fixing the variance of $\varepsilon_{ij}$ since $Var(\lambda\varepsilon_{ij}) = \sigma_e^2\lambda^2$.
- Normalizing this variance normalizes the scale of utility.

# Observed Heteroskedasticity

Consider the case where $Var(\varepsilon_{ij}^{B}) = \sigma^2$ and $Var(\varepsilon_{ij}^{C}) = k^2\sigma^2$ :

- We can estimate

$$\begin{aligned} U_{ij} &= x_j\beta + \varepsilon_{ij}^{B} \\ U_{ij} &= x_j\beta + \varepsilon_{ij}^{C} \end{aligned}$$

  becomes:

$$\begin{aligned} U_{ij} &= x_j\beta + \varepsilon_{ij} \\ U_{ij} &= x_j\beta/k + \varepsilon_{ij} \end{aligned}$$

- Some interpret this as saying that in segment $C$ the unobserved factors are $\hat{k}$ times larger.

# Deeper Identification Results

Different ways to look at identification

- Are we interested in non-parametric identification of $V_{ij}$, specifying $f(\varepsilon_i)$?
- Or are we interested in non-parametric identification of $U_{ij}$. (Generally hard).
  - Generally we require a large support (special-regressor) or “completeness” condition.
  - Lewbel (2000) does random utility with additively separable but nonparametric error.
  - Berry and Haile (2015) with non-separable error (and endogeneity).

# Logit

- Logit has closed form choice probabilities

$$s_{ij} = \frac{e^{V_{ij}}}{\sum_k e^{V_{ik}}} \approx \frac{e^{\beta' x_{ij}}}{\sum_k e^{\beta' x_{ik}}}$$

- Approximation arises from the hope that we can approximate $V_{ij} \approx X_{ik}\beta$ with something linear in parameters.
- Expected maximum also has closed form:

$$E[\max_j U_{ij}] = \log\left(\sum_j \exp[V_{ij}]\right) + C$$

# Logit Inclusive Value

- Logit Inclusive Value is helpful for several reasons

$$E[\max_j U_{ij}] = \log\left(\sum_j \exp[V_{ij}]\right) + C$$

- Expected utility of best option (without knowledge of realized $\varepsilon_i$) does not depend on $\epsilon_{ij}$.
- This is a globally concave function in $V_{ij}$ (more on that later).
- Allows simple computation of $\Delta CS$ for consumer welfare.

# Multinomial Logit

Multinomial Logit goes by a lot of names in various literatures

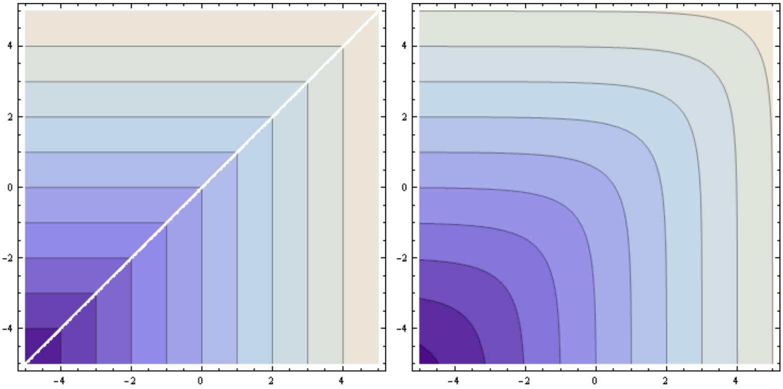
- The problem of multiple choice is often called multiclass classification or softmax regression in other literatures.
- In general these models assume you have individual level data

# Alternative Interpretation

Statistics/Computer Science offer an alternative interpretation

- Sometimes this is called softmax regression.
- Think of this as a continuous/concave approximation to the maximum.
- Consider $\max\{x, y\}$ vs $\log(\exp(x) + \exp(y))$. The exp exaggerates the differences between $x$ and $y$ so that the larger term dominates.
- We can accomplish this by rescaling $k$: $\log(\exp(kx) + \exp(ky))/k$ as $k$ becomes large the derivatives become infinite and this approximates the "hard" maximum.
- $g(1, 2) = 2.31$, but $g(10, 20) = 20.00004$.

# Alternative Interpretation



# Multinomial Logit: Identification

What is actually identified here?

- Helpful to look at the ratio of two choice probabilities

$$\log \frac{s_{ij}(\theta)}{s_{ik}(\theta)} = \mathrm{x}_{\mathrm{ij}}\beta_j - \mathrm{x}_{\mathrm{ik}}\beta_k \rightarrow \mathrm{x}_{\mathrm{i}} \cdot (\beta_j - \beta_k)$$

- We only identify the difference in indirect utilities not the levels.
- This is a feature and not a bug. Why?

