Week 3: Demand Estimation

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Grad IO

Heterogeneity and Endogeneity

Putting it Together

- Now we want to have both price endogeneity and flexible substitution in the same model.
- We are ultimately going with the random coefficients logit model, but we will start with the logit and nested logit.

Basic Idea from Price Endogeneity

$$s_{jt} \quad = \quad \int \frac{\exp[x_{jt}\beta_i]}{1+\sum_k \exp[x_{kt}\beta_i]} f(\beta_i|\theta)$$

- We know prices are set with demand in mind and this can create an endogeneity problem.
- How do we deal with it?
- We would like to instrument in this world but what is the error term exactly?
- An obvious choice might be $\eta_{jt} = (s_{jt}(\theta) - \tilde{s}_{jt})$
- Can we find things that are orthogonal to the error between observed and predicted market shares?
- Do we have the usual IV conditions (exogeneity, relevance, monotonicity, etc.)

- We need to add an unobservable quality term ξ_{jt} to our model

$$\begin{aligned} u_{ijt} &= x_{jt}\beta_i + \xi_{jt} + \varepsilon_{ij} \\ s_{jt} &= \int \frac{\exp[x_{jt}\beta_i + \xi_{jt}]}{1 + \sum_k \exp[x_{kt}\beta_i + \xi_{kt}]} f(\beta_i|\theta) \end{aligned}$$

- The idea is that ξ_{jt} is observed to the firm when prices are set, but not to us the econometricians.
- We call ξ_{jt} a vertical component, because all consumers agree on its value.
- This allows for products j to better than some other product in a way that is not fully explained by differences in x_j and x_k.
- Basically there is something about a BMW that makes it better than a Peugeot in a way that is not fully captured by its mileage, weight, horsepower, etc. that leads to it having higher sales and/or higher prices.

- Think about the plain IIA logit for a minute:

$$
\begin{aligned}
u_{ijt} &= x_{jt}\beta + \xi_{jt} + \varepsilon_{ij} \\
s_{jt} &= \frac{\exp[x_{jt}\beta + \xi_{jt}]}{1 + \sum_k \exp[x_{kt}\beta + \xi_{kt}]}
\end{aligned}
$$

- Take logs

$$
\begin{aligned}
\ln s_{0t} &= -\log\left(1 + \sum_k \exp[x_{kt}\beta + \xi_{kt}]\right) \\
\ln s_{jt} &= [x_{jt}\beta - \alpha p_{jt} + \xi_{jt}] - \log\left(1 + \sum_k \exp[x_{kt}\beta + \xi_{kt}]\right) \\
\ln s_{jt} - \ln s_{0t} &= x_{jt}\beta - \alpha p_{jt} + \xi_{jt}
\end{aligned}
$$

Inversion: IIA Logit

$$\underbrace{\ln s_{jt} - \ln s_{0t}}_{data!} = \underbrace{x_{jt}\beta - \alpha p_{jt} + \xi_{jt}}_{\delta_{jt}}$$

- The LHS is data! The RHS is now a linear IV problem!
- α is the price coefficient on the endogenous variable.
- We know how to solve this. We need instruments that shift p_{jt} but are orthogonal to ξ_{jt}.
- Economic theory tells us how: cost shifters, markup shifters.
- Markups in IIA logit are pretty boring since they only depend on your shares and α.
- If number of products varies across markets, that works. Otherwise you want cost shifters in cross section or time series.

Was that magic?

- No. It was just a nonlinear change of variables from $\eta_{jt} \rightarrow \xi_{jt}$.
- Our (conidtional) moment condition is just that $E[\xi_{jt}|x_{jt}, z_{jt}] = 0$.
- We moved from the space of shares and MLE for the logit to the space of utilities and an IV model.
- We are losing some efficiency – but now we are able to estimate under weaker conditions.

- We do need a technical condition. This only works if the market size $N \to \infty$.
- That is our data/shares we must believe we are observing without any sampling error.
- This is not necessary for the multinomial MLE where shares have some natural sampling variation.
- In our IV/GMM approach we cannot have this sampling error. (Why?).

Inversion: Nested Logit (Berry 1994 / Cardell 1991)

This takes a bit more algebra but not much

$$\underbrace{\ln s_{jt} - \ln s_{0t}}_{data!} = x_{jt}\beta - \alpha p_{jt} - \sigma \underbrace{\log(s_{j|gt})}_{data!} + \xi_{jt}$$

- Same as logit plus an extra term $\log(s_{j|g})$ the within group share.
- We now have a second endogenous parameter.
- If you don't see it – realize we are regressing Y on a function of Y. This should always make you nervous.
- If you forget to instrument for σ you will get $\sigma \to 1$ because of attenuation bias.
- A good instrument for σ is the number of products within the nest. Why?

