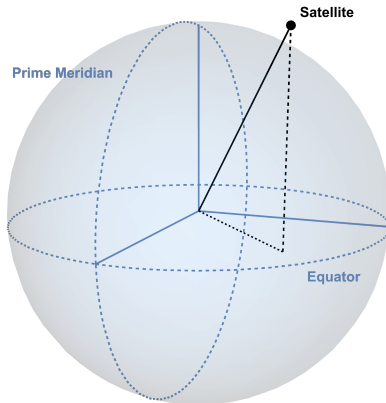# Part 8: Solving
Building a GPS receiver from scratch
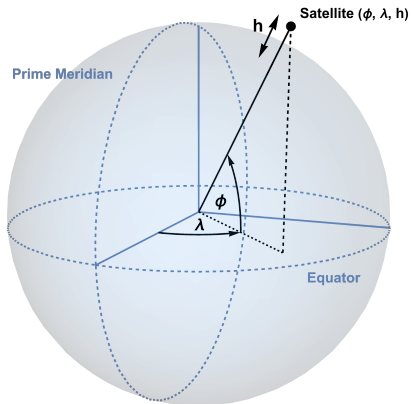
Chris Doble

Topics

1. The pseudorange equation

2. Satellite location and transit time

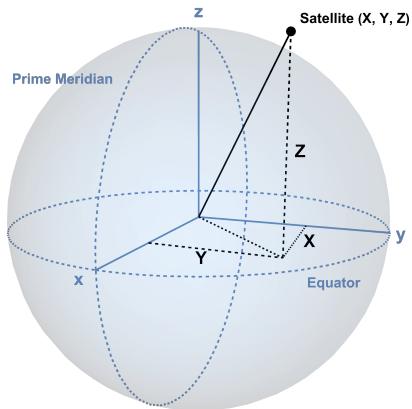3. System of equations
   - Definition
   - Solving
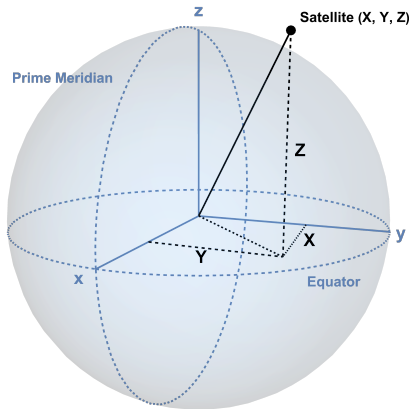
Coordinate system

Transit time T

Transit time $T \Rightarrow$ distance cT
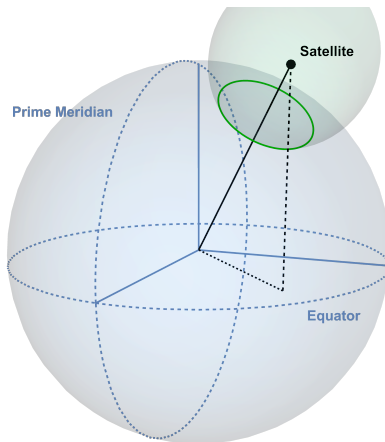
Transit time $T \Rightarrow$ distance cT

The pseudorange equation

$$\sqrt{(X - x)^2 + (Y - y)^2 + (Z - z)^2} = cT$$

where (x, y, z) is our unknown location.

$$\sqrt{(X - x)^2 + (Y - y)^2 + (Z - z)^2} - cT = 0$$

where (x, y, z) is our unknown location.

The pseudorange equation

$$\sqrt{(X - x)^2 + (Y - y)^2 + (Z - z)^2} - c(T - t) = 0$$

where (x, y, z) is our unknown location and t is our clock bias.

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where (x, y, z) is our unknown location and t is our clock bias.