# Multinomial Logit: Identification

As another idea suppose we add a constant $C$ to each $\beta_j$.

$$s_{ij} = \frac{\exp[\mathrm{x_i}(\beta_j + C)]}{\sum_k \exp[\mathrm{x_i}(\beta_k + C)]} = \frac{\exp[\mathrm{x_i} C] \exp[\mathrm{x_i}\beta_j]}{\exp[\mathrm{x_i} C] \sum_k \exp[\mathrm{x_i}\beta_k]}$$

- This has no effect. That means we need to fix a normalization $C$. The most convenient is generally that $C = -\beta_K$.
- We normalize one of the choices to provide a utility of zero.
- We actually already made another normalization. Does anyone know what?

# Multinomial Logit: Identification

The most sensible normalization in demand settings is to allow for an outside option which produces no utility in expectation.

$$s_{ij} = \frac{\exp[\mathrm{x_i}\beta_j]}{1 + \sum_k \exp[\mathrm{x_i}\beta_k]}$$

- Hopefully the choice of outside option is well defined: not buying a yogurt, buying some other used car, etc.
- Now this resembles the binomial logit model more closely.

# Back to Scale of Utility

- Consider $U^*_{ij} = V_{ij} + \varepsilon^*_{ij}$ with $Var(\varepsilon^*) = \sigma^2\pi^2/6$.
- Without changing behavior we can divide by $\sigma$ so that $U_{ij} = V_{ij}/\sigma + \varepsilon_{ij}$ and $Var(\varepsilon^*/\sigma) = Var(\varepsilon) = \pi^2/6$

$$s_{ij} = \frac{e^{V_{ij}/\sigma}}{\sum_k e^{V_{ik}/\sigma}} \approx \frac{e^{\beta^*/\sigma \cdot x_{ij}}}{\sum_k e^{\beta^*/\sigma \cdot x_{ik}}}$$

- Every coefficient $\beta$ is rescaled by $\sigma$. This implies that only the ratio $\beta^*/\sigma$ is identified.
- Coefficients are relative to variance of unobserved factors. More unobserved variance $\longrightarrow$ smaller $\beta$.
- Ratio $\beta_1/\beta_2$ is invariant to the scale parameter $\sigma$.

# Taste Variation

- Logit allows for taste variation across individuals if two conditions are met: individual level data and interact observed characteristics only.
- We often want to allow for something like $U_{ij} = x_j \beta_i - \alpha_i p_j + \varepsilon_{ij}$.
- We might want $\beta_i = \theta / y_i$ where $y_i$ is the income for individual $i$ or $\beta_i = \theta y_i$, etc.
- Can also have $z_{ij}$ such as the distance between $i$ and hospital $j$.
- Cannot have unobserved heterogeneity or heteroskedasticity in $\varepsilon_{ij}$.

# Taste Variation

$$\frac{s_{ij}}{s_{ik}} = \frac{e^{V_{ij}}}{\sum_{k'} e^{V_{ik'}}} \Big/ \frac{e^{V_{ik}}}{\sum_{k'} e^{V_{ik'}}} = \frac{e^{V_{ij}}}{e^{V_{ik}}} = \exp[V_{ij} - V_{ik}].$$

- The ratio of choice probabilities for $j$ and $k$ depends only on $j$ and $k$ and not on any alternative $l$, this is known as independence of irrelevant alternatives.
- For some (Luce (1959)) IIA was an attractive property for axiomatizing choice.
- In fact the logit was derived in the search for a statistical model that satsified various axioms.

# IIA Property

- The well known counterexample: You can choose to go to work on a car $c$ or blue bus $bb$. $P_c = P_{bb} = \frac{1}{2}$ so that $\frac{P_c}{P_{bb}} = 1$.
- Now we introduce a red bus $rb$ that is identical to $bb$. Then $\frac{P_{rb}}{P_{bb}} = 1$ and $P_c = P_{bb} = P_{rb} = \frac{1}{3}$ as the logit model predicts.
- In reality we don't expect painting a bus red would change the number of individuals who drive a car so we would anticipate $P_c = \frac{1}{2}$ and $P_{bb} = P_{rb} = \frac{1}{4}$.
- We may not encounter too many cases where $\rho_{\varepsilon_{ik}, \varepsilon_{ij}} \approx 1$, but we have many cases where this $\rho_{\varepsilon_{ik}, \varepsilon_{ij}} \neq 0$
- What we need is the ratio of probabilities to change when we introduce a third option!