We can't solve for δ_{jt} directly this time. We often exploit a trick when β_i, ν_i is normally distributed:

$$s_{jt} = \int \frac{\exp[\delta_{jt} + x_{jt} \cdot \Sigma \cdot \nu_i]}{1 + \sum_k \exp[\delta_{kt} + x_{kt} \cdot \Sigma \cdot \nu_i]} f(\nu_i|\theta)$$

- This is a $J \times J$ system of equations for each t.
- It is diagonally dominant.
- There is a unique vector ξ_t that solves it for each market t.
- If you can work out $\frac{\partial s_{jt}}{\partial \delta_{kt}}$ (easy) you can solve this using Newton's Method.

Contraction: BLP

BLP actually propose an easy solution to find δ_t. Fix θ and solve for δ. Think about doing this one market at a time:

$$\delta^{(k)}(\theta) = \delta^{(k-1)}(\theta) + \log(\tilde{s}_j) - \log(s_j(\delta_t^{(k-1)}, \theta)$$

- They prove (not easy) that this is a contraction mapping.
- If you keep iterating this equation enough $|\delta^{(k)}(\theta) - \delta^{(k-1)}(\theta)| < \epsilon_{tol}$ you can recover the δ's so that the observed shares and the predicted shares are identical.
- Practical tip: ϵ_{tol} needs to be as small as possible. ($\approx 10^{-13}$).
- Practical tip: Contraction isn't as easy as it looks: $\log(s_j(\delta_t^{(k-1)}, \theta)$ requires computing the numerical integral each time (either via quadrature or monte carlo).

BLP Pseudocode

From the outside, in:

- Outer loop: search over nonlinear parameters θ to minimize GMM objective:

$$\widehat{\theta_{BLP}} = \arg\max_{\theta} (Z'\hat{\xi}(\theta)) W (Z'\hat{\xi}(\theta))'$$

- Inner Loop:
  - Fix θ.
  - Solve for δ so that $s_{jt}(\delta, \theta) = \tilde{s}_{jt}$.
    - Computing $s_{jt}(\delta, \theta)$ requires numerical integration (quadrature or monte carlo).
  - We can do IV-GMM to recover $\hat{\alpha}(\theta), \hat{\beta}(\theta), \hat{\xi}(\theta)$.

$$\delta_{jt} = x_{jt}\beta - \alpha p_{jt} + \xi_{jt}$$

  - Use $\hat{\xi}(\theta)$ to construct moment conditions.
- When we have found $\hat{\theta}_{BLP}$ we can use this to update $W \rightarrow W(\hat{\theta}_{BLP})$ and do 2-stage GMM.

- Now that you have done change of variables to get:

$$\delta_{jt} = x_{jt}\beta - \alpha p_{jt} + \xi_{jt}$$

- We can do IV-GMM to recover $\hat{\alpha}(\theta), \hat{\beta}(\theta), \hat{\xi}(\theta)$.
- Outer Loop update guess θ, solve for δ and repeat.

$$\widehat{\theta_{BLP}} = \arg\max_{\theta} (Z'\hat{\xi}(\theta)) W (Z'\hat{\xi}(\theta))'$$

- When we have found $\hat{\theta}_{BLP}$ we can use this to update $W \rightarrow W(\hat{\theta}_{BLP})$ and do 2-stage GMM.

- BLP give us both a statistical estimator and an algorithm to obtain estimates.
- Plenty of other algorithms exist
  - We could solve for δ using the contraction mapping, using `fsolve` / Newton's Method / Guess and Check (not a good idea!).
  - We could try and consider a non-nested estimator for the BLP problem instead of solving for $\delta(\theta), \xi(\theta)$ we could let $\delta, \xi, \alpha, \beta$ be free parameters.
- We could think about different statistical estimators such as K-step GMM, Continuously Updating GMM, etc.

