Alternating Direction Implicit (ADI) Method

Introduction

The Alternating Direction Implicit (ADI) method is a numerical technique used to solve multi-dimensional partial differential equations (PDEs). It is particularly useful for solving two-dimensional linear elliptic and parabolic PDEs, such as the heat equation. The ADI method is an extension of the implicit finite difference method, designed to improve computational efficiency and stability.

Two-Dimensional Heat Equation

Consider the two-dimensional heat equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

We discretize both time and space. Let $u_{i,j}^n$ denote the value of u at time step n and grid point (x_i, y_j).

Discretization

- Spatial discretization:

$$x_i = i\Delta x, \quad y_j = j\Delta y$$

where $i, j = 0, 1, \ldots, N$.

- Time discretization:

$$t^n = n\Delta t$$

where $n = 0, 1, 2, \ldots$.

ADI Method Steps

1. **Initialization:** Initialize $u_{i,j}^0$ with the initial condition.
2. **First Half-Step (Implicit in x-Direction):**

   From $u_{i,j}^n$ to $u_{i,j}^{n+1/2}$:

$$\frac{u_{i,j}^{n+1/2} - u_{i,j}^n}{\Delta t/2} = \left.\frac{\partial^2 u}{\partial x^2}\right|_{i,j}^{n+1/2} + \left.\frac{\partial^2 u}{\partial y^2}\right|_{i,j}^{n}$$

This leads to:

$$\frac{u_{i,j}^{n+1/2} - u_{i,j}^{n}}{\Delta t/2} = \frac{u_{i+1,j}^{n+1/2} - 2u_{i,j}^{n+1/2} + u_{i-1,j}^{n+1/2}}{(\Delta x)^2} + \frac{u_{i,j+1}^{n} - 2u_{i,j}^{n} + u_{i,j-1}^{n}}{(\Delta y)^2}$$

Rearrange to get a tridiagonal system for $u_{i,j}^{n+1/2}$:

$$-\alpha u_{i-1,j}^{n+1/2} + (1+2\alpha)u_{i,j}^{n+1/2} - \alpha u_{i+1,j}^{n+1/2} = u_{i,j}^{n} + \beta(u_{i,j+1}^{n} - 2u_{i,j}^{n} + u_{i,j-1}^{n})$$

where $\alpha = \frac{\Delta t}{2(\Delta x)^2}$ and $\beta = \frac{\Delta t}{2(\Delta y)^2}$.

3. **Second Half-Step (Implicit in y-Direction):**

   From $u_{i,j}^{n+1/2}$ to $u_{i,j}^{n+1}$:

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n+1/2}}{\Delta t/2} = \left.\frac{\partial^2 u}{\partial x^2}\right|_{i,j}^{n+1/2} + \left.\frac{\partial^2 u}{\partial y^2}\right|_{i,j}^{n+1}$$

   This leads to:

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n+1/2}}{\Delta t/2} = \frac{u_{i+1,j}^{n+1/2} - 2u_{i,j}^{n+1/2} + u_{i-1,j}^{n+1/2}}{(\Delta x)^2} + \frac{u_{i,j+1}^{n+1} - 2u_{i,j}^{n+1} + u_{i,j-1}^{n+1}}{(\Delta y)^2}$$

   Rearrange to get a tridiagonal system for $u_{i,j}^{n+1}$:

$$-\beta u_{i,j-1}^{n+1} + (1+2\beta)u_{i,j}^{n+1} - \beta u_{i,j+1}^{n+1} = u_{i,j}^{n+1/2} + \alpha(u_{i+1,j}^{n+1/2} - 2u_{i,j}^{n+1/2} + u_{i-1,j}^{n+1/2})$$

Example

Let's solve the heat equation on a domain $0 \leq x, y \leq 1$ with boundary conditions $u = 0$ on all boundaries and initial condition $u(x, y, 0) = \sin(\pi x)\sin(\pi y)$.

1. **Discretize the domain:**

   Let $\Delta x = \Delta y = h$ and define the grid points $x_i = ih$ and $y_j = jh$ where $i, j = 0, 1, \ldots, N$ and $h = \frac{1}{N}$.

2. **Initial condition:**

   Set $u_{i,j}^{0} = \sin(\pi x_i)\sin(\pi y_j)$ for $i, j = 1, \ldots, N-1$.

3. **Apply ADI method:**

   - **First half-step (implicit in x-direction):**

     For each j from 1 to $N-1$, solve the tridiagonal system for $u_{i,j}^{n+1/2}$:

$$-\alpha u_{i-1,j}^{n+1/2} + (1+2\alpha)u_{i,j}^{n+1/2} - \alpha u_{i+1,j}^{n+1/2} = u_{i,j}^{n} + \beta(u_{i,j+1}^{n} - 2u_{i,j}^{n} + u_{i,j-1}^{n})$$

- **Second half-step (implicit in y-direction):**
  For each i from 1 to $N-1$, solve the tridiagonal system for $u_{i,j}^{n+1}$:

$$-\beta u_{i,j-1}^{n+1} + (1+2\beta) u_{i,j}^{n+1} - \beta u_{i,j+1}^{n+1} = u_{i,j}^{n+1/2} + \alpha (u_{i+1,j}^{n+1/2} - 2u_{i,j}^{n+1/2} + u_{i-1,j}^{n+1/2})$$

4. **Iterate until the final time step:**

   Repeat the above steps until the desired time is reached.

Summary

- **First half-step:** Implicit in x-direction, solving a tridiagonal system for $u_{i,j}^{n+1/2}$ using values from time step n. - **Second half-step:** Implicit in y-direction, solving a tridiagonal system for $u_{i,j}^{n+1}$ using values from the intermediate step $n+1/2$.

By alternating the direction of implicit treatment, the ADI method reduces the complexity of solving a multi-dimensional PDE, making it more efficient and stable than solving the full implicit problem directly.