# IIA Property

- IIA implies that we can obtain consistent estimates for $\beta$ on any subset of alternatives.
- This means instead of using all $J$ alternatives in the choice set, we could estimate on some subset $S \subset J$.
- This used to be a way to reduce the computational burden of estimation (not clear this is an issue in 2016).
- Sometimes we have choice based samples where we oversample people who choose a particular alternative. Manski and Lerman (1977) show we can get consistent estimates for all but the ASC. This requires knowledge of the difference between the true rate $A_j$ and the choice-based sample rate $S_j$.
- Hausman proposes a specification test of the logit model: estimate on the full dataset to get $\hat{\beta}$, construct a smaller subsample $S^k \subset J$ and $\hat{\beta}^k$ for one or more subsets $k$. If $|\hat{\beta}^k - \hat{\beta}|$ is small enough.

# IIA Property

$$\frac{\partial s_{ij}}{\partial z_{ij}} = s_{ij}(1 - s_{ij})\frac{\partial V_{ij}}{\partial z_{ij}}$$

And Elasticity:

$$\frac{\partial \log s_{ij}}{\partial \log z_{ij}} = s_{ij}(1 - s_{ij})\frac{\partial V_{ij}}{\partial z_{ij}}\frac{z_{ij}}{s_{ij}} = (1 - s_{ij})z_{ij}\frac{\partial V_{ij}}{\partial z_{ij}}$$

With cross effects:

$$\frac{\partial s_{ij}}{\partial z_{ik}} = -s_{ij}s_{ik}\frac{\partial V_{ik}}{\partial z_{ik}}$$

And Elasticity:

$$\frac{\partial \log s_{ij}}{\partial \log z_{ik}} = -s_{ik}z_{ik}\frac{\partial V_{ik}}{\partial z_{ik}}$$

For the linear $V_{ij}$ case we have that $\frac{\partial V_{ij}}{\partial z_{ij}} = \beta_z$.

# Proportional Substitution

Cross elasticity doesn't really depend on $j$.

$$\frac{\partial \log s_{ij}}{\partial \log z_{ik}} = -s_{ik} z_{ik} \underbrace{\frac{\partial V_{ik}}{\partial z_{ik}}}_{\beta_z}.$$

- This leads to the idea of proportional substitution. As option $k$ gets better it proportionally reduces the shares of the all other choices.
- Likewise removing an option $k$ means that $\tilde{s}_{ij} = \frac{s_{ij}}{1-s_{ik}}$ for all other $j$.
- This might be a desirable property but probably not.

# Multinomial Logit: Estimation with Individual Data

Estimation is straightforward via Maximum Likelihood (MLE):

$$\begin{aligned}
L(\mathrm{y}|\mathrm{x},\theta) &= \prod_{i=1}^{N} \underbrace{\frac{n_i!}{\prod_{j=1}^{J} y_{ij}!}}_{C(\mathrm{y})} \prod_{j=1}^{J} s_{ij}(x_{ij},\theta)^{y_{ij}} \\
ll(\mathrm{y}|\mathrm{x},\theta) &= \sum_{i=1}^{N} \log(C(\mathrm{y})) + \sum_{i=1}^{N}\sum_{j=1}^{J} y_{ij} \log(s_{ij}(x_{ij},\theta)) \\
l(\mathrm{y}|\mathrm{x},\theta) &\approx \sum_{i=1}^{N}\sum_{j=1}^{J} y_{ij} \log(s_{ij}(x_{ij},\theta))
\end{aligned}$$

- We can ignore the combinatorial term (with the factorials) since it does not affect the location of the maximum (it is additive and doesn't depend on $\theta$).

# Multinomial Logit: Inclusive Value

To be more specific:

- Let's look a little more closely at what's going on:

$$\sum_{i=1}^{N}\sum_{j=1}^{J} y_{ij}\left[x_{ij}\beta - \underbrace{\log\left(\sum_{k=1}^{K} x_{ik}\beta\right)}_{IV_i(\mathrm{x_i},\theta)}\right]$$

- We call the term on the right the logit inclusive value. It does not depend on $k$ but might vary across choice situations/individuals $i$.
- The point of the inclusive value is to guarantee that $\sum_{k=1} s_{ik}(\mathrm{x_i},\theta) = 1$.
- If we somehow observed $IV_i(\theta)$ we could just do linear regression (in fact we could do this separately for each $K$).

# Multinomial Logit: Estimation with Aggregate Data

Estimation is just like before

- Suppose that all consumers had the same $x_{ij} = x_j$ (Choices depended only on products not on income, education, etc.)
- We can construct $y_j^* = \sum_{i=1}^N y_{ij}$.

$$l(\mathrm{y}|\mathrm{x}, \theta) \approx \sum_{j=1}^{J} y_j^* \log(s_j(\mathrm{x}, \theta))$$

- When each consumer $i$ faces the same choice environment, we can aggregate data into sufficient statistics.