Dube Fox Su (2012)

$$\begin{aligned}
\arg\min_{\theta_2} \quad & \psi'\Omega^{-1}\psi \quad \text{s.t.} \\
\psi &= \xi(\theta_2)'Z \\
\xi_{jt}(\theta) &= \delta_{jt}(\theta_2) - x_{jt}\beta - \alpha p_{jt} \\
\log(S_{jt}) &= \log(s_{jt}(\delta, \theta_2))
\end{aligned} \tag{1}$$

$$\begin{aligned}
\arg \min_{\theta_2, \alpha, \beta, \xi, \psi} \quad & \psi'\Omega^{-1}\psi \quad \text{s.t.} \\
\psi &= \xi' Z \\
\xi_{jt} &= \delta_{jt} - x_{jt}\beta - \alpha p_{jt} \\
\log(S_{jt}) &= \log(s_{jt}(\theta_2, \delta))
\end{aligned} \tag{2}$$

Comparing Approaches

- The original BLP paper and the DFS paper define different algorithms to produce the same statistical estimator.
  - The BLP algorithm is a nested fixed point (NFP) algorithm.
  - The DFS algorithm is a mathematical program with equilibrium constraints (MPEC).
  - The unknown parameters satisfy the same set of first-order conditions. (Not only asymptotically, but in finite sample).
  - $\hat{\theta}_{NFP} \approx \hat{\theta}_{MPEC}$ but for numerical differences in the optimization routine.
- Our choice of algorithm should mostly be about computational convenience.

BLP: NFP Advantages/Disadvantages

- Advantages
  - Concentrate out all of the linear in utility parameters (ξ, δ, β) so that we only search over Σ. When $\dim(\Sigma) = K$ is small (few dimensions of unobserved heterogeneity) this is a big advantage. For $K \leq 3$ this is my preferred approach.
  - When T (number of markets/periods) is large then you can exploit solving in parallel for δ market by market.
- Disadvantages
  - Small numerical errors in contraction can be amplified in the outer loop, $\rightarrow$ tolerance needs to be very tight.
  - Errors in numerical integration can also be amplified in the outer loop $\rightarrow$ must use a large number of draws/nodes.
  - Hardest part is working out the Jacobian via IFT.

$$D\delta_{.t} = \begin{pmatrix} \frac{\partial \delta_{1t}}{\partial \theta_{21}} & \cdots & \frac{\partial \delta_{1t}}{\partial \theta_{2L}} \\ \vdots & \ddots & \vdots \\ \frac{\partial \delta_{Jt}}{\partial \theta_{21}} & \cdots & \frac{\partial \delta_{Jt}}{\partial \theta_{2L}} \end{pmatrix} = - \begin{pmatrix} \frac{\partial s_{1t}}{\partial \delta_{1t}} & \cdots & \frac{\partial s_{1t}}{\partial \delta_{Jt}} \\ \vdots & \ddots & \vdots \\ \frac{\partial s_{Jt}}{\partial \delta_{1t}} & \cdots & \frac{\partial s_{Jt}}{\partial \delta_{Jt}} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial s_{1t}}{\partial \theta_{21}} & \cdots & \frac{\partial s_{1t}}{\partial \theta_{2L}} \\ \vdots & \ddots & \vdots \\ \frac{\partial s_{Jt}}{\partial \theta_{21}} & \cdots & \frac{\partial s_{Jt}}{\partial \theta_{2L}} \end{pmatrix},$$

BLP: MPEC Advantages/Disadvantages

- Advantages
  - Problem scales better in dim(Σ).
  - Because all constraints hold at the optimum only: less impact of numerical error in tolerance or integration.
  - Derivatives are less complicated than $\frac{\partial \delta}{\partial \theta}$ (no IFT).
- Disadvantages
  - We are no longer concentrating out parameters, so there are a lot more of them! Storing the (Hessian) matrix of second derivatives can be difficult on memory.
  - We have to find the derivatives of the shares with respect to all of the parameters β, ξ, θ. (The other derivatives are pretty easy).
  - Parallelizing the derivatives is trickier than NFP case.

BLP Extensions: Demographics

- It is helpful to allow for interactions with consumer demographics (such as income).

$$\alpha_{it} = \overline{\alpha} + \sigma_p \nu_i + \pi_p y_{it}$$

- A few ways to do this:
  - You could just use cross sectional variation in s_{jt} and $\overline{y}_t$ (mean or median income).
  - Better: Draw y_{it} from a geographic specific income distribution. Draw ν_i from a general distribution of unobserved heterogeneity.
- Ex: Nevo (2000) Cereal demand sampled individual level D_i from geographic specific CPS data
- Joint distribution of income, income-squared, age, child at home.

$$\beta_i = \overline{\beta} + \Pi D_i + \sigma \nu_i$$

- with enough observations on the same product it is possible to include fixed effects

$$\delta_{jt}(\Sigma) = x_{jt}\beta - \alpha p_{jt} + \underbrace{\xi_{jt}}_{\xi_j + \xi_t + \Delta\xi_{jt}}$$

- What does ξ_j mean in this context?
- What would ξ_t mean in this context?
- $\Delta\xi_{jt}$ is now the structural error term, this changes our identification strategy a little.

