Interest-Rate Models

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Abstract and Figures

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UK Government bond yields at t…

Interest-Rate Models

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1 Introduction

In this article we will describe some of the main developments in interest-rate modelling since Black & Scholes' (1973) and Merton's (1973) original articles on the pricing of equity derivatives. In particular, we will focus on continuous-time, arbitrage-free models for the full *term structure of interest rates*. Other models which model a limited number of key interest rates or which operate in discrete time (for example, the Wilkie (1995) model) will be considered elsewhere. Additionally, more detailed accounts of *affine term-structure models* and *market models* are given elsewhere in this volume.

Here we will describe the basic principles of arbitrage-free pricing and cover various frameworks for modelling: short-rate models (for example, Vasicek, Cox-Ingersoll-Ross, Hull-White); the Heath-Jarrow-Morton approach for modelling the forward-rate curve; pricing using state-price deflators including the Flesaker-

The article works through various approaches and models in a historical sequence. Partly this is for history's sake, but, more importantly, the older models are simpler and easier to understand. This will allow us to build up gradually to the more up to date, but more complex, modelling techniques.

1.1 Interest rates and prices

One of the first problems one encounters in this field is the variety of different ways of presenting information about the term structure.[1] The expression "yield curve" is often used in a sloppy way with the result that it often means different things to different people: how is the yield defined; is the rate annualised or semi-annual or continuously compounding; does it refer to the yield on a coupon bond or a zero-coupon bond? To avoid further confusion then, we will give some precise definitions.

- We will consider here only default-free government debt. Bonds which involve a degree of credit risk will be dealt with in a separate article.
- The basic building blocks from the mathematical point of view are zero-coupon bonds.[2] In its standard form such a contract promises to pay £1 on a fixed date in the future. Thus we use the notation $D(t,T)$ to represent the value at time t of £1 at time T.[3] The bond price process has the boundary conditions $D(T,T)=1$ and $D(t,T)>0$ for all $t \leq T$.
- A *fixed-income* contract equates to a collection of zero-coupon bonds. For example, suppose it is currently time 0 and the contract promises to pay the fixed amounts $c_1, c_2, \ldots, c_n$ at the fixed times $t_1, t_2, \ldots, t_n$. If we assume that there are no taxes[4] then the fair or market price for this contract at time 0 is
$$P = \sum_{i=1}^{n} c_i D(0, t_i).$$
(This identity follows from a simple, static hedging strategy which involves replicating the coupon bond payments with the payments arising from a portfolio of zero-coupon bonds.) The *gross redemption yield* (or *yield-to-maturity*) is a measure of the average interest rate earned over the term of the contract given the current price P. The gross redemption yield is the

[1] We implicitly assume that readers have gone beyond the assumption that the yield curve is flat!

[2] From the practical point of view it is sensible to start with frequently-traded coupon bonds, the prices of which can be used to back out zero-coupon-bond prices. Zero-coupon bonds do exist in several countries, but they are often relatively illiquid making their quoted prices out of date and unreliable.

[3] Here the $D(\cdot)$ notation uses D for discount bond or discounted price. Common notation used elsewhere is $P(t,T)$ for price and $B(t,T)$ for bond price. Additionally, one or other of the t or T can be found as a subscript to distinguish between the nature of the two variables: t is the dynamic variable, while T is usually static.

[4] Alternatively we can assume that income and capital gains are taxed on the same mark-to-market basis.

$$P = \hat{P}(\delta) = \sum_{i=1}^{n} c_i e^{-\delta t_i}.$$

If the c_i are all positive then the solution to this equation is unique.

δ as found above is a continuously compunding rate of interest. However, the gross redemption yield is usually quoted as an annual (that is, we quote $i = \exp(\delta) - 1$) or semi-annual rate ($i^{(2)} = 2[\exp(\delta/2) - 1]$) depending on the frequency of contracted payments.[5]

- The *spot-rate curve* at time t refers to the set of gross redemption yields on zero-coupon bonds. The spot rate at time t for a zero-coupon bond maturing at time T is denoted by $R(t,T)$ which is the solution to $D(t,T) = \exp[-R(t,T)(T-t)]$: that is,

$$R(t,T) = \frac{-1}{(T-t)} \log D(t,T).$$

- The instantaneous, *risk-free* rate of interest is the very short-maturity spot rate

$$r(t) = \lim_{T \to t} R(t,T).$$

This gives us the *money-market account* or *cash account* $C(t)$ which invests only at this risk-free rate. Thus $C(t)$ has the stochastic differential equation (SDE)

$$dC(t) = r(t)C(t)dt$$

with solution

$$C(t) = C(0) \exp\left[\int_0^t r(u)du\right].$$

- Spot rates refer to the *on-the-spot* purchase of a zero-coupon bond. In contrast, *forward rates* give us rates of interest which refer to a future period of investment. Standard contracts will refer to both the future delivery and maturity dates. Thus $F(t, T_1, T_2)$ is used to denote the (continuously compounding) rate which will apply between times T_1 and T_2 as determined by a contract entered into at time t. The standard contract also requires that the value of the contract at time t is zero. Thus, a simple no-arbitrage

[5]Thus $\exp(\delta) \equiv 1 + i \equiv \left(1 + \frac{1}{2} i^{(2)}\right)^2$.

3

... Below we shall briefly review the properties of that model. See Luenberger (1998) and especially Cairns (2004) for more detail and further references. What we eventually aim at, is to use the solution to the problem described in the previous section as input for the determination of the short rates in the framework of the recombinant binomial model. ...
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... A result on its characteristic function can be found in Section 3 of [12]. Alternatively, a result on the joint conditional MGF of (∫ ,) can be found in Theorem 4.8 of [9] . We state here an simplified result that suits our purpose: ...

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... In addition, the hedging results under a variable interest rate environment simulated from the discretised Cox-Ingersoll-Ross (CIR) model are also provided in the table. This interest rate model has the advantages of allowing for mean reversion and also avoiding negative interest rates (Cairns 2004 While it is difficult to predict how long the current low interest rate levels would continue, it can be envisaged that higher interest rates with more fluctuations would reduce the hedge effectiveness to even a greater extent. Accordingly, interest rate swaps and government bonds may be added to the hedging scheme to mitigate the impact of interest rate risk (Tsai et al. 2011). ...

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April 2003 · Mathematical Finance

Andrew Cairns

In this paper we propose a new family of term-structure models based on the Flesaker and Hughston positive-interest framework [B. Flesaker and L. P. Hughston, Positive interest, Risk 9, No. 1, 46–49 (1996)]. The models are Markov and time homogeneous, with correlated Ornstein-Uhlenbeck processes as state variables. We provide a theoretical analysis of the one-factor model and a thorough empirical ... [Show full abstract]

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January 2003

Keiichi Tanaka

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