# Multinomial Logit: Estimation with Aggregate Data

Aggregation is probably the most important property of the logit:

- Instead of individual data, or a single group we might have multiple groups: if prices only change once per week, we can aggregate all of the week's sales into one "observation".
- Likewise if we only observe that an individual is within one of five income buckets – there is no loss from aggregating our data into these five buckets.
- All of this depends on the precise form of $s_j(\mathrm{x_i}, \theta)$. When it doesn't change across observations: we can aggregate.
- It functions as if we have a representative consumer up to $\varepsilon_i$.
- We can use this idea to go from individual level to market demand: $q_j(\mathrm{x_i}) = N_i s_{ij}(\theta)$.

# Multinomial Logit: Elasticity

An important output from a demand system are elasticities

- An important element in $\mathrm{x_i}$ are prices $[p_1, \ldots, p_J]$
- Helpful to write $u_{ij} = x_j\beta - \alpha p_j$ (assumes aggregation!).

$$\frac{\partial q_j}{\partial p_k} = -N \cdot \alpha \left( I[j = k]s_j - \sum_{k=1}^{K} s_j s_k \right)$$

- This implies that $\eta_{jj} = \frac{\partial q_j}{\partial p_j}\frac{p_j}{q_j} = -\alpha p_j(1 - s_j)$.
- The price elasticity is increasing in own price! (Why is this a bad idea?)
- $\eta_{jk} = \frac{\partial q_j}{\partial p_k}\frac{p_k}{q_j} = -\alpha p_k s_k$.
- The cross price elasticity doesn't depend on which product $j$ you are talking about!

# Multinomial Logit: IIA

The multinomial logit is frequently criticized for producing unrealistic substitution patterns

- Suppose we got rid of a product $k$ then $s_j^{(1)} = s_j^{(0)} \frac{1}{1-s_k}$.
- Substitution is just proportional to your pre-existing shares $s_j$
- No concept of "closeness" of competition!

# Can we do better?

Multinomial Probit?

- The probit has $\varepsilon_i \sim N(0, \Sigma)$.
- If $\Sigma$ is unrestricted, then this can produce relatively flexible substitution patterns.
- Flexible is relative: still have normal tails, only pairwise correlations, etc.
- It might be that $\rho_{12}$ is large if $1, 2$ are similar products.
- Much more flexible than Logit

Downside

- $\Sigma$ has potentially $J^2$ parameters (that is a lot)!
- Maybe $J * (J-1)/2$ under symmetry. (still a lot).
- Each time we want to compute $s_j(\theta)$ we have to simulate an integral of dimension $J$.
- I wouldn't do this for $J \geq 5$.

# Relaxing IIA

Let's make $\varepsilon_{ij}$ more flexible than IID. Hopefully still have our integrals work out.

$$u_{ij} = x_{ij}\beta + \varepsilon_{ij}$$

- One approach is to allow for a block structure on $\varepsilon_{ij}$ (and consequently on the elasticities).
- We assign products into groups $g$ and add a group specific error term

$$u_{ij} = x_{ij}\beta + \eta_g + \varepsilon_{ij}$$

- The trick putting a distribution on $\eta_g + \varepsilon_{ij}$ so that the integrals still work out.
- Do not try this at home: it turns out the required distribution is known as GEV and the resulting model is known as the nested logit.

# Nested Logit

A traditional (and simple) relaxation of the IIA property is the Nested Logit. This model is often presented as two sequential decisions.

- First consumers choose a category (following an IIA logit).
- Within a category consumers make a second decision (following the IIA logit).
- This leads to a situation where while choices within the same nest follow the IIA property (do not depend on attributes of other alternatives) choices among different nests do not!

# Alternative Interpretation

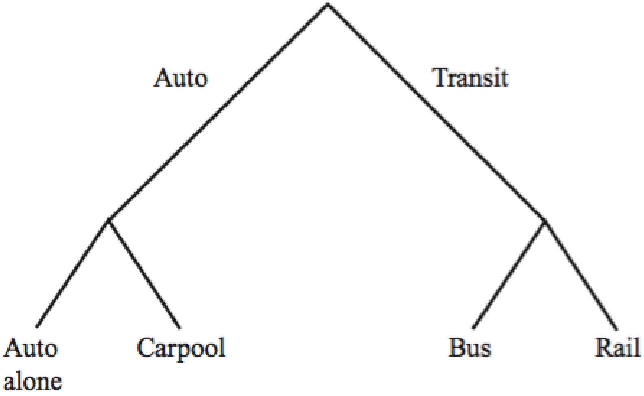


Figure 4.1. Tree diagram for mode choice.