Suppose we had additional data on behavior of individuals (in addition to aggregate market).

- Examples:
  - For some customers have answer to “Which car would you have purchased if the car you bought was not available?”
  - Demographic data on purchasers of a single brand.
  - Full individual demographic and choice data.

Extensions: Micro Data (Petrin 2002), (microBLP 2004)

- Previously we had moment conditions from orthogonality of structural error (ξ) and (X, Z) in order to form our GMM objective.

$$E[\xi_{jt}|x_{jt}, z_{jt}] = 0 \rightarrow E[\xi'[Z\ X]] = 0$$

- We can incorporate additional information using “micro-moments” or additional moment conditions to match the micro data.
  - $Pr($ i buys j $|y_i \in [0, \$20K]) = c_1$
  - $Cov(d_i, s_{ijt}) = c_2$
  - Construct an additional error term ζ_1, ζ_2 and interact that with instruments to form additional moment conditions.
  - Econometrics get tricky when we have a different number of observations for $E[\zeta'[XZ]] = 0$ and $E[\xi'[XZ]] = 0$.
  - May not be able to get covariance of moments taken over different sets of observations!

Alternative: Vertical Model (Bresnahan 1987)

- Imagine everyone agreed on the quality of the products offered for sale.
- The only thing people disagree on is willingness to pay for quality

$$U_{ij} = \overline{u} + \delta_j - \alpha_i p_j$$

- How do we estimate?
  - Sort goods from $p_1 < p_2 < p_3 \ldots < p_J$.
    It must be that $\delta_1 < \delta_2 < \ldots < \delta_J$. Why?
  - Normalize o.g. to 0 so that $0 > \delta_1 - \alpha_i p_1$ or $\alpha_i > \delta_1 / p_1$.
  - $s_0 = F(\infty) - F(\frac{\delta_1}{p_1}) = 1 - F(\frac{\delta_1}{p_1})$ where $F(\cdot)$ is CDF of α_i.
  - In general choose j IFF:

$$\frac{\delta_{j+1} - \delta_j}{p_{j+1} - p_j} < \alpha_i < \frac{\delta_j - \delta_{j-1}}{p_j - p_{j-1}}$$

$$s_j = F\left(\frac{\delta_{j+1} - \delta_j}{p_{j+1} - p_j}\right) - F\left(\frac{\delta_j - \delta_{j-1}}{p_j - p_{j-1}}\right)$$

Estimation

- Choose parameters θ of $F(\cdot)$ in order to best match s_j.
  - Can do MLE $\arg\max_\theta \sum_j \tilde{s}_j \log s_j(\theta)$.
  - Can do least squares $\sum_j (\tilde{s}_j - s_j(\theta))^2$.
  - Can do IV/GMM if I have an instrument for price. $\delta_j = x_j \beta + \xi_j$.
  - Extremely easy when $F \sim \exp(\lambda)$.
- What about elasticities?
  - When I change the price of j it can only affect (s_{j-1}, s_j, s_{j+1}).
  - We have set all of the other cross-price elasticities to be zero.
  - If a luxury car and a truck have similar prices, this can create strange substitution patterns.

Pure Characteristics Model: Berry Pakes (2001/2007)

$$u_{ij} = \delta_j + \sum_k \nu_{ik} x_{jk} + \xi_j + \underbrace{\sigma_i \epsilon_{ij}}_{\to 0}$$

- Can think of this like random coefficients model where we take the variance of ϵ to zero.
- Can think of this a vertical model, with vertical tastes over several characteristics.
  - PCs: everyone prefers more Mhz, more RAM, and more storage but differ in WTP.
  - Possible that there is no PC specific ϵ.
- Advantages
  - Logit error means there is always some substitution to all other goods.
  - Reality may be you only compete with a small number of competitors.
  - Allows for crowding in the product space.
- Disadvantage: no closed form for s_j, so estimation is extremely difficult.
- Minjae Song (Homotopy) and Che-Lin Su (MPCC) have made progress using two different approaches.

Adding Supply

Supply

- Economic theory gives us some additional powerful restrictions.
- We may want to impose $MR = MC$.
- Alternatively, we can ask – what is a good instrument for demand? something from another equation (ie: supply).

Some setup

We can break up the parameter space into three parts:

- θ_1: linear exogenous demand parameters,
- θ_2: parameters including price and random coefficients (endogenous / nonlinear)
- θ_3: linear exogenous supply parameters.