Topics

Transmission time

Transmission time

The user shall correct the time received from the SV with the equation (in seconds)

$$t \quad = \quad t_{sv} - \Delta t_{sv} \qquad (1)$$

where

$$t \quad = \quad \text{GPS system time (seconds)},$$

$$t_{sv} \quad = \quad \text{effective SV PRN code phase time at message transmission time (seconds)},$$

$$\Delta t_{sv} \quad = \quad \text{SV PRN code phase time offset (seconds)}.$$

The SV PRN code phase offset is given by

$$\Delta t_{sv} \quad = \quad a_{f0} + a_{f1}(t - t_{oc}) + a_{f2}(t - t_{oc})^2 + \Delta t_r \quad (2)$$

$$\vdots$$

Thus, the user who utilizes the L1 C/A signal only shall modify the code phase offset in accordance with paragraph 20.3.3.3.1 with the equation

$$(\Delta t_{SV})_{L1C/A} \quad = \quad \Delta t_{SV} \; - \; T_{GD}$$

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$$t_{sv} = \text{TOW} \times 6\,\text{s}$$

$$t_{sv} = \text{TOW} \times 6\,\text{s} + \text{PRN count} \times 1\,\text{ms}$$

Reception time

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3. Calculate the number of seconds since GPS started operating
4. Calculate the remainder when divided by the number of seconds in a GPS week

Location

$\mu = 3.986005 \times 10^{14}$ meters^3/sec^2

$\dot{\Omega}_e = 7.2921151467 \times 10^{-5}$ rad/sec

$A = \left(\sqrt{A}\,\right)^2$

$n_0 = \sqrt{\dfrac{\mu}{A^3}}$

$t_k = t - t_{oe}$*

$n = n_0 + \Delta n$

$M_k = M_0 + n t_k$

$E_0 = M_k$

$E_j = E_{j-1} + \dfrac{M_k - E_{j-1} + e \sin E_{j-1}}{1 - e \cos E_{j-1}}$

$E_k = E_j$

$v_k = 2 \tan^{-1} \left(\sqrt{\dfrac{1+e}{1-e}} \tan \dfrac{E_k}{2} \right)$

$\Phi_k = v_k + \omega$

$\delta u_k = c_{us}\sin 2\Phi_k + c_{uc}\cos 2\Phi_k$
$\delta r_k = c_{rs}\sin 2\Phi_k + c_{rc}\cos 2\Phi_k$
$\delta i_k = c_{is}\sin 2\Phi_k + c_{ic}\cos 2\Phi_k$

$u_k = \Phi_k + \delta u_k$

$r_k = A(1 - e\cos E_k) + \delta r_k$

$i_k = i_0 + \delta i_k + (IDOT)\, t_k$

$x_k' = r_k \cos u_k$
$y_k' = r_k \sin u_k$

$\Omega_k = \Omega_0 + (\dot{\Omega} - \dot{\Omega}_e)\, t_k - \dot{\Omega}_e\, t_{oe}$

$x_k = x_k'\cos\Omega_k - y_k'\cos i_k \sin\Omega_k$
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Topics

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2. Satellite location and transit time