# Nested Logit

Utility looks basically the same as before:

$$U_{ij} = V_{ij} + \underbrace{\eta_{ig} + \widetilde{\varepsilon_{ij}}}_{\varepsilon_{ij}(\lambda_g)}$$

- We add a new term that depends on the group $g$ but not the product $j$ and think about it as varying unobservably over individuals $i$ just like $\varepsilon_{ij}$.
- Now $\varepsilon_i \sim F(\varepsilon)$ where $F(\varepsilon) = \exp[-\sum_{g=G}^{G} \left( \sum_{j \in J_g} \exp[-\varepsilon_{ij}/\lambda_g] \right)^{\lambda_g}$. This is no longer Type I EV but GEV.
- The key is the addition of the $\lambda_g$ parameters which govern (roughly) the within group correlation.
- This distribution is a bit cooked up to get a closed form result, but for $\lambda_g \in [0, 1]$ for all $g$ it is consistent with random utility maximization.

# Nested Logit

The nested logit choice probabilities are:

$$P_{ij} = \frac{e^{V_{ij}/\lambda_g} \left( \sum_{k \in J_g} e^{V_{ik}/\lambda_g} \right)^{\lambda_g - 1}}{\sum_{h=1}^{G} \left( \sum_{k \in J_h} e^{V_{ik}/\lambda_h} \right)^{\lambda_h}}$$

Within the same group $g$ we have IIA and proportional substitution

$$\frac{P_{ij}}{P_{ik}} = \frac{e^{V_{ij}/\lambda_g}}{e^{V_{ik}/\lambda_g}}$$

But for different groups we do not:

$$P_{ij} = \frac{e^{V_{ij}/\lambda_g} \left( \sum_{k \in J_g} e^{V_{ik}/\lambda_g} \right)^{\lambda_g - 1}}{e^{V_{ik}/\lambda_h} \left( \sum_{k \in J_h} e^{V_{ik}/\lambda_h} \right)^{\lambda_h - 1}}$$

# Nested Logit

We can take the probabilities and re-write them slightly with the substitution that $\lambda_g \cdot \underbrace{\log\left(\sum_{k \in J_g} e^{V_{ik}}\right)}_{IV_{ig}}$.

$$\begin{aligned}
P_{ij} &= \frac{e^{V_{ij}/\lambda_g}}{\left(\sum_{k \in J_g} e^{V_{ik}/\lambda_g}\right)} \cdot \frac{\left(\sum_{k \in J_g} e^{V_{ik}/\lambda_g}\right)^{\lambda_g}}{\sum_{h=1}^{G} \left(\sum_{k \in J_h} e^{V_{ik}/\lambda_h}\right)^{\lambda_h}} \\
&= \underbrace{\frac{e^{V_{ij}/\lambda_g}}{\left(\sum_{k \in J_g} e^{V_{ik}/\lambda_g}\right)}}_{P_{ij|g}} \cdot \underbrace{\frac{e^{\lambda_g IV_{ig}}}{\sum_{h=1}^{G} e^{\lambda_h IV_{ih}}}}_{P_{ig}}
\end{aligned}$$

This is the decomposition into two logits that leads to the "sequential logit" story.

# Nested Logit : Notes

- $\lambda_g = 1$ is the simple logit case (IIA)
- $\lambda_g \rightarrow 0$ implies that all consumers stay within the nest.
- $\lambda < 0$ or $\lambda > 1$ can happen and usually means something is wrong. These models are not generally consistent with RUM. (If you report one in your paper I will reject it).
- $\lambda$ is often interpreted as a correlation parameter and this is almost true but not exactly!
- There are other extensions: overlapping nests, or three level nested logit.
- In general the hard part is understanding what the appropriate nesting structure is ex ante. Maybe for some problems this is obvious but for many not.

# Nested Logit

In practice we end up with the following:

$$s_{ij} = s_{ij|g}(\theta) s_{ig}(\theta)$$

- Because the nested logit can be written as the within group share $s_{ij|g}$ and the share of the group $s_{ig}$ we often explain this model as sequential choice
- First you pick a category, then you pick a product within a category.
- This is a sometimes helpful (sometimes unhelpful) way to think about this.
- We can also think about this as putting a block structure on the covariance matrix of $\varepsilon_i$
- You need to assign products to categories before you estimate and you can't make mistakes!