Supply Side

Consider the multi-product Bertrand FOCs:

$$\begin{aligned}\arg\max_{p\in\mathcal{J}_f} \pi_f(\mathbf{p}) &= \sum_{j\in\mathcal{J}_f}(p_j-c_j)\cdot s_j(\mathbf{p})+\kappa_{fg}\sum_{k\in\mathcal{J}_g}(p_k-c_k)\cdot s_k(\mathbf{p})\\ 0 &= s_j(\mathbf{p})+\sum_{k\in\mathcal{J}_f}(p_k-c_k)\frac{\partial s_k}{\partial p_j}(\mathbf{p})\end{aligned}$$

It is helpful to define the ownership matrix $\Omega_{(j,k)}(\mathbf{p})=-\frac{\partial s_j}{\partial p_k}(\mathbf{p})$:

$$A(\kappa)_{(j,k)}=\left\{\begin{array}{ll}1 & \text{for } (j,k)\in\mathcal{J}_f \text{ for any } f\\ 0 & \text{o.w}\end{array}\right\}$$

We can re-write the FOC in matrix form where $\odot$ denotes Hadamard product (element-wise):

$$\begin{aligned}s(\mathbf{p}) &= (A\odot\Omega(\mathbf{p}))\cdot(\mathbf{p}-\mathbf{mc}),\\ \mathbf{mc} &= \mathbf{p}-\underbrace{(A\odot\Omega(\mathbf{p}))^{-1}s(\mathbf{p})}_{\eta(\mathbf{p},\mathbf{s},\theta_2)}.\end{aligned}$$

Recover implied markups/ marginal costs, and assume a functional form for $mc_{jt}(x_{jt}, w_{jt})$.

$$\begin{aligned}\widehat{\mathbf{mc}}(\theta) &= \mathbf{p} - \Omega(\mathbf{p},\theta)^{-1} q(\mathbf{p},\theta)\\ f(mc_{jt}) &= [x_{jt}\,, w_{jt}]\gamma + \omega_{jt}\end{aligned}$$

Which we can solve for ω_{jt}:

$$\omega_{jt} = f(\mathbf{p} - \Omega(\mathbf{p},\theta)^{-1} q(\mathbf{p},\theta)) - x_{jt}\gamma_1 - w_{jt}\gamma_2$$

- $f(\cdot)$ is usually $\log(\cdot)$ or identity.
- I can use this to form additional moments: $E[\omega'_{jt} Z^s_{jt}] = 0.$

Simultaneous Supply and Demand

(a) For each market t: solve $\mathcal{S}_{jt} = s_{jt}(\delta_{\cdot t}, \theta_2)$ for $\widehat{\delta}_{\cdot t}(\theta_2)$.

(b) For each market t: use $\widehat{\delta}_{\cdot t}(\theta_2)$ to construct $\eta_{\cdot t}(\mathbf{q_t}, \mathbf{p_t}, \widehat{\delta}_{\cdot t}(\theta_2), \theta_2)$

(c) For each market t: Recover $\widehat{mc}_{jt}(\widehat{\delta}_{\cdot t}(\theta_2), \theta_2) = p_{jt} - \eta_{jt}(\widehat{\delta}_{\cdot t}(\theta_2), \theta_2)$

(d) Stack up $\widehat{\delta}_{\cdot t}(\theta_2)$ and $\widehat{mc}_{jt}(\widehat{\delta}_{\cdot t}(\theta_2), \theta_2)$ and use linear IV-GMM to recover $[\widehat{\theta}_1(\theta_2), \widehat{\theta}_3(\theta_2)]$ following the recipe in Appendix

(e) Construct the residuals:

$$
\begin{aligned}
\widehat{\xi}_{jt}(\theta_2) &= \widehat{\delta}_{jt}(\theta_2) - x_{jt}\widehat{\beta}(\theta_2) + \alpha p_{jt} \\
\widehat{\omega}_{jt}(\theta_2) &= \widehat{mc}_{jt}(\theta_2) - [x_{jt}\ w_{jt}]\,\widehat{\gamma}(\theta_2)
\end{aligned}
$$

(f) Construct sample moments

$$
\begin{aligned}
g_n^D(\theta_2) &= \frac{1}{N}\sum_{jt} Z_{jt}^{D\prime}\,\widehat{\xi}_{jt}(\theta_2) \\
g_n^S(\theta_2) &= \frac{1}{N}\sum_{jt} Z_{jt}^{S\prime}\,\widehat{\omega}_{jt}(\theta_2)
\end{aligned}
$$