3. System of equations
   - Definition
   - Solving

Topics

Definition

$$\sqrt{(X_1 - x)^2 + (Y_1 - y)^2 + (Z_1 - z)^2} - c(T_1 - t) = 0$$

$$\sqrt{(X_2 - x)^2 + (Y_2 - y)^2 + (Z_2 - z)^2} - c(T_2 - t) = 0$$
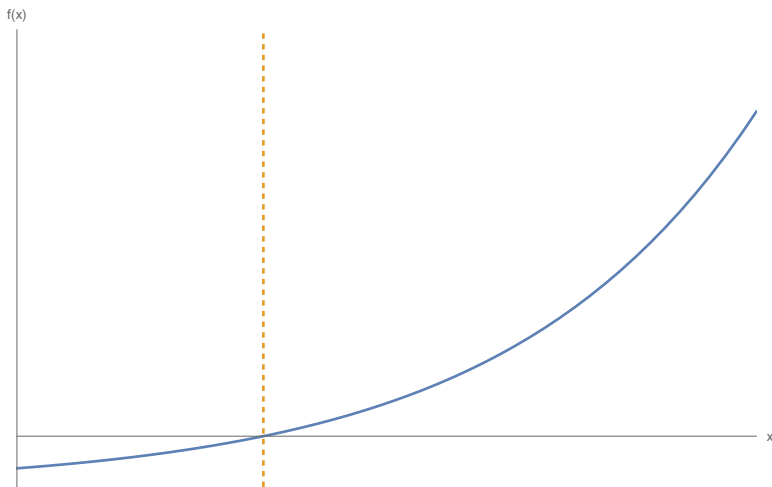
$$\sqrt{(X_3 - x)^2 + (Y_3 - y)^2 + (Z_3 - z)^2} - c(T_3 - t) = 0$$

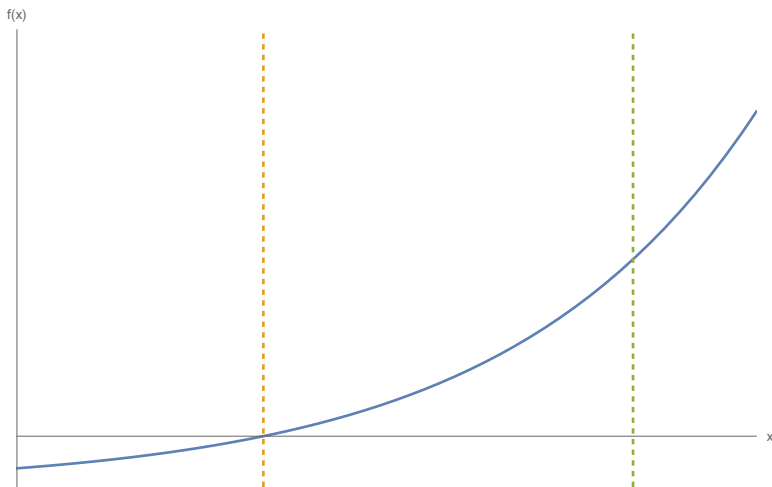$$\sqrt{(X_4 - x)^2 + (Y_4 - y)^2 + (Z_4 - z)^2} - c(T_4 - t) = 0$$
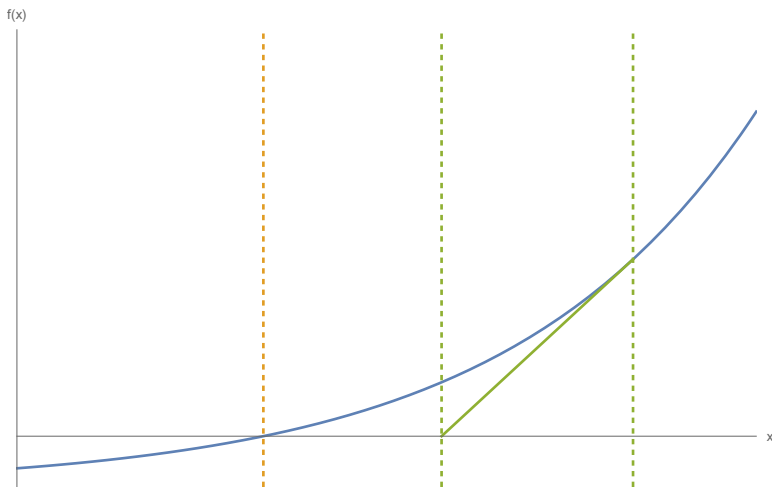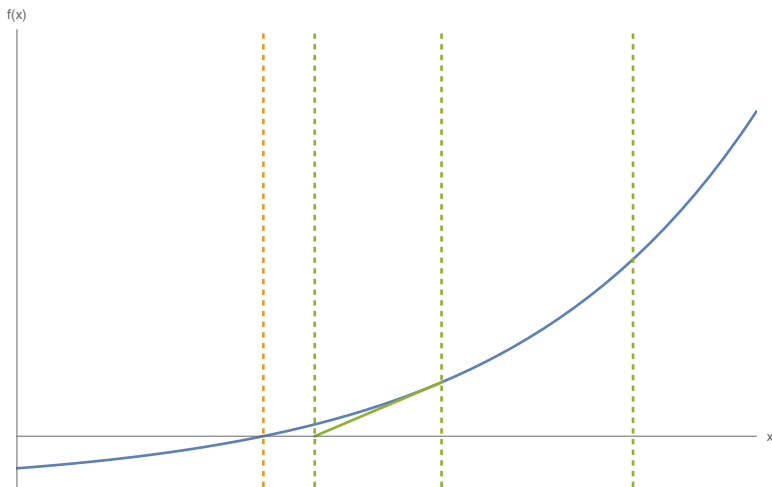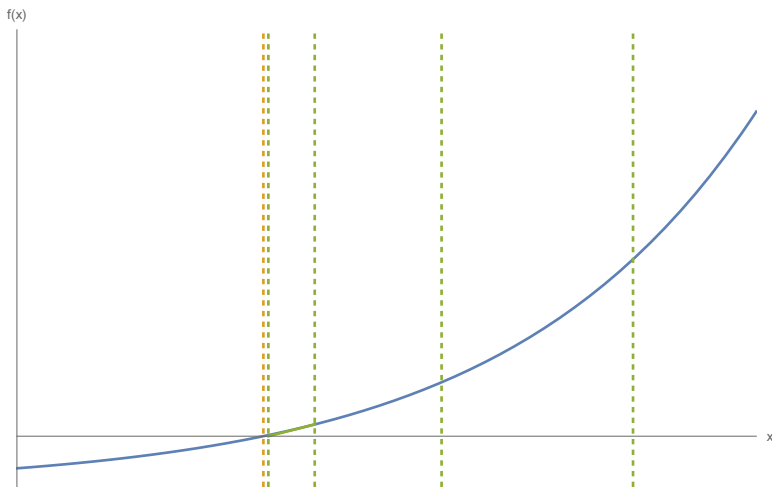
Topics