# Nested Logit

How does it actually look?

$$
\begin{aligned}
IV_{ig}(\theta) &= \log\left(\sum_{k\in G}\exp[x_k\beta/(1-\lambda_g)]\right) = E_\varepsilon[\max_{j\in G} u_{ij}] \\
s_{ij|g}(\theta) &= \frac{\exp[x_j\beta/(1-\lambda_g)]}{\sum_{k\in G}\exp[x_k\beta/(1-\lambda_g)]} \\
s_{ig}(\theta) &= \frac{\exp[IV_{ig}]^{1-\lambda_g}}{\sum_h \exp[IV_{ih}]^{1-\lambda_h}}
\end{aligned}
$$

# Nested Logit

How does it actually look?

$$\log\left(\frac{s_{ij|g}(\theta)}{s_{ik|g}(\theta)}\right) = (x_j - x_k)\cdot\frac{\beta}{1-\lambda_g}$$

- We are back to having the IIA property but now within the group $G$.
- We also have IIA across groups $g, h$
- $\lambda_g$ and $\alpha$ govern the elasticities, which also have a block structure.
- Sometimes people refer to this as the product of two logits
- In the old days people used to estimate by fitting sequential IIA logit models – this is consistent but inefficient – you shouldn't do this today!
- Estimation happens via MLE. This can be tricky because the model is non-convex. It helps to substitute $\tilde{\beta} = \beta/(1-\lambda_g)$

# Parametric Identification (Ackerberg and Rysman RJE)

Look at derivatives:

$$\begin{aligned}
\frac{\partial s_{j|g}}{\partial X_j} &= \beta_1 s_{j|g}(1 - s_{j|g}) \\
\frac{\partial s_g}{\partial X} &= (1-\lambda)\beta_1 s_g (1 - s_g) \\
\frac{\partial s_g}{\partial J} &= \frac{1-\lambda}{J} s_g (1 - s_g)
\end{aligned}$$

- We get $\beta$ by changing $x_j$ within group
- We get nesting parameter $\lambda$ by varying $X$
- We don't have any parameters left to explain changing number of products $J$.

# Nested Logit

There are more potential generalizations though they are less frequently used:

- You can have multiple levels of nesting: first I select a size car (compact, mid-sized, full-sized) then I select a manufacturer, finally a car.
- You can have potentially overlapping nests: Yogurt brands are one nest, Yogurt flavors are a second nest. This way strawberry competes with strawberry and/or Dannon substitutes for Dannon.

# Mixed Logit

We relax the IIA property by mixing over various logits:

$$
\begin{aligned}
u_{ijt} &= x_j\beta + \mu_{ij} + \varepsilon_{ij} \\
s_{ij} &= \int \frac{\exp[x_j\beta + \mu_{ij}]}{1 + \sum_k \exp[x_k\beta + \mu_{ik}]} f(\mu_i|\theta)
\end{aligned}
$$

- Each individual draws a vector $\mu_i$ of $\mu_{ij}$ (separately from $\varepsilon$).
- Conditional on $\mu_i$ each person follows an IIA logit model.
- However we integrate (or mix) over many such individuals giving us a mixed logit or heirarchical model (if you are a statistician)
- In practice these are not that different from linear random effects models you have learned about previously.
- It helps to think about fixing $\mu_i$ first and then integrating out over $\varepsilon_i$

# Mixed/ Random Coefficients Logit

As an alternative, we could have specified an error components structure on $\varepsilon_i$.

$$U_{ij} = \beta x_{ij} + \underbrace{\nu_i z_{ij} + \varepsilon_{ij}}_{\tilde{\varepsilon}_{ij}}$$

- The key is that $\nu_i$ is unobserved and mean zero. But that $x_{ij}, z_{ij}$ are observed per usual and $\varepsilon_{ij}$ is IID Type I EV.
- This allows for a heteroskedastic structure on $\varepsilon_i$, but only one which we can project down onto the space of $z$.

An alternative is to allow for individuals to have random variation in $\beta_i$:

$$U_{ij} = \beta_i x_{ij} + \varepsilon_{ij}$$

Which is the random coefficients formulation (these are the same model).

# Mixed/ Random Coefficients Logit

- Kinds of heterogeneity
  - We can allow for there to be two types of $\beta_i$ in the population (high-type, low-type). latent class model.
  - We can allow $\beta_i$ to follow an independent normal distribution for each component of $x_{ij}$ such as $\beta_i = \overline{\beta} + \nu_i \sigma$.
  - We can allow for correlated normal draws using the Cholesky root of the covariance matrix.
  - Can allow for non-normal distributions too (lognormal, exponential). Why is normal so easy?
- The structure is extremely flexible but at a cost.
- We generally must perform the integration numerically.
- High-dimensional numerical integration is difficult. In fact, integration in dimension 8 or higher makes me very nervous.
- We need to be parsimonious in how many variables have unobservable heterogeneity.