(g) Construct GMM objective $Q_n(\theta_2) = \begin{bmatrix} g_n^d(\theta_2) \\ g_n^s(\theta_2) \end{bmatrix}' W \begin{bmatrix} g_n^d(\theta_2) \\ g_n^s(\theta_2) \end{bmatrix}$

Some different definitions:

$$\begin{aligned} y_{jt}^{D} &:= \hat{\delta}_{jt}(\theta_2) + \alpha p_{jt} &&= (x_{jt}\ v_{jt})'\beta + \xi_t =: x_{jt}^{D\prime}\beta + \xi_{jt} \\ y_{jt}^{S} &:= \widehat{mc}_{jt}(\theta_2) &&= (x_{jt}\ w_{jt})'\gamma + \omega_t =: x_{jt}^{S\prime}\gamma + \omega_{jt} \end{aligned} \tag{3}$$

Stacking the system across observations yields:[1]

$$\underbrace{\begin{bmatrix} y_D \\ y_S \end{bmatrix}}_{2N\times 1} = \underbrace{\begin{bmatrix} X_D & 0 \\ 0 & X_S \end{bmatrix}}_{2N\times(K_1+K_3)} \underbrace{\begin{bmatrix} \beta \\ \gamma \end{bmatrix}}_{(K_1+K_3)\times 1} + \underbrace{\begin{bmatrix} \xi \\ \omega \end{bmatrix}}_{2N\times 1} \tag{4}$$

[1] Note: we cannot perform independent regressions unless we are willing to assume that $Cov(\xi_{jt}, \omega_{jt}) = 0$.

Instruments and Identification

- Once we have $\delta_{jt}(\theta)$ identification of linear parameters is pretty straightforward

$$\delta_{jt}(\theta) = x_{jt}\beta - \alpha p_{jt} + \xi_j + \xi_t + \Delta\xi_{jt}$$

- This is either basic linear IV or panel linear IV.
- How are σ taste parameters identified?
  - Consider increasing the price of j and measuring substitution to other products k, k' etc.
  - If sales of k increase with p_j and $(x_j^{(1)}, x_k^{(1)})$ are similar then we increase the σ that corresponds to $x^{(1)}$.
  - Price is the most obvious to vary, but sometimes this works for other characteristics (like distance).
  - Alternative: vary the set of products available to consumers by adding or removing an option.

- Recall the nested logit, where there are two separate endogeneity problems
  - Price: this is the familiar one!
  - Nonlinear characteristics σ this is the other one.
- We are doing nonlinear GMM: Start with $E[\xi_{jt}|x_{jt}, z_{jt}] = 0$ use $E[\xi' [ZX]] = 0$.
  - In practice this means that for valid instruments (x, z) any function $f(x, z)$ is also a valid instrument $E[\xi_{jt} f(x_{jt}, z_{jt})] = 0$.
  - We can use $x, x^2, x^3, \ldots$ or interactions $x \cdot z, x^2 \cdot z^2, \ldots$.
  - What is a reasonable choice of $f(\cdot)$?
  - Where does z come from?

Exclusion Restrictions

$$\begin{aligned}
\delta_{jt}(\mathcal{S}_t, \tilde{\theta}_2) &= [x_{jt}, v_{jt}]\beta - \alpha p_{jt} + \xi_{jt} \\
f(p_{jt} - \eta_{jt}(\theta_2, \mathbf{p}, \mathbf{s})) &= h(x_{jt}, w_{jt}; \theta_3) + \omega_{jt}
\end{aligned}$$

The first place to look for exclusion restrictions/instruments:

- Something in another equation!
- v_j shifts demand but not supply
- w_j shifts supply but not demand
- If it doesn't shift either is it really relevant?

The equilibrium markup is a function of everything! $\eta_{jt}(\mathbf{p}, \mathbf{s}, \xi_t, \omega_t, x_t, w_t, v_t, \theta_2)$:

- It is literally endogenous (depends on error terms)!
- But lots of potential instruments beyond excluded v_t or w_t.
- Also v_{-j} and w_{-j} and x_{-j}.
- Not p_{-j} or ξ_{-j}, etc.
- The idea is that these instruments shift the marginal revenue curve.
- What is a good choice of $f(x_{-j})$? etc.