The Newton-Raphson method

The Newton-Raphson method

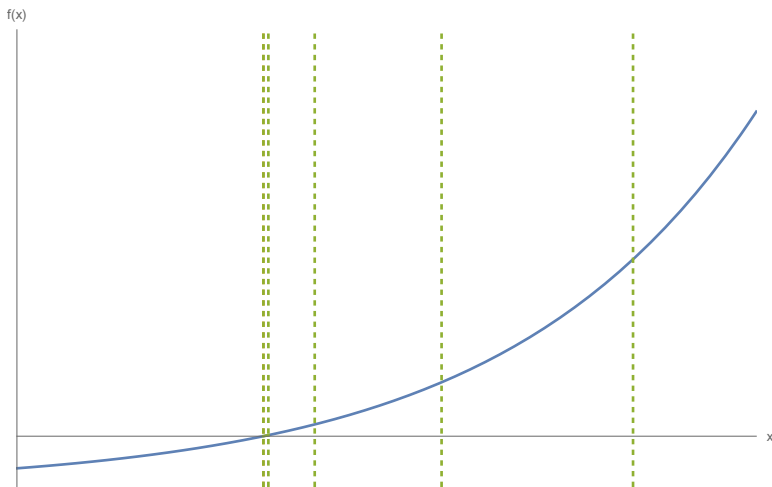# The Newton-Raphson method

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The Gauss-Newton algorithm

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- Convert from ECEF to geodetic coordinates using Bowring's method

Recap

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- We need a system of at least four pseudorange equations to solve for x, y, z, and t
- The Gauss-Newton algorithm is used to estimate x, y, z, and t

Thank you!

`https://github.com/chrisdoble/gps-receiver`