# Mixed Logit

How does it work?

- Well we are mixing over individuals who conditional on $\beta_i$ or $\mu_i$ follow logit substitution patterns, however they may differ wildly in their $s_{ij}$ and hence their substitution patterns.
- For example if we are buying cameras: I may care a lot about price, you may care a lot about megapixels, and someone else may care mostly about zoom.
- The basic idea is that we need to explain the heteroskedasticity of $Cov(\varepsilon_i, \varepsilon_j)$ what random coefficients do is let us use a basis from our $X$'s.
- If our $X$'s are able to span the space effectively, then an RC logit model can approximate any arbitrary RUM (McFadden and Train 2002).
- Of course if you have 1000 products and two random coefficients, you are asking for a lot.

# Mixed/ Random Coefficients Logit

Suppose there is only one random coefficient, and the others are fixed:

- $f(\beta_i \theta) \sim N(\overline{\beta}, \sigma)$.
- We can re-write this as the integral over a transformed standard normal density

$$P_{ij}(\theta) = \int \frac{e^{V_{ij}(\nu_i, \theta)}}{\sum_k e^{V_{ik}(\nu_i, \theta)}} f(\nu_i) \partial \nu$$

- Monte Carlo Integration: Independent Normal Case
  - Draw $\nu_i$ from the standard normal distribution.
  - Now we can rewrite $\beta_i = \overline{\beta} + \nu_i \sigma$
  - For each $\beta_i$ calculate $P_{ij}(\beta_i)$.
  - $\frac{1}{S} \sum_{s=1}^{S} P_{ij} = \widehat{P_j^s}$
- Gaussian Quadrature
  - Or we can draw a non-random set of points $\nu_i$ and corresponding weights $w_i$ and approximate the integral to a high level of polynomial accuracy.

# Quadrature in higher dimensions

- Quadrature is great in low dimensions – but scales badly in high dimensions.
- If we need $N_a$ points to accurately approximate the integral in $d = 1$ then we need $N_a^d$ points in dimension $d$ (using the tensor product of quadrature rules).
- There is some research on quadrature rules that nest and also how to carefully eliminate points so that the number doesn't grow so quickly.
- Try `sparse-grids.de`

# Estimation

How do we actually estimate these models?

- In practice we should be able to do MLE.

$$\max_{\theta} \sum_{i=1}^{N} y_{ij} \log P_{ij}(\theta)$$

- When we are doing IIA logit, this problem is globally convex and is easy to estimate using Newton's Method.
- When doing nested logit or random coefficients logit, it generally is non-convex which can make life difficult.
- The tough part is generally working out what $\frac{\partial \log P_{ij}}{\partial \theta}$ is, especially when we need to simulate to obtain $P_{ij}$.
- It turns out that MSLE actually has consistent problems for fixed $S$. Why?
- Alternative? MSM/MoM type estimators (next time).

# Mixed Logit: Estimation

- Just like before, we do MLE
- One wrinkle–how do we compute the integral?

$$
\begin{aligned}
s_{ij} &= \int \frac{\exp[x_j \beta_i]}{1 + \sum_k \exp[x_k \beta_i]} f(\beta_i|\theta) \\
&= \sum_{s=1}^{ns} w_s \frac{\exp[x_j(\overline{\beta} + \Sigma \nu_{is})]}{1 + \sum_k \exp[x_k(\overline{\beta} + \Sigma \nu_{is})]}
\end{aligned}
$$

- Option 1: Monte Carlo integration. Draw $NS = 1000$ or so samples of $\nu_i$ from the standard normal and set $w_i = \frac{1}{NS}$.
- Option 2: Quadrature. Choose $\nu_i$ and $w_i$ according to a Gaussian quadrature rule. Like `quad` in MATLAB.
- Personally I get nervous about integrals in dimension greater than 5. People routinely have 20 or more though.

# Mixed Logit: Hints

How bad is the simulation error?

- Depends how small your shares are.
- Since you care about $\log s_{jt}$ when shares are small, tiny errors can be enormous.
- Often it is pretty bad.
- I recommend sticking with quadrature at a high level of precision.
- `sparse-grids.de` provide efficient high dimensional quadrature rules.