- Common choices are average characteristics of other products in the same market $f(x_{-j,t})$. BLP instruments
  - Same firm $z_{1jt} = \overline{x}_{-j_f,t} = \frac{1}{|F_j|} \sum_{k \in \mathcal{F}_j} x_{kt} - \frac{1}{|F_j|} x_{jt}$.
  - Other firms $z_{2jt} = \overline{x}_{\cdot t} - \overline{x}_{-j_f,t} - \frac{1}{J} x_{jt}$.
  - Plus regressors $(1, x_{jt})$.
  - Plus higher order interactions
- Technically linearly independent for large (finite) J, but becoming highly correlated.
  - Can still exploit variation in number of products per market or number of products per firm.
- Correlated moments $\rightarrow$ "many instruments".
  - May be inclined to "fix" correlation in instrument matrix directly.

Armstrong (2016): Weak Instruments?

Consider the limit as $J \to \infty$

$$\frac{s_{jt}(\mathbf{p_t})}{\left|\frac{\partial s_{jt}(\mathbf{p_t})}{\partial p_{jt}}\right|} = \frac{1}{\alpha}\frac{1}{1-s_{jt}} \to \frac{1}{\alpha}$$

- Hard to use markup shifting instruments to instrument for a constant.
- How close to the constant do we get in practice?
- Average of x_{-j} seems like an especially poor choice. Why?
- Shows there may still be some power in: products per market, products per firm.
- Convergence to constant extends to mixed logits (see Gabaix and Laibson 2004).
- Suggests that you really need cost shifters.

- Also need instruments for the Σ or σ random coefficient parameters.
- Instead of average of other characteristics $f(x) = \frac{1}{J-1}\sum_{k \neq j} x_k$, can transform as distance to x_j.

$$d_{jt}^k = x_k - x_j$$

- And use this transformed to construct two kinds of IV (Squared distance, and count of local competitors)

$$\begin{aligned} DIV_1 = \quad & \textstyle\sum_{j \in F} d_{jt}^2, & \sum_{j \notin F} d_{jt}^2 \\ DIV_2 = \quad & \textstyle\sum_{j \in F} I[d_{jt} < c] & \sum_{j \notin F} I[d_{jt} < c] \end{aligned}$$

- They choose c to correspond to one standard deviation of x across markets.

Optimal Instruments

- Since any $f(x,z)$ satisfies our orthogonality condition, we can try to choose $f(x,z)$ as a basis to approximate optimal instruments.
- This is challenging in practice – and in fact suffers from a curse of dimensionality.
- This is frequently given as a rationale behind higher order x's.
- When the dimension of x is low – this may still be feasible. $(K \leq 3)$.

How to construct optimal instruments in form of Chamberlain (1987)

$$E\left[\frac{\partial \xi_{jt}}{\partial \theta}|X_t, w_{jt}\right] = \left[\beta, E\left[\frac{\partial \xi_{jt}}{\partial \alpha}|X_t, w_{jt}\right], E\left[\frac{\partial \xi_{jt}}{\partial \sigma}|X_t, w_{jt}\right]\right]$$

Some challenges:

1. p_{jt} depends on X_t, w_t, ξ_t in a highly nonlinear way (no explicit solution!).
2. $E[\frac{\partial \xi_{jt}}{\partial \sigma}|X_t, w_t] = E[[\frac{\partial \mathbf{s_t}}{\partial \boldsymbol{\delta}_\mathbf{t}}]^{-1}[\frac{\partial \mathbf{s_t}}{\partial \boldsymbol{\sigma}}]|X_t, w_t]$ (not conditioned on endogenous p!)

"Feasible" Recipe:

1. Fix $\hat{\theta} = (\hat{\alpha}, \hat{\beta}, \hat{\sigma})$ and draw ξ_t from empirical density
2. Solve fixed point equation for $\widehat{p_{jt}}$
3. Compute necessary Jacobian
4. Average over all values of ξ_t. (Lazy approach: use only $\xi = 0$).

Simplified Version: Reynaert Verboven (2014)

- Optimal instruments are easier to work out if $p = mc$.

$$c = p + \underbrace{\Delta^{-1} s}_{\to 0} = X\gamma_1 + W\gamma_2 + \omega$$

- Linear cost function means linear reduced-form price function.

$$\begin{aligned} E\left[\frac{\partial \xi_{jt}}{\partial \alpha} | z_t\right] &= E[p_{jt}|z_t] = x_{jt}\gamma_1 + w_{jt}\gamma_2 \\ E\left[\frac{\partial \omega_{jt}}{\partial \alpha} | z_t\right] &= 0, \quad E\left[\frac{\partial \omega_{jt}}{\partial \sigma} | z_t\right] = 0 \\ E\left[\frac{\partial \xi_{jt}}{\partial \sigma} | z_t\right] &= E\left[\frac{\partial \delta_{jt}}{\partial \sigma} | z_t\right] \end{aligned}$$

- If we are worried about endogenous oligopoly markups is this a reasonable idea?
- Turns out that the important piece tends to be shape of jacobian for σ_x.