# Even More Flexibility (Fox, Kim, Ryan, Bajari)

Suppose we wanted to nonparametrically estimate $f(\beta_i|\theta)$ instead of assuming that it is normal or log-normal.

$$s_{ij} \quad = \quad \int \frac{\exp[x_j \beta_i]}{1 + \sum_k \exp[x_k \beta_i]} f(\beta_i|\theta)$$

- Choose a distribution $g(\beta_i)$ that is more spread out that $f(\beta_i|\theta)$
- Draw several $\beta_s$ from that distribution (maybe 500-1000).
- Compute $\hat{s}_{ij}(\beta_s)$ for each draw of $\beta_s$ and each $j$.
- Holding $\hat{s}_{ij}(\beta_s)$ fixed, look for $w_s$ that solve

$$\min_w \left( s_j - \sum_{s=1}^{ns} w_s \hat{s}_{ij}(\beta_s) \right)^2 \quad \text{s.t. } \sum_{s=1}^{ns} w_s = 1, \quad w_s \geq 0 \quad \forall s$$

# Even More Flexibility (Fox, Kim, Ryan, Bajari)

- Like other semi-/non- parametric estimators, when it works it is both general and very easy.
- We are solving a least squares problem with constraints: positive coefficients, coefficients sum to 1.
- It tends to produce sparse models with only a small number of $\beta_s$ getting positive weights.
- This is way easier than solving a random coefficients logit model with all but the simplest distributions.
- There is a bias-variance tradeoff in choosing $g(\beta_i)$.
- Incorporating parameters that are not random coefficients loses some of the simplicity.
- I have no idea how to do this with large numbers of fixed effects.

# Convexity and Computation

# Convexity

## An optimization problem is convex if

$$\min_{x} f(\mathrm{x}) \quad s.t. \quad h(\mathrm{x}) \leq 0 \quad A\mathrm{x} = 0$$

- $f(\mathrm{x}), h(\mathrm{x})$ are convex (PSD second derivative matrix)
- Equality Constraint is affine

## Some helpful identities about convexity

- Compositions and sums of convex functions are convex.
- Norms $||$ are convex, max is convex, log is convex
- $\log(\sum_{i=1}^{n} \exp(x_i))$ is convex.
- Fixed Points can introduce non-convexities.
- Globally convex problems have a unique optimum

# Properties of Convex Optimization

- If a program is globally convex then it has a unique minimizer that will be found by convex optimizers.
- If a program is not globally convex, but is convex over a region of the parameter space, then most convex optimization routines find any local minima in the convex hull
- Convex optimization routines are unlikely to find local minima (including the global minimum) if they do not begin in the same convex hull as the optimum (starting values matter!).
- Most good commercial routines are clever about dealing with multiple starting values and handling problems that are well approximated by convex functions.
- Good Routines use information about sparseness of Hessian – this generally determines speed.

# Nested Logit Model

## FIML Nested Logit Model is Non-Convex

$$\min_{\theta} \sum_j q_j \ln P_j(\theta) \quad \text{s.t.} \quad P_j(\theta) = \frac{e^{x_j \beta/\lambda} (\sum_{k \in g_l} e^{x_j \beta/\lambda})^{\lambda-1}}{\sum_{\forall l'} (\sum_{k \in g_l'} e^{x_j \beta/\lambda})^{\lambda}}$$

This is a pain to show but the problem is with the cross term $\frac{\partial^2 P_j}{\partial \beta \partial \lambda}$ because $\exp[x_j \beta/\lambda]$ is not convex.

## A Simple Substitution Saves the Day: let $\gamma = \beta/\lambda$

$$\min_{\theta} \sum_j q_j \ln P_j(\theta) \quad \text{s.t.} \quad P_j(\theta) = \frac{e^{x_j \gamma} (\sum_{k \in g_l} e^{x_j \gamma})^{\lambda-1}}{\sum_{\forall l'} (\sum_{k \in g_l'} e^{x_j \gamma})^{\lambda}}$$

This is much better behaved and easier to optimize.

# Nested Logit Model

| | Original[1] | Substitution[2] | No Derivatives[3] |
|---|---:|---:|---:|
| Parameters | 49 | 49 | 49 |
| Nonlinear $\lambda$ | 5 | 5 | 5 |
| Likelihood | 2.279448 | 2.279448 | 2.27972 |
| Iterations | 197 | 146 | 352 |
| Time | 59.0 s | 10.7 s | 192s |

Discuss Nelder-Meade

[1] KNITRO-AMPL

[2] KNITRO-AMPL

[3] fminunc-MATLAB

# Computing Derivatives

A key aspect of any optimization problem is going to be computing the derivatives (first and second) of the model. There are some different approaches

- Numerical: Often inaccurate and error prone (why?)
- Pencil and Paper: this tends to be mistake prone – but often actually the fastest
- Automatic (AMPL): Software brute forces through a chain rule calculation at every step (limited language).
- Symbolic (Maple/Mathematica): software “knows” derivatives of certain objects and can do its own simplification. (limited language).