Optimal Instruments: Reynaert Verboven (2014)

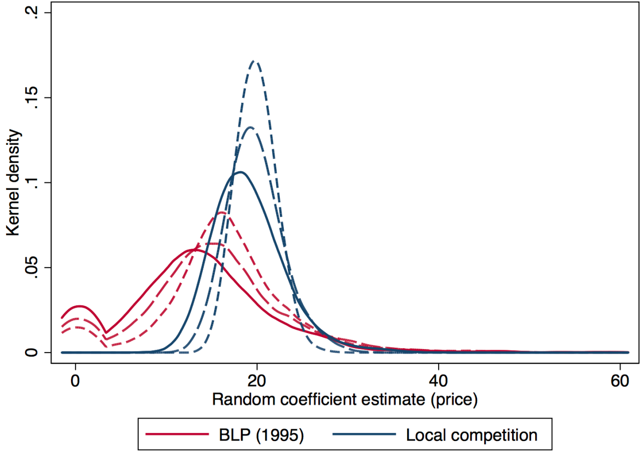
Table 2: Bias and Efficiency with Imperfect Competition

		Single Equation GMM								
			g_{jt}^1			g_{jt}^2			g_{jt}^3	
	True	Bias	St Err	RMSE	Bias	St Err	RMSE	Bias	St Err	RMSE
β^0	2	-0.127	0.899	0.907	-0.155	0.799	0.814	-0.070	0.514	0.519
β^1	2	-0.068	0.899	0.901	0.089	0.766	0.770	-0.001	0.398	0.398
α	-2	0.006	0.052	0.052	0.010	0.049	0.050	0.010	0.043	0.044
σ^1	1	-0.162	0.634	0.654	-0.147	0.537	0.556	-0.016	0.229	0.229
		Joint Equation GMM								
			g_{jt}^1			g_{jt}^2			g_{jt}^3	
	True	Bias	St Err	RMSE	Bias	St Err	RMSE	Bias	St Err	RMSE
β^0	2	-0.095	0.714	0.720	-0.103	0.677	0.685	0.005	0.459	0.459
β^1	2	0.089	0.669	0.675	0.098	0.621	0.628	-0.009	0.312	0.312
α	-2	0.001	0.047	0.047	0.002	0.046	0.046	-0.001	0.043	0.043
σ^1	1	-0.116	0.462	0.476	-0.110	0.418	0.432	0.003	0.133	0.133

Bias, standard errors (St Err) and root mean squared errors (RMSE) are computed from 1000 Monte Carlo replications. Estimates are based on the MPEC algorithm and Sparse Grid integration. The instruments g_{jt}^1, g_{jt}^2 and g_{jt}^3 are defined in section 2.4 and 2.5.

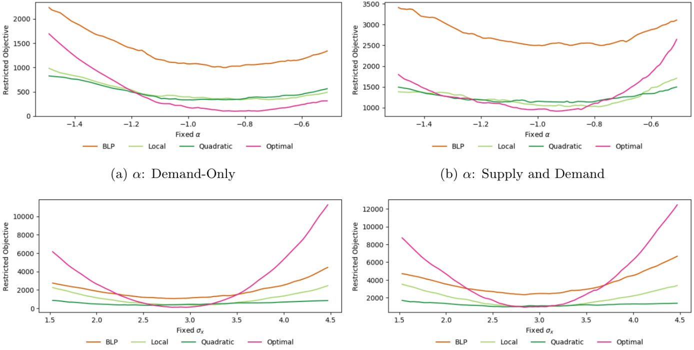
Differentiation Instruments: Gandhi Houde (2016)

Figure 4: Distribution of parameter estimates in small and large samples



IV Comparison: Conlon and Gortmaker (2019)

Figure 2: Profiled GMM Objective with Alternative IV (“Simple” simulation)



(a) α: Demand-Only

(b) α: Supply and Demand

(c) σ_x: Demand-Only

(d) σ_x: Supply and Demand

Each plot profiles the GMM objective $Q(\theta)$ with respect to a single parameter for our “Simple” simulation scenario and a single simulation. We fix either σ_x or α and re-optimize over other parameters and plot the restricted objective in each subplot. The top row profiles the objective over the price parameter α, while the bottom row profiles over the random coefficient σ_x. The left column uses moments from demand alone, while the right column uses both supply and demand moments.